

Final Exam: Practice 2

Name: _____

- Basic Calculators are allowed. Graphic calculators are not allowed.
 - A page of formula is allowed. Only formulas are allowed on the page. The page will be checked during the exam.
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Problem 1

Use the definition of derivatives to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = 1$

$$f(x) = x^2 + 4x + 1$$

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 + 4(x+h) + 1 - (x^2 + 4x + 1)}{h} \\&= \frac{\cancel{x^2} + 2xh + h^2 + \cancel{4x} + 4h + \cancel{1} - \cancel{x^2} - \cancel{4x} - \cancel{1}}{h} \\&= \frac{2xh + h^2 + 4h}{h} = \frac{h(2x + h + 4)}{h} \\&= 2x + h + 4\end{aligned}$$

$$\text{plug } h=0 \text{ in: } f'(x) = 2x + 4$$

$$(*) \text{ tangent line at } x=1: \quad y - f(1) = f'(1)(x-1)$$

$$f(1) = 1^2 + 4 \cdot 1 + 1 = 6; \quad f'(1) = 2 + 4 = 6$$

$$\Rightarrow \boxed{y - 6 = 6(x-1)}$$

Problem 2

Find $f'(x)$.

$$f(x) = \frac{x^8}{2} - \frac{4x^3}{7} - \frac{1}{\sqrt{x}} + \sqrt[3]{x} + 2024x + 1$$

$$f(x) = \frac{x^8}{2} - \frac{4x^3}{7} - x^{-1/2} + x^{1/3} + 2024x + 1$$

$$\Rightarrow f'(x) = \frac{8x^7}{2} - \frac{12x^2}{7} + \frac{1}{2}x^{-3/2} + \frac{1}{3}x^{-2/3} + 2024$$

$$f(x) = (\sqrt[4]{x} + 1)(x + 1)$$

$$f(x) = (x^{1/4} + 1)(x + 1)$$

$$\begin{aligned}\Rightarrow f'(x) &= (x^{1/4} + 1)' \cdot (x + 1) + (x^{1/4} + 1) \cdot (x + 1)' \\ &= \frac{1}{4}x^{-3/4} \cdot (x + 1) + (x^{1/4} + 1) \cdot 1\end{aligned}$$

$$f(x) = \frac{3x + 2}{3x - 2} \text{ (Simplify your answers.)}$$

$$\begin{aligned}f'(x) &= \frac{(3x+2)' \cdot (3x-2) - (3x+2)(3x-2)'}{(3x-2)^2} \\ &= \frac{3(3x-2) - (3x+2) \cdot 3}{(3x-2)^2} = \frac{9x-6-9x-6}{(3x-2)^2} = \frac{-12}{(3x-2)^2}\end{aligned}$$

$$f(x) = x^7 \sin x$$

$$\begin{aligned}f'(x) &= (x^7)' \cdot \sin x + x^7 (\sin x)' \\ &= 7x^6 \cdot \sin x + x^7 \cdot \cos x\end{aligned}$$

$$f(x) = \frac{x}{3 \cos x}$$

$$\begin{aligned}f'(x) &= \frac{(x)'(3 \cos x) - x \cdot (3 \cos x)'}{(3 \cos x)^2} \\ &= \frac{3 \cos x + 3x \sin x}{(3 \cos x)^2}\end{aligned}$$

$$f(x) = \cos^{2024} x$$

$$f'(x) = 2024 \cos^{2023} x \cdot (\cos x)'$$

$$= 2024 \cos^{2023} x \cdot (-\sin x)$$

$$f(x) = \cos(2x^4 + x^2 + 1)$$

$$f'(x) = -\sin(2x^4 + x^2 + 1) \cdot [2x^4 + x^2 + 1]'$$

$$= -\sin(2x^4 + x^2 + 1) \cdot [8x^3 + 2x]$$

$$f(x) = \cos(2\sin x + 3\cos x + x)$$

$$f'(x) = -\sin(2\sin x + 3\cos x + x) \cdot [2\sin x + 3\cos x + x]'$$

$$= -\sin(2\sin x + 3\cos x + x) [2\cos x - 3\sin x + 1]$$

$$f(x) = (\cos x - \sin x)^{100}$$

$$f'(x) = 100 (\cos x - \sin x)^{99} \cdot (\cos x - \sin x)'$$

$$= 100 (\cos x - \sin x)^{99} [-\sin x - \cos x]$$

$$f(x) = 4^x + 6^x - 7 \log_8 x + 9 \ln x - \frac{3 \log_2 x}{3} + \frac{\log_7 x}{2} + x + 1$$

$$f'(x) = 4^x \ln 4 + 6^x \ln 6 - \frac{7}{x \cdot \ln 8} + \frac{9}{x} - \frac{1}{x \ln 2} + \frac{1}{2x \ln 7} + 1$$

$$\begin{aligned} f(x) &= \ln(2x^2 + 3x + \cos x) \\ f'(x) &= \frac{(2x^2 + 3x + \cos x)'}{2x^2 + 3x + \cos x} \\ &= \frac{4x + 3 - \sin x}{2x^2 + 3x + \cos x} \end{aligned}$$

$$\begin{aligned} f(x) &= 3^{\sin x + \cos x} \\ f'(x) &= 3^{\sin x + \cos x} \cdot \ln 3 \cdot (\sin x + \cos x)' \\ &= 3^{\sin x + \cos x} \cdot \ln 3 \cdot (\cos x - \sin x) \end{aligned}$$

$$\begin{aligned} f(x) &= 3^{x^2 \cos x} \\ f'(x) &= 3^{x^2 \cos x} \cdot \ln 3 \cdot (x^2 \cos x)' \\ &= 3^{x^2 \cos x} \cdot \ln 3 \cdot [(x^2)' \cdot \cos x + x^2 (\cos x)'] \\ &= 3^{x^2 \cos x} \cdot \ln 3 \cdot [2x \cos x - x^2 \sin x] \end{aligned}$$

Problem 3

$$y + x^4 y + 3x^3 = 2$$

(a) Find dy/dx or y' by differentiating implicitly.

$$(y + x^4 y + 3x^3)' = (2)'$$

$$\Rightarrow y' + (x^4 y)' + 6x = 0$$

$$\Rightarrow y' + [4x^3 y + x^4 \cdot y'] + 6x = 0$$

$$\Rightarrow y' + x^4 y' = -6x - 4x^3 y$$

$$\Rightarrow y' [1 + x^4] = -6x - 4x^3 y$$

$$\Rightarrow y' = \frac{-6x - 4x^3 y}{1 + x^4}$$

(b) Solve the equation for y as a function of x , and find dy/dx from that equation.

* Solve for y : $y + x^4 y + 3x^3 = 2$

$$\Rightarrow y(1 + x^4) + 3x^3 = 2$$

$$\Rightarrow y(1 + x^4) = 2 - 3x^3$$

$$\Rightarrow y = \frac{2 - 3x^3}{1 + x^4}$$

* Find dy/dx

$$\frac{dy}{dx} = y' = \left(\frac{2 - 3x^3}{1 + x^4} \right)'$$

$$= \frac{(2 - 3x^3)' \cdot (1 + x^4) - (2 - 3x^3)(1 + x^4)'}{(1 + x^4)^2}$$

$$= \frac{-6x(1 + x^4) - (2 - 3x^3)(4x^3)}{(1 + x^4)^2}$$

(c) Write an equation for the tangent line at the point $(0, 2)$

slope at $(0, 2)$: $y' = \frac{-6x - 4x^3 y}{1 + x^4} = 0$

Equation of tangent line:

$$y - 2 = 0 \cdot (x - 0)$$

$$\Rightarrow y - 2 = 0$$

$$\Rightarrow \boxed{y = 2}$$

Problem 4

- (a) Find the local linear approximation of $f(x) = e^x$ at $x_0 = 0$. Notice that $e^0 = 1$

$$f(x) \approx f'(x_0)(x - x_0) + f(x_0)$$

$$\Rightarrow f(x) \approx f'(0)x + f(0)$$

we have: $f(0) = e^0 = 1$

$$f'(x) = (e^x)' = e^x$$

$$\Rightarrow f'(0) = e^0 = 1$$

$$\Rightarrow e^x \approx x + 1$$

- (b) Use the local linear approximation obtained in part (a) to approximate $e^{.01}$

$$e^x \approx x + 1 \quad \Rightarrow \quad e^{.01} \approx .01 + 1 = 1.01$$

Problem 5

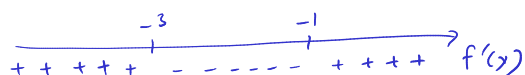
Given that

$$f(x) = x^3 + 6x^2 + 9x + 1$$

Find all the intervals where

- a. $f(x)$ is increasing
- b. $f(x)$ is decreasing
- c. $f(x)$ is concave upward
- d. $f(x)$ is concave downward

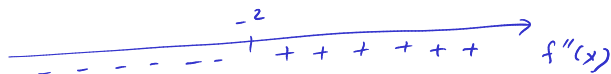
$$\begin{aligned} (*) \quad f'(x) &= 3x^2 + 12x + 9 \\ &= 3(x^2 + 4x + 3) \\ &= 3(x+1)(x+3) \\ f'(x) = 0 &\Rightarrow x = -1, x = -3 \end{aligned}$$



$\Rightarrow f(x)$ are increasing on $(-\infty, -3)$ and $(-1, \infty)$

$$\begin{aligned} (*) \quad f''(x) &= 6x + 12 \\ &= 6(x+2) \\ f''(x) = 0 &\Rightarrow x = -2 \end{aligned}$$

sign chart:



$\Rightarrow f(x)$ is concave downward on $(-\infty, -2)$

$f(x)$ is concave upward on $(-2, \infty)$

Problem 6

Find all the relative extrema of

$$f(x) = 2x^3 - 9x^2 + 12x + 2$$

$$f'(x) = 6x^2 - 18x + 12$$

$$= 6(x - 3x + 2)$$

$$= 6(x - 1)(x - 2)$$

$$f'(x) = 0 \Rightarrow x = 1, x = 2$$

Sign Chart:



Relative max when $f'(x)$ changes from (+) to (-)

\Rightarrow relative max at $x = 1$

$$f(1) = 7$$

Relative min when $f'(x)$ changes from (-) to (+)

\Rightarrow relative min at $x = 2$

$$f(2) = 16 - 36 + 24 + 2$$

$$= 6$$

Problem 7

Find the absolute maximum and absolute minimum of $f(x) = x^3 + 6x^2 + 9x + 1$ on the interval $[-4, 4]$.

$$f'(x) = 3x^2 + 12x + 9$$

$$= 3(x^2 + 4x + 3)$$

$$= 3(x + 1)(x + 3)$$

$$f'(x) = 0 \Rightarrow x = -1; x = -3$$

Evaluating $f(x)$ at these two points $(-1, -3)$

and the ending points $(-4, 4)$:

$$f(-1) = -3$$

$$f(-3) = 1$$

$$f(-4) = -3$$

$$f(4) = 197$$

The absolute min is the smallest of these 4 numbers, which is -3 .

The absolute max is the largest of these 4 numbers, which is 197 .

Problem 8

The given equation has one (real) solution. Approximate the solution by Newton's method.

$$x^3 + 5x + 2 = 0$$

$$\text{we have : } f'(x) = 3x^2 + 5$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 + 5x_n + 2}{3x_n^2 + 5}$$

$$x_0 = 0 \Rightarrow x_1 = x_0 - \frac{x_0^3 + 5x_0 + 2}{3x_0^2 + 5}$$

$$= -0.4$$

$$x_1 = x_0 - \frac{x_0^3 + 5x_0 + 2}{3x_0^2 + 5} = -0.4$$

$$x_2 = x_1 - \frac{x_1^3 + 5x_1 + 2}{3x_1^2 + 5} = -0.3883$$

$$x_3 = x_2 - \frac{x_2^3 + 5x_2 + 2}{3x_2^2 + 5} = -0.3883$$

since it is convergent to -0.3883 , the solution is -0.3883

Problem 9

Find the following

$$\int (2x^4 - x^3 + 2x + 1) dx$$

$$= \frac{2x^5}{5} - \frac{x^4}{4} + x^2 + x + C$$

$$\int \left(\sqrt[5]{x} - 3x + \frac{2}{x} + 1 \right) dx$$

$$= \int \left(x^{1/5} - 3x + \frac{2}{x} + 1 \right) dx$$

$$= \frac{x^{6/5}}{6/5} - \frac{3x^2}{2} + 2 \ln|x| + x + C$$

$$\int \left(3^x + 2 \sin x - 3 \cos x + 3x + 1 \right) dx$$

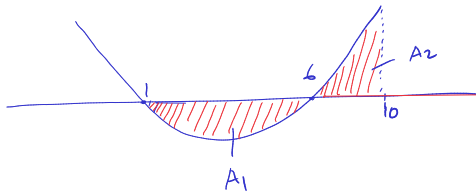
$$= \frac{3^x}{\ln 3} - 2 \cos x - 3 \sin x + \frac{3x^2}{2} + x + 1$$

$$\begin{aligned}
 & \int (x^2 + x + 1)(x^2 + 2) dx \\
 &= \int (x^4 + x^3 + x^2 + 2x^2 + 2x + 2) dx \\
 &= \int (x^4 + x^3 + 3x^2 + 2x + 2) dx \\
 &= \frac{x^5}{5} + \frac{x^4}{4} + \frac{3x^3}{3} + \frac{2x^2}{2} + 2x + C
 \end{aligned}$$

Problem 10

Calculate the area between $f(x) = x^2 - 7x + 6$ and x-axis bounded by $x = 1$ and $x = 10$

$$\begin{aligned}
 f(x) &= x^2 - 7x + 6 \\
 &= (x-1)(x-6)
 \end{aligned}$$



$$A = A_1 + A_2$$

$$A_1 = - \int_1^6 (x^2 - 7x + 6) dx$$

$$A_2 = \int_6^{10} (x^2 - 7x + 6) dx$$

$$A_1 = - \int_1^6 (x^2 - 7x + 6) dx = \int_6^1 (x^2 - 7x + 6) dx$$

$$= \left. \frac{x^3}{3} - \frac{7x^2}{2} + 6x \right|_6^1$$

$$= \left(\frac{1^3}{3} - \frac{7 \cdot 1^2}{2} + 6 \cdot 1 \right) - \left(\frac{6^3}{3} - \frac{7 \cdot 6^2}{2} + 6 \cdot 6 \right)$$

$$= -20.833$$

$$A_2 = \int_6^{10} (x^2 - 7x + 6) dx$$

$$= \left. \frac{x^3}{3} - \frac{7x^2}{2} + 6x \right|_6^{10}$$

$$= \left(\frac{10^3}{3} - \frac{7 \cdot 10^2}{2} + 6 \cdot 10 \right) - \left(\frac{6^3}{3} - \frac{7 \cdot 6^2}{2} + 6 \cdot 6 \right)$$

$$= 61.333$$

$$\begin{aligned} \Rightarrow A &= A_1 + A_2 \\ &= 20.833 + 61.333 \\ &= 82.1666 \end{aligned}$$