Exam 1 - Practice 1

Name:

Notice: Calculators are not allowed.

Some formulas:

- The derivative of $f(x)$ is defined by the formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- An equation of the tangent line at $x = a$ is

$$y - f(a) = f'(a)(x - a)$$

Problem 1.

Find the following limits.

$$\lim_{x \to 1} \frac{x^2 + 4x + 3}{x^2 + 5x + 4}$$

plug $x=1$ in

$$= \frac{1^2 + 4.1 + 3}{1^2 + 5.1 + 4} = \frac{8}{10} = \frac{4}{5}$$

$$\lim_{x \to -1} \frac{x^2 + 4x + 3}{x^2 + 5x + 4}$$

[Notice: cannot plug $x = -1$ in]

$$= \lim_{x \to -1} \frac{(x+1)(x+3)}{(x+1)(x+4)}$$

$$= \lim_{x \to -1} \frac{x+3}{x+4} = \frac{-1+3}{-1+4} = \frac{2}{3}$$

$$\lim_{x \to 1} \frac{2x^2 + 4x + 3}{3x^2 + 5x + 6}$$

plug $x = 1$ in

$$= \frac{2+4+3}{3+5+6} = \frac{9}{14}$$

$$\lim_{x \to \infty} \frac{2x^2 + 4x + 3}{3x^2 + 5x + 6}$$

$$= \lim_{x \to \infty} \frac{2x^2}{3x^2} = \frac{2}{3}$$

[Notice: when $x \to \infty$, we can drop the smaller terms]

$$\lim_{x \to \infty} \frac{2x^2 + 4x + 3}{3x^3 + 5x + 6}$$

$$= \lim_{x \to \infty} \frac{2x^2}{3x^3} = \lim_{x \to \infty} \frac{2}{3x} = 0$$

$$\lim_{x\to\infty} \frac{2x^5+4x+3}{3x^3+5x+6}$$

$$= \lim_{x\to\infty} \frac{2x^5}{3x^3} = \lim_{x\to\infty} \frac{2x^2}{3} = \infty$$

$$\lim_{x\to 1} \frac{\sin 3x}{\sin 5x} = \frac{\sin 3}{\sin 5}$$

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \frac{3x}{5x} = \frac{3}{5}$$

[Notice: when $x \to 0$, $\sin kx \approx kx$]

$$\lim_{x\to 0} \frac{x^2+\sin 3x}{x+\sin 5x}$$

$$= \lim_{x\to 0} \frac{x^2+3x}{x+5x} = \lim_{x\to 0} \frac{x(x+3)}{6x}$$

$$= \lim_{x\to 0} \frac{x+3}{6} = 3/6$$

Problem 2

Find values of x, if any, at which the function is not continuous.

a. $f(x) = x^2 + \frac{x}{3} + 2024$

No values of x found because this function continuous for any x.

b. $f(x) = x^2 + \frac{3}{x-1} + 2024$

$f(x)$ is not continuous at $x = 1$

c. $f(x) = \frac{3}{2x+5} + \frac{x-1}{x^2-5x+6}$

$f(x)$ is not continuous when $2x+5=0$ or $x^2-5x+6=0$

$2x+5=0 \iff \boxed{x = -5/2}$

$x^2-5x+6=0 \iff (x-2)(x-3)=0$

$\iff \boxed{x=2, \quad x=3}$

Problem 3.

Find a value of the constant k, if possible, that will make the function continuous everywhere.

a.

$$f(x) = \begin{cases} x-2, & x \leq 2 \\ kx^2 + k, & x > 2 \end{cases}$$

We need $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

OR $2-2 = 4k + k$

$\Rightarrow 5k = 0 \Rightarrow \boxed{k=0}$

b.

$$f(x) = \begin{cases} x^2 + x + 4, & x \le 0 \\ -9x + k^2, & x > 0 \end{cases}$$

we need

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x)$$

$$\Rightarrow \quad 0^2 + 0 + 4 = k^2 \quad \Rightarrow \quad \boxed{k = 2, \quad k = -2}$$

Problem 4.

a. Use the definition of derivatives to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = 1$

$$f(x) = 2x^2 - 3x + 4$$

$$\frac{f(x+h) - f(x)}{h} = \frac{2(x+h)^2 - 3(x+h) + 4 - [2x^2 - 3x + 4]}{h}$$

$$= \frac{2(x^2 + 2xh + h^2) - \cancel{3x} - 3h + \cancel{4} - 2x^2 + \cancel{3x} - \cancel{4}}{h}$$

$$= \frac{2x^2 + 4xh + 2h^2 - 3h - 2x^2}{h}$$

$$= \frac{4xh + h^2 - 3h}{h} = \frac{h(4x + h - 3)}{h}$$

$$= 4x + h - 3$$

plus $h = 0$ in, we have:

$$f'(x) = 4x - 3$$

Tangent line at $x=1$: $y - f(1) = f'(1)(x-1)$

$f(1) = 2-3+4 = 3$, $f'(1) = 4-3 = 1$

$\Rightarrow \quad y - 3 = 1\cdot(x-1)$

$\Rightarrow \quad y = x - 1 + 3$

$\Rightarrow \quad \boxed{y = x + 2}$

b. Use the definition of derivatives to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = 0$

$$f(x) = \frac{1}{x+1}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} = \frac{\frac{x+1 - (x+h+1)}{(x+h+1)\cdot(x+1)}}{h} = \frac{\frac{\cancel{x}+\cancel{1} - \cancel{x} - h - \cancel{1}}{(x+h+1)\cdot(x+1)}}{h}$$

$$= \frac{-\cancel{h}}{(x+h+1)(x+1)} \cdot \frac{1}{\cancel{h}}$$

plus $h=0$ in, we have: $f'(x) = \dfrac{-1}{(x+1)(x+1)} = -\dfrac{1}{(x+1)^2}$

the equation of the tangent line at $x=0$ is

$f(0) = \dfrac{1}{1+0} = 1$

$f'(0) = -\dfrac{1}{(1+0)^2} = -1$

$y - f(0) = f'(0)(x-0)$

$y - 1 = -1\cdot(x)$

$\Rightarrow \quad \boxed{y = -x + 1}$