Marginal Analysis

$\mathcal{P}$: capital P

p : lowercase p

Example :

q: the number of units produced

* price function: $p(q)$

* cost fuction: $C(q)$

* Average cost function : $A(q) = \dfrac{C(q)}{q}$

* Revenue function: $R(q) = q \cdot p(q)$

* Profit function: $\mathcal{P}(q) = R(q) - C(q)$

$$= q \cdot p(q) - C(q)$$

* Marginal cost function is the derivative of the cost function,

$C'(q)$.

+ $C'(q)$ presents the change of the cost to produce 1 more item.

For example: $\boxed{C'(100)} \approx \underbrace{C(101) - C(100)}$

The change in cost to produce 1 more item $(100 \to 101)$

$$C'(1000) \approx \underbrace{C(1001) - C(1000)}$$

(*) Marginal Revenue: $R'(q)$

(*) Marginal Profit: $P'(q)$

Example:

Adam Goodman determines that when q thousand units of his product are produced each month, they will all be sold at a price of $p(q) = 22.2 - 1.2q$ dollars per unit. The total cost of producing the q units will be $C(q) = 0.4q^2 + 3q + 40$ thousand dollars.

a. How many units should Adam produce to maximize profit? What is the maximum profit he can expect?

$$P(q) = R(q) - C(q)$$

$$= q \cdot p(q) - C(q)$$

$$= q \cdot (22.2 - 1.2q) - (.4q^2 + 3q + 40)$$

$$= 22.2q - 1.2q^2 - .4q^2 - 3q - 40$$

$$\boxed{P(q) = -1.6q^2 + 19.2q - 40}$$

(*) Step 1: Find $P'(q)$

$$P'(q) = -1.6 * 2 \cdot q + 19.2$$

$$P'(q) = -3.2q + 19.2$$

(*) Step 2: Solve for $P'(q) = 0$

$$-3.2q + 19.2 = 0$$

$$\Rightarrow \quad 3.2q = 19.2$$

$$\Rightarrow \quad q = \frac{19.2}{3} = 6$$

Step 3: Find $P''(q)$ and test the size of $P''(6)$

$$P''(q) = (-3.2q + 19.2)'$$

$$P''(q) = -3.2$$

$$\Rightarrow \quad P''(6) = -3.2 < 0$$

$P(q)$ will be maximized at $q = 6$. The maximum profit is

$$P(6) = -1.6(6)^2 + 19.2(6) - 40 = 17.6$$

$$\boxed{P(q) = -1.6q^2 + 19.2q - 40}$$

Notice: (Marginal Analysis for maximum profit)

$$P(q) = R(q) - C(q)$$

$$P'(q) = [R(q) - C(q)]' = R'(q) - C'(q)$$

① $P(q)$ is maximized when $P'(q) = 0$ or

$$R'(q) - C'(q) = 0$$

$$\boxed{R'(q) = C'(q)}$$

marginal revenue

marginal cost

② $P(q)$ is maximized when $\underbrace{P''(q) < 0}$

b. How many units should Adam produce to minimize the average cost per unit of production $A(q) = \dfrac{C(q)}{q}$? What is the minimal average cost?

$$A(q) = \frac{C(q)}{q} = \frac{.4q^2 + 3q + 40}{q}$$

q : letter q

9 : number nine

$$A(q) = \frac{.4q^2}{q} + \frac{3q}{q} + \frac{40}{q} = .4q + 3 + \frac{40}{q}$$

$$A'(q) = .4 - \frac{40}{q^2}$$

$$\left[\left(\frac{40}{q}\right)' = (40 \cdot q^{-1})' = 40 \cdot (-1)\, q^{-2} = \frac{40}{q^2}\right]$$

$$A'(q) = 0$$

$$\Rightarrow \quad .4 - \frac{40}{q^2} = 0$$

$$\Rightarrow \quad \frac{40}{q^2} = .4 \quad \Rightarrow \quad q^2 = \frac{40}{.4} = 100$$

$$\Rightarrow \quad \boxed{q = 10}$$

$$A''(q) = \left[.4 - \frac{40}{q^2}\right]' = \left(-\frac{40}{q^2}\right)'$$

$$= \left(-40 \cdot q^{-2}\right)' = \left(-40 \cdot (-2) \cdot q^{-3}\right)$$

$$= 80\, q^{-3} = \frac{80}{q^3}$$

$\Rightarrow A''(10) = \frac{80}{10^3} > 0 \Rightarrow A(q)$ is minimized at $q = 10$

and the minimum is $A(10) = \frac{.4q^2 + 3q + 40}{q}$

$$= \frac{.4\,(10)^2 + 3 \cdot 10 + 40}{10} = \boxed{11}$$

* Notice: (Marginal Analysis for Average cost)

$$A(q) = \frac{C(q)}{q}$$

$A(q)$ is minimized when $A'(q) = 0$

$$\left[\frac{C(q)}{q}\right]' = 0$$

$$\frac{C'(q) \cdot q - C(q) \cdot (q)'}{q^2} = 0 \qquad (q') = 1$$

$$C'(q) \cdot q - C(q) = 0$$

$$\Rightarrow C'(q) \cdot q = C(q)$$

$$\Rightarrow C'(q) = \underbrace{\frac{C(q)}{q}}_{A(q)}$$

$$\boxed{\underset{\text{marginal cost}}{C'(q)} = \underset{\text{Average cost}}{A(q)}}$$

This means the average cost is minimized when the marginal cost is the same as the average cost

Assignment 23

Adam Goodman determines that when q thousand units of his product are produced each month, they will all be sold at a price of $p(q) =$ [redacted] dollars per unit. The total cost of producing the q units will be $C(q) =$ [redacted] thousand dollars.

a. How many units should Adam produce to maximize profit? What is the maximum profit he can expect?

b. How many units should Adam produce to minimize the average cost per unit of production $A(q) = \dfrac{C(q)}{q}$? What is the minimal average cost?

Use the following functions

1. $p(q) = 49 - q;\ C(q) = \frac{1}{8}q^2 + 4q + 200$

2. $p(q) = 37 - 2q;\ C(q) = 3q^2 + 5q + 75$

3. $p(q) = 180 - 2q;\ C(q) = q^3 + 5q + 162$