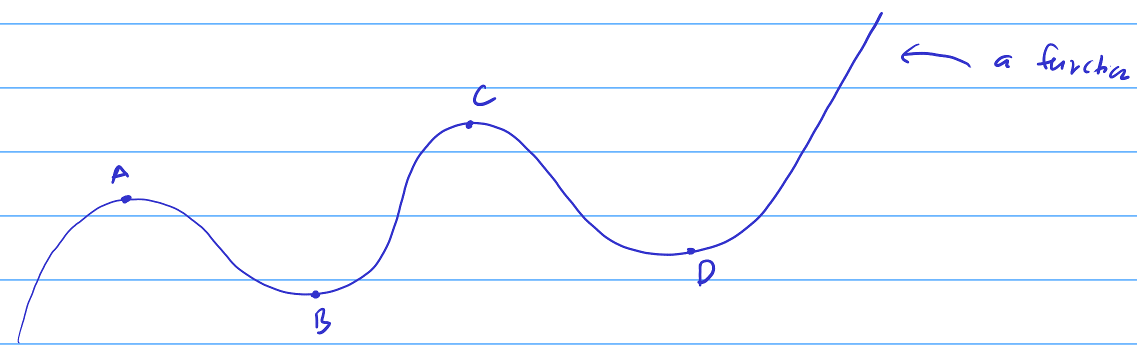
Application of Derivatives



A and C are local maximum

B and D are local minimum

A, B, C, D are local extrema

⊛ Second Derivative test for local extrema.

① If $f'(c) = 0$ and $f''(c) > 0$ then the point $(c, f(c))$ is a local minimum of $f(x)$

② If $f'(c) = 0$ and $f''(c) < 0$ then the point $(c, f(c))$ is a local maximum of $f(x)$

⊛ Second derivative :

⊕ The second derivative of $f(x)$ is the the derivative of the first derivative, $f'(x)$.

⊕ The second derivative is denoted by $f''(c)$

Example

$$f(x) = x^7$$

(The 1^{st} derivative) $f'(x) = 7x^6$

(2^{nd} derivative) $f''(x) = (7x^6)' = 7 \cdot 6 \cdot x^5$

$$= 42x^5$$

Example : Find $f''(x)$

① $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = (3x^2)' = 6x$

② $f(x) = 2x$

$f'(x) = (2x)' = 2$

$f''(x) = (2)' = 0$

③ $f(x) = 3^x$

$\Rightarrow f'(x) = 3^x \cdot \ln 3$

$\Rightarrow f''(x) = (3^x \cdot \ln 3)' = (3^x)' \cdot \ln 3$

scalar

$$= 3^x \cdot \ln 3 \cdot \ln 3$$

Example

The per-day cost function for the manufacture of portable MP3 players is given by

$$C(x) = 128,000 + 30x + x^3,$$ cost

where x is the number of MP3 players manufactured per day. Assume that the company cannot manufacture more than 100 MP3 players per day. How many MP3 players should be manufactured in order to minimize average cost?

x : The number of items produced , $x > 0$ and x is a real number

cost function $C(x) = 128,000 + 30x + x^3$

$x \leq 100$

To produce x items, it costs $C(x)$ (USD). On average,

it costs $\dfrac{C(x)}{x}$ to produce 1 item.

The average cost function $\dfrac{C(x)}{x}$. Let call this function $f(x)$

$$f(x) = \frac{C(x)}{x} = \frac{128,000 + 30x + x^3}{x}, \quad \boxed{0 < x \leq 100}$$

we need to find x that minimizes $f(x)$

Step 1 : Find $f'(x)$

$$f(x) = \frac{128,000 + 30x + x^3}{x} = \frac{128,000}{x} + \frac{30x}{x} + \frac{x^3}{x}$$

$$= 128,000 \cdot x^{-1} + 30 + x^2$$

$\Rightarrow \quad f'(x) = 128{,}000 \cdot (-1) \cdot x^{-1-1} + 2x$

$f'(x) = -128{,}000\, x^{-2} + 2x$

Step 2: Solve for $f'(x) = 0$

$-128{,}000 \cdot x^{-2} + 2x = 0$

$\Rightarrow \quad 2x = 128{,}000 \cdot x^{-2}$

$\Rightarrow \quad 2x \cdot x^{2} = 128{,}000 \cdot \underbrace{x^{-2} \cdot x^{2}}_{=1}$

Notice: $x^{k} \cdot x^{-k} = 1$

$\Rightarrow \quad 2x^{3} = 128{,}000$

$\Rightarrow \quad x^{3} = \frac{128{,}000}{2} = 64{,}000$

$\Rightarrow \quad x^{3} = 64{,}000$

$\Rightarrow \quad x = \sqrt[3]{64{,}000} = 40$

$\Rightarrow \quad \boxed{x = 40}$

$\boxed{f'(40) = 0}$

Step 3: Find $f''(x)$ and determine the size of $f''(40)$

$f'(x) = -128{,}000 \cdot x^{-2} + 2x$

$f''(x) = \left(-128{,}000\, x^{-2} + 2x\right)'$

$$f''(x) = -128{,}000 \cdot (-2) \cdot x^{-2-1} + 2$$

$$= 256{,}000\, x^{-3} + 2$$

$$\boxed{f''(x) = \frac{256{,}000}{x^3} + 2}$$

plus $x = 40$, $f''(40) = \frac{256{,}000}{40^3} + 2 > 0$

$\Rightarrow$ The the minimum is at $\boxed{x = 40}$

Assignment 21

The per-day cost function for the manufacture of portable MP3 players is given by

$$C(x) = 686,000 + 30x + x^3,$$

where x is the number of MP3 players manufactured per day. Assume that the company cannot manufacture more than 100 MP3 players per day. How many MP3 players should be manufactured in order to minimize average cost?

Example :

A company estimates that if it sets the price of an item at p dollars, then it can sell $q = 2,700 - p^2$, where $0 \leq p \leq 50$, items per year. What price will bring in the greatest annual revenue?

Revenue function: (price for 1 item) * (the number items sold)

$$R(p) = p * q$$

$$R(p) = p * (2700 - p^2) \qquad \boxed{0 \leq p \leq 50}$$

Notice we have a function of p (not x)

Step 1: Find $R'(p)$

$$R(p) = p * (2700 - p^2)$$

$$\Rightarrow R(p) = p \cdot 2700 - p^3$$

$$R'(p) = 2700 - 3p^2$$

Step 2: Solve for $R'(p) = 0$

$$2700 - 3p^2 = 0$$

$$3p^2 = 2700$$

$$p^2 = \frac{2700}{3} = 900$$

$$p = \sqrt{900}$$

$$\boxed{p = 30}$$

Step 3: Find $R''(p)$ and test the sign of $R''(30)$

$$R'(p) = 2700 - 3p^2$$

$$R''(p) = -6p$$

$$\Rightarrow R''(30) = -6 \cdot 30 < 0$$

This means the revenue is maximized at $\boxed{p = 30}$

Assignment 22

① A company estimates that if it sets the price of an item at p dollars, then it can sell $q = 300,000 - 10p^2$, where $0 \leq p \leq 150$, items per year. What price will bring in the greatest annual revenue?

② A company estimates that if it sets the price of an item at p dollars, then it can sell $q = 1,200 - p^2$, where $0 \leq p \leq 30$, items per year. What price will bring in the greatest annual revenue?

③ (Extra Credits)

A bus company will charter a bus that holds 50 people to groups of 35 or more. If a group contains exactly 35 people, each person pays \$60. In large groups, everybody's fare is reduced by \$1 for each person in excess of 35. Determine the size of the group for which the bus company's revenue will be greatest.