Final Exam: Practice 1

Name:

- *Basic Calculators are allowed. Graphic calculators are not allowed.*
- *A page of formula is allowed. Only formulas are allowed on the page. The page will be checked during the exam.*

Problem 1

Use the definition of derivatives to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = 1$

$$f(x) = 2x^2 - 3x + 4$$

$$\frac{f(x+h) - f(x)}{h} = \frac{2(x+h)^2 - 3(x+h) + 4 - \left[2x^2 - 3x + 4\right]}{h}$$

$$= \frac{2(x^2 + 2xh + h^2) - 3x - 3h + 4 - 2x^2 + 3x - 4}{h}$$

$$= \frac{2x^2 + 4xh + 2h^2 - 3h - 2x^2}{h}$$

$$= \frac{4xh + 2h^2 - 3h}{h} = \frac{h(4x + 2h - 3)}{h}$$

$$= 4x + 2h - 3$$

plus $h = 0$ in, we have:

$$f'(x) = 4x - 3$$

Tangent line at $x = 1$: $\quad y - f(1) = f'(1)(x-1)$

$f(1) = 2 - 3 + 4 = 3$, $\quad f'(1) = 4 - 3 = 1$

$\Rightarrow \quad y - 3 = 1 \cdot (x-1)$

$\Rightarrow \quad y = x - 1 + 3$

$\Rightarrow \quad \boxed{y = x + 2}$

Problem 2

Find $f'(x)$.

$$f(x) = \frac{2x^3}{3} + \frac{x^5}{7} - \frac{2}{\sqrt[3]{x}} + \frac{5}{6\sqrt{x}} + 2024$$

$f(x) = \frac{2}{3}x^3 + \frac{1}{7}x^5 - 2x^{-1/3} + \frac{5}{6}x^{-1/2} + 2024$

$\Rightarrow f'(x) = \frac{2}{3} \cdot 3x^2 + \frac{1}{7} \cdot 5x^4 - 2\left(-\frac{1}{3}\right)x^{-4/3} + \frac{5}{6} \cdot \left(-\frac{1}{2}\right) \cdot x^{-3/2}$

$= 2x^2 + \frac{5}{7}x^4 + \frac{2}{3}x^{-4/3} - \frac{5}{12}x^{-3/2}$

Formulas

$(x^n)' = n \cdot x^{n-1}$

$\sqrt[k]{x} = x^{1/k}$

$\sqrt[k]{x^n} = x^{n/k}$

$\frac{1}{x^k} = x^{-k}$

$$f(x) = (\sqrt{x} + 1)(x+1)$$

$f(x) = (x^{1/2} + 1)(x+1)$

product rule

$(f \cdot g)' = f' \cdot g + f \cdot g'$

$\Rightarrow f'(x) = (x^{1/2} + 1)' \cdot (x+1) + (x^{1/2} + 1)(x+1)'$

$= \frac{1}{2}x^{-1/2} \cdot (x+1) + (x^{1/2} + 1) \cdot 1$

$$f(x) = \frac{x-1}{x+1} \quad \text{(Simplify your answers.)}$$

$f'(x) = \dfrac{(x-1)' \cdot (x+1) - (x-1)(x+1)'}{(x+1)^2}$

Quotient rule

$= \dfrac{x+1 - (x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$

$$f(x) = x\sin x$$

$(\sin x)' = \cos x$

$\Rightarrow f'(x) = x' \cdot \sin x + x \cdot (\sin x)'$

$= \sin x + x \cdot \cos x$

$$f(x) = \frac{x}{\tan x}$$

$$(\tan x)' = \sec^2 x$$

$$f'(x) = \frac{(x)' \cdot \tan x - x \cdot (\tan x)'}{(\tan x)^2}$$

$$= \frac{\tan x - x \cdot \sec^2 x}{(\tan x)^2}$$

$$f(x) = \cos^{2024} x$$

$$(\cos x)' = -\sin x$$

$$\left(\left[g(x)\right]^n\right)' = n \cdot \left(g(x)\right)^{n-1} \cdot g'(x)$$

$$f'(x) = 2024 \ \cos^{2023} x \cdot (\cos x)'$$

$$= 2024 \ \cos^{2023} x \cdot (-\sin x)$$

$$f(x) = \cos(3x^2 + x + 1)$$

$$\left[\cos(g(x))\right]' = \left[-\sin g(x)\right] \cdot g'(x)$$

$$f'(x) = \left[-\sin(3x^2 + x + 1)\right] \cdot (3x^2 + x + 1)'$$

$$= \left[-\sin(3x^2 + x + 1)\right] \cdot (6x + 1)$$

$$f(x) = \tan\left(\cos x + \sqrt{x}\right)$$

$$\left[\tan(g(x))\right]' = \left[\sec^2 g(x)\right] \cdot g'(x)$$

$$f'(x) = \left[\sec^2(\cos x + \sqrt{x})\right] \cdot (\cos x + \sqrt{x})'$$

$$= \left[\sec^2(\cos x + \sqrt{x})\right] \cdot \left(-\sin x + \frac{1}{2} x^{-1/2}\right)$$

$$f(x) = \Big(\cos x + \sin x\Big)^{2024}$$

$$f'(x) = 2024\,(\cos x + \sin x)^{2023} \cdot (\cos x + \sin x)'$$

$$= 2024\,(\cos x + \sin x)^{2023}\,(-\sin x + \cos x)$$

$$f(x) = 2024^x + 7^x - 2\log_9 x + 3\ln x - \frac{3\log_2 x}{5} + \frac{\log_7 x}{3} + 2024$$

$$f'(x) = 2024^x \cdot \ln 2024 + 7^x \cdot \ln 7 - \frac{2}{x\cdot \ln 9} + \frac{3}{x} - \frac{3}{5x\cdot \ln 2} + \frac{1}{3x\cdot \ln 7}$$

$$(\log_b x)' = \frac{1}{x\cdot \ln b}$$
$$(\ln x)' = \frac{1}{x}$$
$$(b^x)' = b^x \cdot \ln b$$
$$(e^x)' = e^x$$

$$f(x) = \log_7\Big(\sqrt{x} + x^2 + x + 1\Big)$$

$$f(x) = \frac{(\sqrt{x} + x^2 + x + 1)'}{(\sqrt{x} + x^2 + x + 1)\cdot \ln 7}$$

$$= \frac{\frac{1}{2}x^{-1/2} + 2x + 1}{(\sqrt{x} + x^2 + x + 1)\ln 7}$$

$$\Big[\log_b g(x)\Big]' = \frac{g'(x)}{g(x)\cdot \ln b}$$

$$f(x) = e^{\sin x + \tan x + 2x^3}$$

$$f'(x) = e^{\sin x + \tan x + 2x^3} \cdot (\sin x + \tan x + 2x^3)'$$

$$= e^{\sin x + \tan x + 2x^3} \cdot (\cos x + \sec^2 x + 6x^2)$$

$$\Big[e^{g(x)}\Big]' = e^{g(x)} \cdot g'(x)$$

$$f(x) = e^{x\sin x}$$

$$f'(x) = e^{x\cdot \sin x} \cdot (x\cdot \sin x)'$$

$$= e^{x\cdot \sin x} \cdot \Big[(x)'\cdot \sin x + x\cdot (\sin x)'\Big]$$

$$= e^{x\sin x}\Big[\sin x + x\cos x\Big]$$

Problem 3

$$y + xy - 2x^3 = 1$$

(a) Find dy/dx or y' by differentiating implicitly.

$$(y + xy - 2x^3)' = (1)'$$

$$\Rightarrow \quad y' + (xy)' - 6x^2 = 0$$

$$\Rightarrow \quad y' + [x'y + x \cdot y'] - 6x^2 = 0$$

$$\Rightarrow \quad y' + y + xy' - 6x^2 = 0$$

$$\Rightarrow \quad y' + xy' = 6x^2 - y$$

$$\Rightarrow \quad y'[1 + x] = 6x^2 - y$$

$$\Rightarrow \quad y' = \frac{6x^2 - y}{1 + x}$$

(b) Solve the equation for y as a function of x, and find dy/dx from that equation.

Solve for y:

$$y + xy = 1 + 2x^3$$

$$\Rightarrow \quad y(1 + x) = 1 + 2x^3$$

$$\Rightarrow \quad y = \frac{1 + 2x^3}{1 + x}$$

Find y'

$$y' = \frac{(1 + 2x^3)' \cdot (1+x) - (1 + 2x^3)(1+x)'}{(1+x)^2}$$

$$y' = \frac{6x^2(1+x) - (1 + 2x^3)}{(1+x)^2}$$

(c) Write an equation for the tangent line at the point $(0, 1)$

The slope at $(0, 1)$ is $\quad y' = \dfrac{6x^2 - y}{1 + x} = -1 \quad$ (we can use either formulas for y')

The tangent line at $(0, 1)$ is

$$y - 1 = -1 \cdot (x - 0)$$

$$\Rightarrow \quad \boxed{y = -x + 1}$$

Problem 4

(a) Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$

$$f(x) \approx f'(x_0) (x - x_0) + f(x_0)$$

$$\Rightarrow \quad \boxed{f(x) = f'(1)(x - 1) + f(1)}$$

we have:
$$f'(x) = (\sqrt{x})' = (x^{1/2})'$$
$$= \frac{1}{2} \cdot x^{-1/2}$$

$$\Rightarrow \quad f'(1) = \frac{1}{2} \cdot 1^{-1/2} = \frac{1}{2}$$

$$f(1) = \sqrt{1} = 1$$

$$\Rightarrow \quad f(x) \approx \frac{1}{2} \cdot (x - 1) + 1$$

$$\Rightarrow \quad f(x) \approx \frac{1}{2} x + \frac{1}{2}$$

$$\Rightarrow \quad \boxed{\sqrt{x} \approx \frac{1}{2} x + \frac{1}{2}}$$

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{1.1}$

we have:
$$\sqrt{x} \approx \frac{1}{2} x + \frac{1}{2} \cdot$$

Plug in $x = 1.1$:

$$\sqrt{1.1} \approx \frac{1}{2} \cdot 1.1 + \frac{1}{2} = 1.05$$

Problem 5

Given that

$$f(x) = x^3 - 3x^2 + 1$$

Find all the intervals where

    a. $f(x)$ is increasing
    b. $f(x)$ is decreasing
    c. $f(x)$ is concave upward
    d. $f(x)$ is concave downward

$$f(x) = x^3 - 3x^2 + 1$$

: Find $f'(x)$ and factor it

$$f'(x) = 3x^2 - 6x$$

$$= 3x(x-2)$$

: Solve $f'(x) = 0$

$$3x(x-2) = 0$$

$$x = 0, \quad x = 2$$

: Get the sign chart of $f'(x)$

$x < 0$: plug in a number $x < 0$, say $x = -10$

$$f'(-10) = 3(-10)(-10-2) > 0 \quad (+)$$

$0 < x < 2$: plug in $x = 1$

$$f'(1) = 3(1)(1-2) < 0 \quad (-)$$

$x > 2$: plug in $x = 3$

$$f'(3) = 3(3)(3-2) > 0 \quad (+)$$

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        0              2
 ──────────┼───────────┼──────────→ x
  + +  + +       - - - -  + + + +
                              f'(x)
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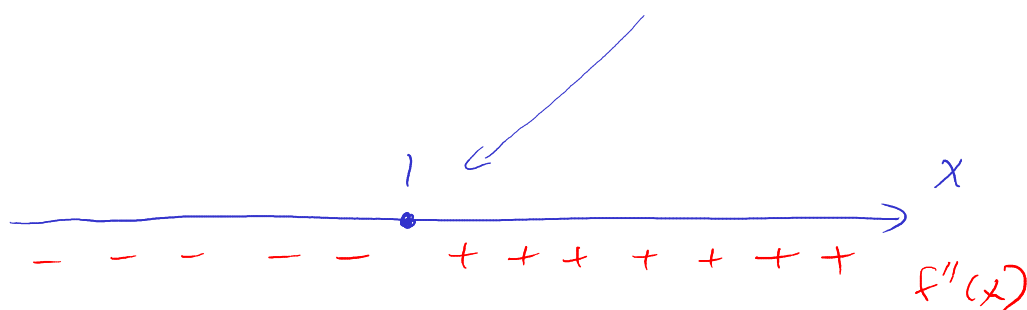ⓐ f(x) is increasing on positive intervals of f'(x) :

(-∞, 0) and (2, ∞)

ⓑ f(x) is decreasing on negative intervals of f'(x) :

(0, 2)

For concavity :

$$f''(x) = (3x^2 - 6x)' = 6x - 6$$

$$= 6(x - 1)$$

$$f''(x) = 0 \quad (\Rightarrow) \quad \boxed{x = 1}$$

```
              1
 ──────────────┼──────────────────→ x
  - - -  - - -    + + + + + + + +
                              f''(x)
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f(x) is concave upward on positive intervals of f''(x)

(1, ∞)

f(x) is concave down on negative intervals of

Problem 6

Find all the relative extrema of

$$f(x) = x^4 - 12x^3$$

$$f'(x) = 4x^3 - 36x^2 = 4x^2(x-9)$$

Set $f'(x) = 0$ to find stationary points:

$$\Rightarrow \quad 4x^2(x-9) = 0$$

$$\Rightarrow \quad 4x^2 = 0 \quad \text{or} \quad x-9 = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 9$$

(*) Sign chart of $f'(x)$

Since $f'(x)$ change sign from $(-)$ to $(+)$ at $x = 9$

$(9, f(9))$ is a relative minimum.

$$f(9) = 9^4 - 12 \cdot 9^3 = -2187$$

There is no realative maximum.

Problem 7

Find the absolute maximum and absolute minimum of $f(x) = x^3 - 6x^2 + 9x + 1$ on the interval [-5, 7].

we have : $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

$$= 3(x-1)(x-3)$$

$f'(x) = 0 \quad (\Rightarrow) \quad x = 1, \quad x = 3$

we have $\quad\quad f(1) = 5 \quad\quad\quad\quad\quad\quad\quad f(-5) = -319$

$\quad\quad\quad\quad\quad f(3) = 1 \quad\quad\quad\quad\quad\quad\quad f(7) = 113$

$\Rightarrow$ The absolute max is : $f(7) = 113$

The absolute min is : $f(-5) = -319$

Problem 8

The given equation has one (real) solution. Approximate the solution by Newton's method.

$$x^3 - 2x - 2 = 0$$

we have: $\qquad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$f'(x) = 3x^2 - 2$

$\Rightarrow \qquad x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n - 2}{3x_n^2 - 2}$

choose $\quad x_0 = 1, \quad$ we have

n	x_n
0	$x_0 = 1$
1	$x_1 = x_0 - \dfrac{x_0^3 - 2x_0 - 2}{3x_0^2 - 2}$ $= 4$
	$x_2 = 2.82$
	$x_3 = 2.14$
	$x_4 = 1.892$
	$x_5 = 1.772$
	$x_6 = 1.769$
	$x_7 = 1.769$
	$x_8 = 1.769$

Since x_n is converging, we stop here.

The solution is $x \approx 1.769$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln |x| + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = - \cos x + C$$

$$\int b^x \, dx = \frac{b^x}{\ln b} + C$$

$$\int e^x \, dx = e^x + C$$

Formulas :

$$\int dx = x + C$$

$$\int k \, dx = k x + C$$

$$\int x \, dx = \frac{x^2}{2} + C$$

Problem 9

Find the following

$$\int \left(x^7 - 2x^6 + 2x + 2024 \right) dx$$

$$= \quad \frac{x^8}{8} - \frac{2 \cdot x^7}{7} + x^2 + 2024x + C$$

$$\int \left(\sqrt{x} + x + \frac{1}{x} \right) dx$$

$$= \quad \int \left(x^{1/2} + x + \frac{1}{x} \right) dx$$

$$= \quad \frac{x^{1/2 + 1}}{1/2 + 1} + \frac{x^2}{2} + \ln |x| + C$$

$$= \quad \frac{x^{3/2}}{3/2} + \frac{x^2}{2} + \ln |x| + C$$

$$\int \left(2^x + 2\sin x - 3\cos x + 1 \right) dx$$

$$= \quad \frac{2^x}{\ln 2} \; - \; 2\cos x \; - \; 3\sin x \; + x \; + C$$
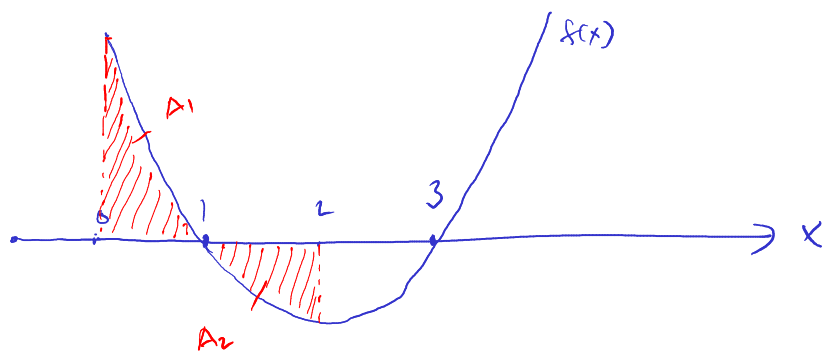
$$\int (x+1)(x+2)dx$$

$$= \int \left(x^2 + 3x + 2 \right) dx$$

$$= \frac{x^3}{3} + \frac{3x^2}{2} + 2x + C$$

Problem 10

Calculate the area between $f(x) = x^2 - 4x + 3$ and x-axis bounded by $x = 0$ and $x = 2$

$$f(x) = x^2 - 4x + 3$$

$$= (x-1)(x-3)$$

The area $= A_1 + A_2$

$$A_1 = \int_0^1 f(x)\, dx$$

$$A_2 = -\int_1^2 f(x)\, dx$$

$$A_1 = \int_0^1 (x^2 - 4x + 3)\, dx = \left. \frac{x^3}{3} - 2x^2 + 3x \right|_0^1 = \frac{4}{3}$$

$$A_2 = -\int_1^2 (x^2 - 4x + 3)\, dx = \int_2^1 (x^2 - 4x + 3)\, dx$$

$$= \left. \frac{x^3}{3} - 2x^2 + 3x \right|_2^1 = \frac{4}{3} - \left(\frac{2^3}{3} - 2 \cdot 2^2 + 3 \cdot 2 \right)$$

$$= \frac{4}{3} - \frac{8}{3} + 2 = \frac{6}{3} - \frac{4}{3} = \frac{2}{3}$$

$$\Rightarrow \text{Area} = A_1 + A_2 = \frac{4}{3} + \frac{2}{3} = 2$$