Rule 10: Chain Rule for exponential function.

Exponential rule: $\left(b^{x}\right)' = b^{x} \cdot \ln b$

Chain rule for exponential: Replace x by $g(x)$

① $\left[b^{g(x)}\right]' = b^{g(x)} \cdot \ln b \cdot g'(x)$

special case when $b = e = 2.71828\ldots$

② $\left[e^{g(x)}\right]' = e^{g(x)} \cdot g'(x)$

Example: Find $f'(x)$

① $f(x) = 2^{x} \Rightarrow f'(x) = 2^{x} \cdot \ln 2$

② $f(x) = 2^{x^2} \Rightarrow f'(x) = 2^{x^2} \cdot \ln 2 \cdot (x^2)'$

$\Rightarrow f'(x) = 2^{x^2} \cdot \ln 2 \cdot (2x)$

③ $f(x) = 3^{x^3+10x}$

$f'(x) = 3^{x^3+10x} \cdot \ln 3 \cdot (x^3+10x)'$

$= 3^{x^3+10x} \cdot \ln 3 \cdot (3x^2+10)$

④ $f(x) = 10^{x^5 + 2x^2 + 3}$

$f'(x) = 10^{x^5 + 2x^2 + 3} \cdot \ln 10 \cdot (x^5 + 2x^2 + 3)'$

$= 10^{x^5 + 2x^2 + 3} \cdot \ln 10 \cdot (5x^4 + 4x)$

⑤ $f(x) = e^{x^6 + 10x + 1}$

$f'(x) = e^{x^6 + 10x + 1} \cdot \underbrace{\ln e}_{=1} \cdot (x^6 + 10x + 1)'$

$= e^{x^6 + 10x + 1} \cdot (6x^5 + 10)$

⑥ $f(x) = 7^{x \cdot \ln x}$

use product rule for this

$f'(x) = 7^{x \cdot \ln x} \cdot \ln 7 \cdot (x \cdot \ln x)'$

$f'(x) = 7^{x \cdot \ln x} \cdot \ln 7 \cdot \left[\underbrace{(x)'}_{=1} \cdot \ln x + x \cdot \underbrace{(\ln x)'}_{=1/x} \right]$

$f'(x) = 7^{x \cdot \ln x} \cdot \ln 7 \left[\ln x + x \cdot \frac{1}{x} \right]$

$= \boxed{7^{x \ln x} \cdot \ln 7 \cdot (\ln x + 1)}$

⑦ $f(x) = 2024^{\frac{e^x}{x}}$

use quotient rule for this

$$f'(x) = 2024^{e^x/x} \cdot \ln 2024 \cdot \left[\frac{e^x}{x}\right]'$$

$$f'(x) = 2024^{e^x/x} \cdot \ln 2024 \cdot \left[\frac{(e^x)' \cdot x - e^x \cdot (x)'}{x^2}\right]$$

$[(e^x)' = e^x]$

$$\Rightarrow f'(x) = 2024^{e^x/x} \cdot \ln 2024 \cdot \left[\frac{e^x \cdot x - e^x \cdot 1}{x^2}\right]$$

Assignment 19: Find $f'(x)$

① $f(x) = 10^x$

② $f(x) = 3^{x^3 + 2x}$

③ $f(x) = 6^{x^2 + 4x + 10}$

④ $f(x) = e^{3x^2 + 5x + 6}$

⑤ $f(x) = e^{\sqrt{x} + 1}$

⑥ $f(x) = e^{e^x}$

⑦ $f(x) = 8^{(x^2+1)\cdot(x^3+2)}$

(8) $f(x) = 20^{x^2 \cdot 3^x}$

(9) $f(x) = 17^{4^x \cdot \ln x}$

(10) $f(x) = 9^{(x^2+1)\log_7 x}$

(11) $f(x) = 9^{x^2+1 / \ln x}$

(12) $f(x) = 6^{e^x / x^2 + x}$

Extra credit:

(13) $f(x) = e^{(x^2+1)^{2024}}$

We are doing Assignment 19. Please see the assignment on Canvas in note 16.pdf

(10) $f(x) = 9^{(x^2+1)\log_7 x}$

$f'(x) = 9^{(x^2+1)\log_7 x} \cdot \ln 9 \cdot \left[(x^2+1) \cdot \log_7 x \right]'$ (product)

$= 9^{(x^2+1)\log_7 x} \cdot \ln 9 \left[(x^2+1)' \cdot \log_7 x + (x^2+1)(\log_7 x)' \right]$

$= 9^{(x^2+1)\log_7 x} \cdot \ln 9 \left[2x \cdot \log_7 x + (x^2+1) \frac{1}{x \cdot \ln 7} \right]$

(11) $f(x) = 9^{x^2+1/\ln x}$

$f'(x) = 9^{x^2+1/\ln x} \cdot \ln 9 \cdot \left(\frac{x^2+1}{\ln x} \right)'$ (quotient rule)

$= 9^{x^2+1/\ln x} \cdot \ln 9 \left[\frac{(x^2+1)' \cdot \ln x - (x^2+1) \cdot (\ln x)'}{(\ln x)^2} \right]$

$= 9^{x^2+1/\ln x} \cdot \ln 9 \cdot \left[\frac{2x \cdot \ln x - (x^2+1) \cdot 1/x}{(\ln x)^2} \right]$

Rule 11 Chain rule for log

log function: $(\log_b x)' = \frac{1}{x \cdot \ln b}$

chain rule for log: $\left[\log_b [g(x)]\right]' = \frac{g'(x)}{g(x) \cdot \ln b}$

Special case: $b = e = 2.718\ldots$ $\left[\ln g(x)\right]' = \frac{g'(x)}{g(x)}$

Example: Find $f'(x)$

① $f(x) = \log_{13} x$

$f'(x) = \frac{1}{x \cdot \ln 13}$

② $f(x) = \log_{13} \underbrace{(x^2 + 10x)}_{g(x)}$

$f'(x) = \frac{g'(x)}{g(x) \cdot \ln 13} = \frac{(x^2+10x)'}{(x^2+10x)\ln 13}$

$= \boxed{\frac{2x+10}{(x^2+10x)\cdot \ln 13}}$

③ $f(x) = \log_7 (x^3 + 2x)$

$$f'(x) = \frac{(x^3+2x)'}{(x^3+2x)\cdot \ln 7} = \frac{3x^2+2}{(x^3+2x)\cdot \ln 7}$$

④ $f(x) = \log_9 (e^x + x)$

$$f'(x) = \frac{(e^x+x)'}{(e^x+x)\cdot \ln 9} = \frac{e^x+1}{(e^x+x)\ln 9}$$

⑤ $f(x) = \log_6 \left[\frac{x^2+1}{x^2+2}\right]$

$$= \log_6 (x^2+1) - \log_6 (x^2+2)$$

$$f'(x) = \left[\log_6 (x^2+1)\right]' - \left[\log_6 (x^2+2)\right]'$$

$$= \frac{(x^2+1)'}{(x^2+1)\cdot \ln 6} - \frac{(x^2+2)'}{(x^2+2)\cdot \ln 6}$$

$$= \frac{2x}{(x^2+1)\ln 6} - \frac{2x}{(x^2+2)\ln 6}$$

notice:

$\log_b \left(\frac{x}{y}\right)$

$= \log_b x - \log_b y$

⑤ $f(x) = \log_4 \left[(x^2+1) \cdot (x^2+2) \right]$

$f(x) = \log_4 (x^2+1) + \log_4 (x^2+2)$

$f'(x) = \dfrac{(x^2+1)'}{(x^2+1) \cdot \ln 4} + \dfrac{(x^2+2)'}{(x^2+2) \ln 4}$

$f'(x) = \dfrac{2x}{(x^2+1) \ln 4} + \dfrac{2x}{(x^2+2) \ln 4}$

Assignment 20 : Find $f'(x)$

① $f(x) = \log_8 x$

② $f(x) = \log_9 (x^2 + 10)$

③ $f(x) = \log_{10} (x^3 + 2x + 1)$

④ $f(x) = \ln (x^{10} + 9x + 2024)$

⑤ $f(x) = \ln (e^x + 2^x + 4^x)$

⑥ $f(x) = \log_3 \dfrac{x+1}{x+2}$

⑦ $f(x) = \log_9 \left(\dfrac{x}{x^2+1} \right)$

(8) $f(x) = \ln\left[\dfrac{x^2+x+1}{e^x+x+1}\right]$

(9) $f(x) = \log_6\left[(x^2+2^x)\cdot(x^3+3^x)\right]$

(10) $f(x) = \ln\left[(\ln x + x)\cdot(x^2+1)\right]$

(11) Extra credits:

$f(x) = \ln\left[\ln(\ln x)\right]$