Basic properties of complex numbers $z = x + iy$ are not mentioned here. Based on the definition of the absolute value (norm) of the complex number: $|z| = (x^2 + y^2)^{1/2}$, two interesting properties are listed below:

1. Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$
2. $|z_1 - z_2| \geq ||z_1| - |z_2||$

The first property can be verified by thinking of the complex number as a vector in $\mathbb{R}^2$ with the real part and complex part being respectively the first and second coordinates. The second property can be proved as follows:

Proof. If we can show both $|z_1 - z_2| \geq |z_1| - |z_2|$ and $|z_1 - z_2| \geq |z_2| - |z_1|$, then we are done. From the triangle inequality, we have:

$$|z_1| = |z_2 + z_1 - z_2| \leq |z_2| + |z_1 - z_2| \Rightarrow |z_1| - |z_2| \leq |z_1 - z_2|$$
$$|z_2| = |z_1 + z_2 - z_1| \leq |z_1| + |z_2 - z_1| \Rightarrow |z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2|$$

□

Roughly speaking, for a complex function $f : \mathbb{C} \to \mathbb{C}$, if $\frac{f(z+h)-f(z)}{h}$ converges as $h \to 0$, then we say f is holomorphic. Note that h can go to 0 from different directions, i.e., from the real axis, the complex axis, or many more. Being a holomorphic function in an open set Ω enables a lot of miracle to happen, such as:

1. Contour integration: for closed paths, the contour integration of f equals 0, independent of the parametrization.
2. Regularity: f is differentiable infinitely many times, and it has a convergent power series (Taylor series expansion).
3. Analytic continuation: if f and g agree on a (possibly tiny) open subset of Ω, then they agree on all Ω.

Note that the complex space is complete, which implies that a sequence converges if and only if it is a Cauchy sequence.

We recall several basic properties of a set here.

Definition 1. *(Interior Point)*
For a set Ω, $z \in \Omega$ is called the interior point if there exists $r > 0$ such that the disk $D_r(z) := w \in \mathbb{C} : |z - w| < r$ is contained in Ω.

Definition 2. *(Open Set)*
A set Ω is called open if every point in it is an interior point.

Definition 3. *(Closed Set)*
A set Ω is called closed if its complement is an open set.

Definition 4. *(Connected Open Set)*
A open set Ω is called connected if it is not possible to find two disjoint open set U_1 and U_2 such that $\Omega = U_1 \cup U_2$.

Definition 5. *(Region)*
A region is an open connected set.

Now, we properly define the holomorphic here.

Definition 6. *(Holomorphic Function)*
Let f be a complex function on an open set Ω. f is holomorphic at the point $z \in \Omega$ if $\frac{f(z+h)-f(z)}{h}$ converges, as h converges to 0. Its limited is denoted as $f'(z)$. f is called a holomorphic function if it is holomorphic at every point point in Ω.

Example 1. *$f(z) = z$ is holomorphic because $\frac{f(z+h)-f(z)}{h} = \frac{z+h-z}{h} = 1$*

Example 2. *$f(z) = \bar{z}$ is not holomorphic because $\frac{f(z+h)-f(z)}{h} = \frac{\bar{z}+\bar{h}-\bar{z}}{h} = \frac{\bar{h}}{h}$. If h is real, then $f'(z) = 1$, but if h is imaginary, $f'(z) = -1$.*

It is clear from Definition 6 that f is holomorphic at a point $z \in \Omega$ if and only if there exists a complex number a such that

$$f(z + h) - f(z) - ah = h\psi(h), \tag{1}$$

where ψ is a function defined for small h and $\lim_{h\to 0} \psi(h) = 0$. (1) can be used to show the following properties.

Proposition 1. *If f and g are holomorphic in Ω, then*

1. *$f+g$ is holomorphic*
2. *fg is holomorphic*
3. *$(f+g)' = f' + g'$*
4. *$(fg)' = f'g + fg'$*

The notion of complex differentiability differs significantly from the real differentiability. As seen from Example 2, if we interpret the complex function $f(z) = \bar{z}$ as a real function with two coordinates, then $f(z) = f(x, -y)$, which is differentiable in the real sense. Now, we build the link between real and complex functions through the Cauchy-Riemann equation.

Suppose f is holomorphic at $z_0 = x_0 + iy_0$, and let $h = h_1 + ih_2$. Consider $f(z) = f(x, y)$. If $h_1 = 0$, we have

$$\lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

Similarly, if $h_2 = 0$, we have

$$\lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} = \frac{\partial f}{\partial x}.$$

Since f is holomorphic at z_0, we have the first Cauchy-Riemann equation as follows:

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}. \tag{2}$$

If we write the holomorphic function as $f(z) = u(x, y) + iv(x, y)$, then we have the following relation:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}. \end{aligned} \tag{3}$$

We also have the following results from the Cauchy-Riemann equation

Proposition 2. *If f is holomorphic at z_0, then*

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad \textit{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0) \tag{4}$$

Proof. For $z = x + iy$, we know $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$. Then, we can define the following two differential operators

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}. \end{aligned}$$

From (2), we have the desired result $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$.

For the second result, first observe

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\frac{\partial f}{\partial x}(z_0) + \frac{1}{2i}\frac{\partial f}{\partial y}(z_0) = \frac{\partial f}{\partial x}(z_0)$$

Then, using the operator $\frac{\partial}{\partial z}$ defined above, we have

$$\frac{\partial u}{\partial z}(z_0) = \frac{1}{2}\frac{\partial u}{\partial x}(z_0) + \frac{1}{2i}\frac{\partial u}{\partial y}(z_0) = \frac{1}{2}(\frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)) = \frac{1}{2}\frac{\partial f}{\partial x}(z_0),$$

which directly leads to the second result. $\square$

So far, we have assumed f to be holomorphic and deduced the relation between its real and imaginary parts. In the next theorem, we show that the converse is also true, which completes the circle.

Theorem 1. *Suppose the complex function $f = u(x, y) + iv(x, y)$ is defined on an open set Ω. If u and v are differentiable and satisfy the Cauchy-Riemann equation (3) on Ω, then f is holomorphic on Ω, and $f'(z) = \partial f/\partial z$.*

Proof. Since u and v are differentiable, we have

$$u(x+h_1, y+h_2) - u(x, y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\psi_1(h)$$
$$v(x+h_1, y+h_2) - v(x, y) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\psi_2(h),$$

where $\psi_1(h), \psi_2(h) \to 0$ as $|h| \to 0$, and $h = h_1 + ih_2$. Using (3), we have:

$$\begin{aligned} f(z+h) - f(z) &= f(x + f_1 + iy + ih_2) - f(x+iy) \\ &= u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) - u(x,y) - iv(x,y) \\ &= \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\psi_1(h) + \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\psi_2(h) \\ &= \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + |h|\psi_1(h) + |h|\psi_2(h), \end{aligned}$$

which implies

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) + \frac{|h|\psi_1(h) + |h|\psi_2(h)}{h} \\ &\to \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) \\ &= 2\frac{\partial u}{\partial z} \\ &= \frac{\partial f}{\partial z}. \end{aligned}$$

□

Definition 7. *(Power Series)*
A power series is an expansion of the form $\sum_{n=0}^{\infty} a_n z^n$.

Definition 8. *(Absolute Convergence)*
A power series is said to converge absolutely if $\sum_{n=0}^{\infty} |a_n z^n|$ converges.

Definition 9. *(Analytic Function)*
We say that a function is analytic in an open set Ω if it has a convergent power series expansion in Ω

We give the following result about the convergence of a power series without proof.

Theorem 2. *(Convergence of Power Series)*
Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \leq R \leq \infty$ such that:

1. *If $|z| < R$, the series converges absolutely.*
2. *If $|z| > R$, the series diverges.*

The radius of convergence R is given by the Hadmard's formula as follows:

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}. \tag{5}$$

Power series provide a very important class of analytic functions that are particularly simple to manipulate, as shown in the following theorem.

Theorem 3. *The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ define a holomorphic function in its disk of convergence. The derivative of f is also a power series obtained by differentiating term be term, i.e. $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$. Additionally, $f'(z)$ has the same radius of convergence.*

Proof. We aim to show the following:

$$\lim_{h\to 0}|\frac{f(z+h)-f(z)}{h}-g(z)|=0,$$

where $g(z):=\sum_{n=0}^{\infty} na_n z^{n-1}$. Only main steps are shown here. The intuition is to separate the infinite sum of f into a finite sum and a tail part, because the finite sum is essentially a polynomial, and its derivative is obtained by differentiating term by term.

Define the following terms

$$\begin{aligned} S_N &:= \sum_{n=0}^{N} a_n z^n \\ E_N &:= \sum_{n=N+1}^{\infty} a_n z^n \\ S_N' &:= \sum_{n=0}^{N} na_n z^{n-1}. \end{aligned}$$

Then,

$$\begin{aligned} \lim_{h\to 0}|\frac{f(z+h)-f(z)}{h}-g(z)| &= \lim_{h\to 0}|\frac{S_N(z+h)-S_N(z)}{h}+\frac{E_N(z+h)-E_N(z)}{h}+S_N'-S_N'-g(z)| \\ &\le \lim_{h\to 0}\{|\frac{S_N(z+h)-S_N(z)}{h}-S_N'|+|\frac{E_N(z+h)-E_N(z)}{h}|+|S_N'-g(z)|\}. \end{aligned}$$

The first term vanishes because polynomials are differentiable, and the derivate is obtained term by term. The last term vanishes because $\lim_{N\to\infty} S_N' = g(z)$. The second term also vanishes based on the following two facts

1. $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+...+ab^{n-1}+b^{n-1})$
2. $\sum_{n=N+1}^{\infty} na_n z^{n-1}$ is the tail of a convergence sequence and thus vanishes.

Finally, f' and f have the same radius of convergence from the Hadmard's formula and the fact that $\lim_{n\to\infty} n^{1/n}=1$.

□

Now, we change our focus to integration of complex functions along curves. First, we give several definitions

Definition 10. *(Primitive)*
A primitive of f on Ω is a function F that is holomorphic on Ω and such that $F'(z)=f(z)$ for all $z\in\Omega$.

Definition 11. *(Parametrized Curve)*
A parametrized curve is a function $z(t)$ which maps a closed interval $[a,b]\subset\mathbb{R}$ to the complex plane.

The integral of f along a curve γ is then defined as

$$\int_\gamma f(z)dz := \int_a^b f(z(t))z'(t)dt.$$

In order for this definition to make sense, one can also show that it is independent of the choice of the parametrization. By definition, the length of the smooth curve γ is

$$\text{length}(\gamma)=\int_a^b |z'(t)|dt.$$

We may then develop an useful bound, called the M-L bound, for the integration over curves:

$$|\int_\gamma f(z)dz| = |\int_a^b f(z(t))z'(t)dt| \le \int_a^b |f(z(t))z'(t)|dt \le \sup_{z\in\gamma}|f(z(t))|\int_a^b |z'(t)|dt = \sup_{z\in\gamma}|f(z(t))|\text{length}(\gamma). \tag{6}$$

Assuming f has a primitive, many useful results arise.

Theorem 4. *If f has a primitive F in Ω, and γ is a curve in Ω that has end points w_1 and w_2, then*

$$\int_\gamma f(z)dz = F(w_1)-F(w_2)$$

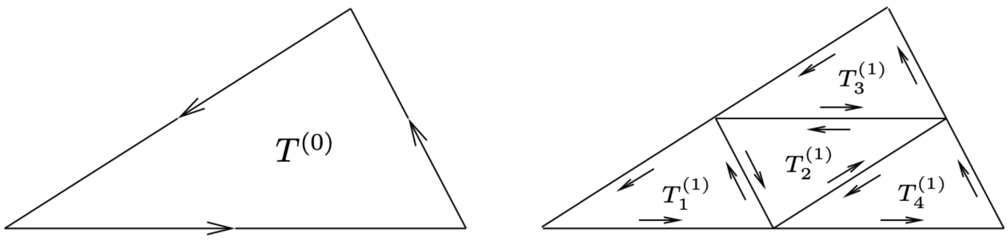

Figure 1: Bisecting triangle in the proof of Goursat's Theorem.

Proof. The proof is quite straightforward. By definition,

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b F'(z(t))z'(t)dt = \int_a^b \frac{d}{dt}F(z(t))dt = F(z(b)) - F(z(a)) = F(w_1) - F(w_2).$$

□

Next, we state several corollaries from this theorem, the proofs of which are very simple, so they are omitted.

Corollary 1. *If γ is a closed curve in an open set Ω, and f has a primitive in Ω, then*

$$\int_\gamma f(z)dz = 0. \tag{7}$$

Corollary 2. *If f is holomorphic in a region Ω, and $f' = 0$, then f is constant.*

Corollary 3. *If f has a primitive in Ω, then the integral does not depend on the choice of the path.*

Corollary 1 gives us one direction, and we would like to show the converse, namely if we know that (7) for some types of curves γ, then a primitive for f exists. We start with the Goursat's theorem.

Theorem 5. *(Goursat) If f is holomorphic in an open set Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)dz = 0.$$

Proof. The proof is based on this brilliant idea of recursively bisecting the triangle, as shown in Fig. 1. The subscript counts the number of triangles, and the superscript represents the iteration. Only main steps of the proof are shown here.

In the first iteration, we have:

$$\begin{aligned}\int_{T^{(0)}} f(z)dz &= \int_{T_1^{(1)}} f(z)dz + \int_{T_1^{(2)}} f(z)dz + \int_{T_1^{(3)}} f(z)dz + \int_{T_1^{(4)}} f(z)dz \\ &\leq 4\int_{T_j^{(1)}} f(z)dz,\end{aligned}$$

for some j that gives the maximum integral value, and we name this triangle $T^{(1)}$. Repeating this process for k times, we obtain

$$|\int_{T^{(0)}} f(z)dz| \leq 4^k |\int_{T^{(k)}} f(z)dz|.$$

We may prove that there is a unique z_0 in every triangle $\{T^{(n)}\}_{n=1,\ldots,k}$. Since f is holomorphic at z_0, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z - z_0)(z - z_0),$$

where $\psi(z - z_0) \to 0$ as $z \to z_0$. The constant $f(z_0)$ and $f'(z_0)(z - z_0)$ has primitives, so they have zero integral value over the closed curve $T^{(k)}$. Finally, $|\int_{T^{(k)}} f(z)dz|$ can be bounded by the M-L bound (6) and the following two facts:

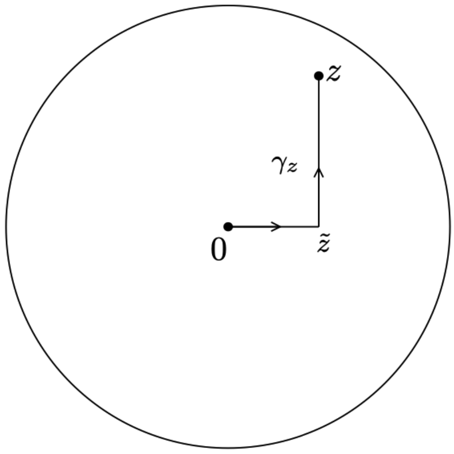

Figure 2: Open disk with the choice of curve γ_z

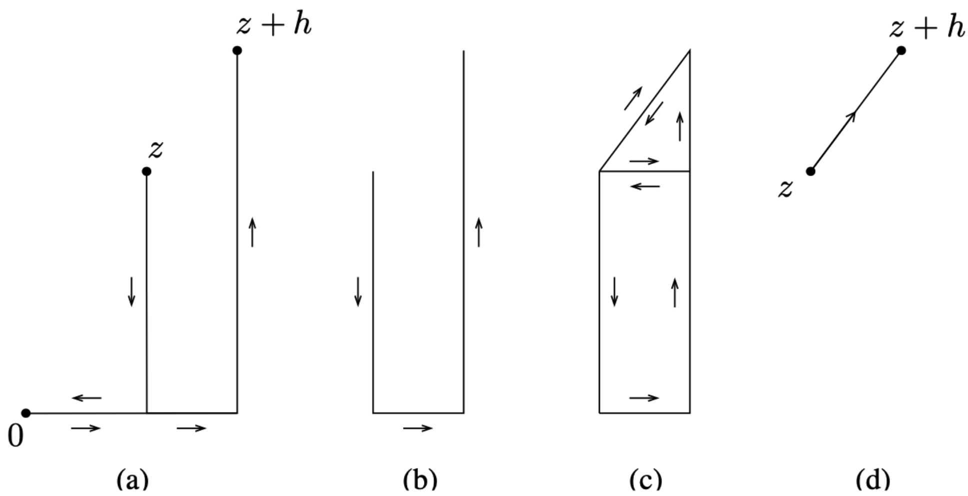

Figure 3: Relation between γ_z and γ_{z+h}.

1. perimeter of $T^{(k)} = \frac{1}{2^k} \times$ perimeter of $T^{(0)}$
2. diameter of $T^{(k)} = \frac{1}{2^k} \times$ diameter of $T^{(0)}$

Then, it is straightforward to show $\lim_{k\to\infty} 4^k |\int_{T^{(k)}} f(z)dz| = 0$, which completes the proof.

□

Next, we prove the existence of primitive in a disk as a consequence of Goursat's Theorem.

Theorem 6. *(Local Existence of Primitives)*
A holomorphic function in an open disk has a primitive in that disk.

Proof. We only show main steps. Consider the open disk D centered at the origin, and choose a point $z \in D$ at random. Constructing the curve γ_z as shown in Fig 2. Define

$$F(z) := \int_{\gamma_z} f(w)dw.$$

We claim that $F(z)$ is holomorphic in the open disk D and $F'(z) = f(z)$, which means that F is the primitive of f in D. Now choose another point $z + h \in D$. Following the steps shown in Fig 3, which uses the Goursat's Theorem, we have

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)dw - \int_{\gamma_z} f(w)dw = \int_\eta f(w)dw,$$

where η is the curve shown in Fig. 3(d). Since f is holomorphic in D, it is continuous at w. Then, $f(w) = f(z) + \psi(w)$, where $\psi(w) \to 0$ as $z \to w$. With the fact that $f(z)$ has a primitive and Theorem 4, we have

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

which completes the proof. □

This theorem says that locally, every holomorphic function has a primitive. Now, we can use the previous result to get

Theorem 7. *(Cauchy's Theorem for a Disk)*
If f is a holomorphic function in a disk, then

$$\int_\gamma f(z)dz = 0, \tag{8}$$

for every closed curve γ in that disk.

Proof. Since f is a holomorphic function in a disk, by previous result, f has a primitive in that disk. Then, by Corollary 1, we have the desired result. □

Corollary 4. *If f is holomorphic in an open set that contains a circle C and its interior, then*

$$\int_C f(w)dw = 0.$$

Proof. We can slightly enlarge the disk with boundary circle C such that f is still holomorphic in it. Then, we can apply Cauchy's theorem to conclude the proof. □