Exam 2 is scheduled for Tuesday, Nov 12.

Review is on Thursday, Nov 7

(*) Chain Rule on Exp./Log functions

(1) $$\left[b^{g(x)} \right]' = b^{g(x)} \cdot \ln b \cdot g'(x)$$

[Note: outside function is exp. and the inside is $g(x)$]

Example:

(1) $f(x) = 2024^{\cos x} \Rightarrow f'(x) = 2024^{\cos x} \cdot \ln 2024 \cdot (\cos x)'$

$$= 2024^{\cos x} \cdot \ln 2024 \; (-\sin x)$$

(2) $f(x) = e^{x^2+x+1}$

$$\Rightarrow f'(x) = e^{x^2+x+1} \cdot \ln e \cdot (x^2+x+1)'$$

$$= e^{x^2+x+1} \cdot (2x+1)$$

[Note: $\ln e = 1$]

② $$\left(\log_6 [\, g(x) \,]\right)' = \frac{g'(x)}{g(x) \cdot \ln 6}$$

special case:

$$\left(\ln [g(x)]\right)' = \frac{g'(x)}{g(x)}$$

<u>Example</u> Find $f'(x)$

① $$f(x) = \log_7 (\underbrace{\sqrt{x} + x^3}_{g(x)})$$

$$\Rightarrow f'(x) = \frac{(\sqrt{x} + x^3)'}{(\sqrt{x} + x^3) \ln 7}$$

$$= \frac{\frac{1}{2} \cdot x^{-1/2} + 3x^2}{(\sqrt{x} + x^3) \cdot \ln 7}$$

<u>Practice</u> : Find $f'(x)$

① $$f(x) = e^{\sin x + \cos x + x^7}$$

$$\Rightarrow f'(x) = e^{\sin x + \cos x + x^7} \cdot (\cos x - \sin x + 7x^6)$$

② $$f(x) = \log_9 (x^3 + x^2 + \sin x)$$

$$f'(x) = \frac{(x^3 + x^2 + \sin x)'}{(x^3 + x^2 + \sin x) \cdot \ln 9} = \frac{3x^2 + 2x + \cos x}{(x^3 + x^2 + \sin x) \cdot \ln 9}$$

③ $f(x) = \ln(\sin x \cdot \cos x)$

$$f'(x) = \frac{(\sin x \cdot \cos x)'}{\sin x \cdot \cos x} = \frac{(\sin x)' \cdot \cos x + \sin x \cdot (\cos x)'}{\sin x \cdot \cos x}$$

$$= \frac{\cos^2 x - \sin^2 x}{\sin x \cdot \cos x}$$

Section 2.7 : Implicit Differentiation

$y = f(x) = x^x$ $\quad f'(x) = ?$

usually the equation of a function is in the form

$$y = f(x)$$

then y' or $\frac{dy}{dx}$ can be called explicit differentiation

But some time a function can be in the form

$$f(x, y) = 0$$

For example:

$$x^3 + y^3 + 1 = 0$$

In this case the function y is given implicitly.

If we want to find y' in this case then we can either

① Solve y by itself explicitly then find y' explicitly.

$$x^3 + y^3 + 1 = 0$$

$$\Rightarrow \quad y^3 = -x^3 - 1$$

$$\Rightarrow \quad y = \sqrt[3]{-x^3-1} = (-x^3-1)^{1/3}$$

$$\Rightarrow \quad \boxed{y' = \frac{1}{3}(-x^3-1)^{-2/3} \cdot (-3x^2)}$$

② OR we can find y' implicitly also called implicit differentiation. we do not have to get y by itself

$$x^3 + y^3 + 1 = 0$$

$$(x^3 + y^3 + 1)' = (0)'$$

$$\Rightarrow \quad (x^3)' + (y^3)' + (1)' = 0$$

$$\Rightarrow \quad 3x^2 + 3y^2 \cdot y' = 0$$

$\Rightarrow \quad 3y^2 \cdot y' = -3x^2$

$\Rightarrow \quad y' = \dfrac{-3x^2}{3y^2}$

$\Rightarrow \quad \boxed{y' = \dfrac{-x^2}{y^2}}$

(*) Remark: Implicit differentiation can be used to find y' even when y is not a function of x.

Explicit differentiation may not be practical to find y' when y is not a function of x.

Example: $x^2 + y^2 - 1 = 0$

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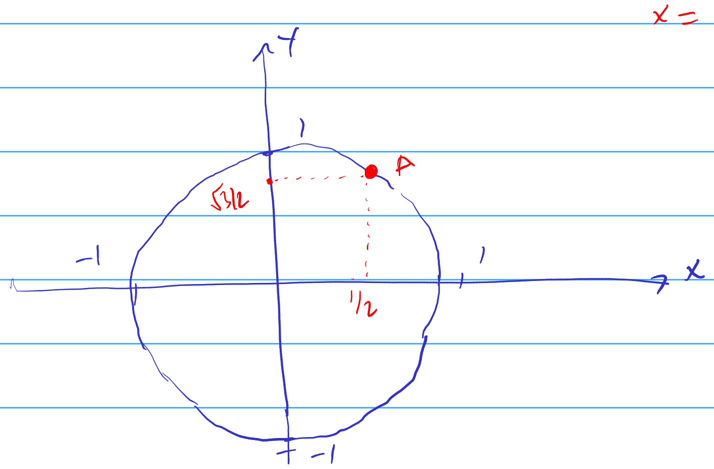
y is not a function of x b/c one input x can produce more than 1 output y.

Say $x = 0$, $y = 1$ or $y = -1$

$x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$



Let find y' implicitly :

$$x^2 + y^2 - 1 = 0$$

$$\Rightarrow (x^2 + y^2 - 1)' = 0$$

$$\Rightarrow 2x + 2y \cdot y' = 0$$

$$\Rightarrow 2y \cdot y' = -2x$$

$$\Rightarrow y' = \frac{-2x}{2y}$$

$$\Rightarrow y' = -\frac{x}{y}$$

(2) write an equation of the tangent line to the graph at the point $A(\underbrace{1/2}_{x_0}, \underbrace{\sqrt{3}/2}_{y_0})$

Formula : In general, an equation of the tangent line at the point (x_0, y_0) is

$$y - y_0 = y'_{\text{evaluated at } (x_0, y_0)} \cdot (x - x_0)$$

In this problem, $x_0 = 1/2$; $y_0 = \sqrt{3}/2$

$$y'_{\text{evaluated at } x_0, y_0} = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$$

So an equation is:

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right)$$

Given the equation

$$x^3 + y^3 = 3xy$$

1. Use implicit differentiation to find dy/dx or y'

$$(x^3 + y^3)' = (3xy)'$$

$$3x^2 + 3y^2 \cdot y' = (3x)' \cdot y + 3x \cdot y'$$

$$3x^2 + 3y^2 \cdot y' = 3y + 3x \cdot y'$$

(now we set y' by itself)

$$\Rightarrow \quad 3y^2 \cdot y' - 3x \cdot y' = 3y - 3x^2$$

$$\Rightarrow \quad y'(3y^2 - 3x) = 3y - 3x^2$$

$$\Rightarrow \quad y' = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}$$

2. Find an equation for the tangent line at the point $(3/2, 3/2)$

$\downarrow x_0$ $\quad \downarrow y_0$

$y - y_0 = y'_{\text{evaluated at } (x_0, y_0)} \cdot (x - x_0)$

The slope $= y'$ evaluated at $(3/2, 3/2)$

$$= \frac{3/2 - (3/2)^2}{(3/2)^2 - 3/2} = -1$$

The equation is

$$y - 3/2 = -1\,(x - 3/2)$$

$$\Rightarrow \quad y = -x + 3$$

Practice : $x + xy - 2x^3 = 2$

(1) Find $\frac{dy}{dx}$ or y' using implicit diff.

(2) Write the equation of the tangent line at $(1, 3)$

(1) $(x + xy - 2x^3)' = (2)'$

$(x)' + (xy)' - (2x^3)' = 0$

$\Rightarrow \quad 1 + \underline{x'} \cdot y + x \cdot y' - 6x^2 = 0$

$\Rightarrow \quad 1 + y + \underline{xy'} - 6x^2 = 0$

$$\Rightarrow \quad x y' = -1 - y + 6x^2$$

$$\Rightarrow \quad y' = \frac{-1 - y + 6x^2}{x}$$

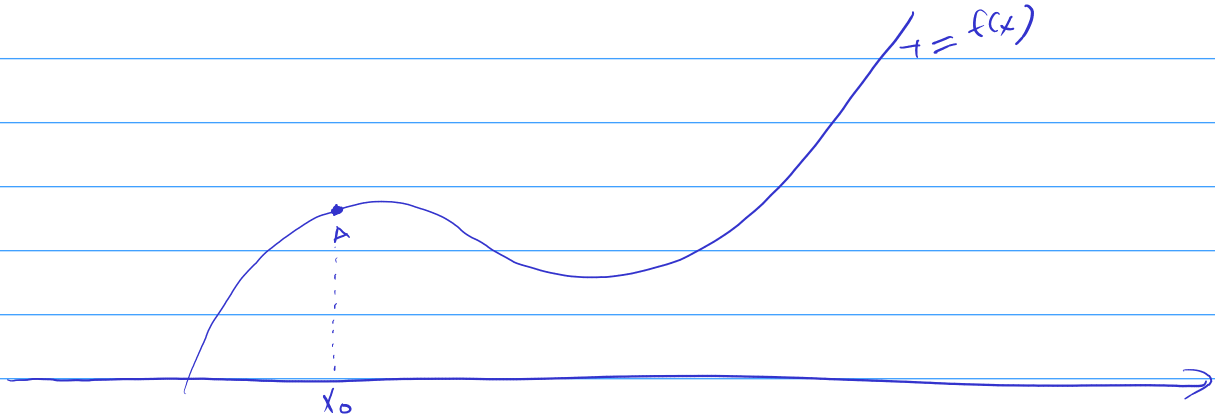
② $y - 3 = y'$ evaluated at $(1, 3) \cdot (x - 1)$

y' evaluated at $(1, 3) = \dfrac{-1 - 3 + 6}{1} = 2$

$$\Rightarrow \quad \boxed{y - 3 = 2(x - 1)}$$

$$y = 2x + 1$$

2.9. Local Linear Approximation



when $x \approx x_0$, $f(x) \approx$ tangent line at $x = 0$

OR $\quad f(x) \approx \underbrace{f'(x_0) \cdot (x - x_0) + f(x_0)}_{\text{linear function}}$

Thus, $f'(x_0)(x - x_0) + f(x_0)$ is called a local linear approx. of $f(x)$ at $x = x_0$

Example :

(a) Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$

$$f(x) \approx f'(x_0)(x - x_0) + f(x_0)$$

$$\Rightarrow \boxed{f(x) = f'(1)(x-1) + f(1)}$$

we have: $f'(x) = (\sqrt{x})' = (x^{1/2})'$

$$= 1/2 \cdot x^{-1/2}$$

$$\Rightarrow f'(1) = 1/2 \cdot 1^{-1/2} = 1/2$$

$f(1) = \sqrt{1} = 1$

$$\Rightarrow f(x) \approx 1/2 \cdot (x-1) + 1$$

$$\Rightarrow f(x) \approx \frac{1}{2}x + \frac{1}{2}$$

$$\Rightarrow \boxed{\sqrt{x} \approx \frac{1}{2}x + \frac{1}{2}}$$

$\sqrt{1.1}$ $\frac{1}{2} \cdot 1.1 + 1/2 =$

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{1.1}$

$\sqrt{1.1} \approx \frac{1}{2} \cdot 1.1 + 1/2 = 1.05$