Once we have a probability space $(X, \mathcal{F}, \mathbb{P})$, it is natural to ask how would the probability measure be defined if we transfer the state X through a measurable map (random variable) from X to Y to a new measurable space $(Y, \mathcal{G})$, i.e., $f^{-1}(B) = \{x \in X : f(x) \in \mathcal{B}\} \in \mathcal{F}, \forall B \in \mathcal{G}$. The probability measure is induced by the following definition of a pushforward.

Definition 1. *(Pushforward Measure)*

Given two measurable spaces (X_1, Σ_1) and (X_2, Σ_2), a probability measure $\mathbb{P} : \Sigma_1 \to [0, 1]$, and a measurable map $f : X_1 \to X_2$, we may define the pushforward measure $\mathbb{Q} : \Sigma_2 \to [0, 1]$ as follows:

$$\mathbb{Q}(B) \equiv f_*\mathbb{P}(B) := \mathbb{P}(f^{-1}(B)), \quad \forall B \in \Sigma_2$$

An useful change of variable theorem directly follows:

Theorem 1. *(Change of variable)*

Assuming the setup in Definition 1. A measurable function g on X_2 is integrable with respect to the pushforward measure $\mathbb{Q}$ if and only if the comoposition $g \circ f$ is integrable with respect to the measure $\mathbb{P}$. The integrals also coincide, i.e., :

$$\int_{X_2} g(x)\mathbb{Q}(dx) = \int_{X_1} g \circ f(x)\mathbb{P}(dx). \tag{1}$$

Proof. We show the change of variable formula (1), which then implies the integrability condition. By definition of the pushfoward measure, we have:

$$\mathbb{Q}(B) = \mathbb{P}(f^{-1}(B)) = \mathbb{P}(\{x \in X : f(x) \in B\}), \quad \forall B \in \Sigma_2,$$

which directly gives the change of variable formula (1). The integrability condition then follows. □

After considering the change of measure from a measurable map, let's see what happen to the probability density function, as defined below. But before doing that, we need another results, namely the Radon-Nikodym theorem, to properly define the density raised from two absolutely continuous measures.

Definition 2. *(Absolutely Continuous)*

Let μ and ν be two measures defined on the measurable space (E, Σ). Then, the measure ν is said to be absolutely continuous with respect to the measure μ if for any $B \in \Sigma$, $\mu(B) = 0$ implies $\nu(B) = 0$. Then, we write $\nu \ll \mu$.

Theorem 2. *(Radon-Nikodym theorem)*

Let μ and ν be two σ-finite measures on a measurable space (E, Σ). If $\nu \ll \mu$, then there exists a measurable function $f : E \to [0, \infty)$ such that for any $B \in \Sigma$:

$$\nu(B) = \int_B f(x)\mu(dx).$$

The function f is called the Radon-Nikodym derivative and is denoted as:

$$f \equiv \frac{d\nu}{d\mu}.$$

The proof is omitted here. Now, we may properly define the probability density function:

Definition 3. *(Probability Density Function)*

Consider a random variable $X : \Omega \to \mathbb{R}$ defined on the probability space $(\Omega, \mathcal{F}, \nu)$ and maps to the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_L)$, where μ_L is the Lebesgue measure. If $\nu \ll \mu_L$, then the Radon-Nikodym derivative f is called the probability density function of X.

Assume that π and π^* are respectively the probability density functions for the random variables X_1 and X_2, respectively defined on $(\Omega_1, \mathcal{F}_1, \mathbb{P})$ and $(\Omega_2, \mathcal{F}_2, \mathbb{Q})$. Additionally, there is a measurable function $f : \Omega_1 \to \Omega_2$. Then, the probability of any set $A \in \mathcal{F}$ under the measure $\mathbb{P}$ should be the same as $\mathbb{Q}(f(A))$. Recall:

$$\mathbb{P}(A) = \int_A \pi(x)\mu_L(dx)$$

$$\mathbb{Q}(A) = \int_{f(A)} \pi^*(x)\mu_L(dx)$$

To compensate for the change in the differential volume over which we integrate, the probability density function π^* needs to be scaled by the change in volumes raised from the transformation f, which is quantified by the determinant of the Jacobian matrix, i.e.:

$$\pi^*(x) = \pi(f^{-1}(x))|\nabla f^{-1}(x)|^{-1}.$$