$$y = f(x) = x^3 - 9x^2 + 1$$

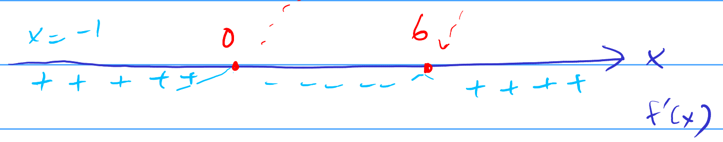
(*) <u>Increasing / decreasing</u>

$$f'(x) = 3x^2 - 18x = 0$$

$$\Rightarrow \quad 3x(x-6) = 0$$

$$\Rightarrow \quad x = 0, \quad x = 6$$

sign chart of $f'(x)$



$x<0$:

$x=-1$

$\Rightarrow f'(-1) = 3(-1)(-1-6)$

$= (+)$

$0<x<6$

$x = 4$

$f'(4) = 3(4)(4-6)$

$(-)$

$f(x)$ increasing on $x<0$ and $x>6$

$f(x)$ decreasing on $0<x<6$

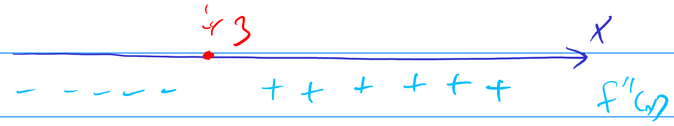
(*) Concavity:

$$f''(x) = 6x - 18 = 0$$

$$\Rightarrow \quad 6(x-3) = 0$$

$$\Rightarrow \quad x = 3$$

sign chart for $f''(x)$

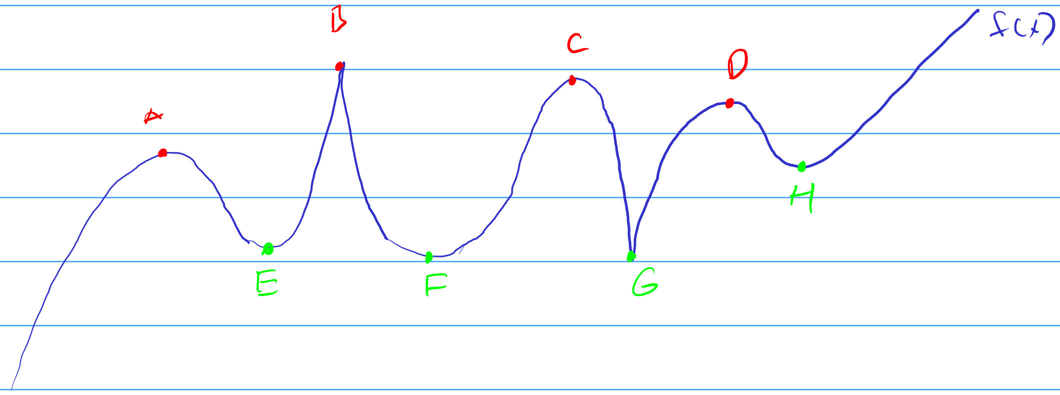


$f(x)$ concave down : $x < 3$

$f(x)$ concave up $x > 3$

Section: 3.2 : Analysis of functions II : Relative Extrema ; Graphing polynomials

(*) Relative Extrema.



Relative maxima : A, B, C, D

Relative minima : E, F, G, H

All of these points are called relative extrema

(*) Critical points :

x_0 is a critical point of $f(x)$ if either

$f'(x_0) = 0$, or

$f'(x_0)$ DNE ($f(x)$ is not differentiable at x_0)

If $f'(x_0) = 0$, we call x_0 is a stationary point.

we have the following result:

A relative extrema occurs at critical points where $f'(x)$ changes the sign:

If $f'(x)$ changes the sign from $(-)$ to $(+)$ at $x = x_0$ then $(x_0, f(x_0))$ is a relative minimum.

If $f'(x)$ changes the sign from $(+)$ to $(-)$ at $x = x_0$ then $(x_0, f(x_0))$ is a relative maxima.

Example: Find all relative extrema of

$$f(x) = x^3 - 9x^2 + 1$$

Step 1: Find all critical points of $f(x)$

All points where $f'(x)$ DNE and

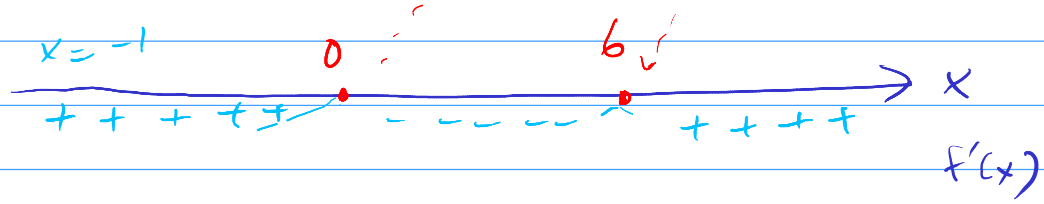
$f'(x) = 0$ (stationary points)

$$f'(x) = 3x^2 - 18x = 0$$

$x = 0, \quad x = 6$ (from the previous example)

<u>Step 2</u>: Draw the sign chart and get the relative extrema.

Sign chart of $f'(x)$



relative maximum $(0, f(0)) = (0, 1)$

relative minimum $(6, f(6)) = (6, -107)$

$f(x) = x^3 - 9x^2 + 1 \Rightarrow f(0) = 1$

$$f(6) = 6^3 - 9 \cdot 6^2 + 1$$

$$= -107$$

Example : $f(x) = 3x^{5/3} - 15x^{2/3}$.

Find all critical points

Step 1 : Find all critical points

$$f'(x) = 3 \cdot \frac{5}{3} x^{5/3 - 1} - 15 \cdot 2/3 \cdot x^{2/3 - 1}$$

$$= 5x^{2/3} - 10x^{-1/3}$$

$$= 5\sqrt[3]{x^2} - 10 \cdot \frac{1}{\sqrt[3]{x}}$$

⊛ when $f'(x)$ DNE : $x = 0$

⊛ when $f'(x) = 0$ (Stationary points)

$$5\sqrt[3]{x^2} - \frac{10}{\sqrt[3]{x}} = 0$$

$$\Rightarrow \frac{5\sqrt[3]{x^2}}{1} = \frac{10}{\sqrt[3]{x}}$$

$$\Rightarrow 5 \cdot \sqrt[3]{x^2} \cdot \sqrt[3]{x} = 10$$

$$\Rightarrow \sqrt[3]{x^3} = \frac{10}{5} \Rightarrow \boxed{x = 2}$$

7–14 Locate the critical points and identify which critical points are stationary points. ■

7. $f(x) = 4x^4 - 16x^2 + 17$ **8.** $f(x) = 3x^4 + 12x$

3.2.4 THEOREM (*Second Derivative Test*) *Suppose that f is twice differentiable at the point x_0.*

(*a*) *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.*

(*b*) *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.*

(*c*) *If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.*

Find the relative extrema of $f(x) = 3x^5 - 5x^3$.

29–32 Find the relative extrema using both first and second derivative tests. ■

29. $f(x) = 1 + 8x - 3x^2$ **30.** $f(x) = x^4 - 12x^3$

33–42 Use any method to find the relative extrema of the function f. ■

33. $f(x) = x^4 - 4x^3 + 4x^2$ **34.** $f(x) = x(x-4)^3$

35. $f(x) = x^3(x+1)^2$ **36.** $f(x) = x^2(x+1)^3$

37. $f(x) = 2x + 3x^{2/3}$ **38.** $f(x) = 2x + 3x^{1/3}$

39. $f(x) = \dfrac{x+3}{x-2}$ **40.** $f(x) = \dfrac{x^2}{x^4+16}$

7–16 Find the absolute maximum and minimum values of f on the given closed interval, and state where those values occur. ■

7. $f(x) = 4x^2 - 12x + 10;\ [1, 2]$

8. $f(x) = 8x - x^2;\ [0, 6]$

9. $f(x) = (x - 2)^3;\ [1, 4]$

10. $f(x) = 2x^3 + 3x^2 - 12x;\ [-3, 2]$

5–8 The given equation has one real solution. Approximate it by Newton's Method. ■

5. $x^3 - 2x - 2 = 0$

6. $x^3 + x - 1 = 0$

7. $x^5 + x^4 - 5 = 0$

8. $x^5 - 3x + 3 = 0$