Applications of Linear Functions

1. Supply, demand and Equilibrium

Linear function

$y = mx + b$

output input

Supply Functions

A linear supply function is written as:

output

$$Q_s = -c + dp$$

input

where:

- Q_s = quantity supplied
- p = price
- c = fixed quantity adjustment
- d = increase in supply for each unit increase in price

constant

Q_s is an increasing function of p

Demand Functions

A linear **demand** function is written as:

output

$$Q_d = a - bp$$

input

where:

- Q_d = quantity demanded
- p = price
- a = maximum quantity demanded when the price is zero
- b = decrease in demand for each unit increase in price $(b > 0)$

Q_d is a decreasing function of p

Equilibrium

Equilibrium occurs when **quantity demanded equals quantity supplied**:

$$\cancel{q_d = q_s} \qquad Q_d = Q_s$$

At equilibrium, there is no surplus or shortage in the market.

Problem:

(Demand) You own a small convenience store. You can sell 90 packs of gum per week if they are marked at $2 each, but only 30 each week if they are marked at $3/pack.

(Supply) Your gum supplier is prepared to sell you 20 packs each week if they are marked at $1/pack and 100 each week if they are marked at $3/pack.

a. Write the demand and supply functions.

Demand

	P	Qd	
(P_1)	2	90	(Q_1)
(P_2)	3	30	(Q_2)

Formula : The equation of the line passing through (P_1, Q_1) and (P_2, Q_2) is

$$Q_d = \frac{Q_2 - Q_1}{P_2 - P_1} \cdot (P - P_1) + Q_1$$

$$\Rightarrow \quad Q_d = \frac{30 - 90}{3 - 2} \cdot (P - 2) + 90$$

$$\Rightarrow \quad Q_d = -60 \, (P - 2) + 90$$

$$\Rightarrow \quad Q_d = -60P + 120 + 90$$

$$\Rightarrow \boxed{Q_d = -60P + 210} \quad (\text{Demand function})$$

Supply :

(Supply) Your gum supplier is prepared to sell you 20 packs each week if they are marked at $1/pack and 100 each week if they are marked at $3/pack.

	P	Q_s	
(P_1)	1	20	(Q_1)
(P_2)	3	100	(Q_2)

using the same formula :

$$Q_s = \frac{Q_2 - Q_1}{P_2 - P_1} \cdot (P - P_1) + Q_1$$

$$\Rightarrow Q_s = \frac{100 - 20}{3 - 1} \cdot (P - 1) + 20$$

$$Q_s = 40 (P - 1) + 20$$

$$Q_s = 40P - 40 + 20$$

$$\boxed{Q_s = 40P - 20} \quad (\text{supply function})$$

b. Find the equilibrium point. For supply to equal demand, the packs of gum must be priced at how much apiece?

$$Q_d = Q_s$$

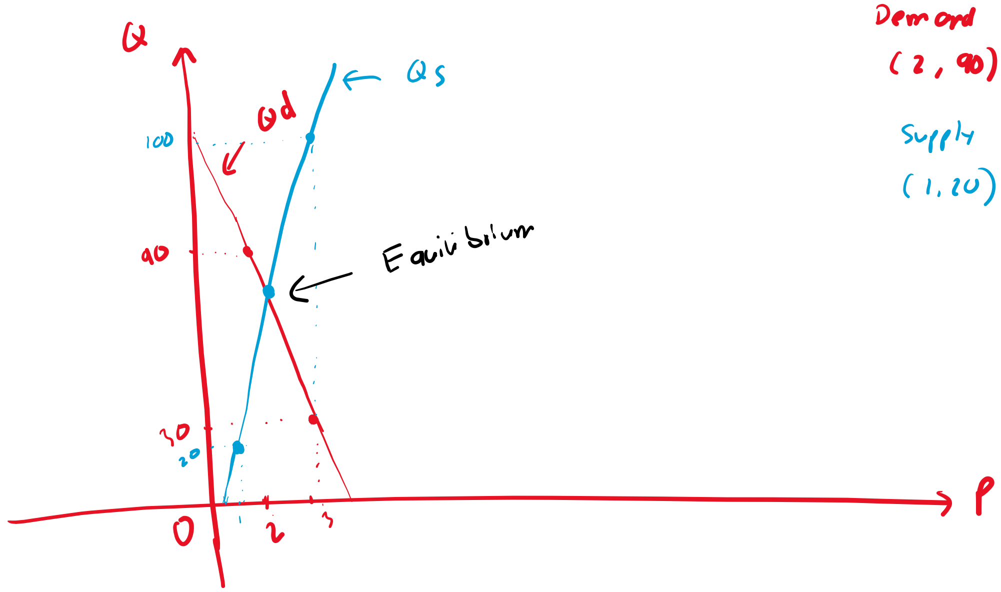
$$-60P + 210 = 40P - 20$$

$$210 + 20 = 40P + 60P$$

$$230 = 100P$$

$$\Rightarrow P = \frac{230}{100} = \boxed{2.3}$$

c. Graph both the demand and supply function on the same axis.



Demand

(2, 90) and (3, 30)

Supply

(1, 20) and (3, 100)

You Try:

The demand for your college newspaper is 2,000 copies each week if the paper is given away free of charge and drops to 800 each week if the charge is $1.50 per copy.

However, the university is prepared to supply only 600 copies per week free of charge but will supply 1,500 each week at $1.50 per copy.

a. Write the demand and supply functions.

b. Find the equilibrium point. For supply to equal demand, the newspaper must be priced at how much apiece?

c. Graph both the demand and supply function on the same axis.

2. Cost – Revenue and Break-Even

Cost Function

The **total cost** of producing goods is the sum of **fixed costs** and **variable costs**.

A linear **cost function** is written as:

$$\overset{\text{output}}{C} = \underbrace{m\overset{\text{input}}{q}}_{\text{variable cost}} + \underbrace{b}_{\text{fixed cost}}$$

where:

- C = total cost
- q = quantity produced
- m = variable cost per unit
- b = fixed cost

C is an increasing function of q

Revenue

Revenue is the total income from selling goods.

Revenue is calculated as:

$$\overset{\text{output}}{R} = pq$$

where:

- R = total revenue
- p = price per unit
- q = quantity sold ← input variable

R is also increasing

Profit

Profit represents the financial gain from producing and selling goods. It is calculated as **total revenue minus total cost**.

The profit function is written as:

$$\text{Profit} = R - C$$

where:

- profit
- R = total revenue
- C = total cost

Break-even point

The **break-even point** is the production level at which **profit equals zero**. At this point, total revenue is exactly equal to total cost.

Mathematically:

$$P = 0$$

or : Cost = Revenue.

m

Problem: An anticlot drug can be made for \$10 per unit. The total cost to produce $\underbrace{100}_{q}$ units is \$1500.

a. Assuming that the cost function is linear, find its rule/equation.

$$C = mq + b$$

$$m = 10 \Rightarrow C = 10q + b$$

when $q = 100$, $C = 1500$

Plug in: $1500 = 10 * 100 + b$

$\Rightarrow 1500 = 1000 + b$

$\Rightarrow 1500 - 1000 = b \Rightarrow \boxed{b = 500}$

The equation is: $\boxed{C = 10q + 500}$

b. What are the fixed costs? $\Rightarrow$ $b = 500$

You Try: A product can be made for \$120 per unit. The total cost to provided 100 units is \$15,800.

a. Assuming that the cost function is linear, find its rule.

b. What are the fixed costs?

Problem:
A bicycle manufacturer experiences fixed monthly costs of \$124,992 and variable costs of \$52 per standard model bicycle produced. The bicycles sell for \$100 each.

b m p

a. Find the cost function

$C = mq + b$

$\Rightarrow \quad C = 52q + 124992$

b. Find the revenue function

$R = p \cdot q \quad \Rightarrow \quad R = 100q$

c. Graph and label the cost and revenue functions on the same set of axes. Label the break-even point.

Break - even : Revenue = Cost

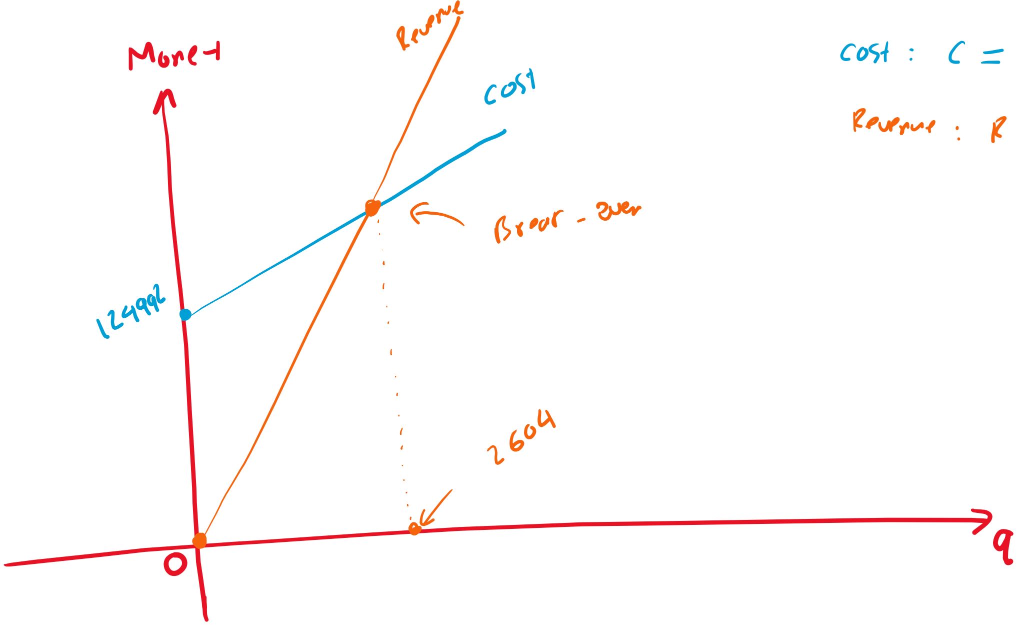
$R = C$

$100q = 52q + 124992$

$\Rightarrow 100q - 52q = 124992$

$\Rightarrow 48q = 124992$

$\Rightarrow q = \frac{124992}{48} = 2604$



Cost : $C = 52q + 124992$

Revenue : $R = 100q$

d. Find the profit function

$$\text{Profit} = R - C$$
$$= 100q - (52q + 124992)$$
$$\Rightarrow \text{Profit} = 100q - 52q - 124992$$
$$= \boxed{48q - 124992}$$

e. How much profit will they make by producing and selling 2000 bicycles?

$$q = 2000 \Rightarrow \text{Profit} = 48 \times 2000 - 124992$$
$$= -28992$$

f. How many bicycles must be produced and sold in order to obtain a profit of \$100,000?

$$\text{Profit} = 100\,000 \Rightarrow 100\,000 = 48q - 124992$$
$$\Rightarrow 100\,000 + 124992 = 48q$$
$$\Rightarrow 224992 = 48q$$
$$\Rightarrow q = \frac{224992}{48} = \boxed{4687.33}$$

You try:
A company manufactures a 65-inch smart TV that sells to retailers for $550. The costs can be described by a linear cost function with fixed costs of $213,000 and a variable cost per item of $250.

a. Find the cost function

b. Find the revenue function

c. Graph and label the cost and revenue functions on the same set of axes. Label the break-even point.

d. Find the profit function

e. How much profit will they make by producing and selling 5000 TVs?

f. How many units must be produced and sold in order to obtain a profit of $50,000?

3. **Simple Interest**

General Simple Interest Model

The total amount under **simple interest** is given by:

$$\boxed{A = P + Prt}$$

A is a linear function of t

where

- A = total amount after t years
- P = principal (initial amount)
- r = annual interest rate (decimal)
- t = time in years

👉 This is a **linear equation in** t.

- **Intercept**: P (amount when $t = 0$)
- **Slope**: Pr (interest earned per year)

Problem

You deposit \$2,000 in a savings account that earns **4% simple interest per year**.

1. Write a linear equation that gives the total amount A after t years.
2. How much money will be in the account after 5 years?

1 $P = 2000$, $r = 4\% = .04$

$A = P + Prt = 2000 + 2000 * .04 * t$

$\Rightarrow \boxed{A = 2000 + 80t}$

2 $t = 5 \Rightarrow A = 2000 + 80 * 5 = \boxed{2400}$

You try

You deposit \$3,500 in a savings account that earns **3% simple interest per year**.

1. Write a linear equation that gives the total amount A after t years.
2. How much money will be in the account after 6 years?

Problem

You invest \$1,800 at **5% simple interest**.

1. Write a linear equation for the total amount A after t years.
2. How long will it take for the investment to reach \$2,070?

$P = 1800, \quad r = 5\% = .05$

1 $\quad A = P + Prt$

$= 1800 + 1800 \times .05\, t$

$\boxed{A = 1800 + 90t}$

2 $\quad A = 2070 \Rightarrow 1800 + 90t = 2070$

$\Rightarrow 90t = 2070 - 1800$

$\Rightarrow 90t = 270$

$\Rightarrow t = \frac{270}{90} = 3$

$\boxed{t = 3}$

You try

You invest \$2,400 at **4% simple interest**.

1. Write a linear equation for the total amount A after t years.
2. How long will it take for the investment to reach \$2,880?

Problem

Two simple-interest investments are available:

- Investment A: $1,000 at 5% simple interest
- Investment B: $1,500 at 3% simple interest

1. Write a linear equation for the total amount A for each investment.
2. Which investment grows faster? Explain using slopes.
3. Will the two investments ever have the same value? If so, when?

You try

Two simple-interest investments are available:

- Investment A: $1,200 at **4% simple interest**
- Investment B: $2,000 at **2% simple interest**

1. Write a linear equation for the total amount A for each investment.
2. Which investment grows faster? Explain using slopes.
3. Will the two investments ever have the same value? If so, when?

4. **Depreciation and Appreciation**

Depreciation (Linear Decrease)

A value **depreciates linearly** if it decreases by the **same dollar amount each year**.

Model

$$\boxed{V = b - mt \quad (m > 0)}$$

Interpretation

- **Intercept** b: initial value
- **Slope** $-m$: loss in value per year

Appreciation (Linear Increase)

A value **appreciates linearly** if it increases by the **same dollar amount each year**.

Model

$$\boxed{V = b + mt \quad (m > 0)}$$

Interpretation

- **Intercept** b: starting value
- **Slope** m: amount of increase per year

Problem: (**Depreciation**) Suppose that the manufacturer's suggested retail price (MSRP) on a 2021 Ford F150 XL pickup truck is $38,345 and that in three years it is worth $26,120.

a. Assuming linear depreciation, find the depreciation function for this vehicle.

b. At what rate is the car depreciating?

c. What will the truck be worth in 6 years?

d. When will the vehicle be worth half of its original value?

You try

A company purchases a laptop system for $18,000. After 2 years, the system is worth $13,200.

a. Assuming linear depreciation, find the value function.

b. Find the depreciation rate per year.

c. What will the system be worth after 5 years?

d. When will the system be worth half of its original value?

Problem: (appreciation)

A house increases in value in an approximately linear fashion from $222,000 to $300,000 in 6 years.

a. Find the appreciation function that gives the value of the house in year x.

b. At what rate is the house appreciating?

c. If the house continues to appreciate at this rate, what will it be worth 12 years from now?

d. When will the house be worth double its original value?

You try

A piece of artwork increases in value from $8,500 to $11,500 over 5 years.

a. Find the appreciation function that gives the value of the artwork in year x.

b. At what rate is the artwork appreciating?

c. What will the artwork be worth after 10 years?

d. When will the artwork be worth triple its original value?

5. **Line of best fit (Excel)**