# Final Exam: Practice 2

*Name:*

- *Basic Calculators are allowed. Graphic calculators are not allowed.*
- *A page of formula is allowed. Only formulas are allowed on the page. The page will be checked during the exam.*

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**Problem 1**

Use the definition of derivatives to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = 1$

$$f(x) = 2x^2 - 3x + 4$$

$$\frac{f(x+h) - f(x)}{h} = \frac{2(x+h)^2 - 3(x+h) + 4 - [2x^2 - 3x + 4]}{h}$$

$$= \frac{2(x^2 + 2xh + h^2) - \cancel{3x} - 3h + \cancel{4} - 2x^2 + \cancel{3x} - \cancel{4}}{h}$$

$$= \frac{2x^2 + 4xh + 2h^2 - 3h - 2x^2}{h}$$

$$= \frac{4xh + 2h^2 - 3h}{h} = \frac{h(4x + 2h - 3)}{h}$$

$$= 4x + 2h - 3$$

plus $h=0$ in, we have:

$$f'(x) = 4x - 3$$

Tangent line at $x=1$ : $y - f(1) = f'(1)(x-1)$

$f(1) = 2-3+4 = 3, \quad f'(1) = 4-3 = 1$

$\Rightarrow \quad y - 3 = 1\cdot(x-1)$

$\Rightarrow \quad y = x - 1 + 3$

$\Rightarrow \quad \boxed{y = x + 2}$

**Problem 2**

Find $f'(x)$.

$$f(x) = \frac{2x^3}{3} + \frac{x^5}{7} - \frac{2}{\sqrt[3]{x}} + \frac{5}{6\sqrt{x}} + 2024$$

$f(x) = \frac{2}{3}x^3 + \frac{1}{7}x^5 - 2x^{-1/3} + \frac{5}{6}x^{-1/2} + 2024$

$\Rightarrow f'(x) = \frac{2}{3}\cdot 3x^2 + \frac{1}{7}\cdot 5x^4 - 2\left(-\frac{1}{3}\right)x^{-4/3} + \frac{5}{6}\cdot\left(-1/2\right)\cdot x^{-3/2}$

$= 2x^2 + \frac{5}{7}x^4 + \frac{2}{3}x^{-4/3} - \frac{5}{12}x^{-3/2}$

Formulas

$(x^n)' = n\cdot x^{n-1}$

$\sqrt[k]{x} = x^{1/k}$

$\sqrt[k]{x^n} = x^{n/k}$

$\frac{1}{x^k} = x^{-k}$

$$f(x) = (\sqrt{x} + 1)(x + 1)$$

$f(x) = (x^{1/2} + 1)(x+1)$

product rule

$\Rightarrow f'(x) = (x^{1/2}+1)'\cdot(x+1) + (x^{1/2}+1)(x+1)'$

$= \frac{1}{2}x^{-1/2}\cdot(x+1) + (x^{1/2}+1)\cdot 1$

product rule

$(f\cdot g)' = f'\cdot g + f\cdot g'$

$$f(x) = \frac{x-1}{x+1} \text{ (Simplify your answers.)}$$

$f'(x) = \frac{(x-1)'\cdot(x+1) - (x-1)(x+1)'}{(x+1)^2}$

Quotient rule

$= \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$

$$f(x) = x\sin x$$

$\Rightarrow f'(x) = x'\cdot\sin x + x\cdot(\sin x)'$

$= \sin x + x\cdot\cos x$

$(\sin x)' = \cos x$

$$f(x) = \frac{x}{\tan x}$$

$$(\tan x)' = \sec^2 x$$

$$f'(x) = \frac{(x)' \cdot \tan x - x \cdot (\tan x)'}{(\tan x)^2}$$

$$= \frac{\tan x - x \cdot \sec^2 x}{(\tan x)^2}$$

$$f(x) = \cos^{2024} x$$

$$(\cos x)' = -\sin x$$

$$\left( [g(x)]^n \right)' = n \cdot (g(x))^{n-1} \cdot g'(x)$$

$$f'(x) = 2024 \cos^{2023} x \cdot (\cos x)'$$

$$= 2024 \cos^{2023} x \cdot (-\sin x)$$

$$f(x) = \cos(3x^2 + x + 1)$$

$$[\cos(g(x))]' = [-\sin g(x)] \cdot g'(x)$$

$$f'(x) = [-\sin(3x^2 + x + 1)] \cdot (3x^2 + x + 1)'$$

$$= [-\sin(3x^2 + x + 1)] \cdot (6x + 1)$$

$$f(x) = \tan\left(\cos x + \sqrt{x}\right)$$

$$[\tan(g(x))]' = [\sec^2 g(x)] \cdot g'(x)$$

$$f'(x) = [\sec^2(\cos x + \sqrt{x})] \cdot (\cos x + \sqrt{x})'$$

$$= [\sec^2(\cos x + \sqrt{x})] \cdot \left(-\sin x + \frac{1}{2} x^{-1/2}\right)$$

$$f(x) = \Big(\cos x + \sin x\Big)^{2024}$$

$$f'(x) = 2024\,(\cos x + \sin x)^{2023} \cdot (\cos x + \sin x)'$$

$$= 2024\,(\cos x + \sin x)^{2023}\,(-\sin x + \cos x)$$

$$f(x) = 2024^x + 7^x - 2\log_9 x + 3\ln x - \frac{3\log_2 x}{5} + \frac{\log_7 x}{3} + 2024$$

$$f'(x) = 2024^x \cdot \ln 2024 + 7^x \cdot \ln 7 - \frac{2}{x \cdot \ln 9} + \frac{3}{x} - \frac{3}{5x \cdot \ln 2} + \frac{1}{3x \cdot \ln 7}$$

$$(\log_b x)' = \frac{1}{x \cdot \ln b}$$

$$(\ln x)' = \frac{1}{x}$$

$$(b^x)' = b^x \cdot \ln b$$

$$(e^x)' = e^x$$

$$f(x) = \log_7\Big(\sqrt{x} + x^2 + x + 1\Big)$$

$$f(x) = \frac{(\sqrt{x} + x^2 + x + 1)'}{(\sqrt{x} + x^2 + x + 1) \cdot \ln 7}$$

$$= \frac{\frac{1}{2}x^{-1/2} + 2x + 1}{(\sqrt{x} + x^2 + x + 1)\ln 7}$$

$$\left[\log_b g(x)\right]' = \frac{g'(x)}{g(x) \cdot \ln b}$$

$$f(x) = e^{\sin x + \tan x + 2x^3}$$

$$f'(x) = e^{\sin x + \tan x + 2x^3} \cdot (\sin x + \tan x + 2x^3)'$$

$$= e^{\sin x + \tan x + 2x^3} \cdot (\cos x + \sec^2 x + 6x^2)$$

$$\left[e^{g(x)}\right]' = e^{g(x)} \cdot g'(x)$$

$$f(x) = e^{x \sin x}$$

$$f'(x) = e^{x \cdot \sin x} \cdot (x \cdot \sin x)'$$

$$= e^{x \cdot \sin x} \cdot \left[(x)' \cdot \sin x + x \cdot (\sin x)'\right]$$

$$= e^{x \sin x}\left[\sin x + x \cos x\right]$$

**Problem 3**

$$y + xy - 2x^3 = 1$$

(a) Find $dy/dx$ or $y'$ by differentiating implicitly.

$$(y + xy - 2x^3)' = (1)'$$

$$\Rightarrow y' + (xy)' - 6x^2 = 0$$

$$\Rightarrow y' + [x' \cdot y + x \cdot y'] - 6x^2 = 0$$

$$\Rightarrow y' + y + xy' - 6x^2 = 0$$

$$\Rightarrow y' + xy' = 6x^2 - y$$

$$\Rightarrow y'[1 + x] = 6x^2 - y$$

$$\Rightarrow y' = \frac{6x^2 - y}{1 + x}$$

(b) Solve the equation for y as a function of x, and find $dy/dx$ from that equation.

⊛ Solve for $y$:

$$y + xy = 1 + 2x^3$$

$$\Rightarrow y(1 + x) = 1 + 2x^3$$

$$\Rightarrow y = \frac{1 + 2x^3}{1 + x}$$

⊛ Find $y'$

$$y' = \frac{(1 + 2x^3)' \cdot (1 + x) - (1 + 2x^3)(1 + x)'}{(1 + x)^2}$$

$$y' = \frac{6x^2(1 + x) - (1 + 2x^3)}{(1 + x)^2}$$

(c) Write an equation for the tangent line at the point (0, 1)

The slope at $(0, 1)$ is $y' = \dfrac{6x^2 - y}{1 + x} = -1$ (we can use either formulas for $y'$)

The tangent line at $(0, 1)$ is

$$y - 1 = -1 \cdot (x - 0)$$

$\Rightarrow \boxed{y = -x + 1}$

**Problem 4**

(a) Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$

$$f(x) \approx f'(x_0)(x - x_0) + f(x_0)$$

$$\Rightarrow \boxed{f(x) = f'(1)(x-1) + f(1)}$$

we have: $f'(x) = (\sqrt{x})' = (x^{1/2})'$

$$= 1/2 \cdot x^{-1/2}$$

$$\Rightarrow f'(1) = 1/2 \cdot 1^{-1/2} = 1/2$$

$f(1) = \sqrt{1} = 1$

$$\Rightarrow f(x) \approx 1/2 \cdot (x - 1) + 1$$

$$\Rightarrow f(x) \approx \frac{1}{2}x + \frac{1}{2}$$

$$\Rightarrow \boxed{\sqrt{x} \approx \frac{1}{2}x + \frac{1}{2}}$$

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{1.1}$

we have: $\sqrt{x} \approx \frac{1}{2}x + \frac{1}{2}$.

plug in $x = 1.1$ :

$$\sqrt{1.1} \approx \frac{1}{2} \cdot 1.1 + 1/2 = 1.05$$

## Problem 5

Given that

$$f(x) = x^3 - 3x^2 + 1$$

Find all the intervals where

a. $f(x)$ is increasing
b. $f(x)$ is decreasing
c. $f(x)$ is concave upward
d. $f(x)$ is concave downward

$$f(x) = x^3 - 3x^2 + 1$$

Step 1 : Find $f'(x)$ and factor it

$$f'(x) = 3x^2 - 6x$$

$$= 3x(x-2)$$

Step 2 : Solve $f'(x) = 0$

$$3x(x-2) = 0$$

$$\underbrace{x = 0}, \quad \underbrace{x = 2}$$

Step 3 : Get the sign chart of $f'(x)$

$$\begin{array}{ccccc} & 0 & & 2 & \\ \hline + + + + & \bullet & - - - - & \bullet & + + + + \end{array} \longrightarrow x$$

$f'(x)$

$x < 0$ : plug in a number $x < 0$, say $x = -10$

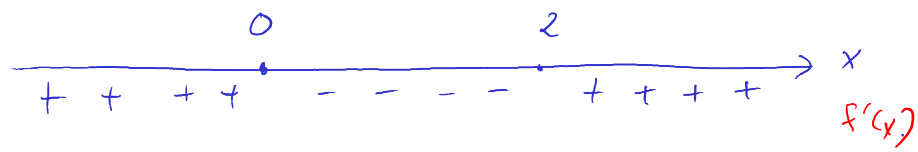
$$f'(-10) = 3(-10)(-10-2) > 0 \qquad (+)$$

$0 < x < 2$ : plug in $x = 1$

$$f'(1) = 3 \cdot (1)(1-2) < 0 \qquad (-)$$

$x > 2$ : plug in $x = 3$

$$f'(3) = 3(3)(3-2) > 0 \qquad (+)$$




(a) f(x) is increasing on positive intervals of $f'(x)$ :

$(-\infty, 0)$ and $(2, \infty)$

(b) f(x) is decreasing on negative intervals of $f'(x)$ :

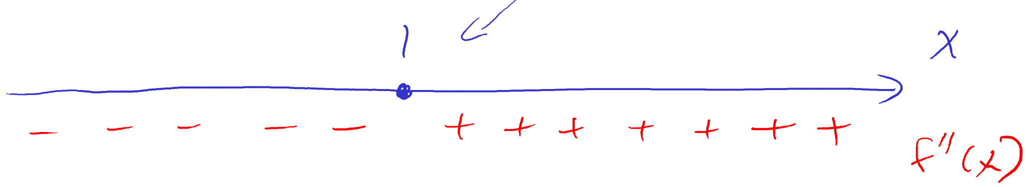
$(0, 2)$

For concavity :

$$f''(x) = (3x^2 - 6x)' = 6x - 6$$

$$= 6(x-1)$$

$$f''(x) = 0 \iff \boxed{x = 1}$$




f(x) is concave upward on postive intervals of $f''(x)$

$(1, \infty)$

f(x) is concave down on negative intervals of

## Problem 6

Find all the relative extrema of

$$f(x) = x^4 - 12x^3$$

$f'(x) = 4x^3 - 36x^2 = 4x^2(x-9)$

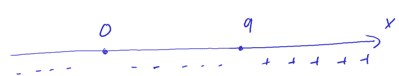
Set $f'(x) = 0$ to find stationary points:

$\Rightarrow \quad 4x^2(x-9) = 0$

$\Rightarrow \quad 4x^2 = 0$ or $x - 9 = 0$

$\Rightarrow \quad x = 0$ or $x = 9$

(*) Sign chart of $f'(x)$




Since $f'(x)$ change sign from $(-)$ to $(+)$ at $x = 9$

$(9, f(9))$ is a relative minimum.

$f(9) = 9^4 - 12 \cdot 9^3 = -2187$

There is no realative maximum.

## Problem 7

Find the absolute maximum and absolute minimum of $f(x) = x^3 - 6x^2 + 9x + 1$ on the interval [-5, 7].

we have : $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

$= 3(x-1)(x-3)$

$f'(x) = 0 \iff x = 1, \quad x = 3$

we have $f(1) = 5$ $\qquad f(-5) = -319$

$f(3) = 1$ $\qquad f(7) = 113$

$\Rightarrow$ The absolute max is : $f(7) = 113$

The absolute min is : $f(-5) = -319$

**Problem 8**

The given equation has one (real) solution. Approximate the solution by Newton's method.

$$x^3 - 2x - 2 = 0$$

we have : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$f'(x) = 3x^2 - 2$

$\Rightarrow \quad x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n - 2}{3x_n^2 - 2}$

choose $x_0 = 1$, we have

| $n$ | $x_n$ |
| --- | --- |
| 0 | $x_0 = 1$ |
| 1 | $x_1 = x_0 - \dfrac{x_0^3 - 2x_0 - 2}{3x_0^2 - 2}$ <br> $= 4$ |
| | $x_2 = 2.82$ |
| | $x_3 = 2.14$ |
| | $x_4 = 1.842$ |
| | $x_5 = 1.772$ |
| | $x_6 = 1.769$ |
| | $x_7 = 1.769$ |
| | $x_8 = 1.769$ |

Since $x_n$ is converging, we stop here.

The solution is $x \approx 1.769$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int b^x \, dx = \frac{b^x}{\ln b} + C$$

$$\int e^x \, dx = e^x + C$$

Formulas :

$$\int dx = x + C$$

$$\int k \, dx = kx + C$$

$$\int x \, dx = \frac{x^2}{2} + C$$

## Problem 9

Find the following

$$\int \left(x^7 - 2x^6 + 2x + 2024\right) dx$$

$$= \frac{x^8}{8} - \frac{2 \cdot x^7}{7} + x^2 + 2024x + C$$

$$\int \left(\sqrt{x} + x + \frac{1}{x}\right) dx$$

$$= \int \left(x^{1/2} + x + \frac{1}{x}\right) dx$$

$$= \frac{x^{1/2+1}}{1/2+1} + \frac{x^2}{2} + \ln|x| + C$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^2}{2} + \ln|x| + C$$

$$\int \left( 2^x + 2\sin x - 3\cos x + 1 \right) dx$$

$$= \frac{2^x}{\ln 2} - 2\cos x - 3\sin x + x + C$$

$$\int (x+1)(x+2)dx$$

$$= \int \left(x^2 + 3x + 2\right) dx$$

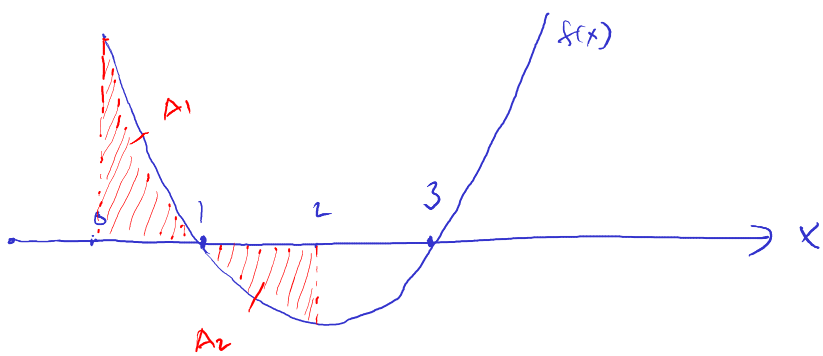
$$= \frac{x^3}{3} + \frac{3x^2}{2} + 2x + C$$

**Problem 10**

Calculate the area between $f(x) = x^2 - 4x + 3$ and x-axis bounded by $x = 0$ and $x = 2$

$$f(x) = x^2 - 4x + 3$$

$$= (x-1)(x-3)$$




$$A_1 = \int_0^1 f(x)\,dx$$

The area $= A_1 + A_2$

$$A_2 = -\int_1^2 f(x)\,dx$$

$$A_1 = \int_0^1 (x^2 - 4x + 3)\,dx = \frac{x^3}{3} - 2x^2 + 3x \Big|_0^1 = 4/3$$

$$A_2 = -\int_1^2 (x^2 - 4x + 3)\,dx = \int_2^1 (x^2 - 4x + 3)\,dx$$

$$= \frac{x^3}{3} - 2x^2 + 3x \Big|_2^1 = 4/3 - \left(\frac{2^3}{3} - 2.2^2 + 3.2\right)$$

$$= \frac{4}{3} - \frac{8}{3} + 2 = \frac{6}{3} - \frac{4}{3} = 2/3$$

$\Rightarrow$ Area $= A_1 + A_2 = 4/3 + 2/3 = 2$