Mathematics Notation for SCMA 645

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1 Background

1.1 Sets and Ordered Sets

An unordered **set** is a collection of items that could numbers or objects. When explicitly listing the **elements** of a set, use curly brackets:

$$L = \{\text{Personal}, \text{Car}, \text{Home}, \text{Farm}, \text{Commercial}\}.$$

The symbol $\in$ means "is an element of." For example, Personal $\in L$. Here is another example of a set:

$$T = \{3, 2, 5, 6\}.$$

The order in which elements are listed is not important.

We can indicate a subset of a set by the elements satisfying a certain criterion using a colon (":"), which means "such that." For example,

$$U = \{t \in T : t > 4\},$$

which says that U is the set of elements t from the set T that are greater than 4. Therefore, $U = \{5, 6\}$.

An **ordered set** is a list of items that is indicated using parentheses. So, {red, green, blue} is the same as {green, blue, red}, but (red, green, blue) is not the same as (green, blue, red).

1.2 Vectors

A **vector** is an ordered set of numbers. When explicitly providing the elements of a vector, use parentheses. We usually write them in a column, and do not use commas to separate the elements:

$$v = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 9 \\ 2 \end{pmatrix}.$$

The **transpose** of a vector is a row vector and is denoted using ^{T}:

$$v^T = \begin{pmatrix} 2 & 4 & 6 & 9 & 2 \end{pmatrix}.$$

The **inner product** of two vectors is the sum-product (the sum of the item-wise products of the vectors). If $a^T = \begin{pmatrix} 1 & 0 & 4 & 5 & 5 \end{pmatrix}$, then the inner product of a and v is

$$a^T v = \begin{pmatrix} 1 & 0 & 4 & 5 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 6 \\ 9 \\ 2 \end{pmatrix} = 1 \times 2 + 0 \times 4 + 4 \times 6 + 5 \times 9 + 5 \times 2 = 81.$$

1.3 Indexing

An **index variable** is a letter usually written as a subscript indicating a generic element of a set. Suppose we have a set of decision variables, one for each element in L. We could write the list of decision variables in any of the following ways:

1. $$x_{\text{Personal}}, x_{\text{Car}}, x_{\text{Home}}, x_{\text{Farm}}, x_{\text{Commercial}},$$

2. $$x_i, i \in \{\text{Personal}, \text{Car}, \text{Home}, \text{Farm}, \text{Commercial}\},$$

3. $$x_i, i \in L.$$

In the last two examples, i is an index variable. In the last example, L is an index set.

Suppose we have another set of decision variables for every combination of elements from T and L. The set of elements containing every combination of T and L is called the *cartesian product* and is denoted $T \times L$. We could write the set of decision variables in the following ways:

1. $$\begin{array}{l} y_{3,\text{Personal}}, y_{3,\text{Car}}, y_{3,\text{Home}}, y_{3,\text{Farm}}, y_{3,\text{Commercial}}, \\ y_{2,\text{Personal}}, y_{2,\text{Car}}, y_{2,\text{Home}}, y_{2,\text{Farm}}, y_{2,\text{Commercial}}, \\ y_{5,\text{Personal}}, y_{5,\text{Car}}, y_{5,\text{Home}}, y_{5,\text{Farm}}, y_{5,\text{Commercial}}, \\ y_{6,\text{Personal}}, y_{6,\text{Car}}, y_{6,\text{Home}}, y_{6,\text{Farm}}, y_{6,\text{Commercial}}. \end{array}$$

2. $$y_{ij}, i \in \{3, 2, 5, 6\}, j \in \{\text{Personal}, \text{Car}, \text{Home}, \text{Farm}, \text{Commercial}\},$$

3. $$y_{ij}, i \in T, j \in L.$$

4.
$$y_{ij}, (i,j) \in T \times L.$$

We could also define a set of decision variables z for a subset of the combinations of elements from T and L.

$$\{y_{ij} : (i,j) \in T \times L, i < 3 \text{ and } j \in \{\text{Home}, \text{Farm}\}\} = \{y_{2,\text{Home}}, y_{2,\text{Farm}}\}.$$

In addition to simplifying the representation of collections of decision variables, index variables are useful for writing sums of decision variables. For example,

$$\sum_{i \in L} x_i = x_{\text{Personal}} + x_{\text{Car}} + x_{\text{Home}} + x_{\text{Farm}} + x_{\text{Commercial}}.$$

In this example, i is an index variable and L is an index set for the sum. The symbol $\sum$ is the Greek letter "S", indicating a sum. Another example is,

$$\begin{aligned} \sum_{i \in T} \sum_{j \in L} y_{ij} \quad = \quad & y_{3,\text{Personal}} + y_{3,\text{Car}} + y_{3,\text{Home}} + y_{3,\text{Farm}} + y_{3,\text{Commercial}} \\ & + y_{2,\text{Personal}} + y_{2,\text{Car}} + y_{2,\text{Home}} + y_{2,\text{Farm}} + y_{2,\text{Commercial}} \\ & + y_{5,\text{Personal}} + y_{5,\text{Car}} + y_{5,\text{Home}} + y_{5,\text{Farm}} + y_{5,\text{Commercial}} \\ & + y_{6,\text{Personal}} + y_{6,\text{Car}} + y_{6,\text{Home}} + y_{6,\text{Farm}} + y_{6,\text{Commercial}}. \end{aligned}$$

Note that we do not need to use the same index variable for a set each time we use it. We used i to index over L for the x variables, and we used j to index over L for the y variables. We can also write

$$\sum_{i=1}^{4} z_i = z_1 + z_2 + z_3 + z_4.$$

where 1 is the lower limit and 4 is the upper limit for the index variable i in the sum. We can use index variables to indicate sets of equations and inequalities. For example,

$$x_i + \sum_{j \in L} y_{ij} \geq 1, i \in T$$

is the same as

$$\begin{aligned} x_3 + y_{3,\text{Personal}} + y_{3,\text{Car}} + y_{3,\text{Home}} + y_{3,\text{Farm}} + y_{3,\text{Commercial}} &\geq 1, \\ x_2 + y_{2,\text{Personal}} + y_{2,\text{Car}} + y_{2,\text{Home}} + y_{2,\text{Farm}} + y_{2,\text{Commercial}} &\geq 1, \\ x_5 + y_{5,\text{Personal}} + y_{5,\text{Car}} + y_{5,\text{Home}} + y_{5,\text{Farm}} + y_{5,\text{Commercial}} &\geq 1, \\ x_6 + y_{6,\text{Personal}} + y_{6,\text{Car}} + y_{6,\text{Home}} + y_{6,\text{Farm}} + y_{6,\text{Commercial}} &\geq 1. \end{aligned}$$

Also,
$$z_i \geq 0, i = 1, 2, 3, 4,$$

is the same as

$$\begin{aligned} z_1 &\geq 0, \\ z_2 &\geq 0, \\ z_3 &\geq 0, \\ z_4 &\geq 0. \end{aligned}$$

Note that for a set of equations, the same index variable *cannot* be used in both a sum and as an index for equations.

1.4 Matrix-Vector Notation

A **matrix** is an array of numbers. The rows and columns of a matrix are vectors.

$$M = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 5 & 8 & 2 \end{pmatrix}$$

We can write the system of equations

$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 &= 0, \\ x_1 - 2x_2 &= 1, \\ 3x_2 - 3x_4 &= 4. \end{aligned}$$

as

$$Ax = b$$

where $A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & -2 & 0 & 0 \\ 0 & 3 & 0 & -3 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$, and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$.

2 Exercises

1. Provide a matrix representation of the following system of equations.

$$\begin{aligned} x_1 - x_2 &= 4, \\ 2x_1 + x_2 &= 6, \\ x_1 + 3x_2 &= 8. \end{aligned}$$

2. Write out all of the terms in each of the following sums:

(a) $\sum_{i=1}^{5} z_i$

(b) $\sum_{i=1}^{5} z_{ij}$

(c) $\sum_{j=0}^{4} \sum_{k=6}^{8} x_{jk}$

(d) $\sum_{j \in S} y_j$ where $S = \{2, 3, 6, 8\}$

3. Give the third term in each sum:

$$(a) \quad \sum_{i=4}^{8} z_i$$
$$(b) \quad \sum_{q=100}^{200} z_{iq}$$

4. Does $\sum_{i \in P} \sum_{j \in Q} x_{ij} = \sum_{j \in P} \sum_{i \in Q} x_{ji}$ for any definition of P and Q?

5. Let $A = \{-1, 0, 1, 2, 3, 4\}$, $B = \{-1, 0, 2, 3, 5\}$, and $C = \{(i, j) : i \in A, j \in B, i + j = 3\}$.

   (a) List all of the elements in C.

   (b) Write out the following inequalities.

   $$x_i + x_j \leq 1, (i, j) \in C$$

   (c) Write out the following equations.

   i. $\sum_{(i,j) \in C} x_{ij} = 1$

   ii. $\sum_{(i,j) \in C : i < j} x_{ij} = 1$

   iii. $\sum_{i \in A} \sum_{j \in B} x_{ij} = 1$

   (d) Evaluate the following expression.

   $$\sum_{(i,j) \in C} i^2$$

6. Put the following linear program in written-out notation:

$$\min \sum_{(i,j) \in A} c_{ij} x_{ij}, \tag{1}$$

$$\text{s.t.} \sum_{j \in V : (i,j) \in A} x_{ij} - \sum_{j \in V : (j,i) \in A} x_{ji} = b_i, i \in V, \tag{2}$$

$$\ell_{ij} \leq x_{ij} \leq u_{ij}, (i, j) \in A, \tag{3}$$

where $V = \{1, 2, 3, 4, 5\}$ and $A = \{(1, 3), (1, 4), (2, 3), (2, 5), (3, 4), (3, 5)\}$.