1.5. Continuity.

⊛ Continuous functions

A function $f(x)$ is continuous if $f(x)$ is continuous at any point.

Example: $f(x) = \sqrt[3]{3x}$

$f(x) = x + 1$

⊛ Some basic continuous function:

① Polynomial functions

Ex: $f(x) = 6x^3 + 5x + 1$

② Exponential functions

$f(x) = 3^x + 10$

③ Sine and cosine functions

$f(x) = \sin 10x + \cos x$

Notice: ⊕ tangent functions are not cont. $f(x) = \tan x = \frac{\sin x}{\cos x}$

⊕ Rational function are not always continuous.

$f(x) = \dfrac{x+1}{x+2}$ is not cont. b/c it is not cont at $x = -2$

$f(x) = \dfrac{x+1}{x^2+2}$ is a continuous function.

(*) continuity on intervals

A function $f(x)$ is continuous on an interval (a, b) if $f(x)$ is continuous at any point on (a, b)

Ex: $f(x) = \dfrac{x+1}{x+2}$ is cont. on $(0, 5)$

is not cont. on $(-10, 10)$

(*) All basic functions are continuous on their domains.

1. polynomials
2. Radical functions
3. Logarithmic / exponential functions
4. Trig. functions

Example: $f(x) = \dfrac{x+1}{x+2}$

Domain: $x \neq -2$ [interval: $(-\infty, -2) \cup (-2, \infty)$]

$f(x)$ is cont. when $x \neq -2$

(*) A result on a limit of this function.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad ; \quad \lim_{x \to 0} \frac{x}{\sin x} = 1$$

This means $\sin x \approx x$ when x near 0

Example: Calculate

① $$\lim_{x \to 0} \frac{\sin 2x}{2x} = \lim_{2x \to 0} \frac{\sin 2x}{2x} = 1$$

② $$\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2$$

$$= 1 \cdot 2 = 2$$

③ $$\lim_{x \to 0} \frac{x}{\sin 3x} = \lim_{x \to 0} \underbrace{\frac{3x}{\sin 3x}}_{\to 1} \cdot \frac{1}{3} = \frac{1}{3}$$

④ $$\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \to 0} \frac{3x}{5x} = 3/5$$

[notice $\sin 3x \approx 3x$, $\sin 5x \approx 5x$]

⑤ $$\lim_{x \to 0} \frac{x + \sin 5x}{2x + \sin x} = \lim_{x \to 0} \frac{x + 5x}{2x + x} = \lim_{x \to 0} \frac{6x}{3x} = 2$$

(6) $$\lim_{x \to 0} \frac{x^2 + \sin 2x}{x^2 + \sin 5x} = \lim_{x \to 0} \frac{x^2 + 2x}{x^2 + 5x}$$

$$= \lim_{x \to 0} \frac{x(x+2)}{x(x+5)}$$

$$= \lim_{x \to 0} \frac{x+2}{x+5} = \frac{0+2}{0+5} = 2/5$$

Practice: Find

(1) $$\lim_{x \to 0} \frac{5x}{\sin 20x}$$

(3) $$\lim_{x \to 0} \frac{2x^3 + \sin x}{x^3 - \sin 10x}$$

(2) $$\lim_{x \to 0} \frac{\sin x}{\sin 2x + x}$$