Exam 2 - Practice 1

Notice:

- *Calculators are not allowed.*
- *Exam 2 is scheduled for Tuesday, Nov 12.*
- *A page of formula is allowed. Only formulas are allowed on the page. The page will be checked during the exam.*

Problem 1.

Find $f'(x)$.

$$f(x) = \frac{2x^3}{3} + \frac{x^5}{7} - \frac{2}{\sqrt[3]{x}} + \frac{5}{6\sqrt{x}} + 2024$$

$f(x) = \frac{2}{3}x^3 + \frac{1}{7}x^5 - 2x^{-1/3} + \frac{5}{6}x^{-1/2} + 2024$

$\Rightarrow f'(x) = \frac{2}{3}\cdot 3x^2 + \frac{1}{7}\cdot 5x^4 - 2\left(-\frac{1}{3}\right)x^{-4/3} + \frac{5}{6}\cdot\left(-\frac{1}{2}\right)\cdot x^{-3/2}$

$= 2x^2 + \frac{5}{7}x^4 + \frac{2}{3}x^{-4/3} - \frac{5}{12}x^{-3/2}$

Formulas

$(x^n)' = n\cdot x^{n-1}$

$\sqrt[k]{x} = x^{1/k}$

$\sqrt[k]{x^n} = x^{n/k}$

$\frac{1}{x^k} = x^{-k}$

$$f(x) = (\sqrt{x} + 1)(x + 1)$$

$f(x) = (x^{1/2} + 1)(x+1)$

product rule

$\Rightarrow f'(x) = (x^{1/2}+1)'\cdot(x+1) + (x^{1/2}+1)(x+1)'$

$= \frac{1}{2}x^{-1/2}\cdot(x+1) + (x^{1/2}+1)\cdot 1$

product rule

$(f\cdot g)' = f'\cdot g + f\cdot g'$

$$f(x) = \frac{x-1}{x+1} \text{ (Simplify your answers.)}$$

$f'(x) = \frac{(x-1)'\cdot(x+1) - (x-1)(x+1)'}{(x+1)^2}$

Quotient rule

$= \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$

$$f(x) = x\sin x$$

$\Rightarrow$ $f'(x) = x' \cdot \sin x + x \cdot (\sin x)'$

$= \sin x + x \cdot \cos x$

$(\sin x)' = \cos x$

$$f(x) = \frac{x}{\tan x}$$

$f'(x) = \dfrac{(x)' \cdot \tan x - x \cdot (\tan x)'}{(\tan x)^2}$

$= \dfrac{\tan x - x \cdot \sec^2 x}{(\tan x)^2}$

$(\tan x)' = \sec^2 x$

$$f(x) = \cos^{2024} x$$

$f'(x) = 2024 \cos^{2023} x \cdot (\cos x)'$

$= 2024 \cos^{2023} x \cdot (-\sin x)$

$(\cos x)' = -\sin x$

$\left([g(x)]^n\right)' = n \cdot (g(x))^{n-1} \cdot g'(x)$

$$f(x) = \cos(3x^2 + x + 1)$$

$f'(x) = \left[-\sin(3x^2 + x + 1)\right] \cdot (3x^2 + x + 1)'$

$= \left[-\sin(3x^2 + x + 1)\right] \cdot (6x + 1)$

$\left[\cos(g(x))\right]' = \left[-\sin g(x)\right] \cdot g'(x)$

$$f(x) = \tan\left(\cos x + \sqrt{x}\right)$$

$$\left[\tan(g(x))\right]' = \left[\sec^2 g(x)\right] \cdot g'(x)$$

$$f'(x) = \left[\sec^2\left(\cos x + \sqrt{x}\right)\right] \cdot \left(\cos x + \sqrt{x}\right)'$$

$$= \left[\sec^2\left(\cos x + \sqrt{x}\right)\right] \cdot \left(-\sin x + \frac{1}{2} x^{-1/2}\right)$$

$$f(x) = \left(\cos x + \sin x\right)^{2024}$$

$$f'(x) = 2024\left(\cos x + \sin x\right)^{2023} \cdot \left(\cos x + \sin x\right)'$$

$$= 2024\left(\cos x + \sin x\right)^{2023}\left(-\sin x + \cos x\right)$$

$$f(x) = 2024^x + 7^x - 2\log_9 x + 3\ln x - \frac{3\log_2 x}{5} + \frac{\log_7 x}{3} + 2024$$

$$f'(x) = 2024^x \cdot \ln 2024 + 7^x \cdot \ln 7 - \frac{2}{x \cdot \ln 9} + \frac{3}{x} - \frac{3}{5x \cdot \ln 2} + \frac{1}{3x \cdot \ln 7}$$

$$(\log_b x)' = \frac{1}{x \cdot \ln b}$$

$$(\ln x)' = \frac{1}{x}$$

$$(b^x)' = b^x \cdot \ln b$$

$$(e^x)' = e^x$$

$$f(x) = \log_7\left(\sqrt{x} + x^2 + x + 1\right)$$

$$\left[\log_b g(x)\right]' = \frac{g'(x)}{g(x) \cdot \ln b}$$

$$f(x) = \frac{\left(\sqrt{x} + x^2 + x + 1\right)'}{\left(\sqrt{x} + x^2 + x + 1\right) \cdot \ln 7}$$

$$= \frac{\frac{1}{2} x^{-1/2} + 2x + 1}{\left(\sqrt{x} + x^2 + x + 1\right) \ln 7}$$

$$f(x) = e^{\sin x + \tan x + 2x^3}$$

$$\left[e^{g(x)} \right]' = e^{g(x)} \cdot g'(x)$$

$$f'(x) = e^{\sin x + \tan x + 2x^3} \cdot \left(\sin x + \tan x + 2x^3 \right)'$$

$$= e^{\sin x + \tan x + 2x^3} \cdot \left(\cos x + \sec^2 x + 6x^2 \right)$$

$$f(x) = e^{x \sin x}$$

$$f'(x) = e^{x \cdot \sin x} \cdot (x \cdot \sin x)'$$

$$= e^{x \cdot \sin x} \cdot \left[(x)' \cdot \sin x + x \cdot (\sin x)' \right]$$

$$= e^{x \sin x} \left[\sin x + x \cos x \right]$$

Problem 2

$$y + xy - 2x^3 = 1$$

(a) Find dy/dx or y' by differentiating implicitly.

$$(y + xy - 2x^3)' = (1)'$$

$$\Rightarrow \quad y' + (xy)' - 6x^2 = 0$$

$$\Rightarrow \quad y' + [x' \cdot y + x \cdot y'] - 6x^2 = 0$$

$$\Rightarrow \quad y' + y + xy' - 6x^2 = 0$$

$$\Rightarrow \quad y' + xy' = 6x^2 - y$$

$$\Rightarrow \quad y'[1 + x] = 6x^2 - y$$

$$\Rightarrow \quad y' = \frac{6x^2 - y}{1 + x}$$

(b) Solve the equation for y as a function of x, and find dy/dx from that equation.

$$y + xy - 2x^3 = 1$$

* Solve for y:

$$y + xy = 1 + 2x^3$$

$$\Rightarrow \quad y(1 + x) = 1 + 2x^3$$

$$\Rightarrow \quad y = \frac{1 + 2x^3}{1 + x}$$

(x) Find y'

$$y' = \frac{(1+2x^3)' \cdot (1+x) - (1+2x^3)(1+x)'}{(1+x)^2}$$

$$y' = \frac{6x^2(1+x) - (1+2x^3)}{(1+x)^2}$$

Problem 3

Given the equation

$$x^3 + y^3 = 3xy$$

1. Use implicit differentiation to find dy/dx or y'

$$(x^3 + y^3)' = (3xy)'$$

$$3x^2 + 3y^2 \cdot y' = (3x)' \cdot y + 3x \cdot y'$$

$$3x^2 + 3y^2 \cdot y' = 3y + 3x \cdot y'$$

(now we get y' by itself)

$$\Rightarrow 3y^2 \cdot y' - 3x \cdot y' = 3y - 3x^2$$

$$\Rightarrow y'(3y^2 - 3x) = 3y - 3x^2$$

$$\Rightarrow y' = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}$$

2. Find an equation for the tangent line at the point $(3/2, 3/2)$

$$y - y_0 = y'_{\text{evaluated at } (x_0, y_0)} \cdot (x - x_0)$$

The slope $= y'$ evaluated at $(3/2, 3/2)$

$$= \frac{3/2 - (3/2)^2}{(3/2)^2 - 3/2} = -1$$

The equation is

$$y - 3/2 = -1\,(x - 3/2)$$

$$\Rightarrow y = -x + 3$$

Problem 4

(a) Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$

$$f(x) \approx f'(x_0)(x - x_0) + f(x_0)$$

$$\Rightarrow \boxed{f(x) = f'(1)(x-1) + f(1)}$$

we have: $f'(x) = (\sqrt{x})' = (x^{1/2})'$

$$= 1/2 \cdot x^{-1/2}$$

$$\Rightarrow \quad f'(1) = \tfrac{1}{2} \cdot 1^{-1/2} = \tfrac{1}{2}$$

$$f(1) = \sqrt{1} = 1$$

$$\Rightarrow \quad f(x) \approx \tfrac{1}{2} \cdot (x - 1) + 1$$

$$\Rightarrow \quad f(x) \approx \frac{1}{2} x + \frac{1}{2}$$

$$\Rightarrow \quad \boxed{\sqrt{x} \approx \frac{1}{2} x + \frac{1}{2}}$$

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{1.1}$

we have: $\sqrt{x} \approx \frac{1}{2} x + \frac{1}{2}$.

plug in $x = 1.1$:

$$\sqrt{1.1} \approx \frac{1}{2} \cdot 1.1 + \tfrac{1}{2} = 1.05$$