ARTICULATED ROBOTS

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6. DIFFERENTIAL KINEMATICS II

$$J = \frac{\partial T(\mathbf{q})}{\partial \mathbf{q}} \qquad \dot{X} = J\dot{\mathbf{q}}$$

6.3 Jacobian Matrix

- ✓ Differential kinematics gives the mapping between the joint velocities and the corresponding end-effector's linear and angular velocity, which is described by a matrix, termed geometric Jacobian that depends on the manipulator configuration.
- ✓ The Jacobian constitutes one of the most important tools for manipulator characterization:
  - Singularities
  - Redundancy
  - Inverse kinematics algorithms
  - Mapping between forces applied to the end-effector and resulting torques at the joints (statics)

✓ According forward kinematics

$$X = \left[x, y, z, \alpha, \beta, \gamma\right]^T = T(\mathbf{q}) = \begin{bmatrix} T_1(q_1, q_2, \cdots, q_n) \\ \cdots \\ T_6(q_1, q_2, \cdots, q_n) \end{bmatrix}_{6\times n} \Longrightarrow \dot{X}_{6\times1} = \frac{d}{dt}T(\mathbf{q}_{n\times1}) = \frac{\partial T(\mathbf{q})}{\partial \mathbf{q}}\frac{d\mathbf{q}}{dt} = \frac{\partial T(\mathbf{q})}{\partial \mathbf{q}}\dot{\mathbf{q}}$$

$$\Longrightarrow \boxed{\dot{X}_{6\times1} = \frac{\partial T(\mathbf{q})}{\partial \mathbf{q}}\dot{\mathbf{q}} = J\dot{\mathbf{q}}}$$

where *J* is called the **Jacobian Matrix**, and

$$J = \frac{\partial \mathbf{T}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial T_1}{\partial \mathbf{q}} \\ \frac{\partial T_2}{\partial \mathbf{q}} \\ \vdots \\ \frac{\partial T_6}{\partial \mathbf{q}} \end{bmatrix} = \begin{bmatrix} \frac{\partial T_1}{\partial q_1} & \frac{\partial T_1}{\partial q_2} & \cdots & \frac{\partial T_1}{\partial q_n} \\ \frac{\partial T_2}{\partial q_1} & \frac{\partial T_2}{\partial q_2} & \cdots & \frac{\partial T_2}{\partial q_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial T_6}{\partial q_1} & \frac{\partial T_6}{\partial q_2} & \cdots & \frac{\partial T_6}{\partial q_n} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{1n} \\ J_{21} & J_{22} & \cdots & J_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ J_{61} & J_{62} & \cdots & J_{6n} \end{bmatrix} = \begin{bmatrix} J_1 & \cdots & J_n \end{bmatrix}$$

✓ Jacobian matrix maps joint velocities in joint space into the global velocities in Cartesian space

$$\dot{X}_{6\times1} = J\dot{\mathbf{q}} = \begin{bmatrix} J_1 & \cdots & J_n \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \cdots \\ \dot{q}_n \end{bmatrix} = \sum_{i=1}^{n} J_i\dot{q}_i = \begin{bmatrix} J_{11}\dot{q}_1 + J_{12}\dot{q}_2 + \cdots + J_{1n}\dot{q}_n \\ \cdots \quad \cdots \quad \cdots \\ J_{61}\dot{q}_1 + J_{62}\dot{q}_2 + \cdots + J_{6n}\dot{q}_n \end{bmatrix}$$

Notes on Jacobian Matrix

- ✓ **Jacobian matrix for a manipulator is not unique.** The mapping from the joint velocities to the corresponding end-effector linear and angular velocities is given by **Geometrical Jacobian**.
- ✓ The Geometrical Jacobian can be computed from different ways including using the geometric methods or the definition of Jacobian matrix if a straight forward transformation is available.
- ✓ If the end-effector pose is expressed with reference to a minimal representation in the operational space, it is possible to compute the Jacobian matrix via differentiation of the direct kinematics function with respect to the joint variables. The resulting Jacobian, termed **analytical Jacobian**, in general differs from the geometric one.

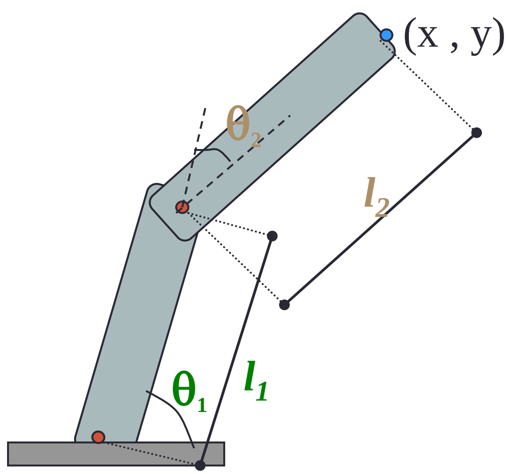
Notes on Jacobian Matrix

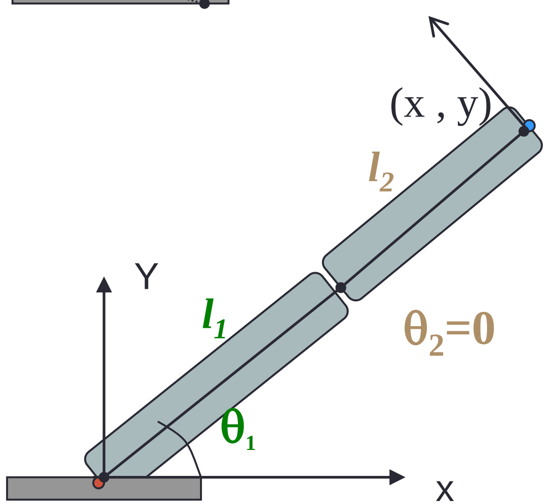
- ✓ Jacobian matrix depends on the frame in which the end-effector velocity is expressed. If it is desired to represent the Jacobian in a different Frame *u*, it is sufficient to know the relative rotation matrix *U*

$$\begin{aligned} {}^{u}\mathbf{v} &= U\mathbf{v} \\ {}^{u}\boldsymbol{\omega} &= U\boldsymbol{\omega} \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} {}^{u}\mathbf{v} \\ {}^{u}\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & U \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & U \end{bmatrix} J\dot{\mathbf{q}} \quad \Rightarrow \quad {}^{u}J = \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & U \end{bmatrix} J$$

EX 6-3-1

Find the linear velocity of the end point of the 2R Planar Arm





$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix} = \begin{bmatrix} T_1(\theta_1, \theta_2) \\ T_2(\theta_1, \theta_2) \end{bmatrix}$$

$$J = \begin{bmatrix} \dfrac{\partial T_1}{\partial \theta_1} & \dfrac{\partial T_1}{\partial \theta_2} \\ \dfrac{\partial T_2}{\partial \theta_1} & \dfrac{\partial T_2}{\partial \theta_2} \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Geometric Jacobian

Definition of the geometric Jacobian matrix implies the decoupling of the linear velocity and angular velocity in the operational space.

$$\dot{X} = \begin{bmatrix} \mathbf{v} \\ \text{----} \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{p}} \\ \text{----} \\ \boldsymbol{\omega} \end{bmatrix} = J\dot{\mathbf{q}} = \begin{bmatrix} J_D \\ \text{----} \\ J_R \end{bmatrix} \dot{\mathbf{q}} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = J_D \dot{\mathbf{q}} \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} = J_R \dot{\mathbf{q}}$$

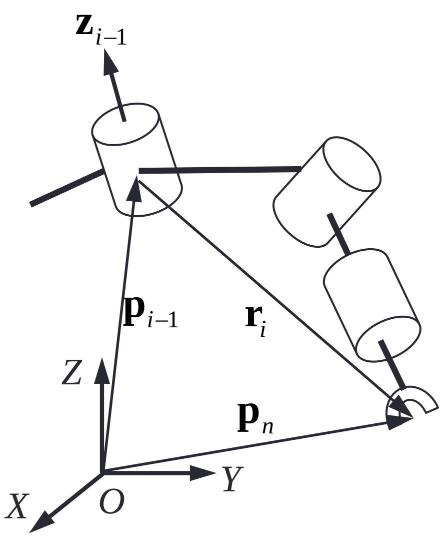
$$J_D(\mathbf{q}) = \begin{bmatrix} J_{D1} & \dots & J_{Dn} \end{bmatrix}^T$$ - **Displacement Jacobian matrix**

$$J_R(\mathbf{q}) = \begin{bmatrix} J_{R1} & \dots & J_{Rn} \end{bmatrix}^T$$ - **Rotation Jacobian matrix**

For the contribution **to the linear velocity** of the end-effector *with* *D-H convention:*

$$\mathbf{v}_n = J_D \dot{\mathbf{q}} = J_{D1}\dot{q}_1 + \cdots + J_{Dn}\dot{q}_n = \sum_{i=1}^{n} J_{Di}\dot{q}_i \qquad (note: \dot{q}_i = \bar{\sigma}_i \dot{\theta}_i + \sigma_i \dot{d}_i)$$

✓ Each term of the summation represents the contribution of the velocity of single Joint *i* to the end-effector's linear velocity when all the other joints are still.



✓ If Joint *i* is prismatic ($\dot{q}_i = \sigma_i \dot{d}_i$), the contribution of joint *i* is

$$J_{Di}\dot{q}_i = \mathbf{z}_{i-1}\dot{d}_i = \sigma_i \mathbf{z}_{i-1}\dot{q}_i \quad \Rightarrow \quad J_{Di} = \sigma_i \mathbf{z}_{i-1}$$

✓ If Joint *i* is revolute ($\dot{q}_i = \bar{\sigma}_i \dot{\theta}_i$), the contribution to the linear velocity is

$$J_{Di}\dot{q}_i = \boldsymbol{\omega}_i^{i-1} \times \mathbf{r}_i = \bar{\sigma}_i \mathbf{z}_{i-1}\dot{\theta}_i \times \left(\mathbf{p}_n - \mathbf{p}_{i-1}\right)$$

$$\Rightarrow \quad J_{Di} = \bar{\sigma}_i \mathbf{z}_{i-1} \times \left(\mathbf{p}_n - \mathbf{p}_{i-1}\right)$$

Therefore, $$\boxed{J_{Di} = \bar{\sigma}_i \mathbf{z}_{i-1} \times \left(\mathbf{p}_n - \mathbf{p}_{i-1}\right) + \sigma_i \mathbf{z}_{i-1}}$$

The contribution **to the angular velocity** of the end-effector n *with <u>D-H convention:</u>*

$$\boldsymbol{\omega}_n = \sum_{i=1}^{n} \boldsymbol{\omega}_i^{i-1} = \sum_{i=1}^{n} J_{Ri}\dot{q}_i$$

✓ If Joint *i* is prismatic ($\dot{q}_i = \sigma_i \dot{d}_i$), the contribution of Joint *i* is

$$J_{Ri}\dot{q}_i = \mathbf{0} \quad \Rightarrow \quad J_{Ri} = \mathbf{0}$$

✓ If Joint *i* is revolute ($\dot{q}_i = \bar{\sigma}_i \dot{\theta}_i$), the contribution to the angular velocity is

$$J_{Ri}\dot{q}_i = \boldsymbol{\omega}_i^{i-1} = \mathbf{z}_{i-1}\dot{\theta}_i = \bar{\sigma}_i \mathbf{z}_{i-1}\dot{q}_i \quad \Rightarrow \quad J_{Ri} = \bar{\sigma}_i \mathbf{z}_{i-1}$$

Therefore, $\boxed{J_{Ri} = \bar{\sigma}_i \mathbf{z}_{i-1}}$

In summary $\boxed{J = \begin{bmatrix} J_{Di} \\ J_{Ri} \end{bmatrix} = \bar{\sigma}_i \begin{bmatrix} \mathbf{z}_{i-1} \times (\mathbf{p}_n - \mathbf{p}_{i-1}) \\ \mathbf{z}_{i-1} \end{bmatrix} + \sigma_i \begin{bmatrix} \mathbf{z}_{i-1} \\ \mathbf{0} \end{bmatrix}}$

Columns of a Jacobian Matrix

$$J_i = \begin{bmatrix} J_{Di} \\ J_{Ri} \end{bmatrix} = \bar{\sigma}_i \begin{bmatrix} \mathbf{z}_{i-1} \times (\mathbf{p}_n - \mathbf{p}_{i-1}) \\ \mathbf{z}_{i-1} \end{bmatrix} + \sigma_i \begin{bmatrix} \mathbf{z}_{i-1} \\ \mathbf{0} \end{bmatrix}$$

- ✓ $\mathbf{z}_{i-1}$ is given by the third column of the rotation matrix R_{i-1}, i.e.,

$$\mathbf{z}_{i-1} = {}^0R_1(q_1)\,{}^1R_2(q_2)\ldots{}^{i-2}R_{i-1}(q_{i-1})\mathbf{Z}_0 = R_{i-1}\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \qquad \text{(6-11)}$$

- ✓ $\mathbf{p}_n$ is the absolute position of the end-effector.
- ✓ $\mathbf{p}_{i-1}$ is given by the first three elements of the fourth column of the homogeneous transformation matrix. It can be obtained from

$$\begin{bmatrix} \mathbf{p}_{i-1} \\ 1 \end{bmatrix} = {}^0T_1(q_1)\,{}^1T_2(q_2)\ldots{}^{i-2}T_{i-1}(q_{i-1})\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T \qquad \text{(6-12)}$$

Computation of Jacobian

- ✓ Jacobian matrix has different representations depending on the configuration of coordinate systems under DH convention.
- ✓ If all joints of a robot are revolute, the geometric Jacobian matrix is given by

$$J = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_n - \mathbf{p}_0) & \mathbf{z}_1 \times (\mathbf{p}_n - \mathbf{p}_1) & \cdots & \mathbf{z}_{n-1} \times (\mathbf{p}_n - \mathbf{p}_{n-1}) \\ \mathbf{z}_0 & \mathbf{z}_1 & \cdots & \mathbf{z}_{n-1} \end{bmatrix}$$

- ✓ If a column is corresponds to a prismatic joint, the Jacobian becomes

$$J = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_n - \mathbf{p}_0) & \cdots & \boxed{\begin{matrix} \mathbf{z}_i \\ \mathbf{0} \end{matrix}} & \cdots & \mathbf{z}_{n-1} \times (\mathbf{p}_n - \mathbf{p}_{n-1}) \\ \mathbf{z}_0 & \cdots & & \cdots & \mathbf{z}_{n-1} \end{bmatrix}$$

Computation of Jacobian

- ✓ If the origin of the frame 3 is at the wrist point for a 6-link robotic arm under DH notation, we will have $\mathbf{p}_3=\mathbf{p}_4=\mathbf{p}_5=\mathbf{p}_6$, the Jacobian matrix can then be further simplified to

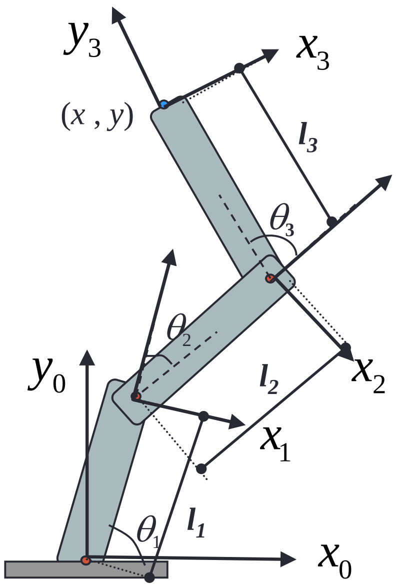
$$J = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_n - \mathbf{p}_0) & \mathbf{z}_1 \times (\mathbf{p}_n - \mathbf{p}_1) & \mathbf{z}_2 \times (\mathbf{p}_n - \mathbf{p}_2) & 0 & 0 & 0 \\ \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 & \mathbf{z}_4 & \mathbf{z}_5 \end{bmatrix}$$

- ✓ If the origin of the frame 3 is not at the wrist point for a 6-link robotic arm under DH notation, we will have $\mathbf{p}_2= \mathbf{p}_3$ and $\mathbf{p}_4=\mathbf{p}_5=\mathbf{p}_6$. the Jacobian matrix can then be simplified to

$$J = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_n - \mathbf{p}_0) & \mathbf{z}_1 \times (\mathbf{p}_n - \mathbf{p}_1) & \mathbf{z}_2 \times (\mathbf{p}_n - \mathbf{p}_2) & \mathbf{z}_3 \times (\mathbf{p}_n - \mathbf{p}_2) & 0 & 0 \\ \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 & \mathbf{z}_4 & \mathbf{z}_5 \end{bmatrix}$$

EX 6-3-2

Find the Jacobian Matrix of the 3R Planar Arm: 1) using the direct differentiation of forward kinematics 2) *using DH notations*



$$l_1 = l_2 = l_3 = 1$$

No.	a_i	α_i	d_i	θ_i
0	l_1	0	l_1	θ_1
1	l_2	0	l_2	θ_2
2	l_3	0	l_3	θ_3

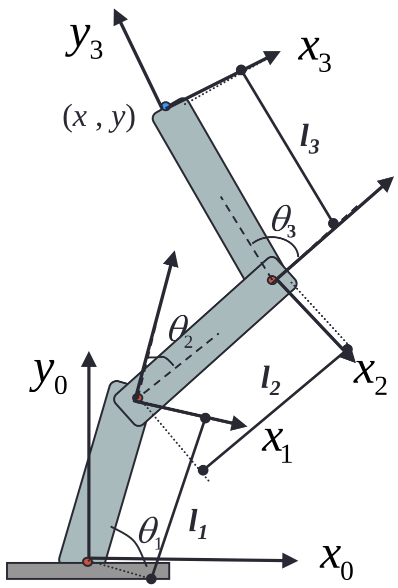
Method 1: *u*sing the direct differentiation of forward kinematics

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} l_1c_1 + l_2c_{12} + l_3c_{123} \\ l_1s_1 + l_2s_{12} + l_3s_{123} \\ 0 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ 0 \end{bmatrix} \Longrightarrow$$

$$J_D = \begin{bmatrix} \frac{\partial T_1}{\partial \theta_1} & \frac{\partial T_1}{\partial \theta_2} & \frac{\partial T_1}{\partial \theta_3} \\ \frac{\partial T_2}{\partial \theta_1} & \frac{\partial T_2}{\partial \theta_2} & \frac{\partial T_1}{\partial \theta_3} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\omega}_n = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} \Longrightarrow J_R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Method 2: *using DH notations*



$$l_1 = l_2 = l_3 = 1$$

No.	a_i	α_i	d_i	θ_i
0	l_1	0	l_1	θ_1
1	l_2	0	l_2	θ_2
2	l_3	0	l_3	θ_3

$$\mathbf{z}_0 = \mathbf{z}_1 = \mathbf{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_1 = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ l_1 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} l_1 c_1 + l_1 c_{12} \\ l_1 s_1 + l_1 s_{12} \\ l_1 + l_2 \end{bmatrix}$$

$$\mathbf{p}_n = \mathbf{p}_3 = \begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_1 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_1 s_{123} \\ l_1 + l_2 + l_3 \end{bmatrix}$$

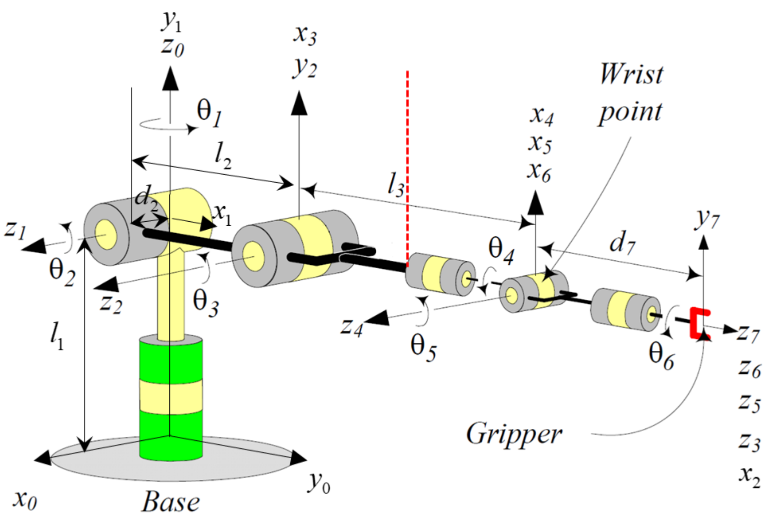
$$J_D = \left[\mathbf{z}_0 \times (\mathbf{p}_n - \mathbf{p}_0) \quad \mathbf{z}_1 \times (\mathbf{p}_n - \mathbf{p}_1) \quad \mathbf{z}_2 \times (\mathbf{p}_n - \mathbf{p}_2) \right]$$

$$= \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \end{bmatrix}$$

$$J_R = \left[\mathbf{z}_0 \quad \mathbf{z}_1 \quad \mathbf{z}_2 \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Ex 6-3-3

Find the Jacobian Matrix of the 6-DoF Manipulator. Without losing generality, the calculation is performed based on wrist point, i.e. $\mathbf{p}_n = \mathbf{p}_6$.



The manipulator has a RPR spherical wrist

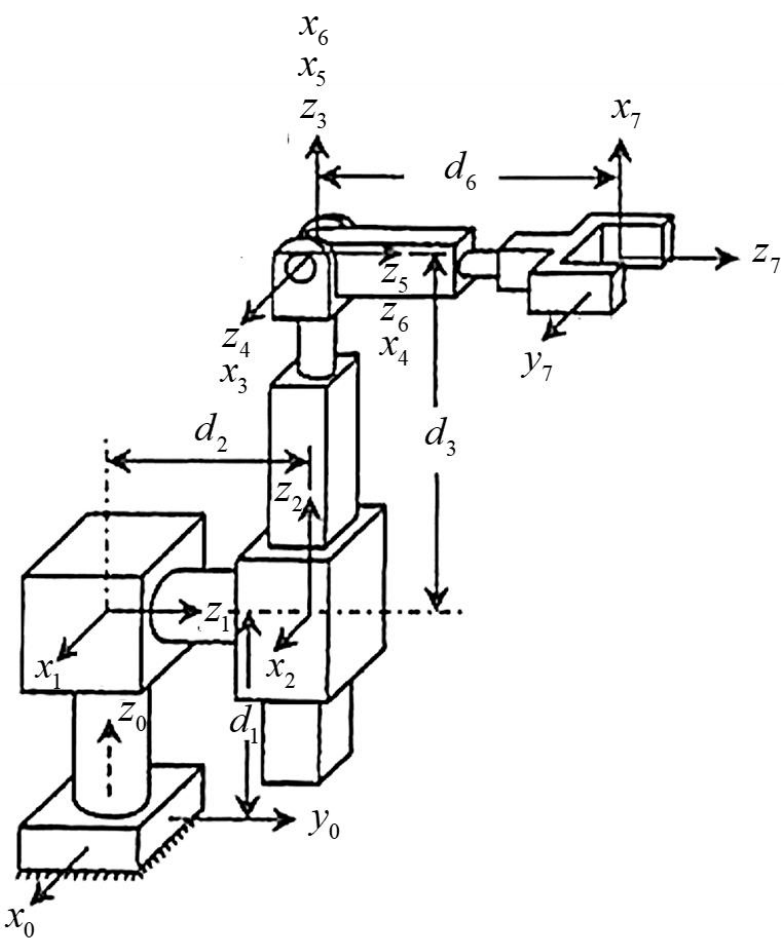
$$\mathbf{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 \end{bmatrix} \quad \mathbf{p}_2 = \mathbf{p}_3 = \begin{bmatrix} d_2 s_1 + l_2 c_1 c_2 \\ l_2 s_1 c_2 - d_2 c_1 \\ l_1 + l_2 s_2 \end{bmatrix}$$

$$\mathbf{p}_4 = \mathbf{p}_5 = \mathbf{p}_6 = \mathbf{p}_n = \begin{bmatrix} d_2 s_1 + l_2 c_1 c_2 + l_3 c_1 s_{23} \\ l_2 s_1 c_2 - d_2 c_1 + l_3 s_1 s_{23} \\ l_1 + l_2 s_2 - l_3 c_{23} \end{bmatrix}$$

$$\mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{z}_1 = \mathbf{z}_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix} \quad \mathbf{z}_3 = \begin{bmatrix} c_1 s_{23} \\ s_1 s_{23} \\ -c_{23} \end{bmatrix} \quad \mathbf{z}_4 = \begin{bmatrix} s_1 c_4 - c_1 s_4 c_{23} \\ -c_1 c_4 - s_1 s_4 c_{23} \\ -s_4 s_{23} \end{bmatrix} \quad \mathbf{z}_5 = \begin{bmatrix} s_1 s_4 s_5 + c_1 s_5 (c_4 c_{23} + s_{23}) \\ s_1 s_4 c_5 + c_1 c_5 (c_4 c_{23} + s_{23}) \\ s_4 s_5 s_{23} - c_5 c_{23} \end{bmatrix}$$

$$J = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_n - \mathbf{p}_0) & \mathbf{z}_1 \times (\mathbf{p}_n - \mathbf{p}_1) & \mathbf{z}_2 \times (\mathbf{p}_n - \mathbf{p}_2) & \mathbf{z}_3 \times (\mathbf{p}_n - \mathbf{p}_3) & 0 & 0 \\ \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 & \mathbf{z}_4 & \mathbf{z}_5 \end{bmatrix}$$

Ex 6-3-4 *Find the Jacobian matrix of the Stanford manipulator*



No.	a_i	α_i	d_i	θ_i
0	0	-90º	d_1	θ_1
1	0	90º	d_2	θ_2
2	0	0	$d_3(+d_0)$	0
3	0	90º	0	θ_4(90º)
4	0	90º	0	θ_5(90º)
5	0	0	0	θ_6

Ex 6-3-4 *Find the Jacobian matrix of the Stanford manipulator*

```
Ti(:,:,1) =

[ cos(th1),   0, -sin(th1),   0]
[ sin(th1),   0,  cos(th1),   0]
[        0,  -1,         0,  d1]
[        0,   0,         0,   1]

Ti(:,:,2) =

[ cos(th2), 0,  sin(th2),   0]
[ sin(th2), 0, -cos(th2),   0]
[        0, 1,         0,  d2]
[        0, 0,         0,   1]

Ti(:,:,3) =

[ 1, 0, 0,  0]
[ 0, 1, 0,  0]
[ 0, 0, 1, d3]
[ 0, 0, 0,  1]

Ti(:,:,4) =

[ cos(th4), 0,  sin(th4), 0]
[ sin(th4), 0, -cos(th4), 0]
[        0, 1,         0, 0]
[        0, 0,         0, 1]

Ti(:,:,5) =

[ cos(th5), 0,  sin(th5), 0]
[ sin(th5), 0, -cos(th5), 0]
[        0, 1,         0, 0]
[        0, 0,         0, 1]

Ti(:,:,6) =

[ cos(th6), -sin(th6), 0, 0]
[ sin(th6),  cos(th6), 0, 0]
[        0,         0, 1, 0]
[        0,         0, 0, 1]
```

$$
{}^0T_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^0T_2 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & -d_2s_1 \\ s_1c_2 & c_1 & s_1s_2 & d_2c_1 \\ -s_2 & 0 & c_2 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^0T_3 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & d_3c_1s_2 - d_2s_1 \\ s_1c_2 & c_1 & s_1s_2 & d_2c_1 + d_3s_1s_2 \\ -s_2 & 0 & c_2 & d_1 + d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^0T_4 = \begin{bmatrix} c_1c_2c_4 - s_1s_4 & c_1s_2 & c_1c_2s_4 + s_1c_4 & X \\ s_1c_2c_4 + c_1s_4 & s_1s_2 & s_1c_2s_4 - c_1c_4 & X \\ -s_2c_4 & c_2 & -s_2s_4 & X \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0T_5 = \begin{bmatrix} c_1s_2s_5 - s_1s_4c_5 + c_1c_2c_4c_5 & c_1c_2s_4 + s_1c_4 & c_1c_2c_4s_5 - s_1s_4s_5 - c_1s_2c_5 & X \\ s_1s_2s_5 + c_1s_4c_5 + s_1c_2c_4c_5 & s_1c_2s_4 - c_1c_4 & s_1c_2c_4s_5 + c_1s_4s_5 - s_1s_2c_5 & X \\ c_2s_5 - s_2c_4c_5 & -s_2s_4 & -c_2c_5 - s_2c_4s_5 & X \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Ex 6-3-4 *Find the Jacobian matrix of the Stanford manipulator*

$$\bar{\sigma} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$
$$\sigma = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} \quad \mathbf{p}_3 = \mathbf{p}_4 = \mathbf{p}_5 = \mathbf{p}_6 = \mathbf{p}_n = \begin{bmatrix} d_3 c_1 s_2 - d_2 s_1 \\ d_2 c_1 + d_3 s_1 s_2 \\ d_1 + d_3 c_2 \end{bmatrix}$$

$$\mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{z}_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad \mathbf{z}_2 = \mathbf{z}_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix} \quad \mathbf{z}_4 = \begin{bmatrix} c_1 c_2 s_4 + s_1 c_4 \\ s_1 c_2 s_4 - c_1 c_4 \\ -s_2 s_4 \end{bmatrix} \quad \mathbf{z}_5 = \begin{bmatrix} c_1 c_2 c_4 s_5 - s_1 s_4 s_5 - c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 - s_1 s_2 c_5 \\ -c_2 c_5 - s_2 c_4 s_5 \end{bmatrix}$$

$$\mathbf{z}_0 \times \mathbf{p}_6 = \begin{bmatrix} -d_2 c_1 - d_3 s_1 s_2 \\ d_3 c_1 s_2 - d_2 s_1 \\ 0 \end{bmatrix} \quad \mathbf{z}_1 \times (\mathbf{p}_6 - \mathbf{p}_1) = \begin{bmatrix} d_3 c_1 c_2 \\ d_3 s_1 c_2 \\ -d_3 s_2 \end{bmatrix}$$

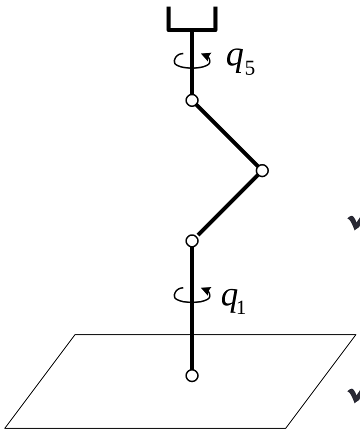
$$J = \bar{\sigma}_i \begin{bmatrix} \mathbf{z}_{i-1} \times (\mathbf{p}_n - \mathbf{p}_{i-1}) \\ \mathbf{z}_{i-1} \end{bmatrix} + \sigma_i \begin{bmatrix} \mathbf{z}_{i-1} \\ \mathbf{0} \end{bmatrix} \Rightarrow$$

$$J = \begin{bmatrix} \mathbf{z}_0 \times \mathbf{p}_6 & \mathbf{z}_1 \times (\mathbf{p}_6 - \mathbf{p}_1) & \mathbf{z}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{0} & \mathbf{z}_3 & \mathbf{z}_4 & \mathbf{z}_5 \end{bmatrix}$$

Note: in practical application, the calculation of *J* is not for wrist point but the point on the end-effector where $\mathbf{p}_n = \mathbf{p}_7$

6.4 Kinematic Singularities

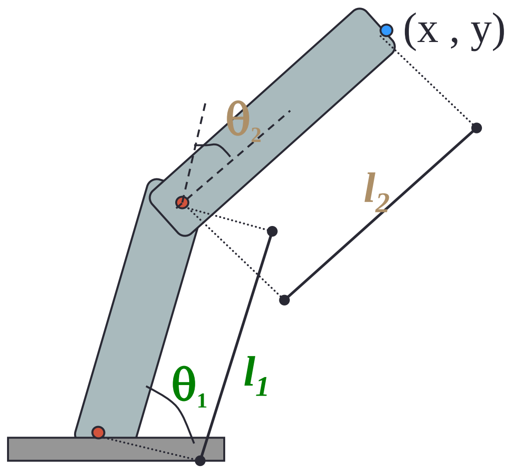
The Jacobian is, in general, a function of the configuration **q**; those configurations **at which J is rank-deficient** are termed kinematic singularities.



- ✓ Singularities represent configurations at which **mobility of the structure is reduced**, i.e., it is not possible to impose an arbitrary motion to the end-effector.
- ✓ When the structure is at a singularity, **infinite solutions** to the inverse kinematics problem may exist.
- ✓ In the neighborhood of a singularity, **small velocities in the operational space may cause large velocities** in the joint space.
- ✓ Mathematically, singularity configurations can be found by calculating the conditions that make Rank(**J**) < min{6,n}:

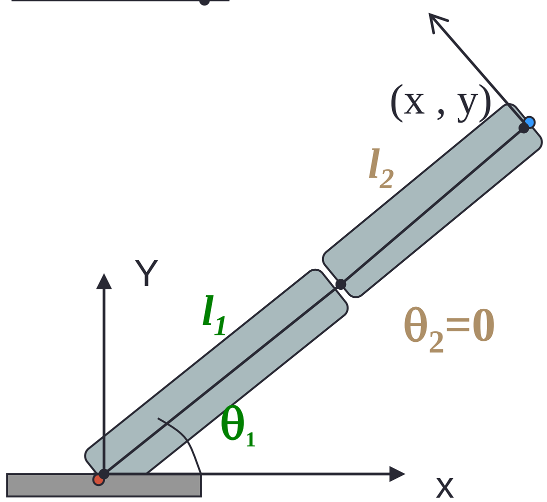
$$|\mathbf{J}| = 0 \quad \text{or} \quad \left|\mathbf{J}\mathbf{J}^T\right| = 0$$

EX 6-4-1 *Find the singularities of of the 2R Planar Arm*



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) \end{bmatrix} = \begin{bmatrix} T_1(\theta_1,\theta_2) \\ T_2(\theta_1,\theta_2) \end{bmatrix}$$

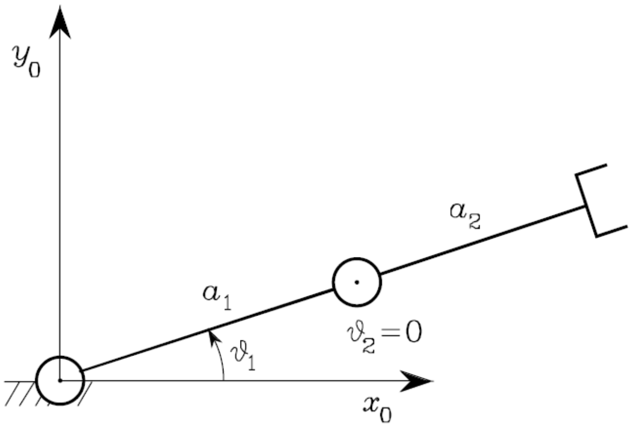
$$\mathbf{J} = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix}$$



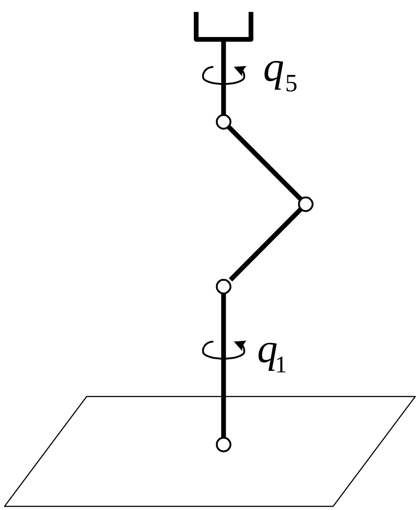
$$\det(\mathbf{J}) = l_1 l_2 \sin\theta_2 = 0 \quad \Rightarrow$$

$$\theta_2 = 0 \quad \text{or} \quad k\pi$$

Types of Singularities



- ✓ **Boundary singularities** that occur when the manipulator is either outstretched or retracted. They can be avoided on condition that the manipulator is not driven to the boundaries of its reachable workspace.

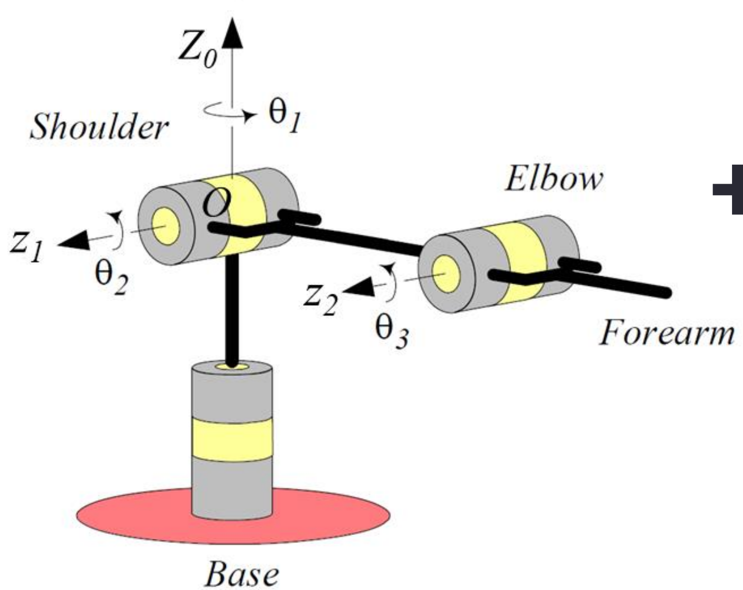


- ✓ **Internal singularities** that occur inside the reachable workspace and are generally **caused by the alignment of two or more axes of motion,** or else by the attainment of **particular end-effector configurations**.
- ✓ Internal singularities constitute a serious problem, as they can be encountered anywhere in the reachable workspace for a planned path in the operational space.

Singularity Decoupling

- ✓ Computation of internal singularities via the Jacobian determinant may be tedious and of no easy solution for complex structures.
- ✓ For manipulators having a spherical wrist, it is possible to split the problem of singularity computation into two separate problems:

arm singularities + ***wrist singularities***



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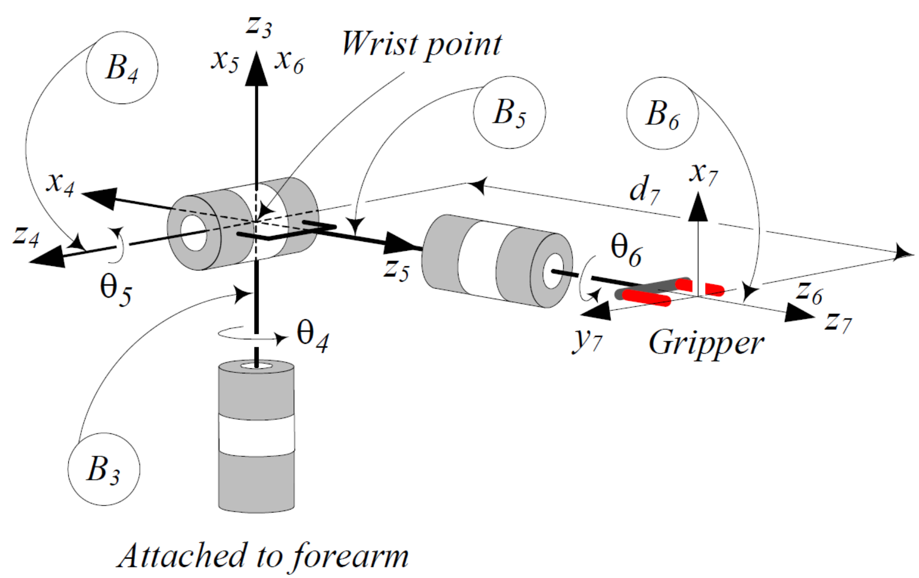


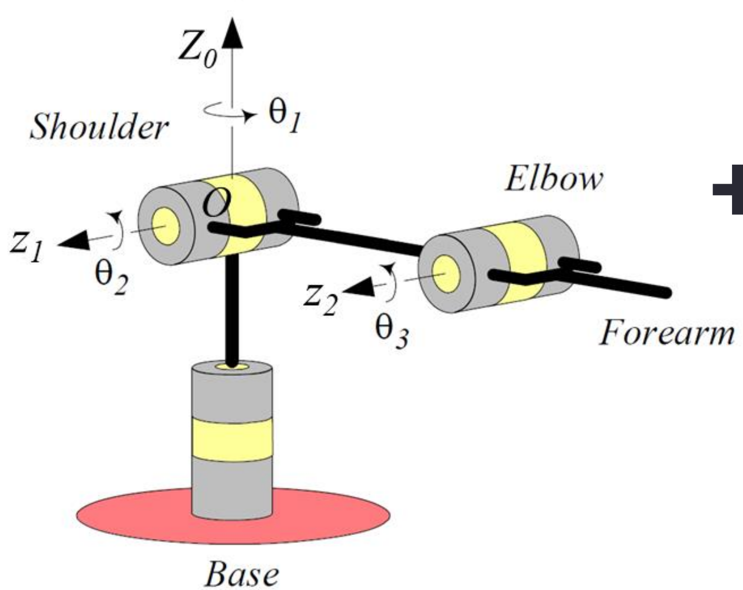
✓ For the sake of simplicity, consider the case n = 6; the Jacobian can be partitioned into (3 × 3) blocks as follows:

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_{11} & \boldsymbol{J}_{12} \\ \boldsymbol{J}_{21} & \boldsymbol{J}_{22} \end{bmatrix}$$

For a spherical wrist $\boldsymbol{J}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \Rightarrow \det(\boldsymbol{J}) = \det(\boldsymbol{J}_{11})\det(\boldsymbol{J}_{22})$

$$\det(\boldsymbol{J}_{11}) = 0 \qquad \det(\boldsymbol{J}_{22}) = 0$$

arm singularities ***wrist singularities***



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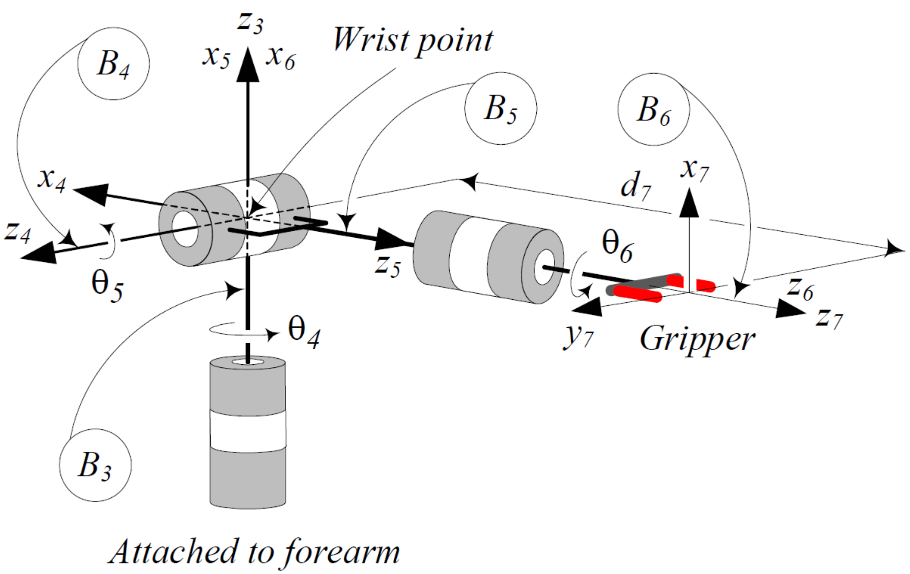
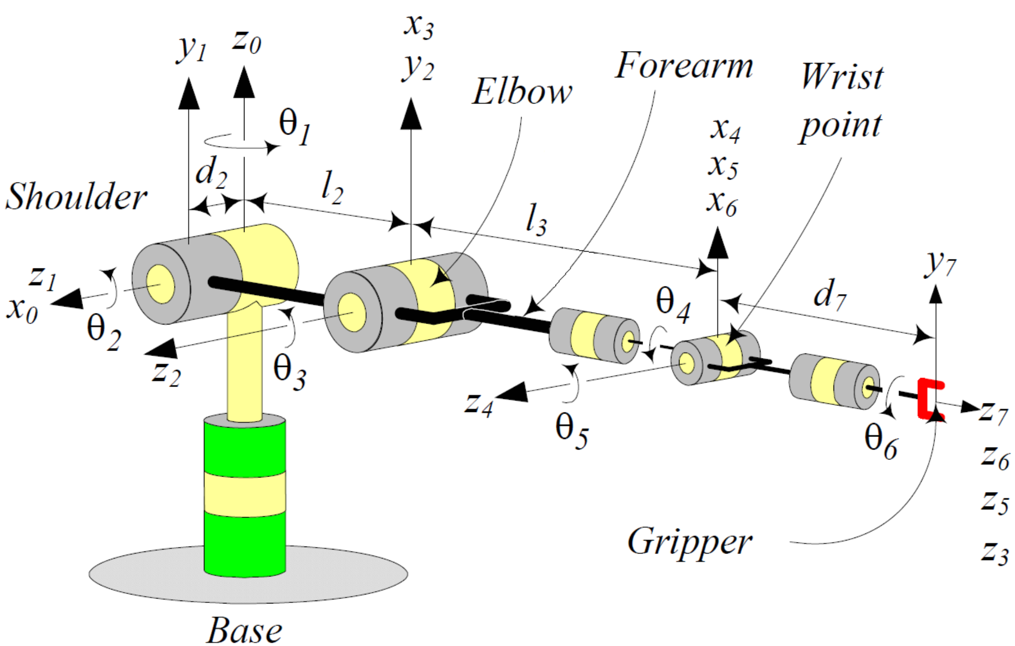
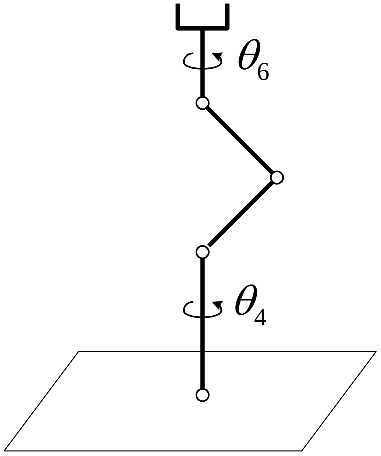


Wrist Singularities

- ✓ It can be recognized that the wrist is at a singular configuration whenever the unit vectors $\boldsymbol{z}_3$, $\boldsymbol{z}_4$, $\boldsymbol{z}_5$ are linearly dependent since $\boldsymbol{J}_{22} = \begin{bmatrix} \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{bmatrix}$.
- ✓ The wrist kinematic structure reveals that a singularity occurs when $\boldsymbol{z}_3$ and $\boldsymbol{z}_5$ are aligned, i.e., whenever

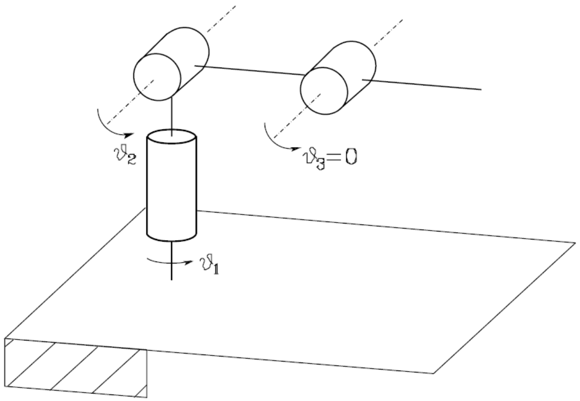
$$\theta_5 = 0 \quad \text{or} \quad \pi$$





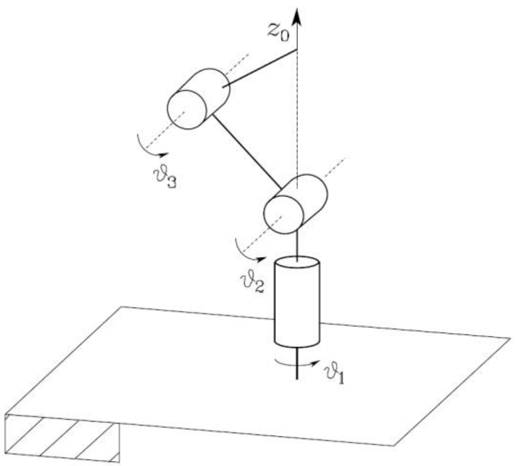
Arm Singularities

✓ Arm singularities are characteristic of a specific manipulator structure.



Elbow singularity: $\theta_3 = 0 \quad \text{or} \quad \pi$

Conceptually equivalent to the singularity found for the 2R planar arm.

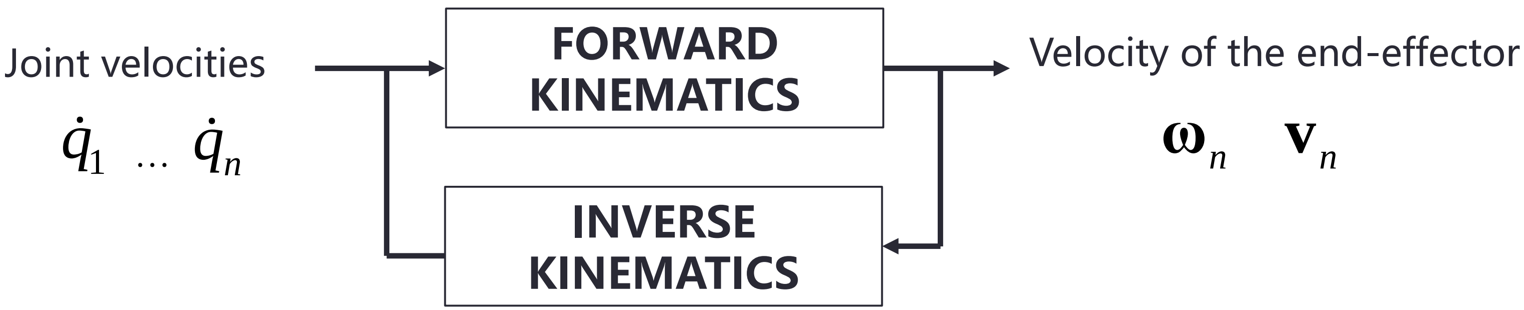


Shoulder singularity: wrist point lies on axis z_0

Unlike the wrist singularities, the arm singularities are well identified in the operational space, and thus they can be suitably avoided in the end-effector trajectory planning stage.

6.4 Inverse Velocity Kinematics

✓ The determination of the time rate of joint variables of a manipulator if Cartesian velocity of end-effector in the global coordinate frame are known.



✓ Solution to the inverse velocity problem can also be given by two ways

- Time derivative of homogeneous transformation matrix ${}^0\dot{T}_n \rightarrow \cdots {}^0\dot{T}_i \rightarrow \cdots {}^0\dot{T}_1$
- Relation between the Cartesian velocity and the time rate of joint variables through Jacobian Matrix, *J*

6.5 Inverse Velocity Kinematics

- ✓ Jacobian matrix provides a solution to the inverse velocity kinematics problem. It also maps the trajectory in the operational space into the joint space.

$$\dot{X}_{6\times 1} = J\dot{\mathbf{q}} = \begin{bmatrix} J_D \\ J_R \end{bmatrix}\dot{\mathbf{q}} \quad \Rightarrow \quad \boxed{\dot{\mathbf{q}} = J^{-1}\dot{X}}$$

- ✓ If the initial manipulator posture q(0) is known, joint positions can be computed by integrating velocities over time,

$$\mathbf{q}(t) = \int_0^t \dot{\mathbf{q}}(\tau)d\tau + \mathbf{q}(0)$$

- ✓ The integration can be performed in discrete time by resorting to numerical techniques using Euler integration method

$$\boxed{\mathbf{q}(t_{k+1}) = \mathbf{q}(t_k) + \dot{\mathbf{q}}(t_k)\Delta t}$$

6.5 Inverse Velocity Kinematics

- ✓ Using derivatives of homogeneous transformation matrix to find the inverse kinematics problem will finally becomes to a problem that needs solving vector algebra equations, which is not systematic. Therefore, the inverse velocity kinematics is usually solved through Jacobian matrix.
- ✓ The existence of solution using Jacobian matrix requires the necessary condition that

$$\text{number of rows of } J \leq \text{number of joint variables}$$

Different Cases

- ✓ **Case 1: *m = n*,** number of rows of Jacobian matrix *m* equals to number of joint variables *n*, *t*he inverse velocity kinematics problem has a unique solution

$$\dot{\mathbf{q}} = J^{-1}\dot{X}$$

- ✓ **Case 2: *m > n*,** the manipulator is under-actuated and not all degrees-of-freedom can be controlled. Solutions do not exist. We may only find a solution to minimize the end-effector's location error.

$$\dot{\mathbf{q}} = (J^T W J)^{-1} J^T W \dot{X} \qquad (W = diag(w_1, \cdots w_n))$$

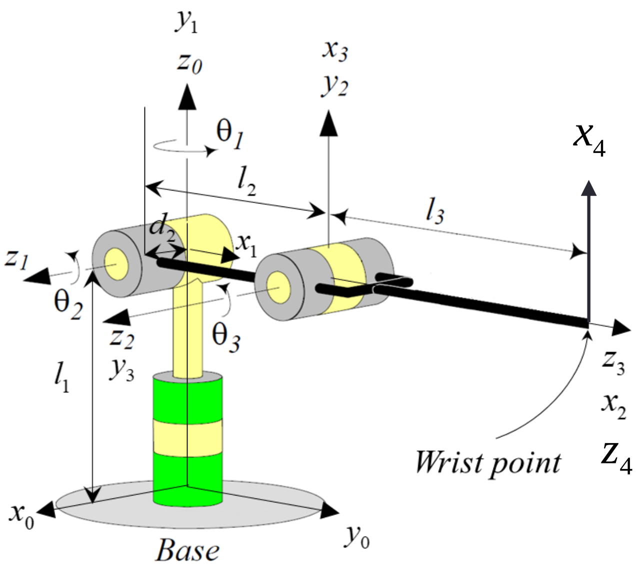
- ✓ **Case 3: *m < n*,** the manipulator is **redundant** and infinite number of solutions exist. Unique solution may be found under certain optimal criteria. If the kinetic energy of the manipulator is required,

$$\dot{\mathbf{q}} = W^{-1} J^T (J W^{-1} J^T)^{-1} \dot{X}$$

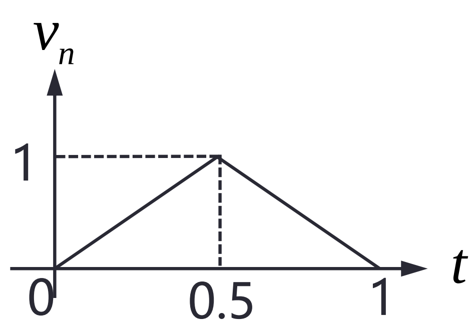
where *W* is positive definite matrix

EX 6-5-1

Perform trajectory planning for the articulated manipulator in operational space and determine joint space velocities. The end point moves along a straight line in Cartesian space starting from P_1=(0.1, 1, 1) to P_2=(-1, -0.5, 1.5) with a speed profile given in the figure.



L1 = 1; L2 = 1; L3 = 1; d2 = 0.1;



1. Trajectory planning in Cartesian space

$$v(t) = \begin{cases} 2t & 0 \le t \le 0.5 \\ 2(1-t) & 0.5 < t \le 1 \end{cases} \qquad p(t) = \begin{cases} t^2 & 0 \le t \le 0.5 \\ -t^2 + 2t - 0.5 & 0.5 < t \le 1 \end{cases}$$

End-effector velocity in Cartesian space

$$V(t) = \vec{K} v(t) = \begin{bmatrix} -2.2 \\ -3 \\ 1 \end{bmatrix} v(t) \qquad \vec{K} = \frac{\vec{P}_2 - \vec{P}_1}{p(1) - p(0)}$$

End-effector position in Cartesian space

$$P(t) = \vec{K}\left[p(t) - p(0)\right] + \vec{P}_1 = \begin{bmatrix} -2.2 \\ -3 \\ 1 \end{bmatrix} p(t) + \begin{bmatrix} 0.1 \\ 1 \\ 1 \end{bmatrix}$$

EX 6-5-1

No.	a_i	α_i	d_i	θ_i
1	0	90º	l_1	θ_1+90º
2	l_2	0	d_2	θ_2
3	0	90º	0	θ_3+90º

P_1=(0.1, 1, 1)
P_2=(-1, -0.5, 1.5)

L1 = 1; L2 = 1; L3 = 1; d2 = 0.1;

Method 1

Find joint space trajectory using inverse kinematics of joint angles

$$P_n(t) = \begin{bmatrix} d_x(t) \\ d_y(t) \\ d_z(t) \end{bmatrix} = P(t)$$

$$\theta_1 = \text{atan2}(d_2, \pm\sqrt{d_x^2 + d_y^2 - d_2^2}) + \text{atan2}(d_y, d_x)$$

$$\theta_2 = \text{atan2}(s, \pm\sqrt{m^2 + n^2 - s^2}) + \text{atan2}(m, n)$$

$$m = 2l_2(d_x c_1 + d_y s_1)$$

$$n = 2l_2(l_1 - d_z)$$

$$s = l_3^2 - l_1^2 - l_2^2 - d_z^2 - (d_x c_1 + d_y s_1)^2 + 2l_1 d_z$$

$$\theta_3 = \text{atan2}(d_x c_1 + d_y s_1 - l_2 c_2, \quad l_1 + l_2 s_2 - d_z) - \theta_2$$

$$\Rightarrow \quad \boldsymbol{\theta}(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{bmatrix}$$

EX 6-5-1

Method 2

$$\dot{\mathbf{q}} = J^{-1}(\mathbf{q})\dot{X} \qquad \mathbf{q}(t_{k+1}) = \mathbf{q}(t_k) + \dot{\mathbf{q}}(t_k)\Delta t$$

Step 1: Determine joint velocities using direct differentiation of joint angels

$$\dot{\boldsymbol{\theta}}(t) = \begin{bmatrix} \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \\ \dot{\theta}_3(t) \end{bmatrix} \Rightarrow \dot{\boldsymbol{\theta}}(t_i) \approx \frac{\boldsymbol{\theta}(t_i) - \boldsymbol{\theta}(t_{i-1})}{\Delta t} \quad \text{where} \quad \Delta t = t_i - t_{i-1}$$

Step 2: Determine joint velocities using inverse of Jacobian matrix

$$\dot{\mathbf{q}} = J^{-1}(\mathbf{q})\dot{X} \Rightarrow \dot{\boldsymbol{\theta}}(t) = J^{-1}(\boldsymbol{\theta}(t))V(t)$$

Step 3: Find joint space trajectory using iteration techniques based on inverse J

$$\mathbf{q}(t_{k+1}) = \mathbf{q}(t_k) + \dot{\mathbf{q}}(t_k)\Delta t$$

$$\dot{\mathbf{q}} = J^{-1}(\mathbf{q})\dot{X}$$

$$\Rightarrow \boldsymbol{\theta}(t_{i+1}) \approx \boldsymbol{\theta}(t_i) - \boldsymbol{\theta}(t_i)\Delta t \qquad \dot{\boldsymbol{\theta}}(t) = J^{-1}(\boldsymbol{\theta}(t))V(t)$$

EX 6-5-1

Jacobian matrix is determined by

$$\mathbf{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 \end{bmatrix} \quad \mathbf{p}_2 = \mathbf{p}_3 = \begin{bmatrix} d_2 s_1 + l_2 c_1 c_2 \\ l_2 s_1 c_2 - d_2 c_1 \\ l_1 + l_2 s_2 \end{bmatrix}$$

$$\mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{z}_1 = \mathbf{z}_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix} \quad \mathbf{p}_n = \begin{bmatrix} d_2 s_1 + l_2 c_1 c_2 + l_3 c_1 s_{23} \\ l_2 s_1 c_2 - d_2 c_1 + l_3 s_1 s_{23} \\ l_1 + l_2 s_2 - l_3 c_{23} \end{bmatrix}$$

$$J = \left[\mathbf{z}_0 \times (\mathbf{p}_n - \mathbf{p}_0) \quad \mathbf{z}_1 \times (\mathbf{p}_n - \mathbf{p}_1) \quad \mathbf{z}_2 \times (\mathbf{p}_n - \mathbf{p}_2) \right] = \begin{bmatrix} d_2 c_1 - l_2 s_1 c_2 - l_3 s_1 s_{23} & c_1 (l_3 c_{23} - l_2 s_2) & l_3 c_1 c_{23} \\ d_2 s_1 + l_2 c_1 c_2 + l_3 c_1 s_{23} & s_1 (l_3 c_{23} - l_2 s_2) & l_3 s_1 c_{23} \\ 0 & l_3 s_{23} + l_2 c_2 & l_3 s_{23} \end{bmatrix}$$