Basic properties of complex numbers $z = x + iy$ are not mentioned here. Based on the definition of the absolute value (norm) of the complex number: $|z| = (x^2 + y^2)^{1/2}$, two interesting properties are listed below:

1. Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$
2. $|z_1 - z_2| \geq ||z_1| - |z_2||$

The first property can be verified by thinking of the complex number as a vector in $\mathbb{R}^2$ with the real part and complex part being respectively the first and second coordinates. The second property can be proved as follows:

Proof. If we can show both $|z_1 - z_2| \geq |z_1| - |z_2|$ and $|z_1 - z_2| \geq |z_2| - |z_1|$, then we are done. From the triangle inequality, we have:

$$|z_1| = |z_2 + z_1 - z_2| \leq |z_2| + |z_1 - z_2| \Rightarrow |z_1| - |z_2| \leq |z_1 - z_2|$$
$$|z_2| = |z_1 + z_2 - z_1| \leq |z_1| + |z_2 - z_1| \Rightarrow |z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2|$$

□

Roughly speaking, for a complex function $f : \mathbb{C} \to \mathbb{C}$, if $\frac{f(z+h)-f(z)}{h}$ converges as $h \to 0$, then we say f is holomorphic. Note that h can go to 0 from different directions, i.e., from the real axis, the complex axis, or many more. Being a holomorphic function in an open set Ω enables a lot of miracle to happen, such as:

1. Contour integration: for closed paths, the contour integration of f equals 0, independent of the parametrization.
2. Regularity: f is differentiable infinitely many times, and it has a convergent power series (Taylor series expansion).
3. Analytic continuation: if f and g agree on a (possibly tiny) open subset of Ω, then they agree on all Ω.

Note that the complex space is complete, which implies that a sequence converges if and only if it is a Cauchy sequence.

We recall several basic properties of a set here.

Definition 1. *(Interior Point)*
For a set Ω, $z \in \Omega$ is called the interior point if there exists $r > 0$ such that the disk $D_r(z) := w \in \mathbb{C} : |z - w| < r$ is contained in Ω.

Definition 2. *(Open Set)*
A set Ω is called open if every point in it is an interior point.

Definition 3. *(Closed Set)*
A set Ω is called closed if its complement is an open set.

Definition 4. *(Connected Open Set)*
A open set Ω is called connected if it is not possible to find two disjoint open set U_1 and U_2 such that $\Omega = U_1 \cup U_2$.

Definition 5. *(Region)*
A region is an open connected set.

Now, we properly define the holomorphic here.

Definition 6. *(Holomorphic Function)*
Let f be a complex function on an open set Ω. f is holomorphic at the point $z \in \Omega$ if $\frac{f(z+h)-f(z)}{h}$ converges, as h converges to 0. Its limited is denoted as $f'(z)$. f is called a holomorphic function if it is holomorphic at every point point in Ω.

Example 1. *$f(z) = z$ is holomorphic because $\frac{f(z+h)-f(z)}{h} = \frac{z+h-z}{h} = 1$*

Example 2. *$f(z) = \bar{z}$ is not holomorphic because $\frac{f(z+h)-f(z)}{h} = \frac{\bar{z}+\bar{h}-\bar{z}}{h} = \frac{\bar{h}}{h}$. If h is real, then $f'(z) = 1$, but if h is imaginary, $f'(z) = -1$.*

It is clear from Definition 6 that f is holomorphic at a point $z \in \Omega$ if and only if there exists a complex number a such that

$$f(z + h) - f(z) - ah = h\psi(h), \tag{1}$$

where ψ is a function defined for small h and $\lim_{h\to 0} \psi(h) = 0$. (1) can be used to show the following properties.

Proposition 1. *If f and g are holomorphic in Ω, then*

1. *$f+g$ is holomorphic*
2. *fg is holomorphic*
3. *$(f+g)' = f' + g'$*
4. *$(fg)' = f'g + fg'$*

The notion of complex differentiability differs significantly from the real differentiability. As seen from Example 2, if we interpret the complex function $f(z) = \bar{z}$ as a real function with two coordinates, then $f(z) = f(x, -y)$, which is differentiable in the real sense. Now, we build the link between real and complex functions through the Cauchy-Riemann equation.

Suppose f is holomorphic at $z_0 = x_0 + iy_0$, and let $h = h_1 + ih_2$. Consider $f(z) = f(x, y)$. If $h_1 = 0$, we have

$$\lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

Similarly, if $h_2 = 0$, we have

$$\lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} = \frac{\partial f}{\partial x}.$$

Since f is holomorphic at z_0, we have the first Cauchy-Riemann equation as follows:

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}. \tag{2}$$

If we write the holomorphic function as $f(z) = u(x, y) + iv(x, y)$, then we have the following relation:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}. \end{aligned} \tag{3}$$

We also have the following results from the Cauchy-Riemann equation

Proposition 2. *If f is holomorphic at z_0, then*

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad \textit{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0) \tag{4}$$

Proof. For $z = x + iy$, we know $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$. Then, we can define the following two differential operators

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}. \end{aligned}$$

From (2), we have the desired result $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$.

For the second result, first observe

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\frac{\partial f}{\partial x}(z_0) + \frac{1}{2i}\frac{\partial f}{\partial y}(z_0) = \frac{\partial f}{\partial x}(z_0)$$

Then, using the operator $\frac{\partial}{\partial z}$ defined above, we have

$$\frac{\partial u}{\partial z}(z_0) = \frac{1}{2}\frac{\partial u}{\partial x}(z_0) + \frac{1}{2i}\frac{\partial u}{\partial y}(z_0) = \frac{1}{2}(\frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)) = \frac{1}{2}\frac{\partial f}{\partial x}(z_0),$$

which directly leads to the second result. □

So far, we have assumed f to be holomorphic and deduced the relation between its real and imaginary parts. In the next theorem, we show that the converse is also true, which completes the circle.

Theorem 1. *Suppose the complex function $f = u(x,y) + iv(x,y)$ is defined on an open set Ω. If u and v are differentiable and satisfy the Cauchy-Riemann equation (3) on Ω, then f is holomorphic on Ω, and $f'(z) = \partial f/\partial z$.*

Proof. Since u and v are differentiable, we have

$$\begin{aligned} u(x+h_1, y+h_2) - u(x,y) &= \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\psi_1(h) \\ v(x+h_1, y+h_2) - v(x,y) &= \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\psi_2(h), \end{aligned}$$

where $\psi_1(h), \psi_2(h) \to 0$ as $|h| \to 0$, and $h = h_1 + ih_2$. Using (3), we have:

$$\begin{aligned} f(z+h) - f(z) &= f(x + f_1 + iy + ih_2) - f(x+iy) \\ &= u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) - u(x,y) - iv(x,y) \\ &= \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\psi_1(h) + \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\psi_2(h) \\ &= \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + |h|\psi_1(h) + |h|\psi_2(h), \end{aligned}$$

which implies

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) + \frac{|h|\psi_1(h) + |h|\psi_2(h)}{h} \\ &\to \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) \\ &= 2\frac{\partial u}{\partial z} \\ &= \frac{\partial f}{\partial z}. \end{aligned}$$

□

Definition 7. *(Power Series)*
A power series is an expansion of the form $\sum_{n=0}^{\infty} a_n z^n$.

Definition 8. *(Absolute Convergence)*
A power series is said to converge absolutely if $\sum_{n=0}^{\infty} |a_n z^n|$ converges.

Definition 9. *(Analytic Function)*
We say that a function is analytic in an open set Ω if it has a convergent power series expansion in Ω

We give the following result about the convergence of a power series without proof.

Theorem 2. *(Convergence of Power Series)*
Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \le R \le \infty$ such that:

1. *If $|z| < R$, the series converges absolutely.*
2. *If $|z| > R$, the series diverges.*

The radius of convergence R is given by the Hadmard's formula as follows:

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}. \tag{5}$$

Power series provide a very important class of analytic functions that are particularly simple to manipulate, as shown in the following theorem.

Theorem 3. *The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ define a holomorphic function in its disk of convergence. The derivative of f is also a power series obtained by differentiating term be term, i.e. $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$. Additionally, $f'(z)$ has the same radius of convergence.*

Proof. We aim to show the following:

$$\lim_{h\to 0} |\frac{f(z+h)-f(z)}{h} - g(z)| = 0,$$

where $g(z) := \sum_{n=0}^{\infty} na_n z^{n-1}$. Only main steps are shown here. The intuition is to separate the infinite sum of f into a finite sum and a tail part, because the finite sum is essentially a polynomial, and its derivative is obtained by differentiating term by term.

Define the following terms

$$S_N := \sum_{n=0}^{N} a_n z^n$$
$$E_N := \sum_{n=N+1}^{\infty} a_n z^n$$
$$S'_N := \sum_{n=0}^{N} na_n z^{n-1}.$$

Then,

$$\begin{aligned}\lim_{h\to 0} |\frac{f(z+h)-f(z)}{h} - g(z)| &= \lim_{h\to 0} |\frac{S_N(z+h)-S_N(z)}{h} + \frac{E_N(z+h)-E_N(z)}{h} + S'_N - S'_N - g(z)| \\ &\leq \lim_{h\to 0} \{|\frac{S_N(z+h)-S_N(z)}{h} - S'_N| + |\frac{E_N(z+h)-E_N(z)}{h}| + |S'_N - g(z)|\}.\end{aligned}$$

The first term vanishes because polynomials are differentiable, and the derivate is obtained term by term. The last term vanishes because $\lim_{N\to\infty} S'_N = g(z)$. The second term also vanishes based on the following two facts

1. $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... + ab^{n-1} + b^{n-1})$
2. $\sum_{n=N+1}^{\infty} na_n z^{n-1}$ is the tail of a convergence sequence and thus vanishes.

Finally, f' and f have the same radius of convergence from the Hadmard's formula and the fact that $\lim_{n\to\infty} n^{1/n} = 1$.

□

Now, we change our focus to integration of complex functions along curves. First, we give several definitions

Definition 10. *(Primitive)*
A primitive of f on Ω is a function F that is holomorphic on Ω and such that $F'(z) = f(z)$ for all $z \in \Omega$.

Definition 11. *(Parametrized Curve)*
A parametrized curve is a function $z(t)$ which maps a closed interval $[a,b] \subset \mathbb{R}$ to the complex plane.

The integral of f along a curve γ is then defined as

$$\int_{\gamma} f(z)dz := \int_a^b f(z(t))z'(t)dt.$$

In order for this definition to make sense, one can also show that it is independent of the choice of the parametrization. By definition, the length of the smooth curve γ is

$$\text{length}(\gamma) = \int_a^b |z'(t)|dt.$$

We may then develop an useful bound, called the M-L bound, for the integration over curves:

$$|\int_{\gamma} f(z)dz| = |\int_a^b f(z(t))z'(t)dt| \leq \int_a^b |f(z(t))z'(t)|dt \leq \sup_{z\in\gamma} |f(z(t))| \int_a^b |z'(t)|dt = \sup_{z\in\gamma} |f(z(t))| \text{length}(\gamma). \tag{6}$$

Assuming f has a primitive, many useful results arise.

Theorem 4. *If f has a primitive F in Ω, and γ is a curve in Ω that has end points w_1 and w_2, then*

$$\int_{\gamma} f(z)dz = F(w_1) - F(w_2)$$

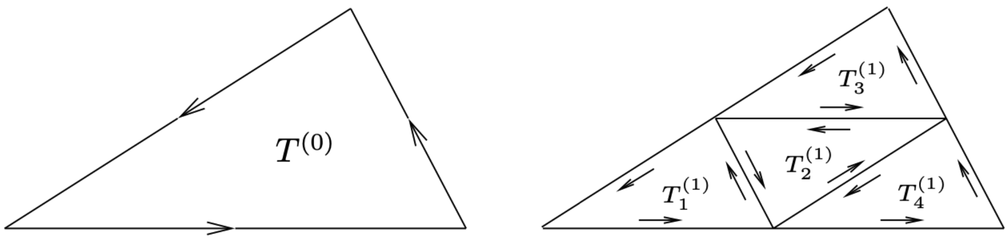

Figure 1: Bisecting triangle in the proof of Goursat's Theorem.

Proof. The proof is quite straightforward. By definition,

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b F'(z(t))z'(t)dt = \int_a^b \frac{d}{dt}F(z(t))dt = F(z(b)) - F(z(a)) = F(w_1) - F(w_2).$$

□

Next, we state several corollaries from this theorem, the proofs of which are very simple, so they are omitted.

Corollary 1. *If γ is a closed curve in an open set Ω, and f has a primitive in Ω, then*

$$\int_\gamma f(z)dz = 0. \tag{7}$$

Example 3. *We will see that $f(z) = \frac{1}{z}$ does not have a primitive using the previous corollary. Parametrize an unit circle γ centered at the origin by $z(\theta) = e^{i\theta}$, $dz = ie^{i\theta}$.*

$$\int_\gamma \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = 2\pi i \neq 0.$$

Corollary 2. *If f is holomorphic in a region Ω, and $f' = 0$, then f is constant.*

Corollary 3. *If f has a primitive in Ω, then the integral does not depend on the choice of the path.*

Corollary 1 gives us one direction, and we would like to show the converse, namely if we know that (7) for some types of curves γ, then a primitive for f exists. We start with the Goursat's theorem.

Theorem 5. *(Goursat)*
If f is holomorphic in an open set Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω, then

$$\int_T f(z)dz = 0.$$

Proof. The proof is based on this brilliant idea of recursively bisecting the triangle, as shown in Fig. 1. The subscript counts the number of triangles, and the superscript represents the iteration. Only main steps of the proof are shown here.

In the first iteration, we have:

$$\begin{aligned}\int_{T^{(0)}} f(z)dz &= \int_{T_1^{(1)}} f(z)dz + \int_{T_1^{(2)}} f(z)dz + \int_{T_1^{(3)}} f(z)dz + \int_{T_1^{(4)}} f(z)dz \\ &\leq 4\int_{T_j^{(1)}} f(z)dz,\end{aligned}$$

for some j that gives the maximum integral value, and we name this triangle $T^{(1)}$. Repeating this process for k times, we obtain

$$|\int_{T^{(0)}} f(z)dz| \leq 4^k |\int_{T^{(k)}} f(z)dz|.$$

We may prove that there is a unique z_0 in every triangle $\{T^{(n)}\}_{n=1,\ldots,k}$. Since f is holomorphic at z_0, we have

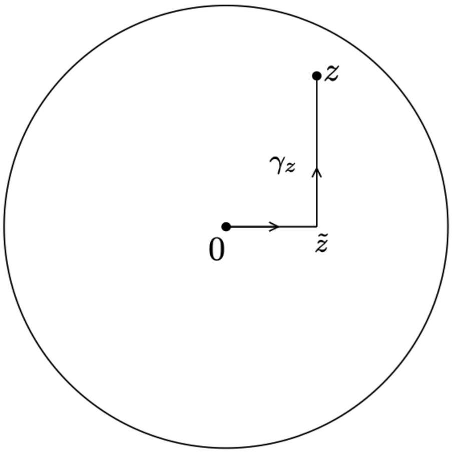

Figure 2: Open disk with the choice of curve γ_z

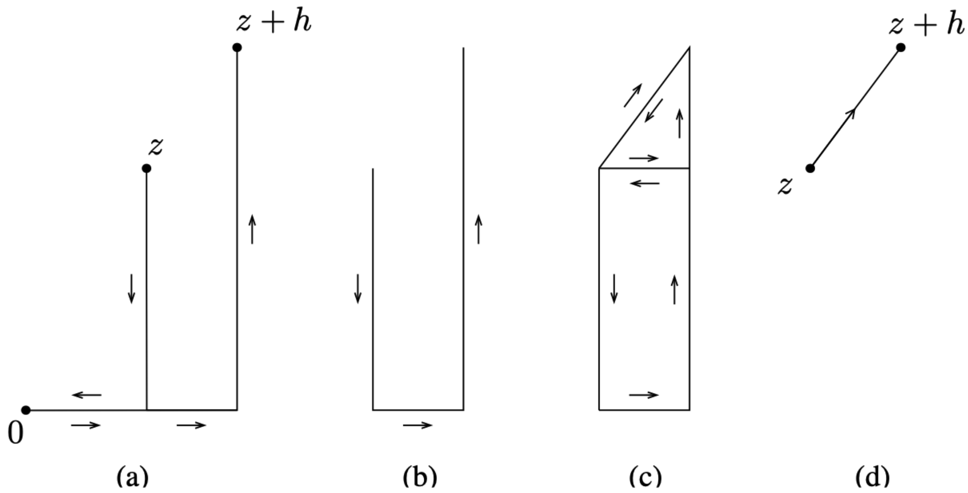

Figure 3: Relation between γ_z and γ_{z+h}.

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z - z_0)(z - z_0),$$

where $\psi(z - z_0) \to 0$ as $z \to z_0$. The constant $f(z_0)$ and $f'(z_0)(z - z_0)$ has primitives, so they have zero integral value over the closed curve $T^{(k)}$. Finally, $|\int_{T^{(k)}} f(z)dz|$ can be bounded by the M-L bound (6) and the following two facts:

1. perimeter of $T^{(k)}$=$\frac{1}{2^k}\times$ perimeter of $T^{(0)}$
2. diameter of $T^{(k)}$=$\frac{1}{2^k}\times$ diameter of $T^{(0)}$

Then, it is straightforward to show $\lim_{k\to\infty} 4^k|\int_{T^{(k)}} f(z)dz| = 0$, which completes the proof.

□

Next, we prove the existence of primitive in a disk as a consequence of Goursat's Theorem.

Theorem 6. *(Local Existence of Primitives)*
A holomorphic function in an open disk has a primitive in that disk.

Proof. We only show main steps. Consider the open disk D centered at the origin, and choose a point $z \in D$ at random. Constructing the curve γ_z as shown in Fig 2. Define

$$F(z) := \int_{\gamma_z} f(w)dw.$$

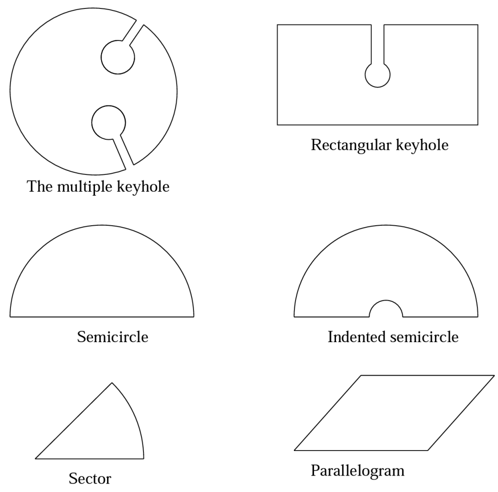

Figure 4: Examples of toy contours.

We claim that $F(z)$ is holomorphic in the open disk D and $F'(z) = f(z)$, which means that F is the primitive of f in D. Now choose another point $z + h \in D$. Following the steps shown in Fig 3, which uses the Goursat's Theorem, we have

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)dw - \int_{\gamma_z} f(w)dw = \int_\eta f(w)dw,$$

where η is the curve shown in Fig. 3(d). Since f is holomorphic in D, it is continuous at w. Then, $f(w) = f(z) + \psi(w)$, where $\psi(w) \to 0$ as $z \to w$. With the fact that $f(z)$ has a primitive and Theorem 4, we have

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

which completes the proof. □

This theorem says that locally, every holomorphic function has a primitive. Now, we can use the previous result to get

Theorem 7. *(Cauchy's Theorem for a Disk)*
If f is a holomorphic function in a disk, then

$$\int_\gamma f(z)dz = 0, \tag{8}$$

for every closed curve γ in that disk.

Proof. Since f is a holomorphic function in a disk, by previous result, f has a primitive in that disk. Then, by Corollary 1, we have the desired result. □

Corollary 4. *If f is holomorphic in an open set that contains a circle C and its interior, then*

$$\int_C f(w)dw = 0.$$

Proof. We can slightly enlarge the disk with boundary circle C such that f is still holomorphic in it. Then, we can apply Cauchy's theorem to conclude the proof. □

Cauchy's theorem also works for other shapes of contour, based on the same argument for the disk as shown in Theorem 7. Some examples of the toy contour is shown in Fig. 4. The important point is that we can define without ambiguity the interior of a contour and construct the polygonal paths in an neighborhood of that contour and its interior, so that the primitive of the holomorphic function f can be defined unambiguously (based on the Goursat's Theorem). For more detail, see the discussion on page 39-41 in the book. We will use the keyhole contour to prove the following theorem. Loosely speaking, if we know the value of a function on the boundary of a circle, we know its value at every point inside the circle.

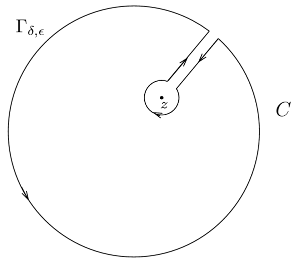

Figure 5: Keyhole contour $\Gamma_{\delta,\epsilon}$.

Theorem 8. *(Cauchy's Integral Formula) Suppose f is a holomorphic function in an open set that contains the closure of a disk D. If C is the boundary circle of this disk with the positive orientation, then*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta,$$

for every point $z \in D$.

Proof. We only show main steps. The proof is based on the keyhole contour Γ (we omitted the subscript, since it is not relevant in this proof) in Fig. 5. Since f and $\frac{1}{\zeta - z}$ are holomorphic functions, we know $\frac{f(\zeta)}{\zeta - z}$ is holomorphic. Then, we have

$$\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \int_C \frac{f(\zeta)}{\zeta - z}d\zeta + \int_{C_\epsilon} \frac{f(\zeta)}{\zeta - z}d\zeta = 0,$$

where C_ϵ represents the small circle with radius ϵ in Fig. 5. We can parametrize C_ϵ as follows:

$$\zeta(\theta) = z + \epsilon e^{i\theta}, \quad d\zeta = i\epsilon e^{i\theta}.$$

Then, it is straightforward to get the result.

□

As a corollary, we obtain further integral formulas expressing the derivatives of f inside the disk in terms of the values of f on the boundary.

Corollary 5. *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta.$$

Proof. The proof is based on induction. Clearly, the formula holds for $n = 0$. Suppose it holds for $n - 1$, then we can show

$$f^{(n)}(z) = \lim_{h\to 0}\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h},$$

by using the following fact: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... + ab^{n-1} + b^{n-1})$. We omitted the following computation. □

We have proved in Theorem 3 that a power series is holomorphic in the interior of the disk of its convergence, now we show the converse.

Theorem 9. *Suppose f is holomorphic in an open set Ω that contains a disk D centered at z_0 and its interior. Then, f has a power series expansion at z_0*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad \forall z \in D,$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Proof. From Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z + z_0 - z_0}d\zeta$$
$$= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)(1 - \frac{z-z_0}{\zeta - z_0})}d\zeta.$$

Note $|\frac{z-z_0}{\zeta-z_0}| < 1$, and we can apply geometric series expansion to $\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}$ as

$$\frac{1}{1-\frac{z-z_0}{\zeta-z_0}} = \sum_{n=0}^{\infty}(\frac{z-z_0}{\zeta-z_0})^n.$$

The result then easily follows. □

Next, we show the analytic continuation result, which was mentioned as one of the motivating points at the very beginning. Roughly speaking, a holomorphic function is determined if we know its value in an appropriate arbitrarily small subset.

Theorem 10. *Suppose f is a holomorphic function in a region (connected and open set) Ω that vanishes on a sequence of distinct points with a limit point in Ω, then f is identically 0.*

Proof. Let the function f vanishes at a sequence of points $\{w_n\}_{n=0}^{\infty}$, and the sequence converges to $z_0 \in \Omega$. We first show that f vanishes on all point in a small disk centered at z_0. From Theorem 9,

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

for all z in the small disk. Suppose that $f(z)$ is not identically 0, then let a_m be the first non-zero coefficient in the power series. Then, we have

$$f(z) = a_m(z-z_0)^m + \sum_{n=m+1}^{\infty} a_n(z-z_0)^n = a_m(z-z_0)^m(1 + g(z - z_0)),$$

where $g \to 0$ as $z \to z_0$. Now take a vanishing point $w_k \neq z_0$ in that disk, we have

$$f(w_k) = a_m(w_k - z_0)^m(1 + g(w_k - z_0)) \neq 0,$$

but this raises a contradiction, because $f(w_k) = 0$ by the choice of w_k. We just proved that the value of f on a small disk around z_0 is identically 0.

Now, we spread this disk to the entire Ω. Define $U := \text{int}\{z \in \Omega : f(z) = 0\}$. U is open by definition. If $\{z_n\}_{n=0}^{\infty}$ is a sequence of point in U with limit z, then $f(z_n) = 0$. By continuity, $f(z) = 0$, implying that $z \in U$. It shows that U is also closed. Now, define a set V to be the complement of U. We have U and V are disjoint open sets that separate Ω. Since Ω is connected, either U or V has to be empty. We showed in the first part of this proof that $z_0 \in U$, so V has to be empty. We have $U = \Omega$, which completes the proof. □

Corollary 6. *If f and g are holomorphic in a region Ω, and $f(z) = g(z)$ for a sequence of distinct points which has the limit also in Ω, then f and g agree on all Ω.*

Proof. It is a direct consequence of the previous theorem. □

Next, we consider singularities, in particular the different kind of point singularities that a holomorphic function can have. In order of increasing severity, these are:

1. removable singularities
2. poles
3. essential singularities

Definition 12. *(Point Singularity/Isolated Singularity)*

A point singularity (or isolated singularity) of a function f is a complex number z_0 such that f is defined in a neighborhood of z_0 but not at the point z_0 itself.

Definition 13. *(Zeros)*

A function has a zero at z_0 if $f(z_0) = 0$. Additionally, z_0 is called the isolated zero if $f(z) \neq 0$ in the neighborhood of z_0.

We use the following theorem to introduce the order of zeros for a holomorphic function.

Theorem 11. *Suppose f is a holomorphic function in a connected open set Ω, and f has a zeros at a point $z_0 \in \Omega$ but does not vanish identically in Ω. Then, there exists a neighborhood $U \subset \Omega$ of z_0, a non-vanishing holomorphic function g on U, and a unique positive integer n such that*

$$f(z) = (z - z_0)^n g(z), \quad \forall z \in U. \tag{9}$$

This unique integer n is denoted as the order of zeros for f.

Proof. Since Ω is open and connected, and f does not vanish identically on Ω, we know that there is a neighborhood $U \subset \Omega$ of z_0 that f is not identically zero in U. Since f is holomorphic in Ω, it has a power series expansion at z_0, and there is a smallest integer n such that $a_n \neq 0$. Then we can write

$$f(z) = (z - z_0)^n [a^n + a_{n+1}(z - z_0) + ...] = (z - z_0)^n g(z),$$

where $g(z)$ is holomorphic and nowhere vanishing since $a_n \neq 0$. This integer n is also unique. Suppose otherwise, $f(z) = (z - z_0)^n g(z) = (z - z_0)^m h(z)$ and $m > n$. Then, $g(z) = (z - z_0)^{m-n} h(z)$. As $z \to z_0$, we have $g(z) = 0$, which is a contradiction. Same reasoning applies for the case of $m < n$. □

The importance of this result is that we can precisely describe the type of singularity possessed by the function $1/f$ at z_0.

Definition 14. *(Deleted Neighborhood)*
The deleted neighborhood of z_0 is defined as an open disk centered at z_0 minus the point z_0. Equivalently, $\{z : 0 < |z - z_0| < r\}$.

Definition 15. *(Poles)*
A function f defined in a deleted neighborhood of z_0 is said to have a pole at z_0 if the function $1/f$, defined to be zero at z_0, is holomorphic in a full neighborhood of z_0.

We have the following theorem, as a direct result from the definition of poles and Theorem 11.

Theorem 12. *If f has a pole at $z_0 \in \Omega$, then in a neighborhood of that point, there exist a non-vanishing holomorphic function h and a unique positive integer n such that*

$$f(z) = (z - z_0)^{-n} h(z). \tag{10}$$

The integer n is called the order of the pole.

Since h also has a power series expansion, we may write f as follows

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + ... + \frac{a_{-1}}{(z - z_0)} + G(z), \tag{11}$$

where $G(z)$ is a holomorphic function in the neighborhood of z_0. In this expression, the summation without the function $G(z)$ is called the principal part of f at the pole z_0, and the coefficient a_{-1} is called the residue of f at that pole. All other terms in the principal part with order strictly greater than 1 has primitives in a deleted neighborhood of z_0. Therefore if $P(z)$ denotes the principal part and C is any circle centered at z_0, we have

$$\frac{1}{2\pi i} \int_C P(z) dz = a_{-1}, \tag{12}$$

Based on (11), we can compute the residue by

$$\text{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} (\frac{d}{dz})^{n-1} (z - z_0)^n f(z). \tag{13}$$

If $f = \frac{n(z)}{d(z)}$ has a simple pole at z_0, using the L'Hopitole's rule, we have a simpler formula to calculate residue

$$\text{res}_{z_0} f = \frac{n(z_0)}{d'(z_0)}. \tag{14}$$

Then, we give the celebrated residue formula.

Theorem 13. *(Residue Formula)*
Suppose that f is holomorphic in an open set containing a circle C and its interior, except for poles at the points $z_1, ..., z_N$ inside C. Then,

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^{N} res_{z_k} f. \tag{15}$$

Proof. We only prove the case when f has a simple pole at z_0. The generalization to the finite number of poles is straightforward.

Consider the same keyhole toy contour as shown in Fig. 5, where the small circle is centered at the pole z_0. As the width of the corridor goes to 0, we have

$$\int_C f(z)dz = \int_{C_\epsilon} f(z)dz.$$

Then, by the Cauchy's integral formula 8 with a constant function $f(z) = a_{-1}$, we have

$$\frac{1}{2\pi i}\int_{C_\epsilon} \frac{a_{-1}}{z - z_0}dz = a_{-1}.$$

For the same reasoning, we have

$$\frac{1}{2\pi i}\int_{C_\epsilon} \frac{a_{-1}}{(z - z_0)^k}dz = 0,$$

for all $k > 1$. Referring to (11), we have

$$\int_C f(z)dz = \int_{C_\epsilon} f(z)dz = 2\pi i a_{-1} = 2\pi i(res)_{z_0} f,$$

since $G(z)$ is holomorphic in the neighborhood of z_0. □

Now, we give a definition of another singularity.

Definition 16. *(Removable Singularity)*

Let f be a function holomorphic in an open set Ω except possibly at one point z_0 in Ω. If we can define f at z_0 in such a way that f becomes holomorphic in all of Ω, we say that z_0 is a removable singularity for f.

Here is an equivalent way of identifying a removable singularity, but we are not going to prove it.

Theorem 14. *f has a removable singularity at z_0 if and only if f is holomorphic in a deleted neighborhood D of z_0 and f is bounded in $D - z_0$.*

Based on this theorem, we will also have an equivalent saying for the pole.

Theorem 15. *Suppose that f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Proof. If z_0 is a pole, then $1/f$ has a zero at z_0, therefore $|f(z)| \to \infty$ as $z \to z_0$. Conversely, if $|f(z)| \to \infty$ as $z \to z_0$, then $1/|f(z)| \to 0$ as $z \to z_0$. So, by previous theorem, $1/f(z)$ has removable singularity at z_0 and vanish at that point. By definition, it means that z_0 is a pole of f. □

Let's make a summary here. Isolated singularities belong to one of the following three categories:

1. Removable singularity (f bounded near z_0)
2. Pole singularity ($|f(z)| \to \infty$ as $z \to z_0$)
3. Essential singularity

By default, any singularity that is not removable or a pole is defined to be an essential singularity. Contrary to the controlled behavior of a holomorphic function near a removable singularity or a pole, it is typical for a holomorphic function to behave erratically near an essential singularity. For example, $e^{1/z}$ has an essential singularity at the origin. It grows to indefinitely as z approaches 0 on the positive real line, while it approaches 0 as z goes to 0 on the negative real axis. Finally, it oscillates rapidly, yet remains bounded, as z approaches the origin on the imaginary axis.

Now, we turn to functions with only isolated singularities that are poles.

Definition 17. *(Meromorphic Functions) A function on an open set Ω is meromorphic if there exists a sequence of points $\{z_1, z_2, ...\}$ that have no limit points in Ω, and such that*

1. *f is holomorphic in $\Omega - \{z_1, z_2, ...\}$.*
2. *f has poles at $\{z_1, z_2, ...\}$.*

We say that $f(z)$ has a pole singularity (removable singularity, essential singularity, respectively) at infinity if $f(1/z)$ has a pole singularity (removable singularity, essential singularity, respectively) at the origin.

Definition 18. *(Meromorphic in the Extended Complex Plane) A meromorphic function in the complex plane that is either holomorphic at infinity or has a pole at infinity is said to be meromorphic in the extended complex plane.*

We won't prove the following theorem.

Theorem 16. *The meromorphic function in the extended complex plane are the rational functions.*

Next, we discuss the argument principle and applications. First notice that the function $\log f(z)$ is multi-valued. In face, $\log f(z) = \log|f(z)| + i \arg f(z)$, where $\arg f(z)$ causes the trouble because it is only determined unambiguously up to an additive integer multiple of 2π. $\log f(z)$ has the aforementioned form because of the following reason. Let $f(z) = re^{i\theta}$, then

$$f(z) = e^{\log f(z)} = e^{Re(\log f(z)) + iIm(\log f(z)} = e^{Re(\log f(z))} e^{iIm(\log f(z)},$$

then, matching the formula, we have

$$\begin{aligned} \log r &= \log|f(z)| = Re(\log f(z)), \\ \theta &= Im(\log f(z)), \end{aligned}$$

which gives us $\log f(z) = Re(\log f(z)) + iIm(\log f(z)) = \log|f(z)| + i \arg f(z)$.

Although $\log f(z)$ is multi-valued, its derivative $f'(z)/f(z)$ is single valued, and $\int_\gamma \frac{f'(z)}{f(z)} dz$ can be interpreted as the net change in the argument of $f(z)$ as z traverses the curve γ. Then, we have the following theorem.

Theorem 17. *(Argument Principle) Suppose f is a meromorphic function in an open set containing a circle C and its interior. If f has no poles and never vanishes on C, then*

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \text{(number of zeros of } f \text{ inside } C\text{)} - \text{(number of poles of } f \text{ inside } C\text{)}, \tag{16}$$

where zeros and poles are counted with multiplicities.

Proof. Suppose f has a zero of order n at z_0, then we may write

$$f(z) = (z - z_0)^n g(z),$$

where $g(z)$ is holomorphic and non-vanishing in the neighborhood around z_0. It implies

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)},$$

which means that $\frac{f'(z)}{f(z)}$ has a simple pole with residue n at z_0. Similarly, we may show that if f has a pole of order n at z_0, then $\frac{f'(z)}{f(z)}$ has a simple pole at with residue $-n$ at z_0. In conclusion, since f is meromorphic, the function f'/f has simple poles at the zeros and poles of f, and the residue is simply the order of the zero of f or the negative of the order of the pole of f. Using the residue formula 13, we have the desired result.

Note that the assumption of the function being meromorphic comes in when we applied the residue formula, which requires the only singularities begin the poles.

□

An application of the argument principle is the Rouche's theorem, which says that holomorphic function can be perturbed slightly without changing the number of its zeros.

Theorem 18. *(Rouche)*

Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If

$$|f(z)| > |g(z)| \quad \forall z \in C,$$

then f and $f + g$ have the same number of zeros inside the circle.

Proof. The idea behind the proof is illustrated in Fig. 6. We can interpret the left hand side of the argument principle as the number of times that $f(z)$ encircles the origin as z travels along the curve γ. We can see from Fig. 6 that by the setup of the theorem, $f + g$ can never include the origin. Thus, by the argument principle, f and $f + g$ have the same number of zeros inside the circle. □

Now, we can state some geometric properties of holomorphic functions when they are considered as mappings.

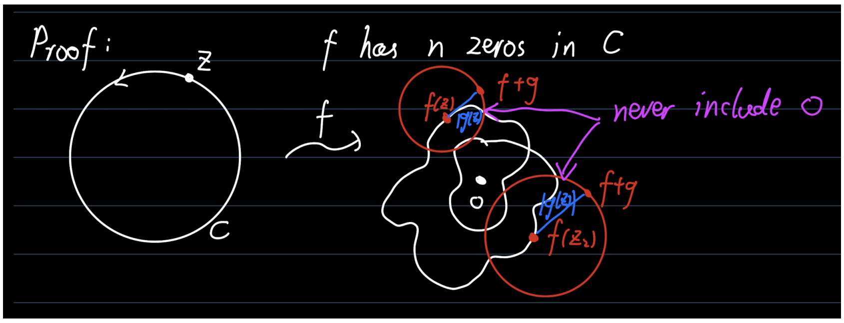

Figure 6: Proof of Rouche's theorem.

Definition 19. *(Open Mapping)*
A mapping is said to be open if it maps open set to open set.

Theorem 19. *(Open Mapping)*
If f is holomorphic and non-constant in a region Ω, than f is open.

Proof. Let w_0 belongs to the range of f, t.e., $f(z_0) = w_0$). We need to show that the neighborhood of w_0 is also contained in the image of f.
Define

$$g(z) := f(z) - w = f(z) - w_0 + w_0 - w =: F(z) + G(z).$$

Find δ such that the set $\{z : |z - z_0| \leq \delta\}$ is contained in Ω and $f(z) \neq w_0$ on the circle $|z - z_0| = \delta$. Find ϵ such that $|f(z) - w_0| \geq \epsilon$ on the circle $|z - z_0| = \delta$. Then, for z on the circle $|z - z_0| = \delta$, we have $|f(z) - w_0| \geq |w - w_0|$ if $|w - w_0| < \epsilon$. From Rouche's theorem 18, we know that $g(z)$ has the same zeros as for $F(z)$. We know that $F(z)$ as a zero at z_0 by assumption, which implies $g(z)$ also has a zero. The proof is complete. □

The next result pertains to the size of a holomorphic function. We refer to the maximum of a holomorphic function in an open set Ω as the maximum of its absolute value IN Ω.

Theorem 20. *(Maximum Modulus Principle) If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.*

Proof. Suppose f attains a maximum at z_0. Since f is a non-constant holomorphic function on Ω, from the open mapping theorem 19, f is an open mapping. Then, there exists an open set $D \subset \Omega$ centered at z_0, such that its image $f(D)$ is open and contains $f(z_0)$. It implies that z_0 is not the maximum, which is a contradiction. □

Corollary 7. *Suppose that Ω is a region with compact closure $\bar{\Omega}$. If f is holomorphic in Ω and continuous in $\bar{\Omega}$, then*

$$\sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \bar{\Omega} - \Omega} |f(z)|$$

Since f is continuous in the compact set $\bar{\Omega}$, $|f|$ attains its maximum in $\bar{\Omega}$.
Now, we switch to find the general form of the Cauchy's theorem. The key is to understand in what regions we can define the primitive of a given holomorphic function. Then, by Corollary 1, we can conclude the Cauchy's theorem. This requires the notion of homotopy and the resulting idea of simple-connectivity.

Definition 20. *(Simply Connected)*
A region Ω in the complex plane is simply connected if any two pair of curves in Ω with the same end-points are homotopic. Loosely speaking, two curves are homotopic if one curve can be deformed into the other by a continuous transformation without ever leaving Ω.

We give the general form of the Cauchy's theorem without proving it.

Theorem 21. *If f is holomorphic in the simply connected region Ω, then*

$$\int_\gamma f(z)dz = 0,$$

for any closed curve γ in Ω.

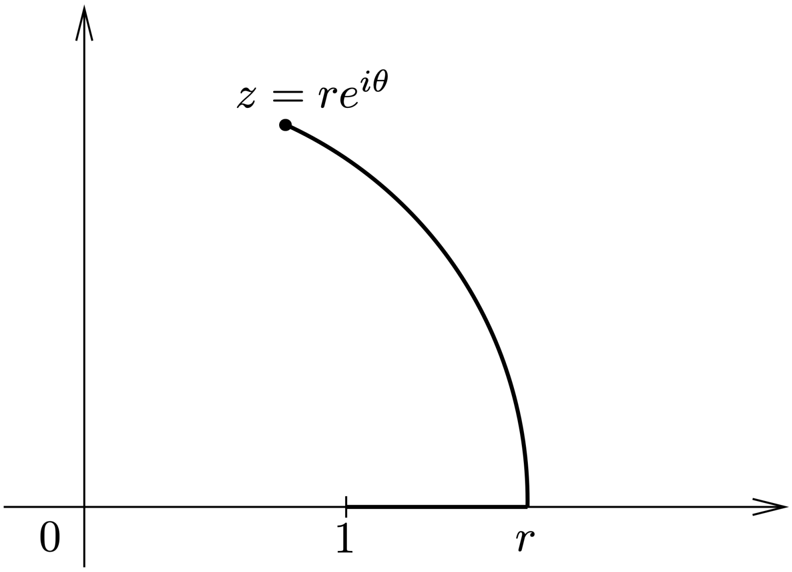

Figure 7: Path of integration for the principal branch of the logarithm.

We have mentioned previously that the function $\log f(z)$ is not single-valued. To make sense of the logarithm as a single-valued function, we must restrict the set on which we define it. This is so-called choice of a branch or sheet of the logarithm. We have the following global definition of a branch of the logarithm function based on simply connected domain.

Theorem 22. *Suppose Ω is simply connected with $1 \in \Omega$ and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_{\Omega}(z)$ so that*

1. *F is holomorphic in Ω.*
2. *$e^{F(z)} = z$ for all $z \in \Omega$.*
3. *$F(r) = log(r)$ whenever r is a real number and near 1*

Proof. We shall construct F as a primitive of $1/z$. The rest of the steps are omitted. □

In the slit plane $\Omega = \mathbb{C} - \{(-\infty, 0]\}$, we have the principal branch of the logarithm $\log z = \log r + i\theta$, where $z = re^{i\theta}$ with $|\theta| < \pi$. Path of integration for the principal branch of the logarithm is shown in Fig. 7. Then,

$$\log z = \int_1^r \frac{1}{x} dx + \int_0^\theta \frac{1}{w} dw = \log r + i\theta.$$

Notice that in general

$$\log(z_1 z_2) \neq \log z_1 + \log z_2.$$

Having defined a logarithm on a simply connected domain, we can take easily take an inter power of a complex number z^n as $z^n = e^{n \log z}$. But what about the complex power? We may use the following theorem.

Theorem 23. *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that*

$$f(z) = e^{g(z)}.$$

Proof. Fix a point z_0 in Ω, and define

$$g(z) := \int_\gamma \frac{f'(w)}{f(w)} dw + c_0,$$

where γ is the curve from z_0 to z, and c_0 is a constant such that $f(z_0) = e^{g(c_0)}$. This definition is independent of the path γ since Ω is simply connected. Since $\frac{f'(w)}{f(w)}$ is holomorphic, we can prove that g is holomorphic using the same technique as in Theorem 6, and

$$g'(z) := \frac{f'(z)}{f(z)}.$$

We also know that $f(z)e^{-g(z)}$ is a constant because

$$\frac{d}{dz}(f(z)e^{-g(z)}) = f'(z)e^{-g(z)} - f(z)g'(z)e^{-g(z)} = 0.$$

Additionally $f(z_0)e^{-g(z_0)} = 0$ implies that $f(z)e^{-g(z)} = 0$ for all $z \in \Omega$. The proof is complete.

$\square$