Exponential / Logarithmic functions

Example :

U.S. Investment Abroad In 1980, direct U.S. business investment abroad was about 13.5 billion dollars. From 1980 through 2010, that investment[12] grew at an average annual rate of 11.24%.

initial amount.

r

a. Make an exponential model that shows the U.S. direct investment abroad A, in billions of dollars, t years after 1980.

b. From 1980, how long did it take for U.S. investments abroad to double?

a. $A(t) = P * (1 + r)^t$

- $A(t)$: amount of money + (years) since 1980
- P: initial amount of investment (principal)
- r: interest rate / growth rate

we have: $P = 13.5$ (billions)

$r = 11.24\% = .1124$

$\Rightarrow$ $\boxed{A(t) = 13.5 * (1.1124)^t}$

(b) we need to find t so that

$A(t) = 13.5 * 2$

$\Rightarrow$ $13.5 * (1.1124)^t = 13.5 * 2$

$$\Rightarrow \quad 1.1124^t = 2$$

$$\Rightarrow \quad t = \log_{1.1124} 2 = \frac{\ln 2}{\ln 1.1124}$$

$$\approx 6.507$$

Example

If tuition at a college is increasing by 6.6% each year, how many years will it take for tuition to double?

$$A(t) = P \cdot (1 + r)^t \qquad r = 6.6\%$$

$A(t)$: tuition at the time t

P: initial tuition

we need to solve for

$$A(t) = 2P$$

$$\Rightarrow \quad P(1 + .066)^t = 2P$$

$$\Rightarrow \quad (1.066)^t = 2$$

$$\Rightarrow \quad t = \log_{1.066} 2 = \frac{\ln 2}{\ln 1.066} \approx 10.845$$

(*) How long does it take for the tuition to triple?

$$A(t) = 3 \cdot P$$

$$\Rightarrow P \cdot 1.066^{t} = 3P$$

$$\Rightarrow 1.066^{t} = 3$$

$$\Rightarrow t = \log_{1.066} 3 = \frac{\ln 3}{\ln 1.066} \approx 17.189$$

Example :

A freezer maintains a constant temperature of 6 degrees Fahrenheit. An ice tray is filled with tap water and placed in the refrigerator to make ice. The difference between the temperature of the water and that of the freezer was sampled each minute and recorded in the table below.

	Time in minutes	0	1	2	3	4	5
d	Temperature difference	69.0	66.3	63.7	61.2	58.8	56.5

Part 1 Test to see that the data are exponential.

Part 2 Find an exponential model for temperature difference.

Part 3 : Rewrite the model in part 2 in term of base e.

Part 4 : when the temperature different reach 30.

Is this a linear growth or decay in temp. difference?

t	d	Change in d
0	69	
1	66.3	$66.3 - 69 = -2.7$
2	63.7	$63.7 - 66.3 = -2.6$
3	61.2	$61.2 - 63.7 = -2.5$
4	58.8	$58.8 - 61.2 = -2.4$
5	56.5	$56.5 - 58.8 = -2.3$

We observe that the differences in d are not the same. This means that the relation between t and d is not linear.

t	d	ratio change in d
0	69	
1	66.3	$66.3 \mid 69 = 0.960869$
2	63.7	$63.7 \mid 66.3 = 0.960784$
3	61.2	$61.2 \mid 63.7 = 0.96075$
4	58.8	$58.8 \mid 61.2 = 0.96078$
5	56.5	$56.5 \mid 58.8 = 0.960884$

We observe that the ratio changes are "almost" the same. So we could use exponential model to model the relation between t and d.

Part 2:

$d = \text{smth} * \text{something}^{t}$

$$d = P \cdot b^t$$

When $t = 0$ and $d = 69$ $\Rightarrow$ $d = P * b^0 = P$

$$69 = P$$

$t = 1$, $d = 66.3$

$$P \cdot b^1 = 66.3$$

$$\Rightarrow 69 \cdot b = 66.3 \Rightarrow b = \frac{66.3}{69}$$

$\Rightarrow \boxed{b \approx .9608}$

$\Rightarrow \boxed{d = 69 * (.9608)^{t}} \leftarrow$

Part 3: Formular to convert the exponental of base b to base e.

$$a \cdot b^{x} = a \cdot e^{(\ln b) \cdot x}$$

(base b) (base e, $e = 2.7182\ldots$)

Apply this formular to: $d = 69 \cdot (.9608)^{t}$

$$= 69 \cdot e^{\ln(.9608) \cdot t}$$

$$\boxed{d = 69 \cdot e^{-0.04t}}$$

Part 4: Solve for $d = 30$

$$\Rightarrow 69 \cdot e^{-.04t} = 30$$

$$\Rightarrow e^{-.04t} = \frac{30}{69}$$

$$\Rightarrow -.04t = \log_e (30/69)$$

[Note: $b^{x} = 4 \Rightarrow x = \log_b 4$]

$$\Rightarrow \quad -.04t = \ln\left(\tfrac{30}{64}\right)$$

$$\Rightarrow \quad t = \frac{\ln(30/64)}{-.04} \approx 20.823$$

Assignment:

t	D
0	32
1	25.6
2	20.48
3	16.38
4	13.11

(1.) Verify that D is an exponential function of t.

(2) Write the equation of the model. (write D as an exponential equation of t).

(3) Rewrite the equation in the base e.

(4) Find t so that D = 1