Exam 3 - Practice 1

Notice:

- *This is a take home exam. Submit the screenshots of your answers and upload to Canvas by Dec 5*

Problem 1.

Given that

$$f(x) = x^3 - 3x^2 + 1$$

Find all the intervals where

a. $f(x)$ is increasing
b. $f(x)$ is decreasing
c. $f(x)$ is concave upward
d. $f(x)$ is concave downward

$$f(x) = x^3 - 3x^2 + 1$$

Step 1 : Find $f'(x)$ and factor it

$$f'(x) = 3x^2 - 6x$$

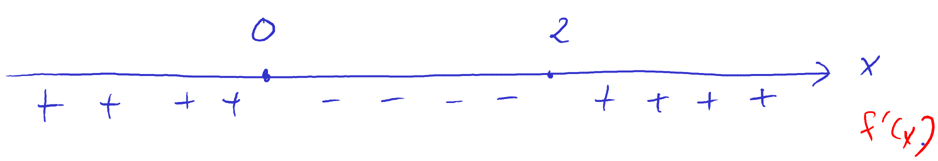
$$= 3x(x-2)$$

Step 2 : Solve $f'(x) = 0$

$$3x(x-2) = 0$$

$$\underbrace{x=0}, \quad \underbrace{x=2}$$

Step 3 : Get the sign chart of $f'(x)$



$x < 0$: plug in a number $x < 0$, say $x = -10$

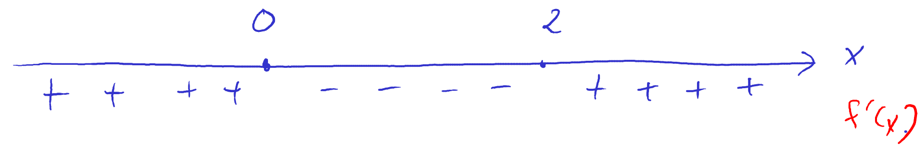
$$f'(-10) = 3(-10)(-10-2) > 0 \quad (+)$$

$0 < x < 2$: plug in $x = 1$

$$f'(1) = 3(1)(1-2) < 0 \quad (-)$$

$x > 2$: plug in $x = 3$

$$f'(3) = 3(3)(3-2) > 0 \quad (+)$$



(a) $f(x)$ is increasing on positive intervals of $f'(x)$:

$(-\infty, 0)$ and $(2, \infty)$

(b) $f(x)$ is decreasing on negative intervals of $f'(x)$:

$(0, 2)$

For concavity:

$$f''(x) = (3x^2 - 6x)' = 6x - 6$$

$$= 6(x-1)$$

$$f''(x) = 0 \iff \boxed{x = 1}$$

$f(x)$ is concave upward on positive intervals of $f''(x)$

$(1, \infty)$

$f(x)$ is concave down on negative intervals of

Problem 2

Find all the relative extrema of

$$f(x) = x^4 - 12x^3$$

$f'(x) = 4x^3 - 36x^2 = 4x^2(x-9)$

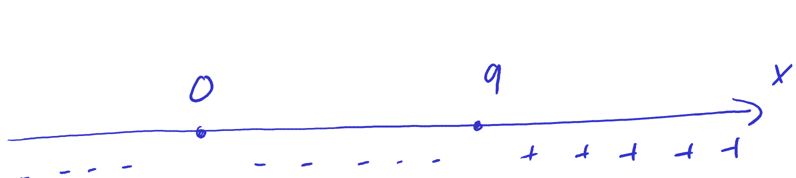
Set $f'(x) = 0$ to find stationary points:

$\Rightarrow \quad 4x^2(x-9) = 0$

$\Rightarrow \quad 4x^2 = 0$ or $x - 9 = 0$

$\Rightarrow \quad x = 0$ or $x = 9$

(*) Sign chart of $f'(x)$



since $f'(x)$ changes sign from $(-)$ to $(+)$ at $x = 9$

$(9, f(9))$ is a relative minimum.

$$f(9) = 9^4 - 12 \cdot 9^3 = -2187$$

There is no realative maximum.

Problem 3

Find an relative extrema of $f(x) = x^4 - 3x^2 + x + 1$ using gradient descent.

using the formula:

$$x_{n+1} = x_n - \lambda \cdot f'(x_n)$$

we have: $f'(x) = 4x^3 - 6x + 1$

choose $\lambda = 0.1$ and $x_0 = 0$

we have:

n	x_n
0	$x_0 = 0$
1	$x_1 = x_0 - \lambda(4x_0^3 - 6x_0 + 1) = -0.1$
	$x_2 = x_1 - \lambda(4x_1^3 - 6x_1 + 1) = -0.2596$
	$x_3 = x_2 - \lambda(4x_2^2 - 6x_2 + 1) = -0.50836$
	$x_4 = x_3 - \lambda(4x_3^2 - 6x_3 + 1) = -1.222167$
	$x_5 = x_4 - \lambda(4x_4^2 - 6x_4 + 1) = -1.325$

	$x_6 = -1.289$
	$x_7 = -1.305$
	$x_8 = -1.2987$
	$x_9 = -1.3017$
	$x_{10} = -1.3004$
	$x_{11} = -1.301$
	$x_{12} = -1.30077$
	$x_{13} = -1.30087$
	$x_{14} = -1.300827$
	$x_{15} = -1.300845$

since x_n is converging, we stop here

a relative extrema is when $x \approx -1.300845$

Problem 4

Find the absolute maximum and absolute minimum of $f(x) = x^3 - 6x^2 + 9x + 1$ on the interval $[5, 7]$.

we have: $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

$$= 3(x-1)(x-3)$$

$f'(x) = 0 \iff x = 1, \; x = 3$

we have $f(1) = 5$ $f(5) = 21$

$f(3) = 1$ $f(7) = 113$

$\Rightarrow$ The absolute max is : $f(7) = 113$

The absolute min is : $f(3) = 1$

Problem 5

The given equation has one (real) solution. Approximate the solution by Newton's method.

$$x^3 - 2x - 2 = 0$$

we have : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$f'(x) = 3x^2 - 2$

$\Rightarrow$ $x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n - 2}{3x_n^2 - 2}$

choose $x_0 = 1$, we have

n	x_n
0	$x_0 = 1$
1	$x_1 = x_0 - \dfrac{x_0^3 - 2x_0 - 2}{3x_0^2 - 2}$ $= 4$
	$x_2 = 2.82$
	$x_3 = 2.14$
	$x_4 = 1.842$
	$x_5 = 1.772$
	$x_6 = 1.769$
	$x_7 = 1.769$
	$x_8 = 1.769$

Since x_n is converging, we stop here.

The solution is $x \approx 1.769$