# Autoregressive model - AR(1)

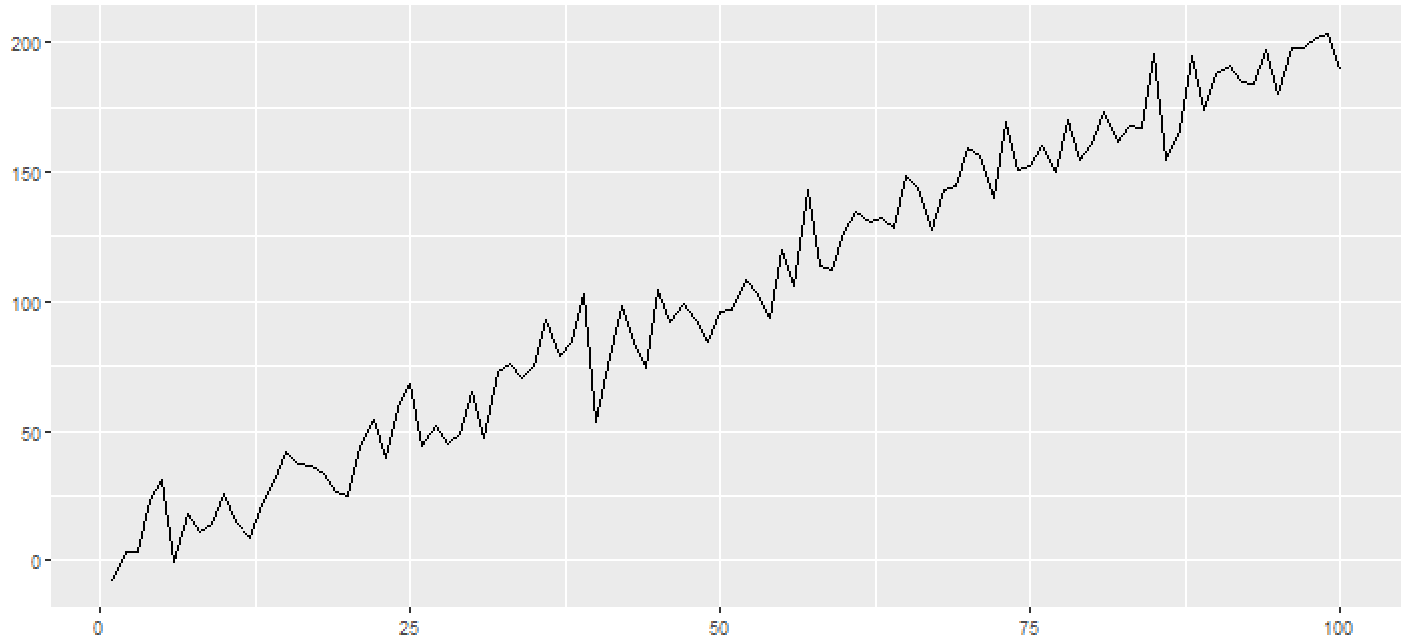
Son Nguyen

# Stationary

- A time series $y_t$ is stationary if
  - $E(y_t) = constant$
  - $Cov(y_t, y_s)$ only depends on the time lag $|t - s|$
- If $y_t$ is stationary then $Var(y_t) = Constant$

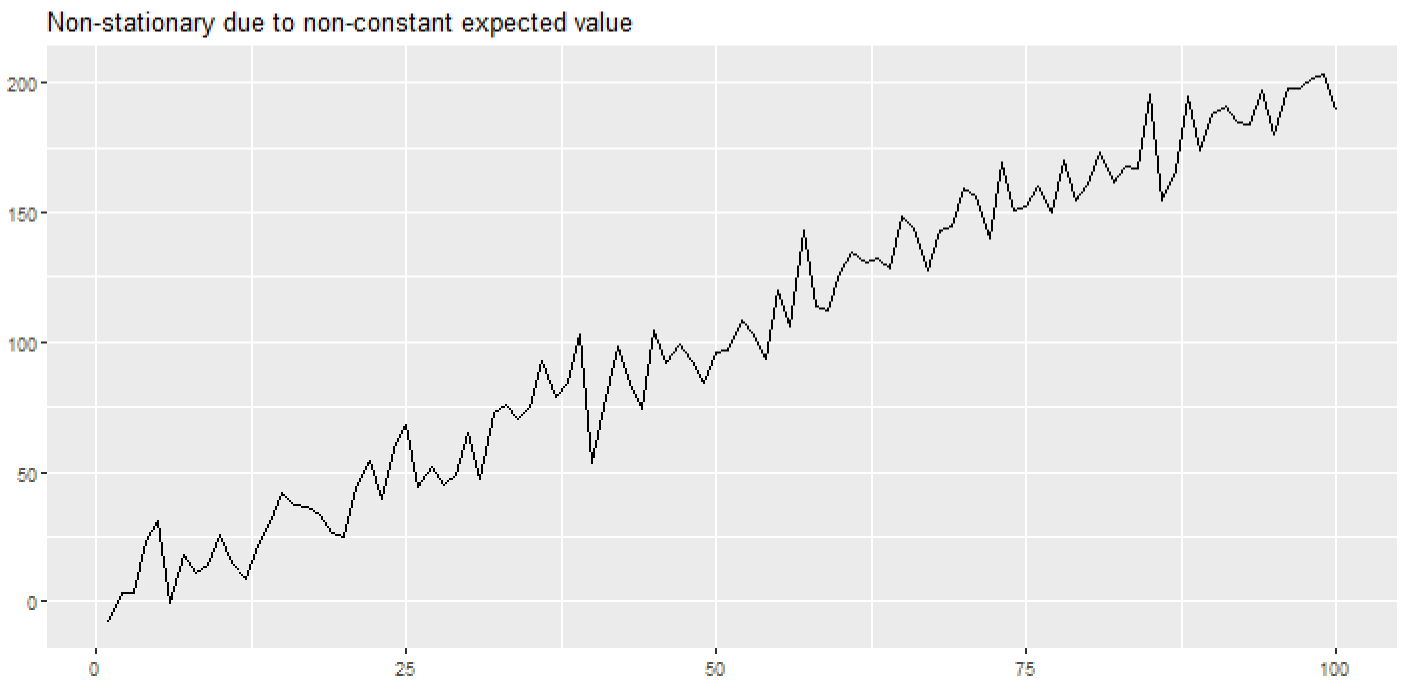
# Example

```
set.seed(30)
n = 100
e ← ts(rnorm(n, sd = 10))
t = c(1:n)
y = 2*t+3+e
library(ggfortify)
autoplot(y) + ggtitle("")
```



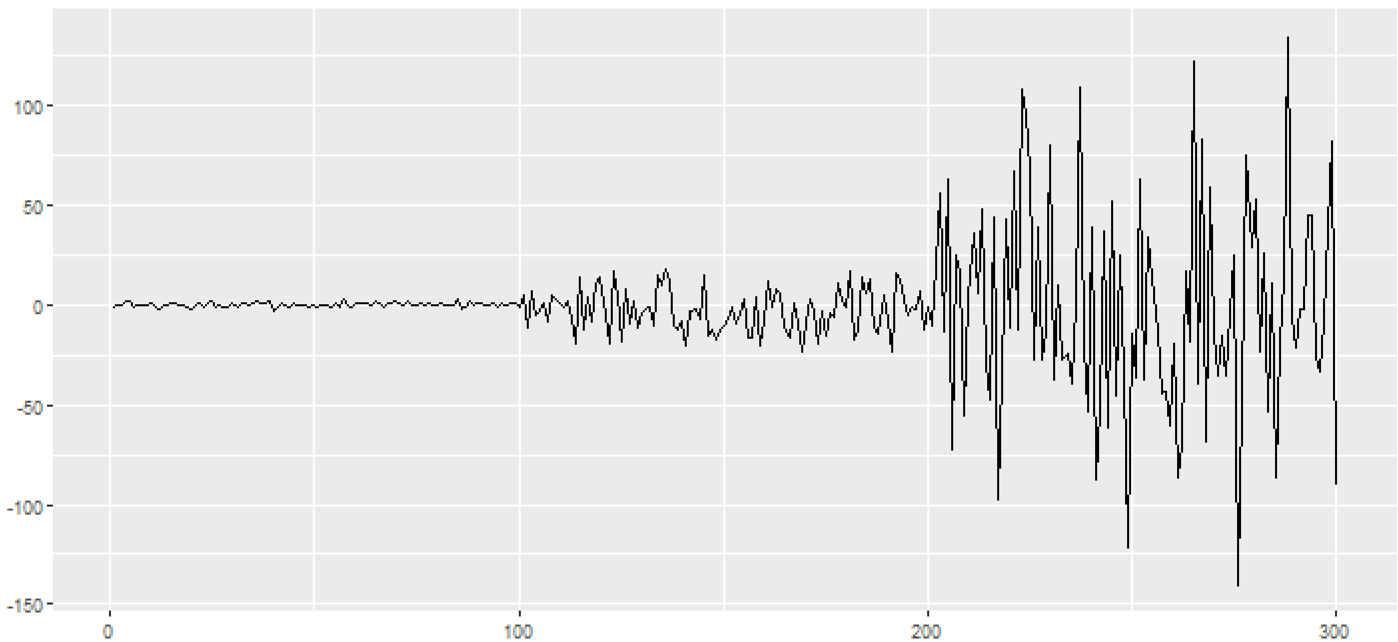
# Example

```r
set.seed(30)
n = 100
e <- ts(rnorm(n, sd = 10))
t = c(1:n)
y = 2*t+3+e
library(ggfortify)
autoplot(y) + ggtitle("Non-stationary due to non-constant expected value")
```



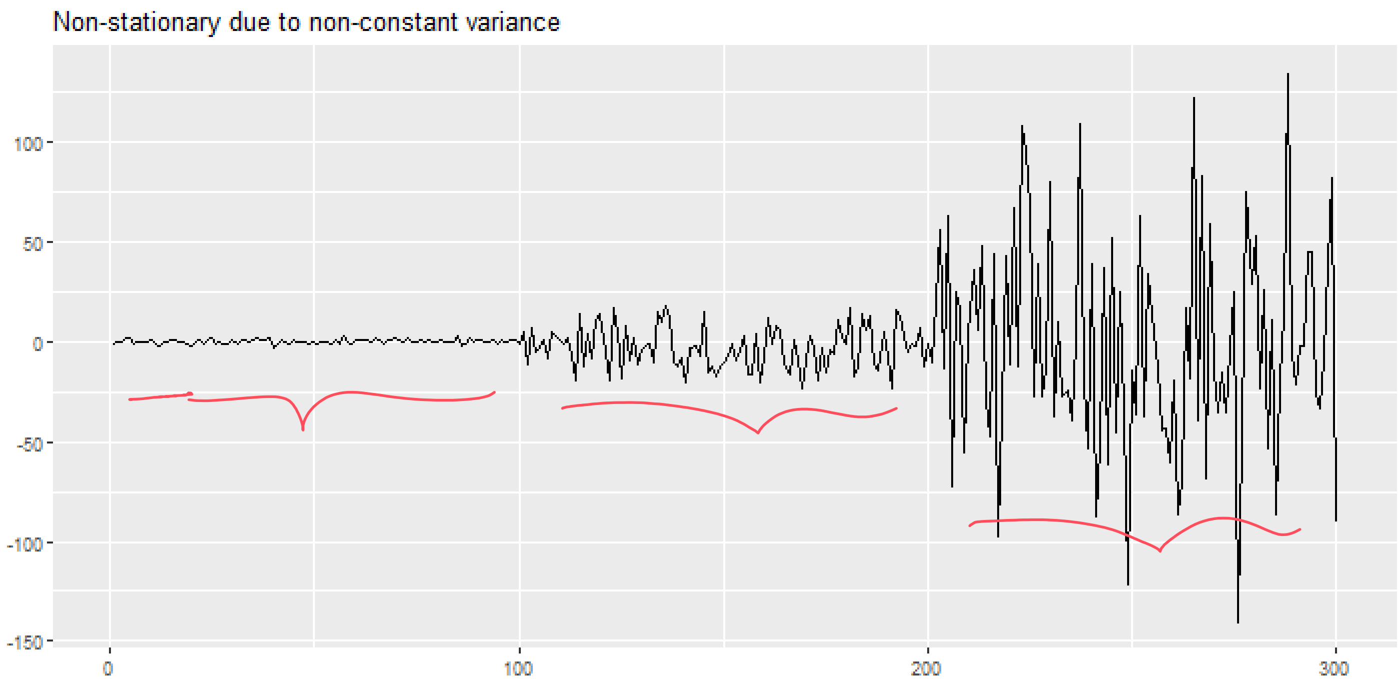
# Example

```r
set.seed(30)
n = 100
e1 <- rnorm(n, sd = 1)
e2 <- rnorm(n, sd = 10)
e3 <- rnorm(n, sd = 50)
y = c(e1,e2,e3)
library(ggfortify)
autoplot(ts(y)) + ggtitle("")
```



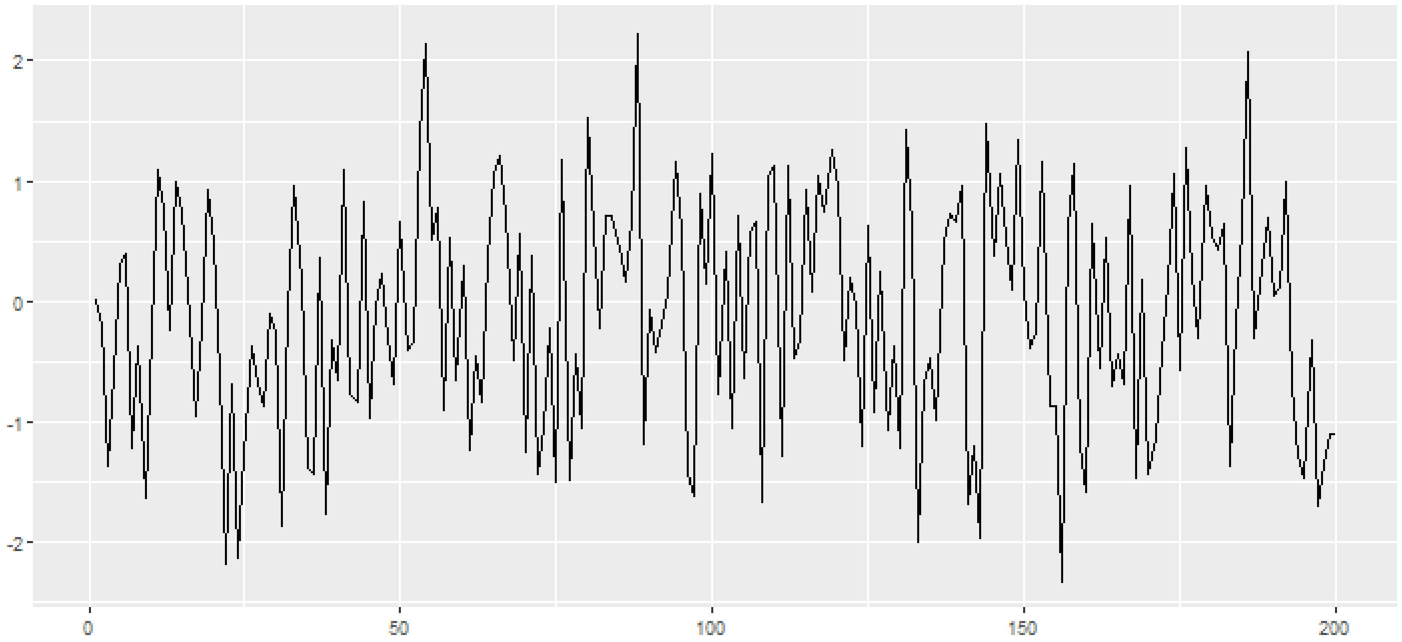
# Example

```r
set.seed(30)
n = 100
e1 <- rnorm(n, sd = 1)
e2 <- rnorm(n, sd = 10)
e3 <- rnorm(n, sd = 50)
y = c(e1,e2,e3)
library(ggfortify)
autoplot(ts(y)) + ggtitle("Non-stationary due to non-constant variance")
```



# Example

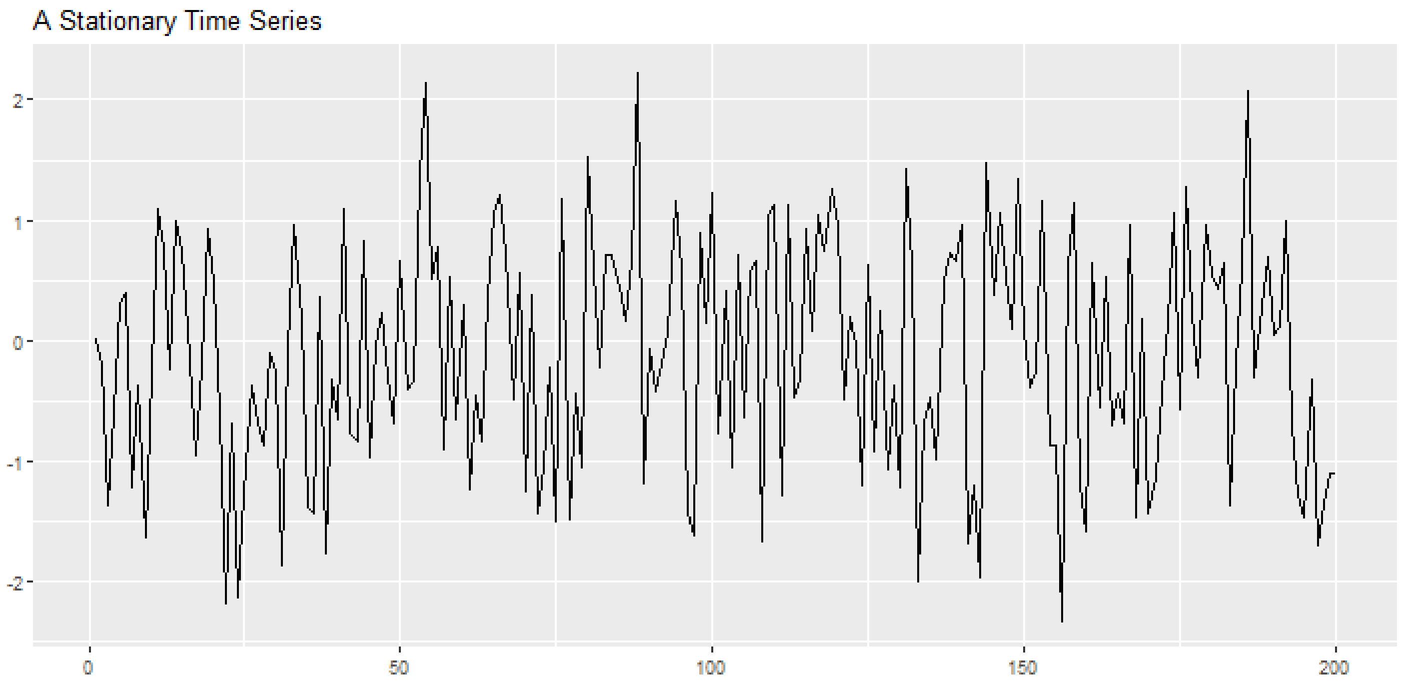
```r
set.seed(10)
y <- ts(rnorm(200))
library(ggfortify)
autoplot(y) + ggtitle("")
```



# Example

- White Noise is stationary

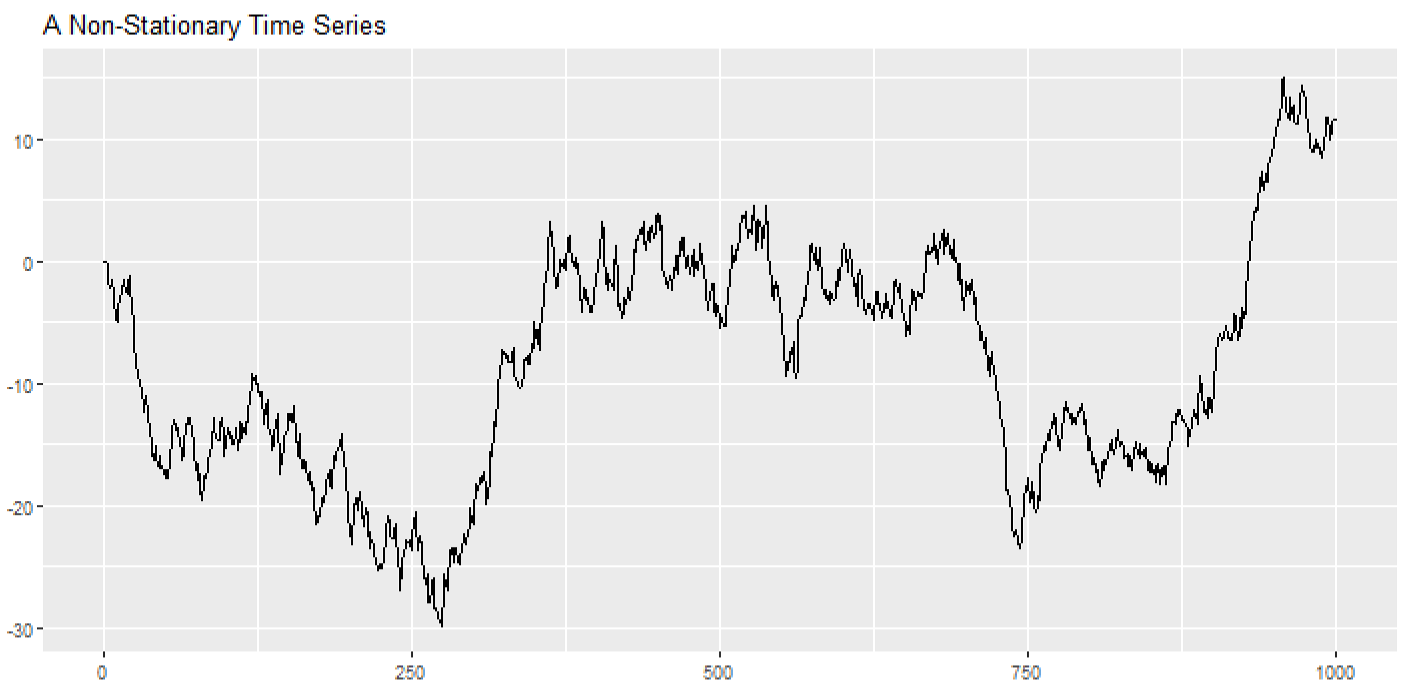
```
set.seed(10)
y ← ts(rnorm(200))
library(ggfortify)
autoplot(y) + ggtitle("A Stationary Time Series")
```



# Example

- Random Walk is not stationary

```
set.seed(10)
y ← arima.sim(list(order=c(0,1,0)), n=1000)
library(ggfortify)
autoplot(y) + ggtitle("A Non-Stationary Time Series")
```



# Autoregressive model

$$y_t = \beta_0 + \beta_1 y_{t-1} + \epsilon_t,$$

$\varepsilon_t \sim (0, \sigma^2)$

- If $\beta_1 > 1$, the series will diverge
- If $\beta_1 = 1$, the series becomes a random walk model.
- If $\beta_1 = 0$, the series becomes a white noise.
- If $|\beta_1| < 1$, the series is convergent and stationary

# Autoregressive model - AR(1)

- A time series $y_t$ is called a *first-order autoregressive model*, or AR(1) if

$$y_t = \beta_0 + \beta_1 y_{t-1} + \epsilon_t,$$

where $|\beta_1| \leq 1$, $\epsilon_t \sim (0, \sigma^2)$ and $\epsilon_{t+k}$ is independent of $y_t$ for any $t > 0$ and $k > 0$.

- Three parameters of the models are $\beta_0, \beta_1$, and $\sigma^2$

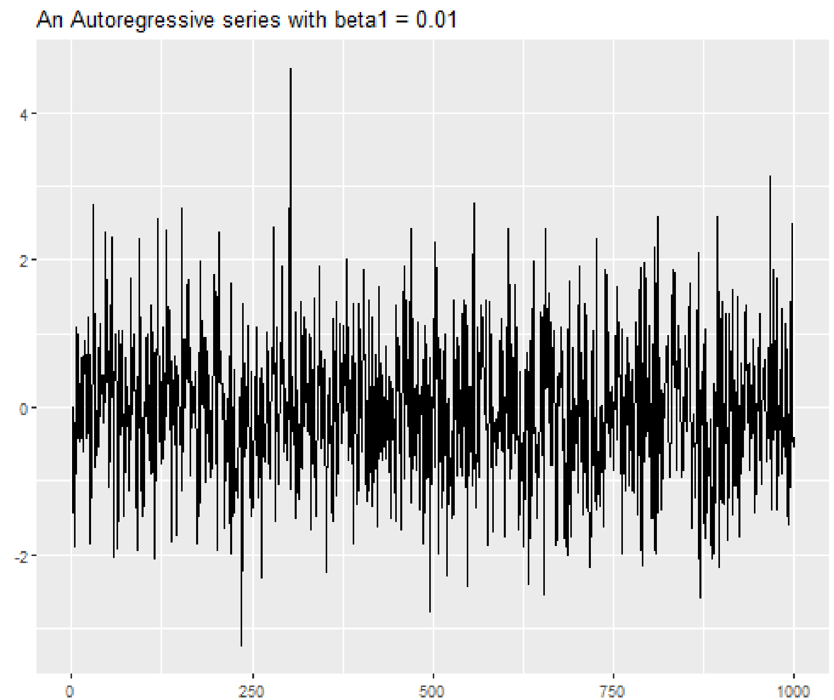
- AR(1) can also be written as

$$y_t - \mu = \beta_1(y_{t-1} - \mu) + \epsilon_t,$$

where $\beta_0 = \mu(1 - \beta_1)$. Here, $\mu$ is the mean of the series.

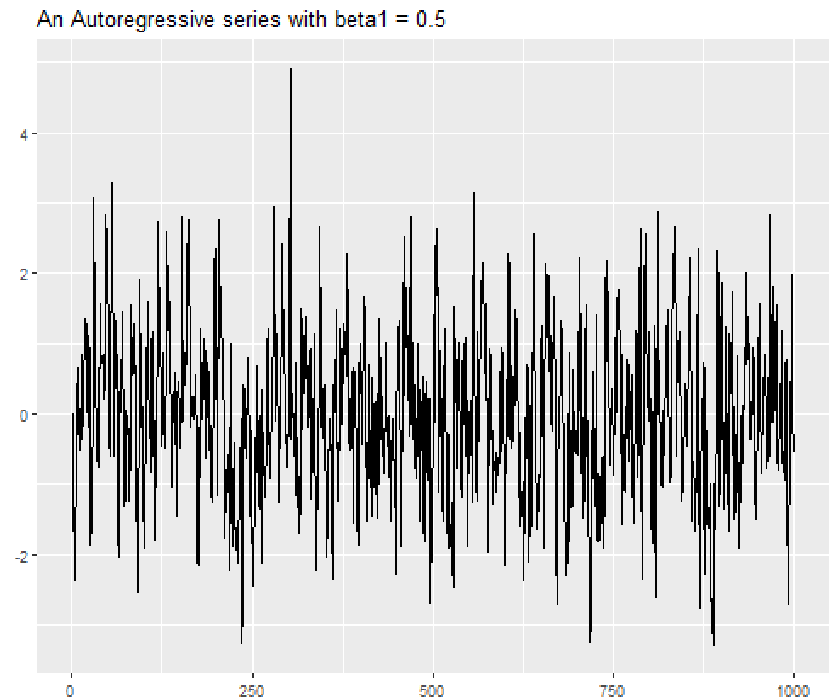
# AR(1)

```r
library(ggfortify)
set.seed(2023)
n = 1000
y = rep(0, n)

y[1] = 0
b0 = 0
b1 = .01
e = rnorm(n, sd = 1)

for (t in 2:n)
{
  y[t] = b0 + b1*y[t-1]+e[t]
}

autoplot(ts(y)) + ggtitle(paste0("An Autoregressi
```



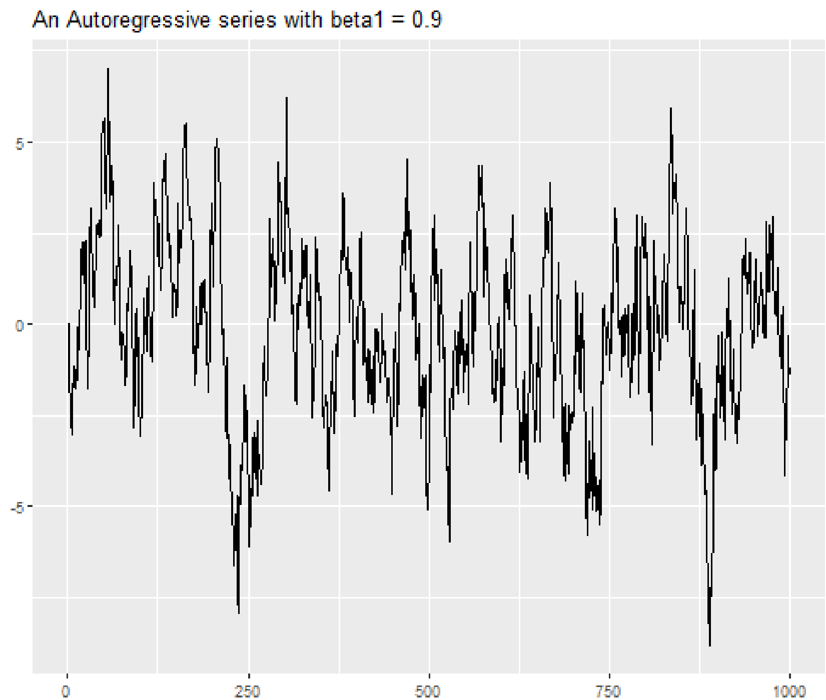
# AR(1)

```
set.seed(2023)
n = 1000
y = rep(0, n)

y[1] = 0
b0 = 0
b1 = .5
e = rnorm(n, sd = 1)

for (t in 2:n)
{
  y[t] = b0 + b1*y[t-1]+e[t]
}

autoplot(ts(y)) + ggtitle(paste0("An Autoregressi
```



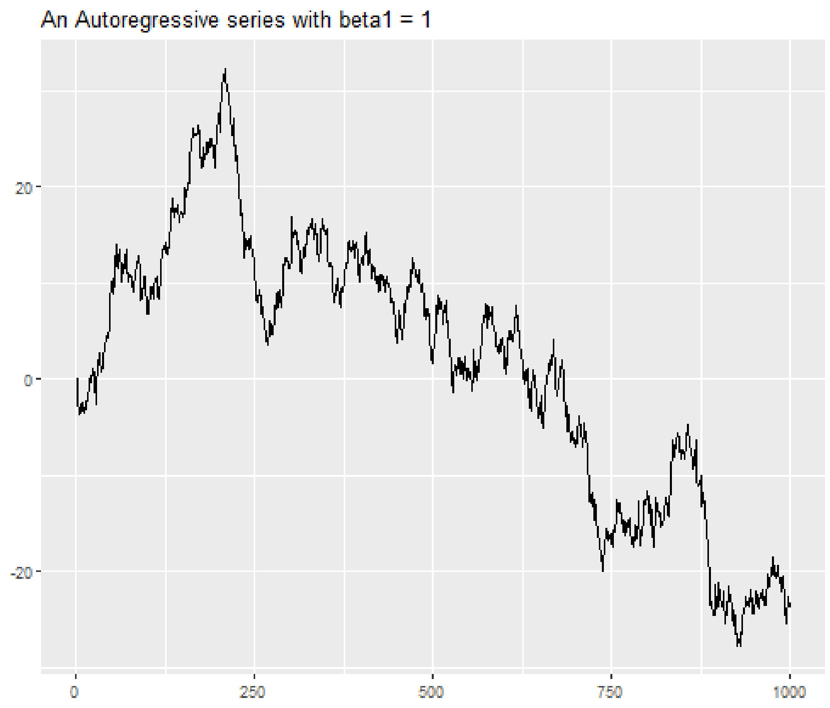
# AR(1)

```r
set.seed(2023)
n = 1000
y = rep(0, n)

y[1] = 0
b0 = 0
b1 = .9
e = rnorm(n, sd = 1)

for (t in 2:n)
{
  y[t] = b0 + b1*y[t-1]+e[t]
}

autoplot(ts(y)) + ggtitle(paste0("An Autoregressi
```



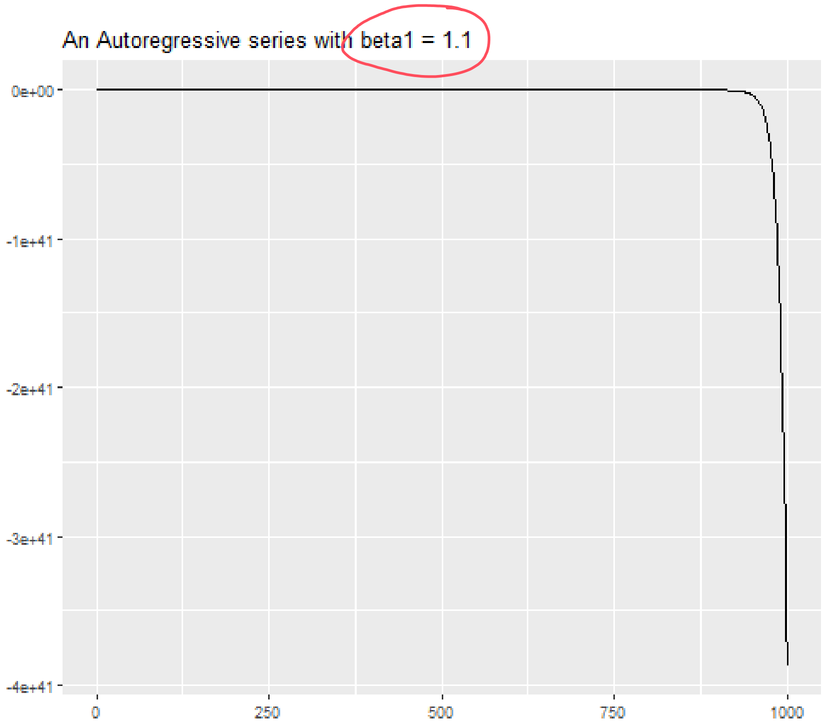
# AR(1)

```
set.seed(2023)
n = 1000
y = rep(0, n)

y[1] = 0
b0 = 0
b1 = 1
e = rnorm(n, sd = 1)

for (t in 2:n)
{
  y[t] = b0 + b1*y[t-1]+e[t]
}

autoplot(ts(y)) + ggtitle(paste0("An Autoregressi
```



# AR(1)

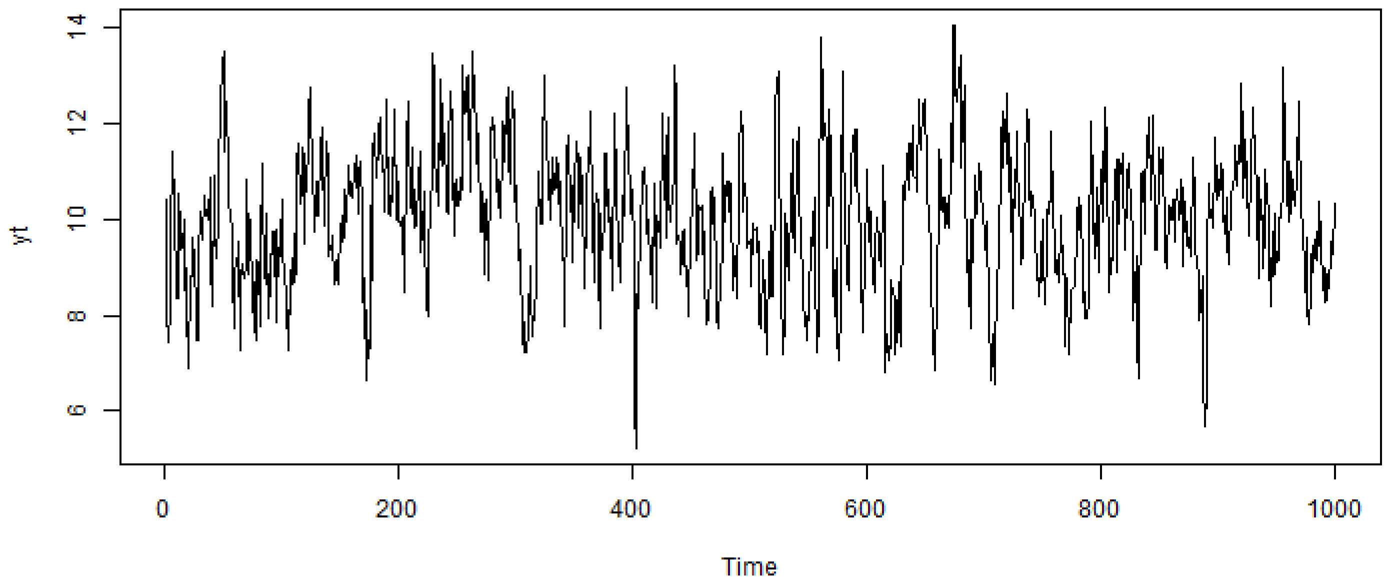
```
set.seed(2023)
n = 1000
y = rep(0, n)

y[1] = 0
b0 = 0
b1 = 1.1
e = rnorm(n, sd = 1)

for (t in 2:n)
{
  y[t] = b0 + b1*y[t-1]+e[t]
}

autoplot(ts(y)) + ggtitle(paste0("An Autoregressi
```



# Simulating AR(1)

- We can conveniently simulate AR(1) using `arima.sim` function

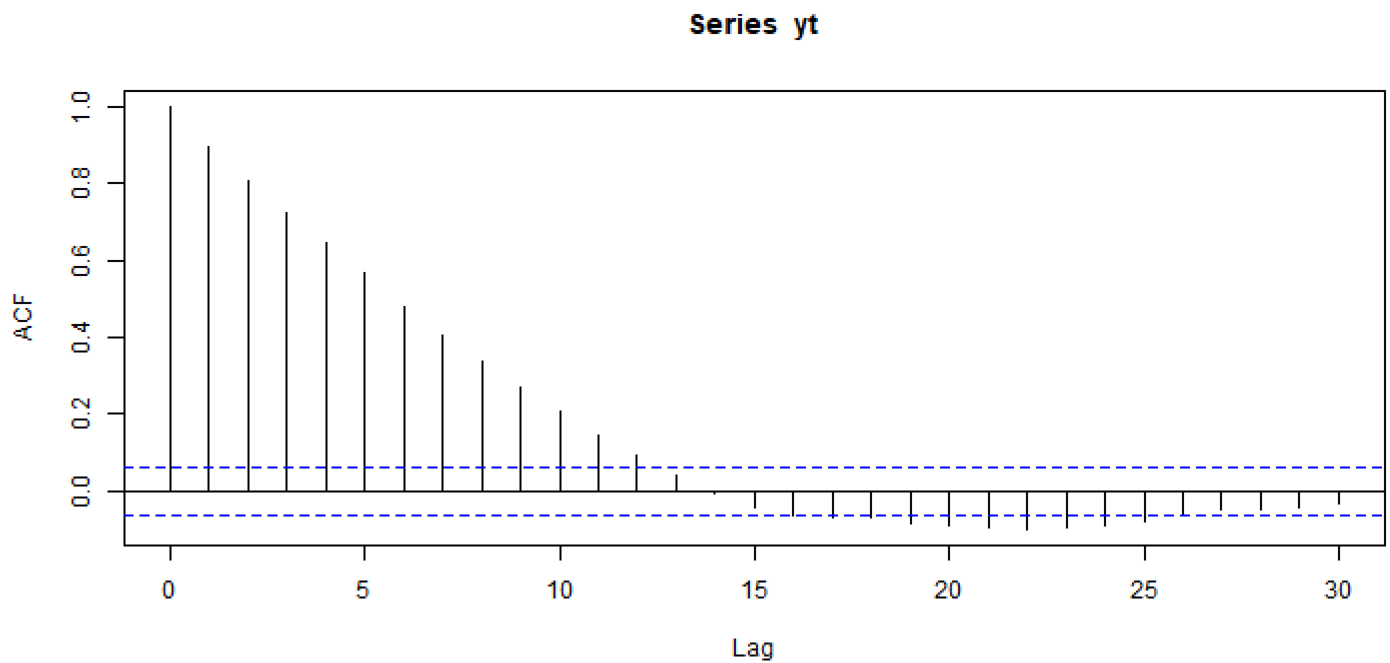
```
yt <- arima.sim(list(order=c(1,0,0), ar=c(.7)), n=1000)
b0 = 10
yt <- yt + b0
plot(yt)
```



# ACF

- For a positive value of $\beta_1$ the ACF exponentially decreases to 0 as the lag increases

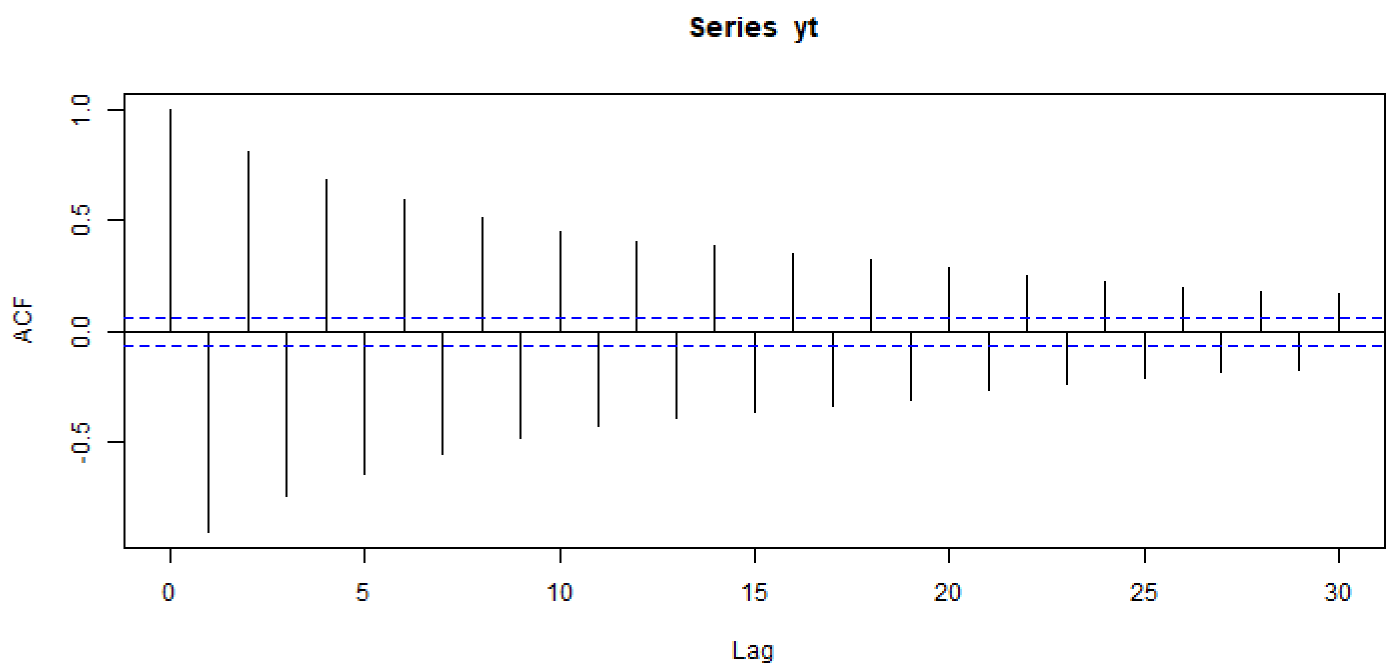
```
yt ← arima.sim(list(order=c(1,0,0), ar=c(.9)), n=1000)
b0 = 10
yt ← yt + b0
acf(yt)
```



# ACF

- For negative $\beta_1$ the ACF also exponentially decays to 0 as the lag increases, but the algebraic signs for the autocorrelations alternate between positive and negative

```
yt ← arima.sim(list(order=c(1,0,0), ar=c(-.9)), n=1000)
b0 = 10
yt ← yt + b0
acf(yt)
```



# Parameter Estimation

- AR(1) is very similar to linear model where $y_{t-1}$ play the roles of the predictor and $y_t$ is the response
- In linear model, the predictor $\boldsymbol{x}$ is assumed to be non-random while the predictor $y_{t-1}$ is non-random in AR(1)
- We estimate $\beta_0$ and $\beta_1$ by minimizing

$$\sum_{t=2}^{T}\Big(y_t - E(y_t|y_{t-1})\Big)^2 = \sum_{t=2}^{T}\Big(y_t - \beta_0 - \beta_1 y_{t-1}\Big)^2$$

- These estimators are called the conditional least squares estimators

# Parameter Estimation

The coefficients are estimated by

$$\hat{\beta}_1 = \frac{\sum_{t=2}^{T}(y_{t-1} - \bar{y})(y_t - \bar{y})}{\sum_{t=2}^{T}(y_t - \bar{y})^2}$$

$$\hat{\beta}_0 = \bar{y}(1 - \hat{\beta}_1)$$

The only parameter left to estimate is the error variance, $\sigma_\epsilon^2$, (error mean is zero), which can be estimated by $s^2$

$$s^2 = \frac{\sum_{t=2}^{T}(e_t - \bar{e})^2}{T - 3}$$

where $e_t = y_t - (\hat{\beta}_0 - \hat{\beta}_1 y_{t-1})$.

# Example

You are given the following six observed values of the autoregressive model of order one time series

$$y_t = \beta_0 + \beta_1 y_{t-1} + \epsilon_t, \text{ with } Var(\epsilon_t) = \sigma^2.$$

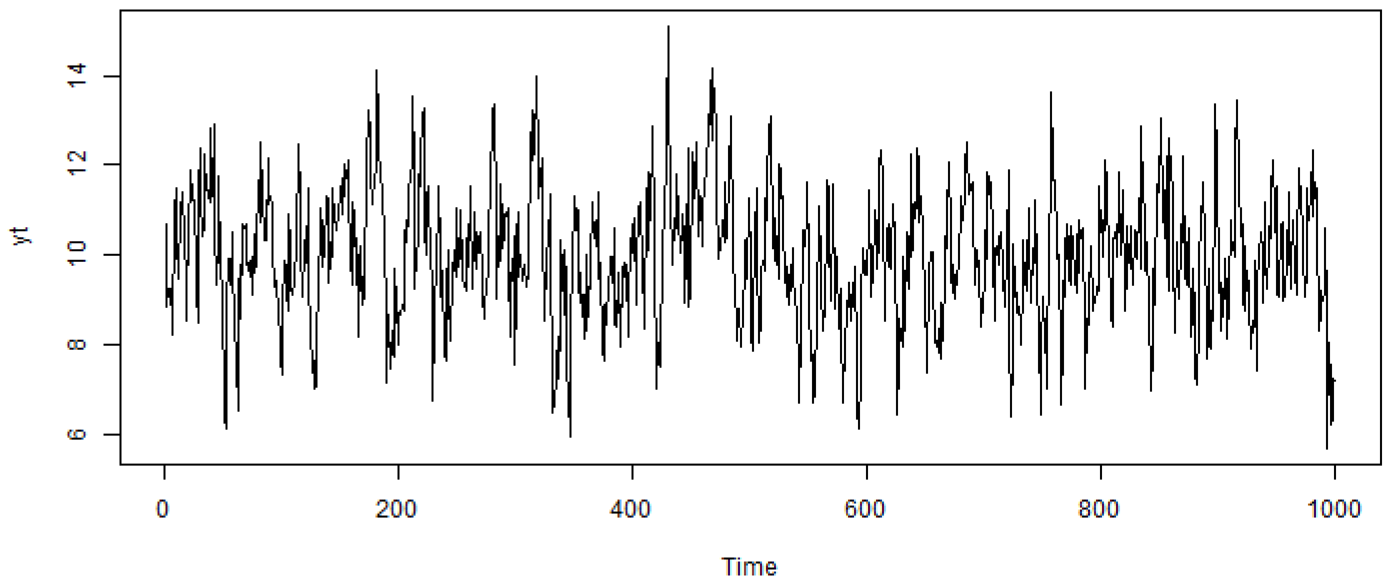
| $t$ | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| $y_t$ | 1 | 3 | 5 | 8 | 13 |

Calculate $\hat{\beta}_1$ using the conditional least squares method.

# Estimating AR(1)

- We can estimate the coefficients of AR(1) using the `arima` function

```
yt ← arima.sim(list(order=c(1,0,0), ar=c(.7)), n=1000)

b0 = 10

yt ← yt + b0

plot(yt)
```



# Estimating AR(1)

- We can estimate the coefficients of AR(1) using the `arima` function

```
arima(yt, order = c(1,0,0))
```

```
##
## Call:
## arima(x = yt, order = c(1, 0, 0))
##
## Coefficients:
##          ar1  intercept
##       0.7132     9.9620
## s.e.  0.0222     0.1115
##
## sigma^2 estimated as 1.028:  log likelihood = -1432.89,  aic = 2871.78
```

- We see that the estimated coefficients are close to the true values.

# Forecasting with AR(1)

- Suppose we have the AR(1) time series with known $\beta_0$ and $\beta_1$. If these parameters are unknown we can estimate them by the formula in the previous slices.
- We use the following formulas to for forecasting

$$\hat{y}_{T+1} = \beta_0 + \beta_1 y_T$$

$$\hat{y}_{T+k} = \mu + \beta_1^k (y_T - \mu)$$

where $\mu = \frac{\beta_0}{1-\beta_1}$.

# Example

You are given

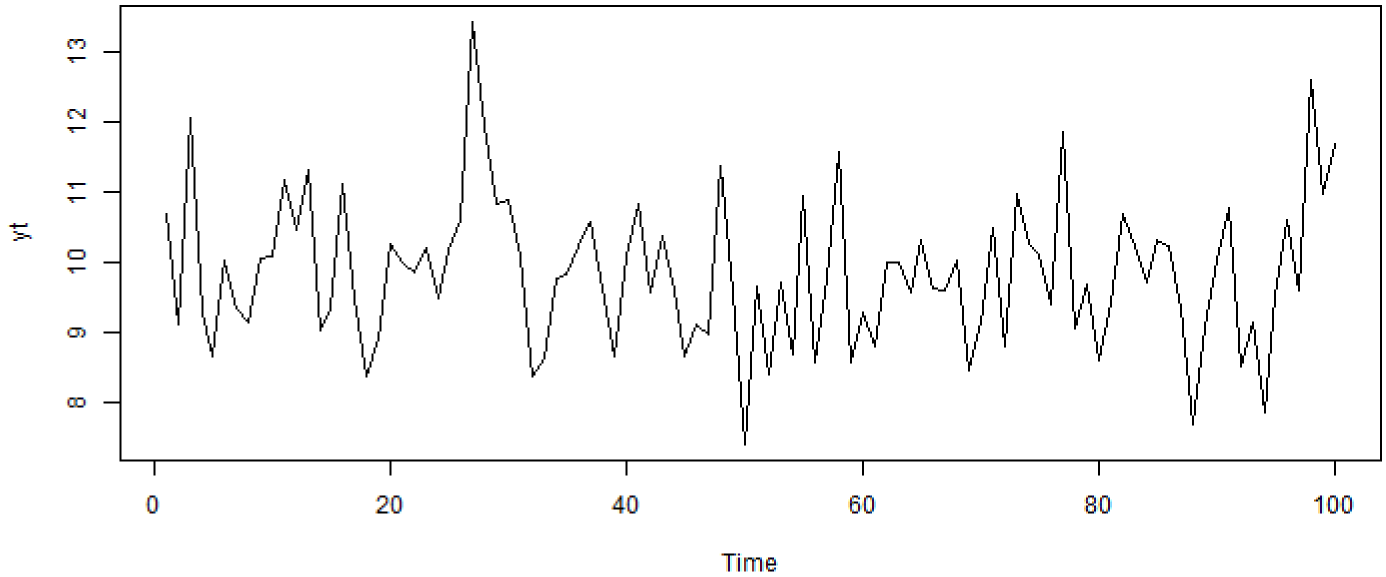
$$y_t = .3y_{t-1} + 4 + \epsilon$$

$$y_T = 7$$

Calculate the three step ahead forecast of $y_{T+3}$

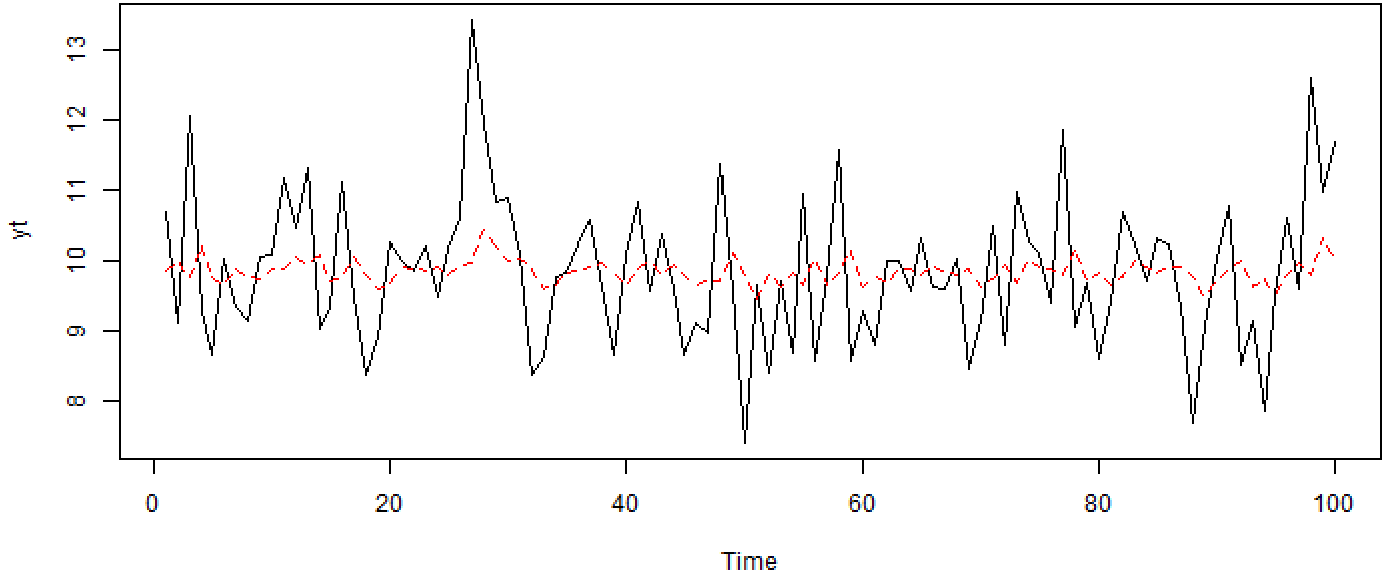
# Forecasting with AR(1)

```
# create an AR(1) series
yt <- arima.sim(list(order=c(1,0,0), ar=c(.2)), n=100)
b0 = 10
yt <- yt + b0
plot(yt)
```



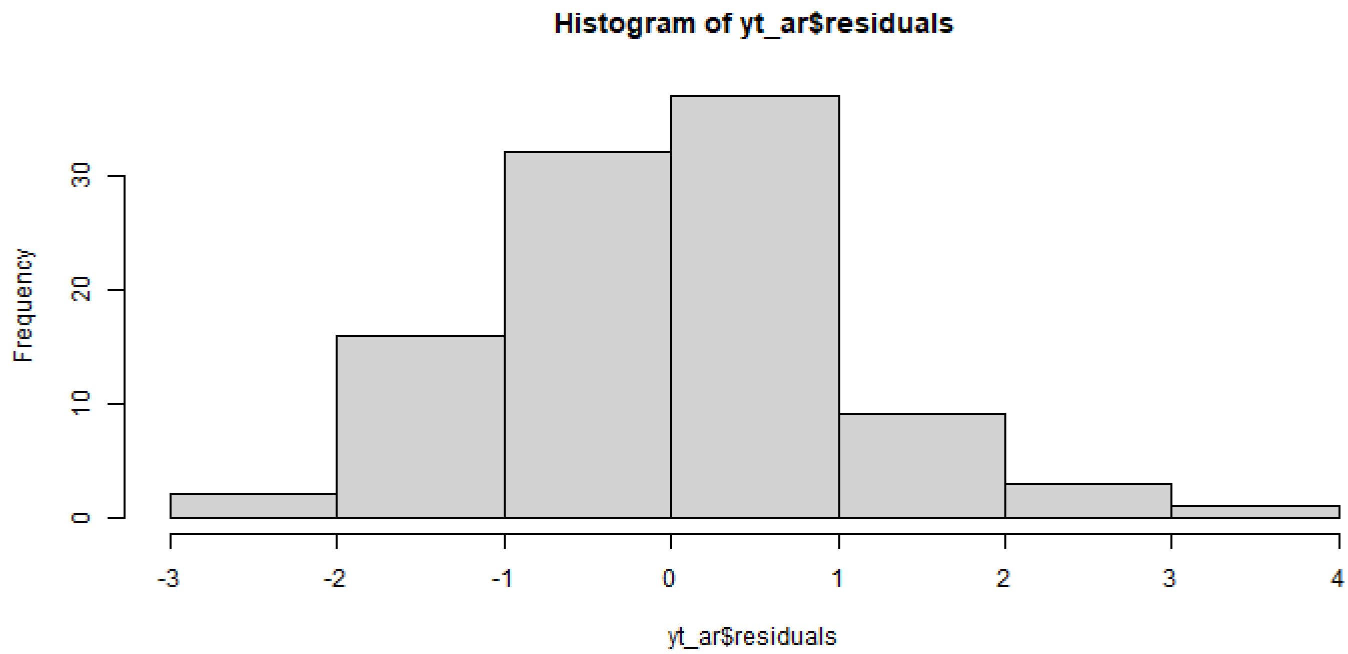
# Forecasting with AR(1)

```
# estimate the series using AR(1) model
yt_ar  = arima(yt, order = c(1,0,0))

# plot the estimated series and the original series
yt_predicted ← yt - yt_ar$residuals

plot(yt)
points(yt_predicted, type = "l",
       col = "red", lty = 2)
```



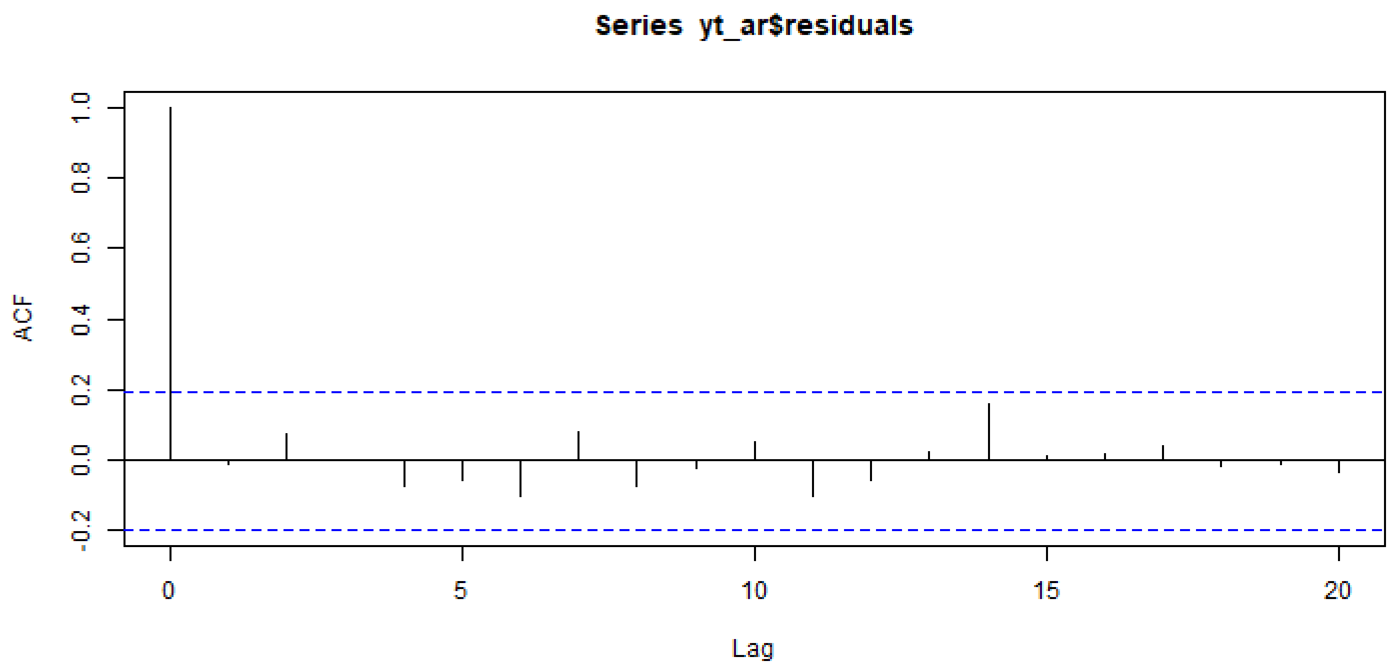
# Forecasting with AR(1)

```
hist(yt_ar$residuals)
```



# Forecasting with AR(1)

```
acf(yt_ar$residuals)
```




- The ACF of the error is similar to that of a white-noise.

# Forecasting with AR(1)

```
ts3_forecasts2 <- forecast(yt_ar, h=5)
plot(ts3_forecasts2)
```