⊛ Chain Rule

② Let $y = g(x) = h(k(x))$. We can find $g'(x)$ as follows.

Let $u = k(x)$.

$$\boxed{g'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

$y = h(u)$ $\quad$ $u = k(x)$

$$y \longleftrightarrow x \quad \Leftrightarrow \quad y \longleftrightarrow u \longleftrightarrow x$$

$$y \longleftrightarrow v \longleftrightarrow u \longleftrightarrow x$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

Example: Find $f'(x)$ or dy/dx

$$y = f(x) = (\sin x)^{2024}$$

Let $\underbrace{u = \sin x}$. Then $y = u^{2024}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left(2024 \cdot u^{2023}\right) \cdot \cos x$$

$$= 2024 \cdot (\sin x)^{2023} \cdot \cos x$$

② $y = (\underbrace{x^2 + x + 1}_{u})^{100}$

$u = x^2 + x + 1 \Rightarrow \frac{du}{dx} = 2x + 1$

$y = u^{100} \Rightarrow \frac{dy}{du} = 100 \cdot u^{99}$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 100 \cdot u^{99} \cdot (2x+1)$

$= 100 (x^2 + x + 1)^{99} \cdot (2x+1)$

③ $y = \cos(\underbrace{x^2 + 3x + 2}_{u})$

$u = x^2 + 3x + 2 \Rightarrow \frac{dy}{du} = 2x + 3$

$y = \cos u \Rightarrow \frac{dy}{du} = -\sin u$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (-\sin u) \cdot (2x+1)$

$= [-\sin(x^2 + 3x + 2)] (2x+1)$

Practice : Find dy/dx

① $y = (\tan x)^{2025}$

② $y = \tan(x^2 + 9x)$

(*) Apply chain rule to power functions

$$y = [g(x)]^n$$

$$\Rightarrow \frac{dy}{dx} = n\,[g(x)]^{n-1} \cdot g'(x)$$

Example : $y = (\tan x)^{2025}$

$$\Rightarrow \frac{dy}{dx} = 2025 \cdot (\tan x)^{2024} \cdot (\tan x)'$$

$$= \boxed{2025 \cdot (\tan x)^{2024} \cdot \sec^2 x}$$

(*) Apply chain rules to trig. function

① $y = \sin(g(x))$

$$\Rightarrow \frac{dy}{dx} = [\cos g(x)] \cdot g'(x)$$

② $y = \cos g(x) \Rightarrow \frac{dy}{dx} = -[\sin g(x)] \cdot g'(x)$

③ $y = \tan g(x) \Rightarrow \frac{dy}{dx} = [\sec^2 g(x)] \cdot g'(x)$

④ $y = \cot g(x) \Rightarrow \frac{dy}{dx} = -[\csc^2 g(x)] \cdot g'(x)$

⑤ $y = \sec g(x)$

$$\Rightarrow \frac{dy}{dx} = [\sec g(x)] \cdot [\tan g(x)] \cdot g'(x)$$

⑤ $y = \csc g(x)$

$$\Rightarrow \frac{dy}{dx} = -[\csc g(x)] \cdot [\cot g(x)] \cdot g'(x)$$

Example : $y = \tan(x^2 + 9x)$

By ③, $$\frac{dy}{dx} = [\sec^2(x^2+9x)] \cdot (x^2+9x)'$$

$$= [\sec^2(x^2+9x)] \cdot (2x+9)$$

Example :

① $y = \sin(\cos x)$

$$\Rightarrow \frac{dy}{dx} = [\cos(\cos x)] \cdot (\cos x)'$$

$$= \cos(\cos x) \cdot -\sin x$$

$$= -\cos(\cos x) \cdot \sin x$$

② $y = \cos(\sqrt{x} + 1)$

$$\frac{dy}{dx} = -[\sin(\sqrt{x}+1)] \cdot (\sqrt{x}+1)'$$

$$= -[\sin(\sqrt{x}+1)] \cdot \left(\frac{1}{2} \cdot x^{-1/2}\right)$$

Practice :

① $y = (\sin x + \cos x)^{100}$

② $y = \sin(\sqrt{x} + x)$

③ $y = \tan(x^3 + x + 1)$

④ $y = \tan(\sin x + x^2)$