Autoregressive model - AR(1)

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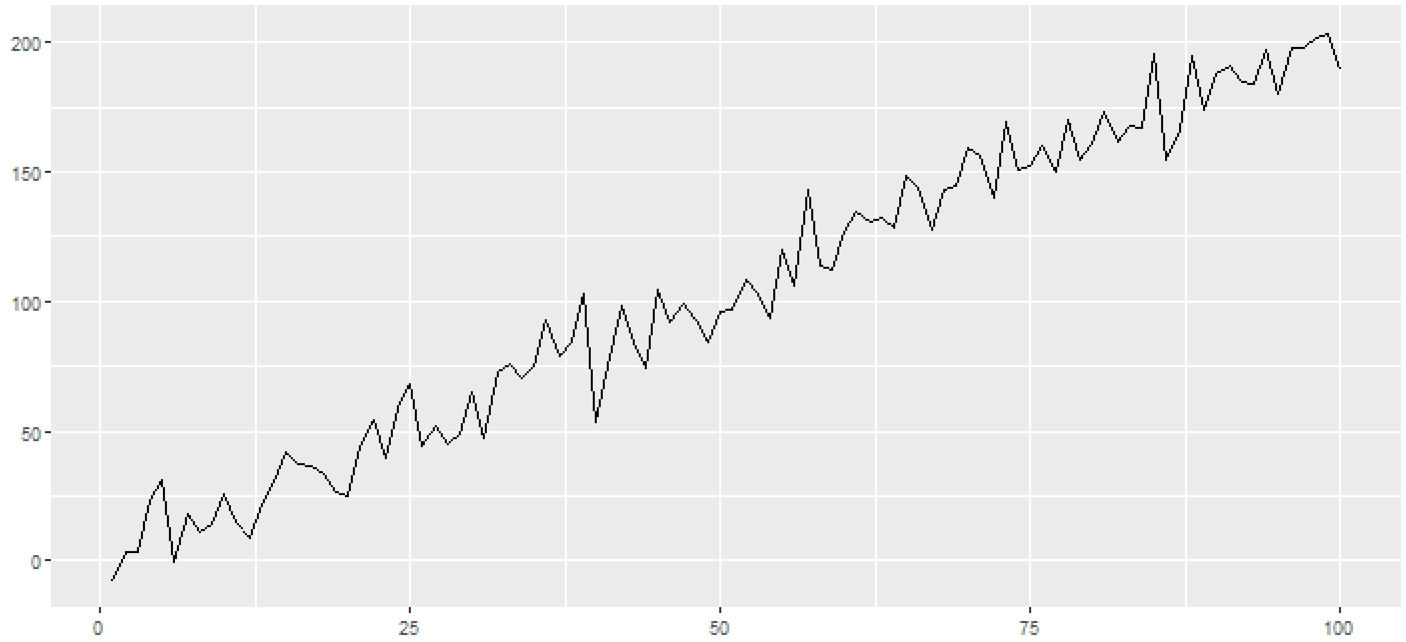
Stationary

- A time series y_t is stationary if
  - $E(y_t) = constant$
  - $Cov(y_t, y_s)$ only depends on the time lag $|t - s|$
- If y_t is stationary then $Var(y_t) = Constant$

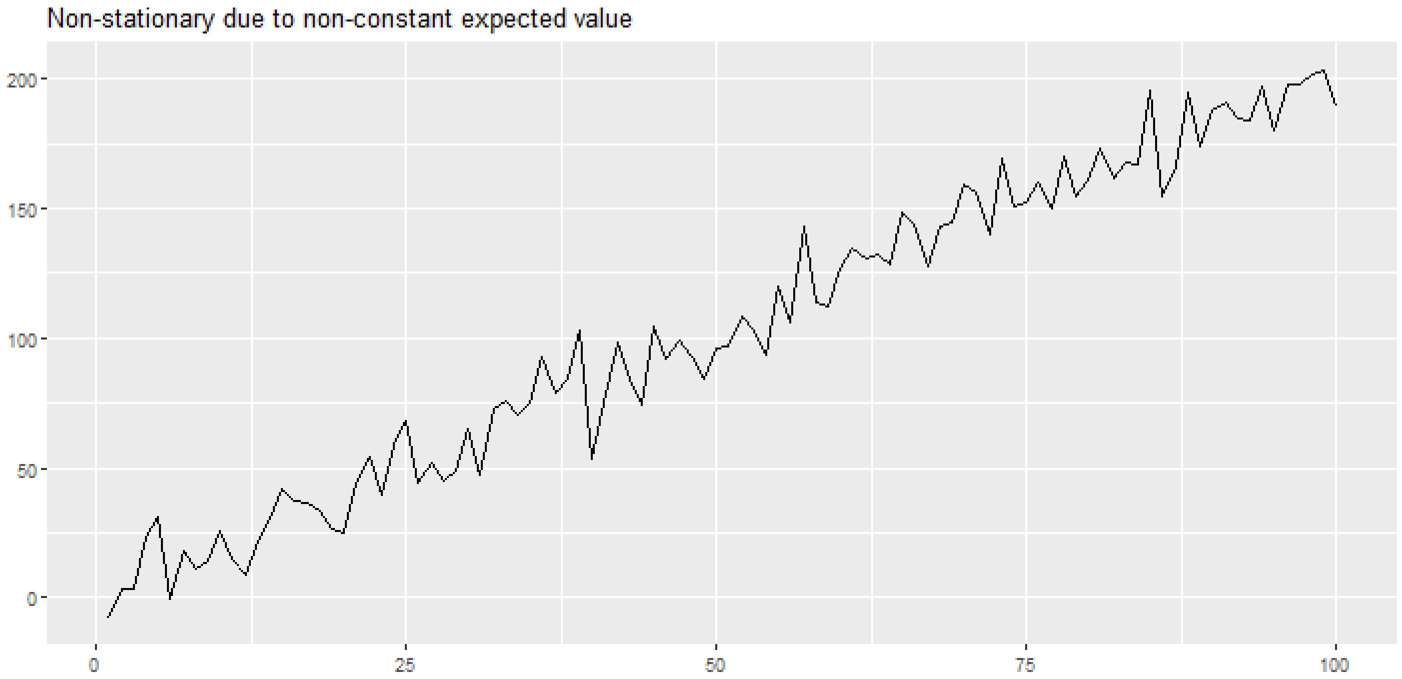
Example

```r
set.seed(30)
n = 100
e <- ts(rnorm(n, sd = 10))
t = c(1:n)
y = 2*t+3+e
library(ggfortify)
autoplot(y) + ggtitle("")
```



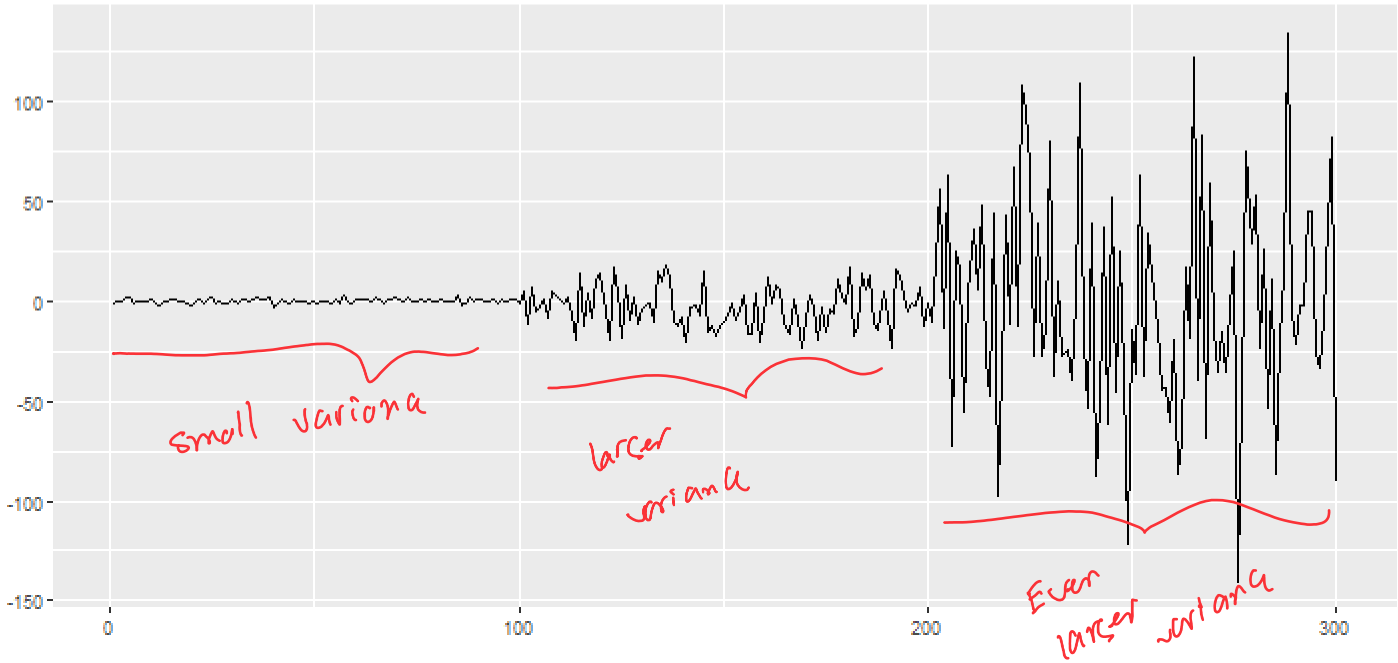
Example

```
set.seed(30)
n = 100
e ← ts(rnorm(n, sd = 10))
t = c(1:n)
y = 2*t+3+e
library(ggfortify)
autoplot(y) + ggtitle("Non-stationary due to non-constant expected value")
```



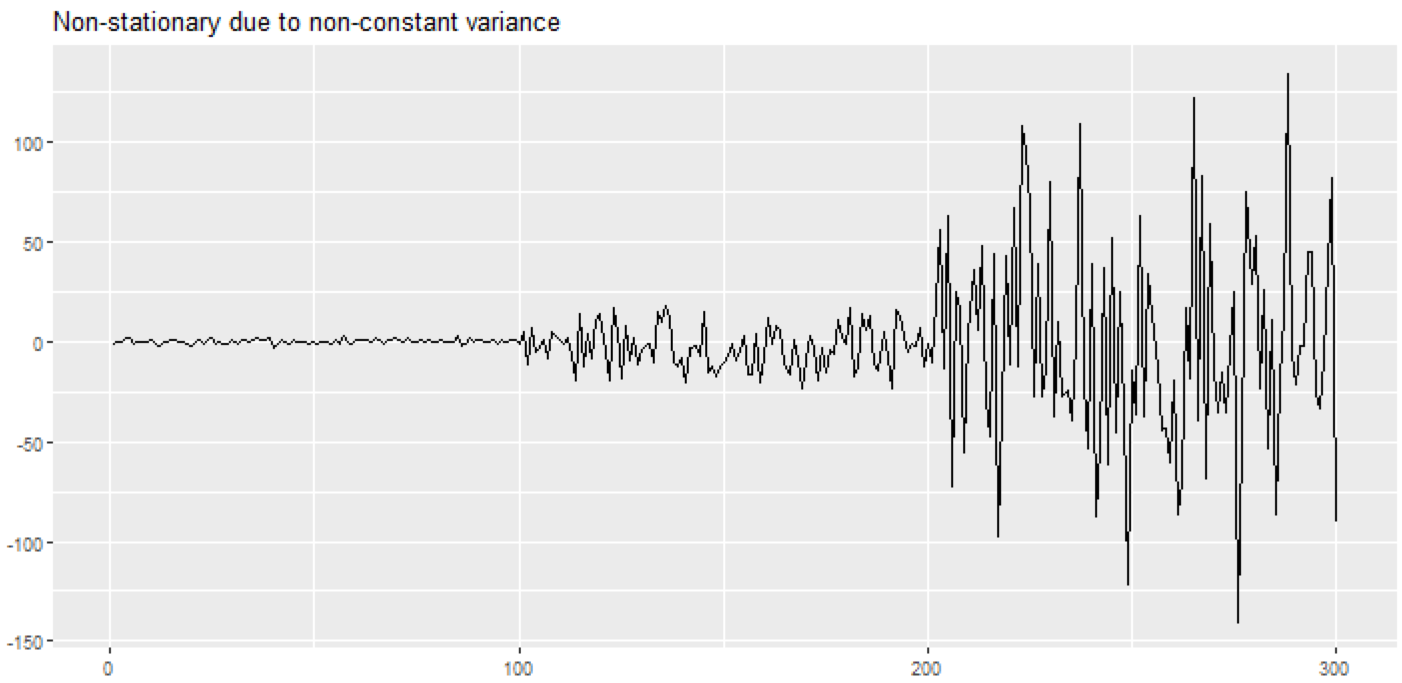
Example

```r
set.seed(30)
n = 100
e1 ← rnorm(n, sd = 1)
e2 ← rnorm(n, sd = 10)
e3 ← rnorm(n, sd = 50)
y = c(e1,e2,e3)
library(ggfortify)
autoplot(ts(y)) + ggtitle("")
```



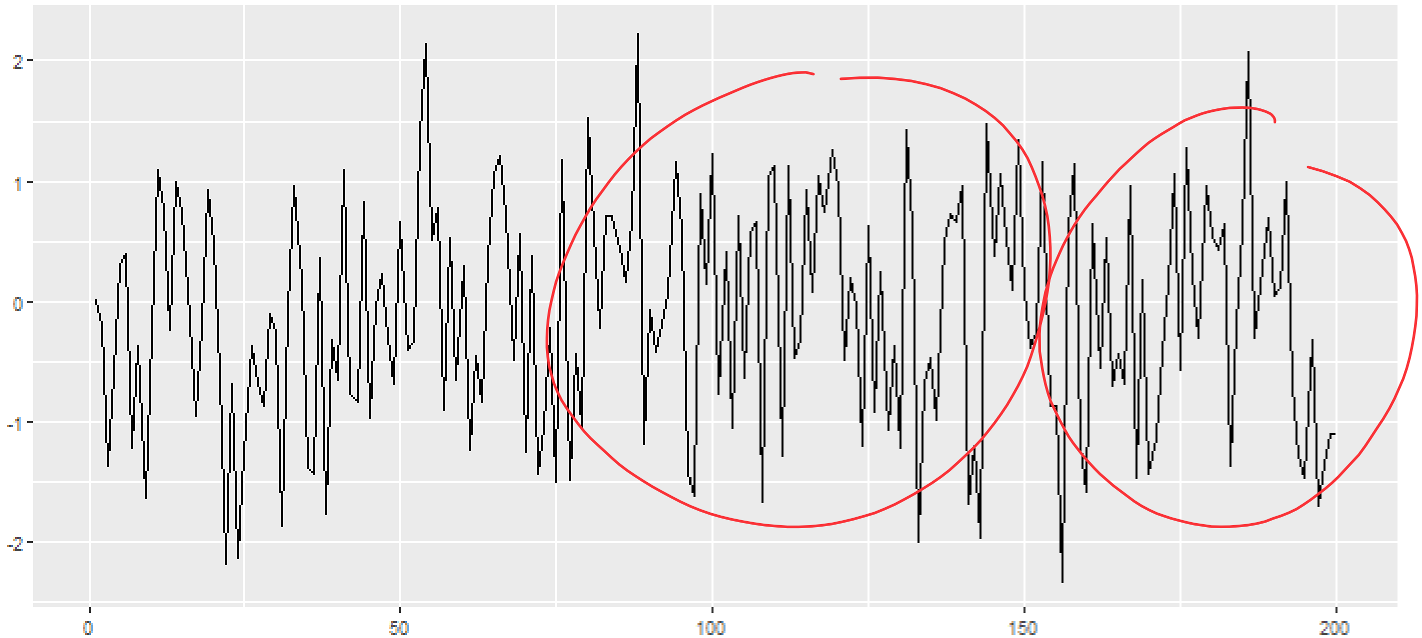
Example

```
set.seed(30)
n = 100
e1 <- rnorm(n, sd = 1)
e2 <- rnorm(n, sd = 10)
e3 <- rnorm(n, sd = 50)
y = ts(c(e1,e2,e3))
library(ggfortify)
autoplot(y) + ggtitle("Non-stationary due to non-constant variance")
```



Example

```
set.seed(10)
y ← ts(rnorm(200))
library(ggfortify)
autoplot(y) + ggtitle("")
```

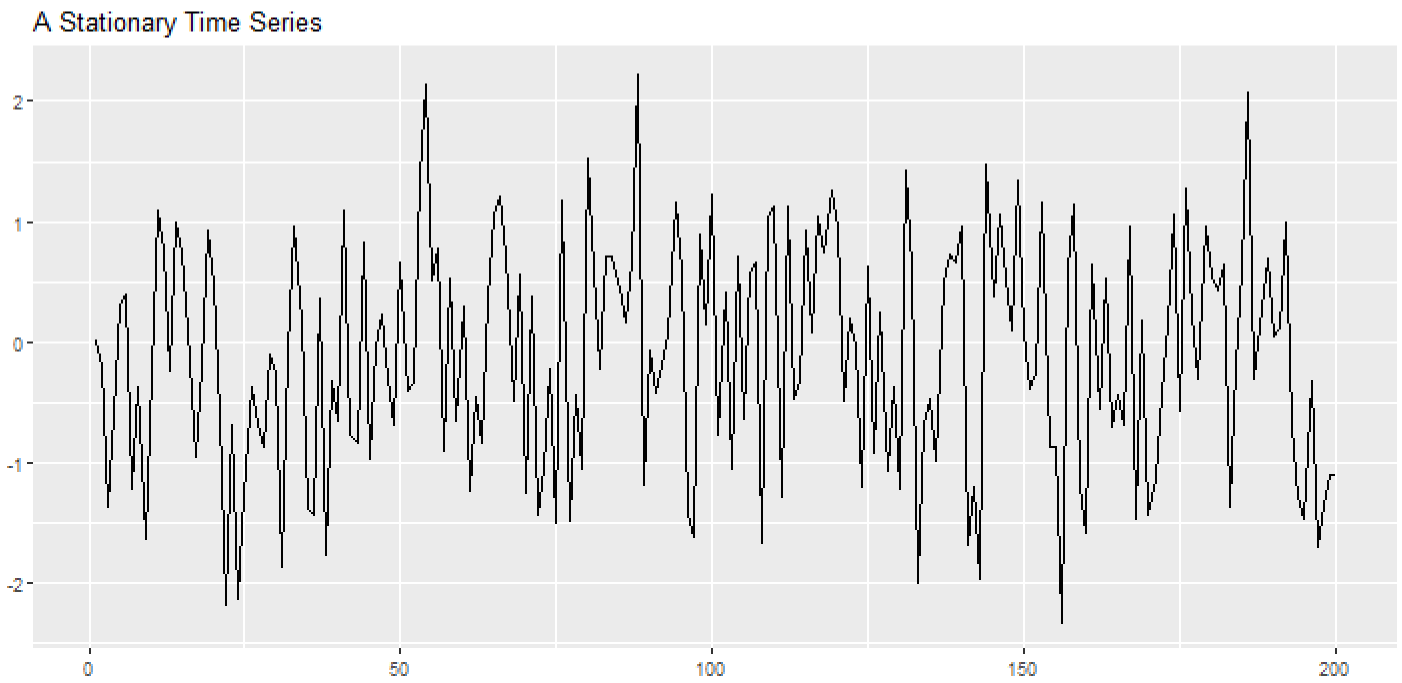


Example

- White Noise is stationary

Example

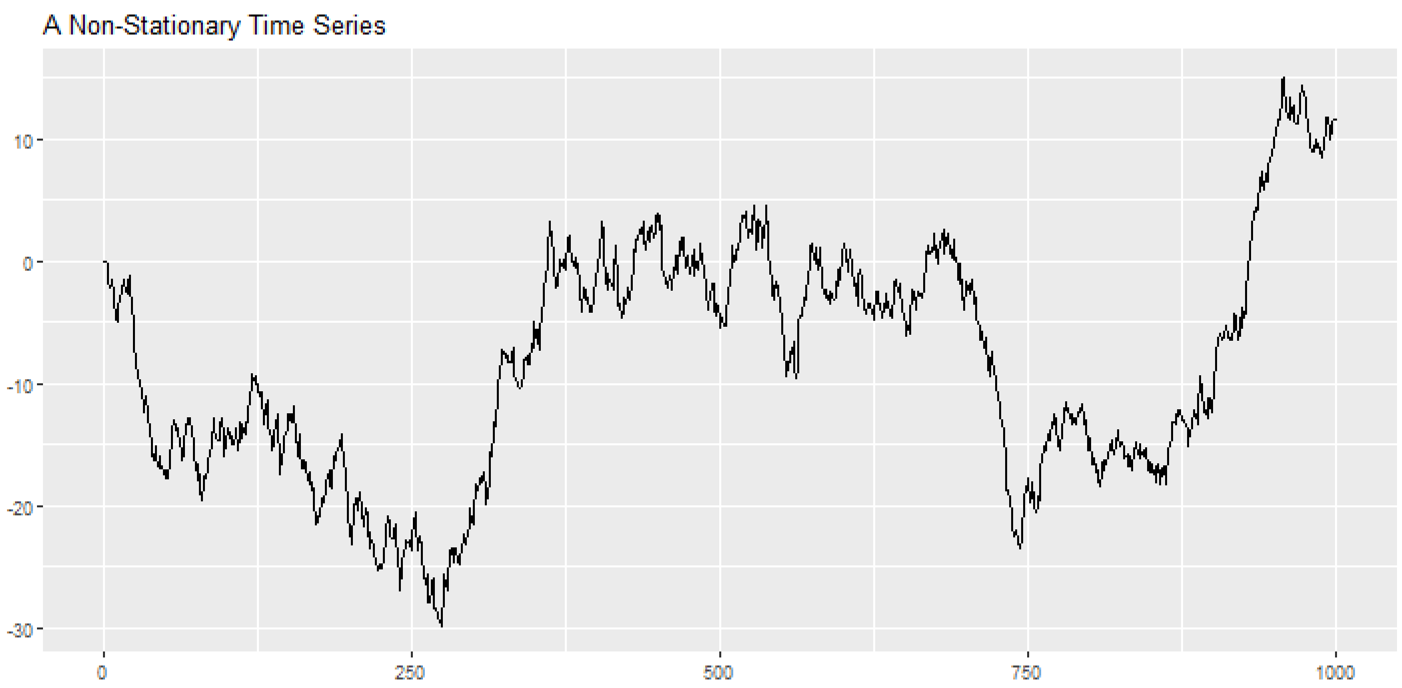
```
set.seed(10)
y ← ts(rnorm(200))
library(ggfortify)
autoplot(y) + ggtitle("A Stationary Time Series")
```



Example

- Random Walk is not stationary

```
set.seed(10)
y ← arima.sim(list(order=c(0,1,0)), n=1000)
library(ggfortify)
autoplot(y) + ggtitle("A Non-Stationary Time Series")
```



- Series with trend or seasonality are not stationary

Autoregressive model

$$y_t = \beta_0 + \beta_1 y_{t-1} + \epsilon_t, \rightarrow AR(1)$$

- If $\beta_1 > 1$, the series will diverge
- If $\beta_1 = 1$, the series becomes a random walk model.
- If $\beta_1 = 0$, the series becomes a white noise.
- If $|\beta_1| < 1$, the series is convergent and stationary

$$Y_t = \beta_0 + \beta_1 Y_{t-2}$$

$$\underbrace{Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 Y_{t-2}}_{AR(2)}$$

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

Autoregressive model - AR(1)

- A time series y_t is called a *first-order autoregressive model*, or AR(1) if

$$y_t = \beta_0 + \beta_1 y_{t-1} + \epsilon_t,$$

where $|\beta_1| \leq 1$, $\epsilon_t \sim (0, \sigma^2)$ and ϵ_{t+k} is independent of y_t for any $t > 0$ and $k > 0$.

- Three parameters of the models are $\beta_0, \beta_1,$ and σ^2

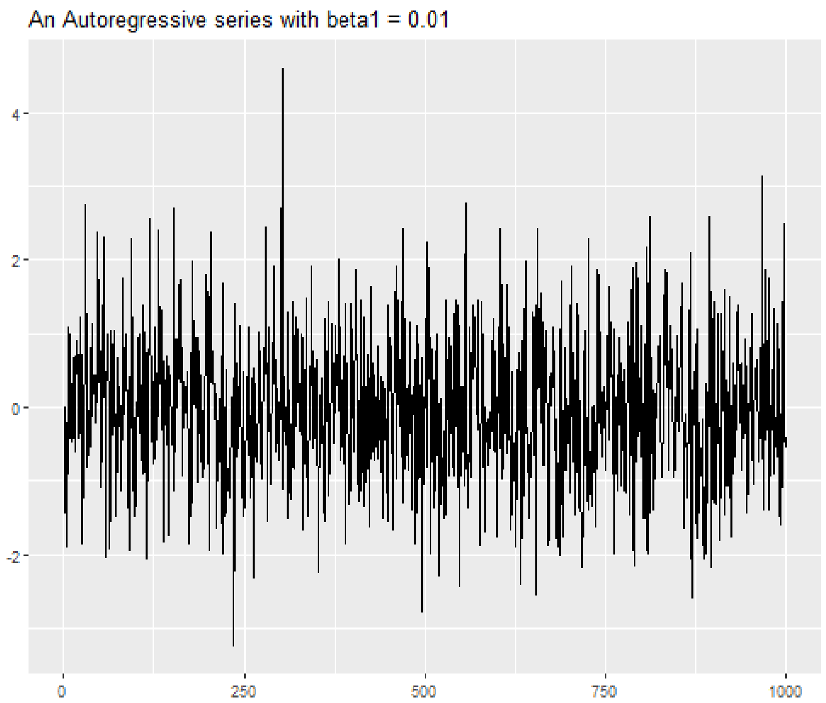
- AR(1) can also be written as

$$y_t - \mu = \beta_1(y_{t-1} - \mu) + \epsilon_t,$$

where $\beta_0 = \mu(1 - \beta_1)$. Here, μ is the mean of the series.

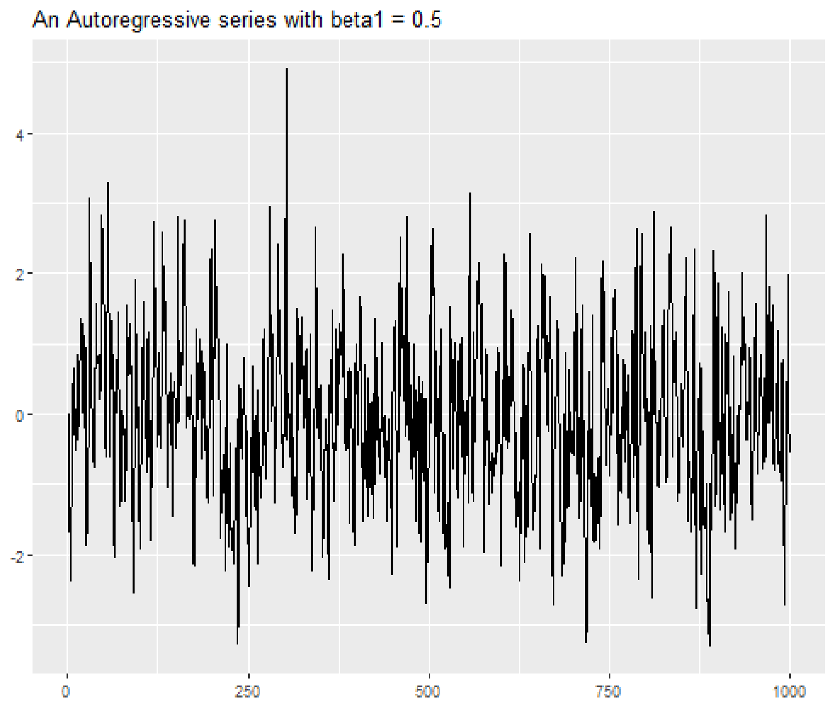
AR(1)

```r
library(ggfortify)
set.seed(2023)
n = 1000
y = rep(0, n)

y[1] = 0
b0 = 0
b1 = .01
e = rnorm(n, sd = 1)

for (t in 2:n)
{
  y[t] = b0 + b1*y[t-1]+e[t]
}

autoplot(ts(y)) + ggtitle(paste0("An Autoregressi
```



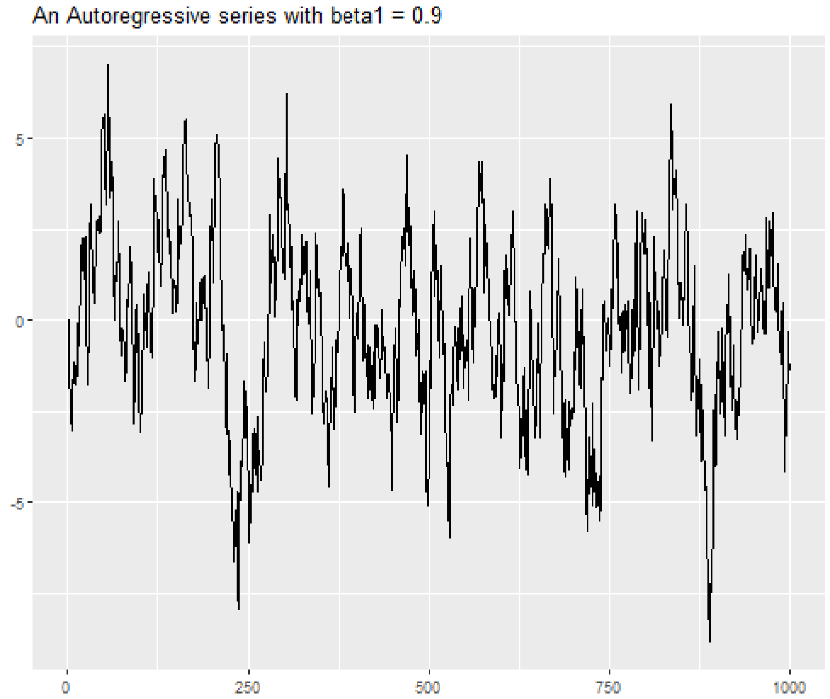
AR(1)

```
set.seed(2023)
n = 1000
y = rep(0, n)

y[1] = 0
b0 = 0
b1 = .5
e = rnorm(n, sd = 1)

for (t in 2:n)
{
  y[t] = b0 + b1*y[t-1]+e[t]
}

autoplot(ts(y)) + ggtitle(paste0("An Autoregressi
```



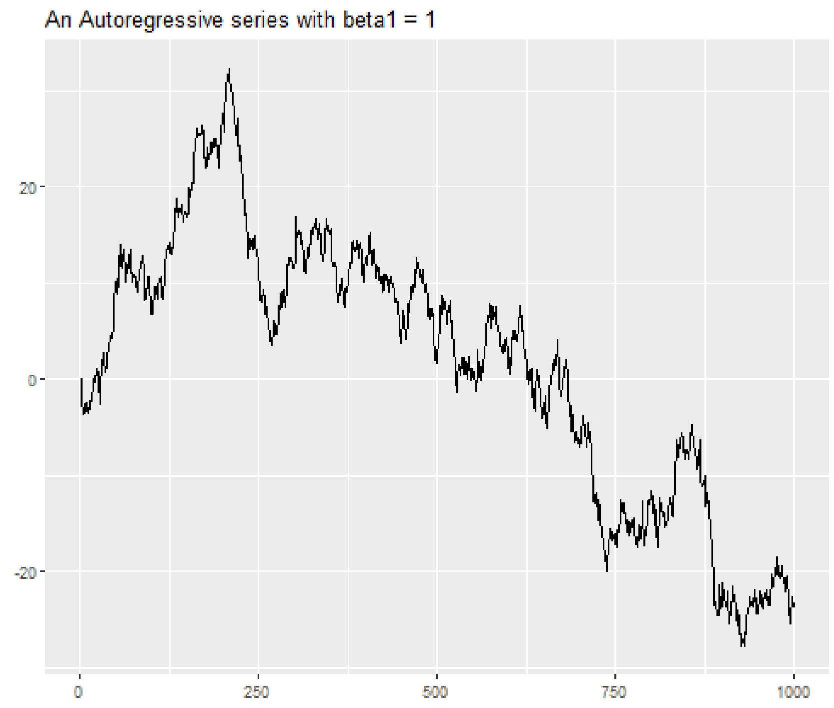
AR(1)

```r
set.seed(2023)
n = 1000
y = rep(0, n)

y[1] = 0
b0 = 0
b1 = .9
e = rnorm(n, sd = 1)

for (t in 2:n)
{
  y[t] = b0 + b1*y[t-1]+e[t]
}

autoplot(ts(y)) + ggtitle(paste0("An Autoregressi
```



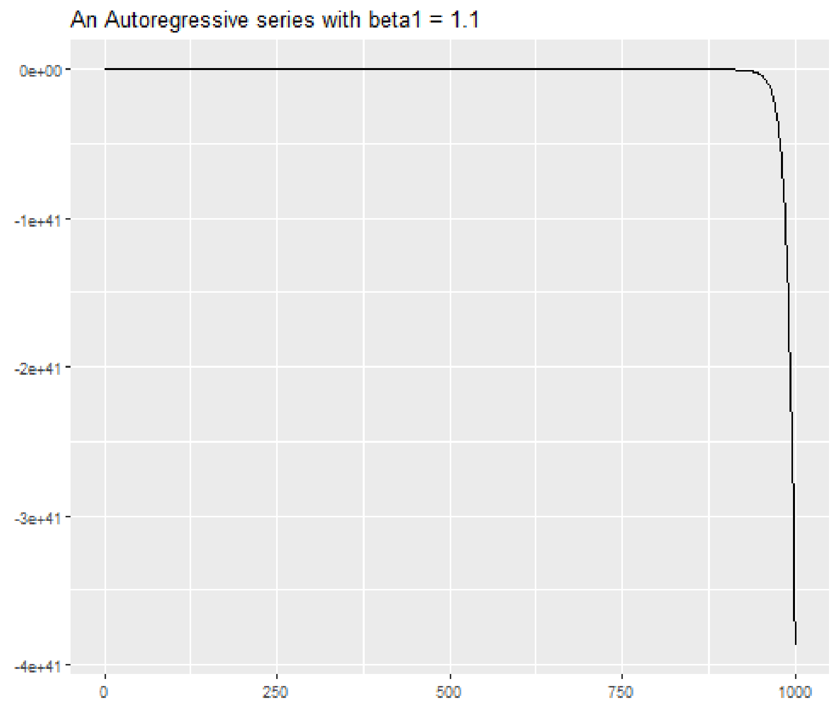
AR(1)

```
set.seed(2023)
n = 1000
y = rep(0, n)

y[1] = 0
b0 = 0
b1 = 1
e = rnorm(n, sd = 1)

for (t in 2:n)
{
  y[t] = b0 + b1*y[t-1]+e[t]
}

autoplot(ts(y)) + ggtitle(paste0("An Autoregressi
```



AR(1)

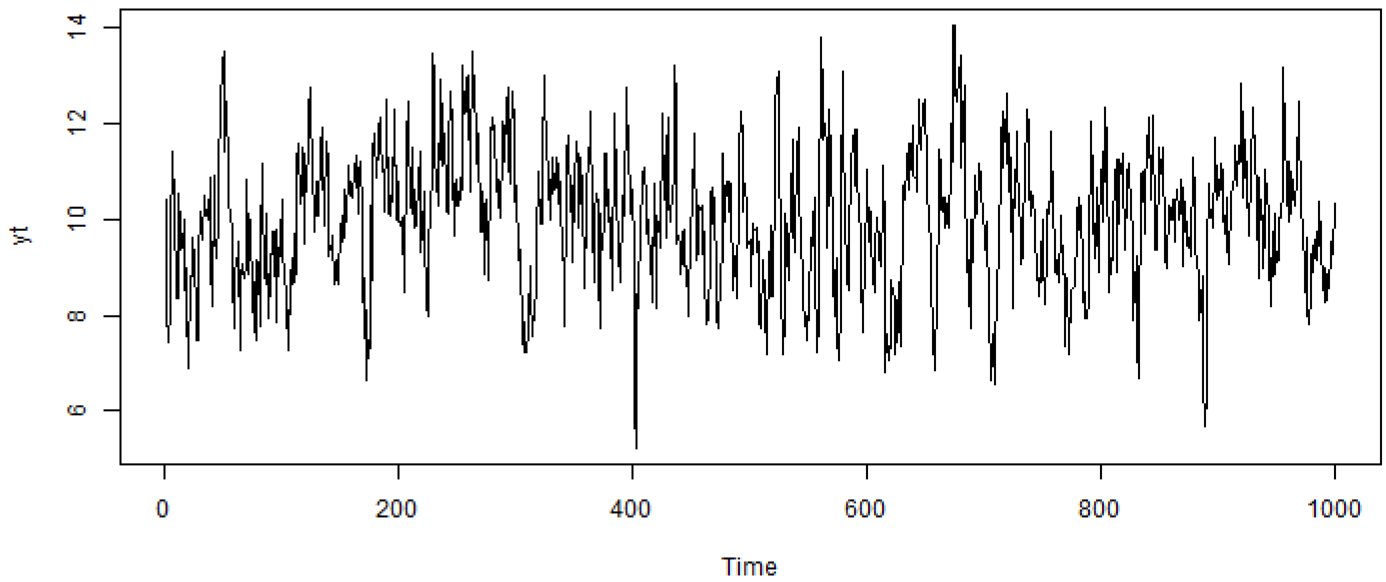
```r
set.seed(2023)
n = 1000
y = rep(0, n)

y[1] = 0
b0 = 0
b1 = 1.1
e = rnorm(n, sd = 1)

for (t in 2:n)
{
  y[t] = b0 + b1*y[t-1]+e[t]
}

autoplot(ts(y)) + ggtitle(paste0("An Autoregressi
```



Simulating AR(1)

- We can conveniently simulate AR(1) using `arima.sim` function

```
yt ← arima.sim(list(order=c(1,0,0), ar=.7), n=1000)
b0 = 10
yt ← yt + b0
plot(yt)
```

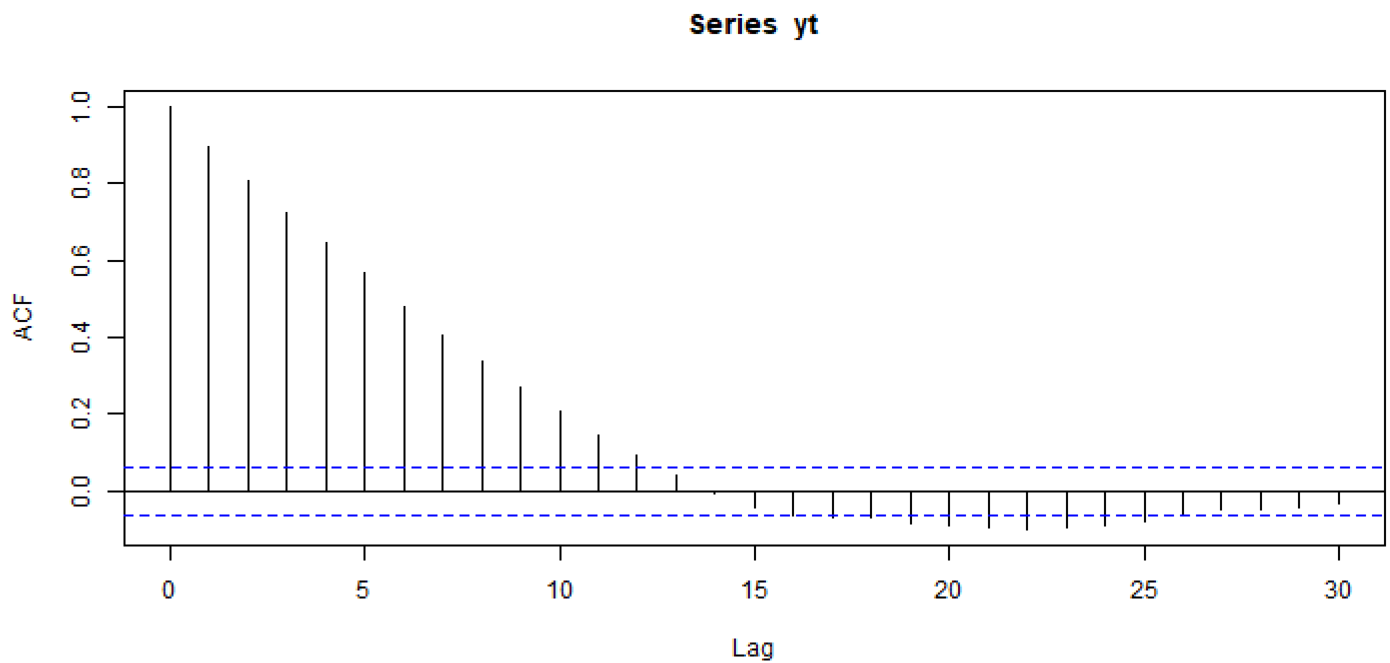


ACF

- For a positive value of β_1 the ACF exponentially decreases to 0 as the lag increases

```
yt ← arima.sim(list(order=c(1,0,0), ar=.9), n=1000)
b0 = 10
yt ← yt + b0
acf(yt)
```

β_1

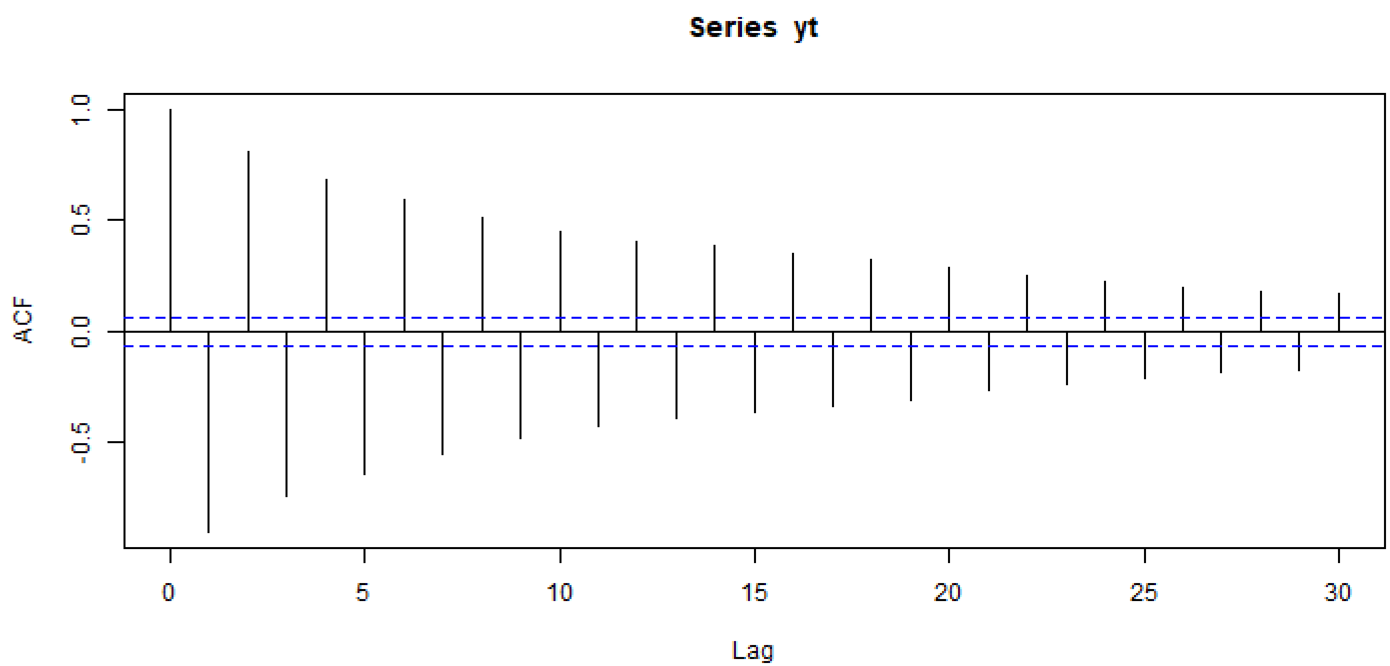


ACF

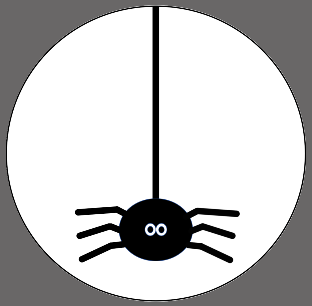
- For negative β_1 the ACF also exponentially decays to 0 as the lag increases, but the algebraic signs for the autocorrelations alternate between positive and negative

```
yt ← arima.sim(list(order=c(1,0,0), ar=-.9), n=1000)
b0 = 10
yt ← yt + b0
acf(yt)
```

β_1



Parameter Estimation



Parameter Estimation

- AR(1) is very similar to linear model where y_{t-1} play the roles of the predictor and y_t is the response
- In linear model, the predictor x is assumed to be non-random while the predictor y_{t-1} is non-random in AR(1)
- We estimate β_0 and β_1 by minimizing

$$\sum_{t=2}^{T}\Big(y_t - E(y_t|y_{t-1})\Big)^2 = \sum_{t=2}^{T}\Big(y_t - \beta_0 - \beta_1 y_{t-1}\Big)^2$$

- These estimators are called the conditional least squares estimators

Parameter Estimation

The coefficients are estimated by

$$\hat{\beta}_1 = \frac{\sum_{t=2}^{T}(y_{t-1} - \bar{y})(y_t - \bar{y})}{\sum_{t=2}^{T}(y_t - \bar{y})^2}$$
$$\hat{\beta}_0 = \bar{y}(1 - \hat{\beta}_1)$$

The only parameter left to estimate is the error variance, σ_ϵ^2, (error mean is zero), which can be estimated by s^2

$$s^2 = \frac{\sum_{t=2}^{T}(e_t - \bar{e})^2}{T - 3}$$

where $e_t = y_t - (\hat{\beta}_0 - \hat{\beta}_1 y_{t-1})$.

Example

You are given the following six observed values of the autoregressive model of order one time series

$$y_t = \beta_0 + \beta_1 y_{t-1} + \epsilon_t, \text{ with } Var(\epsilon_t) = \sigma^2.$$

t	**1**	**2**	**3**	**4**	**5**
y_t	1	3	5	8	13

Calculate $\hat{\beta}_1$ using the conditional least squares method.

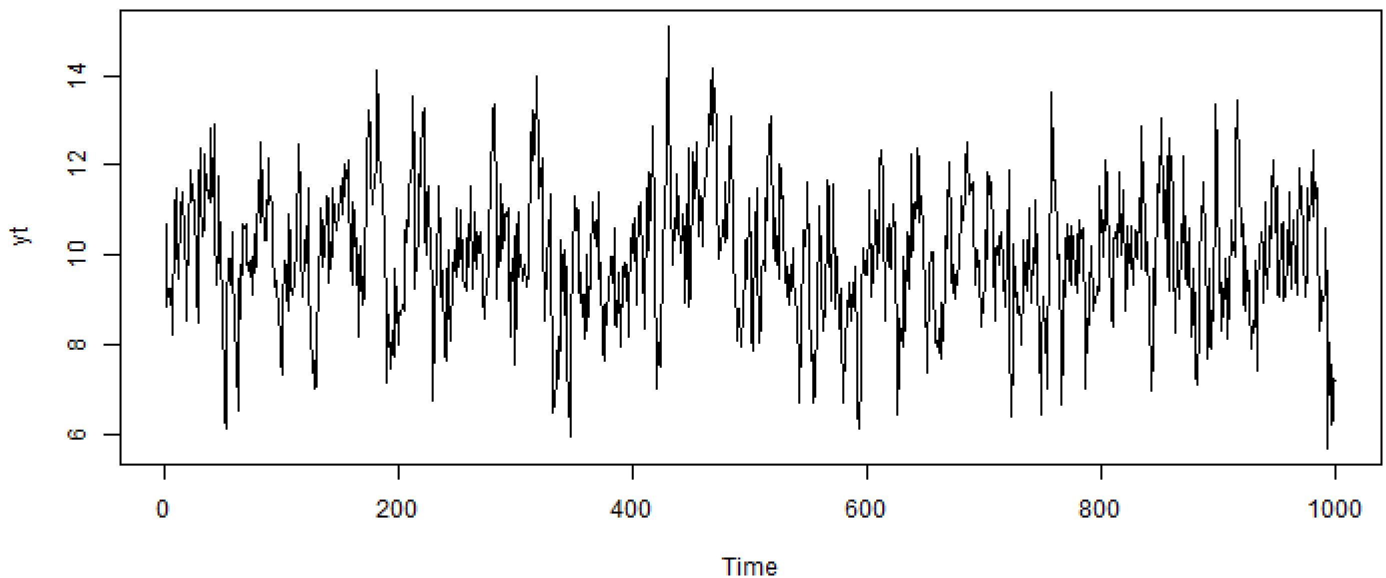
Parameter Estimation in R

- We can estimate the coefficients of AR(1) using the `arima` function

```
yt <- arima.sim(list(order=c(1,0,0), ar=c(.7)), n=1000)

b0 = 10

yt <- yt + b0

plot(yt)
```

β_0

$\beta_1 = .7$



Parameter Estimation in R

- We can estimate the coefficients of AR(1) using the `arima` function

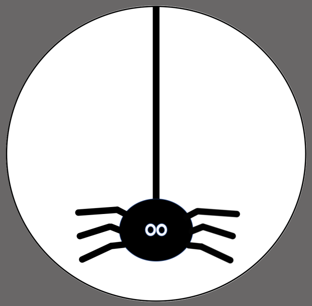
```
arima(yt, order = c(1,0,0))
```

```
##
## Call:
## arima(x = yt, order = c(1, 0, 0))
##
## Coefficients:
##          ar1  intercept
##       0.7132     9.9620
## s.e.  0.0222     0.1115
##
## sigma^2 estimated as 1.028:  log likelihood = -1432.89,  aic = 2871.78
```

$\hat{\beta}_1$ $\hat{\beta}_0$

- We see that the estimated coefficients are close to the true values.

Forecasting with AR(1)



Forecasting with AR(1)

- Suppose we have the AR(1) time series with known β_0 and β_1. If these parameters are unknown we can estimate them by the formula in the previous slices.

- We use the following formulas to for forecasting

$$\hat{y}_{T+1} = \beta_0 + \beta_1 y_T$$

$$\hat{y}_{T+k} = \mu + \beta_1^k (y_T - \mu)$$

where $\mu = \frac{\beta_0}{1-\beta_1}$.

Example

You are given

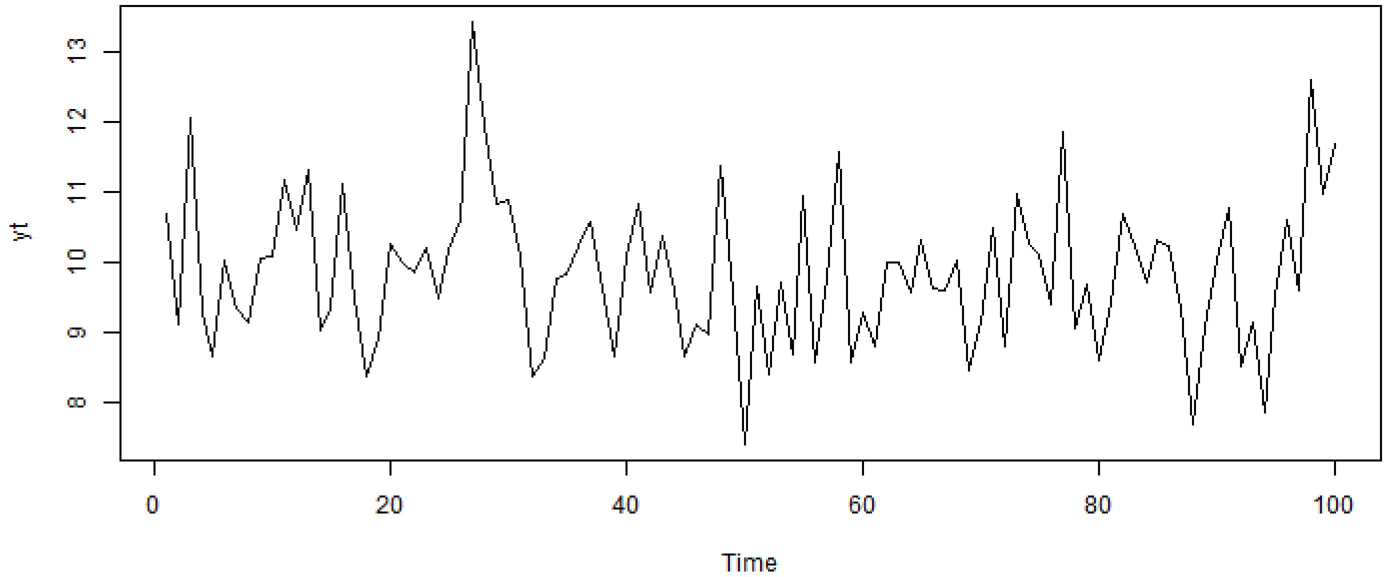
$$y_t = .3y_{t-1} + 4 + \epsilon$$

$$y_T = 7$$

Calculate the three step ahead forecast of y_{T+3}

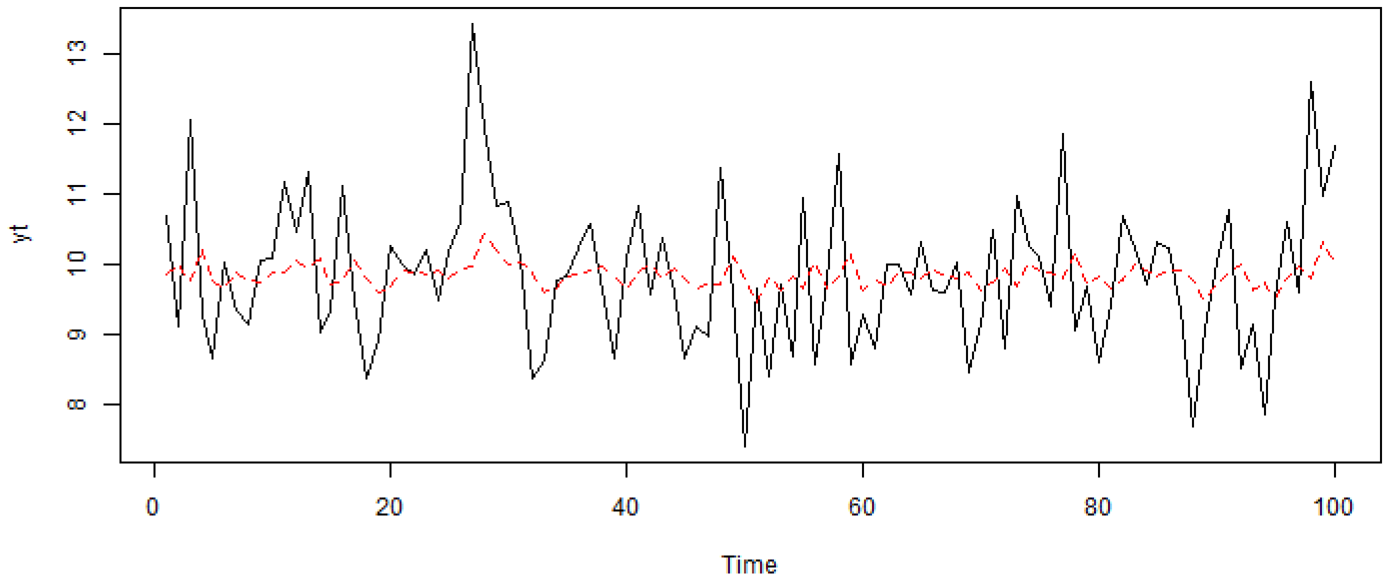
Forecasting with AR(1)

```
# create an AR(1) series
yt <- arima.sim(list(order=c(1,0,0), ar=c(.2)), n=100)
b0 = 10
yt <- yt + b0
plot(yt)
```



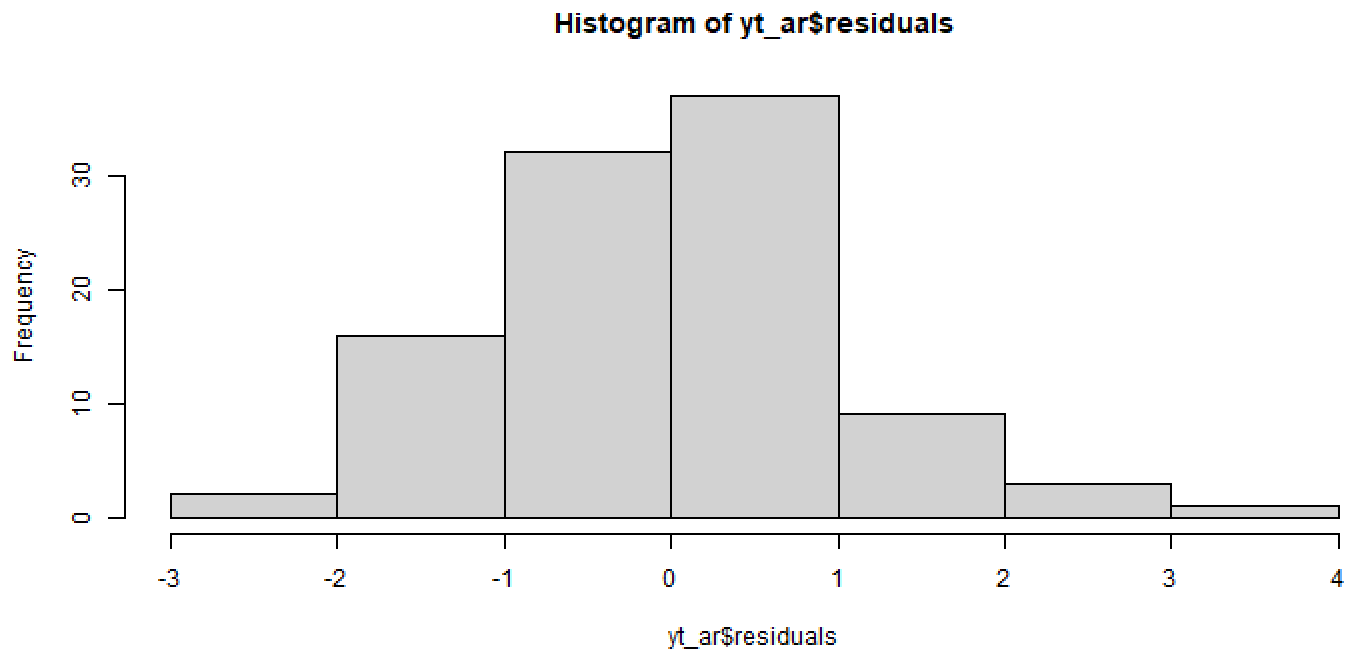
Forecasting with AR(1)

```
# estimate the series using AR(1) model
yt_ar  = arima(yt, order = c(1,0,0))

# plot the estimated series and the original series
yt_predicted <- yt - yt_ar$residuals

plot(yt)
points(yt_predicted, type = "l",
       col = "red", lty = 2)
```



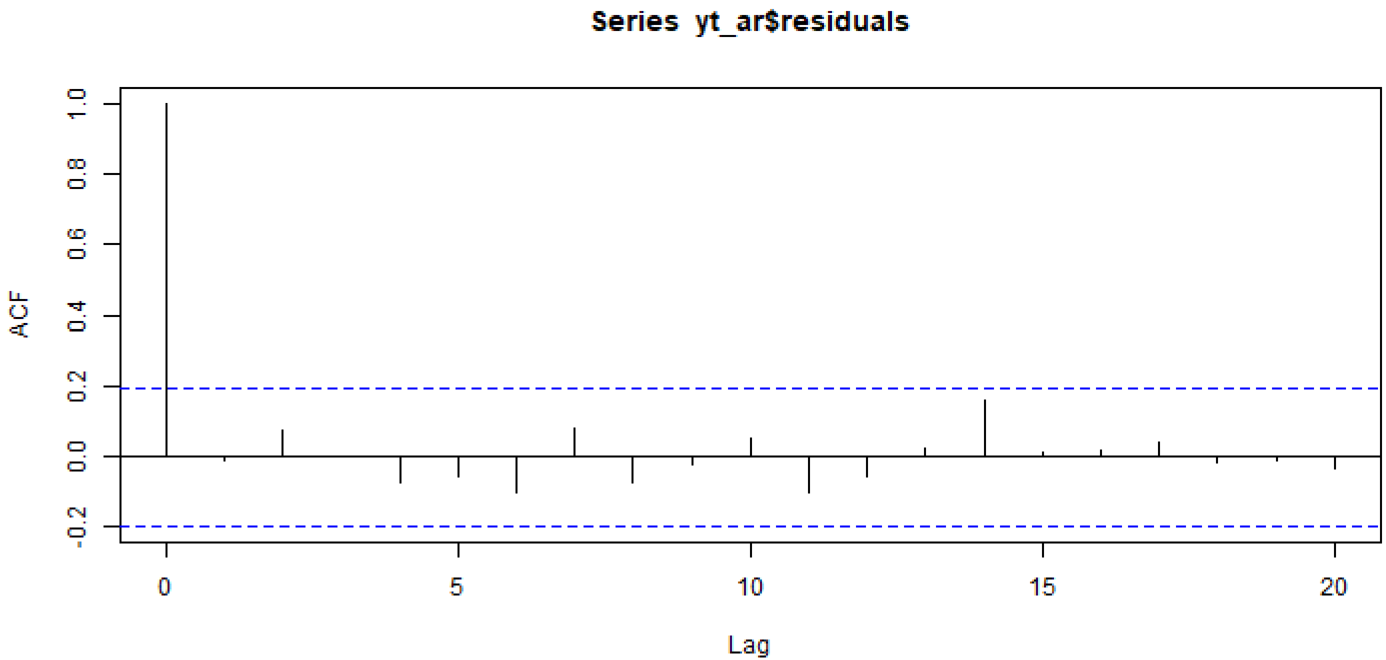
Forecasting with AR(1)

```
hist(yt_ar$residuals)
```



Forecasting with AR(1)

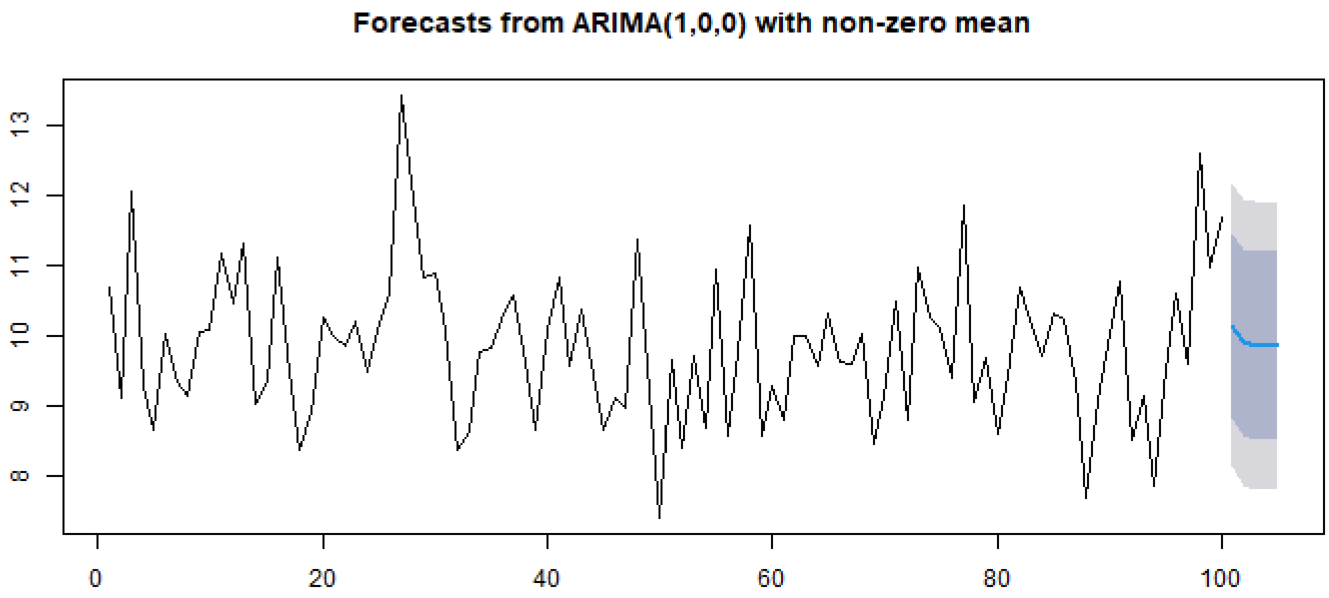
```
acf(yt_ar$residuals)
```



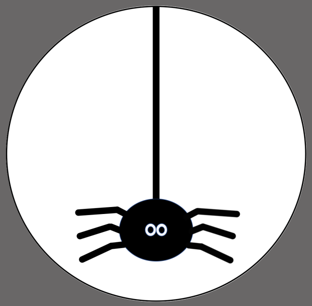
- The ACF of the error is similar to that of a white-noise.

Forecasting with AR(1)

```
ts3_forecasts2 <- forecast(yt_ar, h=5)
plot(ts3_forecasts2)
```



Differencing technique



Differencing technique

- A series should be stationary to be modeled by the AR(1) model.
- If the series is not stationary, one can de-trend the series using the differencing technique.
- The differenced series then can potentially stationary and can be modeled by the AR(1) model for forecasting

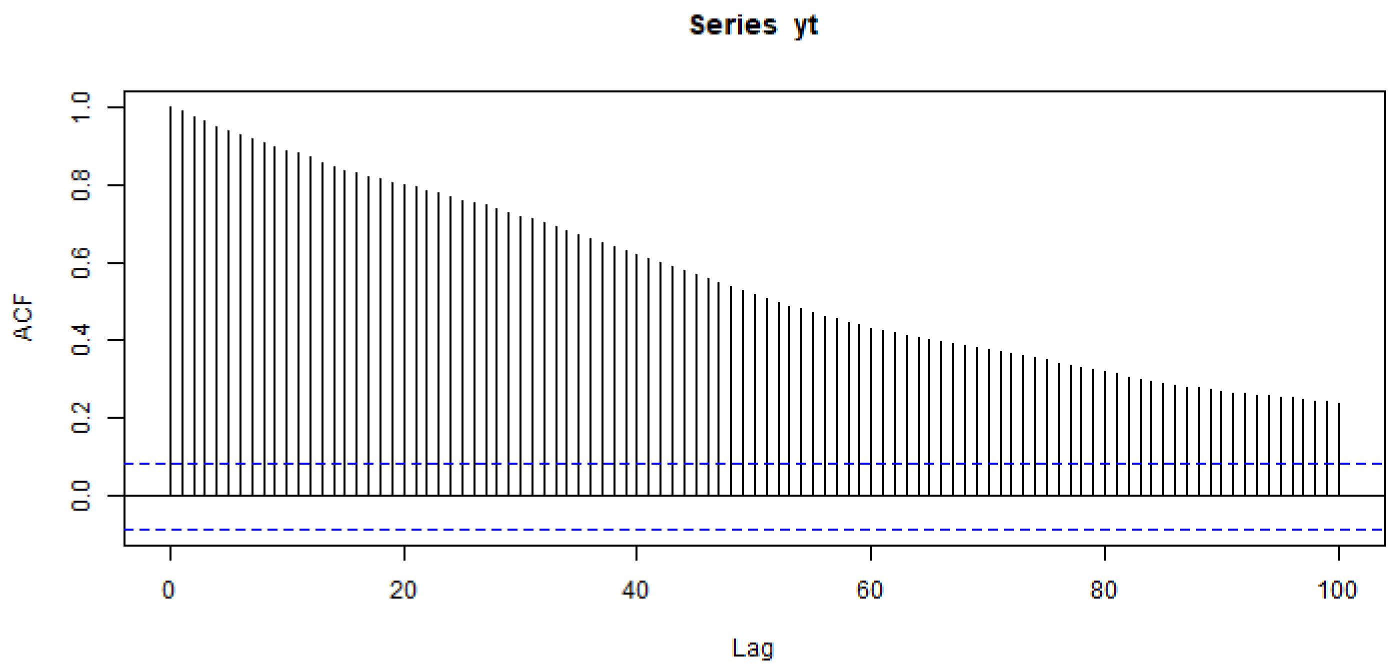
Example

```r
library(quantmod)
library(forecast)
getSymbols("AAPL")
```

```
## [1] "AAPL"
```

```r
yt = AAPL$AAPL.Open
yt <- yt[index(yt) > "2023-01-01"]
```

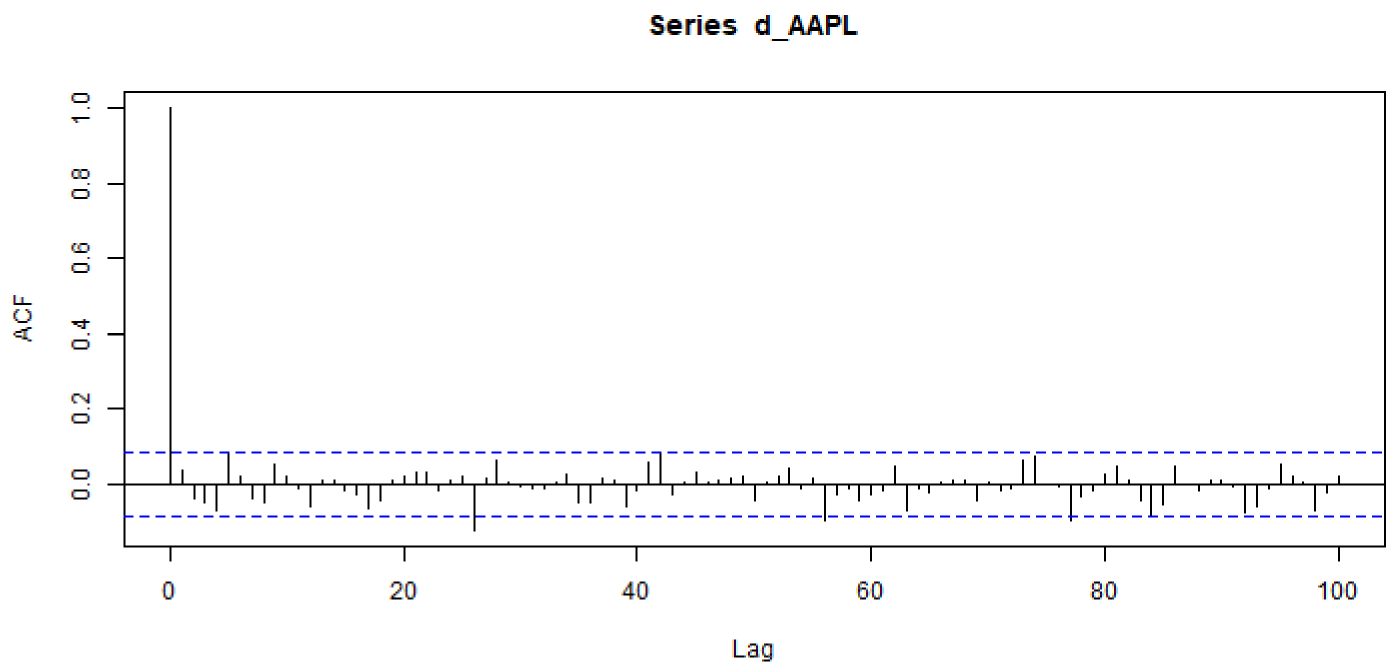
```r
acf(yt, lag.max = 100)
```



- Looking at the ACF, we observe that the correlations does not die out to zero but later go out of the blue lines.
- Thus this series is non-stationary and it is not reasonable to model the stock using the AR model.
- We will use a differencing technique to transform the stock to a stationary series. Consider the difference stock

$$d_n = y_{n+1} - y_n$$

```
d_AAPL = ts(as.numeric(diff(yt))[-1])
acf(d_AAPL, lag.max = 100)
```



- The ACF plot shows that the difference series is stationary and can be model by an AR model.

The ACF plot shows that the difference series is stationary and can be model by an AR model.

```
ar_AAPL = arima(d_AAPL, order = c(1,0,0))
ar_AAPL
```

```
##
## Call:
## arima(x = d_AAPL, order = c(1, 0, 0))
##
## Coefficients:
##          ar1  intercept
##       0.0385     0.2143
## s.e.  0.0433     0.1325
##
## sigma^2 estimated as 8.648:  log likelihood = -1331.23,  aic = 2668.47
```

- Forecast the next observation of d_n.

```
d_n = forecast(ar_AAPL, h = 1)
```

- Notice that $y_{n+1} = y_n + d_n$, we can forecast y_{n+1} using y_n and d_n

```
y_next = d_n$mean + yt[length(yt)]
y_next = as.numeric(y_next)
y_next
```

```
## [1] 244.8857
```