ARTICULATED ROBOTS

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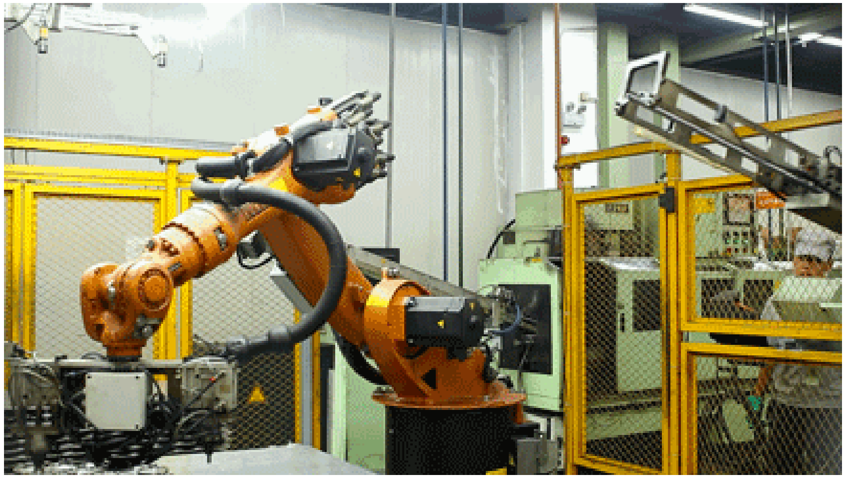
2. ROTATION GEOMETRY

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

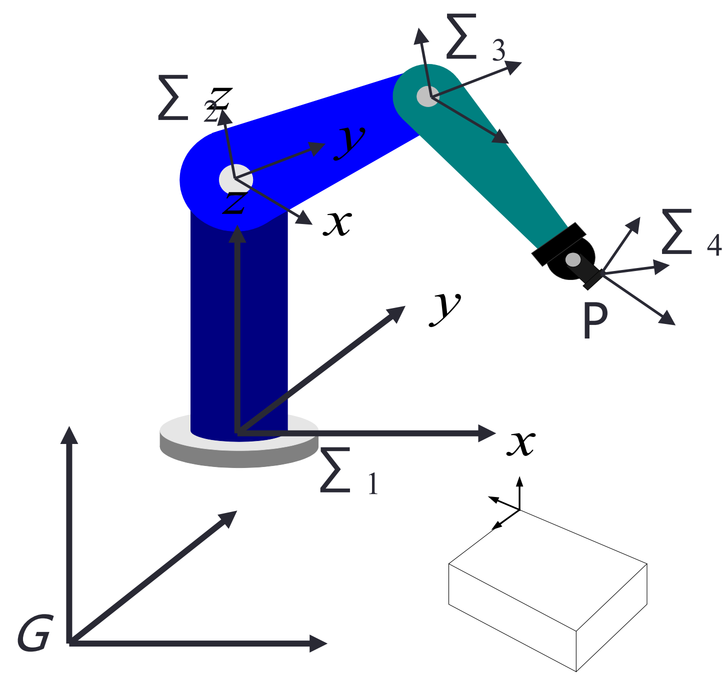
2.1 Position & Orientation

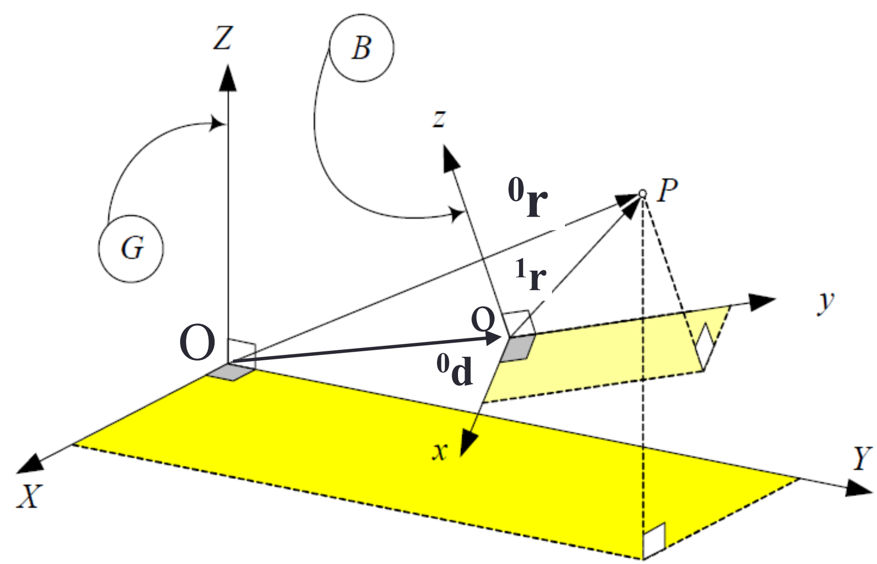
- ✓ In order to manipulate an object in space, it is necessary to describe orientation, position, and their time derivatives of the end-effector or the object manipulated by the end-effector.
- ✓ Analysis of motion is called kinematics. It is the science of geometry and is restricted to a pure geometrical description of motion.



Coordinate Frame

Since links of a manipulator may rotate or translate with respect to each other, body-attached coordinate frames $\sum_1$, $\sum_2$, $\sum_3$, · · · will be established along with the joint axis for each link to find their relative configurations, and within the global fixed reference frame $\sum_0$.

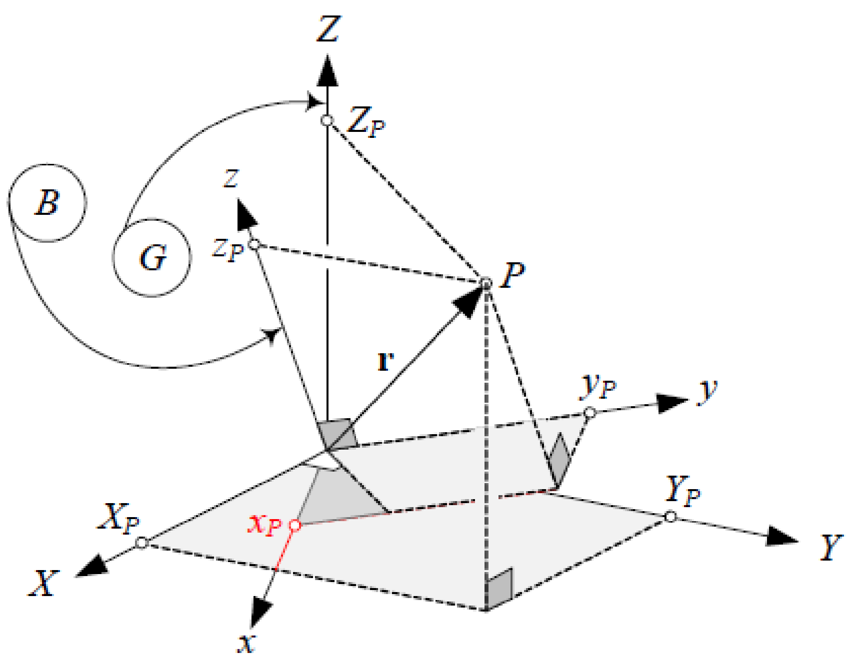




Rotation

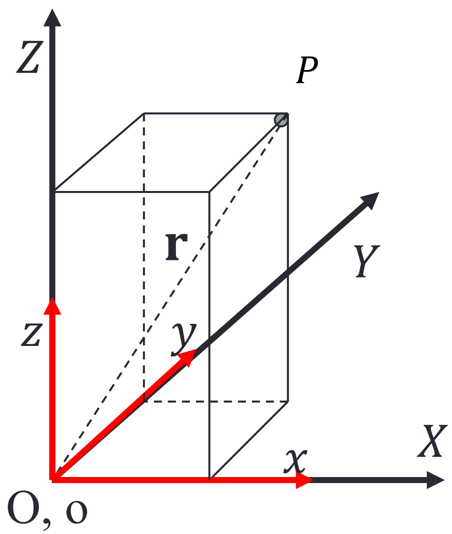
- ✓ Each rigid linkage of a robotic arm is attached with a coordinated system ∑.
- ✓ Linkages rotates with respect to its adjacent ones.

Point P is fixed on a linkage with an attached frame o-xyz and coordinates x_p, y_p, z_p. *How do we determine the position of P after this linkage rotates with respect to frame OXYZ?*



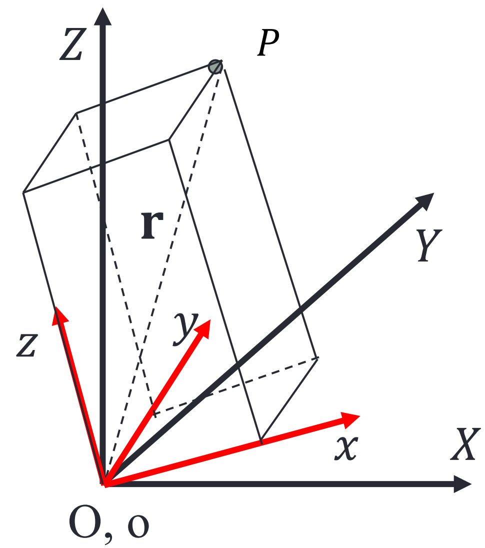
The vector that determines the pose of the rigid body has different representations in different frames after frame oxyz rotates with respect to OXYZ by certain angles.

$$ {}^{0}\mathbf{r} = X\mathbf{I} + Y\mathbf{J} + Z\mathbf{K} $$



O-XYZ: global coordinate frame

$$ {}^{1}\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} $$



o-xyz: body attached coordinate frame

Rotation Transformation Matrix

Coordinates of P in OXYZ are projections of vector ${}^{G}\mathbf{r}$ on each axes of OXYZ, respectively. Alternatively, using the **definition of the inner product** we may also have

$$X = \mathbf{I} \cdot \mathbf{r} = \mathbf{I} \cdot x\mathbf{i} + \mathbf{I} \cdot y\mathbf{j} + \mathbf{I} \cdot z\mathbf{k}$$

$$Y = \mathbf{J} \cdot \mathbf{r} = \mathbf{J} \cdot x\mathbf{i} + \mathbf{J} \cdot y\mathbf{j} + \mathbf{J} \cdot z\mathbf{k}$$

$$Z = \mathbf{K} \cdot \mathbf{r} = \mathbf{K} \cdot x\mathbf{i} + \mathbf{K} \cdot y\mathbf{j} + \mathbf{K} \cdot z\mathbf{k}$$

⇩

$$^{0}\mathbf{r} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \mathbf{I} \cdot \mathbf{i} & \mathbf{I} \cdot \mathbf{j} & \mathbf{I} \cdot \mathbf{k} \\ \mathbf{J} \cdot \mathbf{i} & \mathbf{J} \cdot \mathbf{j} & \mathbf{J} \cdot \mathbf{k} \\ \mathbf{K} \cdot \mathbf{i} & \mathbf{K} \cdot \mathbf{j} & \mathbf{K} \cdot \mathbf{k} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = {}^{0}R_{1}\, {}^{1}\mathbf{r}$$

${}^{0}R_{1}$ is called a **rotation transformation matrix**

$$ {}^{0}R_{1} = \begin{bmatrix} \mathbf{I}\cdot\mathbf{i} & \mathbf{I}\cdot\mathbf{j} & \mathbf{I}\cdot\mathbf{k} \\ \mathbf{J}\cdot\mathbf{i} & \mathbf{J}\cdot\mathbf{j} & \mathbf{J}\cdot\mathbf{k} \\ \mathbf{K}\cdot\mathbf{i} & \mathbf{K}\cdot\mathbf{j} & \mathbf{K}\cdot\mathbf{k} \end{bmatrix} $$

${}^{0}R_{1}$ converts coordinates of P in $\sum_1$ into coordinates in $\sum_0$.

In a same manner, coordinates of P in $\sum_0$ can be converted into $\sum_1$ with

$$ {}^{1}\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{i}\cdot\mathbf{I} & \mathbf{i}\cdot\mathbf{J} & \mathbf{i}\cdot\mathbf{K} \\ \mathbf{j}\cdot\mathbf{I} & \mathbf{j}\cdot\mathbf{J} & \mathbf{j}\cdot\mathbf{K} \\ \mathbf{k}\cdot\mathbf{I} & \mathbf{k}\cdot\mathbf{J} & \mathbf{k}\cdot\mathbf{K} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = {}^{1}R_{0}\,{}^{0}\mathbf{r} $$

${}^{1}R_{0}$ is a **rotation matrix** converting coordinates of P in $\sum_0$ into coordinates in $\sum_1$.

⇩

$$ {}^{1}R_{0} = {}^{0}R_{1}{}^{-1} $$

Properties

1. *R* is an orthogonal matrix because

$$^{1}R_0 = {^{0}R_1}^{-1}$$

2. The rotation transformation matrix is also called a matrix of direction cosines because its elements are all direction cosines of **r** with respect to axes of the coordinate frame.

$$^{1}R_0 = \begin{bmatrix} \mathbf{I}\cdot\mathbf{i} & \mathbf{I}\cdot\mathbf{j} & \mathbf{I}\cdot\mathbf{k} \\ \mathbf{J}\cdot\mathbf{i} & \mathbf{J}\cdot\mathbf{j} & \mathbf{J}\cdot\mathbf{k} \\ \mathbf{K}\cdot\mathbf{i} & \mathbf{K}\cdot\mathbf{j} & \mathbf{K}\cdot\mathbf{k} \end{bmatrix} = \begin{bmatrix} \cos(\mathbf{I},\mathbf{i}) & \cos(\mathbf{I},\mathbf{j}) & \cos(\mathbf{I},\mathbf{k}) \\ \cos(\mathbf{J},\mathbf{i}) & \cos(\mathbf{J},\mathbf{j}) & \cos(\mathbf{J},\mathbf{k}) \\ \cos(\mathbf{K},\mathbf{i}) & \cos(\mathbf{K},\mathbf{j}) & \cos(\mathbf{K},\mathbf{k}) \end{bmatrix}$$

R is also called a direction cosine matrix between coordinate frames 0 and 1

Properties

3. Three columns of R are representations of 3 primary axes of the local frame 1 in 0, **n**, **s**, and **a,** respectively. Therefore, the pose of a rigid body can be described by the relative relations among axes of their coordinate systems.

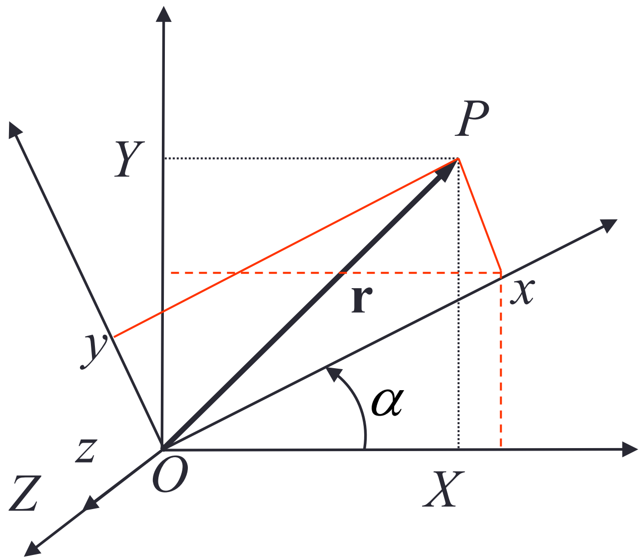
$$\mathbf{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \quad \mathbf{s} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} \quad \mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad {}^0R_1 \triangleq [\mathbf{n} \quad \mathbf{s} \quad \mathbf{a}] = \begin{bmatrix} \mathbf{I}\cdot\mathbf{i} & \mathbf{I}\cdot\mathbf{j} & \mathbf{I}\cdot\mathbf{k} \\ \mathbf{J}\cdot\mathbf{i} & \mathbf{J}\cdot\mathbf{j} & \mathbf{J}\cdot\mathbf{k} \\ \mathbf{K}\cdot\mathbf{i} & \mathbf{K}\cdot\mathbf{j} & \mathbf{K}\cdot\mathbf{k} \end{bmatrix}$$

4. There are only 3 variables in the matrix because it contains 6 constraints.

$$\|\mathbf{n}\| = \|\mathbf{s}\| = \|\mathbf{a}\| = 1 \qquad \mathbf{n}\times\mathbf{s} = \mathbf{a} \qquad \mathbf{s}\times\mathbf{a} = \mathbf{n} \qquad \mathbf{a}\times\mathbf{n} = \mathbf{s}$$

Rotation about Global Axes

Rotation about Z-axis



$$X = x\cos\alpha - y\sin\alpha$$

$$Y = x\sin\alpha + y\cos\alpha$$

$$Z = z$$

$$\Downarrow$$

$$^{0}\mathbf{r} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_Z(\alpha)\,^{1}\mathbf{r}$$

Using definition of direction cosines, the same results can be obtained:

$$R_Z(\alpha) = \begin{bmatrix} \mathbf{I}\cdot\mathbf{i} & \mathbf{I}\cdot\mathbf{j} & \mathbf{I}\cdot\mathbf{k} \\ \mathbf{J}\cdot\mathbf{i} & \mathbf{J}\cdot\mathbf{j} & \mathbf{J}\cdot\mathbf{k} \\ \mathbf{K}\cdot\mathbf{i} & \mathbf{K}\cdot\mathbf{j} & \mathbf{K}\cdot\mathbf{k} \end{bmatrix}$$

$$\begin{array}{lll} \hat{I}\cdot\hat{\imath} = \cos\alpha, & \hat{I}\cdot\hat{\jmath} = -\sin\alpha, & \hat{I}\cdot\hat{k} = 0 \\ \hat{J}\cdot\hat{\imath} = \sin\alpha, & \hat{J}\cdot\hat{\jmath} = \cos\alpha, & \hat{J}\cdot\hat{k} = 0 \\ \hat{K}\cdot\hat{\imath} = 0, & \hat{K}\cdot\hat{\jmath} = 0, & \hat{K}\cdot\hat{k} = 1 \end{array}$$

About X-axis

$$R_X(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

About Y-axis

$$R_Y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

About Z-axis

$$R_Z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

EX 2-1-1

After a vector $^0\mathbf{r}$ = [1, 3, 2]^T in frame 0 rotates with frame 1 about Z-axis by 60 degrees, its position in the global coordinate frame is:

$$^{0}\mathbf{r} = R_Z\,{}^{1}\mathbf{r} = \begin{bmatrix} \cos 60 & \sin 60 & 0 \\ -\sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2.0981 \\ 2.3660 \\ 2 \end{bmatrix}$$

EX 2-1-2

Consider a rigid body B that is continuously turning about the Y-axis of G at a rate of 0.3 rad/ s. Find coordinates of a point P on B in G and its velocity.

1. The rotation transformation matrix of the body is:

$$Q_{Y,\beta} = \begin{bmatrix} \cos 0.3t & 0 & \sin 0.3t \\ 0 & 1 & 0 \\ -\sin 0.3t & 0 & \cos 0.3t \end{bmatrix}$$

2. Coordinates in global and local frame can be related by ${}^{G}\mathbf{r}_P = Q_{Y,\beta}\,{}^{B}\mathbf{r}_P$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos 0.3t & 0 & \sin 0.3t \\ 0 & 1 & 0 \\ -\sin 0.3t & 0 & \cos 0.3t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos 0.3t + z\sin 0.3t \\ y \\ z\cos 0.3t - x\sin 0.3t \end{bmatrix}$$

3. The velocity of P can be found by taking a time derivative of ${}^{G}\mathbf{r}_P = Q_{Y,\beta}\,{}^{B}\mathbf{r}_P$

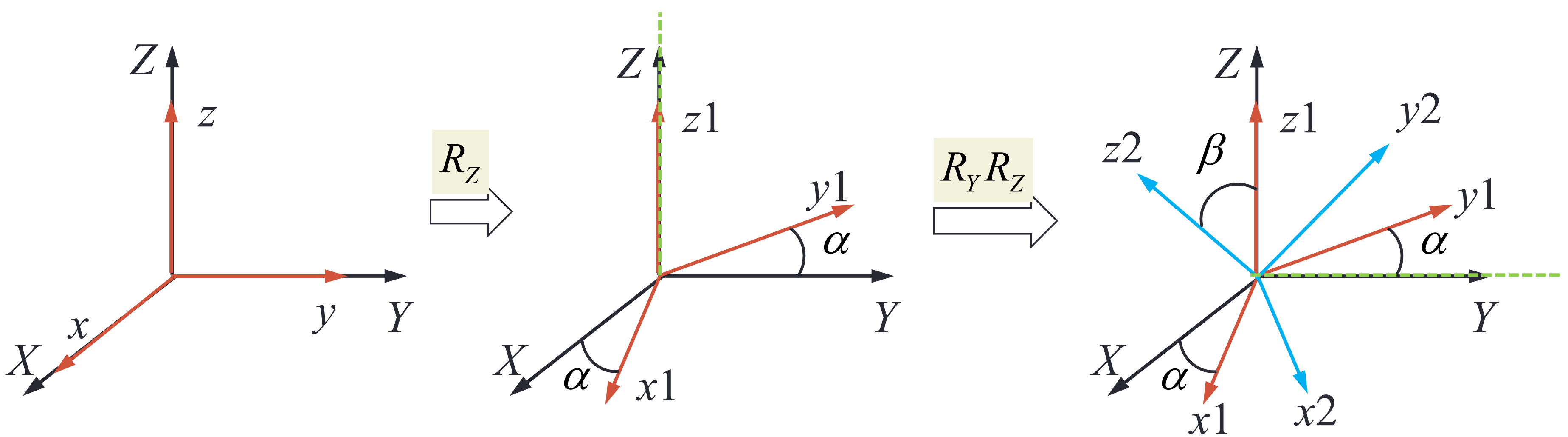
$${}^{G}\mathbf{v}_P = \dot{Q}_{Y,\beta}\,{}^{B}\mathbf{r}_P = 0.3 \begin{bmatrix} z\cos 0.3t - x\sin 0.3t \\ 0 \\ -x\cos 0.3t - z\sin 0.3t \end{bmatrix}$$

Successive Rotation

✓ R_1 and R_2 are two successive rotations **about global axes**. A vector **r** after the first rotation becomes ${}^0\mathbf{r}_1 = R_1\mathbf{r}$. After the second rotation, it becomes ${}^0\mathbf{r}_2 = R_2\,{}^0\mathbf{r}_1$. Then we have

$$\left.\begin{aligned} {}^0\mathbf{r}_2 &= {}^0R_2\,{}^0\mathbf{r}_1 \\ {}^0\mathbf{r}_1 &= {}^0R_1\mathbf{r} \end{aligned}\right\} \Rightarrow {}^0\mathbf{r}_2 = {}^0R_2({}^0R_1\mathbf{r}) = {}^0R_2\,{}^0R_1\mathbf{r} \Rightarrow \boxed{{}^0R = {}^0R_2\,{}^0R_1}$$

pre-multiplied



Successive Rotation

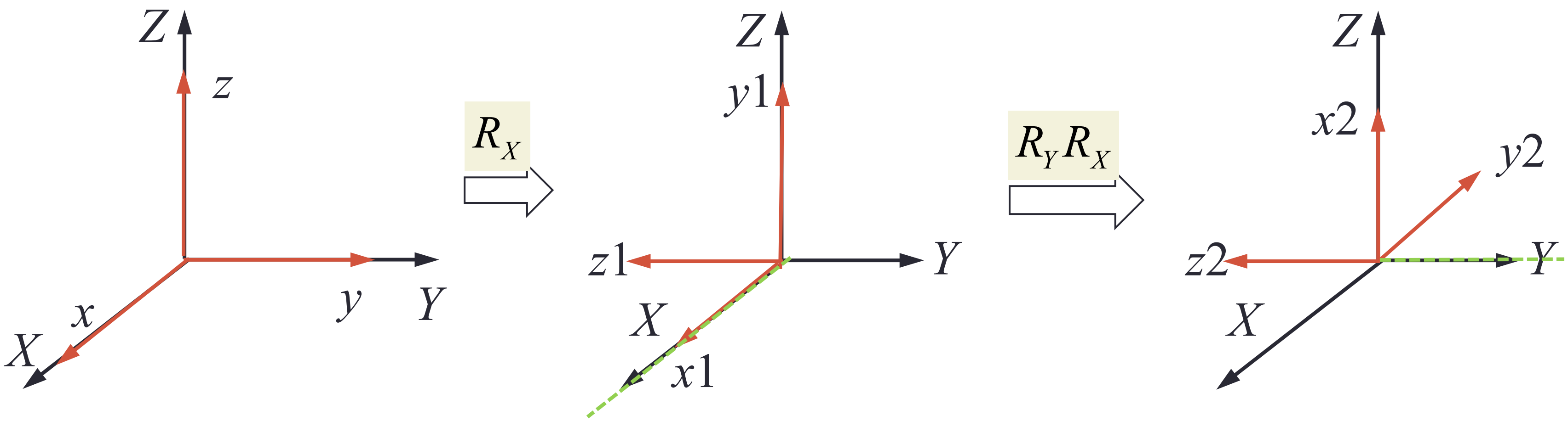
- ✓ The final global position of a point P in a rigid body B with position vector **r**, after a sequence of rotations R_1, R_2, R_3, ..., R_n about the global axes can be found by

$$\boxed{{}^{0}\mathbf{r}_{n} = {}^{0}R_{n}\,\mathbf{r}} \qquad \text{where} \qquad {}^{0}R_{n} = {}^{0}R_{n} \cdots {}^{0}R_{2}\,{}^{0}R_{1}$$

- ✓ *R* should be **pre-multiplied** if the rotation is **about global axes.**
- ✓ Matrix multiplications do not commute. Changing the order of global rotation matrices is equivalent to changing the order of rotations.

EX 2-1-3

Find rotation matrix after successive rotations about X by 90degs and Y by -90degs.



$$^{0}R_1 \triangleq [\mathbf{n} \quad \mathbf{s} \quad \mathbf{a}]$$

$$^{0}R_1 = R_Y(-\frac{\pi}{2})R_X(\frac{\pi}{2})$$

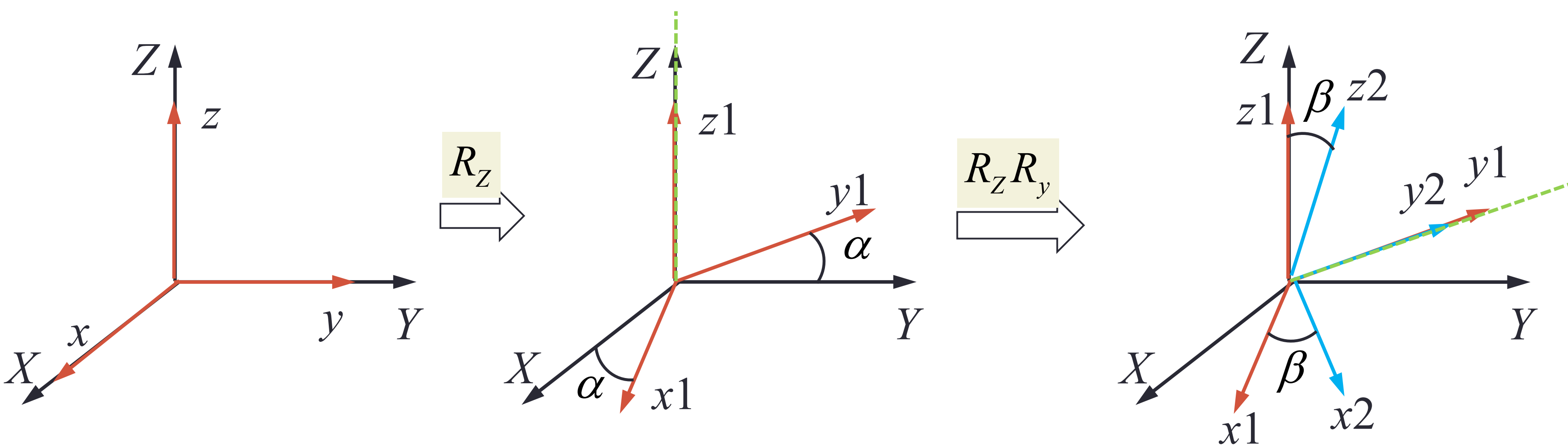
$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Rotation about Local Axes

- ✓ Assume that successive rotations are made in the way that the first rotation is about the primary axis of frame 0 and the second rotation is about the primary axis of frame 1 which is a local frame. A vector **r** in $\sum_0$, $\sum_1$ and $\sum_2$, is represented by ${}^0\mathbf{r}$, ${}^1\mathbf{r}$, and ${}^2\mathbf{r}$, respectively, and

$$\left.\begin{aligned} {}^0\mathbf{r} &= {}^0R_1\,{}^1\mathbf{r} \\ {}^1\mathbf{r} &= {}^1R_2\,{}^2\mathbf{r} \end{aligned}\right\} \Rightarrow {}^0\mathbf{r} = {}^0R_1\,{}^1\mathbf{r} = {}^0R_1({}^1R_2\,{}^2\mathbf{r}) \Rightarrow \boxed{{}^0R_2 = {}^0R_1\,{}^1R_2}$$

post-multiplied.

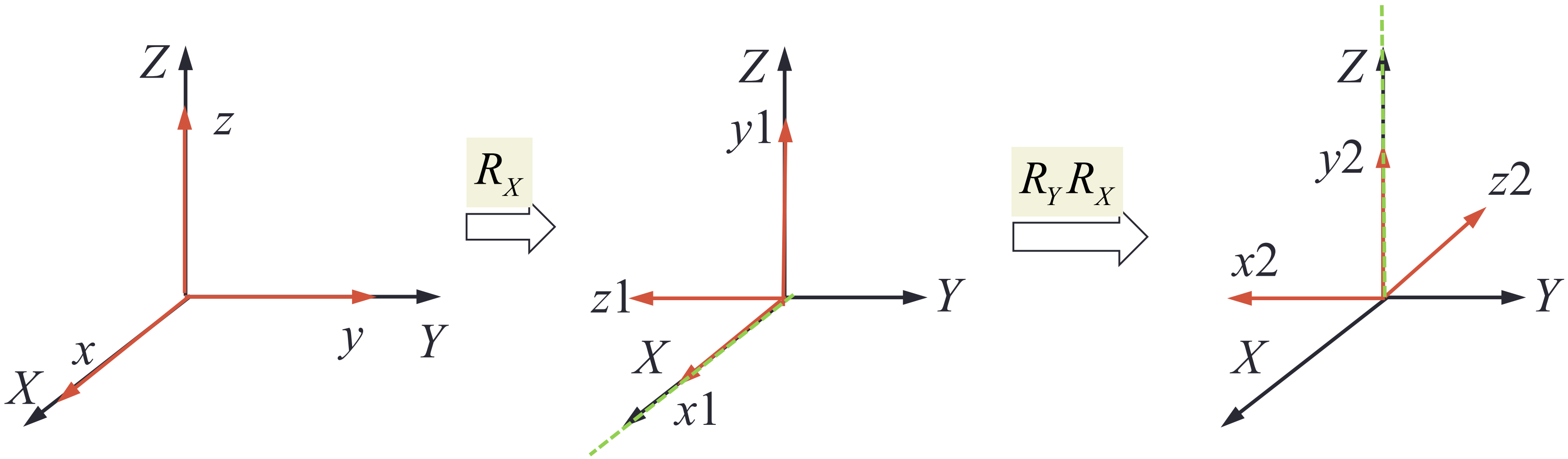


Composition of Rotations

- ✓ The equation implies that if a rotation is about a local frame, the rotation matrix *R* is a relative one and it should be **post-multiplied**.
- ✓ If a rotation is about a global frame, the rotation matrix *R* is an absolute one and it should be **pre-multiplied**.

EX 2-1-4

Find rotation matrix after successive rotations about X by 90degs and y by -90degs.



$$ {}^0R_1 \triangleq [\mathbf{n} \quad \mathbf{s} \quad \mathbf{a}] $$

$$ {}^0R_1 = R_X(\frac{\pi}{2})R_y(-\frac{\pi}{2}) $$

$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} $$

Code Session

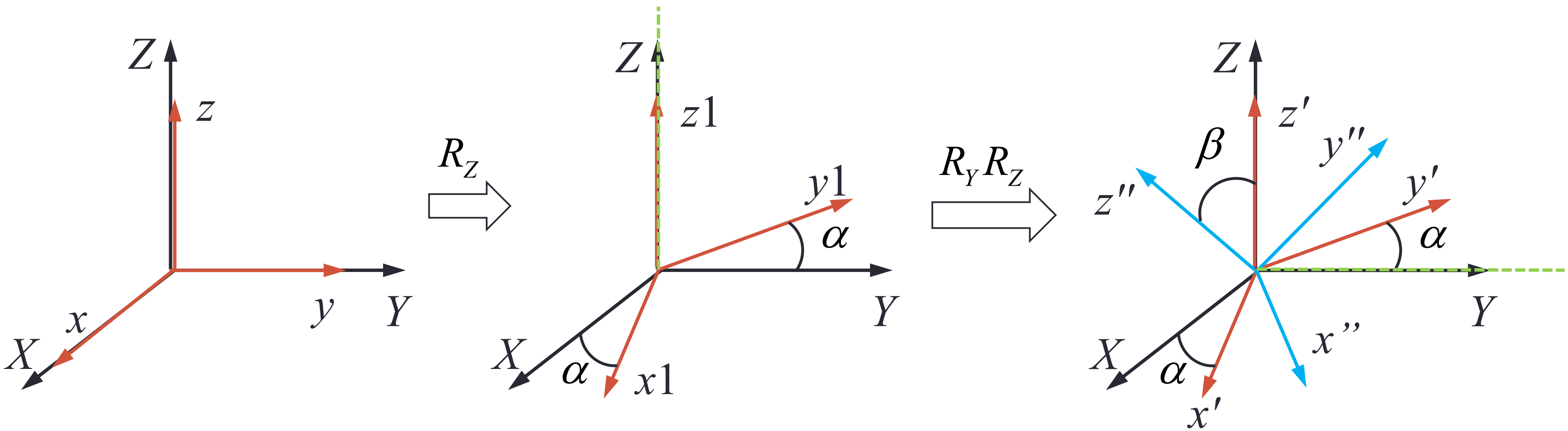
Ch2_1.m

2.2 Representation of Orientation

- ✓ **Rotation matrix** is one of the many models for representing the orientation of a rigid body.
- ✓ Mathematically, a given orientation can be expressed by three successive rotation about axes of a coordinate frame, which implies that any direction cosine matrix can be the product of three fundamental rotation matrices.
- ✓ The orientation can be defined by **three angles** associated with the fundamental rotation matrices.
- ✓ Triple-rotation representation can be made by using **angles about axes of a (fixed) global coordinate frame** and/or **angles about axes of a (moving) local coordinate frame.**

Fixed-angle Representation

- ✓ If triple rotations are made about the primary axes of a global coordinate frame, we can use three **Fixed Angle** representation to describe an orientation.
- ✓ In general, there are 12 different independent combinations of triple rotations about the global axes to transform body coordinates in frame 1 from the coincident position with a global frame 0 to any final orientation.

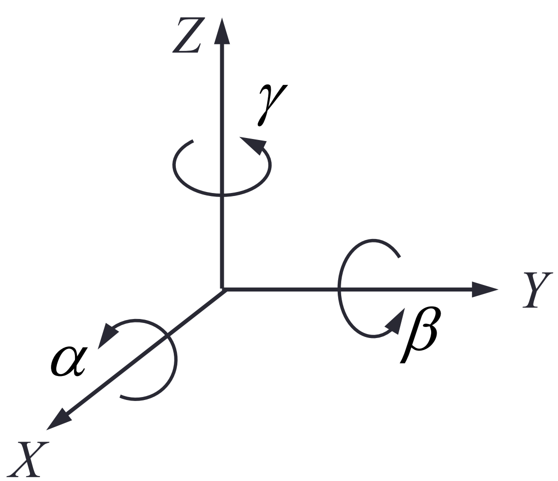


RPY Angels

Triple angles about fixed axes can be utilized to determine the orientation of a rigid body. They are also called global roll-pitch-yaw angles.

α, β, and γ are called

X-Y-Z RPY Angels



$$
\begin{aligned}
{}^{0}R_{1} &= R_Z(\gamma)R_Y(\beta)R_X(\alpha) \qquad (\sin\alpha \to s\alpha, \quad \cos\alpha \to c\alpha) \\
&= \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix} \\
&= \begin{bmatrix} c\alpha c\beta & -s\alpha c\beta & s\beta \\ s\alpha c\gamma + c\alpha s\beta s\gamma & c\alpha c\gamma - s\alpha s\beta s\gamma & -c\beta s\gamma \\ s\alpha s\gamma - c\alpha s\beta c\gamma & c\alpha s\gamma + s\alpha s\beta c\gamma & c\beta c\gamma \end{bmatrix}
\end{aligned}
$$

The 12 fixed-angle representations include

ABC Type	ABA Type
ZXY, ZYX	ZXZ, ZYZ
XYZ, XZY	XYX, XZX
YXZ, YZX	YXY, YZY

EX 2-2-1

A local frame rotates about the global Z-Y-X axes by 30deg, 45deg, and 90deg, subsequently. Determine the orientation by three fixed angles.

$$
{}^{0}R_{1} = R_X(\gamma)R_Y(\beta)R_Z(\alpha)
$$

$$
= \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix}
$$

$$
= \begin{bmatrix} c\alpha c\beta & -s\alpha c\beta & s\beta \\ s\alpha c\gamma + c\alpha s\beta s\gamma & c\alpha c\gamma - s\alpha s\beta s\gamma & -c\beta s\gamma \\ s\alpha s\gamma - c\alpha s\beta c\gamma & c\alpha s\gamma + s\alpha s\beta c\gamma & c\beta c\gamma \end{bmatrix}
$$

$$
\mathrm{R} = \begin{bmatrix} 0.6124 & -0.3536 & 0.7071 \\ 0.6124 & -0.3536 & -0.7071 \\ 0.5000 & 0.8660 & 0 \end{bmatrix}
$$

Euler-angle Representation

- ✓ Euler proved that any two independent orthogonal coordinate frames with a common origin can be related by a sequence of three rotations about the local coordinate axes, where no two successive rotations may be about the same axis.
- ✓ Triple angle rotation made about local axes can also determine the orientation of a rigid body. In this case, these angles are called Euler Angles.
- ✓ In general, there are 12 different independent combinations of triple rotation about local axes.

ABC Type	ABA Type
Zxy, Zyx,	Zxz, Zyz
Xyz, Xzy,	Xyx, Xzx
Yxz, Yzx,	Yxy, Yzy

Z-x-z Euler Angles

A particular case is z-x-z Euler Angle which performs

- ✓ Step1: rotation about the Z-axis by α - Precession
- ✓ Step2: rotation about the local x-axis by β - Nutation
- ✓ Step3: rotation about the local z-axis by γ - Spin

$$
\begin{aligned}
{}^{0}R_{1} &= R_{Z}(\alpha)R_{x}(\beta)R_{z}(\gamma) \\
&= \begin{bmatrix} \mathrm{c}\alpha & -\mathrm{s}\alpha & 0 \\ \mathrm{s}\alpha & \mathrm{c}\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\beta & -\mathrm{s}\beta \\ 0 & \mathrm{s}\beta & \mathrm{c}\beta \end{bmatrix} \begin{bmatrix} \mathrm{c}\gamma & -\mathrm{s}\gamma & 0 \\ \mathrm{s}\gamma & \mathrm{c}\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
{}^{0}R_{1} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{c}\alpha\mathrm{c}\gamma - \mathrm{s}\alpha\mathrm{c}\beta\mathrm{s}\gamma & -\mathrm{c}\alpha\mathrm{s}\gamma - \mathrm{s}\alpha\mathrm{c}\beta\mathrm{c}\gamma & \mathrm{s}\alpha\mathrm{s}\beta \\ \mathrm{s}\alpha\mathrm{c}\gamma + \mathrm{c}\alpha\mathrm{c}\beta\mathrm{s}\gamma & -\mathrm{s}\alpha\mathrm{s}\gamma + \mathrm{c}\alpha\mathrm{c}\beta\mathrm{c}\gamma & -\mathrm{c}\alpha\mathrm{s}\beta \\ \mathrm{s}\beta\mathrm{s}\gamma & \mathrm{s}\beta\mathrm{c}\gamma & \mathrm{c}\beta \end{bmatrix}
$$

EX 2-2-2

A local frame rotates about the X-y-z axes by 90deg, 45deg, and 30deg, subsequently. Determine the orientation by three Euler angles.

$$
{}^{0}R_{1} = R_X(\alpha)R_y(\beta)R_z(\gamma)
$$

$$
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} c\gamma c\beta & \text{-}s\gamma c\beta & s\beta \\ c\alpha s\gamma + s\alpha s\beta c\gamma & c\alpha c\gamma - s\alpha s\beta s\gamma & -s\alpha c\beta \\ s\alpha s\gamma - c\alpha s\beta c\gamma & s\alpha c\gamma + c\alpha s\beta s\gamma & c\alpha c\beta \end{bmatrix}
$$

$$
\mathrm{R} = \begin{bmatrix} 0.6124 & \text{-}0.3536 & 0.7071 \\ 0.6124 & \text{-}0.3536 & \text{-}0.7071 \\ 0.5000 & 0.8660 & 0 \end{bmatrix}
$$

Euler Angles V.S. Fixed Angles

ABC Euler angles $R_{Zyx} = R_Z(\alpha)R_y(\beta)R_x(\gamma)$

CBA fixed angles $R_{XYZ} = R_Z(\gamma)R_Y(\beta)R_X(\alpha)$

ABC Euler-angle representation has the same form as the CBA fixed- angle representation

ABA Euler angles $R_{Zxz} = R_Z(\alpha)R_x(\beta)R_z(\gamma)$

ABA fixed angles $R_{ZXZ} = R_Z(\gamma)R_X(\beta)R_Z(\alpha)$

ABA Euler-angle representation has the same form as the ABA fixed- angle representation

2.3 Triple-angles

1. The triple-angle representation is not unique. An orientation defined by a rotation matrix may correspond to different triple Euler angles or fixed angles. For example

$$R_Z(\gamma \pm \pi) R_Y(-\beta \pm \pi) R_X(\alpha \pm \pi) = R_Z(\gamma) R_Y(\beta) R_X(\alpha)$$

Proof

$$R_Z(\pm\pi + \gamma) R_Y(\pm\pi - \beta) R_X(\pm\pi + \alpha)$$

$$= \begin{bmatrix} \cos(\pm\pi + \lambda) & -\sin(\pm\pi + \lambda) & 0 \\ \sin(\pm\pi + \lambda) & \cos(\pm\pi + \lambda) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\pm\pi - \beta) & 0 & \sin(\pm\pi - \beta) \\ 0 & 1 & 0 \\ -\sin(\pm\pi - \beta) & 0 & \cos(\pm\pi - \beta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pm\pi + \alpha) & -\sin(\pm\pi + \alpha) \\ 0 & \sin(\pm\pi + \alpha) & \cos(\pm\pi + \alpha) \end{bmatrix}$$

$$= \begin{bmatrix} -\cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & -\cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & -\cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & -\cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & -\cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= R_Z(\gamma) R_Y(\beta) R_X(\alpha)$$

- ✓ For ABC or CBA type of fixed angles or Euler angels, in order to obtain unique solution of orientation by triple rotations, the pitch angel β is usually restricted within $(-\pi/2,\ \pi/2)$:

ABC or CBA type: $\boxed{\beta \in (-\frac{\pi}{2},\ \ \frac{\pi}{2})}$ $\quad (\alpha,\beta,\gamma) \in (-\pi,\pi] \times (-\frac{\pi}{2},\frac{\pi}{2}) \times (-\pi,\pi]$

Euler or RPY angles: $\quad (\alpha, \beta, \gamma) \Leftrightarrow (\alpha \pm \pi, -\beta \pm \pi, \gamma \pm \pi)$

- ✓ Similarly, it can be proved that for ABA type of fixed angles or Euler angels, if the pitch angel β is restricted within $(0,\ \pi)$, unique solution of orientation by triple rotations can be obtained.

ABA type: $\boxed{\beta \in (0,\ \ \pi)}$ $\quad (\alpha,\beta,\gamma) \in (-\pi,\pi] \times (0,\pi) \times (-\pi,\pi]$

Euler or RPY angles: $\quad (\alpha, \beta, \gamma) \Leftrightarrow (\alpha \pm \pi, -\beta, \gamma \pm \pi)$

✓ The equivalent RPY angles can be found when a rotation matrix is given.

$$ {}^{0}R_{1} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\beta c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha s\gamma + c\alpha s\beta c\gamma \\ c\beta s\gamma & c\alpha c\gamma + s\alpha s\beta s\gamma & c\alpha s\beta s\gamma - s\alpha c\gamma \\ -s\beta & s\alpha c\beta & c\alpha c\beta \end{bmatrix} $$

r_{ij} indicates the element of row *i* and column *j* of the RPY rotation matrix

If $\beta \neq \pm\frac{\pi}{2}$

$$ \because \beta \in (-\frac{\pi}{2}, \ \frac{\pi}{2}), \quad \therefore \cos\beta > 0 $$

$$ \beta = -\arcsin r_{31} \qquad \alpha = \mathrm{atan2}\left(r_{32}, r_{33}\right) \qquad \gamma = \mathrm{atan2}\left(r_{21}, r_{11}\right) $$

$$ \arctan 2(y,x) = \begin{cases} \arctan(y/x) & x > 0 \\ \arctan(y/x) + \pi & x < 0,\ y \geq 0 \\ \arctan(y/x) - \pi & x < 0,\ y < 0 \\ \pi/2 & x = 0,\ y > 0 \\ -\pi/2 & x = 0,\ y < 0 \\ \text{Undifined} & x = 0,\ y = 0 \end{cases} $$

$\boxed{\text{If } \beta = \pm\frac{\pi}{2},}$ there will be infinite number of solutions for *α* and *γ*.

$$\sin\beta = \pm 1, \quad \cos\beta = 0$$

$$ {}^{0}R_{1} = \begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ \mp 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \pm \mathrm{s}\alpha\mathrm{c}\gamma - \mathrm{c}\alpha\mathrm{s}\gamma & \mathrm{s}\alpha\mathrm{s}\gamma \pm \mathrm{c}\alpha\mathrm{c}\gamma \\ 0 & \mathrm{c}\alpha\mathrm{c}\gamma \pm \mathrm{s}\alpha\mathrm{s}\gamma & \pm\mathrm{c}\alpha\mathrm{s}\gamma - \mathrm{c}\gamma\mathrm{s}\alpha \\ \mp 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\sin(\gamma \mp \alpha) & \pm\cos(\alpha \mp \gamma) \\ 0 & \cos(\alpha \mp \gamma) & -\sin(\alpha \mp \gamma) \\ \mp 1 & 0 & 0 \end{bmatrix} $$

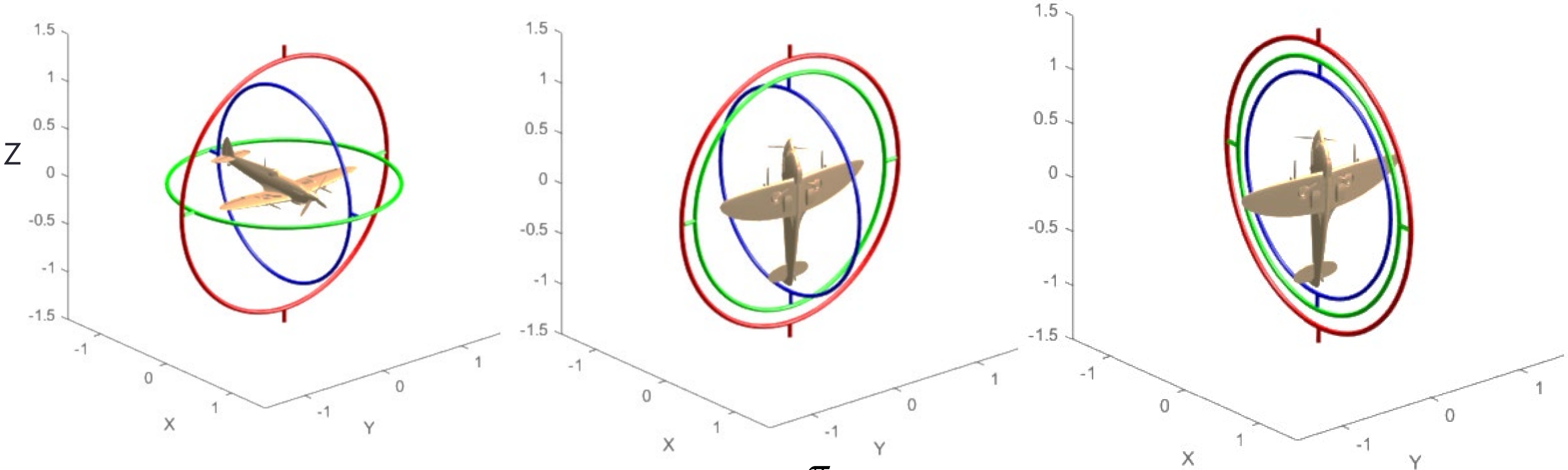
X-Y-Z RPY angles	$\alpha \mp \gamma = \mathrm{atan2}(-r_{23}, r_{22})$	$\beta = \pm\frac{\pi}{2}$

- ✓ ABC or CBA type of triple angle representation of orientations suffers from the problem of many solutions if $\beta = \pm\frac{\pi}{2}$.

- ✓ Similarly, it can be proved that ABA type of triple angle representation will also have problem of many solutions if $\beta = 0 \text{ or } \pi$.

Problems with Triple-angle Representation

2. Certain pitch angle may cause infinite solutions of triple angle representation. This singularity is the well-known **Gimbal Lock problem**



$$\beta = \pm\frac{\pi}{2}$$

The x-axis is align with Z-axis. In this case, the mechanism loses 1 DoF. There are infinite number of configurations of triple angles. This singularity often appears at wrist of a robotic manipulator.

EX 2-2-3

Find Z-x-z Euler angles from a rotation matrix R = $\begin{pmatrix} 0.1268 & -0.9268 & 0.3536 \\ 0.7803 & -0.1268 & -0.6124 \\ 0.6124 & 0.3536 & 0.7071 \end{pmatrix}$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\gamma - s\alpha c\beta s\gamma & -c\alpha s\gamma - s\alpha c\beta c\gamma & s\alpha s\beta \\ s\alpha c\gamma + c\alpha c\beta s\gamma & -s\alpha s\gamma + c\alpha c\beta c\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix}$$

If $\beta = 0, or\ \beta = \pi$

$$R = \begin{bmatrix} \cos(\alpha \pm \gamma) & -\sin(\gamma \pm \alpha) & 0 \\ \sin(\alpha \pm \gamma) & \pm\cos(\alpha \pm \gamma) & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

$$\alpha \pm \gamma = \operatorname{atan2}(r_{21}, r_{11})$$

If $\beta \neq 0\ and\ \beta \neq \pi$

$$\because \beta \in (0, \pi), \quad \therefore \sin\beta > 0$$

$$\beta = \arccos r_{33}$$

$$\alpha = \operatorname{atan2}(r_{13}, -r_{23})$$

$$\gamma = \operatorname{atan2}(r_{31}, r_{32})$$

$$\beta = \arccos 0.7071 = 45\,\deg$$

$$\alpha = \operatorname{atan2}(0.3536, 0.6124) = 30\,\deg$$

$$\gamma = \operatorname{atan2}(0.6124, 0.3536) = 60\,\deg$$