Consider the following stochastic differential equation (SDE)

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{1}$$

where b is drift term, σ is the dispersion, and W_t is a one-dimensional Brownian motion. Here we consider functions b and σ to be Borel-measurable functions from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$. The SDE (1) can be equivalently written in the following integral form:

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

In this section, we are interested in the conditions under which the SDE (1) has strong existence and strong uniqueness. Recall that for an ordinary differential equation (ODE), shown as below:

$$x'(t) = b(t, x(t)), \tag{2}$$

the Picard-Lindelof/Cauchy-Lipschitz theorem states that if the function b is Lipschitz with respect to the state variable, then there exists a solution to the ODE (2). This is to say that there exists a constant K such that $|b(t,x) - b(t,y)| \leq K|x-y|$. For uniqueness of the solution, locally Lipschitz condition is enough. This is to say that for all $R > 0$, there exists a constant K_R, depending on R, such that $|b(t,x) - b(t,y)| \leq K_R|x-y|$ for all $x, y \in [-R, R]$.

We can define similar conditions to show the strong existence and strong uniqueness for the SDE (1). We start by introducing the Gronwall inequality, which will be useful for proving the uniqueness results.

Lemma 1. *(Gronwall Inequality)*

Suppose that the measurable function $f : [0,\infty) \to \mathbb{R}$ satisfies

$$0 \leq f(t) \leq a + b\int_0^t f(s)ds,$$

for some $a, b \in \mathbb{R}$. Then,

$$f(t) \leq ae^{bt}.$$

The proof for this Lemma is omitted, and we introduce the uniqueness result here.

Theorem 1. *(Strong Uniqueness)*

If for all $R > 0$, there exists a constant C_R such that for all $t \geq 0$ and for all $x, y \in [-R, R]$, the following conditions hold for the functions b and σ in the SDE (1)

$$\begin{aligned} |b(t,x) - b(t,y)| &\leq C_R|x-y| \\ |\sigma(t,x) - \sigma(t,y)| &\leq C_R|x-y|, \end{aligned}$$

then strong uniqueness holds for the SDE (1).

Proof. Let X and $\tilde{X}$ be two strong solutions to the SDE (1). Define $Y_t := X_t - \tilde{X}_t$. We would like to compute $\mathbb{E}[Y_t^2]$. From Ito's formula, we have:

$$\begin{aligned} Y_t^2 &= Y_0^2 + 2\int_0^t Y_s dY_s + \int_0^t d\langle Y\rangle_s \\ &= 2\int_0^t Y_s(b(s,X_s) - b(s,\tilde{X}_s))ds + 2\int_0^t Y_s(\sigma(s,X_s) - \sigma(s,\tilde{X}_s))dW_s + \int_0^t (\sigma(s,X_s) - \sigma(s,\tilde{X}_s))^2 ds. \end{aligned} \tag{3}$$

Now, for a fixed $R > 0$, define the stopping time as follows

$$\tau_R = \inf\{t \geq 0 : |X_s| \geq R \quad \text{or} \quad |\tilde{X}_s| \geq R\}.$$

With this localization, we would like to show that the stochastic integral is a martingale. We first show that the integrand is square-integrable.

$$\mathbb{E}\left[\int_0^{t\wedge\tau_R} (Y_s(\sigma(s,X_s) - \sigma(s,\tilde{X}_s)))^2 ds\right] \leq C_R^2 \mathbb{E}\left[\int_0^{t\wedge\tau_R} ((X_s - \tilde{X}_s))^4 ds\right] \leq 16C_R^2R^2T < \infty.$$

Thus, we know $Y_s(\sigma(s,X_s) - \sigma(s,\tilde{X}_s)) \in \mathcal{L}_W^*$, and the stochastic integral in (3) is (locally) a martingale, which implies that its expectation is zero. Then, we (locally) bound the expectation of (3) as follows:

$$
\begin{aligned}
0 \le \mathbb{E}[Y_{t\wedge\tau_R}^2] &= 2\mathbb{E}\left[\int_0^{t\wedge\tau_R} Y_s(b(s,X_s)-b(s,\tilde{X}_s))ds\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_R}(\sigma(s,X_s)-\sigma(s,\tilde{X}_s))^2 ds\right] \\
&\le 2C_R^2\mathbb{E}\left[\int_0^{t\wedge\tau_R}(X_s-\tilde{X}_s)^2 ds\right] + C_R^2\mathbb{E}\left[\int_0^{t\wedge\tau_R}(X_s-\tilde{X}_s)^2 ds\right] \\
&= \tilde{C}\mathbb{E}\left[\int_0^t (X_{s\wedge\tau_R}-\tilde{X}_{s\wedge\tau_R})^2 ds\right] \\
&= \tilde{C}\int_0^t \mathbb{E}\left[(X_{s\wedge\tau_R}-\tilde{X}_{s\wedge\tau_R})^2\right] ds.
\end{aligned}
$$

Now, we are ready to use Lemma 1. In particular, we have $a = 0$, which implies $\mathbb{E}[Y_{t\wedge\tau_R}^2] = 0$. Note $R \to \infty \Rightarrow \tau_R \to \infty \Rightarrow t \wedge \tau_R = t$. Then, by the dominated convergence theorem, we have $\mathbb{E}[Y_t^2] = 0$. This shows that $\{X_t, 0 \le t < \infty\}$ and $\{\tilde{X}_t, 0 \le t < \infty\}$ are modifications of each other. Since both solutions are continuous, we know that they are indistinguishable.

□

Next, we show the existence of the strong solution to the SDE (1), which asks more than locally Lipschitz for the function b and σ.

Theorem 2. *(Strong Existence)*

Assume functions b and σ in the SDE (1) are Lipschitz continuous and satisfy the growth condition as detailed below:

$$
\begin{aligned}
&|b(t,x)-b(t,y)| \le C|x-y| \\
&|\sigma(t,x)-\sigma(t,y)| \le C|x-y| \\
&|b(t,x)|^2 + |\sigma(t,x)|^2 \le C(1+|x|^2),
\end{aligned}
$$

for some constant C, then the strong existence holds for the SDE (1).

Moreover, for every finite $T > 0$, there exists a positive constant $\tilde{C}$, depending only on T and C such that

$$
\mathbb{E}[|X_t|^2] \le \tilde{C}(1+|x|^2)e^{\tilde{C}t}, \quad t \in [0,T]. \tag{4}
$$

We first present the following lemma without prove.

Lemma 2. *Consider the following sequence of stochastic processes $\{X^{(k)}\}_{k\in\mathbb{N}}$. Starting from $X_t^{(0)} := x$, define $X^{(k)}$ recursively as:*

$$
X_t^{(k+1)} := x + \int_0^t b(s, X^{(k)})ds + \int_0^t \sigma(s, X^{(k)})dW_s. \tag{5}
$$

Then, for each $k \in \mathbb{N}$, the process $X_t^{(k)}$ is square integrable in the following sense: for every finite $T > 0$, there exists a positive constant $\tilde{C}$, depending only on T and C such that

$$
\mathbb{E}[|X_t^{(k)}|^2] \le \tilde{C}(1+|x|^2)e^{\tilde{C}t}, \quad t \in [0,T], k \in \mathbb{N}. \tag{6}
$$

Now, we prove Theorem 2.

Proof. The goal is to show that the sequence of stochastic processes $\{X^{(k)}\}_{k\in\mathbb{N}}$ defined recursively in (5) converges to X_t so that the limit of (5) coincides with the SDE (1). Let's start with the following:

$$
\mathbb{E}\left[\max_{t\in[0,T]} |X_t^{(k+1)} - X_t^{(k)}|^2\right] = \mathbb{E}\left[\max_{t\in[0,T]} (B_t + M_t)^2\right] \le 2\mathbb{E}\left[\max_{t\in[0,T]} B_t^2\right] + 2\mathbb{E}\left[\max_{t\in[0,T]} M_t^2\right],
$$

where

$$
\begin{aligned}
B_t &:= \int_0^t b(s, X_s^{(k)}) - b(s, X_s^{(k-1)})ds \\
M_t &:= \int_0^t \sigma(s, X_s^{(k)}) - \sigma(s, X_s^{(k-1)})dW_s.
\end{aligned}
$$

Let's first focus on the term B_t. Using the Cauchy-Schwarz inequality and the Lipschitz condition on the function b, we have:

$$\begin{aligned}\mathbb{E}\left[\max_{t\in[0,T]} B_t^2\right] &= \mathbb{E}\left[\max_{t\in[0,T]} |\int_0^t b(s,X_s^{(k)}) - b(s,X_s^{(k-1)})ds|^2\right] \\ &\le \mathbb{E}\left[\max_{t\in[0,T]} t\int_0^t |b(s,X_s^{(k)}) - b(s,X_s^{(k-1)})|^2 ds\right] \\ &= \mathbb{E}\left[T\int_0^T |b(s,X_s^{(k)}) - b(s,X_s^{(k-1)})|^2 ds\right] \\ &\le TC^2\mathbb{E}\left[\int_0^T |X_s^{(k)} - X_s^{(k-1)}|^2 ds\right].\end{aligned}$$

Next, we turn attention to M_t. From Lemma 2 and the Lipschitz condition on the function σ, it is easy to show that M_t is a square integrable martingale. This fact allows us to use the BDG-inequality, which yields the following result

$$\begin{aligned}\mathbb{E}\left[\max_{t\in[0,T]} M_t^2\right] &\le \tilde{C}\mathbb{E}[\langle M\rangle_T] \\ &= \tilde{C}\mathbb{E}\left[\int_0^T (\sigma(s,X_s^{(k)}) - \sigma(s,X_s^{(k-1)}))^2 ds\right] \\ &\le \tilde{C}C^2\mathbb{E}\left[\int_0^T (X_s^{(k)} - X_s^{(k-1)})^2 ds\right].\end{aligned}$$

Combining these two results, we have:

$$\begin{aligned}\mathbb{E}\left[\max_{t\in[0,T]} |X_t^{(k+1)} - X_t^{(k)}|^2\right] &\le 2C^2(T+\tilde{C})\mathbb{E}\left[\int_0^T |X_s^{(k)} - X_s^{(k-1)}|^2 ds\right] \\ &= L\mathbb{E}\left[\int_0^T |X_s^{(k)} - X_s^{(k-1)}|^2 ds\right] \\ &\le C^*\frac{(Lt)^k}{k!},\end{aligned} \tag{7}$$

where

$$C^* = \mathbb{E}\left[\max_{t\in[0,T]} |X_t^{(1)} - x|^2\right].$$

The last inequality in (7) can be shown by induction. When $k=0$, the result is obvious. Assuming $k=k$ holds, $k=k+1$ can be verified by using the second last relation in (7).

From Markov's inequality, we have:

$$\mathbb{P}\left[\max_{t\in[0,T]} |X_t^{(k+1)} - X_t^{(k)}| \ge 2^{-k}\right] \le 4^k\mathbb{E}\left[\max_{t\in[0,T]} |X_t^{(k+1)} - X_t^{(k)}|^2\right] \le C^*\frac{(4Lt)^k}{k!} \to 0,$$

which implies that $\mathbb{P}\left[\max_{t\in[0,T]} |X_t^{(k+1)} - X_t^{(k)}| \ge 2^{-k}\right]$ is a convergent series. From the Borel-Cantelli Lemma, we have:

$$\sum_{k=1}^{\infty}\mathbb{P}\left[\max_{t\in[0,T]} |X_t^{(k+1)} - X_t^{(k)}| \ge 2^{-k}\right] < \infty \Rightarrow \mathbb{P}\left[\limsup_{k\to\infty}\max_{t\in[0,T]} |X_t^{(k+1)} - X_t^{(k)}| \ge 2^{-k}\right] = 0.$$

In other words, eventually with probability 1, $\max_{t\in[0,T]} |X_t^{(k+1)} - X_t^{(k)}| < 2^{-k}$. It means that, with probability 1, the sequence $\{X_t^{(k)}\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T],\mathbb{R})$. Under the supremum norm (which makes the space complete), the sequence $\{X_t^{(k)}\}_{k\in\mathbb{N}}$ converges.

Now, it remains to show that the continuous limit of $\{X_t^{(k)}\}_{k\in\mathbb{N}}$, denoted as X_t^{∞}, is indeed the strong solution of the SDE (1). First of all, the integrability condition (4) can be shown from (6) plus the Fatou's Lemma. Next, it is obvious that X_t^{∞} is $\mathbb{F}$-adapted and the initial condition holds. From the growth condition and (4), we know that b and σ^2 are almost surely integrable. It remains to check the following:

$$\int_0^t b(s, X_s^{(k)})ds \to \int_0^t b(s, X_s^{\infty})ds$$
$$\int_0^t \sigma(s, X_s^{(k)})dW_s \to \int_0^t \sigma(s, X_s^{\infty})dW_s.$$

The first convergence relation holds because b is Lipschitz continuous (b is uniformly bounded, and $\{X_t^{(k)}(\omega)\}_{k\in\mathbb{N}}$ converges uniformly). The second convergence result will be shown using the DDS theorem. Define $\tilde{M}_t := \int_0^t \sigma(s, X_s^{(k)}) - \sigma(s, X_s^{\infty})dW_s$. It is easy to see that $\tilde{M}_t \in \mathcal{M}_c^2$, and $\langle M\rangle_t \to \infty$. Then, the martingale $\tilde{M}_t$ is a time changed Brownian motion, i.e., $\tilde{M}_t = W_{\langle\tilde{M}\rangle_t}$. Additionally, we know: $\langle\tilde{M}\rangle_t = \int_0^t (\sigma(s, X_s^{(k)}) - \sigma(s, X_s^{\infty}))^2 ds = 0$, because of the Lipschitz continuity of σ and the uniformly convergence of $\{X_t^{(k)}(\omega)\}_{k\in\mathbb{N}}$. Thus, we have $\tilde{M}_t = W_0 = 0$, by definition of a standard Brownian motion. The proof is completed.

□

Theorems 1 and 2 shows general conditions for strong existence and uniqueness for the SDE (1), and the proofs shown above can be generalized to multi-dimensional cases. However, the (locally) Lipschitz conditions are quite strong. Consider the following two SDEs.

Example 1. *(Squared-Bessel Process)*

$$dR_t = mdt + 2\sqrt{R_t}dW_t$$

Example 2. *(Squared Ornstein Uhlenbeck Processe)*

$$dR_t = a(m - R_t)dt + 2\sqrt{R_t}dW_t$$

We may show that the dispersion coefficients in both SDEs are not (locally) Lipschitz. We have shown before that the squared-Bessel process has a weak solution, but we cannot say more. If we can somehow show that the strong uniqueness holds, from a later study about weak solution, it also implies the weak uniqueness (in distribution). This is indeed true! We will see next that in the one-dimensional case, the Lipschitz condition on the dispersion coefficient can be relaxed considerably.

Theorem 3. *(Yamada & Watanabe (1971))*
Suppose that the coefficients of the one dimensional SDE in (1) *satisfy the conditions*

$$\begin{aligned} |b(t,x) - b(t,y)| &\le C|x-y| \\ |\sigma(t,x) - \sigma(t,y)| &\le h(|x-y|), \end{aligned} \tag{8}$$

for every $0 \le t \le \infty$ *and* $x \in \mathbb{R}, y \in \mathbb{R}$, *where* K *is a positive constant and* $h : [0,\infty) \to [0,\infty)$ *is a strictly increasing function with* $h(0) = 0$ *and*

$$\int_0^{\epsilon} h^{-2}(u)du = \infty, \quad \forall \epsilon > 0. \tag{9}$$

Then, strong uniqueness holds for the SDE (1).

Proof. Suppose X_t and $\tilde{X}_t$ are two solutions to the SDE (1). We would like to show $\mathbb{E}[|X_t - \tilde{X}_t] = 0$, which implies that X_t and $\tilde{X}_t$ are modifications to each other, then by continuity, X_t and $\tilde{X}_t$ are indistinguishable.

Unfortunately, the function $|\cdot|$ is not twice differentiable, so that we cannot directly apply Ito's formula to it, like we did in the proof for Theorem 1. Instead, we construct a sequence of twice differentiable functions to approximate $|\cdot|$. From condition (9), we can construct a decreasing sequence $\{a_n\}_{n=0}^{\infty}$ with $a_0 = 1$, $\lim_{n\to\infty} a_n = 0$, and $\int_{a_{n+1}}^{a_n} h^{-2}(u)du = n$. Then, we construct a sequence of continuous functions $\{\phi_n\}_{n=0}^{\infty}$ on $\mathbb{R}$ with support in (a_{n+1}, a_n), so that $0 \le \phi_n(x) \le \frac{2}{nh^2(x)}$ and $\int_{a_{n+1}}^{a_n} \phi_n(x)dx = 1$. Now, we define the function $\psi_n(x)$ as

$$\psi_n(x) := \int_0^{|x|}\int_0^y \phi_n(u)dudy.$$

$\psi_n(x)$ has the following properties:

1. $\psi_n(x) = \psi_n(-x)$
2. $|\psi_n'(x)| = |\int_0^y \phi_n(u)du| \le 1$
   $\psi_n''(x) = \phi_n(x)$
   They implies that $\psi_n(x)$ is twice continuously differentiable.
3. $\psi_n(x) = |x|, \forall |x| \ge 1$

4. $\lim_{n\to\infty} \psi_n(x) = |x|$

Now, we apply Ito's formula as follows:

$$\begin{aligned} d\psi_n(X_t - \tilde{X}_t) =& \psi_n'(X_t - \tilde{x}_t)d(X_t - \tilde{x}_t) + \frac{1}{2}\psi_n''(X_t - \tilde{x}_t)d\langle X_t - \tilde{x}\rangle_t \\ =& \psi_n'(X_t - \tilde{x}_t)(b(t, X_t) - b(t, \tilde{X}_t))dt + \\ & \psi_n'(X_t - \tilde{x}_t)(\sigma(t, X_t) - \sigma(t, \tilde{X}_t))dW_t + \\ & \frac{1}{2}\psi_n''(X_t - \tilde{x}_t)(\sigma(t, X_t) - \sigma(t, \tilde{X}_t))^2 dt. \end{aligned}$$

We claim $\psi_n'(X_t - \tilde{x}_t)(\sigma(t, X_t) - \sigma(t, \tilde{X}_t)) \in \mathcal{L}_W^*$, which can be shown using the localization argument. Then, $\int_0^t \psi_n'(X_s - \tilde{x}_s)(\sigma(s, X_s) - \sigma(s, \tilde{X}_s))dW_s \in \mathcal{M}_c^2$, so its expectation vanishes. We have the following

$$\begin{aligned} \mathbb{E}[\psi_n(X_t - \tilde{X}_t)] =& \mathbb{E}[\int_0^t \psi_n'(X_s - \tilde{x}_s)(b(s, X_s) - b(s, \tilde{X}_s))ds] + \\ & \frac{1}{2}\mathbb{E}[\int_0^t \psi_n''(X_s - \tilde{x}_s)(\sigma(s, X_s) - \sigma(s, \tilde{X}_s))ds] \\ \leq & C\mathbb{E}[\int_0^t |X_s - \tilde{X}_s|ds] + \frac{1}{2}\mathbb{E}[\int_0^t \frac{2}{nh^2(|X_s - \tilde{X}_s|)}h^2(|X_s - \tilde{X}_s|)ds] \\ =& C\int_0^t \mathbb{E}[|X_s - \tilde{X}_s|]ds + \mathbb{E}[\int_0^t \frac{1}{n}ds]. \end{aligned}$$

Taking limit, we have

$$\mathbb{E}[|X_t - \tilde{X}_t|] \leq C\int_0^t \mathbb{E}[|X_s - \tilde{X}_s|]ds.$$

Applying the Gronwall inequality (1), we conclude that $\mathbb{E}[|X_t - \tilde{X}_t|] = 0$.

□

Sometimes the drift term in the SDE (1) can be too complicated to analyze. If we can either upper or lower bound it by another function that is easier to analyze, then we can compare the solutions of these two SDEs as shown in the following results. Note, this result also only works for one-dimensional SDEs.

Theorem 4. *(Comparison Principle)*

Let X_t and $\tilde{X}_t$ be strong solutions to the SDE (1) *with drift term $b(t,x)$ and $\tilde{b}(t,x)$, respectively. Suppose the following conditions hold:*

1. *b, ,$\tilde{b}$, and σ are continuous functions*
2. *$|\sigma(t,x) - \sigma(t,y)| \leq h(|x-y|)$, with h being the function in Theorem 3*
3. *$b(t,x)$ or $\tilde{b}(t,x)$ satisfies the Lipschitz condition.*
4. *The initial condition $x \leq \tilde{x}$.*
5. *$b(t,x) \leq \tilde{b}(t,x)$ for all t and x*

Then, $X_t \leq \tilde{X}_t$ for all t.

Proof. The proof is similar to the previous one, except we would like to show $\mathbb{E}[|X_t - \tilde{X}_t|_+] := \mathbb{E}[\max\{|X_t - \tilde{X}_t|, 0\}] = 0$. Define $\psi_n(x) = \tilde{\psi}_n(x)\mathbb{1}_{x\geq 0}$, where $\tilde{\psi}_n(x)$ is the same twice continuously differentiable function in the proof for Theorem 3. Without loss of generality, we assume that $\tilde{b}(t,x)$ is Lipschitz continuous. We omit first several steps related to the Ito calculus, because they are identical to the previous proof. Taking expectation, we have

$$
\begin{aligned}
\mathbb{E}[\psi_n(X_t-\tilde{X}_t)] =&\mathbb{E}[\int_0^t \psi_n'(X_s-\tilde{x}_s)(b(s,X_s)-\tilde{b}(s,\tilde{X}_s))ds]+\frac{1}{2}\mathbb{E}[\int_0^t \psi_n''(X_s-\tilde{x}_s)(\sigma(s,X_s)-\sigma(s,\tilde{X}_s))ds]\\
=&\mathbb{E}[\int_0^t \psi_n'(X_s-\tilde{x}_s)(b(s,X_s)-\tilde{b}(s,X_s))ds]+\mathbb{E}[\int_0^t \psi_n'(X_s-\tilde{x}_s)(\tilde{b}(s,X_s)-\tilde{b}(s,\tilde{X}_s))ds]+\\
&\frac{1}{2}\mathbb{E}[\int_0^t \psi_n''(X_s-\tilde{x}_s)(\sigma(s,X_s)-\sigma(s,\tilde{X}_s))ds]\\
\leq&\mathbb{E}[\int_0^t \psi_n'(X_s-\tilde{x}_s)(\tilde{b}(s,X_s)-\tilde{b}(s,\tilde{X}_s))ds]+\frac{1}{2}\mathbb{E}[\int_0^t \psi_n''(X_s-\tilde{x}_s)(\sigma(s,X_s)-\sigma(s,\tilde{X}_s))ds]\\
\leq&C\mathbb{E}[\int_0^t |X_s-\tilde{X}_s|ds]+\frac{1}{2}\mathbb{E}[\int_0^t \frac{2}{nh^2(|X_s-\tilde{X}_s|)}h^2(|X_s-\tilde{X}_s|)ds]\\
=&C\int_0^t \mathbb{E}[|X_s-\tilde{X}_s|]ds+\mathbb{E}[\int_0^t \frac{1}{n}ds].
\end{aligned}
$$

Then, using the Gronwall inequality, we have $\mathbb{E}[|X_t-\tilde{X}_t|_+]=0$, which implies $X_t\leq\tilde{X}_t$ for all t almost surely. □