ARTICULATED ROBOTS

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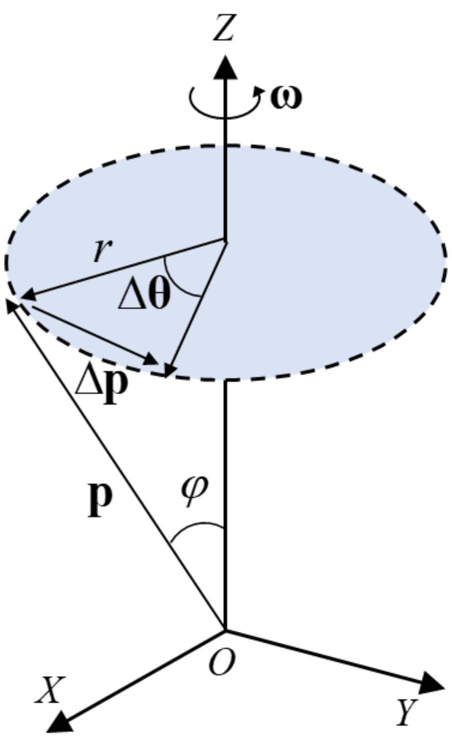
6. DIFFERENTIAL KINEMATICS I

$$\dot{\mathbf{p}} = \boldsymbol{\omega} \times \mathbf{p} = S(\boldsymbol{\omega})\mathbf{p}$$

Differential Kinematics

- ✓ Differential kinematics gives the mapping between the joint velocities and the corresponding end-effector’s linear and angular velocity, which includes knowledge regarding
  - ✓ Representation of velocities
  - ✓ Forward velocity kinematics
  - ✓ Inverse velocity kinematics
  - ✓ Forward acceleration kinematics and statics
  - ✓ In verse acceleration kinematics

6.1 Representation of Angular Velocity



$$|\Delta\boldsymbol{\theta}| = \frac{|\Delta\mathbf{p}|}{|\mathbf{r}|} = \frac{|\Delta\mathbf{p}|}{|\mathbf{p}|\cdot\sin\varphi} \Rightarrow \Delta\mathbf{p} = |\mathbf{p}|\cdot\sin\varphi = \Delta\boldsymbol{\theta}\times\mathbf{p}$$

$$\Rightarrow \mathbf{v} = \dot{\mathbf{p}} = \boldsymbol{\omega}\times\mathbf{p}$$

$$\Downarrow$$

$$\boldsymbol{\omega}\times\mathbf{p} = \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}\times\begin{bmatrix}p_x\\ p_y\\ p_z\end{bmatrix} = \begin{bmatrix}\omega_y p_z - \omega_z p_y\\ \omega_z p_x - \omega_x p_z\\ \omega_x p_y - \omega_y p_x\end{bmatrix} = \begin{bmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{bmatrix}\begin{bmatrix}p_x\\ p_y\\ p_z\end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{v} = \dot{\mathbf{p}} = \boldsymbol{\omega}\times\mathbf{p} = S(\boldsymbol{\omega})\mathbf{p}$$

cross product -> matrix multiplication

O-XYZ **is a inertial frame**

Generally,

$$\frac{d}{dt}(\cdot)_{in} = \frac{d}{dt}(\cdot)_{rot} + \boldsymbol{\omega}\times(\cdot)_{in}$$

Representation of Linear Velocity

Assume that **p** is converted from $^1\mathbf{p}$ in a local frame 1 by rotation matrix *R*

$$^0\mathbf{p} = R\,^1\mathbf{p}$$

The linear velocity of **p** in the global coordinate frame can be given by the derivative of the vector denoting **p**

$$\mathbf{v} = \dot{\mathbf{p}} = \dot{R}\,^1\mathbf{p} = \dot{R}\left(R^{-1}\mathbf{p}\right) = \dot{R}R^T\mathbf{p}$$

According to the property of the rotation matrix

$$RR^T = I \quad \Rightarrow \quad \frac{d}{dt}(RR^T) = \dot{R}R^T + R\dot{R}^T = 0 \quad \Rightarrow \quad \dot{R}R^T = -(\dot{R}R^T)^T$$

Let $S(\boldsymbol{\omega}) = \dot{R}R^T$, apparently, *S* is an asymmetric matrix, and we have

$$\boxed{\mathbf{v} = S(\boldsymbol{\omega})\mathbf{p}}$$

where $S(\boldsymbol{\omega}) = \dot{R}R^T$ $\quad \dot{R} = S(\boldsymbol{\omega})R$

✓ The skew-symmetric matrix $S(\boldsymbol{\omega})$ is defined as

$$\tilde{\boldsymbol{\omega}} = S(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

✓ The operator ~ converts a 3x1 vector to a skew-symmetric matrix.

✓ The operator $^{\vee}$ recovers a vector from the cross-product matrix, i.e.

$$\boldsymbol{\omega} = S^{\vee}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

✓ We have several useful properties:

$$S(\boldsymbol{\omega}) = -S^T(\boldsymbol{\omega}) \qquad \dot{R} = S(\boldsymbol{\omega})R \qquad RS(\boldsymbol{\omega})R^T = S(R\boldsymbol{\omega})$$

Representation of Angular Velocity

- ✓ A rotation can be represented by different triple-rotations about global or local axis. Therefore, the absolute angular velocity is normally different from angular velocities around each axis.

1. Triple-rotation representation of orientation is equivalent to a angle-axis rotation, which implies that the angular velocity can be described by a angular frequency and an axis that the rotation is made about, i.e.

$$\boldsymbol{\omega} = \omega \mathbf{u} \qquad \omega = \sqrt{\omega_X^2 + \omega_Y^2 + \omega_Z^2} \qquad \mathbf{u} = \begin{bmatrix} \omega_X / \sqrt{\omega_X^2 + \omega_Y^2 + \omega_Z^2} \\ \omega_Y / \sqrt{\omega_X^2 + \omega_Y^2 + \omega_Z^2} \\ \omega_Z / \sqrt{\omega_X^2 + \omega_Y^2 + \omega_Z^2} \end{bmatrix}$$

Representation of Angular Velocity

2. The absolute angular velocity can be interpreted by three direction cosines with primary axes of a global frame, $\boldsymbol{\omega} = [\omega_X \quad \omega_Y \quad \omega_Z]^T$. If the rotation is represented by XYZ fixed angle, then

$$S(\boldsymbol{\omega}) = \dot{R}R^T$$

$$= \frac{d}{dt}\left[R_Z(\gamma)R_Y(\beta)R_X(\alpha)\right]\cdot R^T = (\dot{\alpha}R_Z R_Y \frac{\partial R_X}{\partial \alpha} + \dot{\beta}R_Z \frac{\partial R_Y}{\partial \beta} R_X + \dot{\gamma}\frac{\partial R_Z}{\partial \gamma} R_Y R_X)\cdot R^T$$

$$= \begin{bmatrix} 0 & \dot{\alpha}s\beta - \dot{\gamma} & \dot{\beta}c\gamma + \dot{\alpha}c\beta s\gamma \\ \dot{\gamma} - \dot{\alpha}s\beta & 0 & \dot{\beta}s\gamma - \dot{\alpha}c\beta c\gamma \\ -\dot{\beta}c\gamma - \dot{\alpha}c\beta s\gamma & \dot{\alpha}c\beta c\gamma - \dot{\beta}s\gamma & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} = S^{V}(\boldsymbol{\omega}) = \begin{bmatrix} \dot{\alpha}c\beta c\gamma - \dot{\beta}s\gamma \\ \dot{\beta}c\gamma + \dot{\alpha}c\beta s\gamma \\ \dot{\gamma} - \dot{\alpha}s\beta \end{bmatrix} = \begin{bmatrix} c\beta c\gamma & -s\gamma & 0 \\ c\beta s\gamma & c\gamma & 0 \\ -s\beta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

Different triple rotation has an unique matrix transforming local angular velocity to the absolute angular velocity

Representation of Angular Velocity

3. The angular velocity can also be expressed by Euler parameters or unit quaternion representation of a finite rotation

$$^{0}\mathbf{p} = e(t)\,^{1}\mathbf{p}e^{*}(t) \qquad ^{1}\mathbf{p} = e^{*}(t)\,^{0}\mathbf{p}e(t)$$

where $e(t) = e_0 + \mathbf{e} \qquad e^{*}(t) = e^{-1}(t) = e_0 - \mathbf{e}$

$$\left.\begin{array}{l} \Rightarrow \quad \mathbf{v} = \dot{\mathbf{p}} = \dot{e}\,^{1}\mathbf{p}e^{*} + e\mathbf{p}\dot{e}^{*} = \dot{e}e^{*}\mathbf{p} + \mathbf{p}e\dot{e}^{*} \\ ee^{*} = 1 \quad \Rightarrow \quad \dfrac{d}{dt}(ee^{*}) = \dot{e}e^{*} + e\dot{e}^{*} = 0 \end{array}\right\} \Rightarrow \quad \mathbf{v} = \dot{\mathbf{p}} = 2\dot{e}e^{*}\mathbf{p}$$

Therefore, the angular velocity can also be expressed by a quaternion.

$$\boxed{^{0}\boldsymbol{\omega}_1^0 = \boldsymbol{\omega} \triangleq 2\dot{e}e^{*}} \quad \boxed{^{1}\boldsymbol{\omega}_1^0 = e^{*}\boldsymbol{\omega}e = 2e^{*}\dot{e}}$$

where $\boldsymbol{\omega} = <0, \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}> = <0, \quad \mathbf{e}_{\omega}>$

$$\Rightarrow \quad \boldsymbol{\omega} = 2\dot{e}e^* \qquad \text{(using hanmiltonian product)}$$

$$= 2(\dot{e}_0 + \dot{\mathbf{e}})(e_0 - \mathbf{e})$$

$$= 2(\dot{e}_0 e_0 + e_0\dot{\mathbf{e}} - e_0\mathbf{e} + \dot{\mathbf{e}}\cdot\mathbf{e} + \dot{\mathbf{e}}\times\mathbf{e})$$

$$= 2 < \dot{e}_0 e_0 + \dot{\mathbf{e}}\cdot\mathbf{e}, \quad e_0\dot{\mathbf{e}} - e_0\mathbf{e} + \dot{\mathbf{e}}\times\mathbf{e} >$$

$$\frac{d}{dt}(ee^*) = \dot{e}e^* + e\dot{e}^* = 0 \Rightarrow \dot{e}_0 e_0 + \dot{\mathbf{e}}\cdot\mathbf{e} = 0$$

$$\Rightarrow \tilde{\boldsymbol{\omega}} = 2 < 0, \quad e_0\dot{\mathbf{e}} - e_0\mathbf{e} + \dot{\mathbf{e}}\times\mathbf{e} >$$

Expansion of quaternion yields

$$\boldsymbol{\omega} = 2\begin{bmatrix} 0 \\ e_0\dot{e}_1 - e_1\dot{e}_0 + e_2\dot{e}_3 - e_3\dot{e}_2 \\ e_0\dot{e}_2 - e_1\dot{e}_3 - e_2\dot{e}_0 + e_3\dot{e}_1 \\ e_0\dot{e}_3 + e_1\dot{e}_2 - e_2\dot{e}_1 - e_3\dot{e}_0 \end{bmatrix} = 2\begin{bmatrix} e_0 & e_1 & e_2 & e_3 \\ -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix}\begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = 2\begin{bmatrix} \dot{e}_0 & -\dot{e}_1 & -\dot{e}_2 & -\dot{e}_3 \\ \dot{e}_1 & \dot{e}_0 & \dot{e}_3 & -\dot{e}_2 \\ \dot{e}_2 & -\dot{e}_3 & \dot{e}_0 & \dot{e}_1 \\ \dot{e}_3 & \dot{e}_2 & -\dot{e}_1 & \dot{e}_0 \end{bmatrix}\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$\Downarrow$$

$$\boldsymbol{\omega} = 2\begin{bmatrix} e_0 & \mathbf{e} \\ -\mathbf{e} & e_0 I + S(\mathbf{e}) \end{bmatrix}\dot{e} = 2Q(\mathbf{e})\dot{e}$$

$$\Downarrow$$

$$\boldsymbol{\omega} = 2\begin{bmatrix} \dot{e}_0 & \dot{\mathbf{e}}^* \\ -\dot{\mathbf{e}} & \dot{e}_0 I + S(\dot{\mathbf{e}}^*) \end{bmatrix} e = 2Q(\dot{\mathbf{e}}^*)e = 2Q^T(\dot{\mathbf{e}})e$$

$$\boldsymbol{\omega} = 2\dot{e}e^* \Rightarrow$$

$$\dot{e} = \frac{1}{2}\boldsymbol{\omega}e = \frac{1}{2}\begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & -\omega_z & \omega_y \\ \omega_y & \omega_z & 0 & -\omega_x \\ \omega_z & -\omega_y & \omega_x & 0 \end{bmatrix}\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e_0 & -e_1 & -e_2 & -e_3 \\ e_1 & e_0 & e_3 & -e_2 \\ e_2 & -e_3 & e_0 & e_1 \\ e_3 & e_2 & -e_1 & e_0 \end{bmatrix}\begin{bmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\Downarrow$$

$$\dot{e} = \frac{1}{2}Q(\boldsymbol{\omega})\mathbf{e}$$

$$\Downarrow$$

$$\boldsymbol{\omega} = 2Q(\mathbf{e})\dot{e} \Rightarrow$$

$$\dot{e} = \frac{1}{2}Q^{-1}(\mathbf{e})\boldsymbol{\omega} = \frac{1}{2}Q^{T}(\mathbf{e})\boldsymbol{\omega}$$

Time Derivative of Rotation

Given a initial value of *R*(0)=*I*, solving the equation $\dot{R} = S(\boldsymbol{\omega})R$ yields

$$R(t) = e^{St} = I + St + \frac{(St)^2}{2!} + \cdots + \frac{(St)^n}{n!} + \cdots$$

Let ω be the angular frequency, **u** be the unit direction vector of the rotation, and φ be the rotation angle, i.e.

$$\boldsymbol{\omega} = \omega \mathbf{u} \qquad \varphi = \omega t$$

The above equation can be simplified to

$$R(\varphi) = e^{\tilde{\mathbf{u}}\varphi} = I + \tilde{\mathbf{u}} \sin\varphi + \tilde{\mathbf{u}}^2 (1 - \cos\varphi)$$

Euler-Lexell-Rodrigues Equation: ⇕

$$^{0}R_{1} = e^{\tilde{\mathbf{u}}\varphi} = I \cos\varphi + \tilde{\mathbf{u}} \sin\varphi + \mathbf{u}\mathbf{u}^{T} (1 - \cos\varphi)$$

Code Session

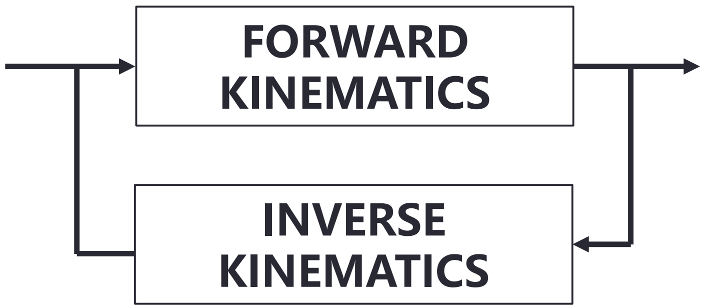
Ch6_1.m

6.2 Forward Velocity Kinematics

✓ The determination of the Cartesian velocity of end-effector in the global coordinate frame if having the time rate of joint variables of a manipulator.

Joint velocities

$\dot{q}_1 \; \dots \; \dot{q}_n$



Velocity of the end-effector

$$\boldsymbol{\omega}_n = \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} \quad \mathbf{v}_n = \begin{bmatrix} v_X \\ v_Y \\ v_Z \end{bmatrix}$$

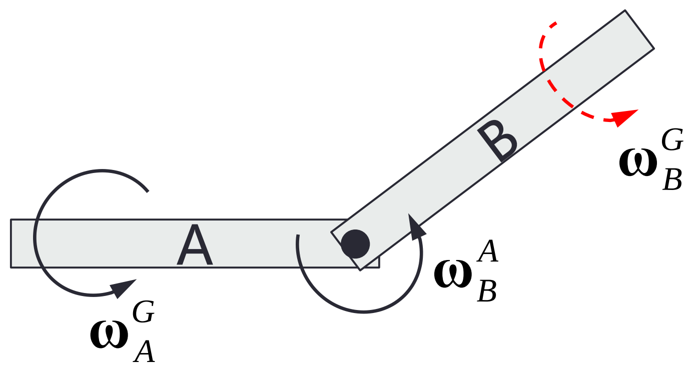
✓ The angular and translational velocity of the end-effector can be represented by two ways:

1）Vectors of angular velocity and translational velocity, $\boldsymbol{\omega}_n \quad \mathbf{v}_n$

2）Time derivative of homogeneous transformation matrix, ${}^0\dot{T}_n$

Angular Velocity of a Rigid Link

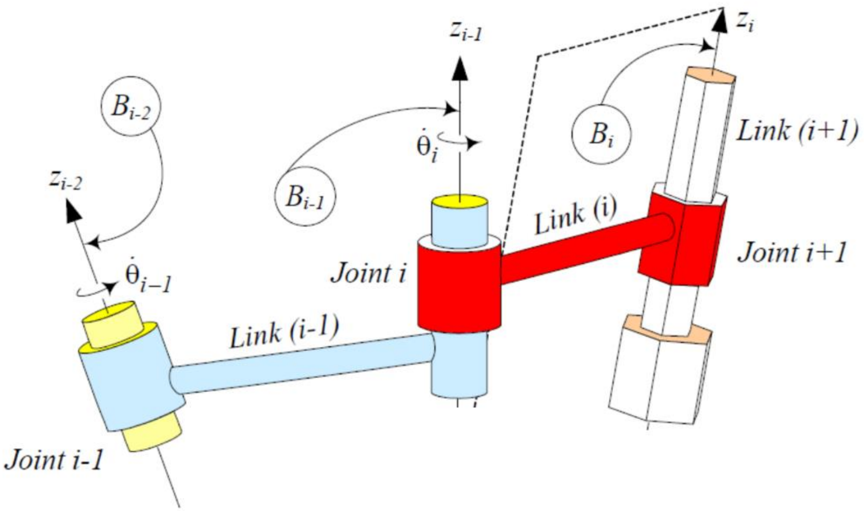
- ✓ Assume that the rigid body B is connected to the rigid body A through a pivot O.
- ✓ Physics tells us a simple conclusion that angular velocity of B comprises two parts: absolute angular velocity of its parent link plus angular velocity of B relative to A.



Angular Velocity of B $\boldsymbol{\omega}_B^G$

$\boldsymbol{\omega}_A^G$ Absolute angular Velocity of A + Relative angular of B w.r.t. A $\boldsymbol{\omega}_B^A$

Consider a generic Link *i* of a manipulator with an open kinematic chain. According to the D–H convention and the forward kinematics of rotation,



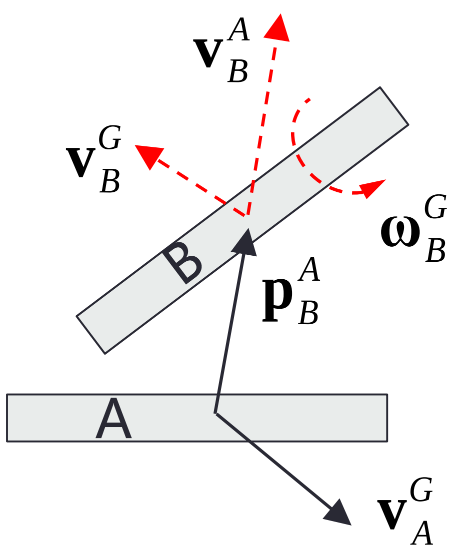
$$
\begin{aligned}
\boldsymbol{\omega}_i &= \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_i^{i-1} \\
&= \boldsymbol{\omega}_{i-1} + R_{i-1}\,{}^{i-1}\boldsymbol{\omega}_i^{i-1}
\end{aligned}
\qquad (6\text{-}1)
$$

$\boldsymbol{\omega}_i$	Global angular velocity of Link *i*
$\boldsymbol{R}_{i-1}$	Pose matrix of Link *i*-1 expressed in global frame
$\boldsymbol{\omega}_i^{i-1}$	Global angular velocity of Link *i rotating about axis i*-1
${}^{i-1}\boldsymbol{\omega}_i^{i-1}$	Relative angular velocity of Link *i rotating about axis i*-1 expressed in local frame

Note: ${}^0\boldsymbol{R}_{i-1} = \boldsymbol{R}_{i-1}$, if the upper-left frame natation is omitted.

Linear Velocity of a Rigid Link

- ✓ Assume that a rigid Link B moves with respect to rigid Link A.
- ✓ It can be concluded that linear velocity of B comprises three parts: absolute velocity of its parent link A, plus relative linear velocity w.r.t A, and plus velocity caused by rotation of B.



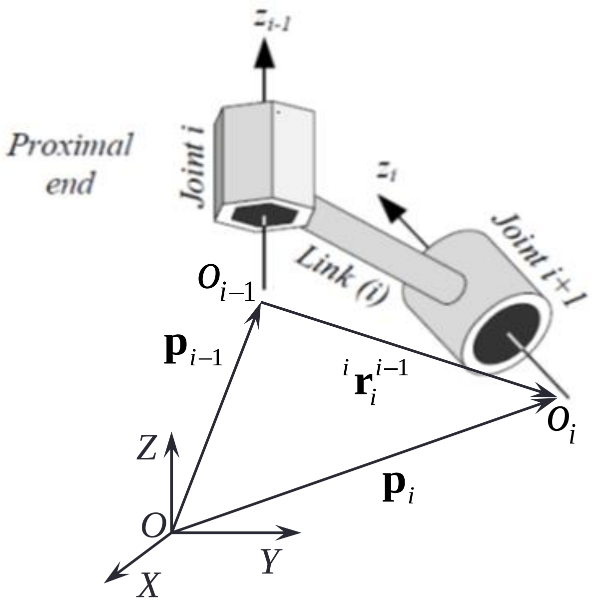
Linear Velocity of B $\quad \mathbf{v}_B^G$

Absolute linear velocity of A $\quad \mathbf{v}_A^G$

Relative linear velocity of B w.r.t. A $\quad \mathbf{v}_B^A$

Velocity caused by rotation of B $\quad \boldsymbol{\omega}_B^G \times \mathbf{p}_B^A$

Consider the generic Link *i* of a manipulator with an open kinematic chain. According to the D–H convention,



$$\dot{\mathbf{p}}_i = \dot{\mathbf{p}}_{i-1} + \mathbf{v}_i^{i-1} + \boldsymbol{\omega}_i \times \mathbf{r}_i^{i-1} \tag{6-2}$$

Relative translation **Relative Rotation**

$\mathbf{p}_i$	Global position of o_i
$\dot{\mathbf{p}}_i$	Global velocity of o_i
$\boldsymbol{\omega}_i$	Global angular velocity of Link *i*
$\mathbf{v}_i^{i-1}$	Relative velocity of o_i w.r.t. o_{i-1} expressed in global frame
$\mathbf{r}_i^{i-1}$	Relative position of o_i w.r.t. o_{i-1} expressed in global frame

This equation can also be produced by calculate the time derivative of position vector

$$\mathbf{p}_i = \mathbf{p}_{i-1} + {}^{0}R_i \, {}^{i}\mathbf{r}_i^{i-1} \quad \Rightarrow \quad \mathbf{v}_i = \dot{\mathbf{p}}_i = \mathbf{v}_{i-1} + \left(R_i \, {}^{i}\dot{\mathbf{r}}_i^{i-1} + \dot{R}_i \, {}^{i}\mathbf{r}_i^{i-1}\right)$$

$$= \mathbf{v}_{i-1} + R_i \, {}^{i}\dot{\mathbf{r}}_i^{i-1} + S(\boldsymbol{\omega}_i) R_i \, {}^{i}\mathbf{r}_i^{i-1}$$

$$= \mathbf{v}_{i-1} + \mathbf{v}_i^{i-1} + \boldsymbol{\omega}_i \times \mathbf{r}_i^{i-1}$$

Velocities of an Robot Link

According to **the D–H convention**, if joint i is a prismatic joint, frame i translates with respect to frame i-1 through d_i and there is no relative rotation between the two frames, i.e.

$$\mathbf{v}_i^{i-1} = \mathbf{z}_{i-1}\dot{d}_i = \sigma_i \mathbf{z}_{i-1}\dot{q}_i \quad \text{and} \quad \boldsymbol{\omega}_i^{i-1} = 0 \qquad (note: \dot{q}_i = \bar{\sigma}_i\dot{\theta}_i + \sigma_i\dot{d}_i)$$

If joint i is a revolute joint, relative rotation is quantified by $\dot{\theta}_i$ and there is no relative translation, i.e.

$$\mathbf{v}_i^{i-1} = 0 \quad \text{and} \quad \boldsymbol{\omega}_i^{i-1} = \mathbf{z}_{i-1}\dot{\theta}_i = \bar{\sigma}_i \mathbf{z}_{i-1}\dot{q}_i$$

Therefore, the linear and angular velocities of a robot link can be determined by

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_i^{i-1} = \boldsymbol{\omega}_{i-1} + \bar{\sigma}_i \mathbf{z}_{i-1}\dot{q}_i \qquad (note: \dot{q}_i = \bar{\sigma}_i\dot{\theta}_i + \sigma_i\dot{d}_i) \qquad \text{(6-3)}$$

$$\mathbf{v}_i = \mathbf{v}_{i-1} + \mathbf{v}_i^{i-1} + \boldsymbol{\omega}_i \times \mathbf{r}_i^{i-1} = \mathbf{v}_{i-1} + \sigma_i \mathbf{z}_{i-1}\dot{q}_i + \boldsymbol{\omega}_i \times \mathbf{r}_i^{i-1} \qquad \text{(6-4)}$$

✓ The computation for the global representations of rotation axis $\mathbf{z}_{i-1}$ of the robot link *i* is not time efficient. However, the representation of the axis vector in frame *i*-1 is always a constant vector **z**=[0 0 1]^T under D-H convention, which implies that the computation procedure can be simplified if the calculation is conducted in frame *i*-1, i.e.

$$\left.\begin{aligned} \boldsymbol{\omega}_i &= \boldsymbol{\omega}_{i-1} + \bar{\sigma}_i \mathbf{z}_{i-1} \dot{q}_i \\ {}^iR_0 &= {}^iR_{i-1}\,{}^{i-1}R_0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} {}^iR_0\boldsymbol{\omega}_i = {}^iR_{i-1}({}^{i-1}R_0\boldsymbol{\omega}_{i-1} + {}^{i-1}R_0\bar{\sigma}_i\mathbf{z}_{i-1}\dot{q}_i) \\ \hat{\boldsymbol{\omega}}_i \triangleq {}^iR_0\boldsymbol{\omega}_i \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \hat{\boldsymbol{\omega}}_i = {}^iR_{i-1}(\hat{\boldsymbol{\omega}}_{i-1} + \bar{\sigma}_i\mathbf{z}\dot{q}_i) \quad (note: \quad \hat{\boldsymbol{\omega}}_0 = 0, \quad \mathbf{z} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T)$$

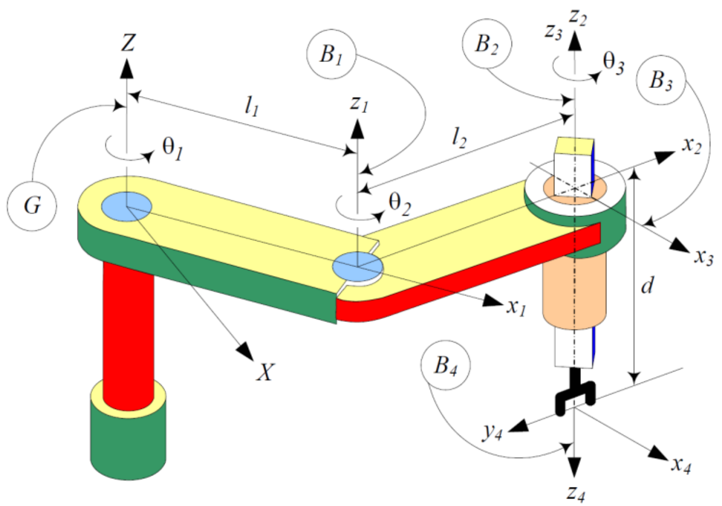
$$\left.\begin{aligned} \mathbf{v}_i &= \mathbf{v}_{i-1} + \sigma_i \mathbf{z}_{i-1}\dot{q}_i + \boldsymbol{\omega}_i \times \mathbf{r}_i^{i-1} \\ {}^iR_0 &= {}^iR_{i-1}\,{}^{i-1}R_0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} {}^iR_0\mathbf{v}_i = {}^iR_{i-1}({}^{i-1}R_0\mathbf{v}_{i-1} + {}^{i-1}R_0\sigma_i\mathbf{z}_{i-1}\dot{q}_i) + {}^iR_0\boldsymbol{\omega}_i \times {}^iR_0\mathbf{r}_i^{i-1} \\ \hat{\mathbf{v}}_i \triangleq {}^iR_0\mathbf{v}_i \end{aligned}\right\} \Rightarrow$$

$$\hat{\mathbf{v}}_i = {}^iR_{i-1}(\hat{\mathbf{v}}_{i-1} + \sigma_i\mathbf{z}\dot{q}_i) + \hat{\boldsymbol{\omega}}_i \times {}^i\mathbf{r}_i^{i-1} \qquad (\hat{\mathbf{v}}_0 = 0, \quad \hat{\mathbf{v}}_i \triangleq {}^iR_0\mathbf{v}_i, \quad {}^i\mathbf{r}_i^{i-1} = \begin{bmatrix} a_i & d_i s\alpha_i & d_i c\alpha_i \end{bmatrix}^T)$$

$${}^iT_{i-1} = \begin{bmatrix} c\theta_i & s\theta_i & 0 & -a_i \\ -c\alpha_i s\theta_i & c\alpha_i c\theta_i & s\alpha_i & -d_i s\alpha_i \\ s\alpha_i s\theta_i & -s\alpha_i c\theta_i & c\alpha_i & -d_i c\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^i\mathbf{r}_{i-1}^i = -{}^i\mathbf{r}_i^{i-1}$$

EX 6-2-1

Find the end-effector's velocity of a SCARA manipulator (L_1=L_2=d=1) under joint velocities of $\left[\dot\theta_1, \dot\theta_2, \dot\theta_3, \dot d\right] = [1, 1, -0.5, 0.1]$ *at* $\left[\theta_1, \theta_2, \theta_3, d\right] = \left[\frac{\pi}{4}, \frac{\pi}{2}, 0, 0.2\right]$



$$ {}^1T_0 = \begin{pmatrix} 0.7071 & 0.7071 & 0 & -1.0000 \\ -0.7071 & 0.7071 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \end{pmatrix} \qquad {}^2T_1 = \begin{pmatrix} 0.0000 & 1.0000 & 0 & -1.0000 \\ -1.0000 & 0.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \end{pmatrix} $$

$$ {}^3T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad {}^4T_3 = \begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & -1.0000 & 0 & 0 \\ 0 & 0 & -1.0000 & 0.2000 \\ 0 & 0 & 0 & 1.0000 \end{pmatrix} $$

$$ \bar\sigma_1 = \bar\sigma_2 = \bar\sigma_3 = 1, \quad \bar\sigma_4 = 0 $$
$$ \sigma_1 = \sigma_2 = \sigma_3 = 0, \quad \sigma_4 = 1 $$

$$ \widehat{\boldsymbol\omega}_i = {}^iR_{i-1}(\widehat{\boldsymbol\omega}_{i-1} + \bar\sigma_i \mathbf{z}\dot q_i) $$

$$ \widehat{\boldsymbol\omega}_1 = {}^1R_0\left(0 + 1\times\begin{bmatrix}0\\0\\1\end{bmatrix}\times 1\right) = \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \widehat{\boldsymbol\omega}_2 = {}^2R_1\left(\begin{bmatrix}0\\0\\1\end{bmatrix} + 1\times\begin{bmatrix}0\\0\\1\end{bmatrix}\times 1\right) = \begin{bmatrix}0\\0\\2\end{bmatrix} $$

$$ \widehat{\boldsymbol\omega}_3 = {}^3R_2\left(\begin{bmatrix}0\\0\\2\end{bmatrix} - 1\times\begin{bmatrix}0\\0\\1\end{bmatrix}\times 0.5\right) = \begin{bmatrix}0\\0\\1.5\end{bmatrix} \qquad \widehat{\boldsymbol\omega}_4 = {}^4R_3\left(\begin{bmatrix}0\\0\\1.5\end{bmatrix} + 0\times\begin{bmatrix}0\\0\\1\end{bmatrix}\times 0.1\right) = \begin{bmatrix}0\\0\\-1.5\end{bmatrix} $$

$$ \boldsymbol\omega_4 \triangleq {}^0R_4\widehat{\boldsymbol\omega}_4 = \begin{bmatrix} -0.7071 & 0.7071 & 0 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix}0\\0\\-1.5\end{bmatrix} = \begin{bmatrix}0\\0\\1.5\end{bmatrix} $$

$$ \widehat{\mathbf v}_1 = {}^1R_0(0+0) + \begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \widehat{\mathbf v}_2 = {}^2R_1\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\2\end{bmatrix}\times\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix} $$

$$ \widehat{\mathbf v}_3 = {}^3R_2\begin{bmatrix}1\\2\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1.5\end{bmatrix}\times\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix} \qquad \widehat{\mathbf v}_4 = {}^4R_3\left(\begin{bmatrix}1\\2\\0\end{bmatrix} + 1\times\begin{bmatrix}0\\0\\1\end{bmatrix}\times 0.1\right) + \begin{bmatrix}0\\0\\-1.5\end{bmatrix}\times\begin{bmatrix}0\\0\\-0.2\end{bmatrix} = \begin{bmatrix}1\\-2\\-0.1\end{bmatrix} $$

$$ \widehat{\mathbf v}_i = {}^iR_{i-1}(\widehat{\mathbf v}_{i-1} + \sigma_i \mathbf{z}\dot q_i) + \widehat{\boldsymbol\omega}_i \times {}^i\mathbf r_i^{i-1} $$

$$ \mathbf v_4 = {}^0R_4\widehat{\mathbf v}_4 = \begin{bmatrix} -0.7071 & 0.7071 & 0 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix}1\\-2\\-0.1\end{bmatrix} = \begin{bmatrix}-2.1213\\-0.7071\\0.1\end{bmatrix} $$

Time Derivative of HTM

✓ Forward velocity kinematics of *n*-link manipulator can be expressed by the time derivative of homogeneous transformation matrix under DH notations.

$$ {}^0T_n = {}^0T_n(q_1, q_1, \cdots, q_n) \Rightarrow {}^0\dot{T}_n = \frac{\partial\, {}^0T_n}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial\, {}^0T_n}{\partial q_n}\dot{q}_n = \sum_{i=1}^{n}\left(\frac{\partial}{\partial q_i}\, {}^0T_n\right)\dot{q}_i $$

$$ {}^0T_n = {}^0T_1(q_1)\cdots {}^0T_1(q_n) \Rightarrow \frac{\partial}{\partial q_i}\, {}^0T_n = {}^0T_{i-1}(q_1, \cdots, q_{i-1})\left[\frac{\mathrm{d}}{\mathrm{d}q_i}\, {}^{i-1}T_i(q_i)\right]{}^iT_n(q_{i+1}, \cdots, q_n) $$

Using DH notations $q_i = \bar{\sigma}_i\theta_i + \sigma_i d_i \Rightarrow$

($\sigma_i = 1$ if joint *i* is prismatic; $\sigma_i = 0$ if the joint is revolute)

$$ \frac{\mathrm{d}}{\mathrm{d}q_i}\, {}^{i-1}T_i(q_i) = \bar{\sigma}_i \frac{\mathrm{d}}{\mathrm{d}\theta_i}\, {}^{i-1}T_i(\theta_i) + \sigma_i \frac{\mathrm{d}}{\mathrm{d}d_i}\, {}^{i-1}T_i(d_i) $$

$$ {}^{i-1}T_i = \begin{bmatrix} c_i & -c\alpha_i s_i & s\alpha_i s_i & a_i c_i \\ s_i & c\alpha_i c_i & -s\alpha_i c_i & a_i s_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}d_i}\, {}^{i-1}T_i(d_i) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} {}^{i-1}T_i(d_i) $$

$$ \Downarrow $$

$$ \frac{d}{d\theta_i}\, {}^{i-1}T_i(\theta_i) = \begin{bmatrix} -s_i & -c\alpha_i c_i & s\alpha_i c_i & -a_i s_i \\ c_i & -c\alpha_i s_i & s\alpha_i s_i & a_i c_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} {}^{i-1}T_i(\theta_i) $$

Let $Q_i = \bar{\sigma}_i \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \sigma_i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$ \frac{\mathrm{d}}{\mathrm{d}q_i}\, {}^{i-1}T_i(q_i) = \bar{\sigma}_i \frac{\mathrm{d}}{\mathrm{d}\theta_i}\, {}^{i-1}T_i(\theta_i) + \sigma_i \frac{\mathrm{d}}{\mathrm{d}d_i}\, {}^{i-1}T_i(d_i) $$

$$ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}q_i}\, {}^{i-1}T_i(q_i) = Q_i\, {}^{i-1}T_i(q_i) $$

$$\frac{\mathrm{d}}{\mathrm{d}q_i}\,{}^{i-1}T_i(q_i) = Q_i\,{}^{i-1}T_i(q_i)$$

$$\frac{\partial}{\partial q_i}\,{}^{0}T_n = {}^{0}T_{i-1}\left[\frac{\mathrm{d}}{\mathrm{d}q_i}\,{}^{i-1}T_i(q_i)\right]{}^{i}T_n$$

$$\Rightarrow \quad \frac{\partial}{\partial q_i}\,{}^{0}T_n = {}^{0}T_{i-1}\left[Q_i\,{}^{i-1}T_i\right]{}^{i}T_n = {}^{0}T_{i-1}Q_i\,{}^{i-1}T_n$$

$${}^{0}\dot{T}_n = \sum_{i=1}^{n}\left(\frac{\partial}{\partial q_i}\,{}^{0}T_n\right)\dot{q}_i$$

$$\Rightarrow \quad {}^{0}\dot{T}_n = \sum_{i=1}^{n}\left({}^{0}T_{i-1}Q_i\,{}^{i-1}T_n\right)\dot{q}_i$$

not computationally efficient

Alternatively,

$$\frac{\mathrm{d}}{\mathrm{d}q_i}\,{}^{i-1}T_i(q_i) = Q_i\,{}^{i-1}T_i(q_i)$$

$${}^{0}\dot{T}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{0}T_{i-1}\,{}^{i-1}T_i\right) = {}^{0}\dot{T}_{i-1}\,{}^{i-1}T_i + {}^{0}T_{i-1}\,{}^{i-1}\dot{T}_i$$

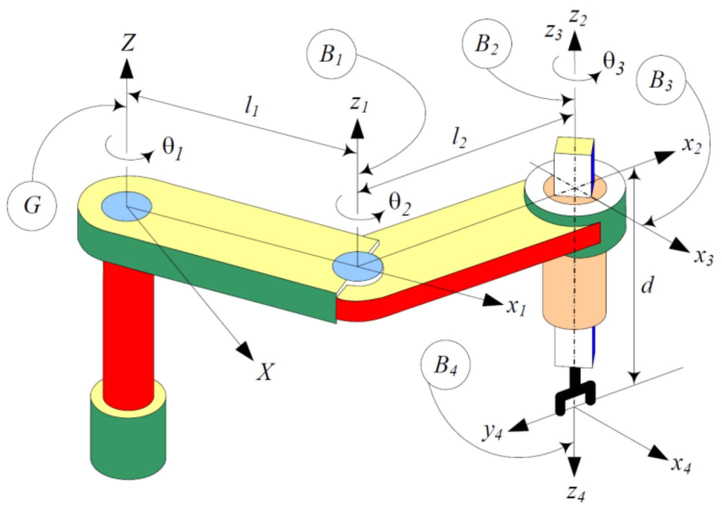
$$\Rightarrow \quad {}^{0}\dot{T}_i = {}^{0}\dot{T}_{i-1}\,{}^{i-1}T_i + {}^{0}T_{i-1}Q_i\,{}^{i-1}T_i\dot{q}_i$$

$$\left({}^{0}\dot{T}_0 = O, \quad {}^{0}T_0 = I\right)$$

This equation is the practically used iteration equation to compute the forward velocity kinematics problem.

EX 6-2-2

Find the end-effector's velocity of a SCARA manipulator ($L_1=L_2=d=1$) under joint velocities of $\left[\dot\theta_1,\dot\theta_2,\dot\theta_3,\dot d\right]=[1,1,-0.5,0.1]$ *at* $[\theta_1,\theta_2,\theta_3,d]=\left[\frac{\pi}{4},\frac{\pi}{2},0,0.2\right]$



$$^0T_1=\begin{bmatrix} c_1 & -s_1 & 0 & l_1c_1\\ s_1 & c_1 & 0 & l_1s_1\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix}\quad ^1T_2=\begin{bmatrix} c_2 & -s_2 & 0 & l_2c_2\\ s_2 & c_2 & 0 & l_2s_2\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix}\quad ^2T_3=\begin{bmatrix} c_3 & -s_3 & 0 & 0\\ s_3 & c_3 & 0 & 0\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix}$$

$$^3T_4=\begin{bmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&-1&d\\ 0&0&0&1\end{bmatrix}\quad Q_1=Q_2=Q_3=\begin{bmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix}\quad Q_4=\begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\end{bmatrix}$$

$$^0T_1=\begin{bmatrix} c_1 & -s_1 & 0 & l_1c_1\\ s_1 & c_1 & 0 & l_1s_1\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix}\quad ^0T_2=\begin{bmatrix} c_{12} & -s_{12} & 0 & l_2c_{12}+l_1c_1\\ s_{12} & c_{12} & 0 & l_2s_{12}+l_1s_1\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix}\quad ^0T_3=\begin{bmatrix} c_{123} & -s_{123} & 0 & l_1c_1+l_2c_{12}\\ s_{123} & c_{123} & 0 & l_1s_1+l_2s_{12}\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix}\quad ^0T_4=\begin{bmatrix} c_{123} & s_{123} & 0 & l_1c_1+l_2c_{12}\\ s_{123} & -c_{123} & 0 & l_1s_1+l_2s_{12}\\ 0&0&-1&d\\ 0&0&0&1\end{bmatrix}$$

$$^0\dot T_i == {}^0\dot T_{i-1}\,{}^{i-1}T_i + {}^0T_{i-1}Q_i\,{}^{i-1}T_i\dot q_i = \left({}^0\dot T_{i-1}+\dot q_i\,{}^0T_{i-1}Q_i\right){}^{i-1}T_i \quad \Rightarrow \quad {}^0\dot T_4 \quad \Rightarrow \quad {}^0\dot R_4,\quad \mathbf{v}$$

EX 6-2-2

Find the end-effector's velocity of a SCARA manipulator (L_1=L_2=d=1) under joint velocities of $\left[\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \dot{d}\right] = [1, 1, -0.5, 0.1]$ *at* $[\theta_1, \theta_2, \theta_3, d] = \left[\frac{\pi}{4}, \frac{\pi}{2}, 0, 0.2\right]$

$$ {}^0\dot{T}_1 = \begin{pmatrix} -0.7071 & -0.7071 & 0 & -0.7071 \\ 0.7071 & -0.7071 & 0 & 0.7071 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad {}^0\dot{T}_2 = \begin{pmatrix} -1.4142 & 1.4142 & 0 & -2.1213 \\ -1.4142 & -1.4142 & 0 & -0.7071 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} $$

$$ {}^0\dot{T}_3 = \begin{pmatrix} -1.0607 & 1.0607 & 0 & -2.1213 \\ -1.0607 & -1.0607 & 0 & -0.7071 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad {}^0\dot{T}_4 = \begin{pmatrix} -1.0607 & -1.0607 & 0 & -2.1213 \\ -1.0607 & 1.0607 & 0 & -0.7071 \\ 0 & 0 & 0 & 0.1000 \\ 0 & 0 & 0 & 0 \end{pmatrix} $$

$$ {}^0\dot{R}_4 = \begin{pmatrix} -1.0607 & -1.0607 & 0 \\ -1.0607 & 1.0607 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{v} = \begin{bmatrix} -2.1213 \\ -0.7071 \\ 0.1 \end{bmatrix} $$

$$ {}^0R_4{}^T = \begin{pmatrix} -0.7071 & 0.7071 & 0 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & -1.0000 \end{pmatrix} $$

$$ \Rightarrow \quad S(\boldsymbol{\omega}) = \dot{R}R^T = \begin{pmatrix} 0 & -1.5000 & 0 \\ 1.5000 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1.5 \end{bmatrix} $$

Code Session

Ch6_2.m