Quadratic Applications

Break-Even Point and Profit Maximization

Example:

Suppose that in a monopoly market the total cost per week of producing a high-tech product is given by $C = 3600 + 100q + 2q^2$. Suppose further that the sales price function for this product is $p = 500 - 2q$.

a. Find the revenue function

$$\text{Revenue} = \text{price} \times \text{quantity} \qquad \leftarrow \quad \text{Should be in -lour formula Sheet.}$$

$$(\Rightarrow) \qquad R = p \cdot q$$

$$R = (500 - 2q) \cdot q$$

b. Find the number of units that will **maximize the revenue.**

Formula : $\qquad y = ax^2 + bx + c$

$\circledast$ $a > 0$: y is minimized   when $x = -\dfrac{b}{2a}$

$\qquad\qquad\qquad\qquad$ y has no maximized value.

$\circledR$ $a < 0$: y is maximized   when $x = -\dfrac{b}{2a}$

$\qquad\qquad\qquad\qquad$ y has no minimized value.

$$R = (500 - 2q) \cdot q$$

$$R = 500q - 2q^2 \qquad (\Rightarrow) \qquad R = -2q^2 + 500q$$

$$[\, a = -2 \, ; \; b = 500, \; c = 0 \,]$$

$\Rightarrow$ R is maximized at $q = -\frac{b}{2a} = \frac{-500}{-4} = 125$

$$\boxed{q = 125}$$ to maximize. R

c. Find the number of units that will give **break-even** for the product

$$\text{Revenue} = \text{cost}$$

$$R = C$$

$$(500 - 2q) \cdot q = 3600 + 100q + 2q^2$$

$$\Leftarrow \quad 500q - 2q^2 = 3600 + 100q + 2q^2$$

$$\Leftarrow \quad 0 = -500q + 2q^2 + 3600 + 100q + 2q^2$$

$$\Leftarrow \quad 0 = 4q^2 - 400q + 3600$$

$$\Leftarrow \quad 0 = 4 \cdot (q^2 - 100q + 900)$$

$$\Leftarrow \quad q^2 - 100q + 900 = 0$$

$$\Leftarrow \quad (q - 90) \cdot (q - 10) = 0$$

$$q - 90 = 0 \quad ; \quad q - 10 = 0$$

$$\boxed{q = 90 \quad ; \quad q = 10}$$

(Notice that we have 2 break-even points)

d. Find the profit function

$$\text{Profit} = R - C$$

$$= (500 - 2q) \cdot q - (2q^2 + 100q + 3600)$$

$$\text{Profit} = -4(q^2 - 100q + 90)$$

e. Find **the maximum profit** and the number of products need to maximize the profit.

$$\text{Profit} = -4q^2 + 400q - 3600$$

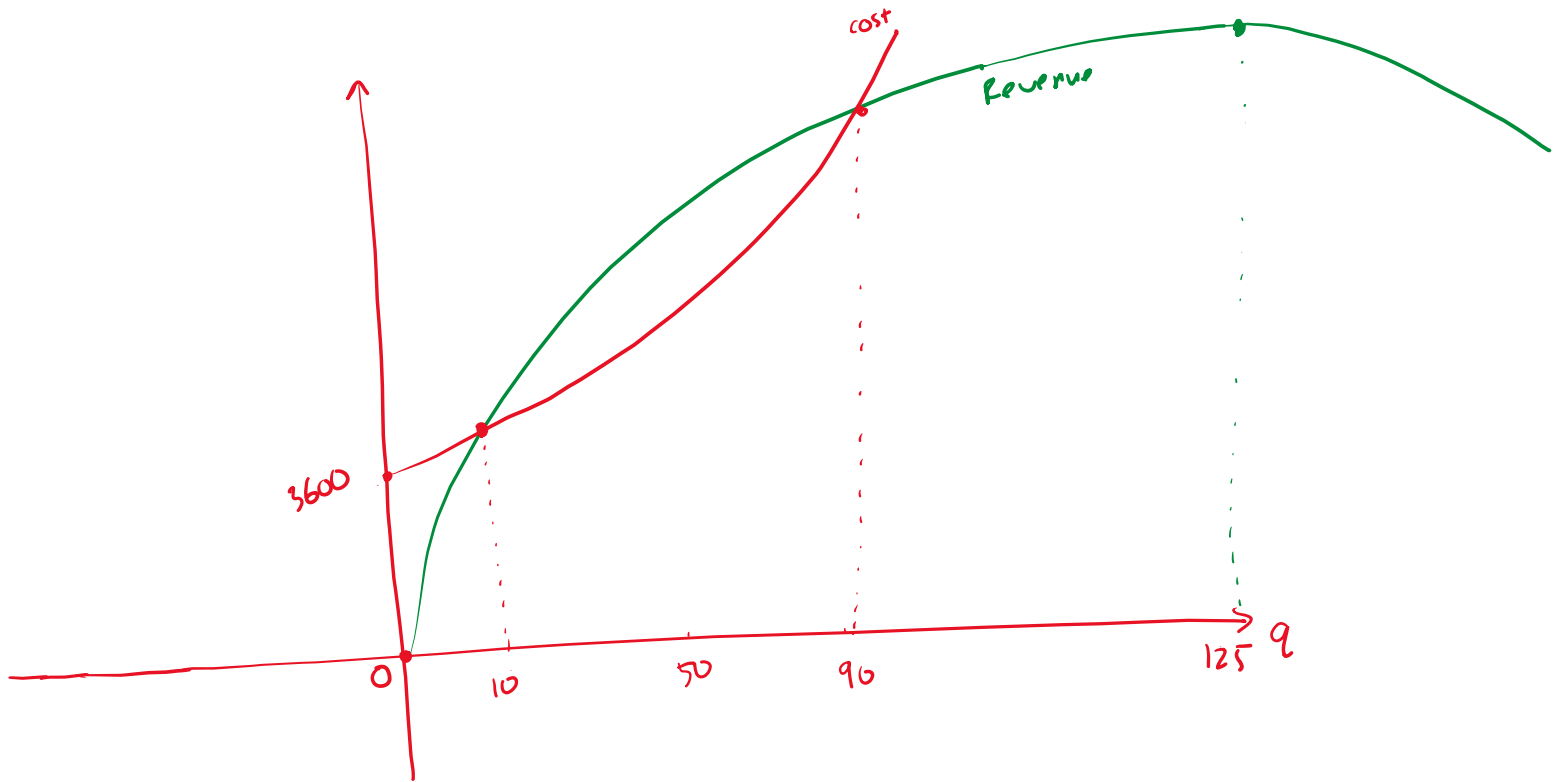$$[a = -4; \quad b = 400; \quad c = -3600]$$

Profit   maximized   at $q = -\dfrac{b}{2a} = \dfrac{-400}{2 \cdot (-4)}$

$$q = 50$$

<u>Notice</u> :   Profit is ALWAYS   maximized   at the   mid-point

of   the two   break-even   points.

f. Graph the profit function and label the vertex, *x*-intercepts, and *y*-intercept.

Key Take Aways

- Maximized Profit is at the mid-point of the two break-even points

- Negative Profit beyond the two break-even points.

- If demand pushes production past the second break-even, the business actually begins to lose money.

- This is because the costs of "stretching" to meet that demand (overtime, extra shipping, machine wear) grow faster than the money coming in.

Real-Life Example

The **"Death by Groupon"** Phenomenon

- In the early 2010s, many small businesses (like bakeries or yoga studios) used Groupon to explode their demand.

- **The Surge:** A bakery might suddenly get 5,000 orders in a weekend.

- **The Second Break-even**: To meet this demand, they had to hire emergency staff, pay overtime, and buy ingredients at retail prices because their wholesale suppliers couldn't scale that fast.

- **The Result:** The cost to make the 5,000th cupcake was actually higher than the discounted price they were paid. Many shops went bankrupt because they successfully reached the "Max Revenue" point but blew right past their "Second Break-even."

You try

Suppose a small electronics company produces a new smart-watch. Through market research, they have determined the Cost Function as '
$C = 3600 + 100q + 2q^2$.

Suppose further that the sales price function for this product is $p = 400 - 4q$.

a. Find the revenue function

b. Find the number of units that will **maximize the revenue.**

c. Find the number of units that will give **break-even** for the product

d. Find the profit function

e. Find **the maximum profit** and the number of products need to maximize the profit.

f. Graph the revenue and cost functions and label the break-even points.

Price Increasing Model

Example: On a certain route, an airline carries 8000 passengers per month, each paying $50. A market survey indicates that for each $1 increase in the ticket price, the airline will lose 100 passengers.

    a. What is the airline's current revenue?

$$R = 8000 * 50 = 400000$$

    b. Create an income (revenue) function if "x" is defined as the number of $1 price increases

<div align="center">

Before   After

$p \rightarrow p + x$

$q \rightarrow q - 100x$

</div>

Before: $p = 50$; $q = 8000$

After: $p = 50 + x$; $q = 8000 - 100x$

Revenue after increasing the price is

$$R = (50 + x) \cdot (8000 - 100x)$$

    c. Find the number of $1 price increases that will maximize the revenue.

we need to find x to maximize

$$R = (50 + x) \cdot (8000 - 100x)$$

$$R = 50 * 8000 - 50 * 100x + 800x - 100x^2$$

$$= -100x^2 + 8000x - 5000x + 400\,000$$

$$= -100x^2 + 3000x + 400\,000$$

$$a = -100 \; ; \quad b = 3000$$

R is maximized at $\quad x = -\dfrac{b}{2a} = \dfrac{-3000}{2 \cdot (-100)} = 15$

d. Find the new ticket price (that will maximize the revenue)

New price $= 50 + 15 = 65$

e. Find the number of passengers at that price in d.

$q = 8000 - 100 \cdot 15 = 8000 - 1500 = 6500$

f. Find the new maximum income (income at that price in d)

New $R = 65 * 6500 = 422500$

You try

An amusement park charges $8 admission and averages 2000 visitors per day. A survey shows that for each $1 increase in the admission price, 100 fewer people would visit the park.

    a. What is the amusement park's current revenue?

    b. Create an income (revenue) function if "x" is defined as the number of $1 admission price increases

    c. Find the number of $1 admission price increases that will maximize the income.

    d. Find the new ticket price (that will maximize the revenue)

    e. Find the number of passengers at that price in d.

    f. Find the new maximum income (income at that price in d)

Supply, Demand, Equilibrium

Example

If the supply function for a commodity is given by $p = 10q^2 + 2q$ and the demand function is given by $p = 150 - 6q^2$, find the point of market equilibrium (Supply equals Demands).

$$10q^2 + 2q = 150 - 6q^2$$

$(=)$ $\quad 10q^2 + 6q^2 + 2q - 150 = 0$

$(=)$ $\quad 16q^2 + 2q - 150 = 0$

$$a = 16 \; ; \quad b = 2 \; ; \quad c = -150$$

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4 \cdot 16 \cdot (-150)}}{2 \times 16}$$

$$= \frac{-2 \pm 98}{32}$$

$$q = \begin{cases} \dfrac{-2-98}{32} = -3.125 \\[4mm] \dfrac{-2+98}{32} = 3 \end{cases}$$

$\Rightarrow$ $\boxed{q = 3}$ (q should always be positive)

You try

If the supply function for a commodity is given by $p = q^2 + 4q$ and the demand function is given by $p = 80 - 3q^2$, find the point of market equilibrium (Supply equals Demands).