Final Exam: Practice 2

Name:

- *Basic Calculators are allowed. Graphic calculators are not allowed.*
- *A page of formula is allowed. Only formulas are allowed on the page. The page will be checked during the exam.*

Problem 1

Use the definition of derivatives to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = 1$

$$f(x) = x^2 + 4x + 1$$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 + 4(x+h) + 1 - (x^2 + 4x + 1)}{h}$$

$$= \frac{\cancel{x^2} + 2xh + h^2 + \cancel{4x} + 4h + \cancel{1} - \cancel{x^2} - \cancel{4x} - \cancel{1}}{h}$$

$$= \frac{2xh + h^2 + 4h}{h} = \frac{h(2x + h^2 + 4)}{h}$$

$$= 2x + h + 4$$

plug $h = 0$ in : $f'(x) = 2x + 4$

(*) tangent line at $x = 1$: $y - f(1) = f'(1)(x-1)$

$f(1) = 1^2 + 4 \cdot 1 + 1 = 6$; $f'(1) = 2 + 4 = 6$

$$\Rightarrow \boxed{y - 6 = 6(x-1)}$$

Problem 2

Find $f'(x)$.

$$f(x) = \frac{x^8}{2} - \frac{4x^3}{7} - \frac{1}{\sqrt{x}} + \sqrt[3]{x} + 2024x + 1$$

$$f(x) = \frac{x^8}{2} - \frac{4x^3}{7} - x^{-1/2} + x^{1/3} + 2024x + 1$$

$$\Rightarrow \quad f'(x) = \frac{8x^7}{2} - \frac{12x^2}{7} + \frac{1}{2}x^{-3/2} + \frac{1}{3}x^{-2/3} + 2024$$

$$f(x) = (\sqrt[4]{x} + 1)(x + 1)$$

$$f(x) = (x^{1/4} + 1)(x + 1)$$

$$\Rightarrow \quad f'(x) = (x^{1/4} + 1)' \cdot (x+1) + (x^{1/4} + 1)(x+1)'$$

$$= 1/4\, x^{-3/4} \cdot (x+1) + (x^{1/4} + 1) \cdot 1$$

$$f(x) = \frac{3x + 2}{3x - 2} \text{ (Simplify your answers.)}$$

$$f'(x) = \frac{(3x+2)' \cdot (3x-2) - (3x+2)(3x-2)'}{(3x-2)^2}$$

$$= \frac{3(3x-2) - (3x+2) \cdot 3}{(3x-2)^2} = \frac{9x - 6 - 9x - 6}{(3x-2)^2} = \frac{-12}{(3x-2)^2}$$

$$f(x) = x^7 \sin x$$

$$f'(x) = (x^7)' \cdot \sin x + x^7 (\sin x)'$$

$$= 7x^6 \cdot \sin x + x^7 \cdot \cos x$$

$$f(x) = \frac{x}{3\cos x}$$

$$f'(x) = \frac{(x)'(3\cos x) - x \cdot (3\cos x)'}{(3\cos x)^2}$$

$$= \frac{3\cos x + 3x \sin x}{(3\cos x)^2}$$

$$f(x) = \cos^{2024} x$$

$$f'(x) = 2024 \cos^{2023} x \cdot (\cos x)'$$

$$= 2024 \cos^{2023} x \cdot (-\sin x)$$

$$f(x) = \cos(2x^4 + x^2 + 1)$$

$$f'(x) = -\sin(2x^4 + x^2 + 1) \cdot [2x^4 + x^2 + 1]'$$

$$= -\sin(2x^4 + x^2 + 1) \cdot [8x^3 + 2x]$$

$$f(x) = \cos\left(2\sin x + 3\cos x + x\right)$$

$$f'(x) = -\sin(2\sin x + 3\cos x + x) \cdot [2\sin x + 3\cos x + x]'$$

$$= -\sin(2\sin x + 3\cos x + x)\, [2\cos x - 3\sin x + 1]$$

$$f(x) = \left(\cos x - \sin x\right)^{100}$$

$$f'(x) = 100(\cos x - \sin x)^{99} \cdot (\cos x - \sin x)'$$

$$= 100(\cos x - \sin x)^{99} [-\sin x - \cos x]$$

$$f(x) = 4^x + 6^x - 7\log_8 x + 9\ln x - \frac{3\log_2 x}{3} + \frac{\log_7 x}{2} + x + 1$$

$$f'(x) = 4^x \ln 4 + 6^x \ln 6 - \frac{7}{x \cdot \ln 8} + \frac{9}{x} - \frac{1}{x \ln 2} + \frac{1}{2x \ln 7} + 1$$

$$f(x) = \ln\left(2x^2 + 3x + \cos x\right)$$

$$f'(x) = \frac{(2x^2 + 3x + \cos x)'}{2x^2 + 3x + \cos x}$$

$$= \frac{4x + 3 - \sin x}{2x^2 + 3x + \cos x}$$

$$f(x) = 3^{\sin x + \cos x}$$

$$f'(x) = 3^{\sin x + \cos x} \cdot \ln 3 \cdot (\sin x + \cos x)'$$

$$= 3^{\sin x + \cos x} \cdot \ln 3 \cdot (\cos x - \sin x)$$

$$f(x) = 3^{x^2 \cos x}$$

$$f'(x) = 3^{x^2 \cdot \cos x} \cdot \ln 3 \cdot (x^2 \cos x)'$$

$$= 3^{x^2 \cos x} \cdot \ln 3 \left[(x^2)' \cdot \cos x + x^2 (\cos x)' \right]$$

$$= 3^{x^2 \cos x} \cdot \ln 3 \left[2x \cos x - x^2 \sin x \right]$$

Problem 3

$$y + x^4y + 3x^3 = 2$$

(a) Find dy/dx or y' by differentiating implicitly.

$$\left(y + x^4 y + 3x^2\right)' = (2)'$$

$$\Rightarrow \quad y' + (x^4 y)' + 6x = 0$$

$$\Rightarrow \quad y' + \left[4x^3 y + x^4 \cdot y'\right] + 6x = 0$$

$$\Rightarrow \quad y' + x^4 y' = -6x - 4x^3 y$$

$$\Rightarrow \quad y'\left[1 + x^4\right] = -6x - 4x^3 y$$

$$\Rightarrow \quad y' = \frac{-6x - 4x^3 y}{1 + x^4}$$

(b) Solve the equation for y as a function of x, and find dy/dx from that equation.

(*) Solve for y :

$$y + x^4 y + 3x^2 = 2$$

$$\Rightarrow \quad y(1 + x^4) + 3x^2 = 2$$

$$\Rightarrow \quad y(1 + x^4) = 2 - 3x^2$$

$$\Rightarrow \quad y = \frac{2 - 3x^2}{1 + x^4}$$

(*) Find dy/dx

$$\frac{dy}{dx} = y' = \left(\frac{2 - 3x^2}{1 + x^4}\right)'$$

$$= \frac{(2 - 3x^2)' \cdot (1 + x^4) - (2 - 3x^2)(1 + x^4)'}{(1 + x^4)^2}$$

$$= \frac{-6x(1 + x^4) - (2 - 3x^2)(4x^3)}{(1 + x^4)^2}$$

(c) Write an equation for the tangent line at the point (0, 2)

slope at $(0, 2)$: $\quad y' = \dfrac{-6x - 4x^3 y}{1 + x^4} = 0$

Equation of tangent line :

$$y - 2 = 0 \cdot (x - 0)$$

$$\Rightarrow \quad y - 2 = 0$$

$$\Rightarrow \quad \boxed{y = 2}$$

Problem 4

(a) Find the local linear approximation of $f(x) = e^x$ at $x_0 = 0$. Notice that $e^0 = 1$

$f(x) \approx f'(x_0)(x - x_0) + f(x_0)$

$\Rightarrow f(x) \approx f'(0)\, x + f(0)$

we have: $f(0) = e^0 = 1$

$f'(x) = (e^x)' = e^x$

$\Rightarrow f'(0) = e^0 = 1$

$\Rightarrow e^x \approx x + 1$

(b) Use the local linear approximation obtained in part (a) to approximate $e^{.01}$

$e^x \approx x + 1 \Rightarrow e^{.01} \approx .01 + 1 = 1.01$

Problem 5

Given that

$$f(x) = x^3 + 6x^2 + 9x + 1$$

Find all the intervals where

a. $f(x)$ is increasing
b. $f(x)$ is decreasing
c. $f(x)$ is concave upward
d. $f(x)$ is concave downward

$$f'(x) = 3x^2 + 12x + 9$$
$$= 3(x^2 + 4x + 3)$$
$$= 3(x+1)(x+3)$$

$f'(x) = 0 \Rightarrow x = -1, \; x = -3$

$$\begin{array}{ccccc} & -3 & & -1 & \\ + + + + + & & - - - - - - & & + + + + \quad f'(x) \end{array}$$

$\Rightarrow$ $f(x)$ are increasing on $(-\infty, -3)$ and $(-1, \infty)$

$$f''(x) = 6x + 12$$
$$= 6(x+2)$$

$f''(x) = 0 \Rightarrow x = -2$

sign chart:

$$\begin{array}{ccc} & -2 & \\ - - - - - - - & & + + + + + + \quad f''(x) \end{array}$$

$\Rightarrow$ $f(x)$ is concave downward on $(-\infty, -2)$

$f(x)$ is concave upward on $(-2, \infty)$

Problem 6

Find all the relative extrema of

$$f(x) = 2x^3 - 9x^2 + 12x + 2$$

$$f'(x) = 6x^2 - 18x + 12$$

$$= 6(x - 3x + 2)$$

$$= 6(x-1)(x-2)$$

$$f'(x) = 0 \Rightarrow x = 1, \; x = 2$$

Sign chart :

	1		2	
$+ + + + +$		$- - - - -$		$+ + + + +$ $f'(x)$

Relative max when $f'(x)$ changes from $(+)$ to $(-)$

$\Rightarrow$ relative max at $x = 1$

$f(1) = 7$

Relative min when $f'(x)$ changes from $(-)$ to $(+)$

$\Rightarrow$ relative min at $x = 2$

$f(2) = 16 - 36 + 24 + 2$

$= 6$

Problem 7

Find the absolute maximum and absolute minimum of $f(x) = x^3 + 6x^2 + 9x + 1$ on the interval [-4, 4].

$$f'(x) = 3x^2 + 12x + 9$$

$$= 3(x^2 + 4x + 3)$$

$$= 3(x+1)(x+3)$$

$$f'(x) = 0 \Leftrightarrow x = -1; \; x = -3$$

Evaluating $f(x)$ at these to points $(-1, -3)$ and the ending points $(-4, 4)$:

$f(-1) = -3$

$f(-3) = 1$

$f(-4) = -3$

$f(4) = 197$

The absolute min is the smallest of these 4 numbers, which is -3.

The absolue max is the largest of these 4 numbers, which is 197

Problem 8

The given equation has one (real) solution. Approximate the solution by Newton's method.

$$x^3 + 5x + 2 = 0$$

we have : $f'(x) = 3x^2 + 5$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 + 5x_n + 2}{3x_n^2 + 5}$$

$$x_0 = 0 \Rightarrow x_1 = x_0 - \frac{x_0^3 + 5x_0 + 2}{3x_0^2 + 5}$$

$$= -0.4$$

$$x_1 = x_0 - \frac{x_0^3 + 5x_0 + 2}{3x_0^2 + 5} = -0.4$$

$$x_2 = x_1 - \frac{x_1^3 + 5x_1 + 2}{3x_1^2 + 5} = -0.3883$$

$$x_3 = x_2 - \frac{x_2^3 + 5x_2 + 2}{3x_2^2 + 5} = -0.3883$$

since it is convergent to -0.3883, the solution is -0.3883

Problem 9

Find the following

$$\int \left(2x^4 - x^3 + 2x + 1\right) dx$$

$$= \frac{2x^5}{5} - \frac{x^4}{4} + x^2 + x + C$$

$$\int \left(\sqrt[5]{x} - 3x + \frac{2}{x} + 1\right) dx$$

$$= \int \left(x^{1/5} - 3x + \frac{2}{x} + 1\right) dx$$

$$= \frac{x^{6/5}}{6/5} - \frac{3x^2}{2} + 2\ln|x| + x + C$$

$$\int \left(3^x + 2\sin x - 3\cos x + 3x + 1\right) dx$$

$$= \frac{3^x}{\ln 3} - 2\cos x - 3\sin x + \frac{3x^2}{2} + x + 1$$

$$\int (x^2 + x + 1)(x^2 + 2)dx$$

$$= \int (x^4 + x^3 + x^2 + 2x^2 + 2x + 2)\, dx$$

$$= \int (x^4 + x^3 + 3x^2 + 2x + 2)\, dx$$

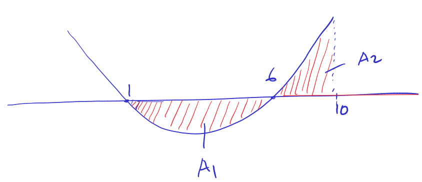
$$= \frac{x^5}{5} + \frac{x^4}{4} + \frac{3x^3}{3} + \frac{2x^2}{2} + 2x + C$$

Problem 10

Calculate the area between $f(x) = x^2 - 7x + 6$ and x-axis bounded by $x = 1$ and $x = 10$

$$f(x) = x^2 - 7x + 6$$

$$= (x-1)(x-6)$$



$$A = A_1 + A_2$$

$$A_1 = -\int_1^6 (x^2 - 7x + 6)\, dx$$

$$A_2 = \int_6^{10} (x^2 - 7x + 6)\, dx$$

$$A_1 = -\int_1^6 (x^2 - 7x + 6)\, dx = \int_6^1 (x^2 - 7x + 6)\, dx$$

$$= \frac{x^3}{3} - \frac{7x^2}{2} + 6x \Big|_6^1$$

$$= \left(\frac{1^3}{3} - \frac{7.1^2}{2} + 6.1 \right) - \left(\frac{6^3}{3} - \frac{7.6^2}{2} + 6.6 \right)$$

$$= -20.833$$

$$A_2 = \int_6^{10} \left(x^2 - 7x + 6 \right) dx$$

$$= \frac{x^3}{3} - \frac{7x^2}{2} + 6x \Big|_6^{10}$$

$$= \left(\frac{10^3}{3} - \frac{7.10^2}{2} + 6.10 \right) - \left(\frac{6^3}{3} - \frac{7.6^2}{2} + 6.6 \right)$$

$$= 61.333$$

$$\Rightarrow \quad A = A_1 + A_2$$

$$= 20.833 + 61.333$$

$$= 82.1666$$