Probability

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SCMA 524: Statistical Fundamentals

What is Probability?

A **probability** is a number between 0 and 1

that measures the likelihood of an event.

- An event that is certain to occur has probability 1.
- An event that is certain not to occur has probability 0.
- An event with probability between 0 and 1 involves uncertainty.
- Events with larger probabilities are more likely to occur.

The Sample Space

- A **sample space** is a set of possible outcomes.
- An **event** is a subset of the sample space.

Example

Rolling a six sided die. The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.
$A = \{2, 4, 6\}$ is the event that you get an even number.

Example

Flipping two coins. The sample space is $\Omega = \{HH, HT, TH, TT\}$.
$A = \{HH, TT\}$ is the event that both coins land with the same side up.

The Complement of an Event

If A is an event then A^c is the event that A does not occur.

Example

Suppose the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.
If $A = \{2, 4, 6\}$ then $A^c = \{1, 3, 5\}$.

Example

Suppose the sample space is $\Omega = \{HH, HT, TH, TT\}$.
If $A = \{HH\}$ then $A^c = \{TT, TH, HT\}$.

Question

Suppose the sample space is $\Omega = \{HH, HT, TH, TT\}$.

If $A = \{HH, TH\}$ then what is A^c?

a) $\{TH, TT\}$

b) $\{HT, TT\}$

c) $\{TT, TH\}$

d) $\{HH, TH\}$

Question

Suppose the sample space is $\Omega = \{HH, HT, TH, TT\}$.

If $A = \{HH, TH\}$ then what is A^c?

a) $\{TH, TT\}$

b) $\{HT, TT\}$

c) $\{TT, TH\}$

d) $\{HH, TH\}$

Answer:

Question

Suppose the sample space is $\Omega = \{HH, HT, TH, TT\}$.

If $A = \{HH, TH\}$ then what is A^c?

a) $\{TH, TT\}$

b) $\{HT, TT\}$

c) $\{TT, TH\}$

d) $\{HH, TH\}$

Answer: b

The Intersection of Two Events

If A and B are events then $A \cap B$ is the event that both A and B occur.

Example

Suppose the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.
If $A = \{2, 4, 6\}$ and $B = \{2, 3, 4\}$ then $A \cap B = \{2, 4\}$.

Example

Suppose the sample space is $\Omega = \{HH, HT, TH, TT\}$.
If $A = \{HH, TT\}$ and $B = \{HT, TH\}$ then $A \cap B = \varnothing = \{\}$.

Question

Suppose the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.

If $A = \{1, 2, 5\}$ and $B = \{2, 3, 5\}$ then what is $A \cap B$?

a) $\{1, 3\}$

b) $\{1, 2, 3, 5\}$

c) $\varnothing$

d) $\{2, 5\}$

Question

Suppose the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.

If $A = \{1, 2, 5\}$ and $B = \{2, 3, 5\}$ then what is $A \cap B$?

a) $\{1, 3\}$

b) $\{1, 2, 3, 5\}$

c) $\varnothing$

d) $\{2, 5\}$

Answer:

Question

Suppose the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.

If $A = \{1, 2, 5\}$ and $B = \{2, 3, 5\}$ then what is $A \cap B$?

a) $\{1, 3\}$

b) $\{1, 2, 3, 5\}$

c) $\varnothing$

d) $\{2, 5\}$

Answer: d

The Union of Two Events

$A \cup B$ is the event that at least one of A or B occur.

Example

Suppose the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.
If $A = \{2, 4, 6\}$ and $B = \{2, 3, 4\}$ then $A \cup B = \{2, 3, 4, 6\}$.

Example

Suppose the sample space is $\Omega = \{HH, HT, TH, TT\}$.
If $A = \{HH, TT\}$ and $B = \{HT, TH\}$ then $A \cup B = \Omega$.

Question

Suppose the sample space is $\Omega = \{\text{North}, \text{South}, \text{East}, \text{West}\}$.

If $A = \{\text{East}, \text{North}\}$ and $B = \{\text{West}, \text{East}\}$ then what is $A \cup B$?

a) $\{\text{North}, \text{East}, \text{West}\}$

b) $\{\text{North}, \text{West}\}$

c) $\{\text{East}\}$

d) $\{\text{East}, \text{West}\}$

Question

Suppose the sample space is $\Omega = \{\text{North}, \text{South}, \text{East}, \text{West}\}$.

If $A = \{\text{East}, \text{North}\}$ and $B = \{\text{West}, \text{East}\}$ then what is $A \cup B$?

a) $\{\text{North}, \text{East}, \text{West}\}$

b) $\{\text{North}, \text{West}\}$

c) $\{\text{East}\}$

d) $\{\text{East}, \text{West}\}$

Answer:

Question

Suppose the sample space is $\Omega = \{\text{North}, \text{South}, \text{East}, \text{West}\}$.

If $A = \{\text{East}, \text{North}\}$ and $B = \{\text{West}, \text{East}\}$ then what is $A \cup B$?

a) $\{\text{North}, \text{East}, \text{West}\}$

b) $\{\text{North}, \text{West}\}$

c) $\{\text{East}\}$

d) $\{\text{East}, \text{West}\}$

Answer: a

The Rule of Complements

If A is an event then $P(A)$ denotes the probability of A.

The probability of A^c is given by

$$P(A^c) = 1 - P(A)$$

Example

If $P(A) = 0.3$ then $P(A^c) = 0.7$

Example

If $P(B) = 0.9$ then $P(B^c) = 0.1$

Mutually Exclusive Events

Events are **mutually exclusive** if at most one of them can occur.

In other words, events A and B are mutually exclusive if $A \cap B = \varnothing$.

Events $A_1, A_2, \ldots, A_n$ are mutually exclusive if $A_i \cap A_j = \varnothing$ for any $i \neq j$.

Example

If $A_1 = \{x, y\}$, $A_2 = \{w, z\}$, and $A_3 = \{j, k\}$
then A_1, A_2, A_3 are mutually exclusive.

Example

If B is an event then B and B^c are mutually exclusive.

Question

Suppose the sample space is $\Omega = \{q, r, s, t, u, v, w, x, y\}$.

Let $J = \{t, w\}$, $K = \{v, u\}$, and $L = \{r, t\}$.

Are J, K, L mutually exclusive?

a) Yes

b) No

Question

Suppose the sample space is $\Omega = \{q, r, s, t, u, v, w, x, y\}$.

Let $J = \{t, w\}$, $K = \{v, u\}$, and $L = \{r, t\}$.

Are J, K, L mutually exclusive?

a) Yes

b) No

Answer:

Question

Suppose the sample space is $\Omega = \{q, r, s, t, u, v, w, x, y\}$.

Let $J = \{t, w\}$, $K = \{v, u\}$, and $L = \{r, t\}$.

Are J, K, L mutually exclusive?

a) Yes

b) No

Answer: b since $J \cap L = \{t\}$

The Addition Rule for Mutually Exclusive Events

If events $A_1, \ldots, A_n$ are mutually exclusive then

$$P\left(A_1 \cup \cdots \cup A_n\right) = P\left(A_1\right) + \cdots + P\left(A_n\right)$$

Example

If A and B are mutually exclusive, $P\left(A\right) = 0.3$, and $P\left(B\right) = 0.2$ then $P\left(A \cup B\right) = 0.3 + 0.2 = 0.5$

Example

If H_1, H_2, H_3 are mutually exclusive $P\left(H_1\right) = 0.4$, $P\left(H_2\right) = 0.2$, and $P\left(H_3\right) = 0.3$ then $P\left(H_1 \cup H_2 \cup H_3\right) = 0.4 + 0.2 + 0.3 = 0.9$

Question

Suppose the sample space is $\Omega = \{q, r, s, t, u, v, w, x, y\}$.

Let $J = \{q, r, s, t, u\}$, $K = \{q, r\}$, $L = \{t, u\}$, and $M = \{s\}$.

Suppose $P(J) = 0.9$ and $P(K) = 0.3$ and $P(L) = 0.4$

What is $P(J^c \cup K \cup L)$?

Question

Suppose the sample space is $\Omega = \{q, r, s, t, u, v, w, x, y\}$.

Let $J = \{q, r, s, t, u\}$, $K = \{q, r\}$, $L = \{t, u\}$, and $M = \{s\}$.

Suppose $P(J) = 0.9$ and $P(K) = 0.3$ and $P(L) = 0.4$

What is $P(J^c \cup K \cup L)$?

Answer:

Question

Suppose the sample space is $\Omega = \{q, r, s, t, u, v, w, x, y\}$.

Let $J = \{q, r, s, t, u\}$, $K = \{q, r\}$, $L = \{t, u\}$, and $M = \{s\}$.

Suppose $P(J) = 0.9$ and $P(K) = 0.3$ and $P(L) = 0.4$

What is $P(J^c \cup K \cup L)$?

Answer: 0.8

$$
\begin{aligned}
J^c &= \{v, w, x, y\} \\
P(J^c) &= 1 - P(J) = 1 - 0.9 = 0.1 \\
P(J^c \cup K \cup L) &= 0.1 + 0.3 + 0.4 \\
P(J^c \cup K \cup L) &= 0.8
\end{aligned}
$$

Question

Suppose the sample space is $\Omega = \{q, r, s, t, u, v, w, x, y\}$.

Let $J = \{q, r, s, t, u\}$, $K = \{q, r\}$, $L = \{t, u\}$, and $M = \{s\}$.

Suppose $P(J) = 0.9$ and $P(K) = 0.3$ and $P(L) = 0.4$

What is the probability of M?

Question

Suppose the sample space is $\Omega = \{q, r, s, t, u, v, w, x, y\}$.

Let $J = \{q, r, s, t, u\}$, $K = \{q, r\}$, $L = \{t, u\}$, and $M = \{s\}$.

Suppose $P(J) = 0.9$ and $P(K) = 0.3$ and $P(L) = 0.4$

What is the probability of M?

Answer:

Question

Suppose the sample space is $\Omega = \{q, r, s, t, u, v, w, x, y\}$.

Let $J = \{q, r, s, t, u\}$, $K = \{q, r\}$, $L = \{t, u\}$, and $M = \{s\}$.

Suppose $P(J) = 0.9$ and $P(K) = 0.3$ and $P(L) = 0.4$

What is the probability of M?

Answer: 0.2

$$
\begin{aligned}
J &= \{q, r, s, t, u\} = K \cup L \cup M \\
P(J) &= P(K) + P(L) + P(M) \\
0.9 &= 0.3 + 0.4 + P(M) \\
0.9 &= 0.7 + P(M) \\
0.2 &= P(M)
\end{aligned}
$$

Exhaustive Events

Events are **exhaustive** if at least one of them must occur.

In other words, A and B are exhaustive if $A \cup B = \Omega$.

Events $A_1, \ldots, A_n$ are exhaustive if $A_1 \cup \cdots \cup A_n = \Omega$.

Example

If $\Omega = \{1, 2, 3, 4, 5\}$ and $A = \{1, 2\}$ and $B = \{2, 3, 4\}$ and $G = \{4, 5\}$ then A, B, G are exhaustive.

Example

If B is an event then B and B^c are exhaustive.

The Law of Total Probability

If $A_1, \ldots, A_n$ are exhaustive and mutually exclusive then

$$P(B) = P(B \cap A_1) + \cdots + P(B \cap A_n)$$

Example

Suppose that $P(B \cap A) = 0.3$ and $P(B \cap A^c) = 0.1$

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A^c) \\ &= 0.3 + 0.1 \\ &= 0.4 \end{aligned}$$

Question

Suppose that $P(B) = 0.65$ and $P(B \cap A) = 0.2$

What is $P(B \cap A^c)$?

Question

Suppose that $P(B) = 0.65$ and $P(B \cap A) = 0.2$

What is $P(B \cap A^c)$?

Answer:

Question

Suppose that $P(B) = 0.65$ and $P(B \cap A) = 0.2$

What is $P(B \cap A^c)$?

Answer: 0.45 since A and A^c are exhaustive and mutually exclusive

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A^c) \\ 0.65 &= 0.2 + P(B \cap A^c) \\ 0.45 &= P(B \cap A^c) \end{aligned}$$

The General Addition Rule

If A and B are events then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example

If $\Omega = \{w, x, y, z\}$ and $A = \{x, y\}$ and $B = \{y, z\}$ and $P(A) = P(B) = 0.5$ and $P(\{y\}) = 0.25$ then

$$\begin{aligned}
A \cup B &= \{x, y, z\} \\
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= P(A) + P(B) - P(\{y\}) \\
&= 0.5 + 0.5 - 0.25 \\
&= 0.75
\end{aligned}$$

Question

Suppose that $P(A) = 0.7$ and $P(B) = 0.8$ and $P(A \cup B) = 0.94$

What is $P(A \cap B)$?

Question

Suppose that $P(A) = 0.7$ and $P(B) = 0.8$ and $P(A \cup B) = 0.94$

What is $P(A \cap B)$?

Answer:

Question

Suppose that $P(A) = 0.7$ and $P(B) = 0.8$ and $P(A \cup B) = 0.94$

What is $P(A \cap B)$?

Answer: 0.56

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
0.94 &= 0.7 + 0.8 - P(A \cap B) \\
P(A \cap B) &= 0.7 + 0.8 - 0.94 \\
P(A \cap B) &= 0.56
\end{aligned}$$

Conditional Probability

- If new information becomes available, probabilities can change.

- **Conditional probability** describes how to revise probabilities on the basis of new information.

- Let A and B be events with probabilities $P(A)$ and $P(B)$

- The **conditional probability** of A given B is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example

Let $\Omega = \{1, 2, 3, 4\}$.

Let $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$, and $A_4 = \{4\}$.

Suppose $P(A_1) = P(A_2) = P(A_3) = P(A_4) = \frac{1}{4}$.

If $G = \{2, 3, 4\}$ and $H = \{1, 2, 3\}$ then

$$\begin{aligned} P(G|H) &= \frac{P(G \cap H)}{P(H)} \\ &= \frac{P(\{2,3\})}{P(\{1,2,3\})} \\ &= \frac{2}{3} \end{aligned}$$

Question

If $P(B) = 0.8$ and $P(A \cap B) = 0.2$ then what is $P(A|B)$?

Question

If $P(B) = 0.8$ and $P(A \cap B) = 0.2$ then what is $P(A|B)$?

Answer:

Question

If $P(B) = 0.8$ and $P(A \cap B) = 0.2$ then what is $P(A|B)$?

Answer: 0.25

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.2}{0.8} = \frac{1}{4} = 0.25 \end{aligned}$$

Question

If $P(A) = 0.5$ and $P(B) = 0.5$ and $P(A \cup B) = 0.7$

then what is $P(A|B)$?

Question

If $P(A) = 0.5$ and $P(B) = 0.5$ and $P(A \cup B) = 0.7$

then what is $P(A|B)$?

Answer:

Question

If $P(A) = 0.5$ and $P(B) = 0.5$ and $P(A \cup B) = 0.7$

then what is $P(A|B)$?

Answer: 0.6

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 0.7 &= 0.5 + 0.5 - P(A \cap B) \\ P(A \cap B) &= 0.3 \end{aligned}$$

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.3}{0.5} = \frac{3}{5} = 0.6 \end{aligned}$$

The Multiplication Rule

If we start with the conditional probability formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

and multiply both sides by $P(B)$, then we get the multiplication rule

$$P(B)\,P(A|B) = P(A \cap B)$$

Example

If $P(J) = \frac{1}{3}$ and $P(K|J) = \frac{3}{4}$ then

$$\begin{aligned} P(J \cap K) &= P(J)\,P(J|K) \\ &= \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4} \end{aligned}$$

Question

If $P(B) = 0.8$ and $P(A|B) = 0.2$ then what is $P(A \cap B)$?

Question

If $P(B) = 0.8$ and $P(A|B) = 0.2$ then what is $P(A \cap B)$?

Answer:

Question

If $P(B) = 0.8$ and $P(A|B) = 0.2$ then what is $P(A \cap B)$?

Answer: 0.16

$$\begin{aligned} P(A \cap B) &= P(B)\,P(A|B) \\ &= (0.8)\,(0.2) = 0.16 \end{aligned}$$

Question

If $P(G|K) = 0.2$, $P(K) = 0.2$, and $P(G) = 0.16$ then what is $P(K|G)$?

Question

If $P(G|K) = 0.2$, $P(K) = 0.2$, and $P(G) = 0.16$ then what is $P(K|G)$?

Answer:

Question

If $P(G|K) = 0.2$, $P(K) = 0.2$, and $P(G) = 0.16$ then what is $P(K|G)$?

Answer: 0.25

$$\begin{aligned} P(K|G) &= \frac{P(K \cap G)}{P(G)} \\ &= \frac{P(K)\,P(G|K)}{P(G)} \\ &= \frac{(0.2)\,(0.2)}{(0.16)} \\ &= \frac{0.04}{0.16} = 0.25 \end{aligned}$$

Question

If $P(G|K) = 0.2$, $P(K) = 0.2$, and $P(G) = 0.16$ then what is $P(K|G)$?

Question

If $P(G|K) = 0.2$, $P(K) = 0.2$, and $P(G) = 0.16$ then what is $P(K|G)$?

Answer:

Question

If $P(G|K) = 0.2$, $P(K) = 0.2$, and $P(G) = 0.16$ then what is $P(K|G)$?

Answer: 0.25

$$
\begin{aligned}
P(K|G) &= \frac{P(K \cap G)}{P(G)} \\
&= \frac{P(K)\,P(G|K)}{P(G)} \\
&= \frac{(0.2)\,(0.2)}{(0.16)} \\
&= \frac{0.04}{0.16} = 0.25
\end{aligned}
$$

Independent Events

- Two events are **independent** if knowledge of of one event provides no information about the probability of the other event.
- Events A and B are independent if $P(A \cap B) = P(A)\,P(B)$
- Events $A_1, A_2, \ldots, A_n$ are independent if

$$P(A_i \cap A_j) = P(A_i)\,P(A_j) \quad \text{for } i \neq j$$

Example

If A, B, C are independent $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{2}{3}$ then

$$\begin{aligned} P(A \cap B \cap C) &= P(A)\,P(B)\,P(C) \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{1}{9} \end{aligned}$$

Question

If A and B are independent, $P(A) = 0.25$, and $P(B) = 0.75$

then what is the probability of $P(A|B)$?

Question

If A and B are independent, $P\left(A\right) = 0.25$, and $P\left(B\right) = 0.75$

then what is the probability of $P\left(A|B\right)$?

Answer:

Question

If A and B are independent, $P(A) = 0.25$, and $P(B) = 0.75$

then what is the probability of $P(A|B)$?

Answer: 0.25

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)\,P(B)}{P(B)} \\ &= P(A) \\ &= 0.25 \end{aligned}$$

Question

If $P(K) = 0.25$, $P(G|K) = 0.2$, and $P(G|K^c) = 0.1$

then what is $P(K|G)$?

Question

If $P(K) = 0.25$, $P(G|K) = 0.2$, and $P(G|K^c) = 0.1$

then what is $P(K|G)$?

Answer:

Question

If $P(K) = 0.25$, $P(G|K) = 0.2$, and $P(G|K^c) = 0.1$

then what is $P(K|G)$?

Answer: 0.4

$$P(G \cap K) = P(K)\,P(G|K) = (0.25)(0.2) = 0.05$$

$$P(G \cap K^c) = P(K^c)\,P(G|K^c) = (0.75)(0.1) = 0.075$$

$$\begin{aligned} P(K|G) &= \frac{P(K \cap G)}{P(G)} \\ &= \frac{P(K \cap G)}{P(G \cap K) + P(G \cap K^c)} \\ &= \frac{0.05}{0.05 + 0.075} = 0.4 \end{aligned}$$

Question

Suppose a patient goes to see a doctor. Let S denote the event that the patient is sick. Only 1% of the population is sick, so $P(S) = 0.01$. The doctor performs a test with 99% reliability, so 99% of sick people test positive and 99% of healthy people test negative. Let T denote the event that the patient tests positive, so $P(T|S) = 0.99$ and $P(T^c|S^c) = 0.99$. What is the probability that the patient is sick, given the patient tests positive?

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Answer:

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Answer:

$$P(S|T) = \frac{P(T \cap S)}{P(T)}$$

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Answer:

$$P(S|T) = \frac{P(T \cap S)}{P(T)} = \frac{P(T \cap S)}{P(T \cap S) + P(T \cap S^c)}$$

Question

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Answer:

$$\begin{aligned} P(S|T) &= \frac{P(T \cap S)}{P(T)} = \frac{P(T \cap S)}{P(T \cap S) + P(T \cap S^c)} \\ &= \frac{P(S)\,P(T|S)}{P(S)\,P(T|S) + P(S^c)\,P(T|S^c)} \end{aligned}$$

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Answer:

$$\begin{aligned} P(S|T) &= \frac{P(T \cap S)}{P(T)} = \frac{P(T \cap S)}{P(T \cap S) + P(T \cap S^c)} \\ &= \frac{P(S)P(T|S)}{P(S)P(T|S) + P(S^c)P(T|S^c)} \\ &= \frac{(0.01)(0.99)}{(0.01)(0.99) + (0.99)(0.01)} \end{aligned}$$

Question

Suppose a patient goes to see a doctor. Let S denote the event that the patient is sick. Only 1% of the population is sick, so $P(S) = 0.01$. The doctor performs a test with 99% reliability, so 99% of sick people test positive and 99% of healthy people test negative. Let T denote the event that the patient tests positive, so $P(T|S) = 0.99$ and $P(T^c|S^c) = 0.99$. What is the probability that the patient is sick, given the patient tests positive?

Answer: 0.5

$$\begin{aligned} P(S|T) &= \frac{P(T \cap S)}{P(T)} = \frac{P(T \cap S)}{P(T \cap S) + P(T \cap S^c)} \\ &= \frac{P(S)\,P(T|S)}{P(S)\,P(T|S) + P(S^c)\,P(T|S^c)} \\ &= \frac{(0.01)\,(0.99)}{(0.01)\,(0.99) + (0.99)\,(0.01)} \\ &= \frac{(0.0095)}{(0.0095) + (0.0095)} = 0.5 \end{aligned}$$

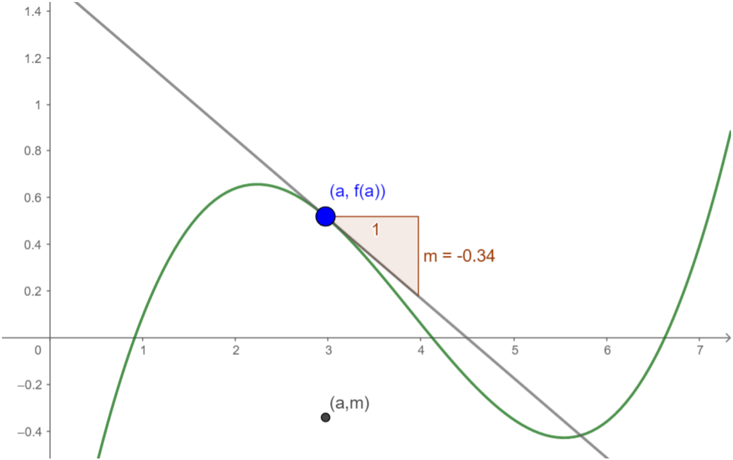
Calculus Review

Daniel Stephenson

SCMA 524: Statistical Fundamentals

The Derivative of a Function

- The **derivative** of a function $f(x)$ at an input value a is the slope of the tangent line to the graph of the function at a.



The Derivative of a Function

- The **derivative** of a function $f(x)$ at an input value a is the slope of the tangent line to the graph of the function at a.

- We write the derivative a function $f(x)$ at an input value c as

$$f'(c) \text{ or } \frac{df}{dx}(c)$$

The Power Rule

- If $f(x) = x^a$ then $f'(x) = a\,x^{a-1}$

Example

If $f(x) = x^3$ then $f'(x) = 3x^2$ so

$$f'(5) = 3(5)^2 = 3(25) = 75$$

Example

If $f(x) = x$ then $f'(x) = 1x^0 = 1 \cdot 1 = 1$ so $f'(5) = 1$

Example

If $f(x) = 1 = x^0$ then $f'(x) = 0x^{-1} = 0$ so $f'(5) = 0$

Question

If $f(x) = x^4$ then what is $\frac{df}{dx}(2)$?

Question

If $f(x) = x^4$ then what is $\frac{df}{dx}(2)$?

Answer:

Question

If $f(x) = x^4$ then what is $\frac{df}{dx}(2)$?

Answer: 32

$$\begin{aligned} f(x) &= x^4 \\ \frac{df}{dx} = f'(x) &= 4x^3 \\ \frac{df}{dx}(2) = f'(2) &= 4\left(2^3\right) = 4(8) = 32 \end{aligned}$$

Question

If $f(x) = x^{-2}$ then what is $f'(2)$?

Question

If $f(x) = x^{-2}$ then what is $f'(2)$?

Answer:

Question

If $f(x) = x^{-2}$ then what is $f'(2)$?

Answer: -0.25

$$
\begin{aligned}
f(x) &= x^{-2} \\
f'(x) &= -2x^{-3} \\
f'(2) &= -2\left(2^{-3}\right) \\
&= -2\left(\frac{1}{2^3}\right) \\
&= -2\left(\frac{1}{8}\right) = -\frac{1}{4} = -0.25
\end{aligned}
$$

Linearity of the Derivative

- If $h(x) = a\,f(x) + b\,g(x)$ then

$$h'(x) = a\,f'(x) + b\,g'(x)$$

Example

If $f(x) = 4x^2 + 5x$ then $f'(x) = 8x + 5$ so

$$f'(5) = 8(5) + 5 = 45$$

Example

If $f(x) = 2x^2 + 3$ then $f'(x) = 2x + 0$ so

$$f'(5) = 4(5) = 20$$

Question

If $f(x) = x^{-1} + x$ then what is $f'(0.5)$?

Question

If $f(x) = x^{-1} + x$ then what is $f'(0.5)$?

Answer:

Question

If $f(x) = x^{-1} + x$ then what is $f'(0.5)$?

Answer: -3

$$
\begin{aligned}
f(x) &= x^{-1} + x \\
f'(x) &= -x^{-2} + 1 \\
f'(0.5) &= -(0.5)^{-2} + 1 \\
&= -\left(\frac{1}{0.5^2}\right) + 1 \\
&= -\left(\frac{1}{0.25}\right) + 1 \\
&= -4 + 1 = -3
\end{aligned}
$$

The Product Rule

- If $h(x) = f(x)\,g(x)$ then

$$h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

Example

If $f(x) = x^2\left(3x + x^2\right)$ then

$$f'(x) = 2x\left(3x + x^2\right) + x^2\left(3 + 2x\right)$$

Example

If $f(x) = x^2(5x)$ then

$$f'(x) = (2x)(5x) + \left(x^2\right)(5)$$

Question

If $f(x) = (2x + x^2)(7x)$ then what is $f'(3)$?

Question

If $f(x) = (2x + x^2)(7x)$ then what is $f'(3)$?

Answer:

Question

If $f(x) = (2x + x^2)(7x)$ then what is $f'(3)$?

Answer: 273

$$
\begin{aligned}
f(x) &= \left(2x + x^2\right)(7x) \\
f'(x) &= (2 + 2x)(7x) + \left(2x + x^2\right)(7) \\
f'(3) &= (2 + 6)\,21 + (6 + 9)\,7 \\
&= (8)\,21 + (15)\,7 \\
&= 224 + 70 = 273
\end{aligned}
$$

The Chain Rule

- If $h(x) = f(g(x))$ then $h'(x) = f'(g(x))\, g'(x)$

Example

If $f(x) = \left(x^2 + 5x\right)^2$ then

The Chain Rule

- If $h(x) = f(g(x))$ then $h'(x) = f'(g(x))\, g'(x)$

Example

If $f(x) = \left(x^2 + 5x\right)^2$ then

$$f'(x) = 2\left(x^2 + 5x\right)\frac{d}{dx}\left[x^2 + 5x\right]$$

The Chain Rule

- If $h(x) = f(g(x))$ then $h'(x) = f'(g(x))\, g'(x)$

Example

If $f(x) = (x^2 + 5x)^2$ then

$$\begin{aligned} f'(x) &= 2\left(x^2 + 5x\right) \frac{d}{dx}\left[x^2 + 5x\right] \\ &= 2\left(x^2 + 5x\right)(2x + 5) \end{aligned}$$

The Chain Rule

- If $h(x) = f(g(x))$ then $h'(x) = f'(g(x))\,g'(x)$

Example

If $f(x) = \left(x^2 + 5x\right)^2$ then

$$\begin{aligned} f'(x) &= 2\left(x^2 + 5x\right)\frac{d}{dx}\left[x^2 + 5x\right] \\ &= 2\left(x^2 + 5x\right)(2x + 5) \end{aligned}$$

Example

If $f(x) = 3(2x + 1)^2$ then

The Chain Rule

- If $h(x) = f(g(x))$ then $h'(x) = f'(g(x))\, g'(x)$

Example

If $f(x) = \left(x^2 + 5x\right)^2$ then

$$\begin{aligned} f'(x) &= 2\left(x^2 + 5x\right) \frac{d}{dx}\left[x^2 + 5x\right] \\ &= 2\left(x^2 + 5x\right)(2x + 5) \end{aligned}$$

Example

If $f(x) = 3(2x + 1)^2$ then

$$f'(x) = 6(2x + 1) \frac{d}{dx}\left[2x + 1\right]$$

The Chain Rule

- If $h(x) = f(g(x))$ then $h'(x) = f'(g(x))\, g'(x)$

Example

If $f(x) = (x^2 + 5x)^2$ then

$$\begin{aligned} f'(x) &= 2\left(x^2 + 5x\right) \frac{d}{dx}\left[x^2 + 5x\right] \\ &= 2\left(x^2 + 5x\right)(2x + 5) \end{aligned}$$

Example

If $f(x) = 3(2x + 1)^2$ then

$$\begin{aligned} f'(x) &= 6(2x + 1) \frac{d}{dx}\left[2x + 1\right] \\ &= 6(2x + 1)(2) = 24x + 12 \end{aligned}$$

Question

If $f(x) = \left(x + x^2\right)^2$ then what is $f'(2)$?

Question

If $f(x) = \left(x + x^2\right)^2$ then what is $f'(2)$?

Answer:

Question

If $f(x) = \left(x + x^2\right)^2$ then what is $f'(2)$?

Answer: 60

$$
\begin{aligned}
f(x) &= \left(x + x^2\right)^2 \\
f'(x) &= 2\left(x + x^2\right)(1 + 2x) \\
f'(2) &= 2(2+4)(1+4) \\
&= 2(6)(5) \\
&= 2(30) \\
&= 60
\end{aligned}
$$

Question

If $f(x) = 0.5\left(3x + x^2\right)^3$ then what is $f'(2)$?

Question

If $f(x) = 0.5\left(3x + x^2\right)^3$ then what is $f'(2)$?

Answer:

Question

If $f(x) = 0.5\left(3x + x^2\right)^3$ then what is $f'(2)$?

Answer: 1050

$$
\begin{aligned}
f(x) &= 0.5\left(3x + x^2\right)^3 \\
f'(x) &= 1.5\left(3x + x^2\right)^2 (3 + 2x) \\
f'(2) &= 1.5\,(6+4)^2\,(3+4) \\
&= 1.5\,(10)^2\,(7) \\
&= 1.5\,(100)\,(7) \\
&= 1050
\end{aligned}
$$

Optimization

- The first order condition for a minimum or maximum of f is

$$f'(x) = 0$$

Example

Suppose $f(x) = x^2 - 6x + 10$. How do we minimize f?

Optimization

- The first order condition for a minimum or maximum of f is

$$f'(x) = 0$$

Example

Suppose $f(x) = x^2 - 6x + 10$. How do we minimize f?

$$f'(x) = 2x - 6 \quad = \quad 0$$

Optimization

- The first order condition for a minimum or maximum of f is

$$f'(x) = 0$$

Example

Suppose $f(x) = x^2 - 6x + 10$. How do we minimize f?

$$\begin{aligned} f'(x) = 2x - 6 &= 0 \\ 2x &= 6 \end{aligned}$$

Optimization

- The first order condition for a minimum or maximum of f is

$$f'(x) = 0$$

Example

Suppose $f(x) = x^2 - 6x + 10$. How do we minimize f?

$$\begin{aligned} f'(x) = 2x - 6 &= 0 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

Optimization

- The first order condition for a minimum or maximum of f is

$$f'(x) = 0$$

Example

Suppose $f(x) = x^2 - 6x + 10$. How do we minimize f?

$$\begin{aligned} f'(x) = 2x - 6 &= 0 \\ 2x &= 6 \\ x &= 3 \\ \\ \operatorname{argmin} f(x) &= 3 \end{aligned}$$

Optimization

- The first order condition for a minimum or maximum of f is

$$f'(x) = 0$$

Example

Suppose $f(x) = x^2 - 6x + 10$. How do we minimize f?

$$\begin{aligned} f'(x) = 2x - 6 &= 0 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} \operatorname{argmin} f(x) &= 3 \\ \min\ f(x) &= (3)^2 - 5(3) + 10 \end{aligned}$$

Optimization

- The first order condition for a minimum or maximum of f is

$$f'(x) = 0$$

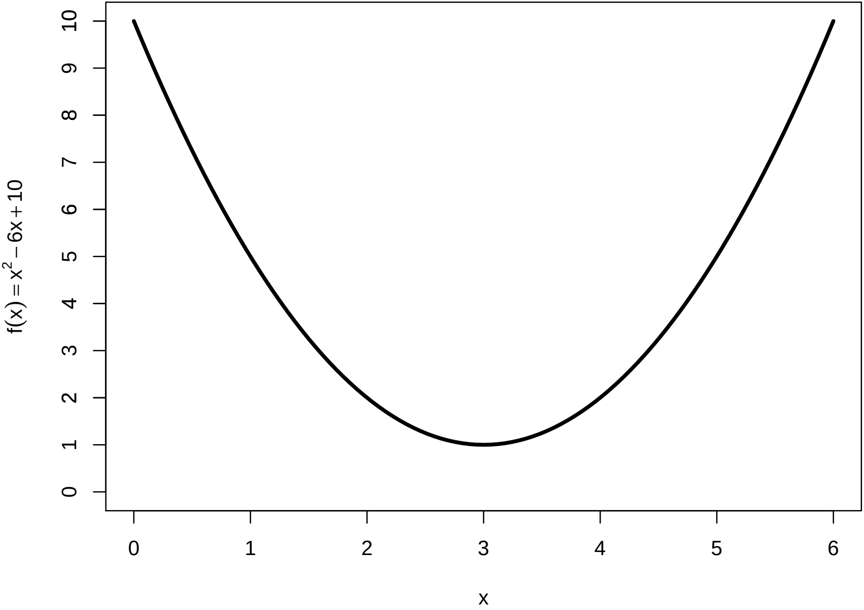
Example

Suppose $f(x) = x^2 - 6x + 10$. How do we minimize f?

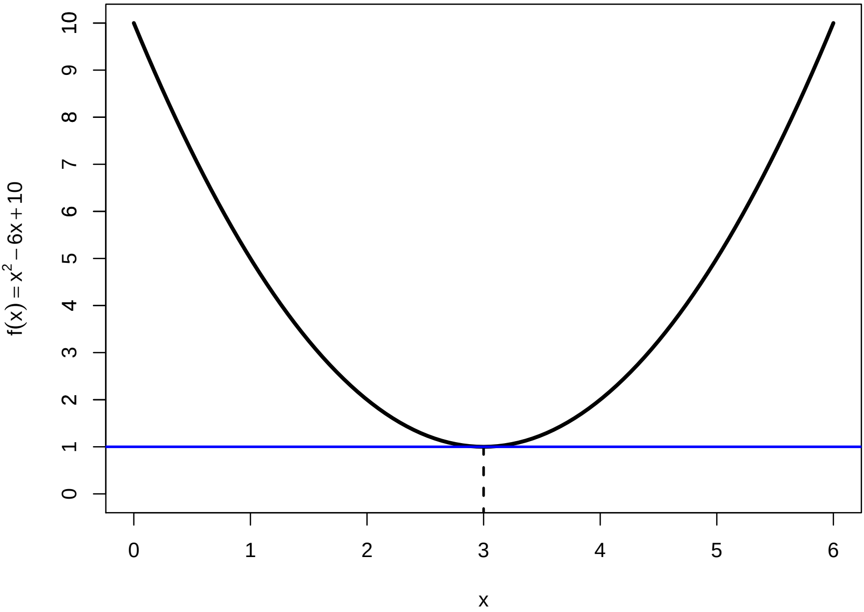
$$\begin{aligned} f'(x) = 2x - 6 &= 0 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} \operatorname{argmin} f(x) &= 3 \\ \min\ f(x) &= (3)^2 - 5(3) + 10 \\ &= 9 - 18 + 10 = 1 \end{aligned}$$

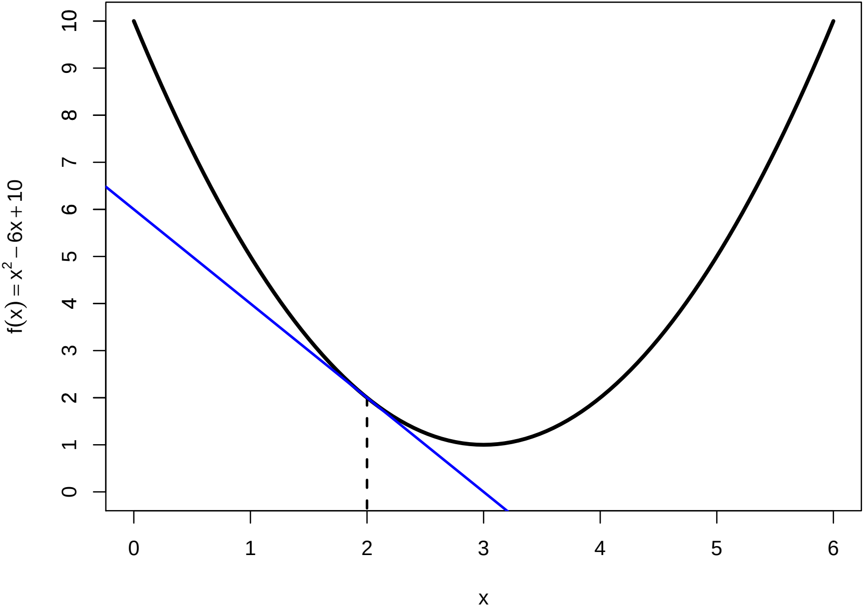
Optimization



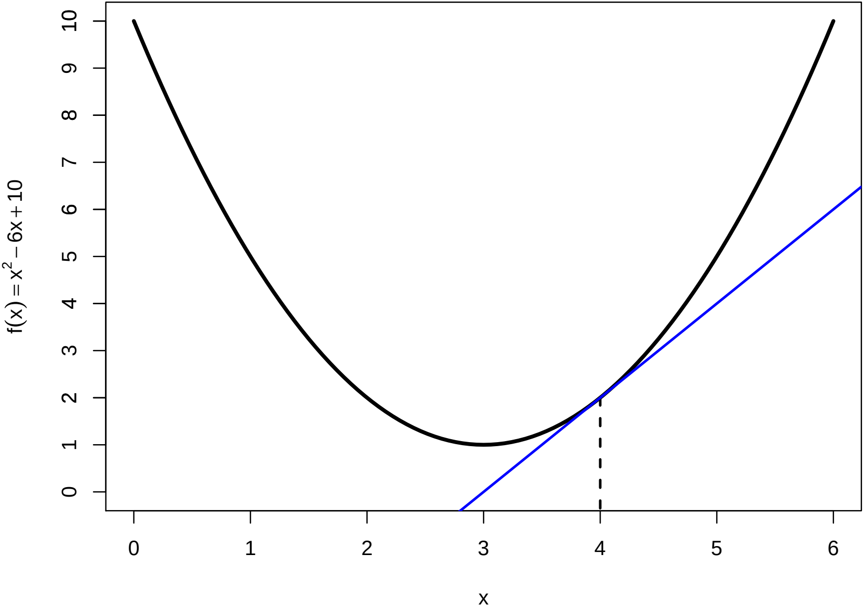
Optimization



Optimization



Optimization



Question

Let $f(x) = 5x^2 + x - 15$

What is $\operatorname{argmin} f(x)$?

Question

Let $f(x) = 5x^2 + x - 15$

What is $\operatorname{argmin} f(x)$?

Answer:

Question

Let $f(x) = 5x^2 + x - 15$

What is $\operatorname{argmin} f(x)$?

Answer: -0.1

$$\begin{aligned} f'(x) = 10x + 1 &= 0 \\ 10x &= -1 \\ x &= -0.1 \end{aligned}$$

Partial Derivatives

- The **partial derivative** of a function of several variables is the derivative with respect to one variable, holding the others constant.
- The partial derivative of $f(x, y)$ with respect to x at a point (a, b) is written as

$$\frac{\partial f}{\partial x}(a, b)$$

Example

If $f(x, y) = x^3 y + 3y^2$ then

$$\begin{aligned} \frac{\partial f}{\partial y}(x, y) &= x^3 + 6y \\ \frac{\partial f}{\partial y}(2, 1) &= 8 + 6 = 14 \end{aligned}$$

Question

If $f(x, y) = x^2 y$ then what is $\dfrac{\partial f}{\partial x}(1, 3)$?

Question

If $f(x, y) = x^2 y$ then what is $\dfrac{\partial f}{\partial x}(1, 3)$?

Answer:

Question

If $f(x, y) = x^2 y$ then what is $\frac{\partial f}{\partial x}(1, 3)$?

Answer: 6

$$\begin{aligned} f(x, y) &= x^2 y \\ \frac{\partial f}{\partial x}(x, y) &= 2xy \\ \frac{\partial f}{\partial x}(1, 3) &= 2(1)(3) = 6 \end{aligned}$$

Question

If $f\left(x, y\right) = x^2 y + 2xy^2$ then what is $\dfrac{\partial f}{\partial y}\left(3, 2\right)$?

Question

If $f(x, y) = x^2 y + 2xy^2$ then what is $\dfrac{\partial f}{\partial y}(3, 2)$?

Answer:

Question

If $f(x, y) = x^2y + 2xy^2$ then what is $\frac{\partial f}{\partial y}(3, 2)$?

Answer: 33

$$
\begin{aligned}
f(x, y) &= x^2y + 2xy^2 \\
\frac{\partial f}{\partial y}(x, y) &= x^2 + 4xy \\
\frac{\partial f}{\partial y}(3, 2) &= 3^2 + 4(3)(2) \\
&= 9 + 24 = 33
\end{aligned}
$$

Optimization

- The first order conditions for a minimum or maximum of $f(x, y)$ are

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

Example

Suppose $f(x, y) = x^2 + y^2 - xy - 9y$. What is argmin $f(x, y)$?

$$\frac{\partial f}{\partial x} = 2x - y \quad = \quad 0$$

Optimization

- The first order conditions for a minimum or maximum of $f(x,y)$ are

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

Example

Suppose $f(x,y) = x^2 + y^2 - xy - 9y$. What is argmin $f(x,y)$?

$$\begin{aligned} \frac{\partial f}{\partial x} = 2x - y &= 0 \\ 2x &= y \end{aligned}$$

Optimization

- The first order conditions for a minimum or maximum of $f(x, y)$ are

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

Example

Suppose $f(x, y) = x^2 + y^2 - xy - 9y$. What is argmin $f(x, y)$?

$$\begin{aligned} \frac{\partial f}{\partial x} = 2x - y &= 0 \\ 2x &= y \end{aligned} \qquad \frac{\partial f}{\partial y} = 2y - x - 9 = 0$$

Optimization

- The first order conditions for a minimum or maximum of $f(x, y)$ are

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

Example

Suppose $f(x, y) = x^2 + y^2 - xy - 9y$. What is argmin $f(x, y)$?

$$\begin{aligned} \frac{\partial f}{\partial x} = 2x - y &= 0 \\ 2x &= y \end{aligned} \qquad \begin{aligned} \frac{\partial f}{\partial y} = 2y - x - 9 &= 0 \\ 2(2x) - x - 9 &= 0 \end{aligned}$$

Optimization

- The first order conditions for a minimum or maximum of $f(x, y)$ are

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

Example

Suppose $f(x, y) = x^2 + y^2 - xy - 9y$. What is argmin $f(x, y)$?

$$\begin{aligned} \frac{\partial f}{\partial x} = 2x - y &= 0 \\ 2x &= y \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} = 2y - x - 9 &= 0 \\ 2(2x) - x - 9 &= 0 \\ 3x &= 9 \end{aligned}$$

Optimization

- The first order conditions for a minimum or maximum of $f(x,y)$ are

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

Example

Suppose $f(x,y) = x^2 + y^2 - xy - 9y$. What is argmin $f(x,y)$?

$$\begin{aligned} \frac{\partial f}{\partial x} = 2x - y &= 0 \\ 2x &= y \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} = 2y - x - 9 &= 0 \\ 2(2x) - x - 9 &= 0 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Optimization

- The first order conditions for a minimum or maximum of $f(x, y)$ are

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

Example

Suppose $f(x, y) = x^2 + y^2 - xy - 9y$. What is argmin $f(x, y)$?

$$\begin{aligned} \frac{\partial f}{\partial x} = 2x - y &= 0 \\ 2x &= y \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} = 2y - x - 9 &= 0 \\ 2(2x) - x - 9 &= 0 \\ 3x &= 9 \\ x &= 3 \\ y &= 2x = 6 \end{aligned}$$

Question

Let $f(x, y) = 4x^2 + 4y^2 - 2xy - 6x - 12y$

Let $(x^*, y^*) = \text{argmin } f(x, y)$

What is y^* ?

Question

Let $f\left(x,y\right)=4x^2+4y^2-2xy-6x-12y$

Let $\left(x^*,y^*\right)=\text{argmin}\ f\left(x,y\right)$

What is y^* ?

Answer:

Question

Let $f(x,y) = 4x^2 + 4y^2 - 2xy - 6x - 12y$

Let $(x^*, y^*) = \text{argmin } f(x,y)$

What is y^* ?

Answer: 1.8

$$\begin{aligned} \frac{\partial f}{\partial x} = 8x - 2y - 6 &= 0 \\ 8x - 6 &= 2y \\ 4x - 3 &= y \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} = 8y - 2x - 12 &= 0 \\ 8y &= 2x + 12 \\ 8(4x - 3) &= 2x + 12 \\ 32x - 24 &= 2x + 12 \\ 30x &= 36 \\ x &= \frac{6}{5} = 1.2 \\ y &= 4x - 3 = 1.8 \end{aligned}$$

Antiderivative

- An **antiderivative** a function f is a function F such that $F'(x) = f(x)$.
- If $f(x) = x^a$ then $F(x) = \frac{1}{a+1}x^{a+1} + C$ where C is a constant.

Example

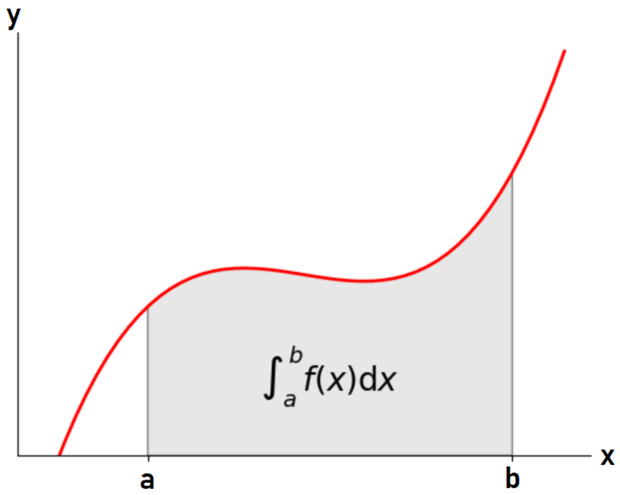
If $f(x) = x^2$ then $F(x) = \frac{1}{3}x^3 + C$.

Example

If $f(x) = x^3 + 5x$ then $F(x) = \frac{1}{4}x^4 + \frac{5}{2}x^2 + C$.

The Definite Integral

- The **definite integral** $\int_a^b f(x)\,dx$ is the area under the graph of f between a and b.



The Fundamental Theorem of Calculus

- If F is an antiderivative of f then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Example

If $f(x) = x^2$ then $F(x) = \frac{1}{3}x^3 + C$ so

The Fundamental Theorem of Calculus

- If F is an antiderivative of f then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Example

If $f(x) = x^2$ then $F(x) = \frac{1}{3}x^3 + C$ so

$$\int_3^6 f(x)\,dx \quad = \quad \Big[F(x)\Big]_3^6 = F(6) - F(3)$$

The Fundamental Theorem of Calculus

- If F is an antiderivative of f then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Example

If $f(x) = x^2$ then $F(x) = \frac{1}{3}x^3 + C$ so

$$\begin{aligned} \int_3^6 f(x)\,dx &= \Big[F(x)\Big]_3^6 = F(6) - F(3) \\ &= \frac{1}{3}(6)^2 - \frac{1}{3}(3)^2 \end{aligned}$$

The Fundamental Theorem of Calculus

- If F is an antiderivative of f then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Example

If $f(x) = x^2$ then $F(x) = \frac{1}{3}x^3 + C$ so

$$\begin{aligned}
\int_3^6 f(x)\,dx &= \Big[F(x)\Big]_3^6 = F(6) - F(3) \\
&= \frac{1}{3}(6)^2 - \frac{1}{3}(3)^2 \\
&= 12 - 3 = 9
\end{aligned}$$

Question

What is $\int_0^2 x^3 dx$?

Question

What is $\int_0^2 x^3 dx$?

Answer:

Question

What is $\int_0^2 x^3 dx$?

Answer: 4

$$
\begin{aligned}
f(x) &= x^3 \\
F(x) &= \frac{1}{4}x^4 + C \\
\int_0^2 x^3 dx &= \Big[F(x)\Big]_0^2 \\
&= F(2) - F(0) \\
&= \frac{1}{4}(2)^4 - \frac{1}{4}(0)^4 \\
&= \frac{1}{4}(16) - 0 = 4
\end{aligned}
$$

Linearity of Integration

- Integration is a linear operation

$$\int_a^b [cf(x) + kg(x)]\,dx = c\int_a^b f(x)\,dx + k\int_a^b g(x)\,dx$$

Example

$$\int_1^2 \left[5x + 6x^2\right] dx \quad = \quad 5\int_1^2 x\,dx + 6\int_1^2 x^2\,dx$$

Linearity of Integration

- Integration is a linear operation

$$\int_a^b \left[cf(x) + kg(x)\right] dx = c\int_a^b f(x)\,dx + k\int_a^b g(x)\,dx$$

Example

$$\begin{aligned}\int_1^2 \left[5x + 6x^2\right] dx &= 5\int_1^2 x\,dx + 6\int_1^2 x^2\,dx \\ &= 5\left[\frac{1}{2}x^2\right]_1^2 + 6\left[\frac{1}{3}x^3\right]_1^2\end{aligned}$$

Linearity of Integration

- Integration is a linear operation

$$\int_a^b [cf(x) + kg(x)]\, dx = c\int_a^b f(x)\, dx + k\int_a^b g(x)\, dx$$

Example

$$\begin{aligned}
\int_1^2 \left[5x + 6x^2\right] dx &= 5\int_1^2 x\, dx + 6\int_1^2 x^2\, dx \\
&= 5\left[\frac{1}{2}x^2\right]_1^2 + 6\left[\frac{1}{3}x^3\right]_1^2 \\
&= \frac{5}{2}\left[x^2\right]_1^2 + \frac{1}{2}\left[x^3\right]_1^2
\end{aligned}$$

Linearity of Integration

- Integration is a linear operation

$$\int_a^b [cf(x) + kg(x)]\, dx = c\int_a^b f(x)\, dx + k\int_a^b g(x)\, dx$$

Example

$$\begin{aligned}
\int_1^2 \left[5x + 6x^2\right] dx &= 5\int_1^2 x\, dx + 6\int_1^2 x^2\, dx \\
&= 5\left[\frac{1}{2}x^2\right]_1^2 + 6\left[\frac{1}{3}x^3\right]_1^2 \\
&= \frac{5}{2}\left[x^2\right]_1^2 + \frac{1}{2}\left[x^3\right]_1^2 \\
&= \frac{5}{2}\left[4 - 1\right] + \frac{1}{2}\left[8 - 1\right]
\end{aligned}$$

Linearity of Integration

- Integration is a linear operation

$$\int_a^b \left[cf(x) + kg(x)\right] dx = c\int_a^b f(x)\,dx + k\int_a^b g(x)\,dx$$

Example

$$\begin{aligned}
\int_1^2 \left[5x + 6x^2\right] dx &= 5\int_1^2 x\,dx + 6\int_1^2 x^2\,dx \\
&= 5\left[\frac{1}{2}x^2\right]_1^2 + 6\left[\frac{1}{3}x^3\right]_1^2 \\
&= \frac{5}{2}\left[x^2\right]_1^2 + \frac{1}{2}\left[x^3\right]_1^2 \\
&= \frac{5}{2}\left[4-1\right] + \frac{1}{2}\left[8-1\right] \\
&= \frac{5}{2}\left[3\right] + \frac{1}{2}\left[7\right] = 11
\end{aligned}$$

Question

If $f(x) = 3$ then what is $\int_2^5 f(x)\,dx$?

Question

If $f(x) = 3$ then what is $\int_2^5 f(x)\, dx$?

Answer:

Question

If $f(x) = 3$ then what is $\int_2^5 f(x)\,dx$?

Answer: 9

$$\begin{aligned}
f(x) &= 3 = 3x^0 \\
F(x) &= 3x + C \\
\int_2^5 3\,dx &= \Big[3x\Big]_2^5 \\
&= 3(5) - 3(2) \\
&= 15 - 6 \\
&= 9
\end{aligned}$$

Question

What is $\int_1^4 (1 + x)\,dx$?

Question

What is $\int_1^4 (1 + x)\, dx$?

Answer:

Question

What is $\int_1^4 (1+x)\,dx$?

Answer: 8

$$
\begin{aligned}
f(x) &= 1+x \\
F(x) &= x+\frac{1}{2}x^2+C \\
\int_1^4 (1+x)\,dx &= \left[x+\frac{1}{2}x^2\right]_2^4 \\
&= \left((4)+\frac{1}{2}(4)^2\right)-\left((2)+\frac{1}{2}(2)^2\right) \\
&= \left(4+\frac{1}{2}(16)\right)-\left(2+\frac{1}{2}(4)\right) \\
&= 12-4=8
\end{aligned}
$$

Additivity of Integration on Intervals

- If $a < b < c$ then

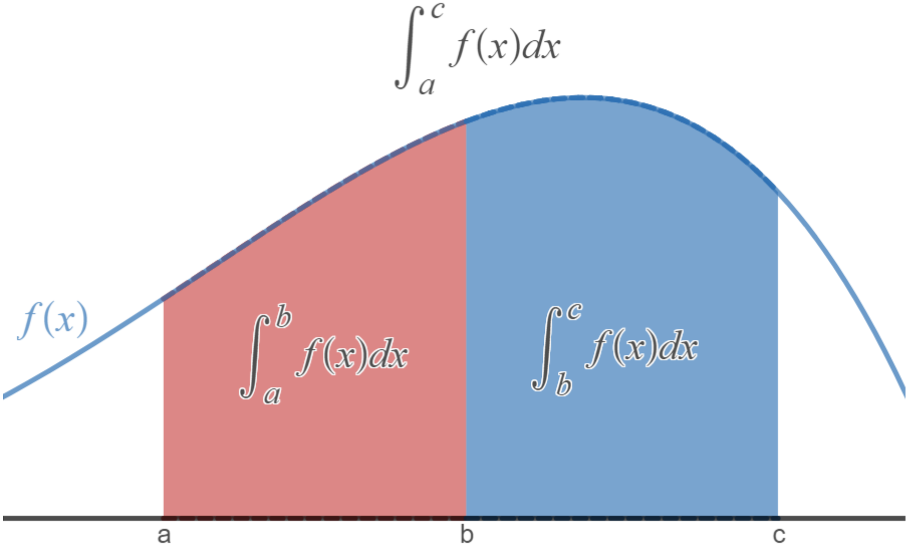
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Example

If $\int_a^b f(x)\,dx = 5$ and $\int_b^c f(x)\,dx = 3$ then

$$\begin{aligned}\int_a^c f(x)\,dx &= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \\ &= 5 + 3 = 8\end{aligned}$$

Additivity of Integration on Intervals



Improper Integrals

- An **improper integral** is the limit of a definite integral as the endpoints approach ∞ or $-\infty$

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\lim_{b\to\infty}\int_a^b f(x)\,dx$$

Improper Integrals

Example

$$\int_1^{\infty} x^{-3} dx = \lim_{b \to \infty} \int_1^{b} x^{-3} dx$$

Improper Integrals

Example

$$\begin{aligned}
\int_1^{\infty} x^{-3} dx &= \lim_{b\to\infty} \int_1^b x^{-3} dx \\
&= \lim_{b\to\infty} \left[-\frac{1}{2}x^{-2}\right]_1^b
\end{aligned}$$

Improper Integrals

Example

$$
\begin{aligned}
\int_1^{\infty} x^{-3} dx &= \lim_{b \to \infty} \int_1^{b} x^{-3} dx \\
&= \lim_{b \to \infty} \left[-\frac{1}{2} x^{-2} \right]_1^{b} \\
&= \lim_{b \to \infty} \left(-\frac{1}{2} b^{-2} \right) - \left(-\frac{1}{2} (1)^{-2} \right)
\end{aligned}
$$

Improper Integrals

Example

$$
\begin{aligned}
\int_1^{\infty} x^{-3} dx &= \lim_{b\to\infty} \int_1^{b} x^{-3} dx \\
&= \lim_{b\to\infty} \left[-\frac{1}{2}x^{-2}\right]_1^b \\
&= \lim_{b\to\infty} \left(-\frac{1}{2}b^{-2}\right) - \left(-\frac{1}{2}(1)^{-2}\right) \\
&= \lim_{b\to\infty} \left(-\frac{1}{2b^2}\right) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}
\end{aligned}
$$

Question

What is $\int_1^\infty x^{-2} dx$?

Question

What is $\int_1^\infty x^{-2} dx$?

Answer:

Question

What is $\int_1^\infty x^{-2}dx$?

Answer: 1

$$\begin{aligned}
\int_1^\infty x^{-2}dx &= \lim_{b\to\infty}\int_1^b x^{-2}dx \\
&= \lim_{b\to\infty}\left[-x^{-1}\right]_1^b \\
&= \lim_{b\to\infty}\left(-b^{-1}\right)-\left(-(1)^{-1}\right) \\
&= \lim_{b\to\infty}\left(-\frac{1}{b}\right)+1=0+1=1
\end{aligned}$$

Question

What is $\int_1^\infty x^{-5} dx$?

Question

What is $\int_1^{\infty} x^{-5} dx$?

Answer:

Question

What is $\int_1^{\infty} x^{-5} dx$?

Answer: 0.25

$$\begin{aligned}
\int_1^{\infty} x^{-5} dx &= \lim_{b\to\infty} \int_1^{b} x^{-5} dx \\
&= \lim_{b\to\infty} \left[-\frac{1}{4}x^{-4}\right]_1^b \\
&= \lim_{b\to\infty} \left(-\frac{1}{4}b^{-4}\right) - \left(-\frac{1}{4}(1)^{-4}\right) \\
&= \lim_{b\to\infty} \left(-\frac{1}{4b^4}\right) + \frac{1}{4} = 0 + \frac{1}{4} = 0.25
\end{aligned}$$

Improper Integrals

Example

$$F(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x^2 & \text{if } 1 < x < 3 \\ 9 & \text{if } x \geq 3 \end{cases}$$

Improper Integrals

Example

$$F(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x^2 & \text{if } 1 < x < 3 \\ 9 & \text{if } x \geq 3 \end{cases} \qquad F'(x) = f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2x & \text{if } 1 < x < 3 \\ 0 & \text{if } x > 3 \end{cases}$$

Improper Integrals

Example

$$F(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x^2 & \text{if } 1 < x < 3 \\ 9 & \text{if } x \geq 3 \end{cases} \qquad F'(x) = f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2x & \text{if } 1 < x < 3 \\ 0 & \text{if } x > 3 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\, dx \quad = \quad \int_{-\infty}^{1} f(x)\, dx + \int_{1}^{3} f(x)\, dx + \int_{3}^{\infty} f(x)\, dx$$

Improper Integrals

Example

$$F(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x^2 & \text{if } 1 < x < 3 \\ 9 & \text{if } x \geq 3 \end{cases} \qquad F'(x) = f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2x & \text{if } 1 < x < 3 \\ 0 & \text{if } x > 3 \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx \\ &= \int_{-\infty}^{1} (0)\,dx + \int_{1}^{3} (2x)\,dx + \int_{3}^{\infty} (0)\,dx \end{aligned}$$

Improper Integrals

Example

$$F(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x^2 & \text{if } 1 < x < 3 \\ 9 & \text{if } x \geq 3 \end{cases} \qquad F'(x) = f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2x & \text{if } 1 < x < 3 \\ 0 & \text{if } x > 3 \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx \\ &= \int_{-\infty}^{1} (0)\,dx + \int_{1}^{3} (2x)\,dx + \int_{3}^{\infty} (0)\,dx \\ &= \int_{1}^{3} 2x\,dx \end{aligned}$$

Improper Integrals

Example

$$F(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x^2 & \text{if } 1 < x < 3 \\ 9 & \text{if } x \geq 3 \end{cases} \qquad F'(x) = f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2x & \text{if } 1 < x < 3 \\ 0 & \text{if } x > 3 \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx \\ &= \int_{-\infty}^{1} (0)\,dx + \int_{1}^{3} (2x)\,dx + \int_{3}^{\infty} (0)\,dx \\ &= \int_{1}^{3} 2x\ dx = \left[x^2\right]_1^3 \end{aligned}$$

Improper Integrals

Example

$$F(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x^2 & \text{if } 1 < x < 3 \\ 9 & \text{if } x \geq 3 \end{cases} \qquad F'(x) = f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2x & \text{if } 1 < x < 3 \\ 0 & \text{if } x > 3 \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\, dx &= \int_{-\infty}^{1} f(x)\, dx + \int_{1}^{3} f(x)\, dx + \int_{3}^{\infty} f(x)\, dx \\ &= \int_{-\infty}^{1} (0)\, dx + \int_{1}^{3} (2x)\, dx + \int_{3}^{\infty} (0)\, dx \\ &= \int_{1}^{3} 2x\ dx = \left[x^2\right]_1^3 = 9 - 1 = 8 \end{aligned}$$

Question

$$F(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x^2 & \text{if } 1 < x < 3 \\ 9 & \text{if } x \geq 3 \end{cases} \qquad F'(x) = f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2x & \text{if } 1 < x < 3 \\ 0 & \text{if } x > 3 \end{cases}$$

What is $\int_2^\infty f(x)\, dx$?

Question

$$F(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x^2 & \text{if } 1 < x < 3 \\ 9 & \text{if } x \geq 3 \end{cases} \qquad F'(x) = f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2x & \text{if } 1 < x < 3 \\ 0 & \text{if } x > 3 \end{cases}$$

What is $\int_2^\infty f(x)\,dx$?

Answer:

Question

$$F(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x^2 & \text{if } 1 < x < 3 \\ 9 & \text{if } x \geq 3 \end{cases} \qquad F'(x) = f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2x & \text{if } 1 < x < 3 \\ 0 & \text{if } x > 3 \end{cases}$$

What is $\int_2^{\infty} f(x)\, dx$?

Answer: 5

$$\begin{aligned} \int_2^{\infty} f(x)\, dx &= \int_2^3 f(x)\, dx + \int_3^{\infty} f(x)\, dx \\ &= \int_1^3 (2x)\, dx + \int_3^{\infty} (0)\, dx \\ &= \int_2^3 2x \; dx = \left[x^2\right]_2^3 = 9 - 4 = 5 \end{aligned}$$

Multiple Integrals

Example

$$\int_0^1 x^2 y^2 dx = \left[\frac{1}{3}x^3 y^2\right]_{x=0}^{x=1}$$

Multiple Integrals

Example

$$\begin{aligned}\int_0^1 x^2y^2dx &= \left[\frac{1}{3}x^3y^2\right]_{x=0}^{x=1} \\ &= \frac{1}{3}\left[x^3y^2\right]_{x=0}^{x=1}\end{aligned}$$

Multiple Integrals

Example

$$\begin{aligned}
\int_0^1 x^2y^2dx &= \left[\frac{1}{3}x^3y^2\right]_{x=0}^{x=1} \\
&= \frac{1}{3}\left[x^3y^2\right]_{x=0}^{x=1} \\
&= \frac{1}{3}\left(1^2y^2 - (0)^2\,y^2\right)
\end{aligned}$$

Multiple Integrals

Example

$$\begin{aligned}
\int_0^1 x^2y^2dx &= \left[\frac{1}{3}x^3y^2\right]_{x=0}^{x=1} \\
&= \frac{1}{3}\left[x^3y^2\right]_{x=0}^{x=1} \\
&= \frac{1}{3}\left(1^2y^2 - (0)^2\,y^2\right) \\
&= \frac{1}{3}y^2
\end{aligned}$$

Multiple Integrals

Example

$$
\begin{aligned}
\int_0^1 x^2 y^2 dx &= \frac{1}{3} y^2 \\
\int_0^3 \int_0^1 x^2 y^2 dx\, dy &= \int_0^3 \left(\int_0^1 x^2 y^2 dx \right) dy \\
&= \int_0^3 \frac{1}{3} y^2 dy = \frac{1}{3} \int_0^3 y^2 \\
&= \frac{1}{3} \left[\frac{1}{3} y^3 \right]_0^3 = \frac{1}{9} \left[y^3 \right]_0^3 \\
&= \frac{1}{9} \left(3^3 - 0^3 \right) \\
&= \frac{1}{9} (27) = 3
\end{aligned}
$$

Question

What is $\int_0^1 \int_1^2 x \, dx \, dy$?

Question

What is $\int_0^1 \int_1^2 x \, dx \, dy$?

Answer:

Question

What is $\int_0^1 \int_1^2 x \, dx \, dy$?

Answer: 1.5

$$\begin{aligned}
\int_1^2 x \, dx &= \left[\frac{1}{2}x^2\right]_{x=1}^{x=2} = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 \\
&= 2 - \frac{1}{2} = \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \int_1^2 x \, dx \, dy &= \int_0^1 \left(\frac{3}{2}\right) dy \\
&= \left[\frac{3}{2}y\right]_0^1 \\
&= \frac{3}{2}(1) - \frac{3}{2}(0) = \frac{3}{2} = 1.5
\end{aligned}$$

Question

What is $\int_0^2 \int_0^1 (x + y)\, dx\, dy$?

Question

What is $\int_0^2 \int_0^1 (x + y)\, dx\, dy$?

Answer:

Question

What is $\int_0^2 \int_0^1 (x+y)\,dx\,dy$?

Answer: 3

$$\begin{aligned}
\int_0^1 (x+y)\,dx &= \left[\frac{1}{2}x^2 + xy\right]_{x=0}^{x=1} \\
&= \frac{1}{2} + y
\end{aligned}$$

$$\begin{aligned}
\int_0^2 \int_0^1 (x+y)\,dx\,dy &= \int_0^2 \left(\frac{1}{2} + y\right) dy \\
&= \left[\frac{1}{2}y + \frac{1}{2}y^2\right]_0^2 \\
&= \left(\frac{1}{2}(2) + \frac{1}{2}(4)\right) - \left(\frac{1}{2}(0) + \frac{1}{2}(0)\right) \\
&= 1 + 2 = 3
\end{aligned}$$

Random Variables

Daniel Stephenson

SCMA 524: Statistical Fundamentals

What is a Random Variable?

- A **random variable** associates a numerical value to every possible outcome of a random process.

Example

Suppose you flip two coins. The sample space is $\Omega = \{HH, HT, TH, TT\}$. Let the random variable X denote the number of heads.

ω	HH	HT	TH	TT
$X(\omega)$	2	1	1	0

If $P(\{HT\}) = P(\{TH\}) = 0.25$ then

$$\begin{aligned} P(X = 1) &= P(\{HT, TH\}) \\ &= P(\{HT\}) + P(\{TH\}) \\ &= 0.25 + 0.25 = 0.5 \end{aligned}$$

Realization of a Random Variable

- The value that a random variable takes is unknown until we observe it. After we observe it, this value is called the **realization**.
- We will often use upper case letters like "X" for random variables and lower case letters like "x" for realizations.

Example

Suppose you flip two coins. The sample space is $\Omega = \{HH, HT, TH, TT\}$. Let the random variable X denote the number of heads.

ω	HH	HT	TH	TT
$X(\omega)$	2	1	1	0

If both coins come up heads, then the realization of X is given by

$$x = X(HH) = 2$$

Probability Distributions

- A **probability distribution** describes the probability of events involving a random variable.
- A random variable is **discrete** if it can only take a finite number of values.
- The probability distribution of a discrete random variable X can be described by a **probability mass function** p_X.

$$p_X(c) = P(X = c)$$

Example

y	0	2	3	5	7
$p_Y(y)$	0.10	0.05	0.05	0.75	0.05

$$P(Y = 2) = p_Y(2) = 0.05$$

The Discrete Uniform Distribution

- A discrete random variable is **uniformly distributed** if each realization has equal probability.

- If X is uniformly distributed over a set $\{a, b, c, \ldots\}$ then we write

$$X \sim U\{a, b, c, \ldots\}$$

Example

If $Z \sim U\{3, 5, 10, 11\}$ then the probability mass function $p_Z(z)$ is

z	3	5	10	11
$p_Z(z)$	0.25	0.25	0.25	0.25

Question

Suppose $Z \sim U\{2, 4, 6, 7, 8\}$

What is $p_Z(8)$?

Question

Suppose $Z \sim U\{2, 4, 6, 7, 8\}$

What is $p_Z(8)$?

Answer:

Question

Suppose $Z \sim U\{2, 4, 6, 7, 8\}$

What is $p_Z(8)$?

Answer: 0.2

z	2	4	6	7	0
$p_W(z)$	0.2	0.2	0.2	0.2	0.2

The Bernoulli Distribution

- A discrete random variable X is **Bernoulli distributed** if

$$\begin{aligned} p_X(1) &= b \\ p_X(0) &= 1-b \end{aligned}$$

- In this case, we write

$$X \sim \text{Bernoulli}(b)$$

Example

If $Y \sim \text{Bernoulli}(0.6)$ then $p_Y(y)$ is given by

y	1	0
$p_Y(y)$	0.6	0.4

Question

Suppose $G \sim \text{Bernoulli}(0.3)$

What is $p_G(0)$?

Question

Suppose $G \sim \text{Bernoulli}\,(0.3)$

What is $p_G\,(0)$?

Answer:

Question

Suppose $G \sim \text{Bernoulli}(0.3)$

What is $p_G(0)$?

Answer: 0.7

g	1	0
$p_G(g)$	0.3	0.7

Cumulative Distribution Functions (CDF)

- A **cumulative distribution function** (CDF) is another way of describing a probability distribution.
- The CDF of a random variable gives probability of observing a realization less than or equal to some value.

$$F_X(c) = P(X \leq c)$$

Example

x	0	1	3	9	12
$p_X(x)$	0.25	0.2	0.4	0.10	0.05

$$F_X(4) = P(X \leq 4) = 0.25 + 0.2 + 0.4 = 0.85$$

Question

x	1	2	3	4	5	6
$p_X(x)$	0.05	0.3	0.2	0.05	0.15	0.25

What is $F_X(4)$?

Question

x	1	2	3	4	5	6
$p_X(x)$	0.05	0.3	0.2	0.05	0.15	0.25

What is $F_X(4)$?

Answer:

Question

x	1	2	3	4	5	6
$p_X(x)$	0.05	0.3	0.2	0.05	0.15	0.25

What is $F_X(4)$?

Answer: 0.6

$$\begin{aligned} F_X(4) &= P(X \leq 4) \\ &= p_X(1) + p_X(2) + p_X(3) + p_X(4) \\ &= 0.05 + 0.3 + 0.2 + 0.05 \\ &= 0.6 \end{aligned}$$

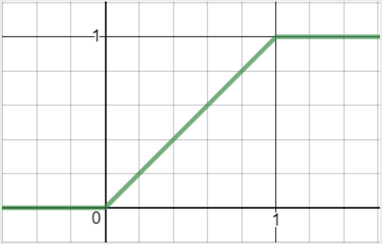
Continuous Random Variables

- A random variable is **continuous** if it has a continuous CDF.
- Continuous random variables can take a continuous range of values.

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ c & \text{if } 0 \le c \le 1 \\ 1 & \text{if } c > 1 \end{cases}$$

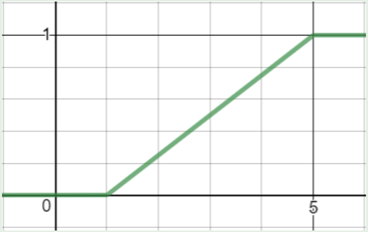
$$F(0.5) = P(X \le 0.5) = 0.5$$



Continuous Random Variables

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ \dfrac{c-1}{4} & \text{if } 1 \leq c \leq 5 \\ 1 & \text{if } c > 5 \end{cases}$$

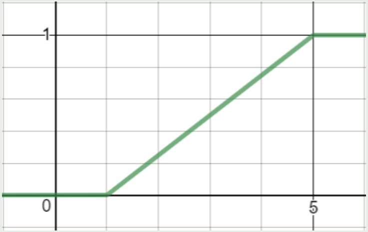


$$P(X \leq 2) + P(2 < X \leq 3) \quad = \quad P(X \leq 3) \qquad \text{(Mutually Exclusive)}$$

Continuous Random Variables

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ \dfrac{c-1}{4} & \text{if } 1 \leq c \leq 5 \\ 1 & \text{if } c > 5 \end{cases}$$

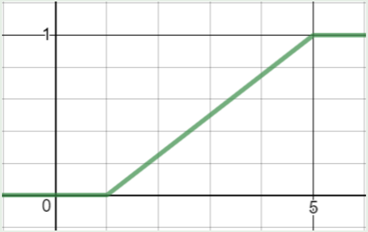


$$\begin{aligned} P(X \leq 2) + P(2 < X \leq 3) &= P(X \leq 3) \qquad \text{(Mutually Exclusive)} \\ F_X(2) + P(2 < X \leq 3) &= F_X(3) \end{aligned}$$

Continuous Random Variables

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ \dfrac{c-1}{4} & \text{if } 1 \leq c \leq 5 \\ 1 & \text{if } c > 5 \end{cases}$$

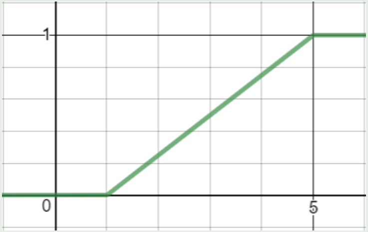


$$\begin{aligned} P(X \leq 2) + P(2 < X \leq 3) &= P(X \leq 3) \qquad \text{(Mutually Exclusive)} \\ F_X(2) + P(2 < X \leq 3) &= F_X(3) \\ P(2 < X \leq 3) &= F_X(3) - F_X(2) \end{aligned}$$

Continuous Random Variables

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ \dfrac{c-1}{4} & \text{if } 1 \le c \le 5 \\ 1 & \text{if } c > 5 \end{cases}$$



$$\begin{aligned} P(X \le 2) + P(2 < X \le 3) &= P(X \le 3) \qquad \text{(Mutually Exclusive)} \\ F_X(2) + P(2 < X \le 3) &= F_X(3) \\ P(2 < X \le 3) &= F_X(3) - F_X(2) \\ &= \frac{1}{2} - \frac{1}{4} = \frac{1}{4} = 0.25 \end{aligned}$$

Question

Suppose $F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ x^2 & \text{if } 0 \leq c \leq 1 \\ 1 & \text{if } c > 1 \end{cases}$

What is $P(0.5 < X \leq 1)$?

Question

Suppose $F_X\left(c\right) = \begin{cases} 0 & \text{if } c < 0 \\ x^2 & \text{if } 0 \leq c \leq 1 \\ 1 & \text{if } c > 1 \end{cases}$

What is $P\left(0.5 < X \leq 1\right)$?

Answer:

Question

Suppose $F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ x^2 & \text{if } 0 \leq c \leq 1 \\ 1 & \text{if } c > 1 \end{cases}$

What is $P(0.5 < X \leq 1)$?

Answer:

$$P(X \leq 0.5) + P(0.5 < X \leq 1) \quad = \quad P(X \leq 1)$$

Question

Suppose $F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ x^2 & \text{if } 0 \leq c \leq 1 \\ 1 & \text{if } c > 1 \end{cases}$

What is $P(0.5 < X \leq 1)$?

Answer:

$$\begin{aligned} P(X \leq 0.5) + P(0.5 < X \leq 1) &= P(X \leq 1) \\ F_X(0.5) + P(0.5 < X \leq 1) &= F_X(1) \end{aligned}$$

Question

Suppose $F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ x^2 & \text{if } 0 \leq c \leq 1 \\ 1 & \text{if } c > 1 \end{cases}$

What is $P(0.5 < X \leq 1)$?

Answer:

$$\begin{aligned} P(X \leq 0.5) + P(0.5 < X \leq 1) &= P(X \leq 1) \\ F_X(0.5) + P(0.5 < X \leq 1) &= F_X(1) \\ P(0.5 < X \leq 1) &= F_X(1) - F_X(0.5) \end{aligned}$$

Question

Suppose $F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ x^2 & \text{if } 0 \leq c \leq 1 \\ 1 & \text{if } c > 1 \end{cases}$

What is $P(0.5 < X \leq 1)$?

Answer:

$$\begin{aligned} P(X \leq 0.5) + P(0.5 < X \leq 1) &= P(X \leq 1) \\ F_X(0.5) + P(0.5 < X \leq 1) &= F_X(1) \\ P(0.5 < X \leq 1) &= F_X(1) - F_X(0.5) \\ &= (1)^2 - (0.5)^2 \end{aligned}$$

Question

Suppose $F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ x^2 & \text{if } 0 \leq c \leq 1 \\ 1 & \text{if } c > 1 \end{cases}$

What is $P(0.5 < X \leq 1)$?

Answer: 0.75

$$\begin{aligned} P(X \leq 0.5) + P(0.5 < X \leq 1) &= P(X \leq 1) \\ F_X(0.5) + P(0.5 < X \leq 1) &= F_X(1) \\ P(0.5 < X \leq 1) &= F_X(1) - F_X(0.5) \\ &= (1)^2 - (0.5)^2 \\ &= 1 - 0.25 = 0.75 \end{aligned}$$

Probability Density Functions (PDF)

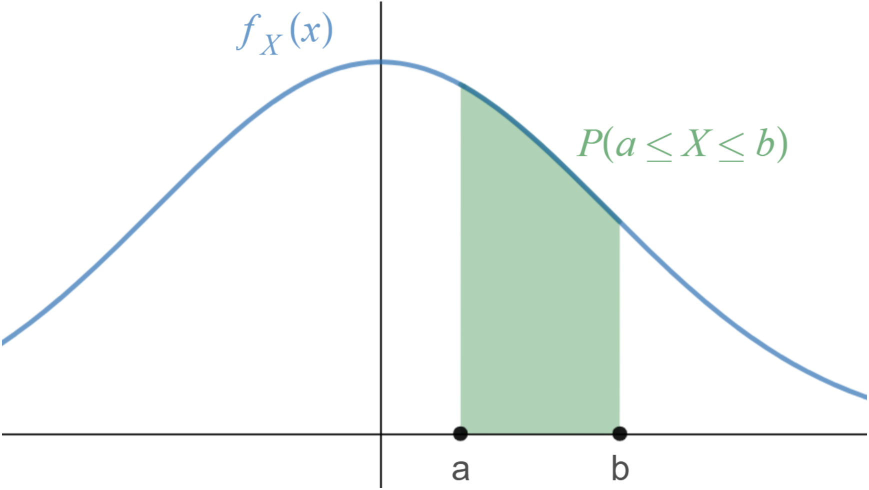
- If X is a continuous random variable, then the PDF of X is the derivative of its CDF.

$$f_X(c) = F'_X(c)$$

- This means that the CDF is an antiderivative of the PDF so

$$\begin{aligned}\int_a^b f_X(x)\,dx &= F(b) - F(a) \\ &= P(X \le b) - P(X \le a) \\ &= P(a \le X \le b)\end{aligned}$$

Probability Density Functions (PDF)



The Continuous Uniform Distribution

A random variable X is **uniformly distributed** on $[a, b]$ if

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

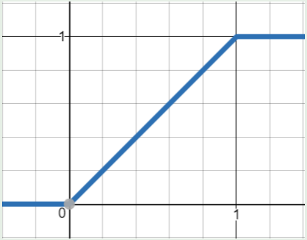
If X is uniformly distributed on $[a, b]$ then we write

$$X \sim U[a, b]$$

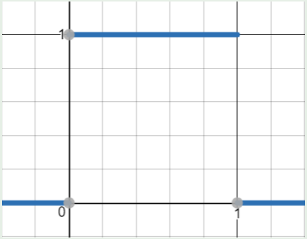
Probability Density Functions (PDF)

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ c & \text{if } 0 \leq c \leq 1 \\ 1 & \text{if } c > 1 \end{cases}$$



$$f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 0 \\ 1 & \text{if } 0 \leq c \leq 1 \\ 0 & \text{if } c > 1 \end{cases}$$



Probability Density Functions (PDF)

Example

$$f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$$

$$P(0.5 < X \leq 1) \quad = \quad \int_{0.5}^{1} f(x)\,dx$$

Probability Density Functions (PDF)

Example

$$f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$$

$$\begin{aligned} P(0.5 < X \leq 1) &= \int_{0.5}^{1} f(x)\,dx \\ &= \int_{0.5}^{1} \frac{1}{2}x\,dx \end{aligned}$$

Probability Density Functions (PDF)

Example

$$f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$$

$$\begin{aligned} P(0.5 < X \leq 1) &= \int_{0.5}^{1} f(x)\,dx \\ &= \int_{0.5}^{1} \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_{0.5}^{1} \end{aligned}$$

Probability Density Functions (PDF)

Example

$$f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$$

$$\begin{aligned} P(0.5 < X \leq 1) &= \int_{0.5}^{1} f(x)\, dx \\ &= \int_{0.5}^{1} \frac{1}{2} x\, dx = \left[\frac{1}{4}x^2\right]_{0.5}^{1} \\ &= \frac{1}{4}(1)^2 - \frac{1}{4}(0.5)^2 \end{aligned}$$

Probability Density Functions (PDF)

Example

$$f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$$

$$\begin{aligned} P(0.5 < X \leq 1) &= \int_{0.5}^{1} f(x)\, dx \\ &= \int_{0.5}^{1} \frac{1}{2}x\, dx = \left[\frac{1}{4}x^2\right]_{0.5}^{1} \\ &= \frac{1}{4}(1)^2 - \frac{1}{4}(0.5)^2 \\ &= \frac{1}{4} - \frac{1}{16} \end{aligned}$$

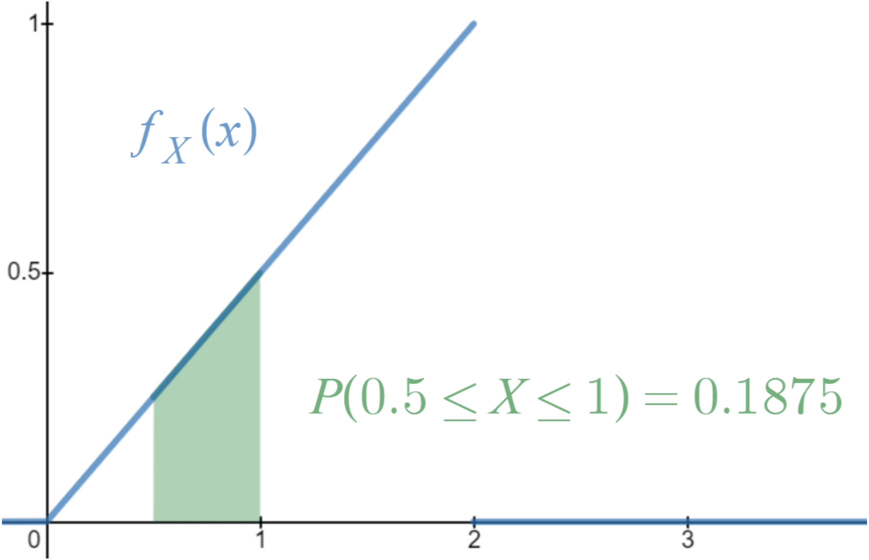
Probability Density Functions (PDF)

Example

$$f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$$

$$\begin{aligned} P(0.5 < X \leq 1) &= \int_{0.5}^{1} f(x)\, dx \\ &= \int_{0.5}^{1} \frac{1}{2} x\, dx = \left[\frac{1}{4}x^2\right]_{0.5}^{1} \\ &= \frac{1}{4}(1)^2 - \frac{1}{4}(0.5)^2 \\ &= \frac{1}{4} - \frac{1}{16} = \frac{3}{16} = 0.1875 \end{aligned}$$

Probability Density Functions (PDF)



Question

Suppose $f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$

What is $P(1 \leq X \leq 1.5)$?

Question

Suppose $f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$

What is $P(1 \leq X \leq 1.5)$?

Answer:

Question

Suppose $f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$

What is $P(1 \leq X \leq 1.5)$?

Answer:

$$P(1 \leq x \leq 1.5) = \int_1^{1.5} f_X(x)\, dx$$

Question

Suppose $f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \le c \le 2 \\ 0 & \text{if } c > 1 \end{cases}$

What is $P(1 \le X \le 1.5)$?

Answer:

$$\begin{aligned} P(1 \le x \le 1.5) &= \int_1^{1.5} f_X(x)\, dx \\ &= \int_1^{1.5} \frac{1}{2}x\, dx = \left[\frac{1}{4}x^2\right]_1^{1.5} \end{aligned}$$

Question

Suppose $f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$

What is $P(1 \leq X \leq 1.5)$?

Answer:

$$\begin{aligned} P(1 \leq x \leq 1.5) &= \int_1^{1.5} f_X(x)\, dx \\ &= \int_1^{1.5} \frac{1}{2} x\, dx = \left[\frac{1}{4}x^2\right]_1^{1.5} \\ &= \frac{1}{4}(1.5)^2 - \frac{1}{4}(1)^2 \end{aligned}$$

Question

Suppose $f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$

What is $P(1 \leq X \leq 1.5)$?

Answer:

$$\begin{aligned} P(1 \leq x \leq 1.5) &= \int_1^{1.5} f_X(x)\, dx \\ &= \int_1^{1.5} \frac{1}{2}x\, dx = \left[\frac{1}{4}x^2\right]_1^{1.5} \\ &= \frac{1}{4}(1.5)^2 - \frac{1}{4}(1)^2 \\ &= \frac{1}{4}\left(\frac{3}{2}\right)^2 - \frac{1}{4} \end{aligned}$$

Question

Suppose $f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$

What is $P(1 \leq X \leq 1.5)$?

Answer:

$$\begin{aligned} P(1 \leq x \leq 1.5) &= \int_1^{1.5} f_X(x)\, dx \\ &= \int_1^{1.5} \frac{1}{2} x\, dx = \left[\frac{1}{4}x^2\right]_1^{1.5} \\ &= \frac{1}{4}(1.5)^2 - \frac{1}{4}(1)^2 \\ &= \frac{1}{4}\left(\frac{3}{2}\right)^2 - \frac{1}{4} \\ &= \frac{9}{16} - \frac{1}{4} \end{aligned}$$

Question

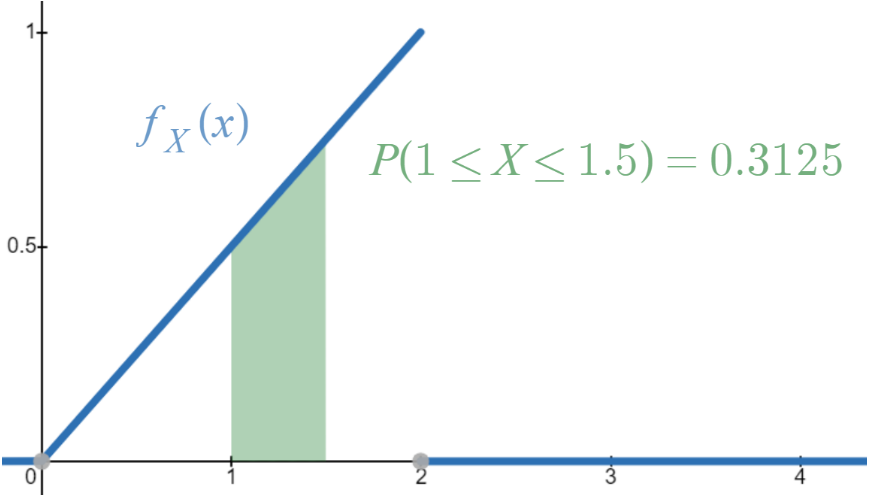
Suppose $f_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 1 \end{cases}$

What is $P(1 \leq X \leq 1.5)$?

Answer: 0.3125

$$\begin{aligned} P(1 \leq x \leq 1.5) &= \int_1^{1.5} f_X(x)\, dx \\ &= \int_1^{1.5} \frac{1}{2} x\, dx = \left[\frac{1}{4}x^2\right]_1^{1.5} \\ &= \frac{1}{4}(1.5)^2 - \frac{1}{4}(1)^2 \\ &= \frac{1}{4}\left(\frac{3}{2}\right)^2 - \frac{1}{4} \\ &= \frac{9}{16} - \frac{1}{4} = \frac{5}{16} = 0.3125 \end{aligned}$$

Probability Density Functions (PDF)



Expected Value

- The **expected value** of a random variable is the probability-weighted average of all its possible realizations.

- The expected value is also called the **population mean**.

- The expected value of a discrete random variable X is

$$\mu_X = E\{X\} = \sum_x x\, p_X(x)$$

Expected Value

Example

y	1	8	12
$p_Y(y)$	0.3	0.2	0.5

$$\mu_Y = E\{Y\} \quad = \quad \sum_y y\, p_Y(y)$$

Expected Value

Example

y	1	8	12
$p_Y(y)$	0.3	0.2	0.5

$$
\begin{aligned}
\mu_Y = E\{Y\} &= \sum_y y\, p_Y(y) \\
&= 1\, p_Y(1) + 8\, p_Y(8) + 12\, p_Y(12)
\end{aligned}
$$

Expected Value

Example

y	1	8	12
$p_Y(y)$	0.3	0.2	0.5

$$
\begin{aligned}
\mu_Y = E\{Y\} &= \sum_y y\, p_Y(y) \\
&= 1\, p_Y(1) + 8\, p_Y(8) + 12\, p_Y(12) \\
&= 1\ (0.3) + 8\ (0.2) + 12\ (0.5)
\end{aligned}
$$

Expected Value

Example

y	1	8	12
$p_Y(y)$	0.3	0.2	0.5

$$\begin{aligned}
\mu_Y = E\{Y\} &= \sum_y y\, p_Y(y) \\
&= 1\, p_Y(1) + 8\, p_Y(8) + 12\, p_Y(12) \\
&= 1\ (0.3) + 8\ (0.2) + 12\ (0.5) \\
&= 0.3 + 1.6 + 6
\end{aligned}$$

Expected Value

Example

y	1	8	12
$p_Y(y)$	0.3	0.2	0.5

$$\begin{aligned}
\mu_Y = E\{Y\} &= \sum_y y\, p_Y(y) \\
&= 1\, p_Y(1) + 8\, p_Y(8) + 12\, p_Y(12) \\
&= 1\ (0.3) + 8\ (0.2) + 12\ (0.5) \\
&= 0.3 + 1.6 + 6 \\
&= 7.9
\end{aligned}$$

Question

x	0	5	6	10
$p_X(x)$	0.3	0.2	0.4	0.1

What is $E\{X\}$?

Question

x	0	5	6	10
$p_X(x)$	0.3	0.2	0.4	0.1

What is $E\{X\}$?

Answer:

Question

x	0	5	6	10
$p_X(x)$	0.3	0.2	0.4	0.1

What is $E\{X\}$?

Answer: 4.4

$$\begin{aligned}
\mu_X = E\{X\} &= \sum_x x\, p_X(x) \\
&= 0\, p_Y(0) + 5\, p_Y(5) + 6\, p_Y(6) + 10\, p_Y(10) \\
&= 0\ (0.3) + 5\ (0.2) + 6\ (0.4) + 10\ (0.1) \\
&= 0 + 1 + 2.4 + 1 \\
&= 4.4
\end{aligned}$$

Expected Value

- The expected value of a continuous random variable X is

$$\mu_X = E\{X\} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \le c \le 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \le c \le 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$\mu_X = E\{X\} \;=\; \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \le c \le 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \le c \le 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$\mu_X = E\{X\} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^2 x\, \frac{1}{2}\, dx$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F_X'(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$\begin{aligned} \mu_X = E\{X\} &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^2 x\, \frac{1}{2}\, dx \\ &= \frac{1}{2} \int_0^2 x\, dx \end{aligned}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \le c \le 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \le c \le 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$\begin{aligned} \mu_X = E\{X\} &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^2 x\, \frac{1}{2}\, dx \\ &= \frac{1}{2}\int_0^2 x\, dx = \frac{1}{2}\left[\frac{1}{2}x^2\right]_0^2 \end{aligned}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \le c \le 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \le c \le 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$\begin{aligned} \mu_X = E\{X\} &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^2 x\, \frac{1}{2}\, dx \\ &= \frac{1}{2}\int_0^2 x\, dx = \frac{1}{2}\left[\frac{1}{2}x^2\right]_0^2 \\ &= \frac{1}{2}\left[\frac{1}{2}(2)^2 - \frac{1}{2}(0)^2\right] \end{aligned}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \le c \le 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F_X'(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \le c \le 2 \\ 0 & \text{if } c > 2 \end{cases}$$

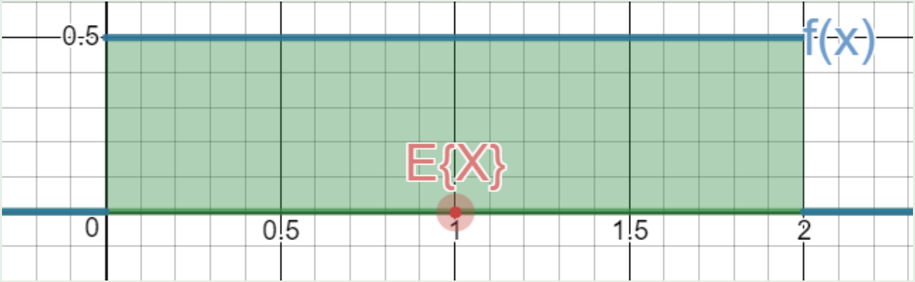
$$\begin{aligned} \mu_X = E\{X\} &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^2 x\, \frac{1}{2}\, dx \\ &= \frac{1}{2}\int_0^2 x\, dx = \frac{1}{2}\left[\frac{1}{2}x^2\right]_0^2 \\ &= \frac{1}{2}\left[\frac{1}{2}(2)^2 - \frac{1}{2}(0)^2\right] \\ &= \frac{1}{2}[2-0] = 1 \end{aligned}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F_X'(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$\mu_X = E\{X\} = 1$$



Question

$$F_X\left(c\right) = \begin{cases} 0 & \text{if } c < 1 \\ c - 1 & \text{if } 1 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases}$$

What is $E\left\{X\right\}$?

Question

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ c - 1 & \text{if } 1 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases}$$

What is $E\{X\}$?

Answer:

Question

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ c - 1 & \text{if } 1 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } 1 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

What is $E\{X\}$?

Answer:

Question

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ c - 1 & \text{if } 1 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } 1 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

What is $E\{X\}$?

Answer:

$$E\{X\} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

Question

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ c - 1 & \text{if } 1 \le c \le 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F_X'(x) = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } 1 \le c \le 2 \\ 0 & \text{if } c > 2 \end{cases}$$

What is $E\{X\}$?

Answer:

$$E\{X\} \quad = \quad \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_1^2 x\, dx$$

Question

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ c - 1 & \text{if } 1 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F_X'(x) = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } 1 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

What is $E\{X\}$?

Answer:

$$\begin{aligned} E\{X\} &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_1^2 x\, dx \\ &= \left[\frac{1}{2}x^2\right]_1^2 \end{aligned}$$

Question

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ c - 1 & \text{if } 1 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } 1 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

What is $E\{X\}$?

Answer:

$$\begin{aligned} E\{X\} &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_1^2 x\, dx \\ &= \left[\frac{1}{2}x^2\right]_1^2 \\ &= \left[\frac{1}{2}(2)^2 - \frac{1}{2}(1)^2\right] \end{aligned}$$

Question

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ c - 1 & \text{if } 1 \le c \le 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F_X'(x) = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } 1 \le c \le 2 \\ 0 & \text{if } c > 2 \end{cases}$$

What is $E\{X\}$?

Answer: 1.5

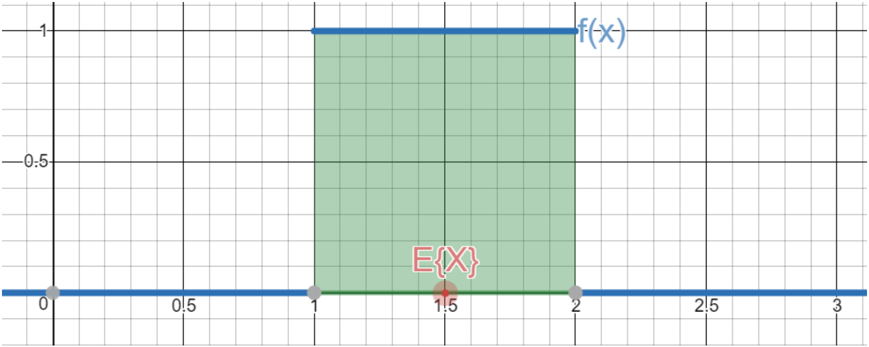
$$\begin{aligned} E\{X\} &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_1^2 x\, dx \\ &= \left[\frac{1}{2}x^2\right]_1^2 \\ &= \left[\frac{1}{2}(2)^2 - \frac{1}{2}(1)^2\right] \\ &= \left[2 - \frac{1}{2}\right] = 1.5 \end{aligned}$$

Question

$$F_X(c) = \begin{cases} 0 & \text{if } c < 1 \\ c - 1 & \text{if } 1 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } 1 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

What is $E\{X\}$?

Answer: 1.5



Expected Value

Example

y	1	2	3
$p_Y(y)$	0.3	0.2	0.5

$$E\left\{Y^2\right\} = \sum_y y^2 \, p_Y(y)$$

Expected Value

Example

y	1	2	3
$p_Y(y)$	0.3	0.2	0.5

$$
\begin{aligned}
E\left\{Y^2\right\} &= \sum_y y^2\, p_Y(y) \\
&= (1)^2\, p_Y(1) + (2)^2\, p_Y(2) + (3)^2\, p_Y(3)
\end{aligned}
$$

Expected Value

Example

y	1	2	3
$p_Y(y)$	0.3	0.2	0.5

$$
\begin{aligned}
E\left\{Y^2\right\} &= \sum_y y^2\, p_Y(y) \\
&= (1)^2\ p_Y(1) + (2)^2\ p_Y(2) + (3)^2\ p_Y(3) \\
&= 1\ (0.3) + 4\ (0.2) + 9\ (0.5)
\end{aligned}
$$

Expected Value

Example

y	1	2	3
$p_Y(y)$	0.3	0.2	0.5

$$
\begin{aligned}
E\left\{Y^2\right\} &= \sum_y y^2\, p_Y(y) \\
&= (1)^2\; p_Y(1) + (2)^2\; p_Y(2) + (3)^2\; p_Y(3) \\
&= 1\;(0.3) + 4\;(0.2) + 9\;(0.5) \\
&= 0.3 + 0.8 + 4.5
\end{aligned}
$$

Expected Value

Example

y	1	2	3
$p_Y(y)$	0.3	0.2	0.5

$$
\begin{aligned}
E\left\{Y^2\right\} &= \sum_y y^2\, p_Y(y) \\
&= (1)^2\; p_Y(1) + (2)^2\; p_Y(2) + (3)^2\; p_Y(3) \\
&= 1\;(0.3) + 4\;(0.2) + 9\;(0.5) \\
&= 0.3 + 0.8 + 4.5 \\
&= 5.6
\end{aligned}
$$

Question

x	0	5	6	10
$p_X(x)$	0.3	0.2	0.4	0.1

If $Y = X^2$ then what is μ_Y ?

Question

x	0	5	6	10
$p_X(x)$	0.3	0.2	0.4	0.1

If $Y = X^2$ then what is μ_Y ?

Answer:

Question

x	0	5	6	10
$p_X(x)$	0.3	0.2	0.4	0.1

If $Y = X^2$ then what is μ_Y ?

Answer: 29.4

$$\begin{aligned}
\mu_Y = E\{Y\} &= E\left\{X^2\right\} \\
&= \sum_x x^2 p_X(x) \\
&= (0)^2 p_Y(0) + (5)^2 p_Y(5) + (6)^2 p_Y(6) + (10)^2 p_Y(10) \\
&= 0(0.3) + 25(0.2) + 36(0.4) + 100(0.1) \\
&= 0 + 5 + 14.4 + 10 \\
&= 29.4
\end{aligned}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F_X'(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$E\left\{X^2\right\} = \int_{-\infty}^{\infty} x^2 \, f_X(x) \, dx$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F_X'(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$E\left\{X^2\right\} = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^2 x^2 \frac{1}{2}\,dx$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \leq c \leq 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \leq c \leq 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$\begin{aligned} E\left\{X^2\right\} &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^2 x^2 \frac{1}{2}\, dx \\ &= \frac{1}{2}\int_0^2 x^2\, dx \end{aligned}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \le c \le 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \le c \le 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$\begin{aligned} E\left\{X^2\right\} &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^2 x^2 \frac{1}{2}\, dx \\ &= \frac{1}{2}\int_0^2 x^2\, dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^2 \end{aligned}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \le c \le 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \le c \le 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$\begin{aligned} E\left\{X^2\right\} &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^2 x^2 \frac{1}{2}\, dx \\ &= \frac{1}{2}\int_0^2 x^2\, dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^2 \\ &= \frac{1}{2}\left[\frac{1}{3}(2)^3 - \frac{1}{2}(0)^3\right] \end{aligned}$$

Expected Value

Example

$$F_X(c) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2}c & \text{if } 0 \le c \le 2 \\ 1 & \text{if } c > 2 \end{cases} \qquad f_X(c) = F'_X(x) = \begin{cases} 0 & \text{if } c < 0 \\ \frac{1}{2} & \text{if } 0 \le c \le 2 \\ 0 & \text{if } c > 2 \end{cases}$$

$$\begin{aligned} E\left\{X^2\right\} &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^2 x^2 \frac{1}{2}\, dx \\ &= \frac{1}{2}\int_0^2 x^2\, dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^2 \\ &= \frac{1}{2}\left[\frac{1}{3}(2)^3 - \frac{1}{2}(0)^3\right] \\ &= \frac{1}{2}\left[\frac{8}{3} - 0\right] = \frac{8}{6} \end{aligned}$$

Linearity of Expectation

- Expectation is a **linear** operation

$$E\{aX + bY\} = aE\{X\} + bE\{Y\}$$

Example

Suppose $Z = 2X + 3Y + 5$.

If $E\{X\} = 4$ and $E\{Y\} = 0.5$ then

$$\begin{aligned}
\mu_Z = E\{Z\} &= E\{2X + 3Y + 5\} \\
&= 2E\{X\} + 3E\{Y\} + 5 \\
&= 2(4) + 3(0.5) + 5 \\
&= 8 + 1.5 + 5 = 14.5
\end{aligned}$$

Question

Let $W = Z + 5Y$.

Suppose $E\{Y\} = 2$ and $E\{Z\} = 3.5$

What is μ_W ?

Question

Let $W = Z + 5Y$.

Suppose $E\{Y\} = 2$ and $E\{Z\} = 3.5$

What is μ_W ?

Answer:

Question

Let $W = Z + 5Y$.

Suppose $E\{Y\} = 2$ and $E\{Z\} = 3.5$

What is μ_W ?

Answer:

$$\mu_W = E\{W\} \quad = \quad E\{Z + 5Y\}$$

Question

Let $W = Z + 5Y$.

Suppose $E\{Y\} = 2$ and $E\{Z\} = 3.5$

What is μ_W ?

Answer:

$$\begin{aligned} \mu_W = E\{W\} &= E\{Z + 5Y\} \\ &= E\{Z\} + 5E\{Y\} \end{aligned}$$

Question

Let $W = Z + 5Y$.

Suppose $E\{Y\} = 2$ and $E\{Z\} = 3.5$

What is μ_W ?

Answer:

$$\begin{aligned} \mu_W = E\{W\} &= E\{Z + 5Y\} \\ &= E\{Z\} + 5E\{Y\} \\ &= 3.5 + 5(2) \end{aligned}$$

Question

Let $W = Z + 5Y$.

Suppose $E\{Y\} = 2$ and $E\{Z\} = 3.5$

What is μ_W ?

Answer: 13.5

$$
\begin{aligned}
\mu_W = E\{W\} &= E\{Z + 5Y\} \\
&= E\{Z\} + 5E\{Y\} \\
&= 3.5 + 5(2) \\
&= 3.5 + 10 = 13.5
\end{aligned}
$$

Variance

- The **variance** of a random variable is the expected squared deviation from the population mean

$$\sigma_X^2 = \mathrm{Var}(X) \quad = \quad E\left\{(X - \mu_X)^2\right\}$$

Variance

- The **variance** of a random variable is the expected squared deviation from the population mean

$$
\begin{aligned}
\sigma_X^2 = \mathrm{Var}\,(X) &= E\left\{(X-\mu_X)^2\right\} \\
&= E\left\{X^2 - 2X\mu_X + \mu_X^2\right\}
\end{aligned}
$$

Variance

- The **variance** of a random variable is the expected squared deviation from the population mean

$$\begin{aligned}\sigma_X^2 = \mathrm{Var}(X) &= E\left\{(X-\mu_X)^2\right\} \\ &= E\left\{X^2 - 2X\mu_X + \mu_X^2\right\} \\ &= E\left\{X^2\right\} - 2\mu_X E\{X\} + \mu_X^2\end{aligned}$$

Variance

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\sigma_X^2 = \mathrm{Var}(X) &= E\left\{(X-\mu_X)^2\right\} \\
&= E\left\{X^2 - 2X\mu_X + \mu_X^2\right\} \\
&= E\left\{X^2\right\} - 2\mu_X E\{X\} + \mu_X^2 \\
&= E\left\{X^2\right\} - 2E\{X\}E\{X\} + E\{X\}^2
\end{aligned}
$$

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&= E\left\{X^2\right\} - 2E\{X\}E\{X\} + E\{X\}^2 \\
&= E\left\{X^2\right\} - 2E\{X\}^2 + E\{X\}^2
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&= E\left\{X^2\right\} - 2\mu_X E\{X\} + \mu_X^2 \\
&= E\left\{X^2\right\} - 2E\{X\}E\{X\} + E\{X\}^2 \\
&= E\left\{X^2\right\} - 2E\{X\}^2 + E\{X\}^2 \\
&= E\left\{X^2\right\} - E\{X\}^2
\end{aligned}
$$

Example

y	1	2	4
$p_Y(y)$	0.3	0.2	0.5

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$$E\{Y\} = 1(0.3) + 2(0.2) + 4(0.5)$$

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$p_Y(y)$	0.3	0.2	0.5

$$
\begin{aligned}
E\{Y\} &= 1(0.3) + 2(0.2) + 4(0.5) \\
&= 0.3 + 0.4 + 2
\end{aligned}
$$

Example

y	1	2	4
$p_Y(y)$	0.3	0.2	0.5

$$\begin{aligned} E\{Y\} &= 1(0.3) + 2(0.2) + 4(0.5) \\ &= 0.3 + 0.4 + 2 = 2.7 \end{aligned}$$

Example

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$$E\left\{Y^2\right\} = 1(0.3) + 4(0.2) + 16(0.5)$$

Example

y	1	2	4
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\end{aligned}
$$

$$
\begin{aligned}
E\left\{Y^2\right\} &= 1(0.3) + 4(0.2) + 16(0.5) \\
&= 0.3 + 0.8 + 8
\end{aligned}
$$

Example

y	1	2	4
$p_Y(y)$	0.3	0.2	0.5

$$\begin{aligned} E\{Y\} &= 1(0.3) + 2(0.2) + 4(0.5) \\ &= 0.3 + 0.4 + 2 = 2.7 \end{aligned}$$

$$\begin{aligned} E\left\{Y^2\right\} &= 1(0.3) + 4(0.2) + 16(0.5) \\ &= 0.3 + 0.8 + 8 = 9.1 \end{aligned}$$

Example

y	1	2	4
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\begin{aligned}
E\{Y\} &= 1(0.3) + 2(0.2) + 4(0.5) \\
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\end{aligned}
$$

$$
\begin{aligned}
E\left\{Y^2\right\} &= 1(0.3) + 4(0.2) + 16(0.5) \\
&= 0.3 + 0.8 + 8 = 9.1
\end{aligned}
$$

$$
\sigma_Y^2 = \mathrm{Var}\{Y\} = E\left\{Y^2\right\} - E\{Y\}^2
$$

Example

y	1	2	4
$p_Y(y)$	0.3	0.2	0.5

$$
\begin{aligned}
E\{Y\} &= 1(0.3) + 2(0.2) + 4(0.5) \\
&= 0.3 + 0.4 + 2 = 2.7
\end{aligned}
$$

$$
\begin{aligned}
E\left\{Y^2\right\} &= 1(0.3) + 4(0.2) + 16(0.5) \\
&= 0.3 + 0.8 + 8 = 9.1
\end{aligned}
$$

$$
\begin{aligned}
\sigma_Y^2 = \mathrm{Var}\{Y\} &= E\left\{Y^2\right\} - E\{Y\}^2 \\
&= 9.1 - (2.7)^2
\end{aligned}
$$

Example

y	1	2	4
$p_Y(y)$	0.3	0.2	0.5

$$
\begin{aligned}
E\{Y\} &= 1(0.3) + 2(0.2) + 4(0.5) \\
&= 0.3 + 0.4 + 2 = 2.7
\end{aligned}
$$

$$
\begin{aligned}
E\left\{Y^2\right\} &= 1(0.3) + 4(0.2) + 16(0.5) \\
&= 0.3 + 0.8 + 8 = 9.1
\end{aligned}
$$

$$
\begin{aligned}
\sigma_Y^2 = \operatorname{Var}\{Y\} &= E\left\{Y^2\right\} - E\{Y\}^2 \\
&= 9.1 - (2.7)^2 \\
&= 9.1 - 7.29
\end{aligned}
$$

Example

y	1	2	4
$p_Y(y)$	0.3	0.2	0.5

$$
\begin{aligned}
E\{Y\} &= 1(0.3) + 2(0.2) + 4(0.5) \\
&= 0.3 + 0.4 + 2 = 2.7
\end{aligned}
$$

$$
\begin{aligned}
E\left\{Y^2\right\} &= 1(0.3) + 4(0.2) + 16(0.5) \\
&= 0.3 + 0.8 + 8 = 9.1
\end{aligned}
$$

$$
\begin{aligned}
\sigma_Y^2 = \mathrm{Var}\{Y\} &= E\left\{Y^2\right\} - E\{Y\}^2 \\
&= 9.1 - (2.7)^2 \\
&= 9.1 - 7.29 = 1.81
\end{aligned}
$$

Variance

Example

$$\mathrm{Var}\,(aX) \;=\; E\left\{(aX)^2\right\} - E\left\{aX\right\}^2$$

Variance

Example

$$\begin{aligned}\operatorname{Var}(aX) &= E\left\{(aX)^2\right\} - E\{aX\}^2 \\ &= E\left\{(aX)^2\right\} - (aE\{X\})^2\end{aligned}$$

Variance

Example

$$\begin{aligned}\mathrm{Var}(aX) &= E\left\{(aX)^2\right\} - E\{aX\}^2 \\ &= E\left\{(aX)^2\right\} - (aE\{X\})^2 \\ &= E\left\{a^2X^2\right\} - a^2E\{X\}^2\end{aligned}$$

Variance

Example

$$\begin{aligned}\mathrm{Var}(aX) &= E\left\{(aX)^2\right\} - E\{aX\}^2 \\ &= E\left\{(aX)^2\right\} - (aE\{X\})^2 \\ &= E\left\{a^2X^2\right\} - a^2E\{X\}^2 \\ &= a^2E\left\{X^2\right\} - a^2(E\{X\})^2\end{aligned}$$

Variance

Example

$$
\begin{aligned}
\mathrm{Var}(aX) &= E\left\{(aX)^2\right\} - E\{aX\}^2 \\
&= E\left\{(aX)^2\right\} - (aE\{X\})^2 \\
&= E\left\{a^2X^2\right\} - a^2E\{X\}^2 \\
&= a^2E\left\{X^2\right\} - a^2(E\{X\})^2 \\
&= a^2\left[E\left\{X^2\right\} - E\{X\}^2\right]
\end{aligned}
$$

Variance

Example

$$
\begin{aligned}
\mathrm{Var}\left(aX\right) &= E\left\{\left(aX\right)^2\right\} - E\left\{aX\right\}^2 \\
&= E\left\{\left(aX\right)^2\right\} - \left(aE\left\{X\right\}\right)^2 \\
&= E\left\{a^2X^2\right\} - a^2E\left\{X\right\}^2 \\
&= a^2E\left\{X^2\right\} - a^2\left(E\left\{X\right\}\right)^2 \\
&= a^2\left[E\left\{X^2\right\} - E\left\{X\right\}^2\right] \\
&= a^2\mathrm{Var}\left(X\right)
\end{aligned}
$$

Question

If $\text{Var}(Y) = 3$ then what is $\text{Var}(5Y)$?

Question

If $\text{Var}(Y) = 3$ then what is $\text{Var}(5Y)$?

Answer:

Question

If $\mathrm{Var}(Y) = 3$ then what is $\mathrm{Var}(5Y)$?

Answer: 75

$$\begin{aligned} \mathrm{Var}(5Y) &= (5)^2\,\mathrm{Var}(Y) \\ &= 25(3) = 75 \end{aligned}$$

Standard Deviation

- The **standard deviation** is the square root of the variance.

$$\mathrm{SD}(x) = \sigma_X = \sqrt{\sigma_X^2} = \sqrt{\mathrm{Var}(X)}$$

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Example

x	0	2
$p_X\left(x\right)$	0.5	0.5

Standard Deviation

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Standard Deviation

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$$\mathrm{SD}(x) = \sigma_X = \sqrt{\sigma_X^2} = \sqrt{\mathrm{Var}(X)}$$

Example

x	0	2
$p_X(x)$	0.5	0.5

$$E\{X\} = 0(0.5) + 2(0.5) = 1$$

Standard Deviation

- The **standard deviation** is the square root of the variance.

$$\mathrm{SD}(x) = \sigma_X = \sqrt{\sigma_X^2} = \sqrt{\mathrm{Var}(X)}$$

Example

x	0	2
$p_X(x)$	0.5	0.5

$$\begin{aligned} E\{X\} &= 0(0.5) + 2(0.5) = 1 \\ E\left\{X^2\right\} &= 0(0.5) + 4(0.5) = 2 \end{aligned}$$

Standard Deviation

- The **standard deviation** is the square root of the variance.

$$\mathrm{SD}(x) = \sigma_X = \sqrt{\sigma_X^2} = \sqrt{\mathrm{Var}(X)}$$

Example

x	0	2
$p_X(x)$	0.5	0.5

$$\begin{aligned} E\{X\} &= 0(0.5) + 2(0.5) = 1 \\ E\left\{X^2\right\} &= 0(0.5) + 4(0.5) = 2 \\ \sigma_X^2 = \mathrm{Var}(X) &= E\left\{X^2\right\} - E\{X\}^2 \end{aligned}$$

Standard Deviation

- The **standard deviation** is the square root of the variance.

$$\mathrm{SD}(x) = \sigma_X = \sqrt{\sigma_X^2} = \sqrt{\mathrm{Var}(X)}$$

Example

x	0	2
$p_X(x)$	0.5	0.5

$$\begin{aligned} E\{X\} &= 0(0.5) + 2(0.5) = 1 \\ E\left\{X^2\right\} &= 0(0.5) + 4(0.5) = 2 \\ \sigma_X^2 = \mathrm{Var}(X) &= E\left\{X^2\right\} - E\{X\}^2 = 2 - (1)^2 = 1 \end{aligned}$$

Standard Deviation

- The **standard deviation** is the square root of the variance.

$$\mathrm{SD}(x) = \sigma_X = \sqrt{\sigma_X^2} = \sqrt{\mathrm{Var}(X)}$$

Example

x	0	2
$p_X(x)$	0.5	0.5

$$\begin{aligned} E\{X\} &= 0(0.5) + 2(0.5) = 1 \\ E\left\{X^2\right\} &= 0(0.5) + 4(0.5) = 2 \\ \sigma_X^2 = \mathrm{Var}(X) &= E\left\{X^2\right\} - E\{X\}^2 = 2 - (1)^2 = 1 \\ \sigma_X = \sqrt{\sigma_X^2} &= \sqrt{1} = 1 \end{aligned}$$

Question

y	1	3
$p_Y(y)$	0.2	0.8

What is $E\{Y\}$?

Question

y	1	3
$p_Y(y)$	0.2	0.8

What is $E\{Y\}$?

Answer:

Question

y	1	3
$p_Y(y)$	0.2	0.8

What is $E\{Y\}$?

Answer: 2.6

$$
\begin{aligned}
E\{Y\} &= 1\,(0.2) + 3\,(0.8) \\
&= 0.2 + 2.4 = 2.6
\end{aligned}
$$

Question

y	1	3
$p_Y(y)$	0.2	0.8

What is $E\left\{Y^2\right\}$?

Question

y	1	3
$p_Y(y)$	0.2	0.8

What is $E\left\{Y^2\right\}$?

Answer:

Question

y	1	3
$p_Y(y)$	0.2	0.8

What is $E\{Y^2\}$?

Answer: 7.4

$$
\begin{aligned}
E\left\{Y^2\right\} &= 1\,(0.2) + 9\,(0.8) \\
&= 0.2 + 7.2 = 7.4
\end{aligned}
$$

Question

y	1	3
$p_Y(y)$	0.2	0.8

What is σ_Y^2?

Question

y	1	3
$p_Y(y)$	0.2	0.8

What is σ_Y^2?

Answer:

Question

y	1	3
$p_Y(y)$	0.2	0.8

$$E\{Y\} = 1(0.2) + 3(0.8) = 0.2 + 2.4 = 2.6$$
$$E\{Y^2\} = 1(0.2) + 9(0.8) = 0.2 + 7.2 = 7.4$$

What is σ_Y^2?

Answer:

Question

y	1	3
$p_Y(y)$	0.2	0.8

$$E\{Y\} = 1(0.2) + 3(0.8) = 0.2 + 2.4 = 2.6$$
$$E\{Y^2\} = 1(0.2) + 9(0.8) = 0.2 + 7.2 = 7.4$$

What is σ_Y^2?

Answer:

$$\sigma_Y^2 = \mathrm{Var}(Y) \quad = \quad E\left\{Y^2\right\} - E\{Y\}^2$$

Question

y	1	3
$p_Y(y)$	0.2	0.8

$$E\{Y\} = 1(0.2) + 3(0.8) = 0.2 + 2.4 = 2.6$$
$$E\{Y^2\} = 1(0.2) + 9(0.8) = 0.2 + 7.2 = 7.4$$

What is σ_Y^2?

Answer:

$$\begin{aligned}\sigma_Y^2 = \mathrm{Var}(Y) &= E\left\{Y^2\right\} - E\{Y\}^2 \\ &= 7.4 - (2.6)^2\end{aligned}$$

Question

y	1	3
$p_Y(y)$	0.2	0.8

$$E\{Y\} = 1(0.2) + 3(0.8) = 0.2 + 2.4 = 2.6$$
$$E\{Y^2\} = 1(0.2) + 9(0.8) = 0.2 + 7.2 = 7.4$$

What is σ_Y^2?

Answer:

$$\begin{aligned}
\sigma_Y^2 = \mathrm{Var}(Y) &= E\left\{Y^2\right\} - E\{Y\}^2 \\
&= 7.4 - (2.6)^2 \\
&= 7.4 - 6.76
\end{aligned}$$

Question

y	1	3
$p_Y(y)$	0.2	0.8

$$E\{Y\} = 1(0.2) + 3(0.8) = 0.2 + 2.4 = 2.6$$
$$E\{Y^2\} = 1(0.2) + 9(0.8) = 0.2 + 7.2 = 7.4$$

What is σ_Y^2?

Answer: 0.64

$$\begin{aligned}\sigma_Y^2 = \mathrm{Var}(Y) &= E\left\{Y^2\right\} - E\{Y\}^2 \\ &= 7.4 - (2.6)^2 \\ &= 7.4 - 6.76 = 0.64\end{aligned}$$

Question

y	1	3
$p_Y(y)$	0.2	0.8

What is σ_Y?

Question

y	1	3
$p_Y(y)$	0.2	0.8

What is σ_Y?

Answer:

Question

y	1	3
$p_Y(y)$	0.2	0.8

$$E\{Y\} = 1(0.2) + 3(0.8) = 0.2 + 2.4 = 2.6$$
$$E\{Y^2\} = 1(0.2) + 9(0.8) = 0.2 + 7.2 = 7.4$$

What is σ_Y?

Answer:

$$\begin{aligned} \sigma_Y^2 = \mathrm{Var}(Y) &= E\left\{Y^2\right\} - E\{Y\}^2 \\ &= 7.4 - (2.6)^2 \\ &= 7.4 - 6.76 = 0.64 \end{aligned}$$

Question

y	1	3
$p_Y(y)$	0.2	0.8

$$E\{Y\} = 1(0.2) + 3(0.8) = 0.2 + 2.4 = 2.6$$
$$E\{Y^2\} = 1(0.2) + 9(0.8) = 0.2 + 7.2 = 7.4$$

What is σ_Y?

Answer: 0.8

$$\begin{aligned}\sigma_Y^2 = \mathrm{Var}(Y) &= E\left\{Y^2\right\} - E\{Y\}^2 \\ &= 7.4 - (2.6)^2 \\ &= 7.4 - 6.76 = 0.64\end{aligned}$$

$$\sigma_Y = \sqrt{\sigma_Y^2} = \sqrt{0.64} = 0.8$$

Joint Probability Distributions

Daniel Stephenson

SCMA 524: Statistical Fundamentals

Joint Probability Mass Functions (PMF)

- A **joint probability distribution** describes the probability of events involving multiple random variables.
- The joint probability distribution of a discrete random variable can be described by a **joint probability mass function**

$$p_{X,Y}(a, b) = P(X = a \cap Y = b)$$

Example

$p_{X,Y}(x,y)$	$y = 0$	$y = 2$	$y = 4$
$x = 1$	0.1	0.2	0
$x = 2$	0.2	0	0.1
$x = 3$	0.1	0.1	0.2

$$P(X = 2 \cap Y = 0) \quad = \quad p_{X,Y}(2, 0) = 0.2$$

Question

Suppose the joint PMF of X and Y is given by

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$
$x=0$	0.15	0.3	0.05
$x=3$	0.1	0.05	0.1
$x=5$	0.05	0.1	0.1

What is $P(X=5 \cap Y=2)$?

Question

Suppose the joint PMF of X and Y is given by

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$
$x=0$	0.15	0.3	0.05
$x=3$	0.1	0.05	0.1
$x=5$	0.05	0.1	0.1

What is $P(X=5 \cap Y=2)$?

Answer:

Question

Suppose the joint PMF of X and Y is given by

$p_{X,Y}(x, y)$	$y = 1$	$y = 2$	$y = 3$
$x = 0$	0.15	0.3	0.05
$x = 3$	0.1	0.05	0.1
$x = 5$	0.05	0.1	0.1

What is $P(X = 5 \cap Y = 2)$?

Answer: 0.1

$$P(X = 5 \cap Y = 2) = p_{X,Y}(5, 2) = 0.1$$

The Discrete Joint Uniform Distribution

- Discrete random variables are **jointly uniformly distributed** if each of their possible joint realizations has equal probability.

Example

If the joint PMF of X and Y is given by

$p_{X,Y}(x, y)$	$y = 3$	$y = 6$
$x = 5$	0.25	0.25
$x = 10$	0.25	0.25

Then X and Y are joint uniformly distributed

Marginal Probability Mass Functions

- If $p_{X,Y}(x,y)$ is the joint probability mass function of X and Y then the **marginal probability mass function** of X is

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

Example

$p_{X,Y}(x,y)$	$y=0$	$y=2$	$y=4$	$p_X(x)$
$x=1$	0.1	0.2	0	0.3
$x=2$	0.2	0	0.1	0.3
$x=3$	0.1	0.1	0.2	0.4
$p_Y(y)$	0.4	0.3	0.3	

$$\begin{aligned} p_X(1) &= p_{X,Y}(1,0) + p_{X,Y}(1,2) + p_{X,Y}(1,4) \\ &= 0.1 + 0.2 + 0 = 0.3 \end{aligned}$$

Question

Suppose the joint PMF of X and Y is given by

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$
$x=0$	0.15	0.3	0.05
$x=3$	0.1	0.05	0.1
$x=5$	0.05	0.1	0.1

What is $P(Y=2)$?

Question

Suppose the joint PMF of X and Y is given by

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$
$x=0$	0.15	0.3	0.05
$x=3$	0.1	0.05	0.1
$x=5$	0.05	0.1	0.1

What is $P(Y=2)$?

Answer:

Question

Suppose the joint PMF of X and Y is given by

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$
$x=0$	0.15	0.3	0.05
$x=3$	0.1	0.05	0.1
$x=5$	0.05	0.1	0.1

What is $P(Y=2)$?

Answer: 0.45

$$\begin{aligned} P(Y=2) = p_Y(2) &= p_{X,Y}(0,2) + p_{X,Y}(3,2) + p_{X,Y}(5,2) \\ &= 0.3 + 0.05 + 0.1 \\ &= 0.45 \end{aligned}$$

Expected Value of Discrete Random Variables

- If X, Y, and Z are discrete random variables and $Z = g(X, Y)$ then
$$E\{Z\} = \sum_x \sum_y g(x, y)\, p_{X,Y}(x, y)$$

Example

$p_{X,Y}(x, y)$	$y = 4$	$y = 5$
$x = 2$	0.2	0.5
$x = 3$	0.0	0.3

$$Z = X^2 Y$$

Expected Value of Discrete Random Variables

- If X, Y, and Z are discrete random variables and $Z = g(X, Y)$ then
$$E\{Z\} = \sum_x \sum_y g(x, y)\, p_{X,Y}(x, y)$$

Example

$p_{X,Y}(x, y)$	$y = 4$	$y = 5$
$x = 2$	0.2	0.5
$x = 3$	0.0	0.3

$$\begin{aligned} Z &= X^2 Y \\ E\{Z\} &= \left(2^2 4\right)(0.2) + \left(2^2 5\right)(0.5) + \\ &\quad \left(3^2 4\right)(0.0) + \left(3^2 5\right)(0.3) \end{aligned}$$

Conditional Probability Mass Functions (PMF)

- A **conditional probability mass function** describes the conditional probability of events involving random variables. The conditional PMF of X given Y is

$$p_{X|Y}(a|b) = P(X = a|Y = b) = \frac{p_{X,Y}(a,b)}{p_Y(b)}$$

Example

$p_{X,Y}(x,y)$	$y=0$	$y=2$	$y=4$
$x=1$	0.1	0.2	0
$x=2$	0.2	0	0.1
$x=3$	0.1	0.1	0.2

$$p_X(1) = 0.1 + 0.2 + 0 = 0.3$$

Conditional Probability Mass Functions (PMF)

- A **conditional probability mass function** describes the conditional probability of events involving random variables. The conditional PMF of X given Y is

$$p_{X|Y}(a|b) = P(X = a|Y = b) = \frac{p_{X,Y}(a,b)}{p_Y(b)}$$

Example

$p_{X,Y}(x,y)$	$y = 0$	$y = 2$	$y = 4$
$x = 1$	0.1	0.2	0
$x = 2$	0.2	0	0.1
$x = 3$	0.1	0.1	0.2

$$\begin{aligned} p_X(1) &= 0.1 + 0.2 + 0 = 0.3 \\ p_{Y|X}(2|1) &= \frac{p_{X,Y}(1,2)}{p_X(1)} = \frac{0.2}{0.3} = \frac{2}{3} \end{aligned}$$

Conditional Expectation of Discrete Random Variables

- The conditional expectation of a discrete random variable X given Y is given by

$$E\{X|Y=c\} = \sum_x x\, p_{X|Y}(x|c)$$

Example

$p_{X,Y}(x,y)$	$y=0$	$y=2$
$x=1$	0.05	0.15
$x=2$	0.5	0.3

Conditional Expectation of Discrete Random Variables

- The conditional expectation of a discrete random variable X given Y is given by

$$E\{X|Y=c\} = \sum_{x} x\, p_{X|Y}(x|c)$$

Example

$p_{X,Y}(x,y)$	$y=0$	$y=2$
$x=1$	0.05	0.15
$x=2$	0.5	0.3

$p_X(1) = 0.2$
$p_X(2) = 0.8$

Conditional Expectation of Discrete Random Variables

- The conditional expectation of a discrete random variable X given Y is given by

$$E\{X|Y=c\} = \sum_x x\, p_{X|Y}(x|c)$$

Example

$p_{X,Y}(x,y)$	$y=0$	$y=2$
$x=1$	0.05	0.15
$x=2$	0.5	0.3

$$p_X(1) = 0.2$$
$$p_X(2) = 0.8$$

$$p_{Y|X}(0|2) = \frac{p_{Y,X}(0,2)}{p_X(2)} = \frac{0.5}{0.8} = \frac{5}{8}$$

Conditional Expectation of Discrete Random Variables

- The conditional expectation of a discrete random variable X given Y is given by

$$E\{X|Y=c\} = \sum_x x\, p_{X|Y}(x|c)$$

Example

$p_{X,Y}(x,y)$	$y=0$	$y=2$
$x=1$	0.05	0.15
$x=2$	0.5	0.3

$$p_X(1) = 0.2$$
$$p_X(2) = 0.8$$

$$\begin{aligned}
p_{Y|X}(0|2) &= \frac{p_{Y,X}(0,2)}{p_X(2)} = \frac{0.5}{0.8} = \frac{5}{8} \\
p_{Y|X}(2|2) &= \frac{p_{Y,X}(2,2)}{p_X(2)} = \frac{0.3}{0.8} = \frac{3}{8}
\end{aligned}$$

Conditional Expectation of Discrete Random Variables

- The conditional expectation of a discrete random variable X given Y is given by

$$E\{X|Y=c\} = \sum_x x\, p_{X|Y}(x|c)$$

Example

$p_{X,Y}(x,y)$	$y=0$	$y=2$
$x=1$	0.05	0.15
$x=2$	0.5	0.3

$$p_X(1) = 0.2$$
$$p_X(2) = 0.8$$

$$\begin{aligned} p_{Y|X}(0|2) &= \frac{p_{Y,X}(0,2)}{p_X(2)} = \frac{0.5}{0.8} = \frac{5}{8} \\ p_{Y|X}(2|2) &= \frac{p_{Y,X}(2,2)}{p_X(2)} = \frac{0.3}{0.8} = \frac{3}{8} \\ E\{Y|X=2\} &= 0\, p_{Y|X}(0|2) + 2\, p_{Y|X}(2|2) \end{aligned}$$

Conditional Expectation of Discrete Random Variables

- The conditional expectation of a discrete random variable X given Y is given by

$$E\{X|Y=c\} = \sum_{x} x\, p_{X|Y}(x|c)$$

Example

$p_{X,Y}(x,y)$	$y=0$	$y=2$
$x=1$	0.05	0.15
$x=2$	0.5	0.3

$$p_X(1) = 0.2$$
$$p_X(2) = 0.8$$

$$\begin{aligned}
p_{Y|X}(0|2) &= \frac{p_{Y,X}(0,2)}{p_X(2)} = \frac{0.5}{0.8} = \frac{5}{8} \\
p_{Y|X}(2|2) &= \frac{p_{Y,X}(2,2)}{p_X(2)} = \frac{0.3}{0.8} = \frac{3}{8} \\
E\{Y|X=2\} &= 0\, p_{Y|X}(0|2) + 2\, p_{Y|X}(2|2) = 0 + \frac{3}{4} = 0.75
\end{aligned}$$

Joint Cumulative Distribution Functions (CDF)

- A **joint cumulative distribution function** (CDF) is another way of describing a joint probability distribution.
- The joint CDF for random variables X, Y is

$$F_{X,Y}(a, b) = P(X \leq a \cap Y \leq b)$$

Example

$p_{X,Y}(x, y)$	$y = 0$	$y = 2$	$y = 4$
$x = 1$	0.1	0.2	0
$x = 2$	0.2	0	0.1
$x = 3$	0.1	0.1	0.2

$$\begin{aligned} F_{X,Y}(2, 2) &= p_{X,Y}(1, 0) + p_{X,Y}(1, 2) + p_{X,Y}(2, 0) + p_{X,Y}(2, 2) \\ &= 0.1 + 0.2 + 0.2 + 0 = 0.5 \end{aligned}$$

Question

Suppose the joint PMF of X and Y is given by

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$
$x=0$	0.15	0.3	0.05
$x=3$	0.1	0.05	0.1
$x=5$	0.05	0.1	0.1

What is $F_{X,Y}(2,2.5)$?

Question

Suppose the joint PMF of X and Y is given by

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$
$x=0$	0.15	0.3	0.05
$x=3$	0.1	0.05	0.1
$x=5$	0.05	0.1	0.1

What is $F_{X,Y}(2, 2.5)$?

Answer:

Question

Suppose the joint PMF of X and Y is given by

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$
$x=0$	0.15	0.3	0.05
$x=3$	0.1	0.05	0.1
$x=5$	0.05	0.1	0.1

What is $F_{X,Y}(2, 2.5)$?

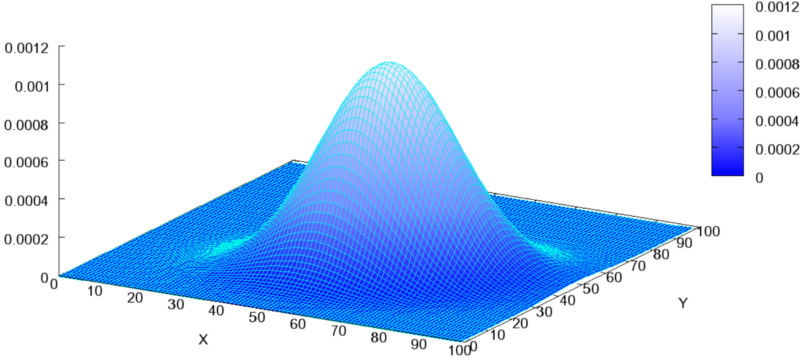
Answer: 0.45

$$
\begin{aligned}
F_{X,Y}(2, 2.5) &= P(X \leq 2 \cap Y \leq 2.5) \\
&= 0.15 + 0.3 \\
&= 0.45
\end{aligned}
$$

Joint Probability Density Functions (PDF)

- The **joint probability density function** (PDF) of continuous random variables X, Y is given by

$$f_{X,Y}(x, y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F_{X,Y}(x, y)$$



The Continuous Joint Uniform Distribution

- Continuous random variables are **joint uniformly distributed** if

$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$

Example

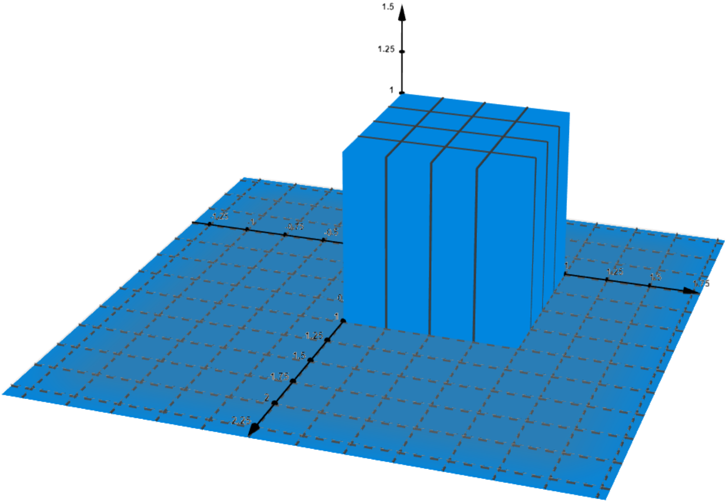
If the joint PDF of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{20} & \text{if } 5 \leq x \leq 10 \text{ and } 0 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Then X and Y are joint uniformly distributed

The Continuous Joint Uniform Distribution

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Joint Probability Density Functions (PDF)

- If $f_{X,Y}$ is the joint PDF of X and Y then

$$P\left(a \leq X \leq b \cap c \leq Y \leq d\right) = \int_a^b \int_c^d f_{X,Y}\left(x, y\right) dy\, dx$$

Example (Link)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0 \leq X \leq 0.5 \cap 0 \leq Y \leq 1) = \int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\, dx$$

Example (Link)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0 \leq X \leq 0.5 \cap 0 \leq Y \leq 1) = \int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\ dx$$

$$\int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy = \int_{y=0}^{y=1} \left(\frac{1}{2}\right) dy \qquad \text{if } 0 \leq x \leq 1$$

Example (Link)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0 \le X \le 0.5 \cap 0 \le Y \le 1) = \int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\, dx$$

$$\begin{aligned} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy &= \int_{y=0}^{y=1} \left(\frac{1}{2}\right) dy \qquad \text{if } 0 \le x \le 1 \\ &= \left[\frac{1}{2} y\right]_{y=0}^{y=1} \end{aligned}$$

Example (Link)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0 \le X \le 0.5 \cap 0 \le Y \le 1) = \int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\; dx$$

$$\begin{aligned} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy &= \int_{y=0}^{y=1} \left(\frac{1}{2}\right) dy \qquad \text{if } 0 \le x \le 1 \\ &= \left[\frac{1}{2} y\right]_{y=0}^{y=1} = \left[\frac{1}{2}(1) - 0(1)\right] \end{aligned}$$

Example (Link)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0 \leq X \leq 0.5 \cap 0 \leq Y \leq 1) = \int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\, dx$$

$$\begin{aligned} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy &= \int_{y=0}^{y=1} \left(\frac{1}{2}\right) dy \qquad \text{if } 0 \leq x \leq 1 \\ &= \left[\frac{1}{2}y\right]_{y=0}^{y=1} = \left[\frac{1}{2}(1) - 0(1)\right] = \frac{1}{2} \end{aligned}$$

Example

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0 \leq X \leq 0.5 \cap 0 \leq Y \leq 1) = \int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\ dx$$

$$\int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy = \frac{1}{2}$$

Example

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0 \leq X \leq 0.5 \cap 0 \leq Y \leq 1) = \int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\ dx$$

$$\int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy = \frac{1}{2}$$

$$\int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\ dx = \int_{x=0}^{x=0.5} \left[\frac{1}{2}\right] dx$$

Example

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0 \le X \le 0.5 \cap 0 \le Y \le 1) = \int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\ dx$$

$$\int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy = \frac{1}{2}$$

$$\int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\ dx = \int_{x=0}^{x=0.5} \left[\frac{1}{2}\right] dx = \left[\frac{1}{2}x\right]_{x=0}^{x=0.5}$$

Example

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

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$$\int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy = \frac{1}{2}$$

$$\begin{aligned} \int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\ dx &= \int_{x=0}^{x=0.5} \left[\frac{1}{2}\right] dx = \left[\frac{1}{2}x\right]_{x=0}^{x=0.5} \\ &= \left[\frac{1}{2}(0.5) - \frac{1}{2}0\right] \end{aligned}$$

Example

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

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$$\begin{aligned} \int_{x=0}^{x=0.5} \int_{y=0}^{y=1} f_{X,Y}(x,y)\, dy\; dx &= \int_{x=0}^{x=0.5} \left[\frac{1}{2}\right] dx = \left[\frac{1}{2}x\right]_{x=0}^{x=0.5} \\ &= \left[\frac{1}{2}(0.5) - \frac{1}{2}0\right] = \frac{1}{4} \end{aligned}$$

Expected Value of Continuous Random Variables

- If X, Y, and Z are continuous random variables

$$E\left\{g\left(X,Y\right)\right\} = \int\int g\left(x,y\right) f_{X,Y}\left(x,y\right) dx\, dy$$

Example

$$
\begin{aligned}
f_{X,Y}(x,y) &= \begin{cases} x+y & \text{if } 0 \leq X \leq 1 \text{ and } 0 \leq Y \leq 1 \\ 0 & \text{otherwise} \end{cases} \\
E\{XY\} &= \int\int x\,y\,f_{X,Y}(x,y)\,dx\,dy
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{X,Y}(x,y) &= \begin{cases} x+y & \text{if } 0 \le X \le 1 \text{ and } 0 \le Y \le 1 \\ 0 & \text{otherwise} \end{cases} \\
E\{XY\} &= \int\int x\,y\,f_{X,Y}(x,y)\,dx\,dy
\end{aligned}
$$

$$
\int x\,y\,f_{X,Y}(x,y)\,dx = \int_0^1 x\,y\,(x+y)\,dx
$$

Example

$$
\begin{aligned}
f_{X,Y}(x,y) &= \begin{cases} x+y & \text{if } 0 \leq X \leq 1 \text{ and } 0 \leq Y \leq 1 \\ 0 & \text{otherwise} \end{cases} \\
E\{XY\} &= \int\int x\,y\,f_{X,Y}(x,y)\,dx\,dy
\end{aligned}
$$

$$
\begin{aligned}
\int x\,y\,f_{X,Y}(x,y)\,dx &= \int_0^1 x\,y\,(x+y)\,dx \\
&= \int_0^1 \left(x^2 y + y^2 x\right) dx
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{X,Y}(x,y) &= \begin{cases} x+y & \text{if } 0 \le X \le 1 \text{ and } 0 \le Y \le 1 \\ 0 & \text{otherwise} \end{cases} \\
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\int x\,y\,f_{X,Y}(x,y)\,dx &= \int_0^1 x\,y\,(x+y)\,dx \\
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&= \left[\frac{1}{3}x^3 y + \frac{1}{2}y^2 x^2\right]_{x=0}^{x=1}
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{X,Y}(x,y) &= \begin{cases} x+y & \text{if } 0 \le X \le 1 \text{ and } 0 \le Y \le 1 \\ 0 & \text{otherwise} \end{cases} \\
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&= \left[\frac{1}{3}x^3 y + \frac{1}{2}y^2 x^2\right]_{x=0}^{x=1} \\
&= \left(\frac{1}{3}(1)\,y + \frac{1}{2}y^2(1)\right) - \left(\frac{1}{3}(0)\,y + \frac{1}{2}y^2(0)\right)
\end{aligned}
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Example

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&= \left(\frac{1}{3}(1)\,y + \frac{1}{2}y^2(1)\right) - \left(\frac{1}{3}(0)\,y + \frac{1}{2}y^2(0)\right) \\
&= \frac{1}{3}y + \frac{1}{2}y^2
\end{aligned}
$$

Example (Link)

$$
\begin{aligned}
f_{X,Y}(x,y) &= \begin{cases} x + y & \text{if } 0 \leq X \leq 1 \text{ and } 0 \leq Y \leq 1 \\ 0 & \text{otherwise} \end{cases} \\
E\{XY\} &= \int\int x\, y\, f_{X,Y}(x,y)\, dx\, dy
\end{aligned}
$$

$$
\int x\, y\, f_{X,Y}(x,y)\, dx = \frac{1}{3}y + \frac{1}{2}y^2
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$$
\begin{aligned}
\int x\,y\,f_{X,Y}(x,y)\,dx &= \frac{1}{3}y + \frac{1}{2}y^2 \\
E\{XY\} &= \int_0^1 \frac{1}{3}y + \frac{1}{2}y^2 dy \\
&= \left[\frac{1}{6}y^2 + \frac{1}{6}y^3\right]_{y=0}^{y=1}
\end{aligned}
$$

Example (Link)

$$f_{X,Y}(x,y) = \begin{cases} x+y & \text{if } 0 \leq X \leq 1 \text{ and } 0 \leq Y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E\{XY\} = \int\int x\, y\, f_{X,Y}(x,y)\, dx\, dy$$

$$\int x\, y\, f_{X,Y}(x,y)\, dx = \frac{1}{3}y + \frac{1}{2}y^2$$

$$\begin{aligned} E\{XY\} &= \int_0^1 \frac{1}{3}y + \frac{1}{2}y^2 dy \\ &= \left[\frac{1}{6}y^2 + \frac{1}{6}y^3\right]_{y=0}^{y=1} \\ &= \left(\frac{1}{6}(1) + \frac{1}{6}(1)\right) - \left(\frac{1}{6}(0) + \frac{1}{6}(0)\right) \end{aligned}$$

Example (Link)

$$
\begin{aligned}
f_{X,Y}(x,y) &= \begin{cases} x+y & \text{if } 0 \le X \le 1 \text{ and } 0 \le Y \le 1 \\ 0 & \text{otherwise} \end{cases} \\
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&= \left[\frac{1}{6}y^2 + \frac{1}{6}y^3\right]_{y=0}^{y=1} \\
&= \left(\frac{1}{6}(1) + \frac{1}{6}(1)\right) - \left(\frac{1}{6}(0) + \frac{1}{6}(0)\right) \\
&= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
\end{aligned}
$$

Marginal Probability Density Functions

- If $f_{X,Y}(x,y)$ is the joint probability density of X and Y then the **marginal probability density** of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

Example

$$f_{X,Y}(x,y) = \begin{cases} 4xy & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

For $0 \le x \le 1$ the marginal PDF of X is

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$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 4xy\,dy = 4x\int_0^1 y\,dy \\ &= 4x\left[\frac{1}{2}y^2\right]_{y=0}^{y=1} \end{aligned}$$

Marginal Probability Density Functions

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$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = \int_0^1 4xy\, dy = 4x \int_0^1 y\, dy \\ &= 4x \left[\frac{1}{2} y^2\right]_{y=0}^{y=1} = 4x \left[\frac{1}{2}(1)^2 - \frac{1}{2}(0)^2\right] = 2x \end{aligned}$$

Conditional Probability Density Functions (PDF)

- If $f_Y(y) > 0$ then the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Example

If $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ then

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)\, f_Y(y)}{f_Y(y)} = f_X(x)$$

Question

$$f(x,y) = \begin{cases} 8xy^3 & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is $f_{X|Y}(0.5 \mid 1)$?

Question

$$f(x,y) = \begin{cases} 8xy^3 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

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What is $f_{X|Y}(\, 0.5 \,|\, 1)$?

Answer:

$$f_{X|Y}(0.5|1) = \frac{f_{X,Y}(0.5,1)}{f_Y(1)}$$

Question

$$f(x, y) = \begin{cases} 8xy^3 & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is $f_{X|Y}(0.5 \mid 1)$?

Answer: 2

$$f_{X|Y}(0.5|1) \quad = \quad \frac{f_{X,Y}(0.5, 1)}{f_Y(1)} = \frac{8(0.5)(1)^3}{2(1)} = \frac{4}{2} = 2$$

Conditional Expectation of Continuous Random Variables

- The conditional expectation of a continuous random variable X given Y is

$$E\{X|Y=c\} = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|c)\, dx$$

Example

$$f_{X|Y}(x|y) = \begin{cases} 1 & \text{if } y \leq x \leq y+1 \\ 0 & \text{otherwise} \end{cases}$$

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E\{X|Y=2\} &= \int_{-\infty}^{\infty} x\, f_{X|Y}(x|2)\, dx
\end{aligned}$$

Conditional Expectation of Continuous Random Variables

- The conditional expectation of a continuous random variable X given Y is

$$E\left\{X|Y=c\right\}=\int_{-\infty}^{\infty} x\, f_{X|Y}\left(x|c\right) dx$$

Example

$$f_{X|Y}\left(x|y\right) = \begin{cases} 1 & \text{if } y \leq x \leq y+1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X|Y}\left(x|2\right) = \begin{cases} 1 & \text{if } 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$E\left\{X|Y=2\right\} = \int_{-\infty}^{\infty} x\, f_{X|Y}\left(x|2\right) dx = \int_{2}^{3} x\, dx = \left[\frac{1}{2}x^2\right]_2^3$$

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$$\begin{aligned} E\{X|Y=2\} &= \int_{-\infty}^{\infty} x\, f_{X|Y}(x|2)\, dx = \int_{2}^{3} x\, dx = \left[\frac{1}{2}x^2\right]_2^3 \\ &= \left[\frac{1}{2}(3)^2 - \frac{1}{2}(2)^2\right] \end{aligned}$$

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f_{X|Y}(x|2) &= \begin{cases} 1 & \text{if } 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases} \\
E\{X|Y=2\} &= \int_{-\infty}^{\infty} x\, f_{X|Y}(x|2)\, dx = \int_{2}^{3} x\, dx = \left[\frac{1}{2}x^2\right]_2^3 \\
&= \left[\frac{1}{2}(3)^2 - \frac{1}{2}(2)^2\right] = \frac{9}{2} - \frac{4}{2} = \frac{5}{2} = 2.5
\end{aligned}$$

The Law of Iterated Expectations

- The expected value of the conditional expectation of X given Y is the same as the unconditional expected value of X.

$$E\Big\{E\{X|Y\}\Big\} = E\{X\}$$

Example

Suppose $p_Y(2) = p_Y(3) = 0.5$ and $E\{X|Y\} = 4 + 5Y$

The Law of Iterated Expectations

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$$E\left\{X\right\} \quad = \quad E\Big\{E\left\{X|Y\right\}\Big\}$$

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Example

Suppose $p_Y(2) = p_Y(3) = 0.5$ and $E\left\{X|Y\right\} = 4 + 5Y$

$$\begin{aligned} E\left\{X\right\} &= E\Big\{E\left\{X|Y\right\}\Big\} \\ &= P(Y=2)\,E\left\{X|Y=2\right\} + P(Y=3)\,E\left\{X|Y=3\right\} \end{aligned}$$

The Law of Iterated Expectations

- The expected value of the conditional expectation of X given Y is the same as the unconditional expected value of X.

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Example

Suppose $p_Y(2) = p_Y(3) = 0.5$ and $E\{X|Y\} = 4 + 5Y$

$$\begin{aligned} E\{X\} &= E\Big\{E\{X|Y\}\Big\} \\ &= P(Y=2)\,E\{X|Y=2\} + P(Y=3)\,E\{X|Y=3\} \\ &= (0.5)\,[4+5(2)] + (0.5)\,[4+5(3)] \end{aligned}$$

The Law of Iterated Expectations

- The expected value of the conditional expectation of X given Y is the same as the unconditional expected value of X.

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Suppose $p_Y(2) = p_Y(3) = 0.5$ and $E\{X|Y\} = 4 + 5Y$

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E\{X\} &= E\Big\{E\{X|Y\}\Big\} \\
&= P(Y=2)\,E\{X|Y=2\} + P(Y=3)\,E\{X|Y=3\} \\
&= (0.5)\,[4 + 5(2)] + (0.5)\,[4 + 5(3)] \\
&= (0.5)\,[14] + (0.5)\,[19] = 16.5
\end{aligned}$$

Taking out Known Terms

- Known terms can come out of the conditional expectation

$$E\left\{f(Y)\,X|Y\right\} = f(Y)\,E\{X|Y\}$$

Example

If $E\{Z|W\} = 2W$ and $E\{W^3\} = 5$ then

Taking out Known Terms

- Known terms can come out of the conditional expectation

$$E\left\{f(Y)X|Y\right\} = f(Y)E\{X|Y\}$$

Example

If $E\{Z|W\} = 2W$ and $E\{W^3\} = 5$ then

$$E\left\{W^2 Z\right\} = E\Big\{E\left\{W^2 Z|W\right\}\Big\} \qquad \text{law of iterated expectations}$$

Taking out Known Terms

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Example

If $E\{Z|W\} = 2W$ and $E\{W^3\} = 5$ then

$$\begin{aligned} E\left\{W^2 Z\right\} &= E\Big\{E\left\{W^2 Z|W\right\}\Big\} && \text{law of iterated expectations} \\ &= E\Big\{W^2 E\{Z|W\}\Big\} && \text{pull out the known term } W^2 \end{aligned}$$

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&= E\left\{W^2(2W)\right\} \\
&= E\left\{2W^3\right\}
\end{aligned}$$

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E\left\{W^2 Z\right\} &= E\left\{E\left\{W^2 Z|W\right\}\right\} && \text{law of iterated expectations} \\
&= E\left\{W^2 E\{Z|W\}\right\} && \text{pull out the known term } W^2 \\
&= E\left\{W^2 (2W)\right\} \\
&= E\left\{2W^3\right\} \\
&= 2E\left\{W^3\right\}
\end{aligned}$$

Taking out Known Terms

- Known terms can come out of the conditional expectation

$$E\left\{f(Y)X|Y\right\} = f(Y)E\{X|Y\}$$

Example

If $E\{Z|W\} = 2W$ and $E\left\{W^3\right\} = 5$ then

$$\begin{aligned}
E\left\{W^2 Z\right\} &= E\left\{E\left\{W^2 Z|W\right\}\right\} && \text{law of iterated expectations} \\
&= E\left\{W^2 E\{Z|W\}\right\} && \text{pull out the known term } W^2 \\
&= E\left\{W^2(2W)\right\} \\
&= E\left\{2W^3\right\} \\
&= 2E\left\{W^3\right\} = 2(5) = 10
\end{aligned}$$

Example

Suppose $E\left\{X|Y\right\} = \sqrt{Y}$ and $E\left\{Y\right\} = 3$

$$E\left\{X\sqrt{Y} + Y\right\}$$

Example

Suppose $E\{X|Y\} = \sqrt{Y}$ and $E\{Y\} = 3$

$$\begin{aligned} & E\left\{X\sqrt{Y} + Y\right\} \\ = \; & E\left\{X\sqrt{Y}\right\} + E\{Y\} \end{aligned}$$

Example

Suppose $E\{X|Y\} = \sqrt{Y}$ and $E\{Y\} = 3$

$$\begin{aligned}
& E\left\{X\sqrt{Y} + Y\right\} \\
= \; & E\left\{X\sqrt{Y}\right\} + E\{Y\} \\
= \; & E\left\{X\sqrt{Y}\right\} + 3
\end{aligned}$$

Example

Suppose $E\{X|Y\} = \sqrt{Y}$ and $E\{Y\} = 3$

$$\begin{aligned} & E\left\{X\sqrt{Y} + Y\right\} \\ = \;& E\left\{X\sqrt{Y}\right\} + E\{Y\} \\ = \;& E\left\{X\sqrt{Y}\right\} + 3 \\ = \;& E\left\{E\left\{X\sqrt{Y}|Y\right\}\right\} + 3 \qquad \text{Law of Iterated Expectations} \end{aligned}$$

Example

Suppose $E\{X|Y\} = \sqrt{Y}$ and $E\{Y\} = 3$

$$\begin{aligned}
& E\left\{X\sqrt{Y} + Y\right\} \\
= \;& E\left\{X\sqrt{Y}\right\} + E\{Y\} \\
= \;& E\left\{X\sqrt{Y}\right\} + 3 \\
= \;& E\left\{E\left\{X\sqrt{Y}|Y\right\}\right\} + 3 \qquad \text{Law of Iterated Expectations} \\
= \;& E\left\{\sqrt{Y}E\{X|Y\}\right\} + 3 \qquad \text{Pull out Known Terms}
\end{aligned}$$

Example

Suppose $E\left\{X|Y\right\} = \sqrt{Y}$ and $E\left\{Y\right\} = 3$

$$\begin{aligned}
& E\left\{X\sqrt{Y} + Y\right\} \\
= \; & E\left\{X\sqrt{Y}\right\} + E\left\{Y\right\} \\
= \; & E\left\{X\sqrt{Y}\right\} + 3 \\
= \; & E\left\{E\left\{X\sqrt{Y}|Y\right\}\right\} + 3 && \text{Law of Iterated Expectations} \\
= \; & E\left\{\sqrt{Y}E\left\{X|Y\right\}\right\} + 3 && \text{Pull out Known Terms} \\
= \; & E\left\{\sqrt{Y}\left(\sqrt{Y}\right)\right\} + 3
\end{aligned}$$

Example

Suppose $E\left\{X|Y\right\} = \sqrt{Y}$ and $E\left\{Y\right\} = 3$

$$\begin{aligned}
& E\left\{X\sqrt{Y} + Y\right\} \\
= \; & E\left\{X\sqrt{Y}\right\} + E\left\{Y\right\} \\
= \; & E\left\{X\sqrt{Y}\right\} + 3 \\
= \; & E\left\{E\left\{X\sqrt{Y}|Y\right\}\right\} + 3 && \text{Law of Iterated Expectations} \\
= \; & E\left\{\sqrt{Y}E\left\{X|Y\right\}\right\} + 3 && \text{Pull out Known Terms} \\
= \; & E\left\{\sqrt{Y}\left(\sqrt{Y}\right)\right\} + 3 \\
= \; & E\left\{Y\right\} + 3
\end{aligned}$$

Example

Suppose $E\left\{X|Y\right\} = \sqrt{Y}$ and $E\left\{Y\right\} = 3$

$$
\begin{aligned}
& E\left\{X\sqrt{Y} + Y\right\} \\
= \;& E\left\{X\sqrt{Y}\right\} + E\left\{Y\right\} \\
= \;& E\left\{X\sqrt{Y}\right\} + 3 \\
= \;& E\left\{E\left\{X\sqrt{Y}|Y\right\}\right\} + 3 && \text{Law of Iterated Expectations} \\
= \;& E\left\{\sqrt{Y}E\left\{X|Y\right\}\right\} + 3 && \text{Pull out Known Terms} \\
= \;& E\left\{\sqrt{Y}\left(\sqrt{Y}\right)\right\} + 3 \\
= \;& E\left\{Y\right\} + 3 \\
= \;& 3 + 3 = 6
\end{aligned}
$$

Independent Random Variables

- X and Y are **independent random variables** if

$$\begin{aligned} F_{X,Y}(a,b) &= F_X(a)\,F_Y(b) \\ p_{X,Y}(a,b) &= p_X(a)\,p_Y(b) \quad \text{if } X \text{ and } Y \text{ are discrete} \\ f_{X,Y}(a,b) &= f_X(a)\,f_Y(b) \quad \text{if } X \text{ and } Y \text{ are continuous} \end{aligned}$$

- If X and Y are independent random variables then

$$E\{XY\} = E\{X\}\,E\{Y\}$$

Independent Random Variables (Link)

Example

Suppose $f_X(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right)$

Suppose $f_Y(y) = \begin{cases} 0.5 & \text{if } 2 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$

If X and Y are independent random variables then

$$\begin{aligned} f_{X,Y}(1,3) &= f_X(1)\, f_Y(3) \\ &= \frac{1}{\pi}\left(\frac{1}{1+(1)^2}\right)(0.5) \\ &= \frac{1}{\pi}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4\pi} \end{aligned}$$

Covariance

- The **covariance** of X and Y is

$$\sigma_{XY} = \mathrm{Cov}\,(X, Y) \quad = \quad E\left\{(X - \mu_X)\,(Y - \mu_Y)\right\}$$

Covariance

- The **covariance** of X and Y is

$$\begin{aligned}\sigma_{XY} = \mathrm{Cov}(X, Y) &= E\{(X - \mu_X)(Y - \mu_Y)\} \\ &= E\{XY - X\mu_Y - \mu_X Y + \mu_X \mu_Y\}\end{aligned}$$

Covariance

- The **covariance** of X and Y is

$$\begin{aligned}
\sigma_{XY} = \mathrm{Cov}(X, Y) &= E\{(X - \mu_X)(Y - \mu_Y)\} \\
&= E\{XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y\} \\
&= E\{XY\} - E\{X\}\mu_Y - \mu_X E\{Y\} + \mu_X\mu_Y
\end{aligned}$$

Covariance

- The **covariance** of X and Y is

$$\begin{aligned}
\sigma_{XY} = \mathrm{Cov}(X, Y) &= E\{(X - \mu_X)(Y - \mu_Y)\} \\
&= E\{XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y\} \\
&= E\{XY\} - E\{X\}\mu_Y - \mu_X E\{Y\} + \mu_X\mu_Y \\
&= E\{XY\} - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y
\end{aligned}$$

Covariance

- The **covariance** of X and Y is

$$\begin{aligned}
\sigma_{XY} = \mathrm{Cov}(X, Y) &= E\{(X - \mu_X)(Y - \mu_Y)\} \\
&= E\{XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y\} \\
&= E\{XY\} - E\{X\}\mu_Y - \mu_X E\{Y\} + \mu_X\mu_Y \\
&= E\{XY\} - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y \\
&= E\{XY\} - \mu_X\mu_Y
\end{aligned}$$

Covariance

- The **covariance** of X and Y is

$$\begin{aligned}
\sigma_{XY} = \mathrm{Cov}(X, Y) &= E\{(X - \mu_X)(Y - \mu_Y)\} \\
&= E\{XY - X\mu_Y - \mu_X Y + \mu_X \mu_Y\} \\
&= E\{XY\} - E\{X\}\mu_Y - \mu_X E\{Y\} + \mu_X \mu_Y \\
&= E\{XY\} - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y \\
&= E\{XY\} - \mu_X \mu_Y \\
&= E\{XY\} - E\{X\} E\{Y\}
\end{aligned}$$

Covariance

- The **covariance** of X and Y is

$$\begin{aligned}\sigma_{XY} = \mathrm{Cov}(X, Y) &= E\{(X-\mu_X)(Y-\mu_Y)\} \\ &= E\{XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y\} \\ &= E\{XY\} - E\{X\}\mu_Y - \mu_X E\{Y\} + \mu_X\mu_Y \\ &= E\{XY\} - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y \\ &= E\{XY\} - \mu_X\mu_Y \\ &= E\{XY\} - E\{X\}E\{Y\}\end{aligned}$$

Example

If X, Y are independent then

$$\begin{aligned}\mathrm{Cov}(X, Y) &= E\{XY\} - E\{X\}E\{Y\} \\ &= E\{X\}E\{Y\} - E\{X\}E\{Y\} = 0\end{aligned}$$

Example

$p_{X,Y}(x,y)$	$y=1$	$y=2$
$x=0$	0.1	0.6
$x=3$	0.3	0

Example

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$p_X(x)$
$x=0$	0.1	0.6	0.7
$x=3$	0.3	0	0.3
$p_Y(y)$	0.4	0.6	

Example

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$p_X(x)$
$x=0$	0.1	0.6	0.7
$x=3$	0.3	0	0.3
$p_Y(y)$	0.4	0.6	

$$E\{X\} = 0.7(0) + 0.3(3) = 0.9$$

Example

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$p_X(x)$
$x=0$	0.1	0.6	0.7
$x=3$	0.3	0	0.3
$p_Y(y)$	0.4	0.6	

$$\begin{aligned} E\{X\} &= 0.7(0) + 0.3(3) = 0.9 \\ E\{Y\} &= 0.4(1) + 0.6(2) = 1.6 \end{aligned}$$

Example

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$p_X(x)$
$x=0$	0.1	0.6	0.7
$x=3$	0.3	0	0.3
$p_Y(y)$	0.4	0.6	

$$\begin{aligned} E\{X\} &= 0.7(0)+0.3(3)=0.9 \\ E\{Y\} &= 0.4(1)+0.6(2)=1.6 \end{aligned}$$

$$\begin{aligned} E\{XY\} &= (0.1)(0)(1)+(0.6)(0)(2)+ \\ &\quad (0.3)(3)(1)+(0)(3)(2) \end{aligned}$$

Example

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$p_X(x)$
$x=0$	0.1	0.6	0.7
$x=3$	0.3	0	0.3
$p_Y(y)$	0.4	0.6	

$$\begin{aligned}
E\{X\} &= 0.7(0)+0.3(3)=0.9\\
E\{Y\} &= 0.4(1)+0.6(2)=1.6
\end{aligned}$$

$$\begin{aligned}
E\{XY\} &= (0.1)(0)(1)+(0.6)(0)(2)+\\
&\quad (0.3)(3)(1)+(0)(3)(2)\\
&= (0.3)(3)=0.9
\end{aligned}$$

Example

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$p_X(x)$
$x=0$	0.1	0.6	0.7
$x=3$	0.3	0	0.3
$p_Y(y)$	0.4	0.6	

$$\begin{aligned}
E\{X\} &= 0.7(0)+0.3(3)=0.9 \\
E\{Y\} &= 0.4(1)+0.6(2)=1.6
\end{aligned}$$

$$\begin{aligned}
E\{XY\} &= (0.1)(0)(1)+(0.6)(0)(2)+ \\
&\quad (0.3)(3)(1)+(0)(3)(2) \\
&= (0.3)(3)=0.9
\end{aligned}$$

$$\sigma_{XY}=\mathrm{Cov}(X,Y) = E\{XY\}-E\{X\}E\{Y\}$$

Example

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$p_X(x)$
$x=0$	0.1	0.6	0.7
$x=3$	0.3	0	0.3
$p_Y(y)$	0.4	0.6	

$$
\begin{aligned}
E\{X\} &= 0.7(0) + 0.3(3) = 0.9 \\
E\{Y\} &= 0.4(1) + 0.6(2) = 1.6
\end{aligned}
$$

$$
\begin{aligned}
E\{XY\} &= (0.1)(0)(1) + (0.6)(0)(2) + \\
&\quad (0.3)(3)(1) + (0)(3)(2) \\
&= (0.3)(3) = 0.9
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{XY} = \mathrm{Cov}(X,Y) &= E\{XY\} - E\{X\}E\{Y\} \\
&= 0.9 - (0.9)(1.6) = -0.54
\end{aligned}
$$

Covariance

Example

If X is a random variable and a is a constant then

$$
\begin{aligned}
\mathrm{Cov}\left(a, X\right) &= E\left\{aX\right\} - E\left\{a\right\}E\left\{X\right\} \\
&= aE\left\{X\right\} - aE\left\{X\right\} \\
&= 0
\end{aligned}
$$

Example

If Y is a random variable then

$$
\begin{aligned}
\mathrm{Cov}\left(Y, Y\right) &= E\left\{YY\right\} - E\left\{Y\right\}E\left\{Y\right\} \\
&= E\left\{Y^2\right\} - E\left\{Y\right\}^2 \\
&= \mathrm{Var}\left(Y\right)
\end{aligned}
$$

Example

If X, Y, and Z are random variables then

$$\operatorname{Cov}(X+Y, Z) = E\{(X+Y)Z\} - E\{X+Y\}E\{Z\}$$

Example

If X, Y, and Z are random variables then

$$\begin{aligned}
\operatorname{Cov}(X+Y, Z) &= E\{(X+Y)Z\} - E\{X+Y\}E\{Z\} \\
&= E\{XZ+YZ\} - (E\{X\}+E\{Y\})E\{Z\}
\end{aligned}$$

Example

If X, Y, and Z are random variables then

$$\begin{aligned}
\mathrm{Cov}\left(X+Y,Z\right) &= E\left\{\left(X+Y\right)Z\right\} - E\left\{X+Y\right\}E\left\{Z\right\} \\
&= E\left\{XZ+YZ\right\} - \left(E\left\{X\right\}+E\left\{Y\right\}\right)E\left\{Z\right\} \\
&= E\left\{XZ\right\} + E\left\{YZ\right\} - E\left\{X\right\}E\left\{Z\right\} - E\left\{Y\right\}E\left\{Z\right\}
\end{aligned}$$

Example

If X, Y, and Z are random variables then

$$\begin{aligned}
\mathrm{Cov}\left(X+Y,Z\right) &= E\left\{(X+Y)\,Z\right\} - E\left\{X+Y\right\}E\left\{Z\right\} \\
&= E\left\{XZ+YZ\right\} - \left(E\left\{X\right\}+E\left\{Y\right\}\right)E\left\{Z\right\} \\
&= E\left\{XZ\right\} + E\left\{YZ\right\} - E\left\{X\right\}E\left\{Z\right\} - E\left\{Y\right\}E\left\{Z\right\} \\
&= E\left\{XZ\right\} - E\left\{X\right\}E\left\{Z\right\} + E\left\{YZ\right\} - E\left\{Y\right\}E\left\{Z\right\}
\end{aligned}$$

Example

If X, Y, and Z are random variables then

$$
\begin{aligned}
\mathrm{Cov}\,(X+Y,Z) &= E\,\{(X+Y)\,Z\} - E\,\{X+Y\}\,E\,\{Z\} \\
&= E\,\{XZ+YZ\} - (E\,\{X\}+E\,\{Y\})\,E\,\{Z\} \\
&= E\,\{XZ\} + E\,\{YZ\} - E\,\{X\}\,E\,\{Z\} - E\,\{Y\}\,E\,\{Z\} \\
&= E\,\{XZ\} - E\,\{X\}\,E\,\{Z\} + E\,\{YZ\} - E\,\{Y\}\,E\,\{Z\} \\
&= \mathrm{Cov}\,(X,Z) + \mathrm{Cov}\,(Y,Z)
\end{aligned}
$$

Covariance

Example

If X, Y are random variables and a, b are constants then

$$\begin{aligned}
\operatorname{Cov}(aX, bY) &= ab \operatorname{Cov}(X, Y) \\
\operatorname{Cov}(X + a, Y + b) &= \operatorname{Cov}(X, Y) \\
\operatorname{Var}(aX + bY) &= a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y) - 2ab \operatorname{Cov}(X, Y)
\end{aligned}$$

Correlation

- The **correlation** of X and Y is

$$\rho_{XY} = \operatorname{Corr}(X, Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sqrt{\sigma_X^2 \sigma_Y^2}} = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X) \operatorname{Var}(Y)}}$$

- Correlation has a value between -1 and 1

    0 is no correlation.

    1 is total positive correlation.

    -1 is total negative correlation.

Question

If $\operatorname{Cov}(Z, W) = 3$ then what is $\operatorname{Cov}(2W, 7Z)$?

Question

If $\text{Cov}(Z, W) = 3$ then what is $\text{Cov}(2W, 7Z)$?

Answer:

Question

If $\text{Cov}(Z, W) = 3$ then what is $\text{Cov}(2W, 7Z)$?

Answer: 42

$$\begin{aligned} \text{Cov}(2W, 7Z) &= (2)(7)\,\text{Cov}(W, Z) \\ &= (14)(3) \\ &= 42 \end{aligned}$$

Question

Suppose $\operatorname{Cov}(W, X) = -2$ and $\operatorname{Var}(X) = 4$ and $\operatorname{Var}(Y) = 6$

What is $\operatorname{Var}\{3X + 4Y\}$?

Question

Suppose $\mathrm{Cov}\,(W, X) = -2$ and $\mathrm{Var}\,(X) = 4$ and $\mathrm{Var}\,(Y) = 6$

What is $\mathrm{Var}\,\{3X + 4Y\}$?

Answer:

Question

Suppose $\text{Cov}(W, X) = -2$ and $\text{Var}(X) = 4$ and $\text{Var}(Y) = 6$

What is $\text{Var}\{3X + 4Y\}$?

Answer: 180

$$\begin{aligned}
\text{Var}(3X + 4Y) &= 3^2 \text{Var}(X) + 4^2 \text{Var}(Y) - 2(3)(4)\ \text{Cov}(X, Y) \\
&= (9)(4) + (16)(6) - (24)(-2) \\
&= 36 + 96 + 48 \\
&= 180
\end{aligned}$$

Question

Suppose $\text{Cov}(W, Z) = 2.5$ and $\text{Cov}(Y, Z) = 3.5$

What is $\text{Cov}(Y - 2W, 3Z)$?

Question

Suppose $\text{Cov}(W, Z) = 2.5$ and $\text{Cov}(Y, Z) = 3.5$

What is $\text{Cov}(Y - 2W, 3Z)$?

Answer:

Question

Suppose $\text{Cov}(W, Z) = 2.5$ and $\text{Cov}(Y, Z) = 3.5$

What is $\text{Cov}(Y - 2W, 3Z)$?

Answer: -4.5

$$\begin{aligned}
\text{Cov}(Y - 2W, 3Z) &= \text{Cov}(Y, 3Z) + \text{Cov}(-2W, 3Z) \\
&= 3\text{Cov}(Y, Z) + (-2)(3)\,\text{Cov}(W, Z) \\
&= 3(3.5) - (6)(2.5) \\
&= 10.5 - 15 \\
&= -4.5
\end{aligned}$$

Question

Suppose $\mathrm{Cov}(X, Y) = 3$ and $\mathrm{Var}(X) = 9$ and $\mathrm{Var}(Y) = 4$

What is ρ_{XY} ?

Question

Suppose $\operatorname{Cov}(X, Y) = 3$ and $\operatorname{Var}(X) = 9$ and $\operatorname{Var}(Y) = 4$

What is ρ_{XY} ?

Answer:

Question

Suppose $\text{Cov}(X, Y) = 3$ and $\text{Var}(X) = 9$ and $\text{Var}(Y) = 4$

What is ρ_{XY} ?

Answer: 0.5

$$
\begin{aligned}
\rho_{XY} = \text{Corr}(X, Y) &= \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} \\
&= \frac{3}{\sqrt{(9)(4)}} \\
&= \frac{3}{\sqrt{36}} \\
&= \frac{3}{6} = 0.5
\end{aligned}
$$

Question

Suppose W and Z are independent random variables.

Suppose $p_W(2) = 0.5$ and $p_W(3) = 0.5$

Suppose $p_Z(4) = 0.1$ and $p_Z(5) = 0.9$

What is $E\left\{Z^2 W\right\}$?

Question

Suppose W and Z are independent random variables.

Suppose $p_W(2) = 0.5$ and $p_W(3) = 0.5$

Suppose $p_Z(4) = 0.1$ and $p_Z(5) = 0.9$

What is $E\left\{Z^2 W\right\}$?

Answer:

Question

Suppose W and Z are independent random variables.

Suppose $p_W(2) = 0.5$ and $p_W(3) = 0.5$

Suppose $p_Z(4) = 0.1$ and $p_Z(5) = 0.9$

What is $E\left\{Z^2 W\right\}$?

Answer: 60.25

$$
\begin{aligned}
E\{W\} &= 0.5\,(2) + 0.5\,(3) = 2.5 \\
E\left\{Z^2\right\} &= 0.1\left(4^2\right) + 0.9\left(5^2\right) = 24.1 \\
E\left\{Z^2 W\right\} &= E\left\{Z^2\right\} E\{W\} \\
&= (24.1)\,(2.5) \\
&= 60.25
\end{aligned}
$$

Question

Suppose $\text{Cov}(X, Y) = 0$ and $\text{Var}(X) = 5$ and $\text{Var}(Y) = 1$

What is $\text{Var}\{2X + 3Y\}$?

Question

Suppose $\operatorname{Cov}(X, Y) = 0$ and $\operatorname{Var}(X) = 5$ and $\operatorname{Var}(Y) = 1$

What is $\operatorname{Var}\{2X + 3Y\}$?

Answer:

Question

Suppose $\mathrm{Cov}(X, Y) = 0$ and $\mathrm{Var}(X) = 5$ and $\mathrm{Var}(Y) = 1$

What is $\mathrm{Var}\{2X + 3Y\}$?

Answer: 29

$$\begin{aligned} \mathrm{Var}(2X + 3Y) &= 2^2\,\mathrm{Var}(X) + 3^2\,\mathrm{Var}(Y) - 2(2)(3)\ \mathrm{Cov}(X, Y) \\ &= (4)(5) + (9)(1) - (12)(0) \\ &= 20 + 9 = 29 \end{aligned}$$

Question

Suppose $\text{Cov}(X, Z) = 4$ and $\text{Cov}(Y, Z) = 5$

What is $\text{Cov}(2X - Y, Z)$?

Question

Suppose $\text{Cov}(X, Z) = 4$ and $\text{Cov}(Y, Z) = 5$

What is $\text{Cov}(2X - Y, Z)$?

Answer:

Question

Suppose $\text{Cov}(X, Z) = 4$ and $\text{Cov}(Y, Z) = 5$

What is $\text{Cov}(2X - Y, Z)$?

Answer: 3

$$
\begin{aligned}
\text{Cov}(2X - Y, Z) &= \text{Cov}(2X, Z) - \text{Cov}(Y, Z) \\
&= 2\text{Cov}(X, Z) - \text{Cov}(Y, Z) \\
&= (2)(4) - 5 \\
&= 8 - 5 = 3
\end{aligned}
$$

Exploring Data

Daniel Stephenson

SCMA 524: Statistical Fundamentals

Random Vectors

- A **random vector** X is a finite list of random variables

$$X = (X_1, X_2, \ldots, X_n)$$

Example

Suppose that you flip a coin and then roll a six sided die twice. When you flip the coin, you count a head as 1 and a tail as 0.

$$X = (X_1, X_2, X_3)$$

$$\begin{aligned} X_1 &\sim U\{0,1\} && \text{(Flip the coin)} \\ X_2 &\sim U\{1,2,3,4,5,6\} && \text{(Roll the die)} \\ X_3 &\sim U\{1,2,3,4,5,6\} && \text{(Roll the die again)} \end{aligned}$$

Samples

- A **sample** x is the realization of a random vector

$$x = (x_1, x_2, \ldots, x_n)$$

Example

Suppose that you roll a six sided die four times.

$$X = (X_1, X_2, X_3, X_4)$$

One possible realization is the sample

$$x = (5, 2, 1, 6)$$

Probability Distributions of Random Vectors

- The joint PMF of a discrete random vector X is

$$
\begin{aligned}
p_X(x) &= p_X(x_1, \ldots, x_n) \\
&= P(X_1 = x_1 \cap X_2 = x_2 \cap \ldots \cap X_n = x_n)
\end{aligned}
$$

Probability Distributions of Random Vectors

- The joint PMF of a discrete random vector X is

$$\begin{aligned} p_X(x) &= p_X(x_1, \ldots, x_n) \\ &= P(X_1 = x_1 \cap X_2 = x_2 \cap \ldots \cap X_n = x_n) \end{aligned}$$

- The joint CDF of a random vector X is

$$\begin{aligned} F_X(x) &= F_X(x_1, \ldots, x_n) \\ &= P(X_1 \leq x_1 \cap X_2 \leq x_2 \cap \ldots \cap X_n \leq x_n) \end{aligned}$$

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- The joint PDF of a continuous random vector X is

$$f_X(x) = f_X(x_1, \ldots, x_n) = \frac{\partial F_X(x)}{\partial x_1 \cdots \partial x_n}$$

Independently and Identically Distributed

- A random vector X is **identically distributed** if

$$X_1, \ldots, X_n \quad \text{are identically distributed}$$

- A random vector X is **independently distributed** if

$$X_1, \ldots, X_n \quad \text{are independent}$$

- A random vector X is **i.i.d.** if it is both independently and identically distributed.

Sample Statisitics

- A **sample statistic** is a measurement that summarizes some aspect of a random sample.
- For example, the **sample mean** $\overline{X}$ is a sample statistic that gives the average of a random sample

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

Sample Statisitics

Example

Suppose that you roll a six sided die twice.

Then $X_1, X_2 \sim i.i.d.\ U\{1, 2, 3, 4, 5, 6\}$.

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$$P\left(\overline{X} = 2\right) = P\left(\frac{X_1 + X_2}{2} = 2\right)$$

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\begin{aligned}
P\left(\overline{X} = 2\right) &= P\left(\frac{X_1 + X_2}{2} = 2\right) \\
&= P\left(X_1 + X_2 = 4\right)
\end{aligned}
$$

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&= \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2
\end{aligned}$$

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&= P\left(X_1+X_2=4\right) \\
&= p_X(1,3)+p_X(3,1)+p_X(2,2) \\
&= \left(\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^2=\frac{3}{36}
\end{aligned}$$

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Sample Statisitics

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\end{aligned}
$$

If $x=(1,5)$ then $\overline{x}=\dfrac{1+5}{2}=\dfrac{6}{2}=3$.

Question

Suppose $Y_1, Y_2, Y_3 \sim i.i.d.\, U\{2, 5\}$

If $y = (2, 5, 5)$ then what is $\overline{y}$?

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If $y = (2, 5, 5)$ then what is $\overline{y}$?

Answer: 4

$$\overline{y} \quad = \quad \frac{2+5+5}{3} = \frac{12}{3} = 4$$

Example

Suppose $Y_1, Y_2, Y_3 \sim i.i.d.\ U\{2,5\}$

$$P\left(\overline{Y} = \frac{7}{3}\right) = P\left(\frac{Y_1 + Y_2 + Y_3}{3} = \frac{7}{3}\right)$$

Example

Suppose $Y_1, Y_2, Y_3 \sim i.i.d.\, U\{2, 5\}$

$$\begin{aligned} P\left(\overline{Y} = \frac{7}{3}\right) &= P\left(\frac{Y_1 + Y_2 + Y_3}{3} = \frac{7}{3}\right) \\ &= P\left(Y_1 + Y_2 + Y_3 = 7\right) \end{aligned}$$

Example

Suppose $Y_1, Y_2, Y_3 \sim i.i.d.\, U\left\{2, 5\right\}$

$$
\begin{aligned}
P\left(\overline{Y} = \frac{7}{3}\right) &= P\left(\frac{Y_1 + Y_2 + Y_3}{3} = \frac{7}{3}\right) \\
&= P\left(Y_1 + Y_2 + Y_3 = 7\right) \\
&= p_Y\left(3, 2, 2\right) + p_Y\left(2, 3, 2\right) + p_Y\left(2, 2, 3\right)
\end{aligned}
$$

Example

Suppose $Y_1, Y_2, Y_3 \sim i.i.d.\, U\{2,5\}$

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&= P\left(Y_1 + Y_2 + Y_3 = 7\right) \\
&= p_Y(3,2,2) + p_Y(2,3,2) + p_Y(2,2,3) \\
&= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3
\end{aligned}$$

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&= p_Y(3,2,2) + p_Y(2,3,2) + p_Y(2,2,3) \\
&= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 \\
&= 3\left(\frac{1}{2}\right)^3
\end{aligned}
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&= P\left(Y_1 + Y_2 + Y_3 = 7\right) \\
&= p_Y(3,2,2) + p_Y(2,3,2) + p_Y(2,2,3) \\
&= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 \\
&= 3\left(\frac{1}{2}\right)^3 = \frac{3}{8}
\end{aligned}$$

Estimators

- An **estimator** is a method of estimating a parameter based on a random sample.
- The **bias** of an estimator is the difference between its expected value and the true value of the parameter.
- An estimator with zero bias is said to be **unbiased**.

Example

If $X_1, \ldots, X_n \sim i.i.d.$ then $\overline{X}$ is an unbiased estimator for μ_X.

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If $X_1, \ldots, X_n \sim i.i.d.$ then $\overline{X}$ is an unbiased estimator for μ_X.

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&= \frac{1}{n}(\mu_X + \cdots + \mu_X) = \frac{1}{n}(n\mu_X) = \mu_X
\end{aligned}
$$

Example

Suppose $Y_1, Y_2, Y_3 \sim i.i.d.\ U\{2, 5\}$

$$E\{Y_i\} = 2(0.5) + 5(0.5)$$

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$$\begin{aligned} E\{Y_i\} &= 2(0.5) + 5(0.5) \\ &= 3.5 \end{aligned}$$

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$$E\left\{\overline{Y}\right\} = E\left\{\frac{Y_1 + Y_2 + Y_3}{3}\right\}$$

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$$
\begin{aligned}
E\left\{\overline{Y}\right\} &= E\left\{\frac{Y_1 + Y_2 + Y_3}{3}\right\} \\
&= \frac{1}{3}E\{Y_1 + Y_2 + Y_3\}
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Variance of the Sample Mean

Example

If $X_1, X_2, \ldots, X_n \sim i.i.d.$ with $E\{X_i\} = \mu$ and $\text{Var}(X_i) = \sigma^2$ then

$$\text{Var}\left(\overline{X}\right) = \text{Var}\left(\frac{1}{n}\left[X_1 + \cdots + X_n\right]\right)$$

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Example

If $X_1, X_2, \ldots, X_n \sim i.i.d.$ with $E\left\{X_i\right\} = \mu$ and $\mathrm{Var}\left(X_i\right) = \sigma^2$ then

$$\begin{aligned}\mathrm{Var}\left(\overline{X}\right) &= \mathrm{Var}\left(\frac{1}{n}\left[X_1 + \cdots + X_n\right]\right) \\ &= \frac{1}{n^2}\mathrm{Var}\left(X_1 + \cdots + X_n\right)\end{aligned}$$

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Variance of the Sample Mean

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If $X_1, X_2, \ldots, X_n \sim i.i.d.$ with $E\{X_i\} = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$ then

$$
\begin{aligned}
\mathrm{Var}\left(\overline{X}\right) &= \mathrm{Var}\left(\frac{1}{n}[X_1 + \cdots + X_n]\right) \\
&= \frac{1}{n^2}\mathrm{Var}(X_1 + \cdots + X_n) \\
&= \frac{1}{n^2}[\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)] \\
&= \frac{1}{n^2}\left[\sigma^2 + \cdots + \sigma^2\right]
\end{aligned}
$$

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\text{Var}\left(\overline{X}\right) &= \text{Var}\left(\frac{1}{n}[X_1 + \cdots + X_n]\right) \\
&= \frac{1}{n^2}\text{Var}(X_1 + \cdots + X_n) \\
&= \frac{1}{n^2}[\text{Var}(X_1) + \cdots + \text{Var}(X_n)] \\
&= \frac{1}{n^2}\left[\sigma^2 + \cdots + \sigma^2\right] \\
&= \frac{1}{n^2}\left[n\sigma^2\right]
\end{aligned}$$

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&= \frac{1}{n^2}\mathrm{Var}(X_1 + \cdots + X_n) \\
&= \frac{1}{n^2}[\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)] \\
&= \frac{1}{n^2}\left[\sigma^2 + \cdots + \sigma^2\right] \\
&= \frac{1}{n^2}\left[n\sigma^2\right] = \frac{\sigma^2}{n}
\end{aligned}
$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4, Y_5 \sim i.i.d.$ with $E\left\{Y_i\right\} = 3$ and $\mathrm{Var}\left(Y_i\right) = 2$

$$\mathrm{Var}\left(\overline{Y}\sqrt{5}\right) \quad = \quad 5\mathrm{Var}\left(\overline{Y}\right)$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4, Y_5 \sim i.i.d.$ with $E\left\{Y_i\right\} = 3$ and $\mathrm{Var}\left(Y_i\right) = 2$

$$
\begin{aligned}
\mathrm{Var}\left(\overline{Y}\sqrt{5}\right) &= 5\mathrm{Var}\left(\overline{Y}\right) \\
&= 5\mathrm{Var}\left(\frac{1}{5}\left[Y_1 + Y_2 + Y_3 + Y_4 + Y_5\right]\right)
\end{aligned}
$$

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&= 5\text{Var}\left(\frac{1}{5}\left[Y_1 + Y_2 + Y_3 + Y_4 + Y_5\right]\right) \\
&= \frac{5}{5^2}\text{Var}\left(Y_1 + Y_2 + Y_3 + Y_4 + Y_5\right)
\end{aligned}
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&= \frac{5}{5^2}\text{Var}\left(Y_1 + Y_2 + Y_3 + Y_4 + Y_5\right) \\
&= \frac{1}{5}\left[\text{Var}(Y_1) + \text{Var}(Y_2) + \text{Var}(Y_3) + \text{Var}(Y_4) + \text{Var}(Y_5)\right]
\end{aligned}$$

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&= \frac{5}{5^2}\text{Var}\left(Y_1 + Y_2 + Y_3 + Y_4 + Y_5\right) \\
&= \frac{1}{5}\left[\text{Var}\left(Y_1\right) + \text{Var}\left(Y_2\right) + \text{Var}\left(Y_3\right) + \text{Var}\left(Y_4\right) + \text{Var}\left(Y_5\right)\right] \\
&= \frac{1}{5}\left[2 + 2 + 2 + 2 + 2\right]
\end{aligned}
$$

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Suppose $Y_1, Y_2, Y_3, Y_4, Y_5 \sim i.i.d.$ with $E\{Y_i\} = 3$ and $\text{Var}(Y_i) = 2$

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&= 5\text{Var}\left(\frac{1}{5}\left[Y_1 + Y_2 + Y_3 + Y_4 + Y_5\right]\right) \\
&= \frac{5}{5^2}\text{Var}\left(Y_1 + Y_2 + Y_3 + Y_4 + Y_5\right) \\
&= \frac{1}{5}\left[\text{Var}(Y_1) + \text{Var}(Y_2) + \text{Var}(Y_3) + \text{Var}(Y_4) + \text{Var}(Y_5)\right] \\
&= \frac{1}{5}\left[2 + 2 + 2 + 2 + 2\right] \\
&= \frac{1}{5}\left[10\right] = 2
\end{aligned}
$$

Sample Variance

- The sample variance S_X^2 is an estimator for the population variance

$$S_X^2 \quad = \quad \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

Sample Variance

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\begin{aligned}
S_X^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i^2 - 2\overline{X}X_i + \overline{X}^2\right)
\end{aligned}
$$

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&= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i^2 - 2\overline{X}X_i + \overline{X}^2\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}X_i^2 - 2\overline{X}\sum_{i=1}^{n}X_i + n\overline{X}^2\right)
\end{aligned}
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S_X^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2 \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i^2-2\overline{X}X_i+\overline{X}^2\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}X_i^2-2\overline{X}\sum_{i=1}^{n}X_i+n\overline{X}^2\right) \\
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\end{aligned}$$

Sample Variance

- The sample variance S_X^2 is an estimator for the population variance

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S_X^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \\
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&= \frac{1}{n-1}\left(\sum_{i=1}^{n}X_i^2 - 2\overline{X}\sum_{i=1}^{n}X_i + n\overline{X}^2\right) \\
&= \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}X_i^2 - 2\overline{X}\frac{1}{n}\sum_{i=1}^{n}X_i + \overline{X}^2\right) \\
&= \frac{n}{n-1}\left(\overline{X^2} - 2\overline{X}^2 + \overline{X}^2\right)
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$$

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&= \frac{n}{n-1}\left(\overline{X^2} - \overline{X}^2\right)
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Sample Variance

- The sample variance S_X^2 is an estimator for the population variance

$$S_X^2 \;=\; \frac{n}{n-1}\left(\overline{X^2} - \overline{X}^2\right)$$

- S_X^2 is an **unbiased estimator** for the population variance σ^2

$$E\left\{S_X^2\right\} = \sigma_X^2 = \mathrm{Var}\left(X\right)$$

Example

Example

Suppose that you roll a six sided die three times. Then $X_1, X_2, X_3 \sim i.i.d.\ U\{1,2,3,4,5,6\}$. If $x = (1,4,1)$ then

$$\overline{x} = \frac{1+4+1}{3} = \frac{6}{3} = 2$$

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\end{aligned}$$

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\end{aligned}$$

Example

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$$\begin{aligned} s_x^2 &= \frac{1}{2}\left[(1-2)^2 + (4-2)^2 + (1-2)^2\right] \\ &= \frac{1}{2}[1+4+1] = \frac{6}{2} = 3 \end{aligned}$$

Question

Suppose $y = (3, 5, 8, 14)$

What is the sample variance of y?

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Answer:

Question

Suppose $y = (3, 5, 8, 14)$

What is the sample variance of y?

Answer:

$$\bar{y} = \frac{3 + 5 + 8 + 14}{4} = \frac{30}{4} = 7.5$$

Question

Suppose $y = (3, 5, 8, 14)$

What is the sample variance of y?

Answer:

$$\begin{aligned}
\overline{y} &= \frac{3+5+8+14}{4} = \frac{30}{4} = 7.5 \\
s_y^2 &= \frac{1}{3}\left[(3-7.5)^2 + (5-7.5)^2 + (8-7.5)^2 + (14-7.5)^2\right]
\end{aligned}$$

Question

Suppose $y = (3, 5, 8, 14)$

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Question

Suppose $y = (3, 5, 8, 14)$

What is the sample variance of y?

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&= \frac{1}{3}\left[(4.5)^2 + (2.5)^2 + (0.5)^2 + (6.5)^2\right] \\
&= \frac{1}{3}[20.25 + 6.25 + 0.25 + 42.25]
\end{aligned}
$$

Question

Suppose $y = (3, 5, 8, 14)$

What is the sample variance of y?

Answer: 23

$$
\begin{aligned}
\bar{y} &= \frac{3+5+8+14}{4} = \frac{30}{4} = 7.5 \\
s_y^2 &= \frac{1}{3}\left[(3-7.5)^2 + (5-7.5)^2 + (8-7.5)^2 + (14-7.5)^2\right] \\
&= \frac{1}{3}\left[(4.5)^2 + (2.5)^2 + (0.5)^2 + (6.5)^2\right] \\
&= \frac{1}{3}[20.25 + 6.25 + 0.25 + 42.25] \\
&= \frac{1}{3}[69] = 23
\end{aligned}
$$

Example

$Y_1, Y_2, Y_3, Y_4 \sim i.i.d. \qquad E\{Y_i\} = 3 \qquad E\{Y_i^2\} = 18$

What is $E\{S_Y^2\}$?

Example

$Y_1, Y_2, Y_3, Y_4 \sim i.i.d. \qquad E\left\{Y_i\right\} = 3 \qquad E\left\{Y_i^2\right\} = 18$

What is $E\left\{S_Y^2\right\}$?

$$E\left\{S_Y^2\right\} = \mathrm{Var}\left(Y_i\right)$$

Example

$Y_1, Y_2, Y_3, Y_4 \sim i.i.d. \qquad E\{Y_i\} = 3 \qquad E\{Y_i^2\} = 18$

What is $E\{S_Y^2\}$?

$$\begin{aligned} E\left\{S_Y^2\right\} &= \text{Var}(Y_i) \\ &= E\left\{Y_i^2\right\} - (E\{Y_i\})^2 \end{aligned}$$

Example

$Y_1, Y_2, Y_3, Y_4 \sim i.i.d. \qquad E\{Y_i\} = 3 \qquad E\{Y_i^2\} = 18$

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$Y_1, Y_2, Y_3, Y_4 \sim i.i.d. \qquad E\{Y_i\} = 3 \qquad E\{Y_i^2\} = 18$

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$$
\begin{aligned}
E\left\{S_Y^2\right\} &= \operatorname{Var}(Y_i) \\
&= E\left\{Y_i^2\right\} - (E\{Y_i\})^2 \\
&= 18 - (3)^2 \\
&= 18 - 9 = 9
\end{aligned}
$$

Sample Standard Deviation

- The sample standard deviation S_X is the square root of the sample variance.

$$\mathrm{stdev}\left(X\right) = S_X = \sqrt{S_X^2} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}$$

Example

Suppose $z = (0, 2, 4)$. Then $\overline{z} = \dfrac{0+2+4}{3} = \dfrac{6}{3} = 2$

$$\begin{aligned} s_z^2 &= \frac{1}{2}\left[(0-2)^2 + (2-2)^2 + (4-2)^2\right] \\ &= \frac{1}{2}\left[4+0+4\right] = \frac{1}{2}\left[8\right] = 4 \\ s_z &= \sqrt{s_z^2} = 2 \end{aligned}$$

The Standard Error of the Sample Mean

- The **standard error of the sample mean** is an estimator for the standard deviation of $\overline{X}$.
- If $X = (X_1, X_2, \ldots, X_n)$ then

$$\mathrm{SE}(\overline{X}) = \frac{S_X}{\sqrt{n}}$$

Example

If $x = (2, 7, 3, 8, 5)$ then $\overline{x} = 5$

$$\begin{aligned}
s_X^2 &= \frac{1}{4}\sum_{i=1}^{5}(x_i - \overline{x})^2 = 6.5 \\
s_X &= \sqrt{s_X^2} \approx 2.54951 \\
\mathrm{SE}(\overline{x}) &= \frac{s_X}{\sqrt{5}} \approx 1.140175
\end{aligned}$$

Example

Suppose $x = (3, 7, 6, 12)$.

$$\bar{x} = \frac{3 + 7 + 6 + 12}{4} = 7$$

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Suppose $x = (3, 7, 6, 12)$.

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\bar{x} &= \frac{3+7+6+12}{4} = 7 \\
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&= \frac{1}{3}\left[(3-7)^2 + (7-7)^2 + (6-7)^2 + (12-7)^2\right]
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&= \frac{1}{3}\left[(3-7)^2 + (7-7)^2 + (6-7)^2 + (12-7)^2\right] \\
&= \frac{1}{3}\left[16 + 0 + 1 + 25\right] = \frac{42}{3} = 14
\end{aligned}
$$

Example

Suppose $x = (3, 7, 6, 12)$.

$$\begin{aligned}
\bar{x} &= \frac{3+7+6+12}{4} = 7 \\
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s_X &= \sqrt{s_X^2} = \sqrt{14}
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s_X &= \sqrt{s_X^2} = \sqrt{14} \\
\mathrm{SE}(\overline{x}) &= \frac{s_X}{\sqrt{4}}
\end{aligned}$$

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\mathrm{SE}(\overline{x}) &= \frac{s_X}{\sqrt{4}} = \frac{\sqrt{14}}{2}
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&= \frac{1}{3}\left[16+0+1+25\right] = \frac{42}{3} = 14 \\
s_X &= \sqrt{s_X^2} = \sqrt{14} \\
\mathrm{SE}(\overline{x}) &= \frac{s_X}{\sqrt{4}} = \frac{\sqrt{14}}{2} \approx 1.8708
\end{aligned}
$$

The Binomial Distribution

- Suppose $X_1, X_2, \ldots, X_n \sim i.i.d.$ Bernoulli (b)

$$\begin{aligned} P(X_i = 1) &= b \\ P(X_i = 0) &= 1 - b \end{aligned}$$

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$$
\begin{aligned}
P(X_i = 1) &= b \\
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$$

- Then $Y = X_1 + X_2 + \cdots + X_n$ has the **Binomial distribution**

$$
p_Y(k) = \binom{n}{k} b^k (1-b)^{n-k} \qquad \text{for } k \in \{1, 2, \ldots, n\}
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$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad n! = (1)(2)\cdots(n)$$

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$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad n! = (1)(2)\cdots(n)$$

- Then we write $Y \sim$ Binomial (n, b)

Example

If $X \sim \text{Binomial}\,(5, 0.2)$ then

$$p_Y\,(k) \;=\; \binom{5}{k}\,(0.2)^k\,(0.8)^{5-k}$$

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\begin{aligned}
p_Y\,(k) &= \binom{5}{k}\,(0.2)^k\,(0.8)^{5-k} \\
p_Y\,(3) &= \binom{5}{3}\,(0.2)^3\,(0.8)^2
\end{aligned}
$$

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If $X \sim \text{Binomial}(5, 0.2)$ then

$$p_Y(k) = \binom{5}{k}(0.2)^k(0.8)^{5-k}$$

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$$\begin{aligned} p_Y(3) &= \binom{5}{3} (0.2)^3 (0.8)^2 = \binom{5}{3} (0.008)(0.64) \\ &= \binom{5}{3} (0.00512) \end{aligned}$$

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&= \binom{5}{3}(0.00512) = \frac{(5!)}{(3!)(2!)}(0.00512)
\end{aligned}
$$

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p_Y\,(3) &= \binom{5}{3}\,(0.2)^3\,(0.8)^2 = \binom{5}{3}\,(0.008)\,(0.64) \\
&= \binom{5}{3}\,(0.00512) = \frac{(5!)}{(3!)\,(2!)}\,(0.00512) \\
&= \frac{(1\cdot 2\cdot 3\cdot 4\cdot 5)}{(1\cdot 2\cdot 3)\,(1\cdot 2)}\,(0.00512)
\end{aligned}$$

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&= \frac{(1\cdot 2\cdot 3\cdot 4\cdot 5)}{(1\cdot 2\cdot 3)\,(1\cdot 2)}(0.00512) \\
&= \frac{(4\cdot 5)}{(1\cdot 2)}(0.00512)
\end{aligned}$$

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&= \frac{(1\cdot 2\cdot 3\cdot 4\cdot 5)}{(1\cdot 2\cdot 3)\,(1\cdot 2)}(0.00512) \\
&= \frac{(4\cdot 5)}{(1\cdot 2)}(0.00512) = (10)\,(0.00512)
\end{aligned}$$

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If $X \sim \text{Binomial}(5, 0.2)$ then

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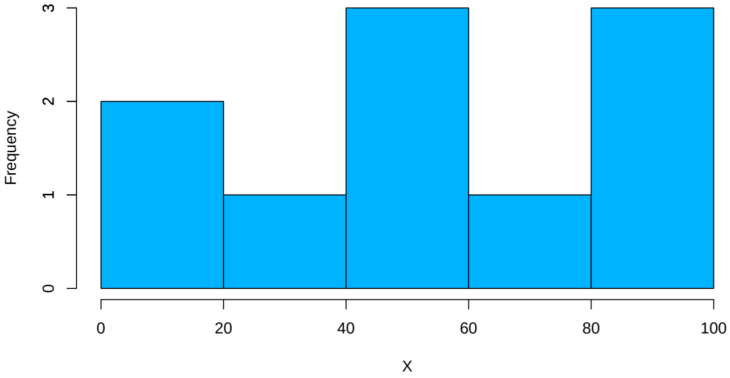
p_Y(3) &= \binom{5}{3} (0.2)^3 (0.8)^2 = \binom{5}{3} (0.008)(0.64) \\
&= \binom{5}{3} (0.00512) = \frac{(5!)}{(3!)(2!)} (0.00512) \\
&= \frac{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)}{(1 \cdot 2 \cdot 3)(1 \cdot 2)} (0.00512) \\
&= \frac{(4 \cdot 5)}{(1 \cdot 2)} (0.00512) = (10)(0.00512) = 0.0512
\end{aligned}$$

Histograms

- A histogram is a graphical display where the data is grouped into ranges and plotted as bars.
- The height of each bar shows how many observations are in each range.

$$x = \{74, 89, 17, 53, 30, 98, 91, 7, 49, 46\}$$



Data

- A **data set** is an array of data where the columns contain variables and the rows contain observations.

Example

Person	Age	Children	Salary
1	35	1	\$65,400
2	61	2	\$62,400
3	35	0	\$63,200
4	37	2	\$52,000
5	32	3	\$81,400

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The sample mean of Age is

$$\frac{1}{5}(35 + 61 + 35 + 37 + 32) = 40$$

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The sample mean of Age is

$$\frac{1}{5}(35+61+35+37+32) = 40$$

The sample variance of age is

$$\frac{1}{4}\left[(35-40)^2+(61-40)^2+(35-40)^2+(37-40)^2+(32-40)^2\right] = 141$$

Question

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What is the sample mean for the number of children?

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5	32	3	\$81,400

What is the sample mean for the number of children?

Answer: 1.6

$$\overline{\text{Children}} = \frac{1}{5}(1+2+0+2+3) = \frac{8}{5} = 1.6$$

Sample Covariance

- The sample covariance between X and Y is

$$S_{XY} = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X} \right) \left(Y_i - \overline{Y} \right)$$

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- The sample covariance between X and Y is

$$
\begin{aligned}
S_{XY} &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(Y_i-\overline{Y}\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_iY_i-\overline{X}Y_i-X_i\overline{Y}+\overline{XY}\right)
\end{aligned}
$$

Sample Covariance

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$$\begin{aligned}
S_{XY} &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(Y_i-\overline{Y}\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_iY_i-\overline{X}Y_i-X_i\overline{Y}+\overline{XY}\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}X_iY_i-\overline{X}\sum_{i=1}^{n}Y_i-\overline{Y}\sum_{i=1}^{n}X_i+n\overline{XY}\right)
\end{aligned}$$

Sample Covariance

- The sample covariance between X and Y is

$$
\begin{aligned}
S_{XY} &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(Y_i-\overline{Y}\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_iY_i-\overline{X}Y_i-X_i\overline{Y}+\overline{X}\overline{Y}\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}X_iY_i-\overline{X}\sum_{i=1}^{n}Y_i-\overline{Y}\sum_{i=1}^{n}X_i+n\overline{X}\overline{Y}\right) \\
&= \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}X_iY_i-\overline{X}\frac{1}{n}\sum_{i=1}^{n}Y_i-\overline{Y}\frac{1}{n}\sum_{i=1}^{n}X_i+\overline{X}\overline{Y}\right)
\end{aligned}
$$

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$$
\begin{aligned}
S_{XY} &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(Y_i-\overline{Y}\right)\\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_iY_i-\overline{X}Y_i-X_i\overline{Y}+\overline{X}\,\overline{Y}\right)\\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}X_iY_i-\overline{X}\sum_{i=1}^{n}Y_i-\overline{Y}\sum_{i=1}^{n}X_i+n\overline{X}\,\overline{Y}\right)\\
&= \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}X_iY_i-\overline{X}\frac{1}{n}\sum_{i=1}^{n}Y_i-\overline{Y}\frac{1}{n}\sum_{i=1}^{n}X_i+\overline{X}\,\overline{Y}\right)\\
&= \frac{n}{n-1}\left(\overline{XY}-\overline{X}\;\overline{Y}-\overline{Y}\;\overline{X}+\overline{X}\;\overline{Y}\right)
\end{aligned}
$$

Sample Covariance

- The sample covariance between X and Y is

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\begin{aligned}
S_{XY} &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_iY_i - \overline{X}Y_i - X_i\overline{Y} + \overline{X}\,\overline{Y}\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}X_iY_i - \overline{X}\sum_{i=1}^{n}Y_i - \overline{Y}\sum_{i=1}^{n}X_i + n\overline{X}\,\overline{Y}\right) \\
&= \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}X_iY_i - \overline{X}\frac{1}{n}\sum_{i=1}^{n}Y_i - \overline{Y}\frac{1}{n}\sum_{i=1}^{n}X_i + \overline{X}\,\overline{Y}\right) \\
&= \frac{n}{n-1}\left(\overline{XY} - \overline{X}\;\overline{Y} - \overline{Y}\;\overline{X} + \overline{X}\;\overline{Y}\right) \\
&= \frac{n}{n-1}\left(\overline{XY} - \overline{X}\;\overline{Y}\right)
\end{aligned}
$$

Sample Covariance

- The sample covariance between X and Y is

$$S_{XY} = \frac{n}{n-1}\left(\overline{XY} - \overline{X}\,\overline{Y}\right)$$

Sample Covariance

- The sample covariance between X and Y is

$$S_{XY} = \frac{n}{n-1}\left(\overline{XY} - \overline{X}\,\overline{Y}\right)$$

- The sample covariance S_{XY} is an unbiased estimator for the population covariance σ_{XY}

$$\begin{aligned} E\{S_{XY}\} &= \sigma_{XY} \\ &= E\{(X-\mu_X)(Y-\mu_Y)\} \\ &= \mathrm{Cov}(X, Y) \end{aligned}$$

Sample Covariance

Example

Person	Age	Children	Height (cm)
1	35	1	176.6
2	61	2	163.4
3	30	0	180.5
Sample Mean	42	1	173.5

The sample covariance between Age and Children is

$$\frac{1}{2}\sum_{i=1}^{3}\left(\text{Age}_i - \overline{\text{Age}}\right)\left(\text{Children}_i - \overline{\text{Children}}\right)$$

$$= \frac{1}{2}\Big[(35-42)(1-1) + (61-42)(2-1) + (30-42)(0-1)\Big]$$

$$= 15.5$$

Question

Person	Age	Children	Height (cm)
1	35	1	176.6
2	61	2	163.4
3	30	0	180.5
Sample Mean	42	1	173.5

What is the sample covariance between Age and Height?

Question

Person	Age	Children	Height (cm)
1	35	1	176.6
2	61	2	163.4
3	30	0	180.5
Sample Mean	42	1	173.5

What is the sample covariance between Age and Height?

Answer:

Question

Person	Age	Children	Height (cm)
1	35	1	176.6
2	61	2	163.4
3	30	0	180.5
Sample Mean	42	1	173.5

What is the sample covariance between Age and Height?

Answer:

$$\overline{\text{Age}} = \frac{35 + 61 + 30}{3} = 42$$

Question

Person	Age	Children	Height (cm)
1	35	1	176.6
2	61	2	163.4
3	30	0	180.5
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Answer:

$$\overline{\text{Age}} = \frac{35+61+30}{3} = 42$$

$$\overline{\text{Height}} = \frac{176.6+163.4+180.5}{3} = 173.5$$

Question

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2	61	2	163.4
3	30	0	180.5
Sample Mean	42	1	173.5

What is the sample covariance between Age and Height?

Answer:

$$\begin{aligned}
\overline{\text{Age}} &= \frac{35+61+30}{3} = 42 \\
\overline{\text{Height}} &= \frac{176.6+163.4+180.5}{3} = 173.5 \\
S_{\text{Age,Height}} &= \frac{1}{2}\Big[(35-42)(176.6-173.5) + \\
&\quad (61-42)(163.4-173.5) + \\
&\quad (30-42)(180.5-173.5)\Big]
\end{aligned}$$

Question

Person	Age	Children	Height (cm)
1	35	1	176.6
2	61	2	163.4
3	30	0	180.5
Sample Mean	42	1	173.5

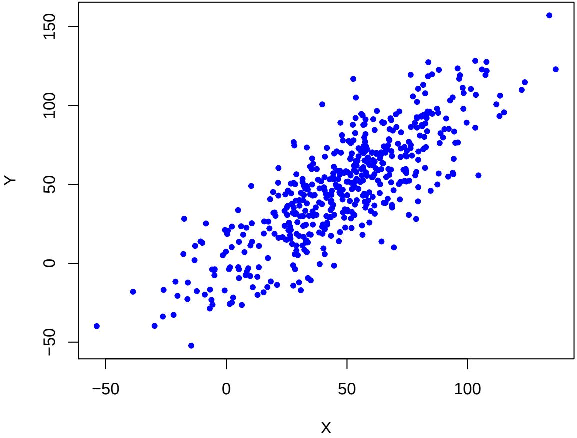
What is the sample covariance between Age and Height?

Answer: -148.8

$$\begin{aligned}
\overline{\text{Age}} &= \frac{35+61+30}{3} = 42 \\
\overline{\text{Height}} &= \frac{176.6+163.4+180.5}{3} = 173.5 \\
S_{\text{Age,Height}} &= \frac{1}{2}\Big[(35-42)(176.6-173.5)+ \\
&(61-42)(163.4-173.5)+ \\
&(30-42)(180.5-173.5)\Big] = -148.8
\end{aligned}$$

Scatter Plots

- A scatter plot is a two-dimensional graph that uses dots to represent the values obtained for two different variables.
- One variable is plotted along the vertical axis and the other plotted along the horizontal axis.



Sample Correlation

- The sample correlation between X and Y is

$$\text{corr}(X, Y) = R_{XY} = \frac{S_{XY}}{S_X\, S_Y} = \frac{\text{cov}(X, Y)}{\text{stdev}(X)\ \text{stdev}(Y)}$$

Example

If $s_X = 6$, $s_Y = 5$, and $s_{XY} = 18$ then

$$\text{corr}(X, Y) = r_{XY} = \frac{18}{(6)(5)} = \frac{18}{30} = 0.6$$

Question

Let $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$ be random samples.

Suppose $s_W^2 = 4$, $s_Z^2 = 9$, and $s_{WZ} = 3$.

What is the sample correlation between W and Z?

Question

Let $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$ be random samples.

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Answer:

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Answer:

$$r_{WZ} = \frac{s_{WZ}}{s_W \, s_Z}$$

Question

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Suppose $s_W^2 = 4$, $s_Z^2 = 9$, and $s_{WZ} = 3$.

What is the sample correlation between W and Z?

Answer:

$$\begin{aligned} r_{WZ} &= \frac{s_{WZ}}{s_W \, s_Z} \\ &= \frac{s_{XY}}{\sqrt{s_W^2}\,\sqrt{s_Z^2}} \end{aligned}$$

Question

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Suppose $s_W^2 = 4$, $s_Z^2 = 9$, and $s_{WZ} = 3$.

What is the sample correlation between W and Z?

Answer:

$$
\begin{aligned}
r_{WZ} &= \frac{s_{WZ}}{s_W \, s_Z} \\
&= \frac{s_{XY}}{\sqrt{s_W^2}\,\sqrt{s_Z^2}} \\
&= \frac{3}{\sqrt{4}\,\sqrt{9}}
\end{aligned}
$$

Question

Let $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$ be random samples.

Suppose $s_W^2 = 4$, $s_Z^2 = 9$, and $s_{WZ} = 3$.

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Answer:

$$\begin{aligned} r_{WZ} &= \frac{s_{WZ}}{s_W\, s_Z} \\ &= \frac{s_{XY}}{\sqrt{s_W^2}\,\sqrt{s_Z^2}} \\ &= \frac{3}{\sqrt{4}\,\sqrt{9}} \\ &= \frac{3}{(2)\,(3)} \end{aligned}$$

Question

Let $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$ be random samples.

Suppose $s_W^2 = 4$, $s_Z^2 = 9$, and $s_{WZ} = 3$.

What is the sample correlation between W and Z?

Answer:

$$
\begin{aligned}
r_{WZ} &= \frac{s_{WZ}}{s_W \, s_Z} \\
&= \frac{s_{XY}}{\sqrt{s_W^2}\,\sqrt{s_Z^2}} \\
&= \frac{3}{\sqrt{4}\,\sqrt{9}} \\
&= \frac{3}{(2)\,(3)} = \frac{1}{2}
\end{aligned}
$$

Question

Let $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$ be random samples.

Suppose $s_W^2 = 4$, $s_Z^2 = 9$, and $s_{WZ} = 3$.

What is the sample correlation between W and Z?

Answer: 0.5

$$
\begin{aligned}
r_{WZ} &= \frac{s_{WZ}}{s_W \, s_Z} \\
&= \frac{s_{XY}}{\sqrt{s_W^2}\,\sqrt{s_Z^2}} \\
&= \frac{3}{\sqrt{4}\,\sqrt{9}} \\
&= \frac{3}{(2)\,(3)} = \frac{1}{2} = 0.5
\end{aligned}
$$

RSTUDIO LAYOUT

Bottom left: console window. Here you can type simple commands after the ">" prompt and R will then execute your command. This is the most important window, because this is where R actually does stuff.

Top left: editor window. Here scripts can be edited and saved.

Top right: environment window. In the environment window you can see which data and values R has in its memory.

Bottom right: files / plots / packages / help window. Here you can open files, view plots (also previous plots), install and load packages or use the help function.

WORKING DIRECTORY

Your working directory is the folder on your computer in which you are currently working. When you ask R to open a certain file, it will look in the working directory for this file, and when you tell R to save a data file or figure, it will save it in the working directory.

Before you start working, set your working directory to where all your data and script files should be stored.

Within RStudio you can go to "Session / Set working directory."

BASIC EXAMPLES OF R COMMANDS

R can be used as a calculator. You can just type your equation in the command window after the ">"

10^2 + 36

If you use brackets and forget to add the closing bracket, the ">" on the command line changes into a "+". The "+" can also mean that R is still busy with some heavy computation. If you want R to quit what it was doing and give back the ">", press ESC.

You can also give numbers a name. By doing so, they become so-called variables which can be used later.

x = 4

You can see that x appears in the workspace window, which means that R now remembers what x is. You can also ask R what x is (just type x and ENTER in the command window):

x

or do calculations with x:

x*5

If you specify x again, R will forget what value it had before. You can also assign a new value to x using the old one.

x = x + 10
x

To remove all variables from R's memory, type:

rm(list=ls())

SCALARS AND VECTORS

Like in many other programs, R organizes numbers in scalars (a single number), vectors (a row of numbers) and matrices (a grid of numbers).

The x you defined before was a scalar. To define a vector with the numbers 3, 4 and 5, you need the function c, which is short for concatenate.

b = c(3,4,5)

If you wanted to compute the mean of all the elements in the vector b from the example above, you could type:

(3+4+5)/3

mean(b)

Within the parenthesis you specify the arguments. Arguments give extra information to the function. In this case, the argument says which vector the mean should computed of (in this case b).

You can compute the sum of 4, 5, 8 and 11 using the function sum by typing:

v = c(4,5,8,11)
sum(v)

to create a vector of the integers from 2 to 5, type:

j = 2:5

to get the sum of the integers from 2 to 5, type:

sum(j)

multiplying j by 3 yields:

j*3

adding 1 one to this yeilds:

j*3 + 1

Many operations in R are "vectorized," meaning that they will automatically be applied to every element of a vector.

To get the lenght of j, type:

length(j)

To create a sequence of numbers, type:

k = seq(from=0, to=0.75, by=0.25)

to add k and j, type:

k + j

to mulitiply k and j, type:

k*j

To get the second element of k, type

k[2]

To change the second element of k to 8, type

k[2] = 8

To check which elemments of j are greater than 3, type

j>3

To get the elements of k where j is greater than 3, type

k[j>3]

GENERATING RANDOM NUMBERS

The function runif is a standard R function which creates random samples from a continuous uniform distribution. To generate 5 random samples from the the continuous uniform distribution on [0,1], type:

runif(5)

Entering the same command again produces 5 new random numbers. Instead of typing the same text again, you can also press the upward arrow key to access previous commands.

If you want 5 random samples from the unuform distribution on [3,8] you can type:

y = runif(n=5, min=3, max=8)

This shows how the same function may have different interfaces and that functions in R can have named arguments (in this case n, min, and max).

RStudio has a nice feature: when you type runif(in the command window and press TAB, RStudio will show the possible arguments.

To compute the sample variance of y, type

var(y)

To compute the square root of the sample variance, type

sqrt(var(y))

To compute the sample standard deviation of y, type

sd(y)

To get y rounded to the nearest 4 decimal places, type

round(y,4)

HELP AND DOCUMENTATION

There is a large amount of free and help available. Some help is automatically installed. To see the documentation for the rnorm function, type

?runif

To see the documentation for the abs function, type

?abs

```
# PLOTS

# R can make graphs. TO plot 50 samples from U[0,1], type

w = runif(50)
plot(w)

# To connect the points with lines, type

lines(w)

# To plot 500 samples from U(5,10), type

g = runif(500, min=5, max=10)
plot(g)

# to compute the sample mean of g type

mean(g)

# to add a red horizontal line at the sample mean, type:

abline(h=mean(g), col='red', lwd=3)

# to create a histogram of g, type

hist(g)

# To generate 500 random samples from U[0,3], type:

b = runif(n=500, min=0, max=3)

# To plot b, type

plot(b)

# To create a histogram of b, type

hist(b)

# To create a scatter-plot with g on the x-axis and b on the y-axis, type

plot(x=g,y=b)

# To add a blue triangle at the sample mean, type
```

```
points(x=mean(g), y=mean(b), col='blue', pch=17, cex=2)

# To plot g in red, type

plot(g, ylim=c(0,17), col='red')

# To add b in blue, type

points(b, col='blue')

# Create a new variable q by typing

q = 10 + 5*b + g

# To see a scatter plot with b on the x-axis and q on the y-axis, type

plot(x=b,y=q)

# to compute the sample covariance between b and q, type

cov(b,q)

# to compute the sample correlation between b and q, type

cor(b,q)



# SCRIPTS

# You can store commands in files called scripts. These scripts typically have file names with the extension .R. You can open an editor window to edit these files by clicking File and New or Open file.

# You can run part of the code by selecting lines and pressing CTRL+ENTER or click Run in the editor window. If you do not select anything, R will run the line your cursor is on. You can run the whole script with CRTL+SHIFT+ENTER.

# Lines beginning with "#" are comments, so they will not run.



# MATRICES

# Matricies are nothing more than 2-dimensional vectors. To define a matrix, type.

mat = matrix(data=c(9,2,3,4,5,6), ncol=3)
mat
```

To see the documentation for the matrix function, type

help(matrix)

Matrix-operations are similar to vector operations. To get the element in the 1st row and 2nd column, type:

mat[1,2]

To get the whole second row, type

mat[2,]

To get the whole third column, type

mat[,3]

To multiply every elelment of the matrix by 2, type

2*mat

To get the mean of the matrix elements, type

mean(mat)

CHARACTER STRINGS

To tell R that something is a character string, you should type the text in quotes

a = 'apple'
b = 'bread'
food = c(a,b)

DATA FRAMES

Data is often organized into data frames. A data frame is a matrix with names above the columns. This is nice, because you can use one of the columns without knowing its position.

To create a data frame, type

people = data.frame(name=c('Will','Emma','Ben'),age=c(19,24,22),gpa=c(78,94.3,91.7))

to get the mean age, type

```
mean(people$age)

# to get the standard deviation of gpa, type

sd(people$gpa)

# to write this data frame to a csv file, type

write.csv(people,file='people.csv',row.names=FALSE)


# READING AND WRITING DATA FILES

# There are many ways to read and write data from within the R environment.

# To read data from the file 'cardata.csv' into a data frame, type

cardata = read.csv('cardata.csv')

# This file contains data from car road tests conducted in 1974

# to calculate the average miles per gallon in this dataset, type

mean(cardata$mpg)

# to create a scatterplot of car weight (1000 lbs) and miles per gallon, type

plot(x=cardata$wt,y=cardata$mpg)

# to attach of the variables in this dataset, type

attach(cardata)

# Now you can access all of these variable just by typing their name

mpg

# to calculate the correlation between weight and miles per gallon, type

cor(wt,mpg)


# FOR LOOPS

# TO avoid having to type the same command over and aver again, you can use a for loop

firstnames = c('Harper','John','Noah','Ava','Mason','Jacab')
```

```r
length(firstnames)

for(i in 1:6) {
  print(i)
  print(firstnames[i])
}


# WRITING YOUR OWN FUNCTIONS

# You can write your own functions using the 'function' keyword

hypotenuse = function(a,b) {
  cSquared = a^2 + b^2
  c = sqrt(cSquared)
  return(c)
}

# Functions you create yourself will work in the same way as built-in R functions.

hypotenuse(3,4)
```

The Normal Distribution

Daniel Stephenson

SCMA 524: Statistical Fundamentals

The Normal Distribution

- A random variable X is **normally distributed** with mean μ and variance σ^2 if the PDF of X is

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The Normal Distribution

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$$\varphi_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

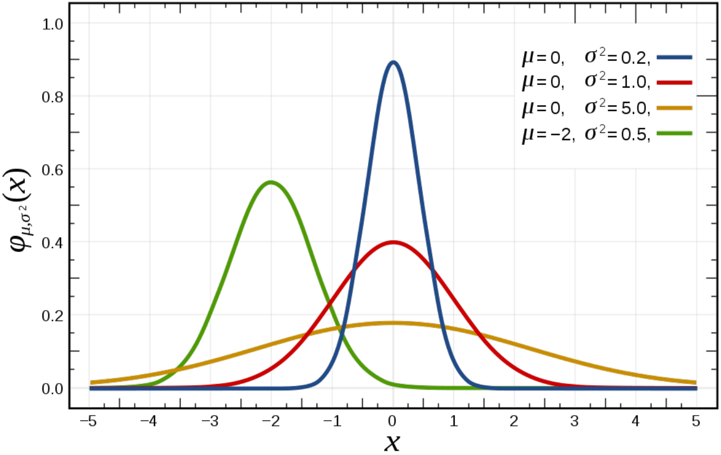
- If X is normally distributed with mean μ and variance σ^2 then we write

$$X \sim N\left(\mu, \sigma^2\right)$$

- The **standard normal distribution** has mean 0 and variance 1.

The Normal Distribution

$$\varphi_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



The Normal Distribution

Example

If $X \sim N\left(\mu, \sigma^2\right)$ then

$$E\left\{cX\right\} = cE\left\{X\right\} = c\mu$$

The Normal Distribution

Example

If $X \sim N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned} E\left\{cX\right\} &= cE\left\{X\right\} = c\mu \\ \mathrm{Var}\left(cX\right) &= c^2\mathrm{Var}\left(X\right) = c^2\sigma^2 \end{aligned}$$

The Normal Distribution

Example

If $X \sim N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned} E\left\{cX\right\} &= cE\left\{X\right\} = c\mu \\ \mathrm{Var}\left(cX\right) &= c^2\mathrm{Var}\left(X\right) = c^2\sigma^2 \\ cX &\sim N\left(c\mu, c^2\sigma^2\right) \end{aligned}$$

The Normal Distribution

Example

If $X \sim N(\mu, \sigma^2)$ then

$$\begin{aligned} E\{cX\} &= cE\{X\} = c\mu \\ \mathrm{Var}(cX) &= c^2\mathrm{Var}(X) = c^2\sigma^2 \\ cX &\sim N\left(c\mu, c^2\sigma^2\right) \end{aligned}$$

$$E\{X + c\} = E\{X\} + c = \mu + c$$

The Normal Distribution

Example

If $X \sim N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned} E\left\{cX\right\} &= cE\left\{X\right\} = c\mu \\ \mathrm{Var}\left(cX\right) &= c^2\mathrm{Var}\left(X\right) = c^2\sigma^2 \\ cX &\sim N\left(c\mu, c^2\sigma^2\right) \end{aligned}$$

$$\begin{aligned} E\left\{X + c\right\} &= E\left\{X\right\} + c = \mu + c \\ \mathrm{Var}\left(X + c\right) &= \mathrm{Var}\left(X\right) = \sigma^2 \end{aligned}$$

The Normal Distribution

Example

If $X \sim N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned}
E\left\{cX\right\} &= cE\left\{X\right\} = c\mu \\
\mathrm{Var}\left(cX\right) &= c^2\mathrm{Var}\left(X\right) = c^2\sigma^2 \\
cX &\sim N\left(c\mu, c^2\sigma^2\right)
\end{aligned}$$

$$\begin{aligned}
E\left\{X + c\right\} &= E\left\{X\right\} + c = \mu + c \\
\mathrm{Var}\left(X + c\right) &= \mathrm{Var}\left(X\right) = \sigma^2 \\
X + c &\sim N\left(\mu + c, \sigma^2\right)
\end{aligned}$$

The Normal Distribution

Example

If $X \sim N\left(\mu, \sigma^2\right)$ then

$$E\left\{\frac{X-\mu}{\sigma}\right\} = \frac{1}{\sigma}\left(E\left\{X\right\}-\mu\right)$$

The Normal Distribution

Example

If $X \sim N\left(\mu, \sigma^2\right)$ then

$$
\begin{aligned}
E\left\{\frac{X-\mu}{\sigma}\right\} &= \frac{1}{\sigma}\left(E\left\{X\right\}-\mu\right) \\
&= \frac{1}{\sigma}\left(\mu-\mu\right)=0
\end{aligned}
$$

The Normal Distribution

Example

If $X \sim N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned} E\left\{\frac{X-\mu}{\sigma}\right\} &= \frac{1}{\sigma}\left(E\left\{X\right\}-\mu\right) \\ &= \frac{1}{\sigma}\left(\mu-\mu\right) = 0 \end{aligned}$$

$$\operatorname{Var}\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}\operatorname{Var}\left(X-\mu\right)$$

The Normal Distribution

Example

If $X \sim N\left(\mu, \sigma^2\right)$ then

$$
\begin{aligned}
E\left\{\frac{X-\mu}{\sigma}\right\} &= \frac{1}{\sigma}\left(E\left\{X\right\}-\mu\right) \\
&= \frac{1}{\sigma}\left(\mu-\mu\right)=0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\left(\frac{X-\mu}{\sigma}\right) &= \frac{1}{\sigma^2}\mathrm{Var}\left(X-\mu\right) \\
&= \frac{1}{\sigma^2}\mathrm{Var}\left(X\right)=\frac{\sigma^2}{\sigma^2}=1
\end{aligned}
$$

The Normal Distribution

Example

If $X \sim N\left(\mu, \sigma^2\right)$ then

$$
\begin{aligned}
E\left\{\frac{X-\mu}{\sigma}\right\} &= \frac{1}{\sigma}\left(E\left\{X\right\}-\mu\right) \\
&= \frac{1}{\sigma}\left(\mu-\mu\right)=0
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Var}\left(\frac{X-\mu}{\sigma}\right) &= \frac{1}{\sigma^2}\operatorname{Var}\left(X-\mu\right) \\
&= \frac{1}{\sigma^2}\operatorname{Var}\left(X\right)=\frac{\sigma^2}{\sigma^2}=1
\end{aligned}
$$

$$
\frac{X-\mu}{\sigma} \sim N\left(0,1\right)
$$

The Normal Distribution

Example

If X and Y are independent with $X \sim N\left(\mu_X, \sigma_X^2\right)$ and $Y \sim N\left(\mu_Y, \sigma_Y^2\right)$

$$E\{aX + bY\} = aE\{X\} + bE\{Y\} = a\mu_X + b\mu_Y$$

The Normal Distribution

Example

If X and Y are independent with $X \sim N\left(\mu_X, \sigma_X^2\right)$ and $Y \sim N\left(\mu_Y, \sigma_Y^2\right)$

$$
\begin{aligned}
E\left\{aX + bY\right\} &= aE\left\{X\right\} + bE\left\{Y\right\} = a\mu_X + b\mu_Y \\
\mathrm{Var}\left(aX + bY\right) &= a^2\mathrm{Var}\left(X\right) + b^2\mathrm{Var}\left(Y\right) + ab\mathrm{Cov}\left(X, Y\right)
\end{aligned}
$$

The Normal Distribution

Example

If X and Y are independent with $X \sim N\left(\mu_X, \sigma_X^2\right)$ and $Y \sim N\left(\mu_Y, \sigma_Y^2\right)$

$$E\{aX + bY\} = aE\{X\} + bE\{Y\} = a\mu_X + b\mu_Y$$

$$\begin{aligned} \mathrm{Var}(aX + bY) &= a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + ab\mathrm{Cov}(X, Y) \\ &= a^2\sigma_X^2 + b^2\sigma_Y^2 + ab(0) \end{aligned}$$

The Normal Distribution

Example

If X and Y are independent with $X \sim N\left(\mu_X, \sigma_X^2\right)$ and $Y \sim N\left(\mu_Y, \sigma_Y^2\right)$

$$E\left\{aX + bY\right\} = aE\left\{X\right\} + bE\left\{Y\right\} = a\mu_X + b\mu_Y$$

$$\begin{aligned}
\mathrm{Var}\left(aX + bY\right) &= a^2\mathrm{Var}\left(X\right) + b^2\mathrm{Var}\left(Y\right) + ab\mathrm{Cov}\left(X, Y\right) \\
&= a^2\sigma_X^2 + b^2\sigma_Y^2 + ab\left(0\right) \\
&= a^2\sigma_X^2 + b^2\sigma_Y^2
\end{aligned}$$

The Normal Distribution

Example

If X and Y are independent with $X \sim N\left(\mu_X, \sigma_X^2\right)$ and $Y \sim N\left(\mu_Y, \sigma_Y^2\right)$

$$\begin{aligned} E\left\{aX + bY\right\} &= aE\left\{X\right\} + bE\left\{Y\right\} = a\mu_X + b\mu_Y \\ \\ \mathrm{Var}\left(aX + bY\right) &= a^2\mathrm{Var}\left(X\right) + b^2\mathrm{Var}\left(Y\right) + ab\mathrm{Cov}\left(X, Y\right) \\ &= a^2\sigma_X^2 + b^2\sigma_Y^2 + ab\left(0\right) \\ &= a^2\sigma_X^2 + b^2\sigma_Y^2 \\ \\ aX + bY &\sim N\left(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2\right) \end{aligned}$$

The Joint Normal Distribution

- Two random variables X and Y are **jointly normal** if

$$\begin{aligned} X &= aU + bV + \mu_X \\ Y &= cU + dV + \mu_Y \\ U, V &\sim i.i.d.\, N(0,1) \end{aligned}$$

The Joint Normal Distribution

- Two random variables X and Y are **jointly normal** if

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- If X and Y are jointly normal then they are **marginally normal**

$$\begin{aligned} X &\sim N\left(\mu_X, \sigma_X^2\right) \\ Y &\sim N\left(\mu_Y, \sigma_Y^2\right) \end{aligned}$$

The Joint Normal Distribution

- Two random variables X and Y are **jointly normal** if

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- If X and Y are jointly normal then they are **marginally normal**

$$\begin{aligned} X &\sim N\left(\mu_X, \sigma_X^2\right) \\ Y &\sim N\left(\mu_Y, \sigma_Y^2\right) \end{aligned}$$

- If X and Y are jointly normal and $\operatorname{Cov}(X, Y) = 0$

  then X and Y are independent.

Question

If W and Z are jointly normal

$E\{W\} = 2$, $E\{Z\} = 5$, and $\mathrm{Cov}(W, Z) = 0$

then what is $E\{ZW\}$?

Question

If W and Z are jointly normal

$E\left\{W\right\} = 2$, $E\left\{Z\right\} = 5$, and $\mathrm{Cov}\left(W, Z\right) = 0$

then what is $E\left\{ZW\right\}$?

Answer:

Question

If W and Z are jointly normal

$E\{W\} = 2$, $E\{Z\} = 5$, and $\text{Cov}(W, Z) = 0$

then what is $E\{ZW\}$?

Answer: 10

W and Z are independent since

they are jointly normal and $\text{Cov}(W, Z) = 0$.

$$E\{ZW\} = E\{Z\} E\{W\} = (2)(5) = 10$$

The Joint Normal Distribution

Example

Suppose X, Y, and Z are jointly normal such that

$$\begin{aligned} X &= U + 2V - W \\ Y &= U + 0.67V + 0.5W \\ Z &= 4U - W + 9 \end{aligned} \qquad U, V, W \sim i.i.d.\, N(0,1)$$

The Joint Normal Distribution

Example

Suppose X, Y, and Z are jointly normal such that

$$\begin{aligned} X &= U + 2V - W \qquad\qquad U, V, W \sim i.i.d.\, N(0,1) \\ Y &= U + 0.67V + 0.5W \\ Z &= 4U - W + 9 \end{aligned}$$

Suppose that
$$\begin{aligned} L_1 &= Y + Z \\ L_2 &= X - 3Y - Z + 1 \\ L_3 &= X + 100 \\ L_4 &= Z - X \end{aligned}$$

The Joint Normal Distribution

Example

Suppose X, Y, and Z are jointly normal such that

$$\begin{aligned} X &= U + 2V - W \qquad\qquad U, V, W \sim i.i.d.\, N(0,1) \\ Y &= U + 0.67V + 0.5W \\ Z &= 4U - W + 9 \end{aligned}$$

Suppose that
$$\begin{aligned} L_1 &= Y + Z \\ L_2 &= X - 3Y - Z + 1 \\ L_3 &= X + 100 \\ L_4 &= Z - X \end{aligned}$$

Then L_1, L_2, L_3, L_4 are jointly normal.

Example

Let $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

Example

Let $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

Let $U_i = \dfrac{X_i - \mu}{\sigma}$

Example

Let $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

Let $U_i = \dfrac{X_i - \mu}{\sigma}$

$$E\left\{U_i\right\} = E\left\{\frac{X_i - \mu}{\sigma}\right\} = \frac{E\left\{X_i\right\} - \mu}{\sigma} = \frac{\mu - \mu}{\sigma} = 0$$

Example

Let $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

Let $U_i = \dfrac{X_i - \mu}{\sigma}$

$$\begin{aligned} E\left\{U_i\right\} &= E\left\{\frac{X_i - \mu}{\sigma}\right\} = \frac{E\left\{X_i\right\} - \mu}{\sigma} = \frac{\mu - \mu}{\sigma} = 0 \\ \mathrm{Var}\left(U_i\right) &= \mathrm{Var}\left(\frac{X_i - \mu}{\sigma}\right) = \frac{\mathrm{Var}\left(X_i\right)}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1 \end{aligned}$$

Example

Let $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

Let $U_i = \dfrac{X_i - \mu}{\sigma}$

$$\begin{aligned} E\left\{U_i\right\} &= E\left\{\frac{X_i - \mu}{\sigma}\right\} = \frac{E\left\{X_i\right\} - \mu}{\sigma} = \frac{\mu - \mu}{\sigma} = 0 \\ \operatorname{Var}\left(U_i\right) &= \operatorname{Var}\left(\frac{X_i - \mu}{\sigma}\right) = \frac{\operatorname{Var}\left(X_i\right)}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1 \end{aligned}$$

Then $U_1, U_2, U_3 \sim i.i.d.\ N(0, 1)$

Example

Let $X_1, X_2, X_3 \sim i.i.d.\ N(\mu, \sigma^2)$

Let $U_i = \dfrac{X_i - \mu}{\sigma}$

$$\begin{aligned} E\{U_i\} &= E\left\{\frac{X_i - \mu}{\sigma}\right\} = \frac{E\{X_i\} - \mu}{\sigma} = \frac{\mu - \mu}{\sigma} = 0 \\ \mathrm{Var}(U_i) &= \mathrm{Var}\left(\frac{X_i - \mu}{\sigma}\right) = \frac{\mathrm{Var}(X_i)}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1 \end{aligned}$$

Then $U_1, U_2, U_3 \sim i.i.d.\ N(0, 1)$

$$\begin{aligned} X_1 &= \sigma U_1 + \mu \\ X_2 &= \sigma U_2 + \mu \\ X_3 &= \sigma U_3 + \mu \end{aligned}$$

So X_1, X_2, X_3 are jointly normal.

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$
\begin{aligned}
\operatorname{Cov}\left(X_i, X_i\right) &= \operatorname{Var}\left(X_i\right) = \sigma^2 \\
\operatorname{Cov}\left(X_i, X_j\right) &= 0 \qquad \text{if } i \neq j
\end{aligned}
$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\begin{aligned}
\operatorname{Cov}\left(X_i, X_i\right) &= \operatorname{Var}\left(X_i\right) = \sigma^2 \\
\operatorname{Cov}\left(X_i, X_j\right) &= 0 \qquad \text{if } i \neq j
\end{aligned}$$

$$\operatorname{Cov}\left(X_1, \overline{X}\right) = \operatorname{Cov}\left(X_1, \frac{1}{3}\left[X_1 + X_2 + X_3\right]\right)$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\begin{aligned} \text{Cov}\left(X_i, X_i\right) &= \text{Var}\left(X_i\right) = \sigma^2 \\ \text{Cov}\left(X_i, X_j\right) &= 0 \qquad \text{if } i \neq j \end{aligned}$$

$$\begin{aligned} \text{Cov}\left(X_1, \overline{X}\right) &= \text{Cov}\left(X_1, \frac{1}{3}\left[X_1 + X_2 + X_3\right]\right) \\ &= \frac{1}{3}\text{Cov}\left(X_1,\ X_1 + X_2 + X_3\right) \end{aligned}$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$
\begin{aligned}
\mathrm{Cov}\left(X_i, X_i\right) &= \mathrm{Var}\left(X_i\right) = \sigma^2 \\
\mathrm{Cov}\left(X_i, X_j\right) &= 0 \qquad \text{if } i \neq j
\end{aligned}
$$

$$
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\mathrm{Cov}\left(X_1, \overline{X}\right) &= \mathrm{Cov}\left(X_1, \frac{1}{3}\left[X_1 + X_2 + X_3\right]\right) \\
&= \frac{1}{3}\mathrm{Cov}\left(X_1,\ X_1 + X_2 + X_3\right) \\
&= \frac{1}{3}\left[\mathrm{Cov}\left(X_1, X_1\right) + \mathrm{Cov}\left(X_1, X_2\right) + \mathrm{Cov}\left(X_1, X_3\right)\right]
\end{aligned}
$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

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\begin{aligned}
\text{Cov}\left(X_i, X_i\right) &= \text{Var}\left(X_i\right) = \sigma^2 \\
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\text{Cov}\left(X_1, \overline{X}\right) &= \text{Cov}\left(X_1, \frac{1}{3}\left[X_1 + X_2 + X_3\right]\right) \\
&= \frac{1}{3}\text{Cov}\left(X_1,\ X_1 + X_2 + X_3\right) \\
&= \frac{1}{3}\left[\text{Cov}\left(X_1, X_1\right) + \text{Cov}\left(X_1, X_2\right) + \text{Cov}\left(X_1, X_3\right)\right] \\
&= \frac{1}{3}\left[\sigma^2 + 0 + 0\right] = \frac{1}{3}\sigma^2
\end{aligned}
$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N\left(10, 5\right)$

What is $\text{Cov}\left(Y_2, \overline{Y}\right)$?

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N(10, 5)$

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$$\text{Cov}\left(Y_2, \overline{Y}\right)$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N(10, 5)$

What is $\text{Cov}\left(Y_2, \overline{Y}\right)$?

$$
\begin{aligned}
& \text{Cov}\left(Y_2, \overline{Y}\right) \\
= \ & \text{Cov}\left(Y_2, \frac{1}{4}\left[Y_1 + Y_2 + Y_3 + Y_4\right]\right)
\end{aligned}
$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N(10, 5)$

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= \ & \text{Cov}\left(Y_2, \frac{1}{4}\left[Y_1 + Y_2 + Y_3 + Y_4\right]\right) \\
= \ & \frac{1}{4}\text{Cov}\left(Y_2,\ Y_1 + Y_2 + Y_3 + Y_4\right)
\end{aligned}
$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N(10, 5)$

What is $\text{Cov}\left(Y_2, \overline{Y}\right)$?

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\begin{aligned}
& \text{Cov}\left(Y_2, \overline{Y}\right) \\
= \quad & \text{Cov}\left(Y_2, \frac{1}{4}\left[Y_1 + Y_2 + Y_3 + Y_4\right]\right) \\
= \quad & \frac{1}{4}\text{Cov}\left(Y_2,\ Y_1 + Y_2 + Y_3 + Y_4\right) \\
= \quad & \frac{1}{4}\left[\text{Cov}\left(Y_2, Y_1\right) + \text{Cov}\left(Y_2, Y_2\right) + \text{Cov}\left(Y_2, Y_3\right) + \text{Cov}\left(Y_2, Y_4\right)\right]
\end{aligned}
$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N(10, 5)$

What is $\mathrm{Cov}\left(Y_2, \overline{Y}\right)$?

$$
\begin{aligned}
& \mathrm{Cov}\left(Y_2, \overline{Y}\right) \\
= \; & \mathrm{Cov}\left(Y_2, \frac{1}{4}\left[Y_1 + Y_2 + Y_3 + Y_4\right]\right) \\
= \; & \frac{1}{4}\mathrm{Cov}\left(Y_2,\ Y_1 + Y_2 + Y_3 + Y_4\right) \\
= \; & \frac{1}{4}\left[\mathrm{Cov}\left(Y_2, Y_1\right) + \mathrm{Cov}\left(Y_2, Y_2\right) + \mathrm{Cov}\left(Y_2, Y_3\right) + \mathrm{Cov}\left(Y_2, Y_4\right)\right] \\
= \; & \frac{1}{4}\left[0 + 5 + 0 + 0\right] = \frac{5}{4} = 1.25
\end{aligned}
$$

Question

Suppose $H_1, H_2, H_3, H_4, H_5 \sim i.i.d.\ N(6, 10)$

What is $\text{Cov}\left(H_2, \overline{H}\right)$?

Question

Suppose $H_1, H_2, H_3, H_4, H_5 \sim i.i.d.\ N(6, 10)$

What is $\mathrm{Cov}\left(H_2, \overline{H}\right)$?

Answer:

Question

Suppose $H_1, H_2, H_3, H_4, H_5 \sim i.i.d.\ N(6, 10)$

What is $\text{Cov}\left(H_2, \overline{H}\right)$?

Answer:

$$\text{Cov}\left(H_2, \overline{H}\right) = \text{Cov}\left(H_2, \frac{H_1 + H_2 + H_3 + H_4 + H_5}{5}\right)$$

Question

Suppose $H_1, H_2, H_3, H_4, H_5 \sim i.i.d.\ N(6, 10)$

What is $\text{Cov}\left(H_2, \overline{H}\right)$?

Answer:

$$\text{Cov}\left(H_2, \overline{H}\right) = \text{Cov}\left(H_2, \frac{H_1 + H_2 + H_3 + H_4 + H_5}{5}\right)$$

$$= \quad \frac{1}{5}\text{Cov}\left(H_2,\ H_1 + H_2 + H_3 + H_4 + H_5\right)$$

Question

Suppose $H_1, H_2, H_3, H_4, H_5 \sim i.i.d.\ N(6, 10)$

What is $\text{Cov}\left(H_2, \overline{H}\right)$?

Answer:

$$
\begin{aligned}
\text{Cov}\left(H_2, \overline{H}\right) &= \text{Cov}\left(H_2, \frac{H_1 + H_2 + H_3 + H_4 + H_5}{5}\right) \\
&= \frac{1}{5}\text{Cov}\left(H_2,\ H_1 + H_2 + H_3 + H_4 + H_5\right) \\
&= \frac{\text{Cov}(H_2, H_1) + \text{Cov}(H_2, H_2) + \text{Cov}(H_2, H_3) + \text{Cov}(H_2, H_4) + \text{Cov}(H_2, H_5)}{5}
\end{aligned}
$$

Question

Suppose $H_1, H_2, H_3, H_4, H_5 \sim i.i.d.\ N(6, 10)$

What is $\text{Cov}\left(H_2, \overline{H}\right)$?

Answer: 2

$$
\begin{aligned}
& \text{Cov}\left(H_2, \overline{H}\right) = \text{Cov}\left(H_2, \frac{H_1 + H_2 + H_3 + H_4 + H_5}{5}\right) \\
= \quad & \frac{1}{5}\text{Cov}\left(H_2,\ H_1 + H_2 + H_3 + H_4 + H_5\right) \\
= \quad & \frac{\text{Cov}\left(H_2, H_1\right) + \text{Cov}\left(H_2, H_2\right) + \text{Cov}\left(H_2, H_3\right) + \text{Cov}\left(H_2, H_4\right) + \text{Cov}\left(H_2, H_5\right)}{5} \\
= \quad & \frac{0 + 10 + 0 + 0 + 0}{5} = \frac{10}{5} = 2
\end{aligned}
$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\text{Cov}\left(X_i, \overline{X}\right) = \frac{1}{3}\sigma^2$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N(\mu, \sigma^2)$

$$\begin{aligned} \text{Cov}\left(X_i, \overline{X}\right) &= \frac{1}{3}\sigma^2 \\ \text{Cov}\left(\overline{X}, \overline{X}\right) &= \text{Cov}\left(\frac{1}{3}\left[X_1 + X_2 + X_3\right], \overline{X}\right) \end{aligned}$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$
\begin{aligned}
\operatorname{Cov}\left(X_i, \overline{X}\right) &= \frac{1}{3}\sigma^2 \\
\operatorname{Cov}\left(\overline{X}, \overline{X}\right) &= \operatorname{Cov}\left(\frac{1}{3}\left[X_1 + X_2 + X_3\right], \overline{X}\right) \\
&= \frac{1}{3}\left[\operatorname{Cov}\left(X_1, \overline{X}\right) + \operatorname{Cov}\left(X_2, \overline{X}\right) + \operatorname{Cov}\left(X_3, \overline{X}\right)\right]
\end{aligned}
$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\begin{aligned}
\text{Cov}\left(X_i, \overline{X}\right) &= \frac{1}{3}\sigma^2 \\
\text{Cov}\left(\overline{X}, \overline{X}\right) &= \text{Cov}\left(\frac{1}{3}\left[X_1 + X_2 + X_3\right], \overline{X}\right) \\
&= \frac{1}{3}\left[\text{Cov}\left(X_1, \overline{X}\right) + \text{Cov}\left(X_2, \overline{X}\right) + \text{Cov}\left(X_3, \overline{X}\right)\right] \\
&= \frac{1}{3}\left[\frac{1}{3}\sigma^2 + \frac{1}{3}\sigma^2 + \frac{1}{3}\sigma^2\right]
\end{aligned}$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\begin{aligned}
\text{Cov}\left(X_i, \overline{X}\right) &= \frac{1}{3}\sigma^2 \\
\text{Cov}\left(\overline{X}, \overline{X}\right) &= \text{Cov}\left(\frac{1}{3}\left[X_1 + X_2 + X_3\right], \overline{X}\right) \\
&= \frac{1}{3}\left[\text{Cov}\left(X_1, \overline{X}\right) + \text{Cov}\left(X_2, \overline{X}\right) + \text{Cov}\left(X_3, \overline{X}\right)\right] \\
&= \frac{1}{3}\left[\frac{1}{3}\sigma^2 + \frac{1}{3}\sigma^2 + \frac{1}{3}\sigma^2\right] \\
&= \frac{1}{3}\sigma^2 = \text{Cov}\left(X_i, \overline{X}\right)
\end{aligned}$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N(10, 5)$

What is $\mathrm{Cov}\left(\overline{Y}, \overline{Y}\right)$?

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N\left(10, 5\right)$

What is $\text{Cov}\left(\overline{Y}, \overline{Y}\right)$?

$$\text{Cov}\left(\overline{Y}, \overline{Y}\right) = \text{Var}\left(\overline{Y}\right)$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N(10, 5)$

What is $\text{Cov}\left(\overline{Y}, \overline{Y}\right)$?

$$\begin{aligned} & \text{Cov}\left(\overline{Y}, \overline{Y}\right) = \text{Var}\left(\overline{Y}\right) \\ = \quad & \text{Var}\left(\frac{1}{4}\left[Y_1 + Y_2 + Y_3 + Y_4\right]\right) \end{aligned}$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N(10, 5)$

What is $\mathrm{Cov}\left(\overline{Y}, \overline{Y}\right)$?

$$
\begin{aligned}
& \mathrm{Cov}\left(\overline{Y}, \overline{Y}\right) = \mathrm{Var}\left(\overline{Y}\right) \\
= \quad & \mathrm{Var}\left(\frac{1}{4}\left[Y_1 + Y_2 + Y_3 + Y_4\right]\right) \\
= \quad & \frac{1}{16}\mathrm{Var}\left(Y_1 + Y_2 + Y_3 + Y_4\right)
\end{aligned}
$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N(10, 5)$

What is $\text{Cov}\left(\overline{Y}, \overline{Y}\right)$?

$$\begin{aligned}
& \text{Cov}\left(\overline{Y}, \overline{Y}\right) = \text{Var}\left(\overline{Y}\right) \\
= \quad & \text{Var}\left(\frac{1}{4}\left[Y_1 + Y_2 + Y_3 + Y_4\right]\right) \\
= \quad & \frac{1}{16}\text{Var}\left(Y_1 + Y_2 + Y_3 + Y_4\right) \\
= \quad & \frac{1}{16}\left[\text{Var}\left(Y_1\right) + \text{Var}\left(Y_2\right) + \text{Var}\left(Y_3\right) + \text{Var}\left(Y_4\right)\right]
\end{aligned}$$

Example

Suppose $Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N\left(10, 5\right)$

What is $\text{Cov}\left(\overline{Y}, \overline{Y}\right)$?

$$
\begin{aligned}
& \text{Cov}\left(\overline{Y}, \overline{Y}\right) = \text{Var}\left(\overline{Y}\right) \\
= \;& \text{Var}\left(\frac{1}{4}\left[Y_1 + Y_2 + Y_3 + Y_4\right]\right) \\
= \;& \frac{1}{16}\text{Var}\left(Y_1 + Y_2 + Y_3 + Y_4\right) \\
= \;& \frac{1}{16}\left[\text{Var}\left(Y_1\right) + \text{Var}\left(Y_2\right) + \text{Var}\left(Y_3\right) + \text{Var}\left(Y_4\right)\right] \\
= \;& \frac{1}{16}\left[5 + 5 + 5 + 5\right] = \frac{20}{16} = \frac{5}{4} = 1.25
\end{aligned}
$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\mathrm{Cov}\left(X_i, \overline{X}\right) = \mathrm{Cov}\left(\overline{X}, \overline{X}\right) = \frac{1}{3}\sigma^2$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\begin{aligned} \mathrm{Cov}\left(X_i, \overline{X}\right) &= \mathrm{Cov}\left(\overline{X}, \overline{X}\right) = \frac{1}{3}\sigma^2 \\ \mathrm{Cov}\left(X_i - \overline{X}, \overline{X}\right) &= \mathrm{Cov}\left(X_i, \overline{X}\right) - \mathrm{Cov}\left(\overline{X}, \overline{X}\right) \end{aligned}$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\begin{aligned}
\operatorname{Cov}\left(X_i, \overline{X}\right) &= \operatorname{Cov}\left(\overline{X}, \overline{X}\right) = \frac{1}{3}\sigma^2 \\
\operatorname{Cov}\left(X_i - \overline{X}, \overline{X}\right) &= \operatorname{Cov}\left(X_i, \overline{X}\right) - \operatorname{Cov}\left(\overline{X}, \overline{X}\right) \\
&= \frac{1}{3}\sigma^2 - \frac{1}{3}\sigma^2 = 0
\end{aligned}$$

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

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\mathrm{Cov}\left(X_i, \overline{X}\right) &= \mathrm{Cov}\left(\overline{X}, \overline{X}\right) = \frac{1}{3}\sigma^2 \\
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&= \frac{1}{3}\sigma^2 - \frac{1}{3}\sigma^2 = 0
\end{aligned}$$

$X_i - \overline{X}$ and $\overline{X}$ are jointly normal and $\mathrm{Cov}\left(X_i - \overline{X}, \overline{X}\right) = 0$.

The Joint Normal Distribution

Example

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\begin{aligned}
\text{Cov}\left(X_i, \overline{X}\right) &= \text{Cov}\left(\overline{X}, \overline{X}\right) = \frac{1}{3}\sigma^2 \\
\text{Cov}\left(X_i - \overline{X}, \overline{X}\right) &= \text{Cov}\left(X_i, \overline{X}\right) - \text{Cov}\left(\overline{X}, \overline{X}\right) \\
&= \frac{1}{3}\sigma^2 - \frac{1}{3}\sigma^2 = 0
\end{aligned}$$

$X_i - \overline{X}$ and $\overline{X}$ are jointly normal and $\text{Cov}\left(X_i - \overline{X}, \overline{X}\right) = 0$.

So $X_i - \overline{X}$ and $\overline{X}$ are independent.

Example

Suppose $Y_1, Y_2, \ldots, Y_{10} \sim i.i.d.\ N(2, 7)$

What is $E\left\{\left(Y_5 - \overline{Y}\right)\overline{Y}\right\}$?

Example

Suppose $Y_1, Y_2, \ldots, Y_{10} \sim i.i.d.\ N(2, 7)$

What is $E\left\{\left(Y_5 - \overline{Y}\right)\overline{Y}\right\}$?

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So $Y_5 - \overline{Y}$ and $\overline{Y}$ are independent

$$E\left\{\left(Y_5 - \overline{Y}\right)\overline{Y}\right\} = E\left\{Y_5 - \overline{Y}\right\} E\left\{\overline{Y}\right\}$$

Example

Suppose $Y_1, Y_2, \ldots, Y_{10} \sim i.i.d.\ N(2, 7)$

What is $E\left\{\left(Y_5 - \overline{Y}\right)\overline{Y}\right\}$?

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So $Y_5 - \overline{Y}$ and $\overline{Y}$ are independent

$$\begin{aligned} E\left\{\left(Y_5 - \overline{Y}\right)\overline{Y}\right\} &= E\left\{Y_5 - \overline{Y}\right\} E\left\{\overline{Y}\right\} \\ &= (2-2)(2) \end{aligned}$$

Example

Suppose $Y_1, Y_2, \ldots, Y_{10} \sim i.i.d.\ N(2,7)$

What is $E\left\{\left(Y_5 - \overline{Y}\right)\overline{Y}\right\}$?

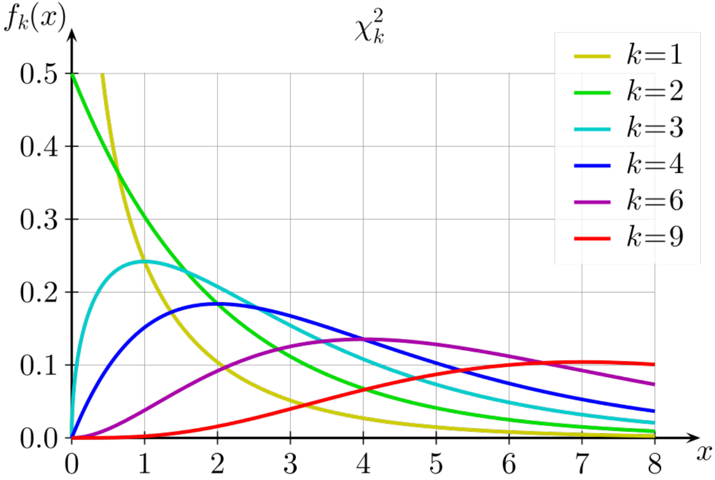
$Y_5 - \overline{Y}$ and $\overline{Y}$ are jointly normal and $\text{Cov}\left(Y_5 - \overline{Y}, \overline{Y}\right) = 0.$

So $Y_5 - \overline{Y}$ and $\overline{Y}$ are independent

$$
\begin{aligned}
E\left\{\left(Y_5 - \overline{Y}\right)\overline{Y}\right\} &= E\left\{Y_5 - \overline{Y}\right\} E\left\{\overline{Y}\right\} \\
&= (2-2)(2) \\
&= (0)(2) = 0
\end{aligned}
$$

The Chi-Squared Distribution

- If $X_1, \ldots, X_k \sim i.i.d.\, N(0,1)$ then $X_1^2 + \cdots + X_k^2 \sim \chi^2(k)$



The Chi-Squared Distribution

- If $X \sim \chi^2(k)$ then $E\{X\} = k$

The Chi-Squared Distribution

- If $X \sim \chi^2(k)$ then $E\{X\} = k$
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  then $X + Y \sim \chi^2(n+m)$

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- If $X \sim \chi^2(k)$ then $E\{X\} = k$
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The Chi-Squared Distribution

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- If X and Y are independent, $X \sim \chi^2(n)$, and $X + Y \sim \chi^2(n+m)$

  then $Y \sim \chi^2(m)$

Example

If X and Y are independent, $X \sim \chi^2(2)$, and $X + Y \sim \chi^2(5)$

then $Y \sim \chi^2(3)$

The Chi-Squared Distribution

Example

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

The Chi-Squared Distribution

Example

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\frac{X_i - \mu}{\sigma} \sim i.i.d.\ N\left(0, 1\right)$$

The Chi-Squared Distribution

Example

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\frac{X_i - \mu}{\sigma} \sim i.i.d.\ N\left(0, 1\right)$$

$$W = \left(\frac{X_1 - \mu}{\sigma}\right)^2 + \left(\frac{X_2 - \mu}{\sigma}\right)^2 + \left(\frac{X_3 - \mu}{\sigma}\right)^2$$

The Chi-Squared Distribution

Example

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

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$$W \sim \chi^2\left(3\right)$$

The Chi-Squared Distribution

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$$W \sim \chi^2(3)$$

$$E\{W\} = 3$$

The Chi-Squared Distribution

Example

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

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$$W \sim \chi^2\left(3\right)$$

$$E\left\{W\right\} = 3$$

$$\mathrm{Var}\left(W\right) = 6$$

Question

Suppose $Y_1, Y_2, Y_3 \sim i.i.d.\ N(7, 25)$

Let $H = \left(\frac{Y_1 - 7}{5}\right)^2 + \left(\frac{Y_2 - 7}{5}\right)^2 + \left(\frac{Y_3 - 7}{5}\right)^2$

What is $\text{Var}(H)$?

Question

Suppose $Y_1, Y_2, Y_3 \sim i.i.d.\ N(7, 25)$

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Answer:

Question

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Answer:

$$\frac{Y_i - 7}{5} \quad \sim \quad N(0, 1)$$

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What is $\mathrm{Var}(H)$?

Answer:

$$\begin{aligned} \frac{Y_i - 7}{5} &\sim N(0, 1) \\ \left(\frac{Y_i - 7}{5}\right)^2 &\sim \chi^2(1) \end{aligned}$$

Question

Suppose $Y_1, Y_2, Y_3 \sim i.i.d.\ N(7, 25)$

Let $H = \left(\frac{Y_1 - 7}{5}\right)^2 + \left(\frac{Y_2 - 7}{5}\right)^2 + \left(\frac{Y_3 - 7}{5}\right)^2$

What is $\mathrm{Var}(H)$?

Answer:

$$
\begin{aligned}
\frac{Y_i - 7}{5} &\sim N(0, 1) \\
\left(\frac{Y_i - 7}{5}\right)^2 &\sim \chi^2(1) \\
H = \sum_{i=1}^{3} \left(\frac{Y_i - 7}{5}\right)^2 &\sim \chi^2(3)
\end{aligned}
$$

Question

Suppose $Y_1, Y_2, Y_3 \sim i.i.d.\ N(7, 25)$

Let $H = \left(\frac{Y_1 - 7}{5}\right)^2 + \left(\frac{Y_2 - 7}{5}\right)^2 + \left(\frac{Y_3 - 7}{5}\right)^2$

What is $\text{Var}(H)$?

Answer: 6

$$\begin{aligned} \frac{Y_i - 7}{5} &\sim N(0, 1) \\ \left(\frac{Y_i - 7}{5}\right)^2 &\sim \chi^2(1) \\ H = \sum_{i=1}^{3} \left(\frac{Y_i - 7}{5}\right)^2 &\sim \chi^2(3) \\ \text{Var}(H) &= (2)(3) = 6 \end{aligned}$$

The t-Distribution

- If Z and V are independent with $Z \sim N(0,1)$ and $V \sim \chi^2(n)$ then

$$T = \frac{Z}{\sqrt{V/n}} \sim t(n)$$

T is said to have a t-distribution with n degrees of freedom.

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- If $T \sim t(n)$ and $n > 1$ then

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- If $T \sim t(n)$ and $n > 1$ then

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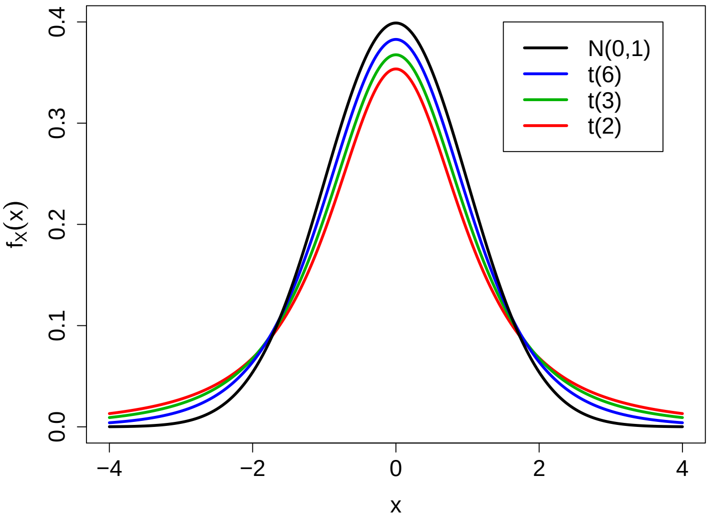
- If $T \sim t(n)$ and $n > 2$ then

$$\text{Var}(T) = \frac{n}{n-2}$$

The t-Distribution



The t-Distribution

Example

If $X_1, X_2, X_3, X_4, X_5, X_6 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

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Example

If $X_1, X_2, X_3, X_4, X_5, X_6 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$Y_i = \frac{X_i - \mu}{\sigma} \sim N\left(0, 1\right)$$

The t-Distribution

Example

If $X_1, X_2, X_3, X_4, X_5, X_6 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned} Y_i &= \frac{X_i - \mu}{\sigma} \sim N(0,1) \\ V &= \sum_{i=1}^{6} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2(6) \end{aligned}$$

The t-Distribution

Example

If $X_1, X_2, X_3, X_4, X_5, X_6 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

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If $Z \sim N(0,1)$ and Z, V are independent then

$$W = \frac{Z}{\sqrt{V/6}} \sim t(6)$$

The t-Distribution

Example

If $X_1, X_2, X_3, X_4, X_5, X_6 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

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If $Z \sim N(0,1)$ and Z, V are independent then

$$\begin{aligned} W &= \frac{Z}{\sqrt{V/6}} \sim t(6) \\ E\{W\} &= 0 \end{aligned}$$

The t-Distribution

Example

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If $Z \sim N(0,1)$ and Z, V are independent then

$$\begin{aligned} W &= \frac{Z}{\sqrt{V/6}} \sim t(6) \\ E\{W\} &= 0 \\ \mathrm{Var}(W) &= \frac{6}{6-2} = 1.5 \end{aligned}$$

Example

Let $Y_1, Y_2, \ldots, Y_{13} \sim i.i.d.\ N(3, 4)$

Example

Let $Y_1, Y_2, \ldots, Y_{13} \sim i.i.d.\ N(3, 4)$

$$\frac{Y_{13} - 3}{2} \sim N(0, 1)$$

Example

Let $Y_1, Y_2, \ldots, Y_{13} \sim i.i.d.\ N(3, 4)$

$$\frac{Y_{13} - 3}{2} \sim N(0, 1) \qquad \sum_{i=1}^{12} \left(\frac{Y_i - 3}{2} \right)^2 \sim \chi^2(12)$$

Example

Let $Y_1, Y_2, \ldots, Y_{13} \sim i.i.d.\ N(3,4)$

$$\frac{Y_{13}-3}{2} \sim N(0,1) \qquad \sum_{i=1}^{12}\left(\frac{Y_i-3}{2}\right)^2 \sim \chi^2(12)$$

Let $H = \dfrac{(Y_{13}-3)}{\sqrt{\frac{1}{12}\sum_{i=1}^{12}(Y_i-3)^2}}$

Example

Let $Y_1, Y_2, \ldots, Y_{13} \sim i.i.d.\ N(3,4)$

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$$\text{Let } H = \frac{(Y_{13}-3)}{\sqrt{\frac{1}{12}\sum_{i=1}^{12}(Y_i-3)^2}} = \frac{\left(\frac{Y_{13}-3}{2}\right)}{\sqrt{\frac{1}{12}\sum_{i=1}^{12}\left(\frac{Y_i-3}{2}\right)^2}}$$

Example

Let $Y_1, Y_2, \ldots, Y_{13} \sim i.i.d.\ N(3,4)$

$$\frac{Y_{13}-3}{2} \sim N(0,1) \qquad \sum_{i=1}^{12}\left(\frac{Y_i-3}{2}\right)^2 \sim \chi^2(12)$$

$$\begin{aligned}
\text{Let } H = \frac{(Y_{13}-3)}{\sqrt{\frac{1}{12}\sum_{i=1}^{12}(Y_i-3)^2}} &= \frac{\left(\frac{Y_{13}-3}{2}\right)}{\sqrt{\frac{1}{12}\sum_{i=1}^{12}\left(\frac{Y_i-3}{2}\right)^2}} \\
&= \frac{N(0,1)}{\sqrt{\chi^2(12)/12}} \sim t(12)
\end{aligned}$$

Example

Let $Y_1, Y_2, \ldots, Y_{13} \sim i.i.d.\ N(3, 4)$

$$\frac{Y_{13} - 3}{2} \sim N(0, 1) \qquad \sum_{i=1}^{12} \left(\frac{Y_i - 3}{2} \right)^2 \sim \chi^2(12)$$

$$\begin{aligned}
\text{Let } H = \frac{(Y_{13} - 3)}{\sqrt{\frac{1}{12} \sum_{i=1}^{12} (Y_i - 3)^2}} &= \frac{\left(\frac{Y_{13} - 3}{2} \right)}{\sqrt{\frac{1}{12} \sum_{i=1}^{12} \left(\frac{Y_i - 3}{2} \right)^2}} \\
&= \frac{N(0, 1)}{\sqrt{\chi^2(12)/12}} \sim t(12) \\
\text{Var}(H) &= \frac{12}{12 - 2}
\end{aligned}$$

Example

Let $Y_1, Y_2, \ldots, Y_{13} \sim i.i.d.\ N(3,4)$

$$\frac{Y_{13}-3}{2} \sim N(0,1) \qquad \sum_{i=1}^{12}\left(\frac{Y_i-3}{2}\right)^2 \sim \chi^2(12)$$

$$\begin{aligned}
\text{Let } H = \frac{(Y_{13}-3)}{\sqrt{\frac{1}{12}\sum_{i=1}^{12}(Y_i-3)^2}} &= \frac{\left(\frac{Y_{13}-3}{2}\right)}{\sqrt{\frac{1}{12}\sum_{i=1}^{12}\left(\frac{Y_i-3}{2}\right)^2}} \\
&= \frac{N(0,1)}{\sqrt{\chi^2(12)/12}} \sim t(12) \\
\operatorname{Var}(H) &= \frac{12}{12-2} = \frac{12}{10} = 1.2
\end{aligned}$$

The F-Distribution

- If U and V are independent with $U \sim \chi^2(n)$ and $V \sim \chi^2(m)$ then

$$Y = \frac{U/n}{V/m} \sim F(n, m)$$

Y has an F-distribution with n and m degrees of freedom.

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- If $Y \sim F(n, m)$ and $m > 2$ then

$$E\{Y\} = \frac{m}{m-2}$$

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  Y has an F-distribution with n and m degrees of freedom.

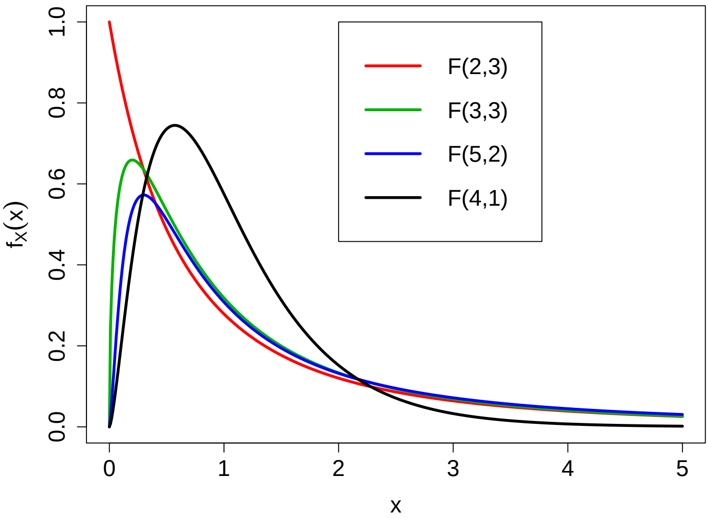
- If $Y \sim F(n, m)$ and $m > 2$ then

$$E\{Y\} = \frac{m}{m-2}$$

- If $Y \sim F(n, m)$ and $m > 4$ then

$$\mathrm{Var}(Y) = \frac{2m^2(n+m-2)}{n(m-2)^2(m-4)}$$

The F-Distribution



Example

Let $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8 \sim i.i.d.\ N(0, 1)$

Example

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Convergence

Daniel Stephenson

SCMA 524: Statistical Fundamentals

Sequences of Random Variables

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Example

Flip a coin repeatedly. Count a head as 1 and a tail as 0.

$$X_1, X_2, X_3, \ldots \sim i.i.d. \; U\{0, 1\}$$

Convergence in Probability

- A sequence of random variables **converges in probability** to c if

$$\lim_{n\to\infty} P\left(|X_n - c| > \varepsilon\right) = 0 \qquad \text{for all } \varepsilon > 0$$

- We write this as $X_n \overset{p}{\to} c$.

- The probability of an "unusual" outcome becomes smaller and smaller as the sequence progresses.

Convergence in Probability

Example

Let $Z_1, Z_2, Z_3, \ldots$ be a sequence of random variables.

Suppose $Z_n \sim U\left[0, \frac{1}{n}\right]$. Let F_n denote the CDF of Z_n.

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Suppose $Z_n \sim U\left[0, \frac{1}{n}\right]$. Let F_n denote the CDF of Z_n.

$$F_n(z) = P(Z_n < z) = \begin{cases} 0 & \text{if } z < 0 \\ \frac{z}{n} & \text{if } z \in \left[0, \frac{1}{n}\right] \\ 1 & \text{if } z > \frac{1}{n} \end{cases}$$

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Therefore $Z_n \xrightarrow{p} 0$.

Convergence in Probability

If $X_n \xrightarrow{p} a$ and $Y_n \xrightarrow{p} b$ then

$$X_n + Y_n \xrightarrow{p} a + b$$

$$X_n Y_n \xrightarrow{p} ab$$

$$\frac{X_n}{Y_n} \xrightarrow{p} \frac{a}{b} \qquad \text{if } b \neq 0$$

Example

If $Q_n \xrightarrow{p} 2$ and $Z_n \xrightarrow{p} 3$ then

$$Q_n + Z_n \xrightarrow{p} 5$$

$$Q_n Z_n \xrightarrow{p} 6$$

$$\frac{Q_n}{Z_n} \xrightarrow{p} \frac{2}{3}$$

The Law of Large Numbers

- If $X_1, X_2, X_3, \ldots$ is an i.i.d. sequence of random variables and

$$E\left\{X_i\right\} = \mu_X$$

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_n$$

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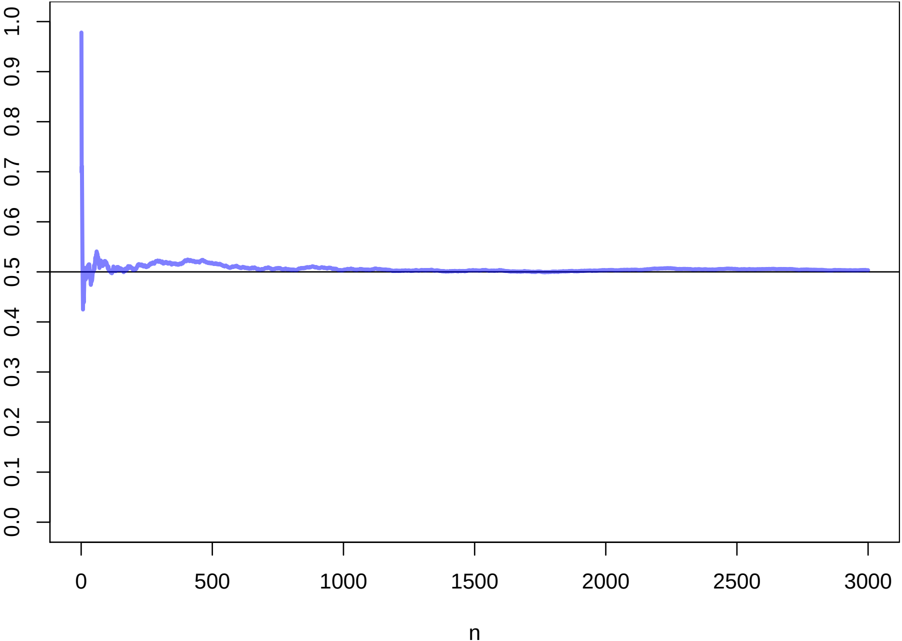
- An estimator is said to be **consistent** if it converges in probability to the quantity being estimated.

- The sample mean $\overline{X}$ is a consistent estimator of the population mean μ_X.

Realizations of $\overline{X}_n$ for $X_i \sim i.i.d.\ U[0,1]$



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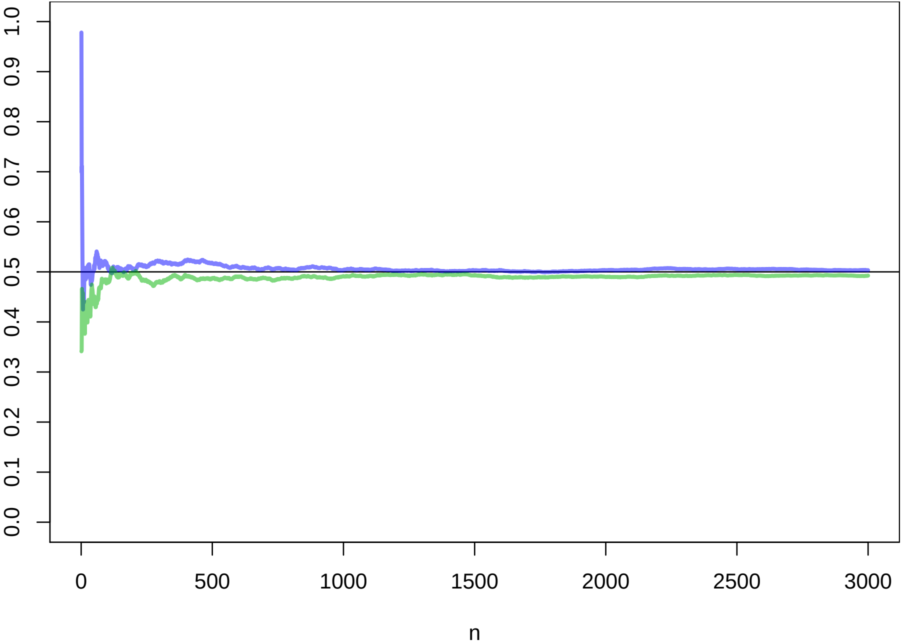

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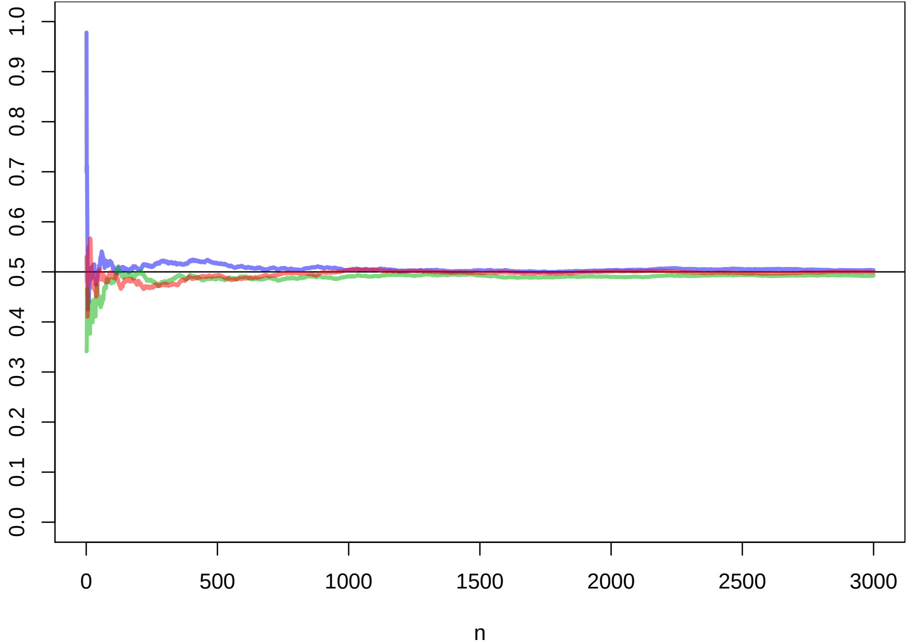

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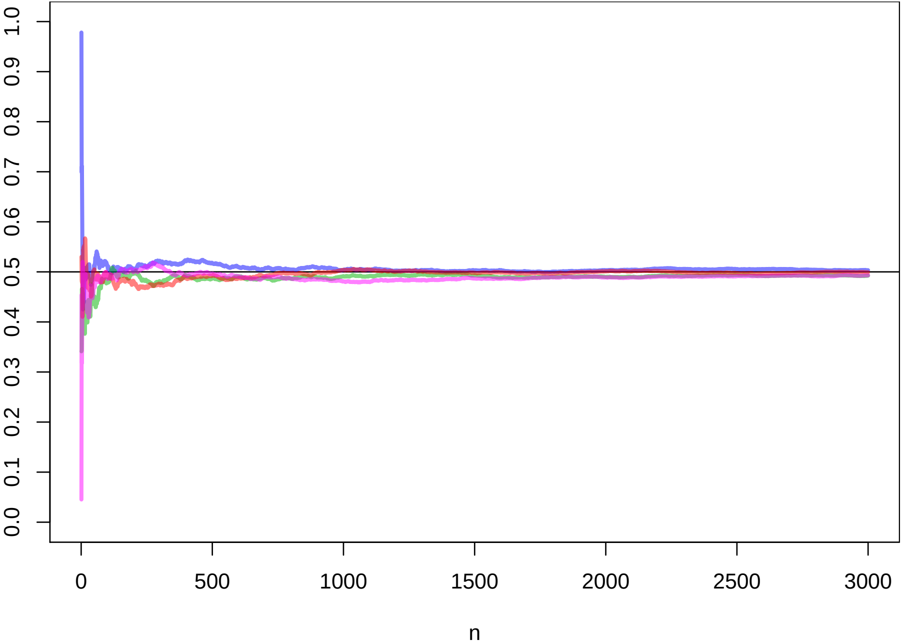

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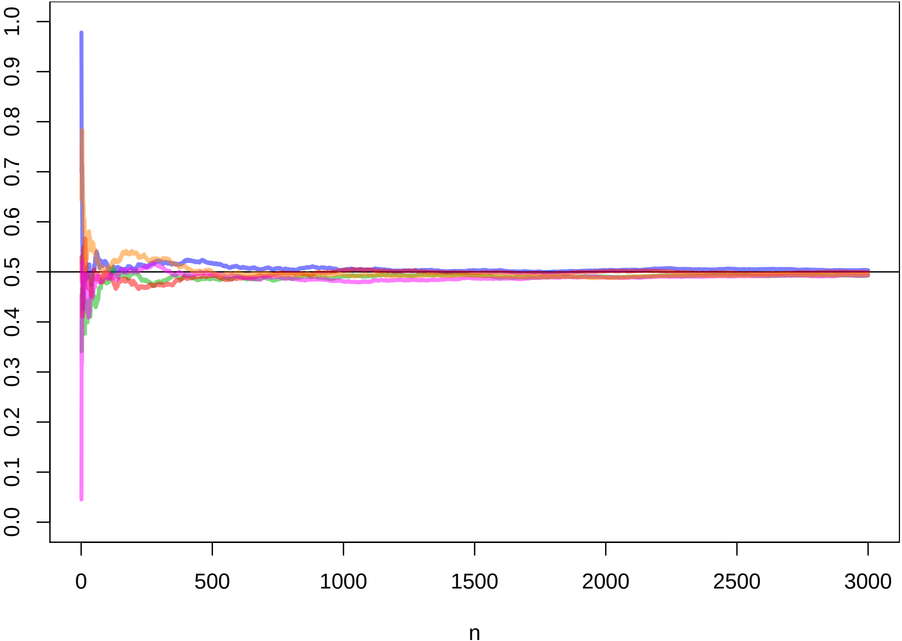

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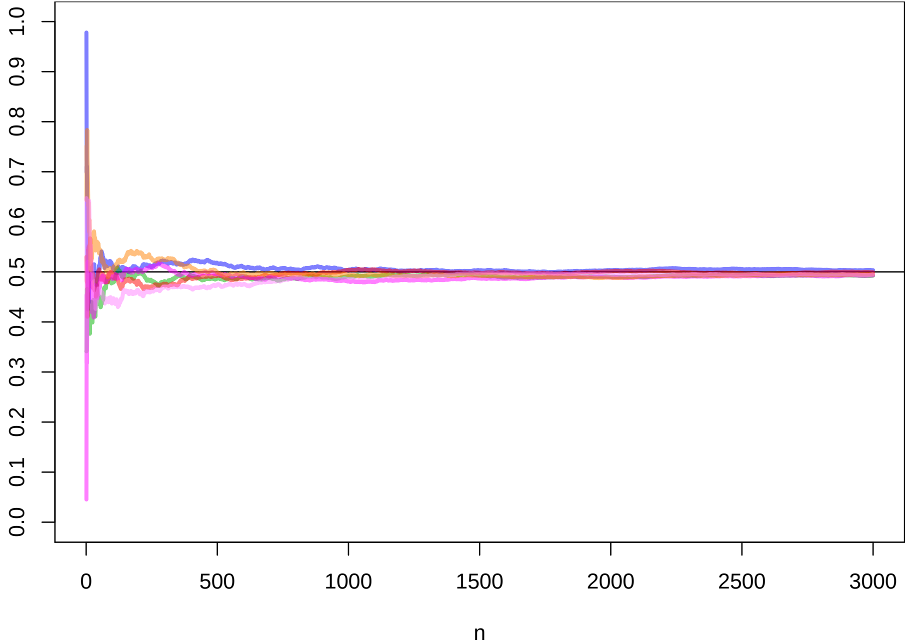

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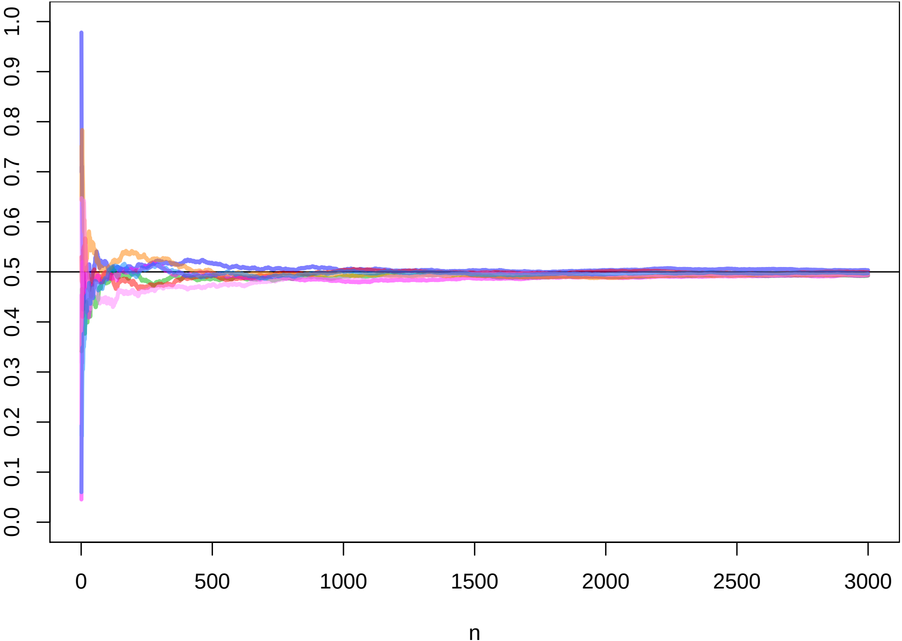

Question

Suppose $Y_1, Y_2, \ldots \sim i.i.d.\ U\{2, 7, 9\}$ and $\overline{Y} \xrightarrow{p} c$.

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Answer:

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Suppose $Y_1, Y_2, \ldots \sim i.i.d.\ U\{2, 7, 9\}$ and $\overline{Y} \xrightarrow{p} c$.

What is c?

Answer: 6

$$\overline{Y} \quad \xrightarrow{p} \quad E\{Y_i\} = \frac{2+7+9}{3} = \frac{18}{3} = 6$$

Question

$Z_1, Z_2, \ldots \sim i.i.d.\ N(5, 3)$

$W_1, W_2, \ldots \sim i.i.d.\ U[4, 10]$

$\overline{Z}\ \overline{W} \xrightarrow{p} k$

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What is k?

Answer: 35

$$\begin{aligned} \overline{Z} &\xrightarrow{p} E\{Z_i\} = 5 \\ \overline{W} &\xrightarrow{p} E\{W_i\} = 7 \\ \overline{Z}\ \overline{W} &\xrightarrow{p} (5)(7) = 35 \end{aligned}$$

Consistency of the Sample Variance

- If $X_1, X_2, X_3, \ldots$ is an i.i.d. sequence of random variables and

$$\begin{aligned}\mathrm{Var}\left(X_i\right) &= E\left\{\left(X_i - \mu_X\right)^2\right\} = \sigma_X^2 \\ S_X^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\end{aligned}$$

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\end{aligned}$$

- The sample variance S_X^2 is a consistent estimator of the population variance σ_X^2.

Consistency of the Sample Standard Deviation

- If $X_1, X_2, X_3, \ldots$ is an i.i.d. sequence of random variables and

$$\begin{aligned} \mathrm{StdDev}\,(X_i) &= \sqrt{\sigma_X^2} = \sigma_X \\ S_X &= \sqrt{S_X^2} \end{aligned}$$

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- The sample standard deviation S_X is a consistent estimator of the population standard deviation σ_X.

Consistency of the Sample Covariance

- If $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are i.i.d. random sequences and

$$\begin{aligned}
\mathrm{Cov}\left(X_i, Y_i\right) &= \sigma_{XY} = E\left\{\left(X_i - \mu_X\right)\left(Y_i - \mu_Y\right)\right\} \\
S_{XY} &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)\left(Y_i - \overline{X}\right)
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- The sample covariance S_{XY} is a consistent estimator of the population covariance σ_{XY}.

Convergence in Distribution

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Convergence in Distribution

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- Then we say X_n **converges in distribution** to Y if

$$\lim_{n\to\infty} F_n(c) \to F_Y(c) \qquad \text{for all } c \in \mathbb{R}$$

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  and we write $X_n \xrightarrow{d} Y$.
- Convergence in distribution is a weaker concept than convergence in probability

  $$\text{If } X_n \xrightarrow{p} Y \text{ then } X_n \xrightarrow{d} Y$$

Convergence in Distribution

If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$ then

$$X_n + Y_n \xrightarrow{d} X + c$$

$$X_n Y_n \xrightarrow{d} Xc$$

$$\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c} \qquad \text{if } c \neq 0$$

Example

If $Q_n \xrightarrow{d} U[0,1]$ and $Z_n \xrightarrow{p} 2$ then

$$Q_n + Z_n \xrightarrow{d} U[2,3]$$

$$Q_n Z_n \xrightarrow{d} U[0,2]$$

$$\frac{Q_n}{Z_n} \xrightarrow{p} U[0,0.5]$$

The Central Limit Theorem

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$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Then $\sqrt{n}\left(\overline{X}_n - \mu\right) \stackrel{d}{\rightarrow} N\left(0, \sigma^2\right)$

The Central Limit Theorem

Example

If $V_1, V_2, \ldots \sim i.i.d.\ U\{1, 2, 3, 4\}$ then

$$E\{V_i\} = \frac{1+2+3+4}{4} = \frac{10}{4} = 2.5$$

The Central Limit Theorem

Example

If $V_1, V_2, \ldots \sim i.i.d.\ U\{1,2,3,4\}$ then

$$
\begin{aligned}
E\{V_i\} &= \frac{1+2+3+4}{4} = \frac{10}{4} = 2.5 \\
E\left\{V_i^2\right\} &= \frac{1+4+9+16}{4} = \frac{30}{4} = 7.5
\end{aligned}
$$

The Central Limit Theorem

Example

If $V_1, V_2, \ldots \sim i.i.d.\ U\left\{1,2,3,4\right\}$ then

$$\begin{aligned}
E\left\{V_i\right\} &= \frac{1+2+3+4}{4} = \frac{10}{4} = 2.5 \\
E\left\{V_i^2\right\} &= \frac{1+4+9+16}{4} = \frac{30}{4} = 7.5 \\
\mathrm{Var}\left(V_i\right) &= E\left\{V_i^2\right\} - E\left\{V_i\right\}^2 = 7.5 - 6.25 = 1.25
\end{aligned}$$

The Central Limit Theorem

Example

If $V_1, V_2, \ldots \sim i.i.d.\ U\{1,2,3,4\}$ then

$$\begin{aligned}
E\{V_i\} &= \frac{1+2+3+4}{4} = \frac{10}{4} = 2.5 \\
E\left\{V_i^2\right\} &= \frac{1+4+9+16}{4} = \frac{30}{4} = 7.5 \\
\mathrm{Var}(V_i) &= E\left\{V_i^2\right\} - E\{V_i\}^2 = 7.5 - 6.25 = 1.25 \\
\sqrt{n}\left(\overline{V}_n - 2.5\right) &\xrightarrow{d} N(0, 1.25)
\end{aligned}$$

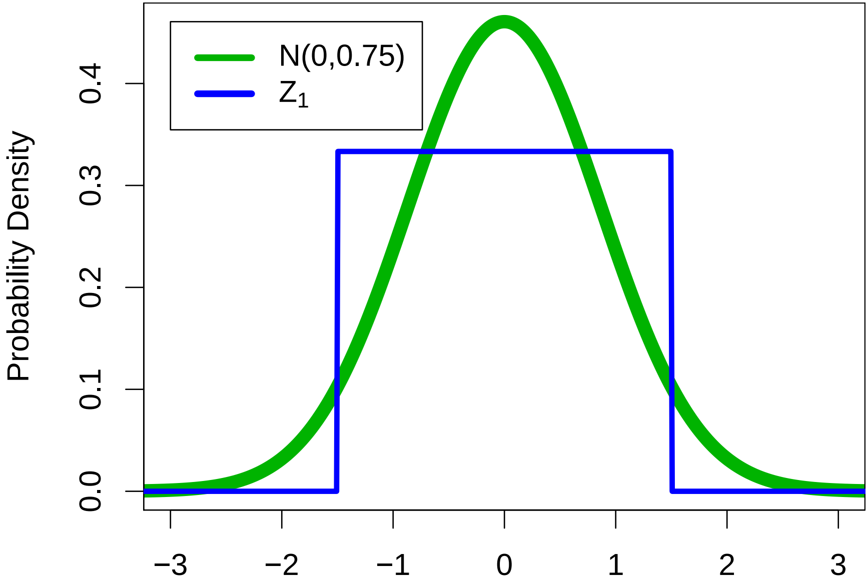
The Central Limit Theorem

Example

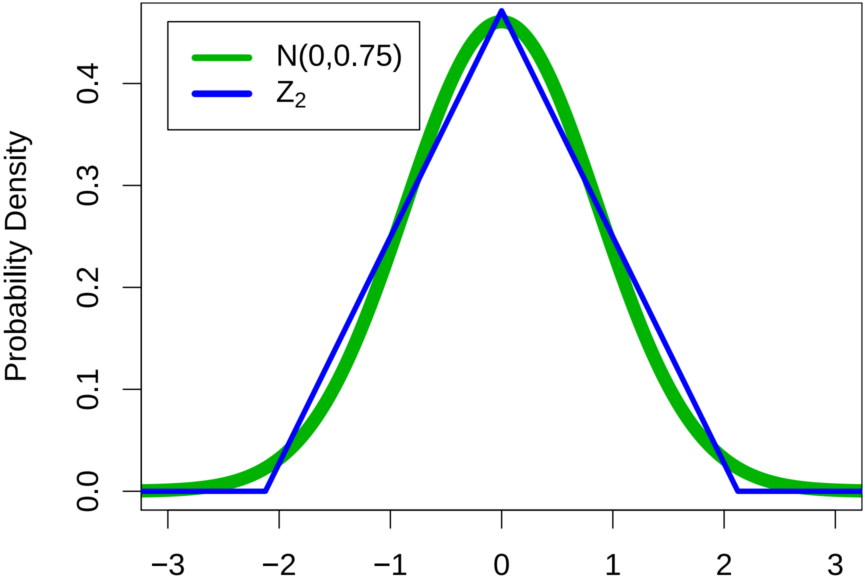
If $V_1, V_2, \ldots \sim i.i.d.\ U\{1,2,3,4\}$ then

$$
\begin{aligned}
E\{V_i\} &= \frac{1+2+3+4}{4} = \frac{10}{4} = 2.5 \\
E\left\{V_i^2\right\} &= \frac{1+4+9+16}{4} = \frac{30}{4} = 7.5 \\
\mathrm{Var}(V_i) &= E\left\{V_i^2\right\} - E\{V_i\}^2 = 7.5 - 6.25 = 1.25 \\
\sqrt{n}\left(\overline{V}_n - 2.5\right) &\xrightarrow{d} N(0, 1.25) \\
\frac{\sqrt{n}\left(\overline{V}_n - 2.5\right)}{\sqrt{1.25}} &\xrightarrow{d} N(0, 1)
\end{aligned}
$$

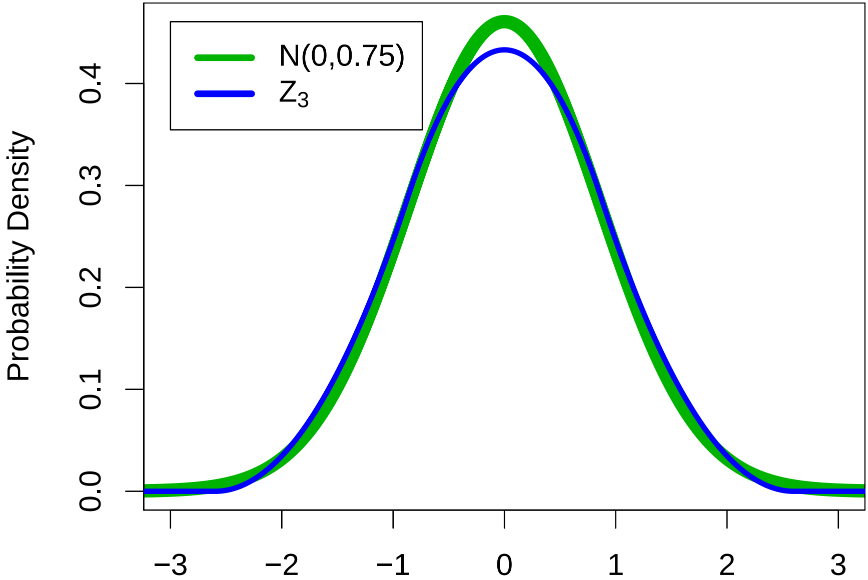
Probability Density of $Z_1 = \sqrt{1}\left(\overline{Y}_1 - 1.5\right)$



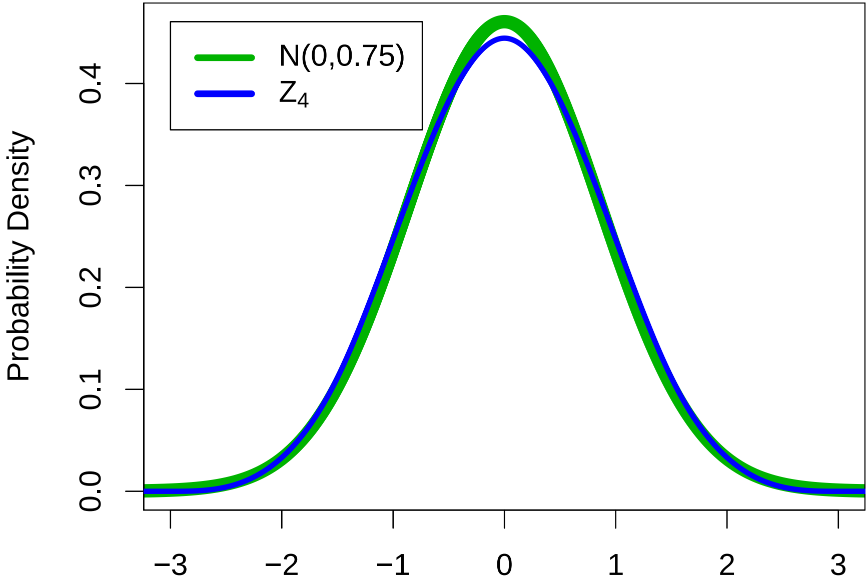
Probability Density of $Z_2 = \sqrt{2}\left(\overline{Y}_2 - 1.5\right)$



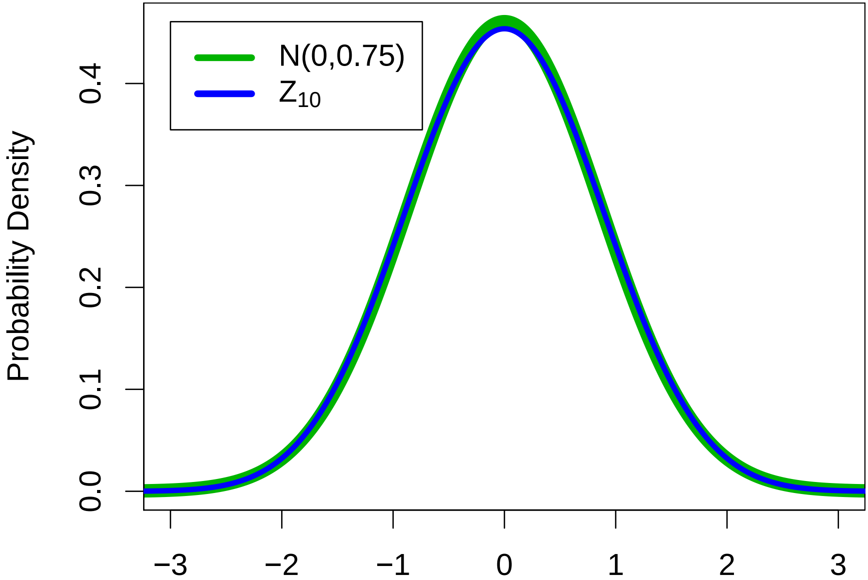
Probability Density of $Z_3 = \sqrt{3}\left(\overline{Y}_3 - 1.5\right)$



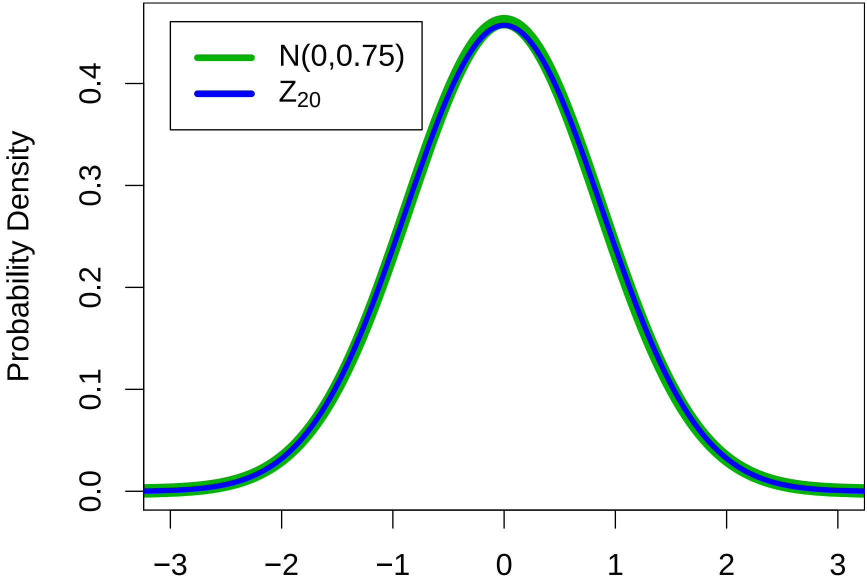
Probability Density of $Z_4 = \sqrt{4}\left(\overline{Y}_4 - 1.5\right)$



Probability Density of $Z_{10} = \sqrt{10}\left(\overline{Y}_{10} - 1.5\right)$



Probability Density of $Z_{20} = \sqrt{20}\left(\overline{Y}_{20} - 1.5\right)$



Sampling Distributions

Daniel Stephenson

SCMA 524: Statistical Fundamentals

Sampling Distributions

- A **sampling distribution** is the probability distribution of a sample statistic.

Example

Suppose $X_1, X_2, X_3, X_4 \sim i.i.d.\ N(2, 5)$.

Since X_1, X_2, X_3, X_4 are jointly normal, $\overline{X}$ is normally distributed.

$$\begin{aligned}
\overline{X} &= \frac{1}{4}(X_1 + X_2 + X_3 + X_4) \\
E\left\{\overline{X}\right\} &= \frac{1}{4}(2 + 2 + 2 + 2) = 2 \\
\mathrm{Var}\left(\overline{X}\right) &= \frac{1}{16}(5 + 5 + 5 + 5) = \frac{20}{16} = 1.25
\end{aligned}$$

The sampling distribution of this sample mean is $\overline{X} \sim N(2, 1.25)$.

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\mathrm{Cov}\left(X_i, X_i\right) = \mathrm{Var}\left(X_i\right) = \sigma^2$$

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$
\begin{aligned}
\operatorname{Cov}\left(X_i, X_i\right) &= \operatorname{Var}\left(X_i\right) = \sigma^2 \\
\operatorname{Cov}\left(X_i, X_j\right) &= 0 \qquad \text{if } i \neq j
\end{aligned}
$$

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned} \mathrm{Cov}\left(X_i, X_i\right) &= \mathrm{Var}\left(X_i\right) = \sigma^2 \\ \mathrm{Cov}\left(X_i, X_j\right) &= 0 \qquad \text{if } i \neq j \end{aligned}$$

$$\mathrm{Cov}\left(X_1, \overline{X}\right) = \mathrm{Cov}\left(X_1, \frac{1}{3}\left[X_1 + X_2 + X_3\right]\right)$$

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$
\begin{aligned}
\operatorname{Cov}\left(X_i, X_i\right) &= \operatorname{Var}\left(X_i\right) = \sigma^2 \\
\operatorname{Cov}\left(X_i, X_j\right) &= 0 \qquad \text{if } i \neq j
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Cov}\left(X_1, \overline{X}\right) &= \operatorname{Cov}\left(X_1, \frac{1}{3}\left[X_1 + X_2 + X_3\right]\right) \\
&= \frac{1}{3}\operatorname{Cov}\left(X_1,\ X_1 + X_2 + X_3\right)
\end{aligned}
$$

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned}
\operatorname{Cov}\left(X_i, X_i\right) &= \operatorname{Var}\left(X_i\right) = \sigma^2 \\
\operatorname{Cov}\left(X_i, X_j\right) &= 0 \qquad \text{if } i \neq j
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\operatorname{Cov}\left(X_1, \overline{X}\right) &= \operatorname{Cov}\left(X_1, \frac{1}{3}\left[X_1 + X_2 + X_3\right]\right) \\
&= \frac{1}{3}\operatorname{Cov}\left(X_1,\ X_1 + X_2 + X_3\right) \\
&= \frac{1}{3}\Big[\operatorname{Cov}\left(X_1, X_1\right) + \operatorname{Cov}\left(X_1, X_2\right) + \operatorname{Cov}\left(X_1, X_3\right)\Big]
\end{aligned}$$

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned}
\text{Cov}\left(X_i, X_i\right) &= \text{Var}\left(X_i\right) = \sigma^2 \\
\text{Cov}\left(X_i, X_j\right) &= 0 \qquad \text{if } i \neq j
\end{aligned}$$

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\text{Cov}\left(X_1, \overline{X}\right) &= \text{Cov}\left(X_1, \frac{1}{3}\left[X_1 + X_2 + X_3\right]\right) \\
&= \frac{1}{3}\text{Cov}\left(X_1,\ X_1 + X_2 + X_3\right) \\
&= \frac{1}{3}\Big[\text{Cov}\left(X_1, X_1\right) + \text{Cov}\left(X_1, X_2\right) + \text{Cov}\left(X_1, X_3\right)\Big] \\
&= \frac{1}{3}\left[\sigma^2 + 0 + 0\right] = \frac{1}{3}\sigma^2
\end{aligned}$$

Sampling Distributions

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\operatorname{Cov}\left(X_i, \overline{X}\right) = \frac{1}{3}\sigma^2$$

Sampling Distributions

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$
\begin{aligned}
\text{Cov}\left(X_i, \overline{X}\right) &= \frac{1}{3}\sigma^2 \\
\text{Cov}\left(\overline{X}, \overline{X}\right) &= \text{Cov}\left(\frac{1}{3}\left[X_1 + X_2 + X_3\right], \overline{X}\right)
\end{aligned}
$$

Sampling Distributions

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\text{Cov}\left(X_i, \overline{X}\right) = \frac{1}{3}\sigma^2$$

$$\begin{aligned}
\text{Cov}\left(\overline{X}, \overline{X}\right) &= \text{Cov}\left(\frac{1}{3}\Big[X_1 + X_2 + X_3\Big], \overline{X}\right) \\
&= \frac{1}{3}\left[\text{Cov}\left(X_1, \overline{X}\right) + \text{Cov}\left(X_2, \overline{X}\right) + \text{Cov}\left(X_3, \overline{X}\right)\right]
\end{aligned}$$

Sampling Distributions

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$
\begin{aligned}
\text{Cov}\left(X_i, \overline{X}\right) &= \frac{1}{3}\sigma^2 \\
\text{Cov}\left(\overline{X}, \overline{X}\right) &= \text{Cov}\left(\frac{1}{3}\left[X_1 + X_2 + X_3\right], \overline{X}\right) \\
&= \frac{1}{3}\left[\text{Cov}\left(X_1, \overline{X}\right) + \text{Cov}\left(X_2, \overline{X}\right) + \text{Cov}\left(X_3, \overline{X}\right)\right] \\
&= \frac{1}{3}\left[\frac{1}{3}\sigma^2 + \frac{1}{3}\sigma^2 + \frac{1}{3}\sigma^2\right]
\end{aligned}
$$

Sampling Distributions

Suppose $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\begin{aligned}
\text{Cov}\left(X_i, \overline{X}\right) &= \frac{1}{3}\sigma^2 \\
\text{Cov}\left(\overline{X}, \overline{X}\right) &= \text{Cov}\left(\frac{1}{3}\left[X_1 + X_2 + X_3\right], \overline{X}\right) \\
&= \frac{1}{3}\left[\text{Cov}\left(X_1, \overline{X}\right) + \text{Cov}\left(X_2, \overline{X}\right) + \text{Cov}\left(X_3, \overline{X}\right)\right] \\
&= \frac{1}{3}\left[\frac{1}{3}\sigma^2 + \frac{1}{3}\sigma^2 + \frac{1}{3}\sigma^2\right] \\
&= \frac{1}{3}\sigma^2 = \text{Cov}\left(X_i, \overline{X}\right)
\end{aligned}$$

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\text{Cov}\left(X_i, \overline{X}\right) = \text{Cov}\left(\overline{X}, \overline{X}\right) = \frac{1}{3}\sigma^2$$

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\begin{aligned}
\text{Cov}\left(X_i, \overline{X}\right) &= \text{Cov}\left(\overline{X}, \overline{X}\right) = \frac{1}{3}\sigma^2 \\
\text{Cov}\left(X_i - \overline{X}, \overline{X}\right) &= \text{Cov}\left(X_i, \overline{X}\right) - \text{Cov}\left(\overline{X}, \overline{X}\right)
\end{aligned}$$

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$
\begin{aligned}
\text{Cov}\left(X_i, \overline{X}\right) &= \text{Cov}\left(\overline{X}, \overline{X}\right) = \frac{1}{3}\sigma^2 \\
\text{Cov}\left(X_i - \overline{X}, \overline{X}\right) &= \text{Cov}\left(X_i, \overline{X}\right) - \text{Cov}\left(\overline{X}, \overline{X}\right) \\
&= \frac{1}{3}\sigma^2 - \frac{1}{3}\sigma^2 = 0
\end{aligned}
$$

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N(\mu, \sigma^2)$

$$\begin{aligned}
\text{Cov}\left(X_i, \overline{X}\right) &= \text{Cov}\left(\overline{X}, \overline{X}\right) = \frac{1}{3}\sigma^2 \\
\text{Cov}\left(X_i - \overline{X}, \overline{X}\right) &= \text{Cov}\left(X_i, \overline{X}\right) - \text{Cov}\left(\overline{X}, \overline{X}\right) \\
&= \frac{1}{3}\sigma^2 - \frac{1}{3}\sigma^2 = 0
\end{aligned}$$

$X_i - \overline{X}$ and $\overline{X}$ are jointly normal and $\text{Cov}\left(X_i - \overline{X}, \overline{X}\right) = 0$.

Sampling Distributions

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$$\begin{aligned}
\text{Cov}\left(X_i, \overline{X}\right) &= \text{Cov}\left(\overline{X}, \overline{X}\right) = \frac{1}{3}\sigma^2 \\
\text{Cov}\left(X_i - \overline{X}, \overline{X}\right) &= \text{Cov}\left(X_i, \overline{X}\right) - \text{Cov}\left(\overline{X}, \overline{X}\right) \\
&= \frac{1}{3}\sigma^2 - \frac{1}{3}\sigma^2 = 0
\end{aligned}$$

$X_i - \overline{X}$ and $\overline{X}$ are jointly normal and $\text{Cov}\left(X_i - \overline{X}, \overline{X}\right) = 0$.

So $X_i - \overline{X}$ and $\overline{X}$ are independent.

Independence of Sample Variance and Sample Mean

If $X_1, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

Then $X_i - \overline{X}$ and $\overline{X}$ are independent so

Independence of Sample Variance and Sample Mean

If $X_1, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

Then $X_i - \overline{X}$ and $\overline{X}$ are independent so

$$\left(X_i - \overline{X}\right)^2 \quad \text{and} \quad \overline{X} \text{ are independent}$$

Independence of Sample Variance and Sample Mean

If $X_1, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

Then $X_i - \overline{X}$ and $\overline{X}$ are independent so

$$\left(X_i - \overline{X}\right)^2 \quad \text{and} \quad \overline{X} \text{ are independent}$$

$$\frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 \quad \text{and} \quad \overline{X} \text{ are independent}$$

Independence of Sample Variance and Sample Mean

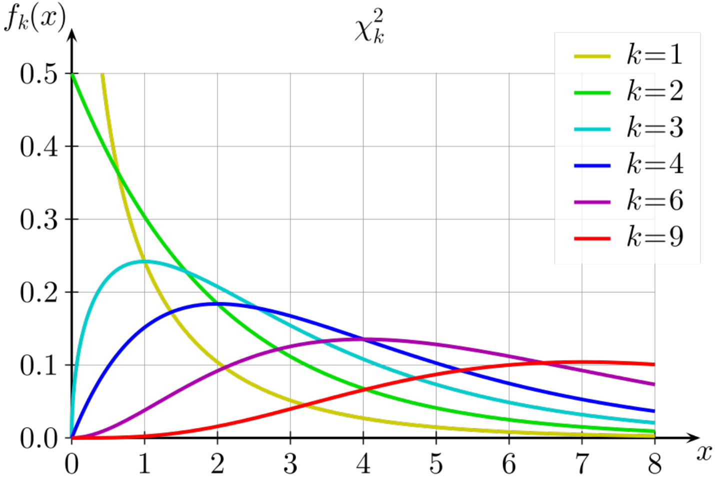
If $X_1, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

Then $X_i - \overline{X}$ and $\overline{X}$ are independent so

$$\begin{aligned} \left(X_i - \overline{X}\right)^2 \quad &\text{and} \quad \overline{X} \text{ are independent} \\ \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \quad &\text{and} \quad \overline{X} \text{ are independent} \\ S_X^2 \quad &\text{and} \quad \overline{X} \text{ are independent} \end{aligned}$$

The Chi-Squared Distribution

- If $X_1, \ldots, X_k \sim i.i.d.\, N(0,1)$ then $X_1^2 + \cdots + X_k^2 \sim \chi^2(k)$



The Chi-Squared Distribution

- If $X \sim \chi^2(k)$ then $E\{X\} = k$
- If $X \sim \chi^2(k)$ then $\mathrm{Var}(X) = 2k$
- If X and Y are independent, $X \sim \chi^2(k)$, and $Y \sim \chi^2(m)$

  then $X + Y \sim \chi^2(k+m)$
- If X and Y are independent, $X \sim \chi^2(k)$, and $X + Y \sim \chi^2(k+m)$

  then $Y \sim \chi^2(m)$

Example

If X and Y are independent, $X \sim \chi^2(2)$, and $X + Y \sim \chi^2(5)$

then $Y \sim \chi^2(3)$

Deviation from the Population Mean

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu_X, \sigma_X^2\right)$ then $\dfrac{X_i - \mu_X}{\sigma_X} \sim N\left(0, 1\right)$

$$\text{Let } W = \frac{1}{\sigma_X^2} \sum_{i=1}^{3} \left(X_i - \mu_X\right)$$

Deviation from the Population Mean

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu_X, \sigma_X^2\right)$ then $\dfrac{X_i - \mu_X}{\sigma_X} \sim N(0,1)$

$$\begin{aligned} \text{Let } W &= \frac{1}{\sigma_X^2} \sum_{i=1}^{3} (X_i - \mu_X) \\ W &= \frac{1}{\sigma_X^2} \left[(X_1 - \mu_X)^2 + (X_2 - \mu_X)^2 + (X_3 - \mu_X)^2 \right] \end{aligned}$$

Deviation from the Population Mean

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu_X, \sigma_X^2\right)$ then $\dfrac{X_i - \mu_X}{\sigma_X} \sim N(0,1)$

$$\begin{aligned}
\text{Let } W &= \frac{1}{\sigma_X^2} \sum_{i=1}^{3} (X_i - \mu_X) \\
W &= \frac{1}{\sigma_X^2} \left[(X_1 - \mu_X)^2 + (X_2 - \mu_X)^2 + (X_3 - \mu_X)^2 \right] \\
&= \frac{(X_1 - \mu_X)^2}{\sigma_X^2} + \frac{(X_2 - \mu_X)^2}{\sigma_X^2} + \frac{(X_3 - \mu_X)^2}{\sigma_X^2}
\end{aligned}$$

Deviation from the Population Mean

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu_X, \sigma_X^2\right)$ then $\dfrac{X_i - \mu_X}{\sigma_X} \sim N(0,1)$

$$
\begin{aligned}
\text{Let } W &= \frac{1}{\sigma_X^2} \sum_{i=1}^{3} (X_i - \mu_X) \\
W &= \frac{1}{\sigma_X^2} \left[(X_1 - \mu_X)^2 + (X_2 - \mu_X)^2 + (X_3 - \mu_X)^2 \right] \\
&= \frac{(X_1 - \mu_X)^2}{\sigma_X^2} + \frac{(X_2 - \mu_X)^2}{\sigma_X^2} + \frac{(X_3 - \mu_X)^2}{\sigma_X^2} \\
&= \left(\frac{X_1 - \mu_X}{\sigma_X} \right)^2 + \left(\frac{X_2 - \mu_X}{\sigma_X} \right)^2 + \left(\frac{X_3 - \mu_X}{\sigma_X} \right)^2
\end{aligned}
$$

Deviation from the Population Mean

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu_X, \sigma_X^2\right)$ then $\dfrac{X_i - \mu_X}{\sigma_X} \sim N\left(0, 1\right)$

$$\begin{aligned}
\text{Let } W &= \frac{1}{\sigma_X^2} \sum_{i=1}^{3} \left(X_i - \mu_X\right) \\
W &= \frac{1}{\sigma_X^2} \left[\left(X_1 - \mu_X\right)^2 + \left(X_2 - \mu_X\right)^2 + \left(X_3 - \mu_X\right)^2\right] \\
&= \frac{\left(X_1 - \mu_X\right)^2}{\sigma_X^2} + \frac{\left(X_2 - \mu_X\right)^2}{\sigma_X^2} + \frac{\left(X_3 - \mu_X\right)^2}{\sigma_X^2} \\
&= \left(\frac{X_1 - \mu_X}{\sigma_X}\right)^2 + \left(\frac{X_2 - \mu_X}{\sigma_X}\right)^2 + \left(\frac{X_3 - \mu_X}{\sigma_X}\right)^2 \\
W &\sim \chi^2\left(3\right)
\end{aligned}$$

Deviation from the Population Mean

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu_X, \sigma_X^2\right)$ then $\dfrac{X_i - \mu_X}{\sigma_X} \sim N(0,1)$

$$
\begin{aligned}
\text{Let } W &= \frac{1}{\sigma_X^2} \sum_{i=1}^{3} (X_i - \mu_X) \\
W &= \frac{1}{\sigma_X^2}\left[(X_1 - \mu_X)^2 + (X_2 - \mu_X)^2 + (X_3 - \mu_X)^2\right] \\
&= \frac{(X_1 - \mu_X)^2}{\sigma_X^2} + \frac{(X_2 - \mu_X)^2}{\sigma_X^2} + \frac{(X_3 - \mu_X)^2}{\sigma_X^2} \\
&= \left(\frac{X_1 - \mu_X}{\sigma_X}\right)^2 + \left(\frac{X_2 - \mu_X}{\sigma_X}\right)^2 + \left(\frac{X_3 - \mu_X}{\sigma_X}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
W &\sim \chi^2(3) \\
E\{W\} &= 3
\end{aligned}
$$

Deviation from the Population Mean

If $X_1, X_2, X_3 \sim i.i.d.\ N\left(\mu_X, \sigma_X^2\right)$ then $\dfrac{X_i - \mu_X}{\sigma_X} \sim N\left(0, 1\right)$

$$
\begin{aligned}
\text{Let } W &= \frac{1}{\sigma_X^2} \sum_{i=1}^{3} \left(X_i - \mu_X\right) \\
W &= \frac{1}{\sigma_X^2} \left[\left(X_1 - \mu_X\right)^2 + \left(X_2 - \mu_X\right)^2 + \left(X_3 - \mu_X\right)^2\right] \\
&= \frac{\left(X_1 - \mu_X\right)^2}{\sigma_X^2} + \frac{\left(X_2 - \mu_X\right)^2}{\sigma_X^2} + \frac{\left(X_3 - \mu_X\right)^2}{\sigma_X^2} \\
&= \left(\frac{X_1 - \mu_X}{\sigma_X}\right)^2 + \left(\frac{X_2 - \mu_X}{\sigma_X}\right)^2 + \left(\frac{X_3 - \mu_X}{\sigma_X}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
W &\sim \chi^2\left(3\right) \\
E\left\{W\right\} &= 3 \\
\text{Var}\left(W\right) &= 6
\end{aligned}
$$

Deviation from the Sample Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \overline{X}$$

Deviation from the Sample Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned}\sum_{i=1}^{n}\left(X_i - \overline{X}\right) &= \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \overline{X} \\ &= \sum_{i=1}^{n} X_i - n\overline{X}\end{aligned}$$

Deviation from the Sample Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned}
\sum_{i=1}^{n}\left(X_i - \overline{X}\right) &= \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \overline{X} \\
&= \sum_{i=1}^{n} X_i - n\overline{X} \\
&= \sum_{i=1}^{n} X_i - n\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)
\end{aligned}$$

Deviation from the Sample Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned}
\sum_{i=1}^{n}\left(X_i - \overline{X}\right) &= \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \overline{X} \\
&= \sum_{i=1}^{n} X_i - n\overline{X} \\
&= \sum_{i=1}^{n} X_i - n\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} X_i = 0
\end{aligned}$$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n} \left(X_i - \overline{X}\right) = 0$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = 0$

Then $Z_i = \frac{X_i - \mu}{\sigma} \sim N\left(0, 1\right)$ and $V_i = Z_i^2 \sim \chi^2\left(1\right)$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\displaystyle\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = 0$

Then $Z_i = \dfrac{X_i - \mu}{\sigma} \sim N\left(0, 1\right)$ and $V_i = Z_i^2 \sim \chi^2\left(1\right)$

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 = \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2$$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = 0$

Then $Z_i = \dfrac{X_i - \mu}{\sigma} \sim N(0,1)$ and $V_i = Z_i^2 \sim \chi^2(1)$

$$\begin{aligned}
\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 &= \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 \\
&= \sum_{i=1}^{n} Z_i^2
\end{aligned}$$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = 0$

Then $Z_i = \dfrac{X_i - \mu}{\sigma} \sim N\left(0,1\right)$ and $V_i = Z_i^2 \sim \chi^2\left(1\right)$

$$\begin{aligned}
\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 &= \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 \\
&= \sum_{i=1}^{n} Z_i^2 \\
&= \sum_{i=1}^{n} V_i
\end{aligned}$$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = 0$

Then $Z_i = \frac{X_i - \mu}{\sigma} \sim N(0,1)$ and $V_i = Z_i^2 \sim \chi^2(1)$

$$\begin{aligned} \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 &= \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 \\ &= \sum_{i=1}^{n} Z_i^2 \\ &= \sum_{i=1}^{n} V_i \sim \chi^2(n) \end{aligned}$$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n} \left(X_i - \overline{X}\right) = 0$ and

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} \left(X_i - \mu\right)^2 \sim \chi^2\left(n\right)$$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = 0$ and

$$
\begin{aligned}
& \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 \sim \chi^2\left(n\right) \\
= \quad & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[X_i - \overline{X} + \overline{X} - \mu\right]^2
\end{aligned}
$$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = 0$ and

$$
\begin{aligned}
& \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right)^2 \sim \chi^2\left(n\right) \\
= & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[X_i-\overline{X}+\overline{X}-\mu\right]^2 \\
= & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[\left(X_i-\overline{X}\right)^2+2\left(X_i-\overline{X}\right)\left(\overline{X}-\mu\right)+\left(\overline{X}-\mu\right)^2\right]
\end{aligned}
$$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = 0$ and

$$\begin{aligned} & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right)^2 \sim \chi^2\left(n\right) \\ = \;& \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[\left(X_i-\overline{X}\right)^2 + 2\left(X_i-\overline{X}\right)\left(\overline{X}-\mu\right) + \left(\overline{X}-\mu\right)^2\right] \end{aligned}$$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = 0$ and

$$
\begin{aligned}
& \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 \sim \chi^2\left(n\right) \\
= & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[\left(X_i - \overline{X}\right)^2 + 2\left(X_i - \overline{X}\right)\left(\overline{X} - \mu\right) + \left(\overline{X} - \mu\right)^2\right] \\
= & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 + \frac{2}{\sigma^2}\left(\overline{X} - \mu\right)\sum_{i=1}^{n}\left(X_i - \overline{X}\right) + \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\overline{X} - \mu\right)^2
\end{aligned}
$$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = 0$ and

$$
\begin{aligned}
& \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 \sim \chi^2\left(n\right) \\
= \; & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[\left(X_i - \overline{X}\right)^2 + 2\left(X_i - \overline{X}\right)\left(\overline{X} - \mu\right) + \left(\overline{X} - \mu\right)^2\right] \\
= \; & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 + \frac{2}{\sigma^2}\left(\overline{X} - \mu\right)\sum_{i=1}^{n}\left(X_i - \overline{X}\right) + \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\overline{X} - \mu\right)^2 \\
= \; & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 + \frac{2}{\sigma^2}\left(\overline{X} - \mu\right)(0) + \frac{n}{\sigma^2}\left(\overline{X} - \mu\right)^2
\end{aligned}
$$

Squared Deviation from the Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then $\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = 0$ and

$$\begin{aligned}
& \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right)^2 \sim \chi^2\left(n\right) \\
= \; & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[\left(X_i-\overline{X}\right)^2 + 2\left(X_i-\overline{X}\right)\left(\overline{X}-\mu\right) + \left(\overline{X}-\mu\right)^2\right] \\
= \; & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2 + \frac{2}{\sigma^2}\left(\overline{X}-\mu\right)\sum_{i=1}^{n}\left(X_i-\overline{X}\right) + \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\overline{X}-\mu\right)^2 \\
= \; & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2 + \frac{2}{\sigma^2}\left(\overline{X}-\mu\right)(0) + \frac{n}{\sigma^2}\left(\overline{X}-\mu\right)^2 \\
= \; & \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2 + \frac{n}{\sigma^2}\left(\overline{X}-\mu\right)^2
\end{aligned}$$

Squared Deviation of Sample Mean from Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned}
\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right)^2 &\sim \chi^2\left(n\right) \\
\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right)^2 &= \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2+\frac{n}{\sigma^2}\left(\overline{X}-\mu\right)^2
\end{aligned}$$

Squared Deviation of Sample Mean from Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2 \quad \sim \quad \chi^2(n)$$

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2 \quad = \quad \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 + \frac{n}{\sigma^2}\left(\overline{X} - \mu\right)^2$$

$$\sqrt{\frac{n}{\sigma^2}\left(\overline{X} - \mu\right)^2} = \sqrt{\frac{n}{\sigma^2}}\left(\overline{X} - \mu\right)$$

Squared Deviation of Sample Mean from Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned}
\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 &\sim \chi^2(n) \\
\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 &= \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2+\frac{n}{\sigma^2}\left(\overline{X}-\mu\right)^2 \\
\sqrt{\frac{n}{\sigma^2}\left(\overline{X}-\mu\right)^2}=\sqrt{\frac{n}{\sigma^2}}\left(\overline{X}-\mu\right) &= \frac{\overline{X}-\mu}{\sqrt{\frac{\sigma^2}{n}}}
\end{aligned}$$

Squared Deviation of Sample Mean from Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$
\begin{aligned}
\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 &\sim \chi^2(n) \\
\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 &= \frac{1}{\sigma^2} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 + \frac{n}{\sigma^2} \left(\overline{X} - \mu\right)^2 \\
\sqrt{\frac{n}{\sigma^2} \left(\overline{X} - \mu\right)^2} = \sqrt{\frac{n}{\sigma^2}} \left(\overline{X} - \mu\right) &= \frac{\overline{X} - \mu}{\sqrt{\frac{\sigma^2}{n}}} = \frac{\overline{X} - \mu}{\mathrm{SD}\left(\overline{X}\right)} \sim N(0, 1)
\end{aligned}
$$

Squared Deviation of Sample Mean from Population Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\begin{aligned}
\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 &\sim \chi^2(n) \\
\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 &= \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2+\frac{n}{\sigma^2}\left(\overline{X}-\mu\right)^2 \\
\sqrt{\frac{n}{\sigma^2}\left(\overline{X}-\mu\right)^2}=\sqrt{\frac{n}{\sigma^2}}\left(\overline{X}-\mu\right) &= \frac{\overline{X}-\mu}{\sqrt{\frac{\sigma^2}{n}}}=\frac{\overline{X}-\mu}{\mathrm{SD}\left(\overline{X}\right)}\sim N(0,1) \\
\frac{n}{\sigma^2}\left(\overline{X}-\mu\right)^2 &\sim \chi^2(1)
\end{aligned}$$

Squared Deviation from the Sample Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 \sim \chi^2\left(n\right) \quad \text{and} \quad \frac{n}{\sigma^2}\left(\overline{X} - \mu\right)^2 \sim \chi^2\left(1\right)$$

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 + \frac{n}{\sigma^2}\left(\overline{X} - \mu\right)^2$$

Squared Deviation from the Sample Mean

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right)^2 \sim \chi^2\left(n\right) \quad \text{and} \quad \frac{n}{\sigma^2}\left(\overline{X}-\mu\right)^2 \sim \chi^2\left(1\right)$$

$$\begin{aligned}
\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right)^2 &= \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2+\frac{n}{\sigma^2}\left(\overline{X}-\mu\right)^2 \\
\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2 &\sim \chi^2\left(n-1\right)
\end{aligned}$$

Variance of the Sample Variance

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then

$$S_X^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$$

Variance of the Sample Variance

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then

$$\begin{aligned} S_X^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \\ \frac{(n-1)\,S_X^2}{\sigma_X^2} &= \frac{1}{\sigma_X^2}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \sim \chi^2\,(n-1) \end{aligned}$$

Variance of the Sample Variance

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then

$$
\begin{aligned}
S_X^2 &= \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 \\
\frac{(n-1)\, S_X^2}{\sigma_X^2} &= \frac{1}{\sigma_X^2} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 \sim \chi^2\,(n-1) \\
\mathrm{Var}\left(\frac{(n-1)\, S_X^2}{\sigma_X^2}\right) &= 2\,(n-1)
\end{aligned}
$$

Variance of the Sample Variance

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then

$$
\begin{aligned}
S_X^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \\
\frac{(n-1)\,S_X^2}{\sigma_X^2} &= \frac{1}{\sigma_X^2}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \sim \chi^2\,(n-1) \\
\mathrm{Var}\left(\frac{(n-1)\,S_X^2}{\sigma_X^2}\right) &= 2\,(n-1) \\
\frac{(n-1)^2}{\sigma_X^4}\mathrm{Var}\left(S_X^2\right) &= 2\,(n-1)
\end{aligned}
$$

Variance of the Sample Variance

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then

$$\begin{aligned}
S_X^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \\
\frac{(n-1)\,S_X^2}{\sigma_X^2} &= \frac{1}{\sigma_X^2}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \sim \chi^2\,(n-1) \\
\mathrm{Var}\left(\frac{(n-1)\,S_X^2}{\sigma_X^2}\right) &= 2\,(n-1) \\
\frac{(n-1)^2}{\sigma_X^4}\mathrm{Var}\left(S_X^2\right) &= 2\,(n-1) \\
\mathrm{Var}\left(S_X^2\right) &= \frac{2\sigma_X^4}{n-1}
\end{aligned}$$

The t-Distribution

- If Z and V are independent with $Z \sim N(0,1)$ and $V \sim \chi^2(k)$ then

$$T = \frac{Z}{\sqrt{V/k}} \sim t(k)$$

T is said to have a t-distribution with n degrees of freedom.

The t-Distribution

- If Z and V are independent with $Z \sim N(0,1)$ and $V \sim \chi^2(k)$ then

$$T = \frac{Z}{\sqrt{V/k}} \sim t(k)$$

  T is said to have a t-distribution with n degrees of freedom.

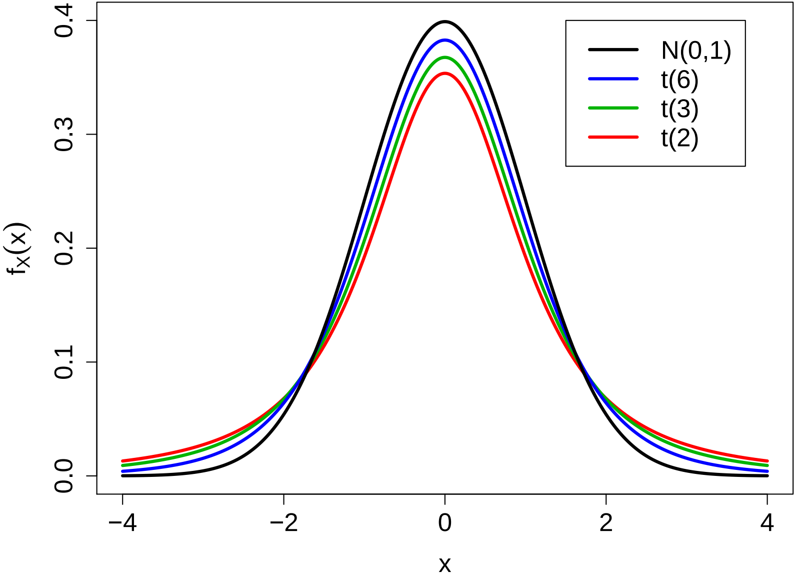
- If $T \sim t(k)$ and $k > 1$ then

$$E\{T\} = 0$$

- If $T \sim t(k)$ and $k > 2$ then

$$\mathrm{Var}(T) = \frac{k}{k-2}$$

The t-Distribution



The t-Statistic

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then S_X^2 and $\overline{X}$ are independent

$$\overline{X} \quad \sim \quad N\left(\mu, \frac{\sigma_X^2}{n}\right)$$

The t-Statistic

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then S_X^2 and $\overline{X}$ are independent

$$\overline{X} \sim N\left(\mu, \frac{\sigma_X^2}{n}\right) \qquad Z = \left(\frac{\overline{X} - \mu}{\sigma_X / \sqrt{n}}\right) \sim N(0,1)$$

The t-Statistic

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then S_X^2 and $\overline{X}$ are independent

$$\overline{X} \sim N\left(\mu, \frac{\sigma_X^2}{n}\right) \qquad Z = \left(\frac{\overline{X} - \mu}{\sigma_X / \sqrt{n}}\right) \sim N(0,1)$$

$$W = \frac{(n-1) S_X^2}{\sigma_X^2} \sim \chi^2 (n-1)$$

The t-Statistic

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then S_X^2 and $\overline{X}$ are independent

$$\overline{X} \;\sim\; N\left(\mu, \frac{\sigma_X^2}{n}\right) \qquad\qquad Z = \left(\frac{\overline{X} - \mu}{\sigma_X / \sqrt{n}}\right) \sim N\left(0, 1\right)$$

$$W \;=\; \frac{\left(n-1\right) S_X^2}{\sigma_X^2} \sim \chi^2\left(n-1\right) \qquad\qquad \mathrm{SE}\left(\overline{X}\right) = \frac{S_X}{\sqrt{n}}$$

The t-Statistic

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then S_X^2 and $\overline{X}$ are independent

$$\overline{X} \sim N\left(\mu, \frac{\sigma_X^2}{n}\right) \qquad Z = \left(\frac{\overline{X} - \mu}{\sigma_X / \sqrt{n}}\right) \sim N(0,1)$$

$$W = \frac{(n-1)\, S_X^2}{\sigma_X^2} \sim \chi^2(n-1) \qquad \mathrm{SE}\left(\overline{X}\right) = \frac{S_X}{\sqrt{n}}$$

$$\frac{\overline{X} - \mu}{\mathrm{SE}\left(\overline{X}\right)} = \frac{\sqrt{n}\left(\overline{X} - \mu\right)}{S_X}$$

The t-Statistic

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then S_X^2 and $\overline{X}$ are independent

$$\overline{X} \sim N\left(\mu, \frac{\sigma_X^2}{n}\right) \qquad Z = \left(\frac{\overline{X} - \mu}{\sigma_X/\sqrt{n}}\right) \sim N\left(0, 1\right)$$

$$W = \frac{\left(n-1\right) S_X^2}{\sigma_X^2} \sim \chi^2\left(n-1\right) \qquad \mathrm{SE}\left(\overline{X}\right) = \frac{S_X}{\sqrt{n}}$$

$$\frac{\overline{X} - \mu}{\mathrm{SE}\left(\overline{X}\right)} = \frac{\sqrt{n}\left(\overline{X} - \mu\right)}{S_X} = \frac{\frac{\sqrt{n}}{\sigma_X}\left(\overline{X} - \mu\right)}{\frac{1}{\sigma_X}\sqrt{S_X^2}}$$

The t-Statistic

If $X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu, \sigma_X^2\right)$ then S_X^2 and $\overline{X}$ are independent

$$\overline{X} \sim N\left(\mu, \frac{\sigma_X^2}{n}\right) \qquad Z = \left(\frac{\overline{X} - \mu}{\sigma_X / \sqrt{n}}\right) \sim N(0,1)$$

$$W = \frac{(n-1)\,S_X^2}{\sigma_X^2} \sim \chi^2(n-1) \qquad \mathrm{SE}\left(\overline{X}\right) = \frac{S_X}{\sqrt{n}}$$

$$\frac{\overline{X} - \mu}{\mathrm{SE}\left(\overline{X}\right)} = \frac{\sqrt{n}\left(\overline{X} - \mu\right)}{S_X} = \frac{\frac{\sqrt{n}}{\sigma_X}\left(\overline{X} - \mu\right)}{\frac{1}{\sigma_X}\sqrt{S_X^2}} = \frac{\left(\frac{\overline{X} - \mu}{\sigma_X / \sqrt{n}}\right)}{\sqrt{\frac{S_X^2}{\sigma_X^2}}}$$

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&= \frac{\left(\frac{\overline{X}-\mu}{\sigma_X/\sqrt{n}}\right)}{\sqrt{\frac{(n-1)S_X^2}{(n-1)\sigma_X^2}}}
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\end{aligned}$$

Example

Suppose $Y_1, Y_2, \ldots, Y_{13} \sim i.i.d.\ N(7, 3)$

Let $H = \dfrac{\overline{Y} - 7}{SE\left(\overline{Y}\right)}$

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$$\text{Var}(H) = \frac{12}{12 - 2} = \frac{12}{10} = 1.2$$

The F-Distribution

- If U and V are independent with $U \sim \chi^2(a)$ and $V \sim \chi^2(b)$ then

$$Y = \frac{U/a}{V/b} \sim F(a, b)$$

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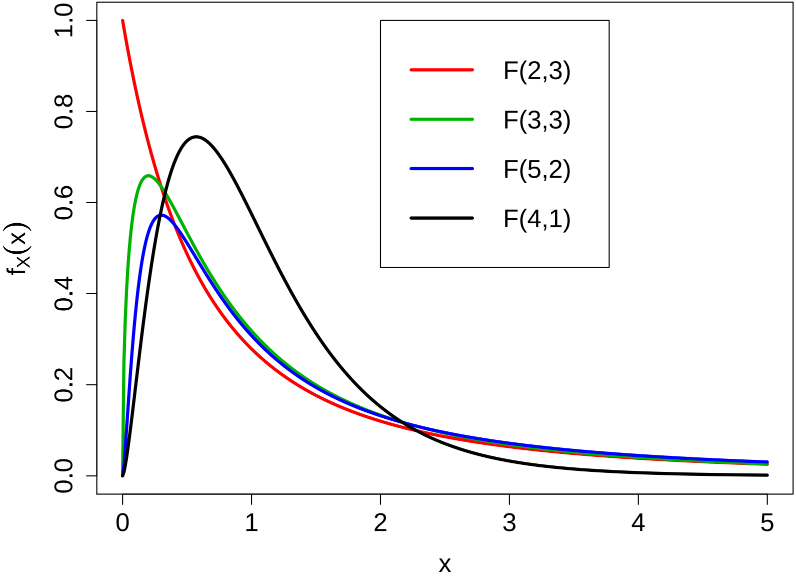
- If $Y \sim F(a, b)$ and $b > 2$ then

$$E\{Y\} = \frac{b}{b-2}$$

- If $Y \sim F(a, b)$ and $b > 4$ then

$$\text{Var}(Y) = \frac{2b^2(a+b-2)}{a(b-2)^2(b-4)}$$

The F-Distribution



The F-Statistic

If $X_1, \ldots, X_n \sim i.i.d.\ N\left(\mu_X, \sigma^2\right)$ and $Y_1, \ldots, Y_m \sim i.i.d.\ N\left(\mu_Y, \sigma^2\right)$

Suppose X_i and Y_j are independent for all i and j.

$$W = \frac{(n-1)\, S_X^2}{\sigma^2} \sim \chi^2\,(n-1) \qquad\qquad H = \frac{(m-1)\, S_Y^2}{\sigma^2} \sim \chi^2\,(m-1)$$

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Example

$X_1, X_2, X_3, X_4, X_5 \sim i.i.d.\ N(7, 9)$

$Y_1, Y_2, Y_3, Y_4 \sim i.i.d.\ N(9, 8)$

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\frac{S_X^2}{S_Y^2} &\sim F(5-1, 4-1) = F(4, 3) \\
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\end{aligned}$$

Hypothesis Testing

Daniel Stephenson

SCMA 524: Statistical Fundamentals

Hypothesis Testing

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6. Compute the value of the test statistic.
7. Reject H_0 if $T \in R$.

One Tailed t-Test

If $X_1, X_2, X_3, X_4, X_5 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\frac{\overline{X} - \mu}{\mathrm{SE}\left(\overline{X}\right)} = \frac{\overline{X} - \mu}{S_X/\sqrt{5}} = \frac{\sqrt{5}\left(\overline{X} - \mu\right)}{S_X} \quad \sim \quad t\left(4\right)$$

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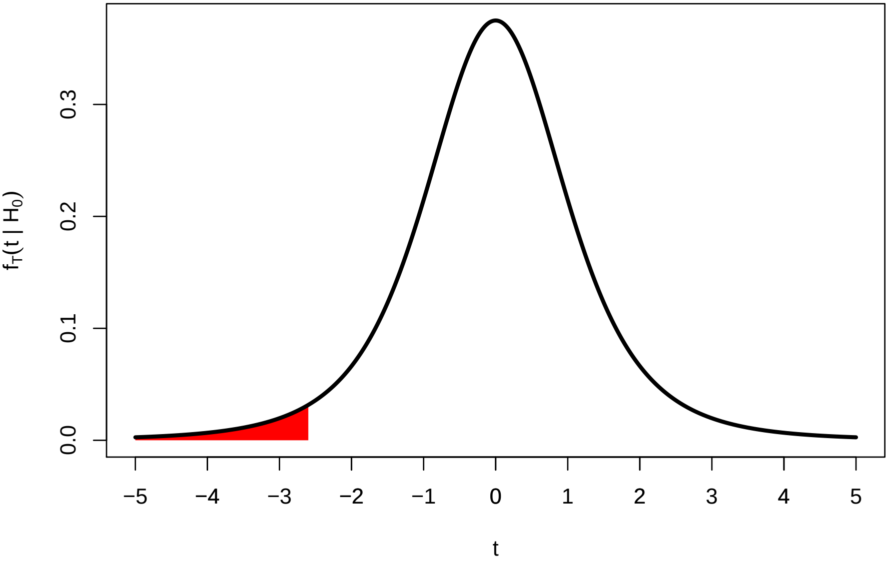
Rejection Region: Reject H_0 if $T < -2.6$

Rejection Region: One Tailed t-Test



One Tailed t-Test

$H_0 : \mu = 2$ $\qquad H_1 : \mu < 2$ $\qquad \mathrm{SE}(\overline{X}) = \dfrac{S_X}{\sqrt{5}}$

$T = \dfrac{\overline{X} - 2}{\mathrm{SE}\left(\overline{X}\right)} \sim t\,(4)$ $\qquad$ Reject H_0 if $T < -2.6$

If $X = (1.5, 2.1, 1.9, 1.5, 0.9)$ then $\overline{X} = 1.58$

One Tailed t-Test

$H_0 : \mu = 2$ $\qquad$ $H_1 : \mu < 2$ $\qquad$ $\text{SE}(\overline{X}) = \frac{S_X}{\sqrt{5}}$

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If $X = (1.5, 2.1, 1.9, 1.5, 0.9)$ then $\overline{X} = 1.58$

$$S_X^2 = \frac{1}{4}\sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2 = 0.212$$

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If $X = (1.5, 2.1, 1.9, 1.5, 0.9)$ then $\overline{X} = 1.58$

$$
\begin{aligned}
S_X^2 &= \frac{1}{4} \sum_{i=1}^{5} \left(X_i - \overline{X}\right)^2 = 0.212 \\
S_X &= \sqrt{S_X^2} \approx 0.4604346
\end{aligned}
$$

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\mathrm{SE}(\overline{X}) &= \frac{S_X}{\sqrt{5}} \approx 0.2059126
\end{aligned}$$

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If $X = (1.5, 2.1, 1.9, 1.5, 0.9)$ then $\overline{X} = 1.58$

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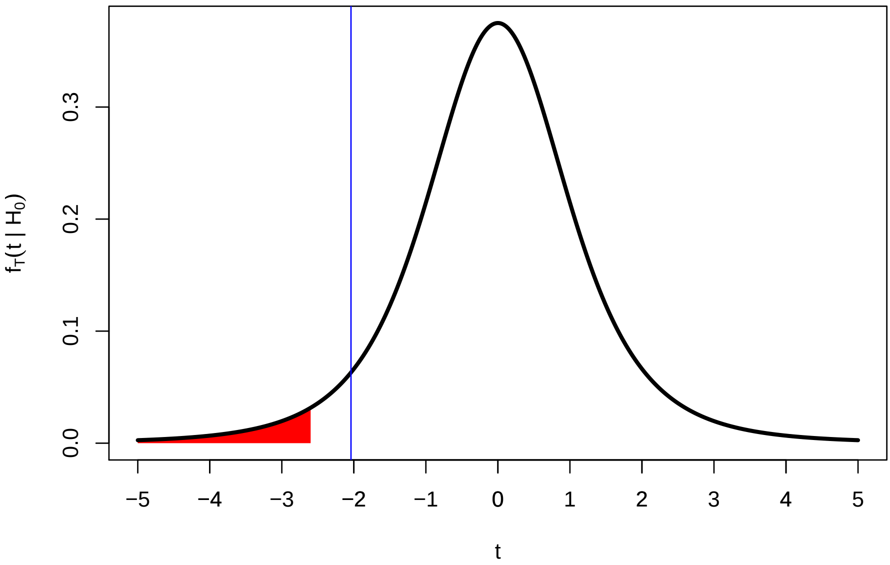
S_X &= \sqrt{S_X^2} \approx 0.4604346 \\
\mathrm{SE}(\overline{X}) &= \frac{S_X}{\sqrt{5}} \approx 0.2059126 \\
\text{t-statistic} = T &= \frac{\overline{X} - 2}{\mathrm{SE}\left(\overline{X}\right)} \approx -2.0397
\end{aligned}
$$

Rejection Region: One Tailed t-Test ($T = -2.0397$)



Question

$H_0 : \mu = 2$ $\qquad$ $H_1 : \mu < 2$ $\qquad$ Reject H_0 if $T < -2.6$

If $X = (3, -7, -3, -5, -3)$ then what is the value of the t-statistic?

(Round your answer to the nearest 4 decimal places)

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$$\overline{X} = \frac{3 - 7 - 3 - 5 - 3}{5} = \frac{-15}{5} = -3$$

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(Round your answer to the nearest 4 decimal places) Answer:

$$\begin{aligned}
\overline{X} &= \frac{3-7-3-5-3}{5} = \frac{-15}{5} = -3 \\
S_X^2 &= \frac{1}{4}\sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2 = 14
\end{aligned}$$

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(Round your answer to the nearest 4 decimal places) Answer:

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S_X &= \sqrt{14}
\end{aligned}
$$

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S_X &= \sqrt{14} \\
\mathrm{SE}(\overline{X}) &= \frac{S_X}{\sqrt{5}} = \frac{\sqrt{14}}{\sqrt{5}}
\end{aligned}
$$

Question

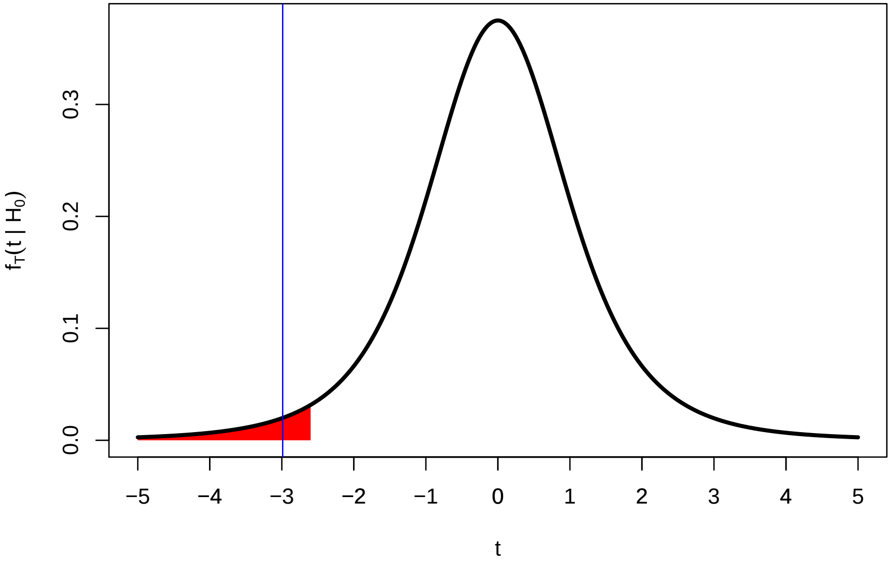
$H_0 : \mu = 2$ $H_1 : \mu < 2$ Reject H_0 if $T < -2.6$

If $X = (3, -7, -3, -5, -3)$ then what is the value of the t-statistic?

(Round your answer to the nearest 4 decimal places) Answer: -2.9881

$$
\begin{aligned}
\overline{X} &= \frac{3-7-3-5-3}{5} = \frac{-15}{5} = -3 \\
S_X^2 &= \frac{1}{4}\sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2 = 14 \\
S_X &= \sqrt{14} \\
\mathrm{SE}(\overline{X}) &= \frac{S_X}{\sqrt{5}} = \frac{\sqrt{14}}{\sqrt{5}} \\
T &= \frac{\overline{X} - 2}{\mathrm{SE}\left(\overline{X}\right)} = \frac{\sqrt{5}\,(-3-2)}{\sqrt{14}} \approx -2.9881
\end{aligned}
$$

Rejection Region: One Tailed t-Test ($T = -2.9881$)



Significance Level: One Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
2.6	0.9700
2.7	0.9730
2.8	0.9756

If the null hypothesis is true then $T \sim t(4)$

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Rejection Region: Reject H_0 if $T < -2.6$

Significance Level:

$$\begin{aligned} P(\text{Reject } H_0 \mid H_0 \text{ is true}) &= P(T < -2.6 | H_0) \\ &= P(T > 2.6 | H_0) \\ &= 1 - P(T < 2.6 | H_0) \\ &\approx 1 - 0.9700 = 0.03 \end{aligned}$$

Example

c	$P(t(3) < c)$	$P(t(4) < c)$	$P(t(5) < c)$	$P(t(6) < c)$
2.2	0.9424	0.9537	0.9605	0.9649
2.3	0.9475	0.9585	0.9651	0.9694
2.4	0.9521	0.9628	0.9692	0.9734
2.5	0.9561	0.9666	0.9726	0.9767

$X_1, X_2, X_3, X_4 \sim i.i.d.\ N(\mu, \sigma^2)$

$H_0 : \mu = 5.2$ $\qquad H_1 : \mu > 5.2$ $\qquad T = \dfrac{\overline{X} - 5.2}{SE\left(\overline{X}\right)}$

Reject H_0 if $T > 2.4$.

What is the significance level of this test?

Example

c	$P\left(t(3) < c\right)$	$P\left(t(4) < c\right)$	$P\left(t(5) < c\right)$	$P\left(t(6) < c\right)$
2.2	0.9424	0.9537	0.9605	0.9649
2.3	0.9475	0.9585	0.9651	0.9694
2.4	0.9521	0.9628	0.9692	0.9734
2.5	0.9561	0.9666	0.9726	0.9767

$X_1, X_2, X_3, X_4 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$

$H_0 : \mu = 5.2$ $\qquad$ $H_1 : \mu > 5.2$ $\qquad$ $T = \dfrac{\overline{X} - 5.2}{SE\left(\overline{X}\right)}$

Reject H_0 if $T > 2.4$.

What is the significance level of this test?

If H_0 is true then: $\qquad T \sim t\left(3\right)$

Example

c	$P(t(3) < c)$	$P(t(4) < c)$	$P(t(5) < c)$	$P(t(6) < c)$
2.2	0.9424	0.9537	0.9605	0.9649
2.3	0.9475	0.9585	0.9651	0.9694
2.4	0.9521	0.9628	0.9692	0.9734
2.5	0.9561	0.9666	0.9726	0.9767

$X_1, X_2, X_3, X_4 \sim i.i.d.\ N(\mu, \sigma^2)$

$H_0 : \mu = 5.2 \qquad H_1 : \mu > 5.2 \qquad T = \dfrac{\overline{X} - 5.2}{SE\left(\overline{X}\right)}$

Reject H_0 if $T > 2.4$.

What is the significance level of this test?

If H_0 is true then: $\qquad T \sim t(3)$

$$\alpha = P(T > 2.4 | H_0)$$

Example

c	$P(t(3) < c)$	$P(t(4) < c)$	$P(t(5) < c)$	$P(t(6) < c)$
2.2	0.9424	0.9537	0.9605	0.9649
2.3	0.9475	0.9585	0.9651	0.9694
2.4	0.9521	0.9628	0.9692	0.9734
2.5	0.9561	0.9666	0.9726	0.9767

$X_1, X_2, X_3, X_4 \sim i.i.d.\ N(\mu, \sigma^2)$

$H_0 : \mu = 5.2$ $\qquad H_1 : \mu > 5.2$ $\qquad T = \dfrac{\overline{X} - 5.2}{SE\left(\overline{X}\right)}$

Reject H_0 if $T > 2.4$.

What is the significance level of this test?

$$\text{If } H_0 \text{ is true then:} \qquad T \sim t(3)$$
$$\alpha = P(T > 2.4 | H_0) \quad = \quad 1 - P(T < 2.4 | H_0)$$

Example

c	$P(t(3) < c)$	$P(t(4) < c)$	$P(t(5) < c)$	$P(t(6) < c)$
2.2	0.9424	0.9537	0.9605	0.9649
2.3	0.9475	0.9585	0.9651	0.9694
2.4	0.9521	0.9628	0.9692	0.9734
2.5	0.9561	0.9666	0.9726	0.9767

$X_1, X_2, X_3, X_4 \sim i.i.d.\ N(\mu, \sigma^2)$

$H_0 : \mu = 5.2$ $\qquad$ $H_1 : \mu > 5.2$ $\qquad$ $T = \dfrac{\overline{X} - 5.2}{SE\left(\overline{X}\right)}$

Reject H_0 if $T > 2.4$.

What is the significance level of this test?

$$\begin{aligned} \text{If } H_0 \text{ is true then:} & \quad T \sim t(3) \\ \alpha = P(T > 2.4 | H_0) &= 1 - P(T < 2.4 | H_0) \\ &\approx 1 - 0.9521 = 0.0479 \end{aligned}$$

Significance Level: One Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $T < c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$\alpha = 0.05 \quad = \quad P(T < c | H_0)$$

Significance Level: One Tailed t-Test

c	$P\left(t(6) < c\right)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $T < c$

If the null hypotheis is true then $T \sim t\left(6\right)$

Suppose we observe $T = -2.2$

$$\alpha \; = \; 0.05 \quad = \quad P\left(T < c | H_0\right) = P\left(T > -c | H_0\right)$$

Significance Level: One Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $T < c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$\begin{aligned}\alpha = 0.05 &= P(T < c|H_0) = P(T > -c|H_0) \\ 0.05 &= 1 - P(T < -c|H_0)\end{aligned}$$

Significance Level: One Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $T < c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$\begin{aligned}
\alpha = 0.05 &= P(T < c|H_0) = P(T > -c|H_0) \\
0.05 &= 1 - P(T < -c|H_0) \\
P(T < -c|H_0) &= 1 - 0.05
\end{aligned}$$

Significance Level: One Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $T < c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$\begin{aligned}
\alpha = 0.05 &= P(T < c | H_0) = P(T > -c | H_0) \\
0.05 &= 1 - P(T < -c | H_0) \\
P(T < -c | H_0) &= 1 - 0.05 = 0.95
\end{aligned}$$

Significance Level: One Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $T < c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$
\begin{aligned}
\alpha = 0.05 &= P(T < c|H_0) = P(T > -c|H_0) \\
0.05 &= 1 - P(T < -c|H_0) \\
P(T < -c|H_0) &= 1 - 0.05 = 0.95 \\
-c &\approx 1.9432
\end{aligned}
$$

Significance Level: One Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $T < c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$
\begin{aligned}
\alpha = 0.05 &= P(T < c|H_0) = P(T > -c|H_0) \\
0.05 &= 1 - P(T < -c|H_0) \\
P(T < -c|H_0) &= 1 - 0.05 = 0.95 \\
-c &\approx 1.9432 \\
c &\approx -1.9432
\end{aligned}
$$

Significance Level: One Tailed t-Test

c	$P\left(t(6) < c\right)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Significance Level: $\alpha = 0.05$

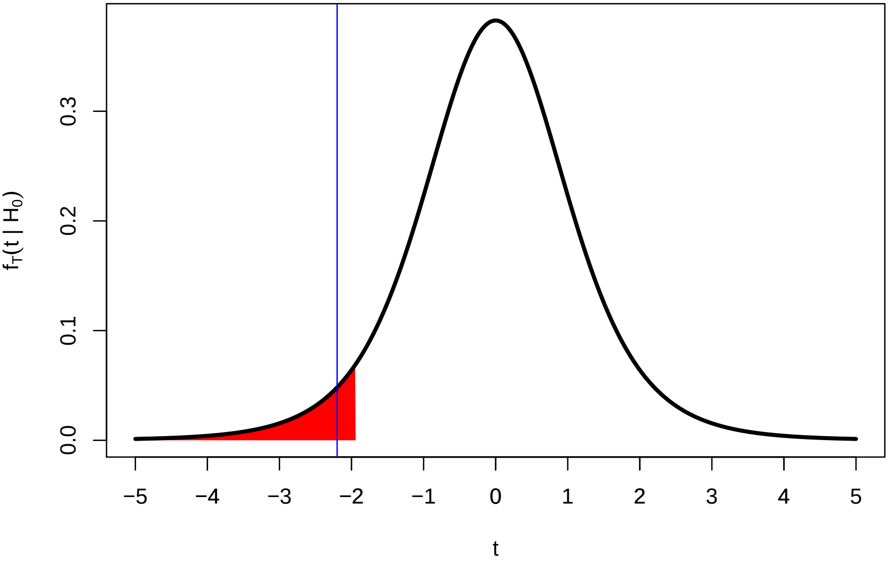
Rejection Region: Reject H_0 if $T < c$

If the null hypotheis is true then $T \sim t\left(6\right)$

Suppose we observe $T = -2.2$

$$\begin{aligned}
\alpha = 0.05 &= P\left(T < c|H_0\right) = P\left(T > -c|H_0\right) \\
0.05 &= 1 - P\left(T < -c|H_0\right) \\
P\left(T < -c|H_0\right) &= 1 - 0.05 = 0.95 \\
-c &\approx 1.9432 \\
c &\approx -1.9432 \\
T = -2.2 < c &\Longrightarrow \text{Reject } H_0
\end{aligned}$$

Rejection Region: One Tailed t-Test



Question

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 0$

Alternative Hypothesis $H_1 : \mu < 0$

Significance Level: $\alpha = 0.04$

Rejection Region: Reject H_0 if $T < c$

What is the value of c?

Question

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 0$

Alternative Hypothesis $H_1 : \mu < 0$

Significance Level: $\alpha = 0.04$

Rejection Region: Reject H_0 if $T < c$

What is the value of c?

Answer:

Question

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 0$

Alternative Hypothesis $H_1 : \mu < 0$

Significance Level: $\alpha = 0.04$

Rejection Region: Reject H_0 if $T < c$

What is the value of c?

Answer:

$$\alpha = 0.04 = P(T < c | H_0)$$

Question

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 0$

Alternative Hypothesis $H_1 : \mu < 0$

Significance Level: $\alpha = 0.04$

Rejection Region: Reject H_0 if $T < c$

What is the value of c?

Answer:

$$\alpha = 0.04 = P(T < c | H_0) = P(T > -c | H_0)$$

Question

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 0$

Alternative Hypothesis $H_1 : \mu < 0$

Significance Level: $\alpha = 0.04$

Rejection Region: Reject H_0 if $T < c$

What is the value of c?

Answer:

$$\begin{aligned} \alpha = 0.04 &= P(T < c|H_0) = P(T > -c|H_0) \\ 0.04 &= 1 - P(T < -c|H_0) \end{aligned}$$

Question

c	$P\left(t(6) < c\right)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 0$

Alternative Hypothesis $H_1 : \mu < 0$

Significance Level: $\alpha = 0.04$

Rejection Region: Reject H_0 if $T < c$

What is the value of c?

Answer:

$$\begin{aligned} \alpha = 0.04 &= P\left(T < c | H_0\right) = P\left(T > -c | H_0\right) \\ 0.04 &= 1 - P\left(T < -c | H_0\right) \\ P\left(T < -c | H_0\right) &= 0.96 \end{aligned}$$

Question

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 0$

Alternative Hypothesis $H_1 : \mu < 0$

Significance Level: $\alpha = 0.04$

Rejection Region: Reject H_0 if $T < c$

What is the value of c?

Answer:

$$\begin{aligned} \alpha = 0.04 &= P(T < c|H_0) = P(T > -c|H_0) \\ 0.04 &= 1 - P(T < -c|H_0) \\ P(T < -c|H_0) &= 0.96 \\ -c &\approx 2.1043 \end{aligned}$$

Question

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 0$

Alternative Hypothesis $H_1 : \mu < 0$

Significance Level: $\alpha = 0.04$

Rejection Region: Reject H_0 if $T < c$

What is the value of c?

Answer: -2.1043

$$\begin{aligned}
\alpha = 0.04 &= P(T < c | H_0) = P(T > -c | H_0) \\
0.04 &= 1 - P(T < -c | H_0) \\
P(T < -c | H_0) &= 0.96 \\
-c &\approx 2.1043 \\
c &\approx -2.1043
\end{aligned}$$

Two Tailed t-Test

If $X_1, X_2, X_3, X_4, X_5, X_6 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\frac{\overline{X} - \mu}{\text{StandardError}\left(\overline{X}\right)} = \frac{\sqrt{5}\left(\overline{X} - \mu\right)}{S_X} \quad \sim \quad t\left(5\right)$$

Two Tailed t-Test

If $X_1, X_2, X_3, X_4, X_5, X_6 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\frac{\overline{X} - \mu}{\text{StandardError}\left(\overline{X}\right)} = \frac{\sqrt{5}\left(\overline{X} - \mu\right)}{S_X} \quad \sim \quad t\left(5\right)$$

Null Hypothesis $H_0 : \mu = 4.5$
Alternative Hypothesis $H_1 : \mu \neq 4.5$

Two Tailed t-Test

If $X_1, X_2, X_3, X_4, X_5, X_6 \sim i.i.d.\ N\left(\mu, \sigma^2\right)$ then

$$\frac{\overline{X} - \mu}{\text{StandardError}\left(\overline{X}\right)} = \frac{\sqrt{5}\left(\overline{X} - \mu\right)}{S_X} \quad \sim \quad t\left(5\right)$$

Null Hypothesis $H_0 : \mu = 4.5$
Alternative Hypothesis $H_1 : \mu \neq 4.5$
Test Statistic: If the null hypotheis is true then

$$\text{t-statistic} \; = \; T \; = \frac{\overline{X} - 4.5}{\text{StandardError}\left(\overline{X}\right)} \sim t\left(5\right)$$

Two Tailed t-Test

If $X_1, X_2, X_3, X_4, X_5, X_6 \sim i.i.d.\ N(\mu, \sigma^2)$ then

$$\frac{\overline{X} - \mu}{\text{StandardError}\left(\overline{X}\right)} = \frac{\sqrt{5}\left(\overline{X} - \mu\right)}{S_X} \quad \sim \quad t(5)$$

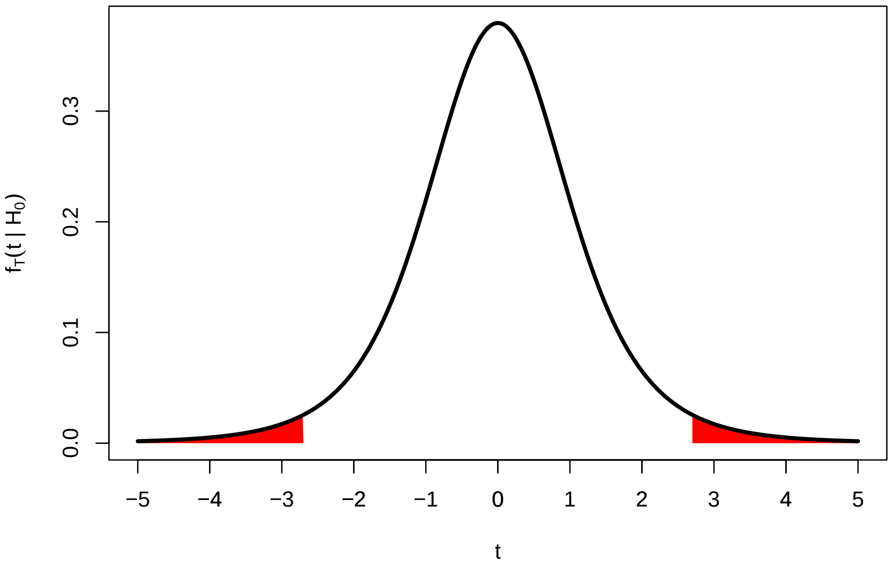
Null Hypothesis $H_0 : \mu = 4.5$
Alternative Hypothesis $H_1 : \mu \neq 4.5$
Test Statistic: If the null hypotheis is true then

$$\text{t-statistic} \;=\; T \;=\; \frac{\overline{X} - 4.5}{\text{StandardError}\left(\overline{X}\right)} \sim t(5)$$

Rejection Region: Reject H_0 if $|T| > 2.7$

$$|T| > 2.7 \quad \Longleftrightarrow \quad T < -2.7 \text{ or } T > 2.7$$

Rejection Region: Two Tailed t-Test



Example

$H_0 : \mu = 4.5$ $H_1 : \mu \neq 4.5$ Reject H_0 if $|T| > 2.7$

If $X = (8, 10, 7, 5, 9, 6)$ then what is the value of the t-statistic?

Example

$H_0 : \mu = 4.5$ $H_1 : \mu \neq 4.5$ Reject H_0 if $|T| > 2.7$

If $X = (8, 10, 7, 5, 9, 6)$ then what is the value of the t-statistic?

$$\overline{X} = \frac{8 + 10 + 7 + 5 + 9 + 6}{6} = 7.5$$

Example

$H_0 : \mu = 4.5$ $H_1 : \mu \neq 4.5$ Reject H_0 if $|T| > 2.7$

If $X = (8, 10, 7, 5, 9, 6)$ then what is the value of the t-statistic?

$$\begin{aligned} \overline{X} &= \frac{8 + 10 + 7 + 5 + 9 + 6}{6} = 7.5 \\ S_X^2 &= \frac{1}{5} \sum_{i=1}^{6} \left(X_i - \overline{X} \right)^2 = 3.5 \end{aligned}$$

Example

$H_0 : \mu = 4.5$ $H_1 : \mu \neq 4.5$ Reject H_0 if $|T| > 2.7$

If $X = (8, 10, 7, 5, 9, 6)$ then what is the value of the t-statistic?

$$\begin{aligned}
\overline{X} &= \frac{8 + 10 + 7 + 5 + 9 + 6}{6} = 7.5 \\
S_X^2 &= \frac{1}{5} \sum_{i=1}^{6} \left(X_i - \overline{X} \right)^2 = 3.5 \\
S_X &= \sqrt{3.5}
\end{aligned}$$

Example

$H_0 : \mu = 4.5$ $H_1 : \mu \neq 4.5$ Reject H_0 if $|T| > 2.7$

If $X = (8, 10, 7, 5, 9, 6)$ then what is the value of the t-statistic?

$$
\begin{aligned}
\overline{X} &= \frac{8+10+7+5+9+6}{6} = 7.5 \\
S_X^2 &= \frac{1}{5}\sum_{i=1}^{6}\left(X_i - \overline{X}\right)^2 = 3.5 \\
S_X &= \sqrt{3.5} \\
\mathrm{SE}(\overline{X}) &= \frac{S_X}{\sqrt{6}} = \frac{\sqrt{3.5}}{\sqrt{6}}
\end{aligned}
$$

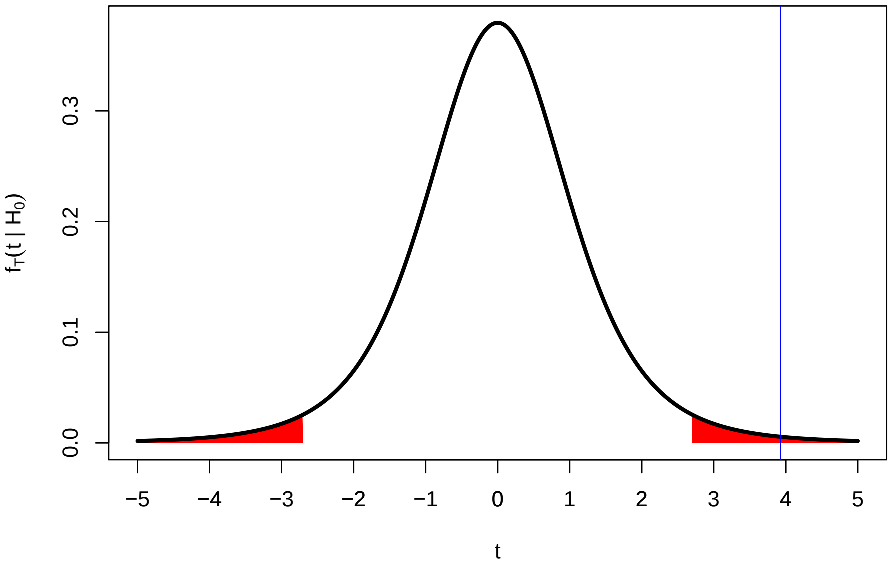
Example

$H_0 : \mu = 4.5$ $\qquad$ $H_1 : \mu \neq 4.5$ $\qquad$ Reject H_0 if $|T| > 2.7$

If $X = (8, 10, 7, 5, 9, 6)$ then what is the value of the t-statistic?

$$
\begin{aligned}
\overline{X} &= \frac{8+10+7+5+9+6}{6} = 7.5 \\
S_X^2 &= \frac{1}{5}\sum_{i=1}^{6}\left(X_i - \overline{X}\right)^2 = 3.5 \\
S_X &= \sqrt{3.5} \\
\mathrm{SE}(\overline{X}) &= \frac{S_X}{\sqrt{6}} = \frac{\sqrt{3.5}}{\sqrt{6}} \\
T &= \frac{\overline{X} - 4.5}{\mathrm{SE}\left(\overline{X}\right)} = \frac{\sqrt{6}\,(7.5 - 4.5)}{\sqrt{3.5}} \approx 3.9279
\end{aligned}
$$

Rejection Region: Two Tailed t-Test ($T = 3.9279$)



Significance Level: Two Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
2.6	0.9700
2.7	0.9730
2.8	0.9756

If the null hypotheis is true then $T \sim t(4)$

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Rejection Region: Reject H_0 if $|T| > 2.6$

Significance Level:

$$P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(|T| > 2.6 | H_0)$$

Significance Level: Two Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
2.6	0.9700
2.7	0.9730
2.8	0.9756

If the null hypotheis is true then $T \sim t(4)$

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Rejection Region: Reject H_0 if $|T| > 2.6$

Significance Level:

$$\begin{aligned} P(\text{Reject } H_0 \mid H_0 \text{ is true}) &= P(|T| > 2.6|H_0) \\ &= P(T > 2.6|H_0) + P(T < -2.6|H_0) \end{aligned}$$

Significance Level: Two Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
2.6	0.9700
2.7	0.9730
2.8	0.9756

If the null hypotheis is true then $T \sim t(4)$

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Rejection Region: Reject H_0 if $|T| > 2.6$

Significance Level:

$$
\begin{aligned}
P(\text{Reject } H_0 \mid H_0 \text{ is true}) &= P(|T| > 2.6|H_0) \\
&= P(T > 2.6|H_0) + P(T < -2.6|H_0) \\
&= 2P(T > 2.6|H_0)
\end{aligned}
$$

Significance Level: Two Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
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Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Rejection Region: Reject H_0 if $|T| > 2.6$

Significance Level:

$$
\begin{aligned}
P(\text{Reject } H_0 \mid H_0 \text{ is true}) &= P(|T| > 2.6 | H_0) \\
&= P(T > 2.6 | H_0) + P(T < -2.6 | H_0) \\
&= 2P(T > 2.6 | H_0) \\
&= 2\left[1 - P(T < 2.6 | H_0)\right]
\end{aligned}
$$

Significance Level: Two Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
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2.5	0.9666
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Significance Level:

$$
\begin{aligned}
P(\text{Reject } H_0 \mid H_0 \text{ is true}) &= P(|T| > 2.6|H_0) \\
&= P(T > 2.6|H_0) + P(T < -2.6|H_0) \\
&= 2P(T > 2.6|H_0) \\
&= 2\left[1 - P(T < 2.6|H_0)\right] \\
&\approx 2\left[1 - 0.9700\right] = 0.06 = \alpha
\end{aligned}
$$

Example

c	$P(t(3) < c)$	$P(t(4) < c)$	$P(t(5) < c)$	$P(t(6) < c)$
2.2	0.9424	0.9537	0.9605	0.9649
2.3	0.9475	0.9585	0.9651	0.9694
2.4	0.9521	0.9628	0.9692	0.9734
2.5	0.9561	0.9666	0.9726	0.9767

$X_1, X_2, X_3, X_4, X_5 \sim i.i.d.\ N(\mu, \sigma^2)$

$H_0 : \mu = 3.55$ $\quad$ $H_1 : \mu \neq 3.55$ $\quad$ $T = \dfrac{\overline{X} - 3.55}{SE(\bar{X})}$

Reject H_0 if $|T| > 2.4$

What is the significance level of this test?

Example

c	$P(t(3) < c)$	$P(t(4) < c)$	$P(t(5) < c)$	$P(t(6) < c)$
2.2	0.9424	0.9537	0.9605	0.9649
2.3	0.9475	0.9585	0.9651	0.9694
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$H_0 : \mu = 3.55$ $\qquad H_1 : \mu \neq 3.55$ $\qquad T = \dfrac{\overline{X} - 3.55}{SE\left(\bar{X}\right)}$

Reject H_0 if $|T| > 2.4$

What is the significance level of this test?

If H_0 is true then: $\qquad T \sim t(4)$

Example

c	$P(t(3) < c)$	$P(t(4) < c)$	$P(t(5) < c)$	$P(t(6) < c)$
2.2	0.9424	0.9537	0.9605	0.9649
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$H_0 : \mu = 3.55$ $\qquad H_1 : \mu \neq 3.55$ $\qquad T = \dfrac{\overline{X} - 3.55}{SE\left(\bar{X}\right)}$

Reject H_0 if $|T| > 2.4$

What is the significance level of this test?

$$\text{If } H_0 \text{ is true then:} \qquad T \sim t(4)$$

$$\alpha = P\left(|T| > 2.4 | H_0\right) \quad = \quad 2P\left(T > 2.4 | H_0\right)$$

Example

c	$P(t(3) < c)$	$P(t(4) < c)$	$P(t(5) < c)$	$P(t(6) < c)$
2.2	0.9424	0.9537	0.9605	0.9649
2.3	0.9475	0.9585	0.9651	0.9694
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$H_0 : \mu = 3.55$ $\qquad H_1 : \mu \neq 3.55$ $\qquad T = \dfrac{\overline{X} - 3.55}{SE\left(\bar{X}\right)}$

Reject H_0 if $|T| > 2.4$

What is the significance level of this test?

$$
\begin{aligned}
\text{If } H_0 \text{ is true then:} \quad & T \sim t(4) \\
\alpha = P(|T| > 2.4 | H_0) &= 2P(T > 2.4 | H_0) \\
\alpha &= 2\left[1 - P(T < 2.4 | H_0)\right]
\end{aligned}
$$

Example

c	$P(t(3) < c)$	$P(t(4) < c)$	$P(t(5) < c)$	$P(t(6) < c)$
2.2	0.9424	0.9537	0.9605	0.9649
2.3	0.9475	0.9585	0.9651	0.9694
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$X_1, X_2, X_3, X_4, X_5 \sim i.i.d.\ N(\mu, \sigma^2)$

$H_0 : \mu = 3.55 \qquad H_1 : \mu \neq 3.55 \qquad T = \dfrac{\overline{X} - 3.55}{SE\left(\bar{X}\right)}$

Reject H_0 if $|T| > 2.4$

What is the significance level of this test?

$$\begin{aligned} \text{If } H_0 \text{ is true then:} &\quad T \sim t(4) \\ \alpha = P(|T| > 2.4|H_0) &= 2P(T > 2.4|H_0) \\ \alpha &= 2[1 - P(T < 2.4|H_0)] \\ \alpha &\approx 2[1 - 0.9628] = 0.0744 \end{aligned}$$

Significance Level: Two Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $|T| > c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$\alpha = 0.05 \quad = \quad P(|T| > c | H_0)$$

Significance Level: Two Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

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Alternative Hypothesis $H_1 : \mu \neq 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $|T| > c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$\alpha = 0.05 \quad = \quad P(|T| > c | H_0) = 2P(T > c | H_0)$$

Significance Level: Two Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
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Alternative Hypothesis $H_1 : \mu \neq 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $|T| > c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$\begin{aligned} \alpha = 0.05 &= P(|T| > c|H_0) = 2P(T > c|H_0) \\ 0.05 &= 2[1 - P(T < c|H_0)] \end{aligned}$$

Significance Level: Two Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
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Alternative Hypothesis $H_1 : \mu \neq 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $|T| > c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$\begin{aligned}
\alpha = 0.05 &= P(|T| > c|H_0) = 2P(T > c|H_0) \\
0.05 &= 2\left[1 - P(T < c|H_0)\right] \\
0.025 &= 1 - P(T < c|H_0)
\end{aligned}$$

Significance Level: Two Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $|T| > c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$\begin{aligned}
\alpha = 0.05 &= P(|T| > c|H_0) = 2P(T > c|H_0) \\
0.05 &= 2[1 - P(T < c|H_0)] \\
0.025 &= 1 - P(T < c|H_0) \\
P(T < c|H_0) &= 1 - 0.025
\end{aligned}$$

Significance Level: Two Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
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2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $|T| > c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$\begin{aligned}
\alpha = 0.05 &= P(|T| > c|H_0) = 2P(T > c|H_0) \\
0.05 &= 2[1 - P(T < c|H_0)] \\
0.025 &= 1 - P(T < c|H_0) \\
P(T < c|H_0) &= 1 - 0.025 = 0.975
\end{aligned}$$

Significance Level: Two Tailed t-Test

c	$P\left(t(6) < c\right)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $|T| > c$

If the null hypotheis is true then $T \sim t\left(6\right)$

Suppose we observe $T = -2.2$

$$\begin{aligned}
\alpha = 0.05 &= P\left(|T| > c|H_0\right) = 2P\left(T > c|H_0\right) \\
0.05 &= 2\left[1 - P\left(T < c|H_0\right)\right] \\
0.025 &= 1 - P\left(T < c|H_0\right) \\
P\left(T < c|H_0\right) &= 1 - 0.025 = 0.975 \\
c &\approx 2.4469
\end{aligned}$$

Significance Level: Two Tailed t-Test

c	$P(t(6) < c)$
1.8744	0.9450
1.9432	0.9500
2.0192	0.9550
2.1043	0.9600
2.2011	0.9650
2.3133	0.9700
2.4469	0.9750

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Significance Level: $\alpha = 0.05$

Rejection Region: Reject H_0 if $|T| > c$

If the null hypotheis is true then $T \sim t(6)$

Suppose we observe $T = -2.2$

$$
\begin{aligned}
\alpha = 0.05 &= P(|T| > c|H_0) = 2P(T > c|H_0) \\
0.05 &= 2[1 - P(T < c|H_0)] \\
0.025 &= 1 - P(T < c|H_0) \\
P(T < c|H_0) &= 1 - 0.025 = 0.975 \\
c &\approx 2.4469 \\
|T| = 2.2 < c &\implies \text{Do not reject } H_0
\end{aligned}
$$

p-value: t-test

- The **p-value** is the probability of obtaining test results at least as extreme is the results actually observed, assuming that the null hypothesis is correct.

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- For a t-test:

$$\text{If } H_1 : \mu < c \text{ and we observe } T = k \text{ then } p = P\left(T < k \mid H_0\right)$$

p-value: t-test

- The **p-value** is the probability of obtaining test results at least as extreme is the results actually observed, assuming that the null hypothesis is correct.

- For a t-test:

$$\text{If } H_1 : \mu < c \text{ and we observe } T = k \text{ then } p = P\left(T < k \mid H_0\right)$$

$$\text{If } H_1 : \mu > c \text{ and we observe } T = k \text{ then } p = P\left(T > k \mid H_0\right)$$

p-value: t-test

- The **p-value** is the probability of obtaining test results at least as extreme is the results actually observed, assuming that the null hypothesis is correct.

- For a t-test:

$$\text{If } H_1 : \mu < c \text{ and we observe } T = k \text{ then } p = P(T < k \mid H_0)$$

$$\text{If } H_1 : \mu > c \text{ and we observe } T = k \text{ then } p = P(T > k \mid H_0)$$

$$\text{If } H_1 : \mu \neq c \text{ and we observe } T = k \text{ then } p = P(|T| > k \mid H_0)$$

p-value: One Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
2.6	0.9700
2.7	0.9730
2.8	0.9756

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Suppose $T|H_0 \sim t(4)$

Rejection Region: Reject H_0 if $T < -2.6$

Suppose we observe $T = -2.8$

Then we reject H_0 since $-2.8 < -2.6$

p-value: $P(T < -2.8|H_0)$

p-value: One Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
2.6	0.9700
2.7	0.9730
2.8	0.9756

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Suppose $T|H_0 \sim t(4)$

Rejection Region: Reject H_0 if $T < -2.6$

Suppose we observe $T = -2.8$

Then we reject H_0 since $-2.8 < -2.6$

p-value: $P(T < -2.8|H_0) \quad = \quad P(T > 2.8|H_0)$

p-value: One Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
2.6	0.9700
2.7	0.9730
2.8	0.9756

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Suppose $T|H_0 \sim t(4)$

Rejection Region: Reject H_0 if $T < -2.6$

Suppose we observe $T = -2.8$

Then we reject H_0 since $-2.8 < -2.6$

$$\textbf{p-value:} \quad P(T < -2.8|H_0) = P(T > 2.8|H_0) = 1 - P(T < 2.8|H_0)$$

p-value: One Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
2.6	0.9700
2.7	0.9730
2.8	0.9756

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu < 2$

Suppose $T|H_0 \sim t(4)$

Rejection Region: Reject H_0 if $T < -2.6$

Suppose we observe $T = -2.8$

Then we reject H_0 since $-2.8 < -2.6$

$$\textbf{p-value:} \quad P(T < -2.8|H_0) \begin{aligned}[t] &= P(T > 2.8|H_0) \\ &= 1 - P(T < 2.8|H_0) \\ &\approx 1 - 0.9756 = 0.0244 \end{aligned}$$

p-value: Two Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
2.6	0.9700
2.7	0.9730
2.8	0.9756

If the null hypotheis is true then $T \sim t(4)$

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Rejection Region: Reject H_0 if $|T| > 2.6$

Suppose we observe $T = -2.7$

Then we reject H_0 since $|-2.7| = 2.7 > 2.6$

p-value: $P(|T| > 2.7|H_0) = 2P(T > 2.7|H_0)$

p-value: Two Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
2.6	0.9700
2.7	0.9730
2.8	0.9756

If the null hypotheis is true then $T \sim t(4)$

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Rejection Region: Reject H_0 if $|T| > 2.6$

Suppose we observe $T = -2.7$

Then we reject H_0 since $|-2.7| = 2.7 > 2.6$

p-value:
$$\begin{aligned} P(|T| > 2.7|H_0) &= 2P(T > 2.7|H_0) \\ &= 2\left[1 - P(T < 2.7|H_0)\right] \end{aligned}$$

p-value: Two Tailed t-Test

c	$P(t(4) < c)$
2.2	0.9537
2.3	0.9585
2.4	0.9628
2.5	0.9666
2.6	0.9700
2.7	0.9730
2.8	0.9756

If the null hypotheis is true then $T \sim t(4)$

Null Hypothesis $H_0 : \mu = 2$

Alternative Hypothesis $H_1 : \mu \neq 2$

Rejection Region: Reject H_0 if $|T| > 2.6$

Suppose we observe $T = -2.7$

Then we reject H_0 since $|-2.7| = 2.7 > 2.6$

p-value:
$$\begin{aligned} P(|T| > 2.7|H_0) &= 2P(T > 2.7|H_0) \\ &= 2\left[1 - P(T < 2.7|H_0)\right] \\ &\approx 2\left[1 - 0.9730\right] = 0.054 \end{aligned}$$

F-Test

$$X_1, X_2, \ldots, X_n \quad \sim \quad i.i.d.\ N\left(\mu_X, \sigma_X^2\right)$$
$$Y_1, Y_2, \ldots, Y_m \quad \sim \quad i.i.d.\ N\left(\mu_Y, \sigma_Y^2\right)$$

Null Hypothesis $H_0 : \sigma_X^2 = \sigma_Y^2$

Alternative Hypothesis $H_1 : \sigma_X^2 > \sigma_Y^2$

F-Test

$$X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu_X, \sigma_X^2\right)$$
$$Y_1, Y_2, \ldots, Y_m \sim i.i.d.\ N\left(\mu_Y, \sigma_Y^2\right)$$

Null Hypothesis $H_0 : \sigma_X^2 = \sigma_Y^2$

Alternative Hypothesis $H_1 : \sigma_X^2 > \sigma_Y^2$

Test Statistic: F-statistic $= T = \dfrac{S_X^2}{S_Y^2}$

$$\text{If } \sigma_X^2 = \sigma_Y^2 \text{ then } T \sim F\,(n-1, m-1)$$

F-Test

$$X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu_X, \sigma_X^2\right)$$

$$Y_1, Y_2, \ldots, Y_m \sim i.i.d.\ N\left(\mu_Y, \sigma_Y^2\right)$$

Null Hypothesis $H_0 : \sigma_X^2 = \sigma_Y^2$

Alternative Hypothesis $H_1 : \sigma_X^2 > \sigma_Y^2$

Test Statistic: F-statistic $= T = \dfrac{S_X^2}{S_Y^2}$

$$\text{If } \sigma_X^2 = \sigma_Y^2 \text{ then } T \sim F\left(n-1, m-1\right)$$

Rejection Region: Reject H_0 if $T > c$

F-Test

$$X_1, X_2, \ldots, X_n \sim i.i.d.\ N\left(\mu_X, \sigma_X^2\right)$$
$$Y_1, Y_2, \ldots, Y_m \sim i.i.d.\ N\left(\mu_Y, \sigma_Y^2\right)$$

Null Hypothesis $H_0 : \sigma_X^2 = \sigma_Y^2$

Alternative Hypothesis $H_1 : \sigma_X^2 > \sigma_Y^2$

Test Statistic: F-statistic $= T = \dfrac{S_X^2}{S_Y^2}$

$$\text{If } \sigma_X^2 = \sigma_Y^2 \text{ then } T \sim F\left(n-1, m-1\right)$$

Rejection Region: Reject H_0 if $T > c$

Significance Level: $\alpha = P\left(T > c | H_0\right)$

Example

$$H_0 : \sigma_X^2 = \sigma_Y^2 \qquad X = (0, 8, 9, 0)$$
$$H_1 : \sigma_X^2 > \sigma_Y^2 \qquad Y = (2, 5, 3, 1, 4)$$

Can we reject H_0 at the 5% significance level?

Example

$$H_0 : \sigma_X^2 = \sigma_Y^2 \qquad X = (0, 8, 9, 0)$$
$$H_1 : \sigma_X^2 > \sigma_Y^2 \qquad Y = (2, 5, 3, 1, 4)$$

Can we reject H_0 at the 5% significance level? $T | H_0 \sim F(3, 4)$

c	$P(F(3,4) < c)$
0.0216	0.005
0.0348	0.010
0.0662	0.025
0.1097	0.050
6.5914	0.950
9.9792	0.975
16.6944	0.990
24.2591	0.995

Example

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$$\alpha = 0.05 = P(T > c|H_0)$$

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$$\alpha = 0.05 = P(T > c|H_0)$$
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$$S_X^2 = \frac{1}{3}\sum_{i=1}^{4}\left(\overline{X} - X_i\right)^2 = 24.25$$

$$S_Y^2 = \frac{1}{4}\sum_{i=1}^{5}\left(\overline{Y} - Y_i\right)^2 = 2.5$$

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$$T = \frac{S_X^2}{S_Y^2} = \frac{24.25}{2.5} = 9.7 > c$$

Example

$$H_0 : \sigma_X^2 = \sigma_Y^2 \qquad X = (0, 8, 9, 0)$$
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24.2591	0.995

$\alpha = 0.05 = P\,(T > c | H_0)$

$P\,(T < c | H_0) = 0.95$

$c \approx 6.5914$

$S_X^2 = \frac{1}{3} \sum_{i=1}^{4} \left(\overline{X} - X_i\right)^2 = 24.25$

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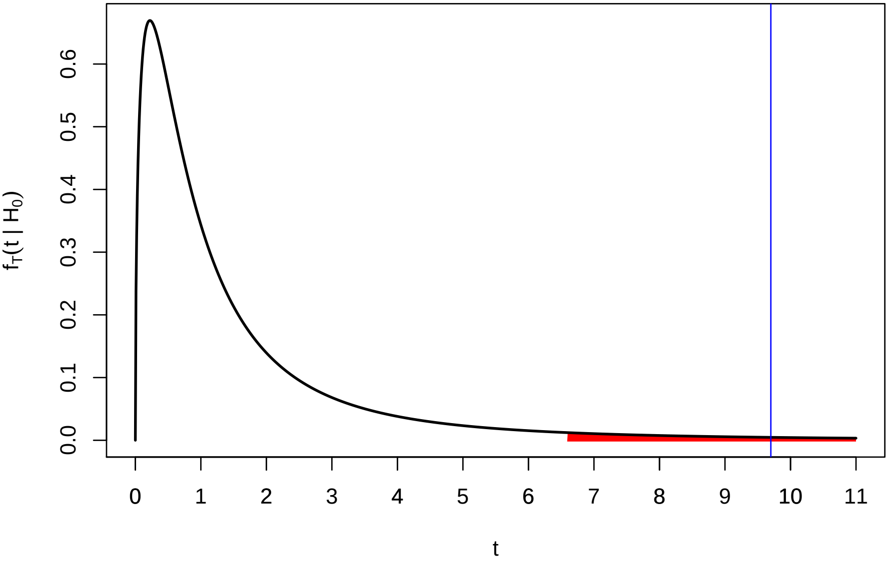
$$T = \frac{S_X^2}{S_Y^2} = \frac{24.25}{2.5} = 9.7 > c$$

We can reject H_0 at the 5% level.

Rejection Region: F-Test



p-value: F-Test

$$H_0 : \sigma_X^2 = \sigma_Y^2 \qquad X = (25, 12, 35, 33)$$
$$H_1 : \sigma_X^2 > \sigma_Y^2 \qquad Y = (4.5, 3.2, 2.6, 0.5)$$

What is the p-value?

c	$P\left(F(3,3) < c\right)$
37	0.9928
38	0.9931
39	0.9933
40	0.9936
41	0.9938
42	0.9940
43	0.9942
44	0.9944
44	0.9946

p-value: F-Test

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$$S_X^2 = \frac{1}{4} \sum_{i=1}^{5} \left(\overline{X} - X_i\right)^2 \approx 108.9167$$

p-value: F-Test

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$$T = \frac{S_X^2}{S_Y^2} \approx \frac{108.9167}{2.78} \approx 39.1787$$

p-value: F-Test

$$H_0 : \sigma_X^2 = \sigma_Y^2 \qquad X = (25, 12, 35, 33)$$
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$$p = P\left(T > 39.1787\right)$$

p-value: F-Test

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p-value: F-Test

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$$p \approx 1 - 0.9933$$

p-value: F-Test

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$$p = P(T > 39.1787)$$
$$p = 1 - P(T < 39.1787)$$
$$p \approx 1 - 0.9933$$
$$p \approx 0.0067$$

Simple Linear Regression

Daniel Stephenson

SCMA 524: Statistical Fundamentals

Simple Linear Regression

- **Simple linear regression** is a method of estimating the relationship between a dependent variable Y and an independent variable X.

- The **simple linear model** is

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$X_1, X_2, \ldots, X_n \sim i.i.d.$$
$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \sim i.i.d.$$
$$E\{\varepsilon_i | X\} = 0$$
$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$
$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$

$$E\{\varepsilon_i\} = E\{E\{\varepsilon_i | X\}\}$$

The Simple Linear Model

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$$\begin{aligned} E\{\varepsilon_i\} &= E\{E\{\varepsilon_i|X\}\} \\ &= E\{0\} \qquad \text{since } E\{\varepsilon_i|X\} = 0 \end{aligned}$$

The Simple Linear Model

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$$E\{X_i \varepsilon_i\} = E\{E\{X_i \varepsilon_i | X\}\}$$

The Simple Linear Model

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$$\begin{aligned} E\{X_i \varepsilon_i\} &= E\{E\{X_i \varepsilon_i | X\}\} \\ &= E\{X_i E\{\varepsilon_i | X\}\} \end{aligned}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\left\{\varepsilon_i | X\right\} = 0$$
$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$

$$\begin{aligned} E\left\{\varepsilon_i\right\} &= E\left\{E\left\{\varepsilon_i | X\right\}\right\} \\ &= E\left\{0\right\} \qquad \text{since } E\left\{\varepsilon_i | X\right\} = 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} E\left\{X_i \varepsilon_i\right\} &= E\left\{E\left\{X_i \varepsilon_i | X\right\}\right\} \\ &= E\left\{X_i E\left\{\varepsilon_i | X\right\}\right\} \\ &= E\left\{X_i \cdot 0\right\} \end{aligned}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$
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The Simple Linear Model

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$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$
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$$E\{X_i \varepsilon_i\} = 0$$

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

The Simple Linear Model

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$$E\left\{\varepsilon_i\right\} = 0$$
$$E\left\{X_i \varepsilon_i\right\} = 0$$

$$\begin{aligned} Y_i &= \alpha + \beta X_i + \varepsilon_i \\ E\left\{Y_i\right\} &= \alpha + \beta E\left\{X_i\right\} + E\left\{\varepsilon_i\right\} \end{aligned}$$

The Simple Linear Model

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$$\begin{aligned} Y_i &= \alpha + \beta X_i + \varepsilon_i \\ E\left\{Y_i\right\} &= \alpha + \beta E\left\{X_i\right\} + E\left\{\varepsilon_i\right\} \\ E\left\{Y_i\right\} &= \alpha + \beta E\left\{X_i\right\} + 0 \end{aligned}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$
$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$
$$E\{\varepsilon_i\} = 0$$
$$E\{X_i \varepsilon_i\} = 0$$

$$\begin{aligned}
Y_i &= \alpha + \beta X_i + \varepsilon_i \\
E\{Y_i\} &= \alpha + \beta E\{X_i\} + E\{\varepsilon_i\} \\
E\{Y_i\} &= \alpha + \beta E\{X_i\} + 0 \\
E\{Y_i\} &= \alpha + \beta E\{X_i\}
\end{aligned}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$
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$$\begin{aligned}
Y_i &= \alpha + \beta X_i + \varepsilon_i \\
E\{Y_i\} &= \alpha + \beta E\{X_i\} + E\{\varepsilon_i\} \\
E\{Y_i\} &= \alpha + \beta E\{X_i\} + 0 \\
E\{Y_i\} &= \alpha + \beta E\{X_i\} \\
\alpha &= E\{Y_i\} - \beta E\{X_i\}
\end{aligned}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\left\{\varepsilon_i | X\right\} = 0$$
$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$
$$E\left\{\varepsilon_i\right\} = 0$$
$$E\left\{X_i \varepsilon_i\right\} = 0$$

$$\alpha = E\left\{Y_i\right\} - \beta E\left\{X_i\right\}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

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$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$

$$E\{\varepsilon_i\} = 0$$

$$E\{X_i \varepsilon_i\} = 0$$

$$\alpha = E\{Y_i\} - \beta E\{X_i\}$$

$$Y_i \quad = \quad \alpha + \beta X_i + \varepsilon_i$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$

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$$\alpha = E\{Y_i\} - \beta E\{X_i\}$$

$$\begin{aligned} Y_i &= \alpha + \beta X_i + \varepsilon_i \\ X_i Y_i &= \alpha X_i + \beta X_i^2 + X_i \varepsilon_i \end{aligned}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\left\{\varepsilon_i | X\right\} = 0$$
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$$\alpha = E\left\{Y_i\right\} - \beta E\left\{X_i\right\}$$

$$\begin{aligned}
Y_i &= \alpha + \beta X_i + \varepsilon_i \\
X_i Y_i &= \alpha X_i + \beta X_i^2 + X_i \varepsilon_i \\
E\left\{X_i Y_i\right\} &= \alpha E\left\{X_i\right\} + \beta E\left\{X_i^2\right\} + E\left\{X_i \varepsilon_i\right\} \\
E\left\{X_i Y_i\right\} &= \alpha E\left\{X_i\right\} + \beta E\left\{X_i^2\right\} + 0
\end{aligned}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\left\{\varepsilon_i | X\right\} = 0$$

$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$

$$E\left\{\varepsilon_i\right\} = 0$$

$$E\left\{X_i \varepsilon_i\right\} = 0$$

$$\alpha = E\left\{Y_i\right\} - \beta E\left\{X_i\right\}$$

$$\begin{aligned} Y_i &= \alpha + \beta X_i + \varepsilon_i \\ X_i Y_i &= \alpha X_i + \beta X_i^2 + X_i \varepsilon_i \\ E\left\{X_i Y_i\right\} &= \alpha E\left\{X_i\right\} + \beta E\left\{X_i^2\right\} + E\left\{X_i \varepsilon_i\right\} \\ E\left\{X_i Y_i\right\} &= \alpha E\left\{X_i\right\} + \beta E\left\{X_i^2\right\} + 0 \end{aligned}$$

$$E\left\{X_i Y_i\right\} = \left(E\left\{Y_i\right\} - \beta E\left\{X_i\right\}\right) E\left\{X_i\right\} + \beta E\left\{X_i^2\right\}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$

$$E\{\varepsilon_i\} = 0$$

$$E\{X_i \varepsilon_i\} = 0$$

$$\alpha = E\{Y_i\} - \beta E\{X_i\}$$

$$\begin{aligned} Y_i &= \alpha + \beta X_i + \varepsilon_i \\ X_i Y_i &= \alpha X_i + \beta X_i^2 + X_i \varepsilon_i \\ E\{X_i Y_i\} &= \alpha E\{X_i\} + \beta E\left\{X_i^2\right\} + E\{X_i \varepsilon_i\} \\ E\{X_i Y_i\} &= \alpha E\{X_i\} + \beta E\left\{X_i^2\right\} + 0 \end{aligned}$$

$$\begin{aligned} E\{X_i Y_i\} &= \left(E\{Y_i\} - \beta E\{X_i\}\right) E\{X_i\} + \beta E\left\{X_i^2\right\} \\ E\{X_i Y_i\} &= E\{Y_i\} E\{X_i\} - \beta E\{X_i\}^2 + \beta E\left\{X_i^2\right\} \end{aligned}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$
$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$
$$E\{\varepsilon_i\} = 0$$
$$E\{X_i \varepsilon_i\} = 0$$

$$\alpha = E\{Y_i\} - \beta E\{X_i\}$$

$$\begin{aligned} Y_i &= \alpha + \beta X_i + \varepsilon_i \\ X_i Y_i &= \alpha X_i + \beta X_i^2 + X_i \varepsilon_i \\ E\{X_i Y_i\} &= \alpha E\{X_i\} + \beta E\left\{X_i^2\right\} + E\{X_i \varepsilon_i\} \\ E\{X_i Y_i\} &= \alpha E\{X_i\} + \beta E\left\{X_i^2\right\} + 0 \end{aligned}$$

$$\begin{aligned} E\{X_i Y_i\} &= \left(E\{Y_i\} - \beta E\{X_i\}\right) E\{X_i\} + \beta E\left\{X_i^2\right\} \\ E\{X_i Y_i\} &= E\{Y_i\} E\{X_i\} - \beta E\{X_i\}^2 + \beta E\left\{X_i^2\right\} \\ E\{X_i Y_i\} - E\{Y_i\} E\{X_i\} &= \beta \left(E\left\{X_i^2\right\} - E\{X_i\}^2\right) \end{aligned}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i|X\} = 0$$
$$E\left\{\varepsilon_i^2|X\right\} = \sigma^2$$
$$E\{\varepsilon_i\} = 0$$
$$E\{X_i\varepsilon_i\} = 0$$

$$\alpha = E\{Y_i\} - \beta E\{X_i\}$$

$$\begin{aligned}
Y_i &= \alpha + \beta X_i + \varepsilon_i \\
X_i Y_i &= \alpha X_i + \beta X_i^2 + X_i\varepsilon_i \\
E\{X_i Y_i\} &= \alpha E\{X_i\} + \beta E\left\{X_i^2\right\} + E\{X_i\varepsilon_i\} \\
E\{X_i Y_i\} &= \alpha E\{X_i\} + \beta E\left\{X_i^2\right\} + 0
\end{aligned}$$

$$\begin{aligned}
E\{X_i Y_i\} &= (E\{Y_i\} - \beta E\{X_i\})\, E\{X_i\} + \beta E\left\{X_i^2\right\} \\
E\{X_i Y_i\} &= E\{Y_i\}\, E\{X_i\} - \beta E\{X_i\}^2 + \beta E\left\{X_i^2\right\} \\
E\{X_i Y_i\} - E\{Y_i\}\, E\{X_i\} &= \beta\left(E\left\{X_i^2\right\} - E\{X_i\}^2\right) \\
\mathrm{Cov}(X_i, Y_i) &= \beta \mathrm{Var}(X_i)
\end{aligned}$$

The Simple Linear Model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

$$E\left\{\varepsilon_i^2 | X\right\} = \sigma^2$$

$$E\{\varepsilon_i\} = 0$$

$$E\{X_i \varepsilon_i\} = 0$$

$$\alpha = E\{Y_i\} - \beta E\{X_i\}$$

$$\begin{aligned}
Y_i &= \alpha + \beta X_i + \varepsilon_i \\
X_i Y_i &= \alpha X_i + \beta X_i^2 + X_i \varepsilon_i \\
E\{X_i Y_i\} &= \alpha E\{X_i\} + \beta E\left\{X_i^2\right\} + E\{X_i \varepsilon_i\} \\
E\{X_i Y_i\} &= \alpha E\{X_i\} + \beta E\left\{X_i^2\right\} + 0
\end{aligned}$$

$$\begin{aligned}
E\{X_i Y_i\} &= \left(E\{Y_i\} - \beta E\{X_i\}\right) E\{X_i\} + \beta E\left\{X_i^2\right\} \\
E\{X_i Y_i\} &= E\{Y_i\} E\{X_i\} - \beta E\{X_i\}^2 + \beta E\left\{X_i^2\right\} \\
E\{X_i Y_i\} - E\{Y_i\} E\{X_i\} &= \beta \left(E\left\{X_i^2\right\} - E\{X_i\}^2\right) \\
\mathrm{Cov}(X_i, Y_i) &= \beta \mathrm{Var}(X_i) \\
\beta &= \frac{\mathrm{Cov}(X_i, Y_i)}{\mathrm{Var}(X_i)} = \frac{\sigma_{XY}}{\sigma_X^2}
\end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$
$y=1$	0.25	0.25
$y=2$	0	0.5

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$E\{X_i\} = (0.25)\,1 + (0.75)\,2 = 1.75$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned} E\{X_i\} &= (0.25)\,1 + (0.75)\,2 = 1.75 \\ E\left\{X_i^2\right\} &= (0.25)\,1 + (0.75)\,4 = 3.25 \end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned} E\{X_i\} &= (0.25)\,1 + (0.75)\,2 = 1.75 \\ E\left\{X_i^2\right\} &= (0.25)\,1 + (0.75)\,4 = 3.25 \\ E\{Y_i\} &= (0.5)\,1 + (0.5)\,2 = 1.5 \end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned}
E\{X_i\} &= (0.25)\,1 + (0.75)\,2 = 1.75 \\
E\left\{X_i^2\right\} &= (0.25)\,1 + (0.75)\,4 = 3.25 \\
E\{Y_i\} &= (0.5)\,1 + (0.5)\,2 = 1.5 \\
E\{X_i Y_i\} &= (0.25)(1)(1) + (0.25)(2)(1) \\
& \quad (0.00)(1)(2) + (0.50)(2)(2) \\
&= 2.75
\end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned} E\{X_i\} &= 1.75 \\ E\left\{X_i^2\right\} &= 3.25 \\ E\{Y_i\} &= 1.5 \\ E\{X_i Y_i\} &= 2.75 \end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned} E\{X_i\} &= 1.75 \\ E\left\{X_i^2\right\} &= 3.25 \\ E\{Y_i\} &= 1.5 \\ E\{X_i Y_i\} &= 2.75 \end{aligned}$$

$$\mathrm{Var}(X_i) = E\left\{X_i^2\right\} - E\{X_i\}^2$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned} E\{X_i\} &= 1.75 \\ E\left\{X_i^2\right\} &= 3.25 \\ E\{Y_i\} &= 1.5 \\ E\{X_i Y_i\} &= 2.75 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X_i) &= E\left\{X_i^2\right\} - E\{X_i\}^2 \\ &= 3.25 - (1.75)^2 = 0.1875 \end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned} E\{X_i\} &= 1.75 \\ E\left\{X_i^2\right\} &= 3.25 \\ E\{Y_i\} &= 1.5 \\ E\{X_i Y_i\} &= 2.75 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X_i) &= E\left\{X_i^2\right\} - E\{X_i\}^2 \\ &= 3.25 - (1.75)^2 = 0.1875 \end{aligned}$$

$$\mathrm{Cov}(X_i, Y_i) = E\{X_i Y_i\} - E\{X_i\} E\{Y_i\}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned} E\{X_i\} &= 1.75 \\ E\left\{X_i^2\right\} &= 3.25 \\ E\{Y_i\} &= 1.5 \\ E\{X_i Y_i\} &= 2.75 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X_i) &= E\left\{X_i^2\right\} - E\{X_i\}^2 \\ &= 3.25 - (1.75)^2 = 0.1875 \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}(X_i, Y_i) &= E\{X_i Y_i\} - E\{X_i\} E\{Y_i\} \\ &= 2.75 - (1.75)(1.5) = 0.125 \end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned}
E\{X_i\} &= 1.75 \\
E\left\{X_i^2\right\} &= 3.25 \\
E\{Y_i\} &= 1.5 \\
E\{X_i Y_i\} &= 2.75 \\
\mathrm{Cov}(X_i, Y_i) &= 0.125 \\
\mathrm{Var}(X_i) &= 0.1875
\end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned}
E\{X_i\} &= 1.75 \\
E\left\{X_i^2\right\} &= 3.25 \\
E\{Y_i\} &= 1.5 \\
E\{X_i Y_i\} &= 2.75 \\
\mathrm{Cov}(X_i, Y_i) &= 0.125 \\
\mathrm{Var}(X_i) &= 0.1875
\end{aligned}$$

$$\beta = \frac{\mathrm{Cov}(X_i, Y_i)}{\mathrm{Var}(X_i)} = \frac{0.125}{0.1875} = \frac{2}{3}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned}
E\{X_i\} &= 1.75 \\
E\left\{X_i^2\right\} &= 3.25 \\
E\{Y_i\} &= 1.5 \\
E\{X_i Y_i\} &= 2.75 \\
\mathrm{Cov}(X_i, Y_i) &= 0.125 \\
\mathrm{Var}(X_i) &= 0.1875
\end{aligned}$$

$$\beta = \frac{\mathrm{Cov}(X_i, Y_i)}{\mathrm{Var}(X_i)} = \frac{0.125}{0.1875} = \frac{2}{3}$$

$$\alpha = E\{Y_i\} - \beta E\{X_i\}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned}
E\{X_i\} &= 1.75 \\
E\left\{X_i^2\right\} &= 3.25 \\
E\{Y_i\} &= 1.5 \\
E\{X_i Y_i\} &= 2.75 \\
\text{Cov}(X_i, Y_i) &= 0.125 \\
\text{Var}(X_i) &= 0.1875
\end{aligned}$$

$$\beta = \frac{\text{Cov}(X_i, Y_i)}{\text{Var}(X_i)} = \frac{0.125}{0.1875} = \frac{2}{3}$$

$$\begin{aligned}
\alpha &= E\{Y_i\} - \beta E\{X_i\} \\
&= 1.5 - \left(\frac{2}{3}\right)(1.75)
\end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned} E\{X_i\} &= 1.75 \\ E\left\{X_i^2\right\} &= 3.25 \\ E\{Y_i\} &= 1.5 \\ E\{X_i Y_i\} &= 2.75 \\ \mathrm{Cov}(X_i, Y_i) &= 0.125 \\ \mathrm{Var}(X_i) &= 0.1875 \end{aligned}$$

$$\beta = \frac{\mathrm{Cov}(X_i, Y_i)}{\mathrm{Var}(X_i)} = \frac{0.125}{0.1875} = \frac{2}{3}$$

$$\begin{aligned} \alpha &= E\{Y_i\} - \beta E\{X_i\} \\ &= 1.5 - \left(\frac{2}{3}\right)(1.75) \\ &= \frac{3}{2} - \left(\frac{2}{3}\right)\left(\frac{7}{4}\right) \end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned} E\{X_i\} &= 1.75 \\ E\left\{X_i^2\right\} &= 3.25 \\ E\{Y_i\} &= 1.5 \\ E\{X_i Y_i\} &= 2.75 \\ \mathrm{Cov}(X_i, Y_i) &= 0.125 \\ \mathrm{Var}(X_i) &= 0.1875 \end{aligned}$$

$$\beta = \frac{\mathrm{Cov}(X_i, Y_i)}{\mathrm{Var}(X_i)} = \frac{0.125}{0.1875} = \frac{2}{3}$$

$$\begin{aligned} \alpha &= E\{Y_i\} - \beta E\{X_i\} \\ &= 1.5 - \left(\frac{2}{3}\right)(1.75) \\ &= \frac{3}{2} - \left(\frac{2}{3}\right)\left(\frac{7}{4}\right) \\ &= \frac{18}{12} - \frac{14}{12} \end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned}
E\{X_i\} &= 1.75 \\
E\left\{X_i^2\right\} &= 3.25 \\
E\{Y_i\} &= 1.5 \\
E\{X_i Y_i\} &= 2.75 \\
\mathrm{Cov}(X_i, Y_i) &= 0.125 \\
\mathrm{Var}(X_i) &= 0.1875
\end{aligned}$$

$$\beta = \frac{\mathrm{Cov}(X_i, Y_i)}{\mathrm{Var}(X_i)} = \frac{0.125}{0.1875} = \frac{2}{3}$$

$$\begin{aligned}
\alpha &= E\{Y_i\} - \beta E\{X_i\} \\
&= 1.5 - \left(\frac{2}{3}\right)(1.75) \\
&= \frac{3}{2} - \left(\frac{2}{3}\right)\left(\frac{7}{4}\right) \\
&= \frac{18}{12} - \frac{14}{12} = \frac{4}{12}
\end{aligned}$$

Example

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.25	0.25	0.5
$y=2$	0	0.5	0.5
$p_X(x)$	0.25	0.75	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$
$$E\{\varepsilon_i | X\} = 0$$

$$\begin{aligned}
E\{X_i\} &= 1.75 \\
E\left\{X_i^2\right\} &= 3.25 \\
E\{Y_i\} &= 1.5 \\
E\{X_i Y_i\} &= 2.75 \\
\text{Cov}(X_i, Y_i) &= 0.125 \\
\text{Var}(X_i) &= 0.1875
\end{aligned}$$

$$\beta = \frac{\text{Cov}(X_i, Y_i)}{\text{Var}(X_i)} = \frac{0.125}{0.1875} = \frac{2}{3}$$

$$\begin{aligned}
\alpha &= E\{Y_i\} - \beta E\{X_i\} \\
&= 1.5 - \left(\frac{2}{3}\right)(1.75) \\
&= \frac{3}{2} - \left(\frac{2}{3}\right)\left(\frac{7}{4}\right) \\
&= \frac{18}{12} - \frac{14}{12} = \frac{4}{12} = \frac{1}{3}
\end{aligned}$$

Question

$p_{XY}(x,y)$	$x=1$	$x=2$
$y=1$	0.1	0.2
$y=2$	0.1	0.6

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\left\{\varepsilon_i | X\right\} = 0$$

What is the value of β?

Question

$p_{XY}(x,y)$	$x=1$	$x=2$
$y=1$	0.1	0.2
$y=2$	0.1	0.6

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

What is the value of β?

Answer:

Question

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.1	0.2	0.3
$y=2$	0.1	0.6	0.7
$p_X(x)$	0.2	0.8	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

What is the value of β?

Answer:

Question

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.1	0.2	0.3
$y=2$	0.1	0.6	0.7
$p_X(x)$	0.2	0.8	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

What is the value of β? Answer:

$$E\{X_i\} = 1.8$$

Question

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.1	0.2	0.3
$y=2$	0.1	0.6	0.7
$p_X(x)$	0.2	0.8	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

What is the value of β? Answer:

$$\begin{aligned} E\{X_i\} &= 1.8 \\ E\left\{X_i^2\right\} &= 3.4 \end{aligned}$$

Question

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.1	0.2	0.3
$y=2$	0.1	0.6	0.7
$p_X(x)$	0.2	0.8	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

What is the value of β?

Answer:

$$\begin{aligned} E\{X_i\} &= 1.8 \\ E\left\{X_i^2\right\} &= 3.4 \\ E\{Y_i\} &= 1.7 \end{aligned}$$

Question

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.1	0.2	0.3
$y=2$	0.1	0.6	0.7
$p_X(x)$	0.2	0.8	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

What is the value of β?

Answer:

$$\begin{aligned} E\{X_i\} &= 1.8 \\ E\left\{X_i^2\right\} &= 3.4 \\ E\{Y_i\} &= 1.7 \\ E\{X_i Y_i\} &= 3.1 \end{aligned}$$

Question

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.1	0.2	0.3
$y=2$	0.1	0.6	0.7
$p_X(x)$	0.2	0.8	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

What is the value of β?

Answer:

$$\begin{aligned} E\{X_i\} &= 1.8 \\ E\left\{X_i^2\right\} &= 3.4 \\ E\{Y_i\} &= 1.7 \\ E\{X_i Y_i\} &= 3.1 \end{aligned}$$

$$\text{Var}(X_i) = E\left\{X_i^2\right\} - E\{X_i\}^2$$

Question

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.1	0.2	0.3
$y=2$	0.1	0.6	0.7
$p_X(x)$	0.2	0.8	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

What is the value of β?

Answer:

$$\begin{aligned} E\{X_i\} &= 1.8 \\ E\left\{X_i^2\right\} &= 3.4 \\ E\{Y_i\} &= 1.7 \\ E\{X_i Y_i\} &= 3.1 \end{aligned}$$

$$\begin{aligned} \text{Var}(X_i) &= E\left\{X_i^2\right\} - E\{X_i\}^2 \\ &= 3.4 - (1.8)^2 = 0.16 \end{aligned}$$

Question

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.1	0.2	0.3
$y=2$	0.1	0.6	0.7
$p_X(x)$	0.2	0.8	

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

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$$\mathrm{Cov}(X_i, Y_i) = E\{X_i Y_i\} - E\{X_i\} E\{Y_i\}$$

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$$\begin{aligned} \mathrm{Cov}(X_i, Y_i) &= E\{X_i Y_i\} - E\{X_i\} E\{Y_i\} \\ &= 3.1 - (1.8)(1.7) = 0.04 \end{aligned}$$

Question

$p_{XY}(x,y)$	$x=1$	$x=2$	$p_Y(y)$
$y=1$	0.1	0.2	0.3
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$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$E\{\varepsilon_i | X\} = 0$$

What is the value of β? Answer: 0.25

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$$\begin{aligned} \mathrm{Cov}(X_i, Y_i) &= E\{X_i Y_i\} - E\{X_i\} E\{Y_i\} \\ &= 3.1 - (1.8)(1.7) = 0.04 \end{aligned}$$

$$\beta = \frac{\mathrm{Cov}(X_i, Y_i)}{\mathrm{Var}(X_i)} = \frac{0.04}{0.16} = 0.25$$

Question

$p_{XY}(x,y)$	$x=1$	$x=2$
$y=1$	0.1	0.2
$y=2$	0.1	0.6

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What is the value of α?

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What is the value of α? Answer:

$$E\{X_i\} = 1.8$$

Question

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$$\begin{aligned} \beta &= 0.25 \\ \alpha &= E\{Y_i\} - \beta E\{X_i\} \\ &= 1.7 - (0.25)(1.8) = 1.25 \end{aligned}$$

Example

$$Y = \alpha + \beta X + \varepsilon$$

$$U_1, U_2, U_3 \sim i.i.d.\ N(0, 1)$$

$$\begin{aligned} X &= 3 + U_1 + 2U_2 \\ Y &= 4 + 2U_2 + U_3 \\ \varepsilon &= U_3 \end{aligned}$$

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Example

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$$\begin{aligned} \alpha &= E\{Y\} - \beta E\{X\} \\ &= 4 - (0.8)(3) \\ &= 4 - 2.4 = 1.6 \end{aligned}$$

Least Squares Estimation

- We can estimate α and β using the realizations x_i and y_i.

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- The **least squares estimates** $\hat{\alpha}$ and $\hat{\beta}$ are

$$\hat{\beta} = \frac{S_{XY}}{S_X}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$$

Consistency of Least Squares Estimation

$\hat{\alpha}$ and $\hat{\beta}$ are **consistent** estimators of α and β

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Consistency of Least Squares Estimation

$\hat{\alpha}$ and $\hat{\beta}$ are **consistent** estimators of α and β

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\hat{\beta} &= \frac{S_{XY}}{S_X^2} \xrightarrow{p} \frac{\sigma_{XY}}{\sigma_X^2} = \beta \\
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Example

i	1	2	3
x_i	1	3	5
y_i	3	5	6

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$$\begin{aligned} \bar{x} &= \frac{1}{3}(1+3+5) = 3 \\ \overline{x^2} &= \frac{1}{3}(1+9+25) = \frac{35}{3} \end{aligned}$$

Example

i	1	2	3
x_i	1	3	5
y_i	3	5	6

$$\begin{aligned}
\bar{x} &= \frac{1}{3}(1+3+5) = 3 \\
\overline{x^2} &= \frac{1}{3}(1+9+25) = \frac{35}{3} \\
\bar{y} &= \frac{1}{3}(3+5+6) = \frac{14}{3}
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\overline{xy} &= \frac{1}{3}(1\cdot 3+3\cdot 5+5\cdot 6) = 16
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S_{XY} &= \frac{3}{2}(\overline{xy} - \bar{y}\bar{x}) = 3
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\end{aligned}
$$

What is the value of $\hat{\beta}$?

Question

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$$
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\end{aligned}
$$

What is the value of $\hat{\beta}$?

Answer:

Question

i	1	2	3
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\begin{aligned}
\bar{x} &= \frac{1}{3}(1+3+5) = 3 \\
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\bar{y} &= \frac{1}{3}(3+5+6) = \frac{14}{3} \\
\overline{xy} &= \frac{1}{3}(1\cdot 3 + 3\cdot 5 + 5\cdot 6) = 16 \\
S_{XY} &= \frac{3}{2}(\overline{xy} - \bar{y}\bar{x}) = 3 \\
S_X^2 &= \frac{3}{2}\left(\overline{x^2} - \bar{x}^2\right) = 4
\end{aligned}
$$

What is the value of $\hat{\beta}$?

Answer: $\hat{\beta} = \dfrac{S_{XY}}{S_X^2} = \dfrac{3}{4} = 0.75$

Multiple Linear Regression

Daniel Stephenson

SCMA 524: Statistical Fundamentals

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  between a single dependent variable Y and
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- The **multiple linear model** is

$$Y_i = \beta_0 + \beta_1 X_{1i} + \cdots + \beta_k X_{ki} + \varepsilon_i$$

$$X_{11}, X_{12}, \ldots, X_{1n} \sim i.i.d.$$

$$\vdots$$

$$X_{k1}, X_{k2}, \ldots, X_{kn} \sim i.i.d.$$

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \sim i.i.d.$$

$$\varepsilon_i | X \sim N\left(0, \sigma_\varepsilon^2\right)$$

Least Squares Estimation

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- Let $\hat{\varepsilon}_i = y_i - \hat{y}_i$ denote the **residual prediction error**.

- The **least squares method** selects the estimates $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_k$

  that minimize the sum of squared residuals.

$$\left(\hat{\alpha}, \hat{\beta}_1, \ldots, \hat{\beta}_k\right) = \underset{a, b_1, \ldots, b_k}{\operatorname{argmin}} \sum_{i=1}^{n} \hat{\varepsilon}_i^2$$

Interpreting Regression Output: Coefficient Estimates

```
> LinearModel = lm(formula = Y ~ X1 + X2)
> summary(LinearModel)

Coefficients:
              Estimate  Std.  Error  t value  Pr(>|t|)
 (Intercept)    0.4879        0.1974    2.472   0.02004  *
 X1             0.4105        0.1803    2.277   0.03095  *
 X2             0.6304        0.1903    3.313   0.00264  **
---
Signif.  codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'  0.1
Residual standard error:  1.019 on 27 degrees of freedom
Multiple R-squared:  0.3709, Adjusted R-squared:  0.3243
F-statistic:  7.96 on 2 and 27 DF, p-value:  0.001916
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Interpreting Regression Output: Standard Errors

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The Standard Error of a Coefficient Estimate

- If $E\left\{\varepsilon_i|X\right\} = 0$ then $\hat{\beta}_j$ is an unbiased estimator for β_j

$$E\left\{\hat{\beta}_j\right\} = \beta_j$$

- The standard error $SE\left(\hat{\beta}_j\right)$ estimates the standard deviation of $\hat{\beta}_j$

- If $\varepsilon_i|X \sim N\left(0, \sigma_\varepsilon^2\right)$ then

$$\frac{\hat{\beta}_j - \beta_j}{SE\left(\hat{\beta}_j\right)} \sim t\left(n - k - 1\right)$$

- n is the number of observations

  k is the number of independent variables.

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_3 X_{3i} + \beta_3 X_{3i} + \varepsilon_i \qquad \text{for } i = 1, \ldots, 10$$

$$\varepsilon_i | X \sim N\left(0, \sigma^2\right)$$

$$\hat{\beta}_2 = 2.7529 \text{ and } SE\left(\hat{\beta}_2\right) = 0.14$$

$$H_0 : \beta_2 = 3$$

Can we reject H_0 at the 5% significance level?

$$H_1 : \beta_2 \neq 3$$

$P\left(t\left(6\right) < c\right)$	c
0.933	1.7319
0.934	1.7428
0.935	1.7538
0.936	1.7650
0.937	1.7764
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&= 2\left(1 - P\left(t\left(6\right) < 1.765\right)\right) \\
&\approx 2\left(1 - 0.936\right) = 0.128
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We can not reject H_0 at the 5% level.

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad \text{for } i = 1, \ldots, 8$$

$H_0 : \beta_1 = 0$ $\qquad$ $\hat{\beta}_1 = 9.5429$ and $SE\left(\hat{\beta}_1\right) = 3.712$

$H_1 : \beta_1 \neq 0$ $\qquad$ What is the p-value?

Table gives the value of c

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$$\text{p-value} = P\left(|t(5)| > 2.5708\right)$$

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$$\begin{aligned}\text{p-value} &= P(|t(5)| > 2.5708) = 2P(t(5) > 2.5708) \\ &= 2[1 - P(t(5) < 2.5708)]\end{aligned}$$

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$$\begin{aligned} \text{p-value} &= P\left(\left|t\left(5\right)\right| > 2.5708\right) = 2P\left(t\left(5\right) > 2.5708\right) \\ &= 2\left[1 - P\left(t\left(5\right) < 2.5708\right)\right] \approx 2\left[1 - 0.975\right] \end{aligned}$$

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad \text{for } i = 1, \ldots, 8$$

$H_0 : \beta_1 = 0$ $\qquad$ $\hat{\beta}_1 = 9.5429$ and $SE\left(\hat{\beta}_1\right) = 3.712$

$H_1 : \beta_1 \neq 0$ $\qquad$ What is the p-value?

Table gives the value of c

$P(t(m) < c)$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$
0.973	3.0832	2.7017	2.5074	2.3904	2.3123	2.2567
0.974	3.1316	2.7382	2.5383	2.4181	2.3380	2.2809
0.975	3.1824	2.7764	2.5706	2.4469	2.3646	2.3060
0.976	3.2359	2.8165	2.6043	2.4770	2.3924	2.3322

$$\text{t-statistic} = \frac{\hat{\beta}_1 - 0}{SE\left(\hat{\beta}_1\right)} = \frac{9.5429}{3.712} \approx 2.5708 \qquad n - k - 1 = 8 - 2 - 1 = 5$$

$$\begin{aligned} \text{p-value} &= P(|t(5)| > 2.5708) = 2P(t(5) > 2.5708) \\ &= 2[1 - P(t(5) < 2.5708)] \approx 2[1 - 0.975] = 2(0.025) = 0.05 \end{aligned}$$

Interpreting Regression Output: t-statistic

```
> LinearModel = lm(formula = Y ~ X1 + X2)
> summary(LinearModel)

Coefficients:
              Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)    0.4879      0.1974    2.472   0.02004  *
 X1             0.4105      0.1803    2.277   0.03095  *
 X2             0.6304      0.1903    3.313   0.00264  **
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
Residual standard error: 1.019 on 27 degrees of freedom
Multiple R-squared: 0.3709, Adjusted R-squared: 0.3243
F-statistic: 7.96 on 2 and 27 DF, p-value: 0.001916
```

Interpreting Regression Output: p-value

```
> LinearModel = lm(formula = Y ~ X1 + X2)
> summary(LinearModel)

Coefficients:
              Estimate  Std.  Error  t value  Pr(>|t|)
 (Intercept)    0.4879       0.1974    2.472   0.02004   *
 X1             0.4105       0.1803    2.277   0.03095   *
 X2             0.6304       0.1903    3.313   0.00264   **
---
Signif.  codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'  0.1
Residual standard error:  1.019 on 27 degrees of freedom
Multiple R-squared:  0.3709, Adjusted R-squared:  0.3243
F-statistic:  7.96 on 2 and 27 DF, p-value:  0.001916
```

Interpreting Regression Output: Significance Codes

```
> LinearModel = lm(formula = Y ~ X1 + X2)
> summary(LinearModel)

Coefficients:
              Estimate  Std.  Error  t value  Pr(>|t|)
 (Intercept)   0.4879       0.1974    2.472   0.02004  *
 X1            0.4105       0.1803    2.277   0.03095  *
 X2            0.6304       0.1903    3.313   0.00264  **
---
Signif.  codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’  0.1
Residual standard error:  1.019 on 27 degrees of freedom
Multiple R-squared:  0.3709, Adjusted R-squared:  0.3243
F-statistic:  7.96 on 2 and 27 DF, p-value:  0.001916
```

The Residual Standard Error

- The **sum of squared residuals** is given by

$$SSR = \sum_{i=1}^{n} \hat{\varepsilon}_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

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$$E\left\{MSR\right\} = E\left\{\hat{\sigma}_\varepsilon^2\right\} = \sigma_\varepsilon^2$$

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- The **residual standard error** is given by

$$RSE = \hat{\sigma}_\varepsilon = \sqrt{\hat{\sigma}_\varepsilon^2} = \sqrt{MSR}$$

Example

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$

$\hat{\beta}_0 = -0.7707 \qquad \hat{\beta}_1 = 0.1051 \qquad \hat{\beta}_2 = -0.7139$

i	y_i	x_{1i}	x_{2i}
1	-1.0	0.5	-0.5
2	-1.2	-0.7	0.4
3	-0.8	-0.3	-0.4
4	0.4	0.6	-0.3
5	0.0	-1.0	-1.0

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$$

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i	y_i	x_{1i}	x_{2i}	$\hat{y}_i$
1	-1.0	0.5	-0.5	-0.3612
2	-1.2	-0.7	0.4	
3	-0.8	-0.3	-0.4	
4	0.4	0.6	-0.3	
5	0.0	-1.0	-1.0	

$$\hat{y}_1 \quad = \quad \hat{\beta}_0 + \hat{\beta}_1 x_{11} + \beta_2 x_{21} \approx -0.3612$$

Example

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$

$\hat{\beta}_0 = -0.7707 \qquad \hat{\beta}_1 = 0.1051 \qquad \hat{\beta}_2 = -0.7139$

i	y_i	x_{1i}	x_{2i}	$\hat{y}_i$	$\hat{\varepsilon}_i$
1	-1.0	0.5	-0.5	-0.3612	-0.6388
2	-1.2	-0.7	0.4		
3	-0.8	-0.3	-0.4		
4	0.4	0.6	-0.3		
5	0.0	-1.0	-1.0		

$$\begin{aligned} \hat{y}_1 &= \hat{\beta}_0 + \hat{\beta}_1 x_{11} + \beta_2 x_{21} \approx -0.3612 \\ \hat{\varepsilon}_1 &= y_1 - \hat{y}_1 \approx -0.6388 \end{aligned}$$

Example

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$

$\hat{\beta}_0 = -0.7707 \qquad \hat{\beta}_1 = 0.1051 \qquad \hat{\beta}_2 = -0.7139$

i	y_i	x_{1i}	x_{2i}	$\hat{y}_i$	$\hat{\varepsilon}_i$	$\hat{\varepsilon}_i^2$
1	-1.0	0.5	-0.5	-0.3612	-0.6388	0.4081
2	-1.2	-0.7	0.4			
3	-0.8	-0.3	-0.4			
4	0.4	0.6	-0.3			
5	0.0	-1.0	-1.0			

$$\begin{aligned} \hat{y}_1 &= \hat{\beta}_0 + \hat{\beta}_1 x_{11} + \beta_2 x_{21} \approx -0.3612 \\ \hat{\varepsilon}_1 &= y_1 - \hat{y}_1 \approx -0.6388 \\ \hat{\varepsilon}_1^2 &\approx (-0.6388)^2 \approx 0.4081 \end{aligned}$$

Question

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$$

$$\hat{\beta}_0 = -0.7707 \qquad \hat{\beta}_1 = 0.1051 \qquad \hat{\beta}_2 = -0.7139$$

i	y_i	x_{1i}	x_{2i}	$\hat{y}_i$	$\hat{\varepsilon}_i$	$\hat{\varepsilon}_i^2$
1	-1.0	0.5	-0.5	-0.3612	-0.6388	0.4081
2	-1.2	-0.7	0.4			
3	-0.8	-0.3	-0.4			
4	0.4	0.6	-0.3			
5	0.0	-1.0	-1.0			

What is the value of $\hat{\varepsilon}_3^2$?

Question

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$$

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i	y_i	x_{1i}	x_{2i}	$\hat{y}_i$	$\hat{\varepsilon}_i$	$\hat{\varepsilon}_i^2$
1	-1.0	0.5	-0.5	-0.3612	-0.6388	0.4081
2	-1.2	-0.7	0.4			
3	-0.8	-0.3	-0.4			
4	0.4	0.6	-0.3			
5	0.0	-1.0	-1.0			

What is the value of $\hat{\varepsilon}_3^2$? Answer:

Question

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \dots, 5$

$\hat{\beta}_0 = -0.7707 \qquad \hat{\beta}_1 = 0.1051 \qquad \hat{\beta}_2 = -0.7139$

i	y_i	x_{1i}	x_{2i}	$\hat{y}_i$	$\hat{\varepsilon}_i$	$\hat{\varepsilon}_i^2$
1	-1.0	0.5	-0.5	-0.3612	-0.6388	0.4081
2	-1.2	-0.7	0.4			
3	-0.8	-0.3	-0.4	-0.4536		
4	0.4	0.6	-0.3			
5	0.0	-1.0	-1.0			

What is the value of $\hat{\varepsilon}_3^2$? Answer:

$$\hat{y}_3 = \hat{\beta}_0 + \hat{\beta}_1 x_{13} + \beta_2 x_{23} \approx -0.4536$$

Question

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$

$\hat{\beta}_0 = -0.7707 \qquad \hat{\beta}_1 = 0.1051 \qquad \hat{\beta}_2 = -0.7139$

i	y_i	x_{1i}	x_{2i}	$\hat{y}_i$	$\hat{\varepsilon}_i$	$\hat{\varepsilon}_i^2$
1	-1.0	0.5	-0.5	-0.3612	-0.6388	0.4081
2	-1.2	-0.7	0.4			
3	-0.8	-0.3	-0.4	-0.4536	-0.3464	
4	0.4	0.6	-0.3			
5	0.0	-1.0	-1.0			

What is the value of $\hat{\varepsilon}_3^2$? Answer:

$$\begin{aligned}
\hat{y}_3 &= \hat{\beta}_0 + \hat{\beta}_1 x_{13} + \beta_2 x_{23} \approx -0.4536 \\
\hat{\varepsilon}_3 &= y_3 - \hat{y}_3 \approx -0.3464
\end{aligned}$$

Question

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$

$\hat{\beta}_0 = -0.7707 \qquad \hat{\beta}_1 = 0.1051 \qquad \hat{\beta}_2 = -0.7139$

i	y_i	x_{1i}	x_{2i}	$\hat{y}_i$	$\hat{\varepsilon}_i$	$\hat{\varepsilon}_i^2$
1	−1.0	0.5	−0.5	−0.3612	−0.6388	0.4081
2	−1.2	−0.7	0.4			
3	−0.8	−0.3	−0.4	−0.4536	−0.3464	0.1199
4	0.4	0.6	−0.3			
5	0.0	−1.0	−1.0			

What is the value of $\hat{\varepsilon}_3^2$? Answer: 0.1199

$$\begin{aligned}
\hat{y}_3 &= \hat{\beta}_0 + \hat{\beta}_1 x_{13} + \beta_2 x_{23} \approx -0.4536 \\
\hat{\varepsilon}_3 &= y_3 - \hat{y}_3 \approx -0.3464 \\
\hat{\varepsilon}_3^2 &\approx (-0.3464)^2 \approx 0.1199
\end{aligned}$$

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$$

$$\hat{\beta}_0 = -0.7707 \qquad \hat{\beta}_1 = 0.1051 \qquad \hat{\beta}_2 = -0.7139$$

i	y_i	x_{1i}	x_{2i}	$\hat{y}_i$	$\hat{\varepsilon}_i$	$\hat{\varepsilon}_i^2$
1	−1.0	0.5	−0.5	−0.3612	−0.6388	0.4081
2	−1.2	−0.7	0.4	−1.1299	−0.0701	0.0049
3	−0.8	−0.3	−0.4	−0.4536	−0.3464	0.1199
4	0.4	0.6	−0.3	−0.4935	0.8935	0.7983
5	0.0	−1.0	−1.0	−0.1619	0.1619	0.0262

$$SSR = \sum_{i=1}^{5} \hat{\varepsilon}_i^2 \quad = \quad (0.4081 + 0.0049 + 0.1199 + 0.7983 + 0.0262) = 1.3575$$

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$$

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1	-1.0	0.5	-0.5	-0.3612	-0.6388	0.4081
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$$SSR = \sum_{i=1}^{5} \hat{\varepsilon}_i^2 \quad = \quad (0.4081 + 0.0049 + 0.1199 + 0.7983 + 0.0262) = 1.3575$$

$$MSR = \hat{\sigma}_\varepsilon^2 \quad = \quad \frac{SSR}{n-k-1} = \frac{SSR}{5-2-1} = \frac{1.297}{2} \approx 0.6788$$

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$$

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$$MSR = \hat{\sigma}_\varepsilon^2 \quad = \quad \frac{SSR}{n-k-1} = \frac{SSR}{5-2-1} = \frac{1.297}{2} \approx 0.6788$$

$$RSE = \hat{\sigma}_\varepsilon \quad = \quad \sqrt{\hat{\sigma}_\varepsilon^2} \approx 0.8239$$

Interpreting Regression Output: Residual Standard Error

```
> LinearModel = lm(formula = Y ~ X1 + X2)
> summary(LinearModel)

Coefficients:
             Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)  -0.7707      0.4724   -1.632     0.244
 X1            0.1051      0.5595    0.188     0.868
 X2           -0.7139      0.8197   -0.871     0.476
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’  0.1
Residual standard error:  0.8239 on 2 degrees of freedom
Multiple R-squared:  0.281, Adjusted R-squared:  -0.438
F-statistic:  0.3908 on 2 and 2 DF, p-value:  0.719
```

Example

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i \qquad i = 1, \ldots, 15$

The sum of squared residuals is 6.1875.

What is the residual standard error?

Example

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i \qquad i = 1, \ldots, 15$

The sum of squared residuals is 6.1875.

What is the residual standard error?

$$MSR = \hat{\sigma}_\varepsilon^2$$

Example

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i \qquad i = 1, \ldots, 15$

The sum of squared residuals is 6.1875.

What is the residual standard error?

$$MSR = \hat{\sigma}_\varepsilon^2 \quad = \quad \frac{SSR}{n - k - 1}$$

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i \qquad i = 1, \ldots, 15$$

The sum of squared residuals is 6.1875.

What is the residual standard error?

$$MSR = \hat{\sigma}_\varepsilon^2 \quad = \quad \frac{SSR}{n-k-1} = \frac{6.1875}{15-3-1}$$

Example

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i \qquad i = 1, \ldots, 15$

The sum of squared residuals is 6.1875.

What is the residual standard error?

$$MSR = \hat{\sigma}_\varepsilon^2 \quad = \quad \frac{SSR}{n-k-1} = \frac{6.1875}{15-3-1} = 0.5625$$

Example

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i \qquad i = 1, \ldots, 15$

The sum of squared residuals is 6.1875.

What is the residual standard error?

$$MSR = \hat{\sigma}_\varepsilon^2 \quad = \quad \frac{SSR}{n-k-1} = \frac{6.1875}{15-3-1} = 0.5625$$

$$\text{residual standard error} = \hat{\sigma}_\varepsilon$$

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i \qquad i = 1, \ldots, 15$$

The sum of squared residuals is 6.1875.

What is the residual standard error?

$$MSR = \hat{\sigma}_\varepsilon^2 \quad = \quad \frac{SSR}{n-k-1} = \frac{6.1875}{15-3-1} = 0.5625$$

$$\text{residual standard error} = \hat{\sigma}_\varepsilon = \sqrt{\hat{\sigma}_\varepsilon^2}$$

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i \qquad i = 1, \ldots, 15$$

The sum of squared residuals is 6.1875.

What is the residual standard error?

$$MSR = \hat{\sigma}_\varepsilon^2 \quad = \quad \frac{SSR}{n-k-1} = \frac{6.1875}{15-3-1} = 0.5625$$

$$\text{residual standard error} = \hat{\sigma}_\varepsilon = \sqrt{\hat{\sigma}_\varepsilon^2} = 0.75$$

Example

```
Coefficients:
             Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)  -2.7136      6.3528   -0.427   0.68211
 X1            2.3875      0.8265    2.889   0.02336  *
 X2            2.2226      0.5376    4.134   0.00438  **
---
Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1
Residual standard error: 1.203 on 7 degrees of freedom
Multiple R-squared: 0.7165, Adjusted R-squared: 0.6354
F-statistic: 8.844 on 2 and 7 DF, p-value: 0.012147
```

What is the sum of squared residuals?

Example

```
Coefficients:
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```

What is the sum of squared residuals?

$$1.203 = \hat{\sigma}_{\varepsilon}$$

Example

```
Coefficients:
             Estimate  Std.  Error  t value  Pr(>|t|)
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Residual standard error:  1.203 on 7 degrees of freedom
Multiple R-squared:  0.7165, Adjusted R-squared:  0.6354
F-statistic:  8.844 on 2 and 7 DF, p-value:  0.012147
```

What is the sum of squared residuals?

$$1.203 = \hat{\sigma}_\varepsilon \quad = \quad \sqrt{MSR}$$

Example

```
Coefficients:
             Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)  -2.7136      6.3528   -0.427   0.68211
 X1            2.3875      0.8265    2.889   0.02336  *
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Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'  0.1
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Multiple R-squared:  0.7165, Adjusted R-squared:  0.6354
F-statistic:  8.844 on 2 and 7 DF, p-value:  0.012147
```

What is the sum of squared residuals?

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Example

```
Coefficients:
              Estimate  Std.  Error  t value  Pr(>|t|)
 (Intercept)   -2.7136       6.3528   -0.427   0.68211
 X1             2.3875       0.8265    2.889   0.02336  *
 X2             2.2226       0.5376    4.134   0.00438  **
---
Signif.  codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'  0.1
Residual standard error:  1.203 on 7 degrees of freedom
Multiple R-squared:  0.7165, Adjusted R-squared:  0.6354
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```

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(1.203)^2 &= \frac{SSR}{7}
\end{aligned}
$$

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SSR &= 7\,(1.203)^2 \approx 10.1305
\end{aligned}
$$

The Coefficient of Determination

- The **sum of squares total** is

$$SST = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

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$$SSE = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$$

- The **coefficient of determination** is

$$R^2 = \frac{SSE}{SST}$$

The coefficient of determination describes the proportion of variation in the dependent variable y_i explained by variation in the independent variables $x_{1i}, \ldots, x_{ik}$.

Example

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$

$\hat{\beta}_0 = -0.7707 \qquad \hat{\beta}_1 = 0.1051 \qquad \hat{\beta}_2 = -0.7139$

y_i	x_{1i}	x_{i2}	$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \beta_2 x_{i2}$
-1.0	0.5	-0.5	-0.3612
-1.2	-0.7	0.4	-1.1299
-0.8	-0.3	-0.4	-0.4536
0.4	0.6	-0.3	-0.4935
0.0	-1.0	-1.0	-0.1619

$\bar{y} = -0.52$

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad i = 1, \ldots, 5$$

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$$SST = \sum_{i=1}^{n} (y_i - \bar{y})^2 = 1.888$$

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$$SST = \sum_{i=1}^{n} (y_i - \bar{y})^2 = 1.888 \qquad SSE = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 \approx 0.5305$$

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$$SST = \sum_{i=1}^{n} (y_i - \bar{y})^2 = 1.888 \qquad SSE = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 \approx 0.5305$$

$$R^2 = \frac{SSE}{SST} \approx \frac{0.5909}{1.297} \approx 0.281$$

Interpreting Regression Output: R^2

```
> LinearModel = lm(formula = Y ~ X1 + X2)
> summary(LinearModel)

Coefficients:
              Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)   -0.7707      0.4724   -1.632     0.244
 X1             0.1051      0.5595    0.188     0.868
 X2            -0.7139      0.8197   -0.871     0.476
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
Residual standard error: 0.8239 on 2 degrees of freedom
Multiple R-squared: 0.281, Adjusted R-squared: -0.438
F-statistic: 0.3908 on 2 and 2 DF, p-value: 0.719
```

The F-statistic

- The F-statistic is given by

$$\text{F-statisitic} = \frac{(\text{Sum of Squared Explained})\,/k}{(\text{Sum of Squares Residuals})\,/(n-k-1)}$$

- The null hypothesis of no explanatory power is

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_k = 0$$

- If the null hypothesis of no explanatory power is true, then

$$\text{F-statistic} \sim F\,(k, n-k-1)$$

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + \varepsilon_i \qquad i = 1, \ldots, 12$$

Sum of Squares Explained $= 11.9211$

Sum of Squared Residuals $= 5.4565$

$H_0 : \beta_1 = \beta_2 = \cdots = \beta_k = 0$

H_1 : at least one $\beta_m \neq 0$

$P(F(4,7) < c)$	c
0.938	3.7369
0.939	3.7651
0.940	3.7939
0.941	3.8233
0.942	3.8533

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$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + \varepsilon_i \qquad i = 1, \ldots, 12$$

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H_1 : at least one $\beta_m \neq 0$

$$\text{F-statistic} \quad = \quad \frac{SSE/k}{SSR/(n-k-1)} = \frac{11.9911/4}{5.4565/7} \approx 3.8233$$

$P\left(F\left(4,7\right) < c\right)$	c
0.938	3.7369
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p-value $\approx$ $P(F(4,7) > 3.8233)$

Example

$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + \varepsilon_i \qquad i = 1, \ldots, 12$

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$$\begin{aligned} \text{p-value} \quad &\approx \quad P(F(4,7) > 3.8233) \\ &= \quad 1 - P(F(4,7) < 3.8233) \end{aligned}$$

Example

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Sum of Squares Explained $= 11.9211$

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$$\text{F-statistic} = \frac{SSE/k}{SSR/(n-k-1)} = \frac{11.9911/4}{5.4565/7} \approx 3.8233$$

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$$\begin{aligned} \text{p-value} &\approx P(F(4,7) > 3.8233) \\ &= 1 - P(F(4,7) < 3.8233) \\ &\approx 1 - 0.941 \end{aligned}$$

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$$\begin{aligned} \text{p-value} \quad &\approx \quad P(F(4,7) > 3.8233) \\ &= \quad 1 - P(F(4,7) < 3.8233) \\ &\approx \quad 1 - 0.941 \\ &= \quad 0.059 \end{aligned}$$

Interpreting Regression Output: F-statistic

```
> LinearModel = lm(formula = Y ~ X1 + X2)
> summary(LinearModel)

Coefficients:
              Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)   0.4879     0.1974     2.472   0.02004   *
 X1            0.4105     0.1803     2.277   0.03095   *
 X2            0.6304     0.1903     3.313   0.00264   **
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
Residual standard error: 1.019 on 27 degrees of freedom
Multiple R-squared: 0.3709, Adjusted R-squared: 0.3243
F-statistic: 7.96 on 2 and 27 DF, p-value: 0.001916
```

Question

```
              Estimate  Std. Error  t value  Pr(>|t|)
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What is the sum of squared residuals?   Answer:

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What is the sum of squared residuals? Answer:

$$1.019 = \hat{\sigma}_{\varepsilon}$$

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```

What is the sum of squared residuals?   Answer:

$$1.019 = \hat{\sigma}_\varepsilon \quad = \quad \sqrt{MSR}$$

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```
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 (Intercept)   0.4879      0.1974    2.472   0.02004  *
 X1            0.4105      0.1803    2.277   0.03095  *
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```

What is the sum of squared residuals? Answer:

$$1.019 = \hat{\sigma}_\varepsilon \quad = \quad \sqrt{MSR} = \sqrt{\frac{SSR}{n-k-1}}$$

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$$1.019 = \hat{\sigma}_\varepsilon \quad = \quad \sqrt{MSR} = \sqrt{\frac{SSR}{n-k-1}} = \sqrt{\frac{SSR}{27}}$$

Question

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What is the sum of squared residuals?   Answer:

$$1.019 = \hat{\sigma}_\varepsilon = \sqrt{MSR} = \sqrt{\frac{SSR}{n-k-1}} = \sqrt{\frac{SSR}{27}}$$

$$(1.019)^2 = \frac{SSR}{27}$$

Question

```
               Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)     0.4879      0.1974    2.472   0.02004  *
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```

What is the sum of squared residuals? Answer: 28.0358

$$
\begin{aligned}
1.019 = \hat{\sigma}_\varepsilon &= \sqrt{MSR} = \sqrt{\frac{SSR}{n-k-1}} = \sqrt{\frac{SSR}{27}} \\
(1.019)^2 &= \frac{SSR}{27} \\
SSR &= 27\,(1.019)^2 \approx 28.0358
\end{aligned}
$$

Example

```
                Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)      0.4879      0.1974    2.472   0.02004  *
 X1               0.4105      0.1803    2.277   0.03095  *
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Multiple R-squared:  0.3709, Adjusted R-squared:  0.3243
F-statistic:  7.96 on 2 and 27 DF, p-value:  0.001916
```

What is the sum of squares explained?

Example

```
              Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)    0.4879      0.1974    2.472   0.02004  *
 X1             0.4105      0.1803    2.277   0.03095  *
 X2             0.6304      0.1903    3.313   0.00264  **
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Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
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```

What is the sum of squares explained?

$$7.96 = \text{F-statistic}$$

Example

```
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 (Intercept)     0.4879        0.1974    2.472   0.02004  *
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Multiple R-squared:  0.3709, Adjusted R-squared:  0.3243
F-statistic:  7.96 on 2 and 27 DF, p-value:  0.001916
```

What is the sum of squares explained?

$$7.96 = \text{F-statistic} \quad = \quad \frac{SSE/k}{SSR/(n-k-1)}$$

Example

```
               Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)     0.4879      0.1974    2.472   0.02004  *
 X1              0.4105      0.1803    2.277   0.03095  *
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Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’  0.1
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Multiple R-squared:  0.3709, Adjusted R-squared:  0.3243
F-statistic:  7.96 on 2 and 27 DF, p-value:  0.001916
```

What is the sum of squares explained?

$$7.96 = \text{F-statistic} \quad = \quad \frac{SSE/k}{SSR/(n-k-1)} = \frac{SSE/2}{28.0358/27}$$

Example

```
               Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)     0.4879      0.1974    2.472   0.02004  *
 X1              0.4105      0.1803    2.277   0.03095  *
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```

What is the sum of squares explained?

$$\begin{aligned} 7.96 = \text{F-statistic} &= \frac{SSE/k}{SSR/(n-k-1)} = \frac{SSE/2}{28.0358/27} \\ SSE &= (7.96)\,(2)\left(\frac{28.0358}{27}\right) \end{aligned}$$

Example

```
              Estimate  Std. Error  t value  Pr(>|t|)
 (Intercept)    0.4879      0.1974    2.472   0.02004  *
 X1             0.4105      0.1803    2.277   0.03095  *
 X2             0.6304      0.1903    3.313   0.00264  **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’  0.1
Residual standard error:  1.019 on 27 degrees of freedom
Multiple R-squared:  0.3709, Adjusted R-squared:  0.3243
F-statistic:  7.96 on 2 and 27 DF, p-value:  0.001916
```

What is the sum of squares explained?

$$\begin{aligned} 7.96 = \text{F-statistic} &= \frac{SSE/k}{SSR/(n-k-1)} = \frac{SSE/2}{28.0358/27} \\ SSE &= (7.96)\,(2)\left(\frac{28.0358}{27}\right) \approx 16.5307 \end{aligned}$$