Exam 1 - Practice 2

Name:

Notice: Calculators are not allowed.

Some formulas:

- The derivative of $f(x)$ is defined by the formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- An equation of the tangent line at $x = a$ is

$$y - f(a) = f'(a)(x - a)$$

Problem 1.

Find the following limits.

$$\lim_{x \to 1} \frac{x^2 - 6x + 5}{x^2 - 3x + 2}$$

$$= \lim_{x \to 1} \frac{(x-1)(x-5)}{(x-1)(x-2)}$$

$$= \lim_{x \to 1} \frac{x-5}{x-2} = \frac{1-5}{1-2} = 4$$

$$\lim_{x \to 2024} \frac{x^2 - 6x + 5}{x^2 - 3x + 2}$$

$$= \frac{2024^2 - 6 \cdot 2024 + 5}{2024^2 - 3 \cdot 2024 + 2} = \frac{2019}{2022}$$

$$\lim_{x \to 0} \frac{2x^6 + 4x^2 + 3}{4x^6 + 5x + 6}$$

$$= \frac{2 \cdot 0^6 + 4 \cdot 0^2 + 3}{4 \cdot 0^6 + 5 \cdot 0 + 6} = 3/6$$

$$\lim_{x \to \infty} \frac{2x^6 + 4x^2 + 3}{4x^6 + 5x + 6}$$

$$= \lim_{x \to \infty} \frac{2 \cdot x^6}{4 \cdot x^6} = 2/4$$

$$\lim_{x \to \infty} \frac{2x^7 + 4x^2 + 3}{4x^6 + 5x + 6}$$

$$= \lim_{x \to \infty} \frac{2x^7}{4 \cdot x^6} = \lim_{x \to \infty} \frac{x}{2} = \infty$$

$$\lim_{x\to\infty} \frac{2x^6+4x^9+3}{4x^7+5x+6}$$

$$= \lim_{x\to\infty} \frac{4x^9}{4x^7} = \lim_{x\to\infty} x^2 = \infty$$

$$\lim_{x\to 1} \frac{\sin x}{\sin 7x}$$

$$= \frac{\sin 1}{\sin 7}$$

$$\lim_{x\to 0} \frac{\sin 30x}{\sin 15x}$$

$$= \lim_{x\to\infty} \frac{30x}{15x} = 2$$

$$\lim_{x\to 0} \frac{x^3+\sin 3x}{2x+4\sin 5x}$$

$$= \lim_{x\to 0} \frac{x^3+3x}{2x+4.5x}$$

$$= \lim_{x\to 0} \frac{x(x^2+3)}{22x}$$

$$= \lim_{x\to 0} \frac{x^2+3}{22} = 3/22$$

Problem 2

Find values of x, if any, at which the function is not continuous.

a. $f(x) = 10x^2 + \frac{x-1}{3} + 2024$

No values of x found because f(x) is continuous for any value of x

b. $f(x) = x^2 + \frac{3(x-2)}{(x-1)(x-3)} + 2024$

f(x) is not continuous when

$(x-1)(x-3) = 0$

or $x = 1, \; x = 3$

c. $f(x) = \frac{3}{2x+5} + \frac{x-1}{x^2-8x+12}$

f(x) is not continuous when

$2x + 5 = 0$ or $x = -5/2$

and

$x^2 - 8x + 12 = 0$

$\Leftrightarrow (x-6)(x-2) = 0 \Leftrightarrow x = 6, \; x = 2$

Problem 3.

Find a value of the constant k, if possible, that will make the function continuous everywhere.

a.

$$f(x) = \begin{cases} 2x - 2, & x \leq 1 \\ kx^2 + k, & x > 1 \end{cases}$$

$2 \cdot 1 - 2 = k + k$

$\Rightarrow k = 0$

b.

$$f(x) = \begin{cases} x^2 + x + 4, & x \leq 3 \\ x^2 + kx + k, & x > 3 \end{cases}$$

$$3^2 + 3 + 4 = 3^2 + 3k + k$$

$$\Rightarrow \quad 4k = 7 \quad \Rightarrow \quad k = 7/4$$

Problem 4.

a. Use the definition of derivatives to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = 1$

$$f(x) = -2x^2 + 4x + 1$$

$$\frac{f(x+h) - f(x)}{h} = \frac{-2(x+h)^2 + 4(x+h) + 1 - (-2x^2 + 4x + 1)}{h}$$

$$= \frac{-2(x^2 + 2xh + h^2) + 4x + 4h + 1 + 2x^2 - 4x - 1}{h}$$

$$= \frac{-2x^2 - 4xh - 2h^2 + 4x + 4h + 1 + 2x^2 - 4x - 1}{h}$$

$$= \frac{-4xh - 2h^2 + 4h}{h} = -4x - 2h + 4$$

plug $h = 0$ in: $\quad f'(x) = -4x + 4$

Equation of the tangent line at $x = 1$

$y - f(1) = f'(1)\,(x-1)$

$f(1) = -2 + 4 + 1 = 3$

$f'(1) = -4 + 4 = 0$

$\Rightarrow y - 3 = 0 \cdot (x-1)$

$\Rightarrow \boxed{y = 3}$

b. Use the definition of derivatives to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = 0$

$$f(x) = \frac{3}{2x+1}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{3}{2(x+h)+1} - \dfrac{3}{2x+1}}{h} = \frac{\dfrac{3(2x+1) - 3\left[2(x+h)+1\right]}{\left[2(x+h)+1\right](2x+1)}}{h}$$

$$= \frac{6x+3 - 3(2x+2h+1)}{(2x+2h+1)(2x+1)} \cdot \frac{1}{h}$$

$$= \frac{6x+3 - 6x - 6h - 3}{(2x+2h+1)(2x+1)} \cdot \frac{1}{h}$$

$$= \frac{-6\cancel{h}}{(2x+2h+1)(2x+1)} \cdot \frac{1}{\cancel{h}} = \frac{-6}{(2x+2h+1)(2x+1)}$$

Plug $h = 0$ in:

$$f'(x) = \frac{-6}{(2x+1)(2x+1)} = \frac{-6}{(2x+1)^2}$$

Equation of tangent line at $x = 0$

$y - f(0) = f'(0)\,(x-0)$

$f(0) = 3$

$f'(0) = -6$

$\Rightarrow y - 3 = -6(x)$

$\Rightarrow \boxed{y = -6x + 3}$