

Consider the following stochastic differential equation (SDE)

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad (1)$$

where b is drift term, σ is the dispersion, and W_t is a one-dimensional Brownian motion. Here we consider functions b and σ to be Borel-measurable functions from $\mathbb{R}_+ \times \mathbb{R}$ to \mathbb{R} . The SDE (1) can be equivalently written in the following integral form:

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

In this section, we are interested in the conditions under which the SDE (1) has strong existence and strong uniqueness. Recall that for an ordinary differential equation (ODE), shown as below:

$$x'(t) = b(t, x(t)), \quad (2)$$

the Picard-Lindelof/Cauchy-Lipschitz theorem states that if the function b is Lipschitz with respect to the state variable, then there exists a solution to the ODE (2). This is to say that there exists a constant K such that $|b(t, x) - b(t, y)| \leq K|x - y|$. For uniqueness of the solution, locally Lipschitz condition is enough. This is to say that for all $R > 0$, there exists a constant K_R , depending on R , such that $|b(t, x) - b(t, y)| \leq K_R|x - y|$ for all $x, y \in [-R, R]$.

We can define similar conditions to show the strong existence and strong uniqueness for the SDE (1). We start by introducing the Gronwall inequality, which will be useful for proving the uniqueness results.

Lemma 1. (Gronwall Inequality)

Suppose that the measurable function $f : [0, \infty) \rightarrow \mathbb{R}$ satisfies

$$0 \leq f(t) \leq a + b \int_0^t f(s)ds,$$

for some $a, b \in \mathbb{R}$. Then,

$$f(t) \leq ae^{bt}.$$

The proof for this Lemma is omitted, and we introduce the uniqueness result here.

Theorem 1. (Strong Uniqueness)

If for all $R > 0$, there exists a constant C_R such that for all $t \geq 0$ and for all $x, y \in [-R, R]$, the following conditions hold for the functions b and σ in the SDE (1)

$$\begin{aligned} |b(t, x) - b(t, y)| &\leq C_R|x - y| \\ |\sigma(t, x) - \sigma(t, y)| &\leq C_R|x - y|, \end{aligned}$$

then strong uniqueness holds for the SDE (1).

Proof. Let X and \tilde{X} be two strong solutions to the SDE (1). Define $Y_t := X_t - \tilde{X}_t$. We would like to compute $\mathbb{E}[Y_t^2]$. From Ito's formula, we have:

$$\begin{aligned} Y_t^2 &= Y_0^2 + 2 \int_0^t Y_s dY_s + \int_0^t d\langle Y \rangle_s \\ &= 2 \int_0^t Y_s(b(s, X_s) - b(s, \tilde{X}_s))ds + 2 \int_0^t Y_s(\sigma(s, X_s) - \sigma(s, \tilde{X}_s))dW_s + \int_0^t (\sigma(s, X_s) - \sigma(s, \tilde{X}_s))^2 ds. \end{aligned} \quad (3)$$

Now, for a fixed $R > 0$, define the stopping time as follows

$$\tau_R = \inf\{t \geq 0 : |X_s| \geq R \text{ or } |\tilde{X}_s| \geq R\}.$$

With this localization, we would like to show that the stochastic integral is a martingale. We first show that the integrand is square-integrable.

$$\mathbb{E} \left[\int_0^{t \wedge \tau_R} (Y_s(\sigma(s, X_s) - \sigma(s, \tilde{X}_s)))^2 ds \right] \leq C_R^2 \mathbb{E} \left[\int_0^{t \wedge \tau_R} ((X_s - \tilde{X}_s))^4 ds \right] \leq 16C_R^2 R^2 T < \infty.$$

Thus, we know $Y_s(\sigma(s, X_s) - \sigma(s, \tilde{X}_s)) \in \mathcal{L}_W^*$, and the stochastic integral in (3) is (locally) a martingale, which implies that its expectation is zero. Then, we (locally) bound the expectation of (3) as follows:

$$\begin{aligned}
0 \leq \mathbb{E}[Y_{t \wedge \tau_R}^2] &= 2\mathbb{E} \left[\int_0^{t \wedge \tau_R} Y_s(b(s, X_s) - b(s, \tilde{X}_s))ds \right] + \mathbb{E} \left[\int_0^{t \wedge \tau_R} (\sigma(s, X_s) - \sigma(s, \tilde{X}_s))^2 ds \right] \\
&\leq 2C_R^2 \mathbb{E} \left[\int_0^{t \wedge \tau_R} (X_s - \tilde{X}_s)^2 ds \right] + C_R^2 \mathbb{E} \left[\int_0^{t \wedge \tau_R} (X_s - \tilde{X}_s)^2 ds \right] \\
&= \tilde{C} \mathbb{E} \left[\int_0^t (X_{s \wedge \tau_R} - \tilde{X}_{s \wedge \tau_R})^2 ds \right] \\
&= \tilde{C} \int_0^t \mathbb{E} \left[(X_{s \wedge \tau_R} - \tilde{X}_{s \wedge \tau_R})^2 \right] ds.
\end{aligned}$$

Now, we are ready to use Lemma 1. In particular, we have $a = 0$, which implies $\mathbb{E}[Y_{t \wedge \tau_R}^2] = 0$. Note $R \rightarrow \infty \Rightarrow \tau_R \rightarrow \infty \Rightarrow t \wedge \tau_R = t$. Then, by the dominated convergence theorem, we have $\mathbb{E}[Y_t^2] = 0$. This shows that $\{X_t, 0 \leq t < \infty\}$ and $\{\tilde{X}_t, 0 \leq t < \infty\}$ are modifications of each other. Since both solutions are continuous, we know that they are indistinguishable. \square

Next, we show the existence of the strong solution to the SDE (1), which asks more than locally Lipschitz for the function b and σ .

Theorem 2. (Strong Existence)

Assume functions b and σ in the SDE (1) are Lipschitz continuous and satisfy the growth condition as detailed below:

$$\begin{aligned}
|b(t, x) - b(t, y)| &\leq C|x - y| \\
|\sigma(t, x) - \sigma(t, y)| &\leq C|x - y| \\
|b(t, x)|^2 + |\sigma(t, x)|^2 &\leq C(1 + |x|^2),
\end{aligned}$$

for some constant C , then the strong existence holds for the SDE (1).

Moreover, for every finite $T > 0$, there exists a positive constant \tilde{C} , depending only on T and C such that

$$\mathbb{E}[|X_t|^2] \leq \tilde{C}(1 + |x|^2)e^{\tilde{C}t}, \quad t \in [0, T]. \quad (4)$$

We first present the following lemma without prove.

Lemma 2. Consider the following sequence of stochastic processes $\{X^{(k)}\}_{k \in \mathbb{N}}$. Starting from $X_t^{(0)} := x$, define $X^{(k)}$ recursively as:

$$X_t^{(k+1)} := x + \int_0^t b(s, X^{(k)})ds + \int_0^t \sigma(s, X^{(k)})dW_s. \quad (5)$$

Then, for each $k \in \mathbb{N}$, the process $X_t^{(k)}$ is square integrable in the following sense: for every finite $T > 0$, there exists a positive constant \tilde{C} , depending only on T and C such that

$$\mathbb{E}[|X_t^{(k)}|^2] \leq \tilde{C}(1 + |x|^2)e^{\tilde{C}t}, \quad t \in [0, T], k \in \mathbb{N}. \quad (6)$$

Now, we prove Theorem 2.

Proof. The goal is to show that the sequence of stochastic processes $\{X^{(k)}\}_{k \in \mathbb{N}}$ defined recursively in (5) converges to X_t so that the limit of (5) coincides with the SDE (1). Let's start with the following:

$$\mathbb{E} \left[\max_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}|^2 \right] = \mathbb{E} \left[\max_{t \in [0, T]} (B_t + M_t)^2 \right] \leq 2\mathbb{E} \left[\max_{t \in [0, T]} B_t^2 \right] + 2\mathbb{E} \left[\max_{t \in [0, T]} M_t^2 \right],$$

where

$$\begin{aligned}
B_t &:= \int_0^t b(s, X_s^{(k)}) - b(s, X_s^{(k-1)})ds \\
M_t &:= \int_0^t \sigma(s, X_s^{(k)}) - \sigma(s, X_s^{(k-1)})dW_s.
\end{aligned}$$

Let's first focus on the term B_t . Using the Cauchy-Schwarz inequality and the Lipschitz condition on the function b , we have:

$$\begin{aligned}
\mathbb{E} \left[\max_{t \in [0, T]} B_t^2 \right] &= \mathbb{E} \left[\max_{t \in [0, T]} \left| \int_0^t b(s, X_s^{(k)}) - b(s, X_s^{(k-1)}) ds \right|^2 \right] \\
&\leq \mathbb{E} \left[\max_{t \in [0, T]} t \int_0^t |b(s, X_s^{(k)}) - b(s, X_s^{(k-1)})|^2 ds \right] \\
&= \mathbb{E} \left[T \int_0^T |b(s, X_s^{(k)}) - b(s, X_s^{(k-1)})|^2 ds \right] \\
&\leq TC^2 \mathbb{E} \left[\int_0^T |X_s^{(k)} - X_s^{(k-1)}|^2 ds \right].
\end{aligned}$$

Next, we turn attention to M_t . From Lemma 2 and the Lipschitz condition on the function σ , it is easy to show that M_t is a square integrable martingale. This fact allows us to use the **BDG-inequality**, which yields the following result

$$\begin{aligned}
\mathbb{E} \left[\max_{t \in [0, T]} M_t^2 \right] &\leq \tilde{C} \mathbb{E}[\langle M \rangle_T] \\
&= \tilde{C} \mathbb{E} \left[\int_0^T (\sigma(s, X_s^{(k)}) - \sigma(s, X_s^{(k-1)}))^2 ds \right] \\
&\leq \tilde{C} C^2 \mathbb{E} \left[\int_0^T (X_s^{(k)} - X_s^{(k-1)})^2 ds \right].
\end{aligned}$$

Combining these two results, we have:

$$\begin{aligned}
\mathbb{E} \left[\max_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}|^2 \right] &\leq 2C^2(T + \tilde{C}) \mathbb{E} \left[\int_0^T |X_s^{(k)} - X_s^{(k-1)}|^2 ds \right] \\
&= L \mathbb{E} \left[\int_0^T |X_s^{(k)} - X_s^{(k-1)}|^2 ds \right] \\
&\leq C^* \frac{(Lt)^k}{k!},
\end{aligned} \tag{7}$$

where

$$C^* = \mathbb{E} \left[\max_{t \in [0, T]} |X_t^{(1)} - x|^2 \right].$$

The last inequality in (7) can be shown by induction. When $k = 0$, the result is obvious. Assuming $k = k$ holds, $k = k + 1$ can be verified by using the second last relation in (7).

From Markov's inequality, we have:

$$\mathbb{P} \left[\max_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}| \geq 2^{-k} \right] \leq 4^k \mathbb{E} \left[\max_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}|^2 \right] \leq C^* \frac{(4Lt)^k}{k!} \rightarrow 0,$$

which implies that $\mathbb{P} \left[\max_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}| \geq 2^{-k} \right]$ is a convergent series. From the Borel-Cantelli Lemma, we have:

$$\sum_{k=1}^{\infty} \mathbb{P} \left[\max_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}| \geq 2^{-k} \right] < \infty \Rightarrow \mathbb{P} \left[\limsup_{k \rightarrow \infty} \max_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}| \geq 2^{-k} \right] = 0.$$

In other words, eventually with probability 1, $\max_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}| < 2^{-k}$. It means that, with probability 1, the sequence $\{X_t^{(k)}\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, T], \mathbb{R})$. Under the supremum norm (which makes the space complete), the sequence $\{X_t^{(k)}\}_{k \in \mathbb{N}}$ converges.

Now, it remains to show that the continuous limit of $\{X_t^{(k)}\}_{k \in \mathbb{N}}$, denoted as X_t^∞ , is indeed the strong solution of the SDE (1). First of all, the integrability condition (4) can be shown from (6) plus the Fatou's Lemma. Next, it is obvious that X_t^∞ is \mathbb{F} -adapted and the initial condition holds. From the growth condition and (4), we know that b and σ^2 are almost surely integrable. It remains to check the following:

$$\begin{aligned}\int_0^t b(s, X_s^{(k)}) ds &\rightarrow \int_0^t b(s, X_s^\infty) ds \\ \int_0^t \sigma(s, X_s^{(k)}) dW_s &\rightarrow \int_0^t \sigma(s, X_s^\infty) dW_s.\end{aligned}$$

The first convergence relation holds because b is Lipschitz continuous (b is uniformly bounded, and $\{X_t^{(k)}(\omega)\}_{k \in \mathbb{N}}$ converges uniformly). The second convergence result will be shown using the [DDS theorem](#). Define $\tilde{M}_t := \int_0^t \sigma(s, X_s^{(k)}) - \sigma(s, X_s^\infty) dW_s$. It is easy to see that $\tilde{M}_t \in \mathcal{M}_c^2$, and $\langle \tilde{M} \rangle_t \rightarrow \infty$. Then, the martingale \tilde{M}_t is a time changed Brownian motion, i.e., $\tilde{M}_t = W_{\langle \tilde{M} \rangle_t}$. Additionally, we know: $\langle \tilde{M} \rangle_t = \int_0^t (\sigma(s, X_s^{(k)}) - \sigma(s, X_s^\infty))^2 ds = 0$, because of the Lipschitz continuity of σ and the uniform convergence of $\{X_t^{(k)}(\omega)\}_{k \in \mathbb{N}}$. Thus, we have $\tilde{M}_t = W_0 = 0$, by definition of a standard Brownian motion. The proof is completed. \square