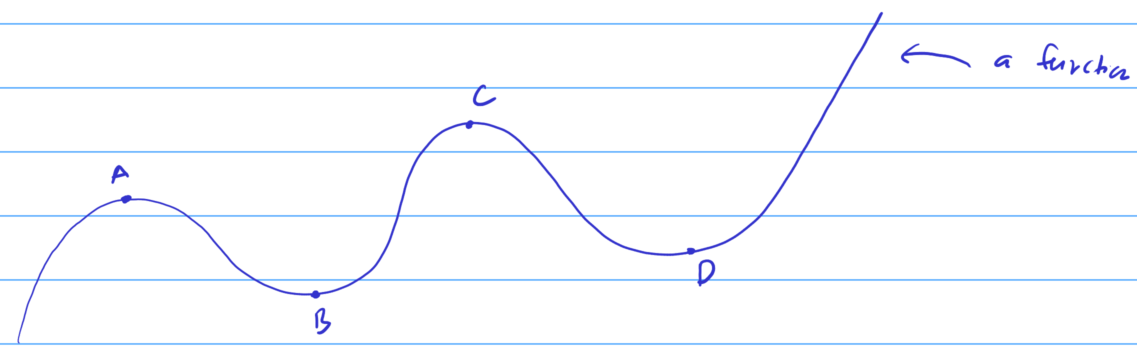
Application of Derivatives



A and C are local maximum

B and D are local minimum

A, B, C, D are local extrema

(*) Second Derivative test for local extrema.

1. If $f'(c) = 0$ and $f''(c) > 0$ then the point $(c, f(c))$ is a local minimum of $f(x)$

2. If $f'(c) = 0$ and $f''(c) < 0$ then the point $(c, f(c))$ is a local maximum of $f(x)$

(*) Second derivative :

(+) The second derivative of $f(x)$ is the the derivative of the first derivative, $f'(x)$.

(+) The second derivative is denoted by $f''(c)$

Example

$$f(x) = x^7$$

(The 1^{st} derivative) $f'(x) = 7x^6$

(2^{nd} derivative) $f''(x) = (7x^6)' = 7 \cdot 6 \cdot x^5$

$$= 42x^5$$

Example : Find $f''(x)$

① $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = (3x^2)' = 6x$

② $f(x) = 2x$

$f'(x) = (2x)' = 2$

$f''(x) = (2)' = 0$

③ $f(x) = 3^x$

$\Rightarrow f'(x) = 3^x \cdot \ln 3$

$\Rightarrow f''(x) = (3^x \cdot \ln 3)' = (3^x)' \cdot \ln 3$

scalar

$$= 3^x \cdot \ln 3 \cdot \ln 3$$

Example

The per-day cost function for the manufacture of portable MP3 players is given by

$$C(x) = 128,000 + 30x + x^3,$$

cost

where x is the number of MP3 players manufactured per day. Assume that the company cannot manufacture more than 100 MP3 players per day. How many MP3 players should be manufactured in order to minimize average cost?

x : The number of items produced , $x > 0$ and x is a real number

cost function $C(x) = 128,000 + 30x + x^3$

$x \leq 100$

To produce x items, it costs $C(x)$ (USD). On average, it costs $\frac{C(x)}{x}$ to produce 1 item.

The average cost function $\frac{C(x)}{x}$. Let call this function $f(x)$

$$f(x) = \frac{C(x)}{x} = \frac{128,000 + 30x + x^3}{x}, \quad \boxed{0 < x \leq 100}$$

we need to find x that minimizes $f(x)$

Step 1 : Find $f'(x)$

$$f(x) = \frac{128,000 + 30x + x^3}{x} = \frac{128,000}{x} + \frac{30x}{x} + \frac{x^3}{x}$$

$$= 128,000 \cdot x^{-1} + 30 + x^2$$

$$\Rightarrow \quad f'(x) = 128{,}000 \cdot (-1) \cdot x^{-1-1} + 2x$$

$$f'(x) = -128{,}000\, x^{-2} + 2x$$

Step 2: Solve for $f'(x) = 0$

$$-128{,}000 \cdot x^{-2} + 2x = 0$$

$$\Rightarrow \quad 2x = 128{,}000 \cdot x^{-2}$$

$$\Rightarrow \quad 2x \cdot x^{2} = 128{,}000 \cdot \underbrace{x^{-2} \cdot x^{2}}_{=1}$$

Notice: $x^{k} \cdot x^{-k} = 1$

$$\Rightarrow \quad 2x^3 = 128{,}000$$

$$\Rightarrow \quad x^3 = \frac{128{,}000}{2} = 64{,}000$$

$$\Rightarrow \quad x^3 = 64{,}000$$

$$\Rightarrow \quad x = \sqrt[3]{64{,}000} = 40$$

$$\Rightarrow \quad \boxed{x = 40}$$

$$\boxed{f'(40) = 0}$$

Step 3: Find $f''(x)$ and determine the size of $f''(40)$

$$f'(x) = -128{,}000 \cdot x^{-2} + 2x$$

$$f''(x) = \left(-128{,}000\, x^{-2} + 2x\right)'$$

$f''(x) = -128{,}000 \cdot (-2) \cdot x^{-2-1} + 2$

$= 256{,}000\, x^{-3} + 2$

$$\boxed{f''(x) = \frac{256{,}000}{x^3} + 2}$$

plus $x = 40$, $f''(40) = \frac{256{,}000}{40^3} + 2 > 0$

$\Rightarrow$ The the minimum is at $\boxed{x = 40}$

Assignment 21

The per-day cost function for the manufacture of portable MP3 players is given by

$$C(x) = 686{,}000 + 30x + x^3,$$

where x is the number of MP3 players manufactured per day. Assume that the company cannot manufacture more than 100 MP3 players per day. How many MP3 players should be manufactured in order to minimize average cost?