Quadratic Applications

Break-Even Point and Profit Maximization

Example:

Suppose that in a monopoly market the total cost per week of producing a high-tech product is given by $C = 3600 + 100q + 2q^2$. Suppose further that the sales price function for this product is

$p = 500 - 2q$.

a. Find the revenue function in term of q.

Revenue = price × quantity

$\Leftrightarrow \quad R = p \cdot q$

$= (500 - 2q) \cdot q$

$\Leftrightarrow \quad R = 500q - 2q^2$

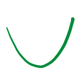
$\Leftrightarrow \quad \boxed{R = -2q^2 + 500q}$

b. Find the number of units that will **maximize the revenue.**

Formula : Maximizing / Minimizing a quadratic.

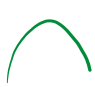
$y = ax^2 + bx + c$

* If $a > 0$: + y is minimized when $x = -\frac{b}{2a}$



+ y has no maximum value.

* If $a < 0$: + y is maximized when $x = -\frac{b}{2a}$



+ y has no minimum value

we have : $R = -2q^2 + 500q$

$a = -2$; $b = 500$; $c = 0$

R is maximized when $q = -\frac{b}{2a} = -\frac{500}{2*(-2)}$

$\Rightarrow \boxed{q = 125}$

c. Find the profit function

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$= -2q^2 + 500q - (3600 + 100q + 2q^2)$$

$$= -2q^2 + 500q - 3600 - 100q - 2q^2$$

$$= \boxed{-4q^2 + 400q - 3600}$$

d. Find the number of units that will give **break-even** for the product

$\underbrace{\text{break-even}}_{\text{Profit} = 0}$

$$\text{profit} = 0$$

$$\Leftrightarrow -4q^2 + 400q - 3600 = 0$$

$$\Leftrightarrow -4 \cdot (q^2 - 100q + 900) = 0$$

$$\Leftrightarrow q^2 - 100q + 900 = 0$$

$$a = 1, \quad b = -100, \quad c = 900$$

$$\Rightarrow q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(quadratic formula)}$$

$$q = \frac{100 \pm \sqrt{(-100)^2 - 4 \cdot 1 \cdot 900}}{2}$$

$$q = \frac{100 \pm 80}{2}$$

$$q = \frac{100 + 80}{2} = 90$$

$$q = \frac{100 - 80}{2} = 10$$

There are 2 break - even points: $q = 10$, $q = 90$

e. Find **the maximum profit** and the number of products need to maximize the profit.

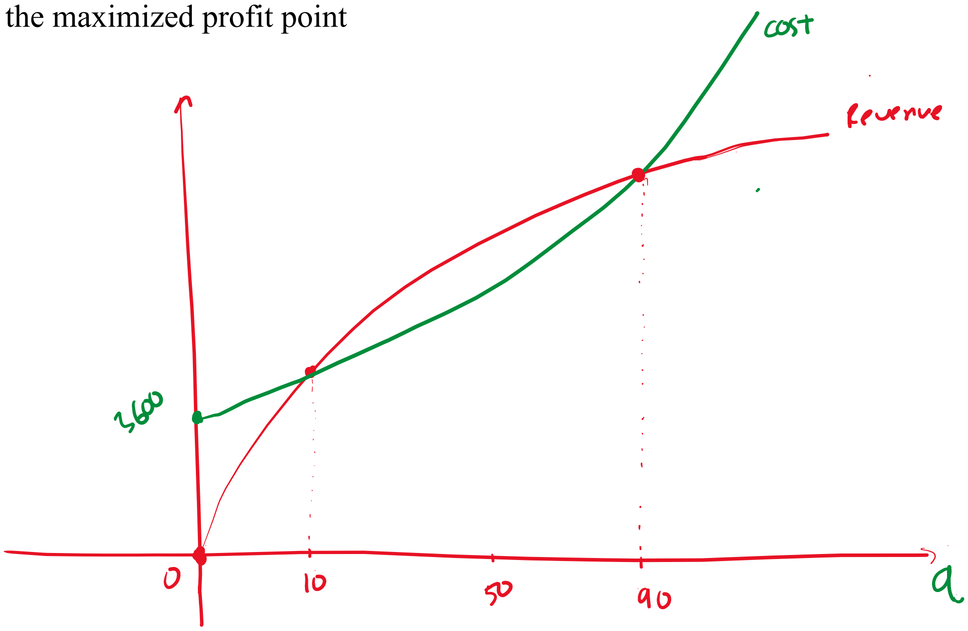
$$\text{Profit} = -4q^2 + 400q - 3600$$

$a = -4$; $b = 400$, $c = -3600$

Profit is maximized when $q = -\frac{b}{2a} = -\frac{400}{2\cdot(-4)} = 50$

Notice: The profit is ALWAYS maximized at the mid-point of the 2 break-even points. $\left(\frac{10+90}{2} = 50\right)$

f. Graph the revenue function and the cost function label the break-even points, fixed cost, and the maximized profit point



Key Take Aways

- Maximized Profit is at the mid-point of the two break-even points

- Negative Profit beyond the two break-even points.

- If demand pushes production past the second break-even, the business actually begins to lose money.

- This is because the costs of "stretching" to meet that demand (overtime, extra shipping, machine wear) grow faster than the money coming in.

Real-Life Example

The **"Death by Groupon"** Phenomenon.

- In the early 2010s, many small businesses (like bakeries or yoga studios) used Groupon to explode their demand.

- **The Surge:** A bakery might suddenly get 5,000 orders in a weekend.

- **The Second Break-even**: To meet this demand, they had to hire emergency staff, pay overtime, and buy ingredients at retail prices because their wholesale suppliers couldn't scale that fast.

- **The Result:** The cost to make the 5,000th cupcake was actually higher than the discounted price they were paid. Many shops went bankrupt because they successfully reached the "Max Revenue" point but blew right past their "Second Break-even."

You try

Suppose a small electronics company produces a new smart-watch. Through market research, they have determined the Cost Function as ‘
$C = 3600 + 100q + 2q^2$.

Suppose further that the sales price function for this product is $p = 400 - 4q$.

a. Find the revenue function
b. Find the number of units that will **maximize the revenue.**
c. Find the profit function
d. Find the number of units that will give **break-even** for the product
e. Find **the maximum profit** and the number of products need to maximize the profit.
f. Graph the revenue function and the cost function label the break-even points and the maximized profit point.

Price Increasing Model

Example: On a certain route, an airline carries 8000 passengers per month, each paying \$50.

A market survey indicates that for each \$1 increase in the ticket price, the airline will lose 100 passengers.

a. What is the airline's current revenue?

Revenue = price x quantity

$= 50 * 8000 = 400,000$

b. Create an income (revenue) function if "x" is defined as the number of \$1 price increases

	current	increasing the price by x
price	50	$50 + x$
Quantity / passengers	8000	$8000 - 100x$
Revenue	$50 \cdot 8000$	$(50 + x) \cdot (8000 - 100x)$

New Revenue $= (50 + x) \cdot (8000 - 100x)$

c. Find the number of \$1 price increases that will maximize the revenue.

we need to find x that maximizes

$(50 + x) \cdot (8000 - 100x)$

$R = 50 * 8000 - 50 * 100x + 8000x - 100x^2$

$R = -100x^2 + 3000x + 400,000$

$a = -100, \quad b = 3000, \quad c = 400,000$

R is maximized when $x = -\frac{b}{2a} = -\frac{3000}{2 * (-100)}$

$$\boxed{x = 15}$$

d. Find the new ticket price (that will maximize the revenue)

New price = current price + x

$= 50 + 15 = \boxed{65}$

e. Find the number of passengers at that price in d.

New number of passengers = $8000 - 100x$

$= 8000 - 100 * 15$

$= 6500$

f. Find the new maximum income (income at that price in d)

new Revenue $= 65 * 6500 = 422500$

You try

An amusement park charges \$8 admission and averages 2000 visitors per day. A survey shows that for each \$1 increase in the admission price, 100 fewer people would visit the park.

a. What is the amusement park's current revenue?

b. Create an income (revenue) function if "x" is defined as the number of \$1 admission price increases

c. Find the number of \$1 admission price increases that will maximize the income.

d. Find the new ticket price (that will maximize the revenue)

e. Find the number of passengers at that price in d.

f. Find the new maximum income (income at that price in d)

Supply, Demand, Equilibrium

Example

If the supply function for a commodity is given by $p = 10q^2 + 2q$ and the demand function is given by $p = 150 - 6q^2$, find the point of market equilibrium (Supply equals Demands).

$$10q^2 + 2q = 150 - 6q^2$$

$$\Leftrightarrow \quad \underline{10q^2} + 2q + \underline{6q^2} - 150 = 0$$

$$\Leftrightarrow \quad 16q^2 + 2q - 150 = 0$$

$$a = 16 \; ; \; b = 2 \; ; \; c = -150$$

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 16 \cdot (-150)}}{2 \cdot 16}$$

$$= \frac{-2 \pm 98}{32} \quad \begin{matrix} \nearrow \dfrac{-2-98}{32} = -3.125 \\ \searrow \dfrac{-2+98}{32} = 3 \end{matrix}$$

Since q is non-negative, $\boxed{q = 3}$

The price at $q = 3$ is $10q^2 + 2q$

$$p = 10 * 3^2 + 2 * 3$$

$$\boxed{p = 96}$$

You try

If the supply function for a commodity is given by $p = q^2 + 4q$ and the demand function is given by $p = 80 - 3q^2$, find the point of market equilibrium (Supply equals Demands).