DEPARTMENT OF MATHEMATICS

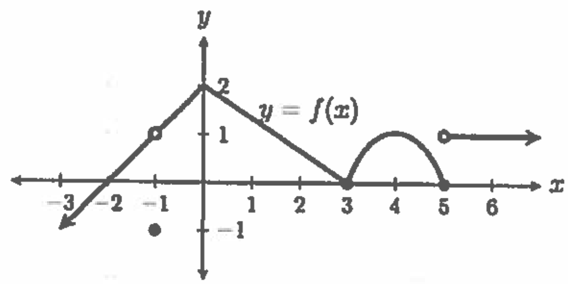
CALCULUS I - MATH 200

SAMPLE FINAL EXAM

DISCLAIMER

a. **The actual final exam contains fewer problems than this sample final exam.**
b. **The actual final exam may contain problems that do not resemble problems on this sample final.**

(1) For the accompanying figure,



circle the correct answer.

(a) $\lim_{x \to 0} f(x)$

(i) 0 (ii) 2 (iii) does not exist

(b) $\lim_{x \to 5^+} f(x)$

i) 0 (ii) 1 (iii) does not exist

(c) $\lim_{x \to 5} f(x)$

i) 0 (ii) 1 (iii) does not exist

(d) $\lim_{x \to -1} f(x)$

i) 1 (ii) −1 (iii) does not exist

(e) $\lim_{h \to 0} \dfrac{f(4+h) - f(4)}{h}$

i) 0 (ii) 1 (iii) 4

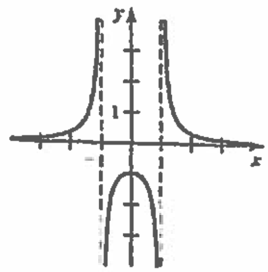
(f) Find all values of x in the interval $[-3, 6]$ where $f(x)$ is not continuous.

i) $-1, 0, 3, 5$ (ii) $-1, 5$ (iii) $-1, 4, 5$

(g) Find all values of x in the interval $[-3, 6]$ where $f(x)$ is not differentiable.

i) $-1, 3, 5$ (ii) $-1, 0, 3, 5$ (iv) $-1, 0, 3, 4, 5$

(2) For the accompanying figure,



circle the correct answer.

(a) $\lim_{x\to 1^+} f(x)$

i) 0 (ii) ∞ (iii) $-\infty$ (iv) does not exist

(b) $\lim_{x\to 1^-} f(x)$

i) 0 (ii) ∞ (iii) $-\infty$ (iv) does not exist

(c) $\lim_{x\to \infty} f(x)$

i) 0 (ii) ∞ (iii) $-\infty$ (iv) does not exist

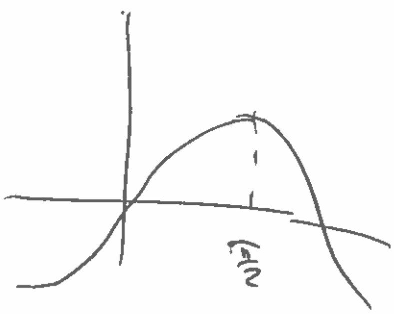
(d) $\lim_{x\to -\infty} f(x)$

ii) 0 (ii) ∞ (iii) $-\infty$ (iv) does not exist

(3) **Find the following limits**

(a) $\lim\limits_{x\to 4^+} \dfrac{x+9}{x^2-16}$ $\quad \lim\limits_{x\to 4^+} \dfrac{x+9}{(x-4)(x+4)} = \dfrac{4+9}{+\varepsilon \cdot 8} = +\infty$

(b) $\lim\limits_{x\to \pi/2^+} \dfrac{x-3}{\sin(x)}$ $\quad = \dfrac{\frac{\pi}{2}-3}{\sin(\frac{\pi}{2})} = \boxed{\dfrac{\frac{\pi}{2}-3}{1}}$



(4) Find the exact value of the following limits. Part (c) and (d) are on the next page.
SHOW WORK

(a) $\lim_{x\to 1} \frac{2x^2-3x+1}{x^2+x-2}$ $\lim_{x\to 1} \frac{(2x-1)\cancel{(x-1)}}{(x+2)\cancel{(x-1)}} = \lim_{x\to 1} \frac{2(1)-1}{1+2} = \boxed{\frac{1}{3}}$

(b) $\lim_{x\to\infty} \frac{\sqrt{9x^2+1}}{x-10} \frac{\sqrt{\frac{1}{x^2}}}{\sqrt{\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{\sqrt{(\frac{1}{x^2})(9x^2+1)}}{(x-10)\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{9+\frac{1}{x^2}}}{1-\frac{10}{x}} = \frac{\sqrt{9+0}}{1-0}$

$= \frac{\sqrt{9}}{1} = \boxed{3}$

(c) $\lim_{x\to 0} \frac{\cos(x)-1}{3x^2}$ $\underset{\text{form}}{\left(\frac{0}{0}\right)}$ $\xrightarrow{L'H}$ $\lim_{x\to 0} \frac{-\sin(x)}{6x}$ $\underset{\text{form}}{\left(\frac{0}{0}\right)}$ $\xrightarrow{L'H}$ $\lim_{x\to 0} \frac{-\cos(x)}{6} = \boxed{\frac{-1}{6}}$

(d) $\lim_{x\to\infty} x^2 \ln\left(1+\frac{4}{x^2}\right)$. $\underset{\text{form}}{(\infty\cdot 0)}$ rewrite $= \lim_{x\to\infty} \frac{\ln\left(1+\frac{4}{x^2}\right)}{\frac{1}{x^2}}$ $\underset{\text{form}}{\left(\frac{0}{0}\right)}$ $\xrightarrow{L'H}$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{4}{x^2}} \cdot -8x^{-3}}{-2x^{-3}} = \lim_{x\to\infty} \frac{4}{1+\frac{4}{x^2}} = \frac{4}{1+0} = \frac{4}{1} = \boxed{4}$$

$f(x+h) = \frac{1}{x+h+2}$

$f(x) = \frac{1}{x+2}$

(5) Show that the derivative of $f(x) = \dfrac{1}{x+2}$ at $x \neq -2$ is $f'(x) = -\dfrac{1}{(x+2)^2}$ using the **limit definition of the derivative . SHOW WORK.**

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left(\frac{x+2}{x+2}\right)\frac{1}{x+h+2} - \frac{1}{x+2}\left(\frac{x+h+2}{x+h+2}\right)}{h}$$

$$\lim_{h \to 0} \frac{x+2 - (x+h+2)}{(x+2)(x+h+2)} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{-\cancel{h}}{(x+2)(x+h+2)} \cdot \frac{1}{\cancel{h}} = \lim_{h \to 0} \frac{-1}{(x+2)(x+h+2)}$$

$$= \frac{-1}{(x+2)(x+2)}$$

$$= \frac{-1}{(x+2)^2} \quad \text{QED}$$

(6) If the equation of the tangent line to the graph of $y = f(x)$ at the point $x = 3$ is $y = 2x + 5$, find $f(3)$ and $f'(3)$.

so $f'(3) = 2$

$f(3) = 2(3) + 5 = 11$

(7) **Find the derivative of the following functions.**

(a) $f(z) = 5z^6 + \frac{3}{z^3} - 7e^{5z} + 3$

$$30z^5 - \frac{9}{z^4} - 35e^{5z}$$

(b) $y = \frac{5x^2}{x^3+2}$

$$\frac{(x^3+2)(10x) - (5x^2)(3x^2)}{(x^3+2)^2} = \frac{10x^4 + 20x - 15x^4}{(x^3+2)^2}$$

$$= \frac{5x(4-x^3)}{(x^3+2)^2}$$

(c) $f(x) = \sqrt{3+\sin(3x)}$

$$\frac{1}{2}(3+\sin(3x))^{-1/2} \cdot 3\cos(3x)$$

$$= \frac{3}{2}\frac{\cos(3x)}{\sqrt{3+\sin(3x}}$$

(d) $h(t) = \sin^{-1}(7t)$

$$\frac{7}{\sqrt{1-(7t)^2}} = \frac{7}{\sqrt{1-49t^2}}$$

(8) Find $\frac{dy}{dx}$ using an appropriate method. **SHOW WORK.**

(a) $x^3y - xy^4 = 6$

$$x^3 \cdot \frac{dy}{dx} + 3x^2y - x \cdot 4y^3\frac{dy}{dx} - y^4 = 0$$

$$\frac{dy}{dx}\left(x^3 - 4xy^3\right) = y^4 - 3x^2y$$

$$\frac{dy}{dx} = \frac{y^4 - 3x^2y}{x^3 - 4xy^3}$$

(b) $y = (2x+1)^{\cos(x)}$

$$\ln(y) = \ln\left((2x+1)^{\cos(x)}\right) = \cos(x) \cdot \ln(2x+1)$$

$$\ln(y) = \cos(x) \cdot \ln(2x+1)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(\cos(x) \cdot \ln(2x+1)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \cos(x) \cdot \frac{1}{2x+1} \cdot 2 + -\sin(x) \cdot \ln(2x+1)$$

$$\frac{dy}{dx} = (2x+1)^{\cos(x)}\left(\frac{2\cos(x)}{2x+1} - \sin(x)\ln(2x+1)\right)$$

(9) **If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume of the water remaining in the tank after t minutes as**

$$V(t) = 5000\left(1 - \frac{1}{40}t\right)^2, \quad \text{for } 0 \le t \le 40$$

Find the rate at which water is draining from the tank after 5 minutes?

$V'(t) = 10000\left(1 - \frac{1}{40}t\right) \cdot -\frac{1}{40} = -250\left(1 - \frac{1}{40}t\right)$

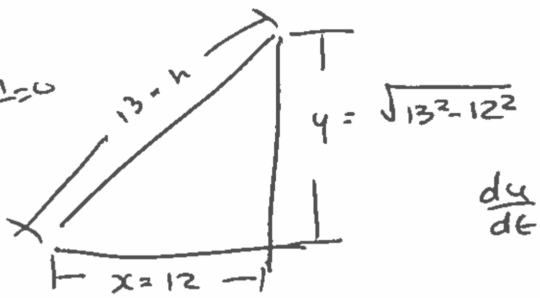
$= -250 + \frac{250}{40}t = -250 + \frac{25}{4}t$

$V'(t) = -250 + \frac{25}{4}t$

$V'(5) = -250 + \frac{25}{4}(5) = -218.75$ gal/min

(10) A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/ sec. How fast is the top of the ladder sliding down the wall then? **SHOW WORK.**

$\frac{dh}{dt} = 0$



$\frac{dy}{dt} = ?$

$\frac{dx}{dt} = 5 ft/sec$

$$x^2 + y^2 = h^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2h\frac{dh}{dt}$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = h \cdot \frac{dh}{dt}$$

$$12(5) + \sqrt{13^2 - 12^2} \cdot \frac{dy}{dt} = 13 \cdot 0$$

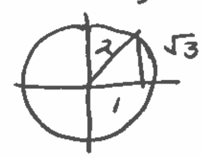
$$\frac{dy}{dt} = \frac{-12 \cdot 5}{\sqrt{13^2 - 12^2}}$$

$$\frac{dy}{dt} = -12 \text{ ft/sec}$$

$y' = 4\sin(x) \cdot -\sin(x) + 4\cos(x) \cdot \cos(x)$

$y' = 4(\cos^2(x) - \sin^2(x))$ $\frac{\pi}{3}$

(11) Find an equation of the tangent line to the graph of $y = 4\sin(x)\cos(x)$ at $x = \frac{\pi}{3}$. **SHOW WORK.**



slope: $y'(\frac{\pi}{3}) = 4(\cos^2(\frac{\pi}{3}) - \sin^2(\frac{\pi}{3}))$

$= 4((\frac{1}{2})^2 - (\frac{\sqrt{3}}{2})^2)$

$= 4(\frac{1}{4} - \frac{3}{4}) = 4 \cdot {}^-\frac{1}{2} = -2$ slope

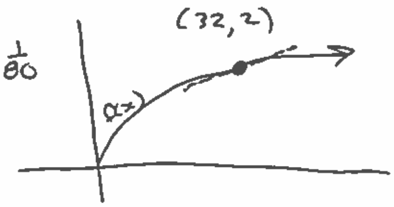
$y - \sqrt{3} = {}^-2(x - \frac{\pi}{3})$

$y = -2x + \frac{\pi}{3} + \sqrt{3}$

$y = -2x + \frac{\pi + 3\sqrt{3}}{3}$

(12) Let $f(x) = x^{\frac{1}{5}}$.

(a) Find the linear approximation to $f(x)$ at $x = 32$. **SHOW WORK.**



$m = f'(x) = \frac{1}{5}x^{-4/5}$ not $m(32) = \frac{1}{80}$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{5(32)^{4/5}}(x - 32)$

$y - 2 = \frac{1}{5 \cdot 16}(x - 32) = \frac{1}{80}(x - 32) + 2 = L(x$

(b) Use the linearization in (a) to approximate $\sqrt[5]{33}$. **SHOW WORK.**

$L(33) = \frac{1}{80}(33 - 32) + 2$

$= \frac{1}{80} \cdot 1 + 2$

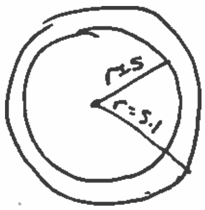
$= \frac{1}{80} + 2$

$= \frac{161}{80} \approx 2.0125$

DATE

PURPOSE

CUE COLUMN



NOTES

#13

Approximate the change in volume when its radius changes from r=5 to r=5.1 ft using the differential. Compare to the exact change in volume ($V = \frac{4}{3}\pi r^3$)

$V = \frac{4}{3}\pi r^3$

$\frac{d}{dr}(V) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)$

$\frac{dV}{dr} = 4\pi r^2 \longrightarrow dV = 4\pi r^2 \, dr$

change in r (.1)

$dV = 4\pi r^2 (.1)$

$dV = 4\pi (5)^2 (.1)$

$dV = 10\pi$ ft³/ft rad

$dV \approx 31.4159$

approx volume change

exact change in volume

$V(5.1) - V(5)$

$\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3$

$\frac{4}{3}\pi(5.1^3 - 5^3)$

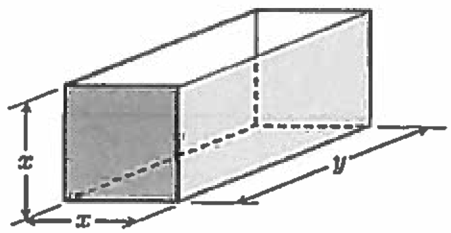
$\frac{4}{3}\pi \cdot (7.651) \approx 32.049$

very

actual volume change

SUMMARY

(14) A metal box with two square ends and an open top is to contain a volume of 36 cubic inches. What dimensions x and y will minimize the total surface area of the metal box? **SHOW WORK.**



$V = l \cdot w h$

$V = y \cdot x \cdot x$

$V = x^2 y$

$36 = x^2 y$

$\boxed{\frac{36}{x^2} = y}$

Substitute

$SA = 2x^2 + 3xy$

$SA = 2x^2 + 3x\left(\frac{36}{x^2}\right)$

$SA = 2x^2 + \frac{108}{x}$

$S' = 4x - \frac{108}{x^2}$

$0 = 4x - \frac{108}{x^2}$

$\frac{108}{x^2} = \frac{4x}{1}$

$108 = 4x^3$

$27 = x^3$

$\boxed{3 = x}$

So $36 = x^2 y$

$36 = (3)^2 y$

$\boxed{4 = y}$

(15) Let $f(x) = x^4 - 18x^2 + 81$. Questions (a)-(e) refer to this function.

(a) Find critical points of f.

$f'(x) = 4x^3 - 36x = 4x(x^2 - 9) = 4x(x-3)(x+3)$

$x = 0 \quad x = -3 \quad x = +3$

(b) Determine intervals on which f is increasing and f is decreasing.

		-3		0		3	
$4x$	$-$		$-$	0	$+$		$+$
$x-3$	$-$		$-$		$-$	0	$+$
$x+3$	$-$	0	$+$		$+$		$+$
	$-$	0	$+$	0	$-$	0	$+$

incr. $(-3, 0) \cup (3, \infty)$

decr $(-\infty, -3) \cup (0, 3)$

(c) Find local maximum and local minimum values of $f(x)$.

$f(0) = 0^4 - 18(0)^2 + 81 = 81$ max $(0, 81)$

$f(-3) = (-3)^4 - 18(-3)^2 + 81 = 0$ min $(-3, 0)$

$f(3) = (3)^4 - 18(3)^2 + 81 = 0$ min $(3, 0)$

(d) Determine intervals on which f is concave up and f is concave down.

$f'' = 12x^2 - 36 = 12(x^2-3)$ $\qquad x = +\sqrt{3}$, $\quad x = -\sqrt{3}$

	$-\sqrt{3}$		$\sqrt{3}$	
$x-\sqrt{3}$	$-$	$-$	0	$+$
$x+\sqrt{3}$	$-$ 0	$+$		$+$
	$+$ 0	$-$	0	$+$

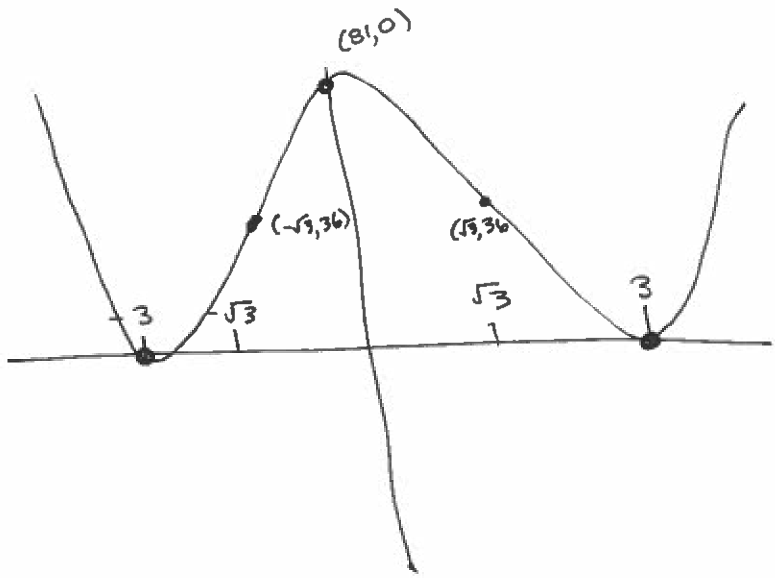
cc up $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

cc down $(-\sqrt{3}, \sqrt{3})$

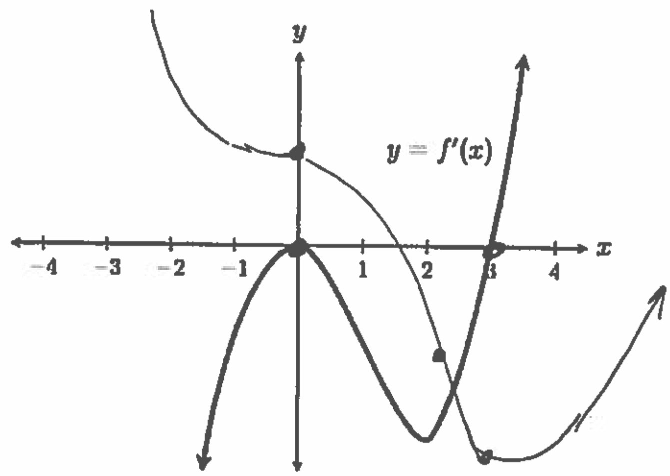
(e) Find inflection point(s).

$x = -\sqrt{3}$ $\qquad f(-\sqrt{3}) = 36$ $\qquad$ inflection $(-\sqrt{3}, 36)$

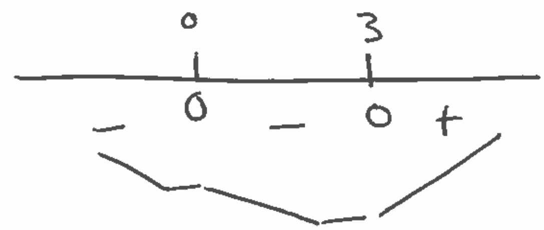
$x = \sqrt{3}$ $\qquad f(\sqrt{3}) = 36$ $\qquad (+\sqrt{3}, 36)$



(16) The graph of the first derivative of a function f is given below.



(a) Find critical point(s) of f.



$x=0$
$x=3$

(b) Determine intervals on which f is increasing and f is decreasing.

$(-\infty,0)\cup(0,3)$ decr

$(3,\infty)$ incr

(c) Find the values of x at which f has local maximum and local minimum values.

$x=3$ min   No max

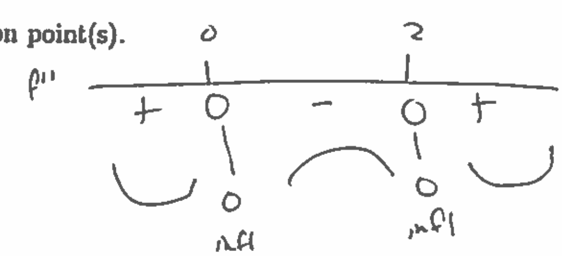
(d) Determine intervals on which f is concave up and f is concave down.

cc up $(-\infty,0)\cup(2,\infty)$

cc down $(0,2)$

(e) Find the values of x at which f has inflection point(s).

$x=2$ inflection
$x=0$ inflection



$$\text{if we use substitution}$$
$$u = 2x$$
$$du = 2dx$$
$$\frac{du}{2} = dx$$
$$\frac{1}{2}\int \sin(u)\,du$$
$$\downarrow$$
$$-\frac{1}{2}\cos(u)$$
$$-\frac{1}{2}\cos(2x)$$

(17) **Find the general antiderivative of the following functions**

(a) $f(x) = \int \sin(2x) - e^{-x} + 3x - 12 \; dx$

$$\downarrow$$

$$-\frac{1}{2}\cos(2x) - e^{-x} + \frac{3x^2}{2} + 12x + C$$

(b) $f(x) = \sqrt{x^3+1}$

$$\frac{d}{dx}\left(\int_0^x \sqrt{\cancel{t}^3+1}\; dt\right) = \sqrt{x^3+1}$$

$$\text{So} \quad \int_0^x \sqrt{t^3+1}\; dt$$

(18) **Find the following indefinite integrals.**

(a) $\int \left(2x^3 + \frac{5}{x} + \frac{1}{x^5} - 3\right) dx$

$= \frac{2x^4}{4} + 5\ln|x| + \frac{1}{-4x^4} - 3x + C$

$= \frac{x^4}{2} + 5\ln|x| - \frac{1}{4x^4} - 3x + C$

(b) $\int \frac{e^t}{\sqrt{e^t+1}}\, dt$

let $u = e^t$

$du = e^t\, dt$

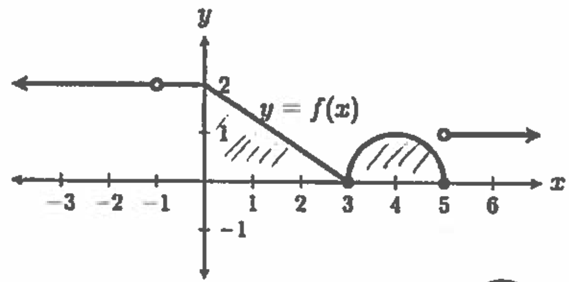
$\frac{du}{e^t} = dt$

$\int \frac{\cancel{e^t}}{u^{1/2}} \frac{du}{\cancel{e^t}}$

$\int u^{-1/2} du$

$\frac{u^{1/2}}{1/2} + C \qquad 2\sqrt{e^t} + C$

(19) The graph of $y = f(x)$ is shown in the figure below. Evaluate $\int_0^5 f(x)\,dx$.



$\frac{1}{2} b \cdot h$ $\quad + \quad$ $\frac{1}{2}\pi r^2$

$\frac{1}{2} \cdot 3 \cdot 2 \quad + \quad \frac{1}{2}\pi(1)^2$

$3 + \frac{\pi}{2}$

(20) Approximate the area bounded by the graph of $f(x) = \sqrt{x^3+1}$ and the x-axis on the interval $[0, 8]$ using midpoint Riemann sum with $n = 4$. Approximate your answer to four decimal places.

$\frac{b-a}{n} = \frac{8-0}{4} = 2$ Δx

$\sum f(x_k^*)\Delta x$

midpoint

$x_k^* = a + (k - \frac{1}{2})\Delta x$

$0 + (k - \frac{1}{2}) \cdot 2$

$2k - 1$

$\sum_{1}^{4} f(2k-1) \cdot 2 = \sum_{1}^{4} \sqrt{(2k-1)^3+1} \cdot 2$

$= 2 \sum_{1}^{4} \sqrt{(2k-1)^3+1}$

$= 2 \cdot (36.47792534) = 72.9559$

(21) Compute the following definite integrals. Exact values only. Calculator answers will not be accepted.

(a) $\int_0^1 \left(5x^4 - x\sqrt{x} + \sqrt{2}\right) dx$

$$\frac{5x^5}{5} - \frac{x^{5/2}}{5/2} + \sqrt{2}\,x \Big|_0^1 \quad \left((1)^5 - \frac{2}{5}(1)^{5/2} + \sqrt{2}(1)\right) - (0)$$

$$1 - \frac{2}{5} + \sqrt{2}$$

$$\boxed{\left(\frac{3}{5} + \sqrt{2}\right)}$$

$u = x^2+5$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$u(2) = 2^2+5 = 9$

$u(0) = 0^2+5 = 5$

(b) $\int_0^2 x\sqrt{x^2+5}\,dx$

$$\int_5^9 \cancel{x}\, u^{1/2} \frac{du}{2\cancel{x}} = \frac{1}{2}\int_5^9 u^{1/2}\,du.$$

$$\frac{1}{2}\frac{u^{3/2}}{3/2}\Big|_5^9 \quad \frac{1}{3}(9)^{3/2} - \frac{1}{3}(5)^{3/2}$$

$$\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}u^{3/2} \quad \frac{1}{3}(3)^3 - \frac{1}{3}(\sqrt{5})^3$$

$$\frac{27}{3} - \frac{1}{3}5\sqrt{5}$$

$$\boxed{9 - \frac{1}{3}5\sqrt{5}}$$

(22) If the population of a bacteria is growing at an instantaneous rate of 2^t million bacteria per hour, what does the definite integral $\int_0^1 2^t\,dt$ represent? If $F(t)$ represents the number of bacteria at time $t \geq 0$, what does $F(0) + \int_0^1 2^t\,dt$ represent?

$\int_0^1 2^t\,dt$ represents the total population of bacteria after 1 hour

$F(0) + \int_0^1 2^t\,dt$ is the beginning number of bacteria plus the total number of bacteria produced in 1 hour