ORF 527 - Stochastic Calculus
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Homework Assignment 5

Due date: May 1st, before 11:59pm

Homework policy: You are welcome to work together, but please submit separate solutions. Please show all your work and write out all proofs in full detail.

The purpose of this homework is to complete the proof that the gene frequency in the Moran model converges to the (weak) solution of the following SDE

$$\mathrm{d}X_t = \frac{1}{\sqrt{2}}\sqrt{X_t(1-X_t)}\mathrm{d}B_t,\ t \geq 0,\ X_0 = x_0 \in \mathbb{R}, \tag{0.1}$$

as the number of individuals N in the population becomes large. For a population of size N, we denote by X_t^N the frequency of gene A at time $t \geq 0$. We assume throughout that X_0^N is non-random and that the limit $x_0 := \lim_{N\to\infty} X_0^N$ exists. Recall that two individuals mate at random time τ_i, $i \in \mathbb{N}^\star$, satisfying

$$\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots, \quad \text{i.i.d. with distribution } \mathcal{E}\left(\binom{N}{2}\right). \tag{0.2}$$

We refer to the Lecture Notes, Chapter 5, Section 4.2. for more information on the model.

Part 1. Existence of limits along subsequences.

The goal of this part is to verify that, on every fixed time interval $[0,T]$ where $T > 0$ is a finite time horizon, the sequence of processes $(X^N)_{N\in\mathbb{N}^\star}$ converges in distribution (up to a subsequence).

(a) Using Assumption (0.2), show that the sequence of random time $(\tau_i)_{i\in\mathbb{N}^\star}$ at which individuals mate corresponds to the arrival times (jump times) of a Poisson process with intensity to be determined.

(b) We further assume that the children will both inherit one of the genes from the parents with probability 1/2. Show that, at the times when two individual mate, the gene frequency X^N either goes up by $1/N$, goes down by $1/N$, or stays the same. Compute the associated probabilities of the three possibilities.

(c) Use the two previous questions to verify that X^N is a martingale.

(d) We assume for now the following statement: for all $C, \delta > 0$, there exists $\eta \in (0,T]$ such that

$$\liminf_{N\to\infty} \mathbb{P}\left(\sup_{0\leq s,t\leq T,\ |t-s|\leq\eta} \left|X_t^N - X_s^N\right| \leq C\right) \geq 1-\delta, \tag{0.3}$$

Use a combination of Prokhorov and Arzelà-Ascoli Theorems to verify that, if the previous statement holds, then we can conclude this part, *i.e.* the sequence of processes $(X^N)_{N\in\mathbb{N}^\star}$ converges in distribution (up to a subsequence).

Hint: as we have seen in class, you should first approximate X^N by a continuous process $\widetilde{X}^N$, and check that this new process $\widetilde{X}^N$ also satisfies condition (0.3). Then, you can use this condition to verify tightness thanks to Arzelà-Ascoli Theorem, and finally conclude by Prokhorov Theorem...

(e) Prove the statement (0.3) assumed in the previous question.

Hint: fix $C, \delta > 0$. You can first view the interval $[0,T]$ as the union of the intervals $[0,\eta], [\eta, 2\eta], \ldots$ and use the "union bound". Recalling that by question (c), X^N is a martingale, you can then use Doob's submartingale inequality (on an appropriate submartingale).

Part 2. Uniqueness for the limiting SDE.

We have proved in class that any limit X (or more rigorously its distribution $\mathbb{P}$) of a subsequence of $(X^N)_{N\in\mathbb{N}^\star}$ solves the martingale problem associated with the SDE (0.1).

(a) Conclude that any such limit X is a weak solution of the SDE (0.1).

(b) Prove that weak uniqueness holds for SDE (0.1).

(c) Conclude that the sequence $(X^N)_{N\in\mathbb{N}^\star}$ converges to the unique weak solution of SDE (0.1).