Section 3.2:

(*) The second derivative test:

Assume that $f'(x)$ and $f''(x)$ exist at $x = x_0$

and $f'(x_0) = 0$ (i.e, x_0 is a stationary point)

then:

1. If $f''(x_0) > 0$ then $(x_0, f(x_0))$ is a relative minimum

2. If $f''(x_0) < 0$ then $(x_0, f(x_0))$ is a relative maximum

3. If $f''(x_0) = 0$ the $(x_0, f(x_0))$ may or may not be a relative extremum.

Example : Find all relative extrema of

$$f(x) = x^3 - 9x^2 + 1$$

$$f'(x) = 3x^2 - 18x = 0$$

$$\Rightarrow \quad 3x(x-6) = 0$$

$$\Rightarrow \quad x = 0, \ \boxed{x = 6} \quad \text{(stationary points)}$$

$$f''(x) = 6x - 18$$

we have ⊛ $f''(0) = 6 \cdot 0 - 18 < 0$

$\Rightarrow$ $(0, f(0))$ is a relative max

⊛ $f''(6) = 6 \cdot 6 - 18 = 18 > 0$

$\Rightarrow$ $(6, f(6))$ is a relative min

Relative max $(0, f(0))$ or $(0, 1)$

Relative min $(6, f(6))$ or $(6, -107)$

Practice: Find all relative extrema using the 2^{nd} derivative test

$$f(x) = -x^3 - 6x^2 + 2$$

Gradient Descent Method.

Given the function $y = f(x)$. A relative extrema of $y = f(x)$ can be found using the following procedure:

Step 1: Make a initial guess x_0

Step 2: Update the guess using

$$x_{n+1} = x_n - \lambda \cdot f'(x_n)$$

where λ is called a learning rate usually from 0 to 1

Example: Use gradient descent to find an relative extremum of $f(x) = x^3 - 9x^2 + 1$

Step 1: Make an initial guess: $x_0 = 10$

Step 2 : updating the guess

$$x_{n+1} = x_n - \eta . f'(x_n)$$

we choose $\eta = .01$

$$f'(x) = 3x^2 - 18x$$

$$\Rightarrow \quad x_{n+1} = x_n - .01 * (3x_n^2 - 18x_n)$$

x_0	10
x_1	$x_0 - .01\,(3x_0^2 - 18 \cdot x_0)$ $= 10 - .01 * (3 \cdot 10^2 - 18 \cdot 10) = 8.8$
x_2	$x_1 - .01\,(3x_1^2 - 18\,x_1)$ $= 8.8 - .01\,(3 . (8.8)^2 - 18 * 8.8)$ $= 8.068$
x_3	$x_2 - .01\,(3x_2^2 - 18\,x_2)$ $= 8.068 - .01\,(3 * (8.068)^2 - 18 * 8.068)$ $= 7.56$
x_4	7.2
x_5	6.94
x_6	6.79
	6.59

$\vdots$

6.00001

An etrema is at $x \approx 6$

<u>Example</u>: $f(x) = x^4 + x^2 + 1$

Find a relative extreme using gradient descent.