ARTICULATED ROBOTS

A/P ZHOU, Chunlin (周春琳)

Institute of Cyber-system and Control

College of Control Science and Engineering, Zhejiang University

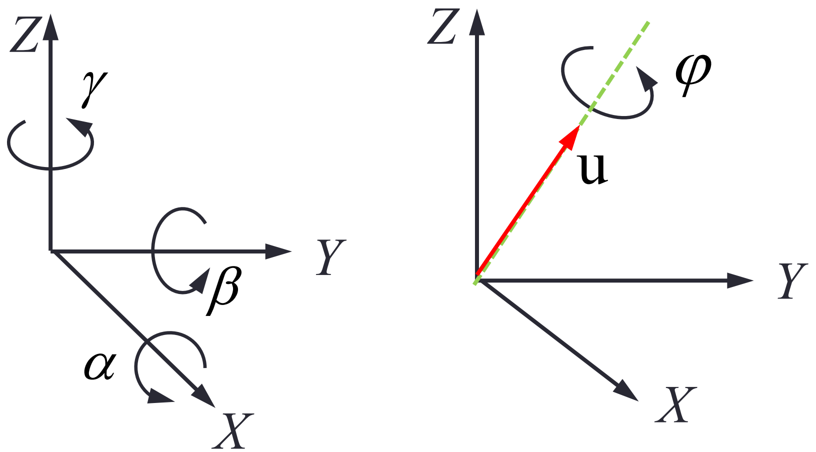
Email: c_zhou@zju.edu.cn

2. ROTATION GEOMETRY II

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

3. Axis-angle Representation

- ✓ An orientation does not have to be the result of triple rotations about primary axes of a coordinate frame.
- ✓ In geometry, **Euler's rotation theorem** states that, in three-dimensional space, any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point (quote from Wikipedia).



- An axis is align with a **unit vector u**
- The rotation is about **u** by *φ.*

3. Axis-angle Representation

- ✓ The unit vector **u** denoted by direction cosines u_1, u_2, and u_3, and

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = 1, \quad {}^0\mathbf{r} = R\,{}^1\mathbf{r}$$

The orientation R can be determined by using Euler-Lexell-Rordrigues formula

$$R_{\mathbf{u}}(\varphi) = (\cos\varphi)\mathbf{I} + (1-\cos\varphi)\mathbf{u}\mathbf{u}^T + (\sin\varphi)S(\mathbf{u})$$

$$S(\mathbf{u}) \triangleq \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} = \frac{1}{2\sin\varphi}(R - R^T)$$

$$\mathbf{u}\times\mathbf{r} = S(\mathbf{u})\mathbf{r} = -\mathbf{r}\times\mathbf{u} = -S(\mathbf{r})\mathbf{u}$$

$$S(\mathbf{u})\mathbf{u} = 0$$

$$\mathbf{r}^T S(\mathbf{u})\mathbf{r} = 0$$

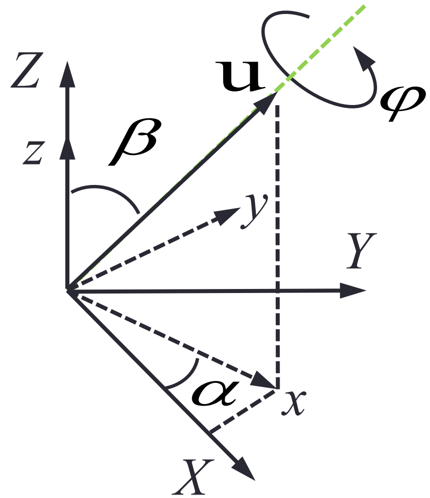
$$S^2(\mathbf{u}) = \mathbf{I} - \mathbf{u}\mathbf{u}^T$$

- Skew symmetric
- The cross-product matrix

- ✓ **Starting from an initial orientation R_0**, the axis-angle rotation becomes

$$R_u(\varphi)R_0$$

Proof: (Hint) A rotation about a specific vector can be achieved by composition of several rotations about primary axis of body attached coordinate frame



1. Rotate about Z by α to make *z*-axis, **u**, and *x*-axis in the same plane;
2. Rotate about local y by β to make z-axis coincide with **u**;
3. Rotate about z-axis (**u**) by φ;
4. Rotate reversely about y-axis by - β and z-axis by - α, respectively.

$$R = R_Z(\alpha)R_y(\beta)R_z(\varphi)R_y(-\beta)R_z(-\alpha)$$

$$\cos\alpha = \frac{u_1}{\sqrt{u_1^2+u_2^2}}$$

$$\sin\alpha = \frac{u_2}{\sqrt{u_1^2+u_2^2}}$$

$$\cos\beta = u_3$$

$$\sin\beta = \sqrt{u_1^2+u_2^2}$$

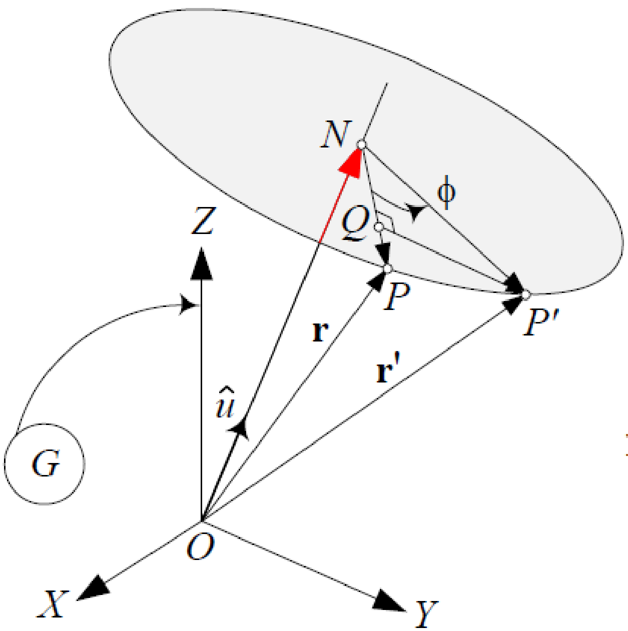
$$\sin\varphi\sin\beta = u_2$$

$$\cos\varphi\sin\beta = u_1$$

$$R\Big|_{\mathrm{v}\varphi=1-\mathrm{c}\varphi} = \begin{bmatrix} \mathrm{c}\varphi+u_1^2\mathrm{v}\varphi & -u_3\mathrm{s}\varphi+u_1u_2\mathrm{v}\varphi & u_2\mathrm{s}\varphi+u_1u_3\mathrm{v}\varphi \\ u_3\mathrm{s}\varphi+u_1u_2\mathrm{v}\varphi & \mathrm{c}\varphi+u_2^2\mathrm{v}\varphi & -u_1\mathrm{s}\varphi+u_2u_3\mathrm{v}\varphi \\ -u_2\mathrm{s}\varphi+u_1u_3\mathrm{v}\varphi & u_1\mathrm{s}\varphi+u_2u_3\mathrm{v}\varphi & \mathrm{c}\varphi+u_3^2\mathrm{v}\varphi \end{bmatrix}$$

$$= (\mathrm{c}\,\varphi)\mathbf{I} + (\mathrm{s}\,\varphi)\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} + (\mathrm{v}\varphi)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$$

$$= (\cos\varphi)\mathbf{I} + (1-\cos\varphi)\mathbf{u}\mathbf{u}^T + (\sin\varphi)S(\mathbf{u})$$



Proof: (another way)

$$\mathbf{r}' = \overrightarrow{ON} + \overrightarrow{NQ} + \overrightarrow{QP'}$$

$$\begin{aligned} \mathbf{r}' &= (\mathbf{r}\cdot\hat{u})\,\hat{u} + \hat{u}\times(\mathbf{r}\times\hat{u})\cos\phi - (\mathbf{r}\times\hat{u})\sin\phi \\ &= (\mathbf{r}\cdot\hat{u})\,\hat{u} + [\mathbf{r} - (\mathbf{r}\cdot\hat{u})\,\hat{u}]\cos\phi + (\hat{u}\times\mathbf{r})\sin\phi \end{aligned}$$

$$\mathbf{r}' = \mathbf{r}\cos\phi + (1-\cos\phi)(\hat{u}\cdot\mathbf{r})\,\hat{u} + (\hat{u}\times\mathbf{r})\sin\phi.$$

$$R_{\mathbf{u}}(\varphi)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}=\begin{bmatrix} \mathrm{c}\varphi+u_1^2\mathrm{v}\varphi & -u_3\mathrm{s}\varphi+u_1u_2\mathrm{v}\varphi & u_2\mathrm{s}\varphi+u_1u_3\mathrm{v}\varphi \\ u_3\mathrm{s}\varphi+u_1u_2\mathrm{v}\varphi & \mathrm{c}\varphi+u_2^2\mathrm{v}\varphi & -u_1\mathrm{s}\varphi+u_2u_3\mathrm{v}\varphi \\ -u_2\mathrm{s}\varphi+u_1u_3\mathrm{v}\varphi & u_1\mathrm{s}\varphi+u_2u_3\mathrm{v}\varphi & \mathrm{c}\varphi+u_3^2\mathrm{v}\varphi \end{bmatrix}$$

For given *R*, the angle-axis representation can be determined by

case 1: $\varphi \in (0,\pi)$ **Two solutions.**

$$\cos\varphi=\frac{1}{2}(r_{11}+r_{22}+r_{33}-1)=\frac{1}{2}\left[tr(R)-1\right]$$

$$\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}=\frac{1}{2\sin\varphi}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

$R_{\mathbf{u}}(\varphi)=R_{-\mathbf{u}}(-\varphi)$

is also a solution

$$(\varphi=\varphi\pm 2k\pi)$$

case 2: $\varphi=0$ **Any non-zero vector u will satisfy.**

case 3: $\varphi=\pi$ **Two solutions.**

$$R_{\mathbf{u}}(\pi)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}=\begin{bmatrix} 2u_1^2\text{-}1 & 2u_1u_2 & 2u_1u_3 \\ 2u_1u_2 & 2u_2^2-1 & 2u_2u_3 \\ 2u_1u_3 & 2u_2u_3 & 2u_3^2\text{-}1 \end{bmatrix}$$

$$\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}=\pm\begin{bmatrix} \sqrt{(r_{11}+1)/2} \\ r_{12}\Big/\sqrt{2(r_{11}+1)} \\ r_{13}\Big/\sqrt{2(r_{11}+1)} \end{bmatrix}$$

EX 2-3-1

A frame ∑1 undergoes three z-x-z Euler rotations (30, 45, 60) deg with respect to ∑0. Find the unique axis-angle representation for this rotation.

$$ {}^{0}R_1 = \begin{bmatrix} c\alpha c\gamma - s\alpha c\beta s\gamma & -c\alpha s\gamma - s\alpha c\beta c\gamma & s\alpha s\beta \\ s\alpha c\gamma + c\alpha c\beta s\gamma & -s\alpha s\gamma + c\alpha c\beta c\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix} = \begin{bmatrix} 0.1268 & -0.9268 & 0.3536 \\ 0.7803 & -0.1268 & -0.6124 \\ 0.6124 & 0.3536 & 0.7071 \end{bmatrix} $$

$$ \cos\varphi = \frac{1}{2}\left[tr(R) - 1\right] \Longrightarrow \varphi = \arccos\frac{tr(R)-1}{2} = 98.42\,\text{deg} $$

$$ S(\mathbf{u}) = \frac{R - R^T}{2\sin\varphi} = \begin{bmatrix} 0 & -0.8629 & -0.1308 \\ 0.8629 & 0 & -0.4882 \\ 0.1308 & 0.4882 & 0 \end{bmatrix} \Longrightarrow \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.4882 \\ -0.1308 \\ 0.8629 \end{bmatrix} $$

$$ R = (\cos\varphi)\mathbf{I} + (1 - \cos\varphi)\mathbf{u}\mathbf{u}^T + (\sin\varphi)S(\mathbf{u}) = \begin{bmatrix} 0.1268 & -0.9268 & 0.3536 \\ 0.7803 & -0.1268 & -0.6124 \\ 0.6124 & 0.3536 & 0.7071 \end{bmatrix} $$

Eigen Value and Eigen Vector of *R*

- ✓ For any vector rotating about itself, the orientation of the vector keeps invariant, i.e. $R\mathbf{u} = \mathbf{u}$, which indicates that $\lambda=1$ is always an eigen value of *R*.
- ✓ Solution to $R\mathbf{u} = \mathbf{u}$ is the rotation axis, i.e. the unique rotation axis is determined by the eigen vector associated with the eigen value $\lambda=1$.

$$R\mathbf{u} = \lambda\mathbf{u} \quad (\lambda \text{ denotes eigen values of } R)$$

The characteristic equation of *R* is $(\lambda - 1)\left[\lambda^2 - \lambda(trR - 1) + 1\right] = 0$

Eigen values of *R* are $\lambda_1 = 1, \quad \lambda_2 = e^{i\varphi}, \quad \lambda_3 = e^{-i\varphi}$

$$\mathbf{u} = eigen\ vector\ (if\ \lambda = 1) \qquad \varphi = \mathrm{atan2}(\mathrm{Im}(\lambda_2), \mathrm{Re}(\lambda_2))$$

EX 2-3-2

A frame ∑1 undergoes three z-x-z Euler rotations (30, 45, 60) deg with respect to ∑0. Find eigen values and eigen vectors of the rotation matrix.

$$
{}^{0}R_1 = \begin{bmatrix} c\alpha c\gamma - s\alpha c\beta s\gamma & -c\alpha s\gamma - s\alpha c\beta c\gamma & s\alpha s\beta \\ s\alpha c\gamma + c\alpha c\beta s\gamma & -s\alpha s\gamma + c\alpha c\beta c\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix} = \begin{bmatrix} 0.1268 & -0.9268 & 0.3536 \\ 0.7803 & -0.1268 & -0.6124 \\ 0.6124 & 0.3536 & 0.7071 \end{bmatrix}
$$

$$
\boldsymbol{\lambda} = \begin{bmatrix} -0.1464 + 0.9892i \\ -0.1464 - 0.9892i \\ 1.0000 \end{bmatrix} \qquad V = \begin{bmatrix} 0.0456 + 0.6154i & 0.0456 - 0.6154i & 0.4882 \\ 0.7010 + 0.0000i & 0.7010 + 0.0000i & -0.1308 \\ 0.0805 - 0.3482i & 0.0805 + 0.3482i & 0.8629 \end{bmatrix}
$$

$$
\varphi = \text{atan2}(0.9892, -0.1464) = 98.4186\,\text{deg}
$$

$$
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.4882 \\ -0.1308 \\ 0.8629 \end{bmatrix}
$$

Code Session

Ch2_3.m

4. Euler Parameters

- ✓ Euler introduced a set of parameters to determine the axis-angle rotation such that e_0 is a scalar， e_1, e_2, e_3 are components of a vector **e** and

$$e_0 = \cos\frac{\varphi}{2}$$

$$\mathbf{e} = e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k} = (\sin\frac{\varphi}{2})\mathbf{u}$$

where

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = e_0^2 + \mathbf{e}^T\mathbf{e} = 1$$

- ✓ The transformation matrix R can be interpreted by Euler parameters as

$$R_{\mathbf{u}}(\varphi) = (e_0^2 - \mathbf{e}^2)\mathbf{I} + 2\mathbf{e}\mathbf{e}^T + 2e_0 S(\mathbf{e})$$

where

$$S(\mathbf{e}) = \frac{1}{4e_0}\left[R_{\mathbf{u}}(\varphi) - R_{\mathbf{u}}^T(\varphi)\right]$$

Proof: utilizing Euler-Lexell-Rordrigues formula

$$\left.\begin{aligned} e_0 &= \cos\frac{\varphi}{2} \\ \mathbf{e} &= (\sin\frac{\varphi}{2})\mathbf{u} \\ e_0^2 + \mathbf{e}^T\mathbf{e} &= 1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \cos\varphi &= 2\cos^2\frac{\varphi}{2} - 1 = 2e_0^2 - (e_0^2 + \mathbf{e}^2) = e_0^2 - \mathbf{e}^2 \\ (1-\cos\varphi)\mathbf{u}\mathbf{u}^T &= \frac{1-\cos\varphi}{\sin^2(\varphi/2)}\mathbf{e}\mathbf{e}^T = 2\mathbf{e}\mathbf{e}^T \\ (\sin\varphi)S(\mathbf{u}) &= \frac{\sin\varphi}{\sin(\varphi/2)}S(\mathbf{e}) = 2\cos\frac{\varphi}{2}S(\mathbf{e}) = 2e_0S(\mathbf{e}) \end{aligned}\right\} \Rightarrow$$

$$R_{\mathbf{u}}(\varphi) = (\cos\varphi)\mathbf{I} + (1-\cos\varphi)\mathbf{u}\mathbf{u}^T + (\sin\varphi)S(\mathbf{u}) = (e_0^2 - \mathbf{e}^2)\mathbf{I} + 2\mathbf{e}\mathbf{e}^T + 2e_0S(\mathbf{e})$$
$$= \begin{bmatrix} 2(e_0^2+e_1^2)-1 & 2(e_1e_2 - e_0e_3) & 2(e_0e_2 + e_1e_3) \\ 2(e_0e_3 + e_1e_2) & 2(e_0^2+e_2^2)-1 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_0e_1 + e_2e_3) & 2(e_0^2+e_3^2)-1 \end{bmatrix}$$

If a set of Euler parameters are given, the rotation axis and angle can be determined

$$\cos\varphi = \frac{1}{2}\left[tr(R) - 1\right], \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{2\sin\varphi}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

If R is given, Euler parameters can be obtained by comparing both sides of

$$R_{\mathbf{u}}(\varphi)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}=\begin{bmatrix} 2(e_0^2+e_1^2)-1 & 2(e_1e_2-e_0e_3) & 2(e_0e_2+e_1e_3) \\ 2(e_0e_3+e_1e_2) & 2(e_0^2+e_2^2)-1 & 2(e_2e_3-e_0e_1) \\ 2(e_1e_3-e_0e_2) & 2(e_0e_1+e_2e_3) & 2(e_0^2+e_3^2)-1 \end{bmatrix}$$

Four sets of solutions can be obtained:

$$e_0=\pm\frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}$$

$$\mathbf{e}=\frac{1}{4e_0}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

$$e_1=\pm\frac{1}{2}\sqrt{1+r_{11}-r_{22}-r_{33}}$$

$$\begin{bmatrix} e_0 \\ e_2 \\ e_3 \end{bmatrix}=\frac{1}{4e_1}\begin{bmatrix} r_{32}+r_{23} \\ r_{21}+r_{12} \\ r_{13}+r_{31} \end{bmatrix}$$

$$e_2=\pm\frac{1}{2}\sqrt{1-r_{11}+r_{22}-r_{33}}$$

$$\begin{bmatrix} e_0 \\ e_1 \\ e_3 \end{bmatrix}=\frac{1}{4e_2}\begin{bmatrix} r_{13}-r_{31} \\ r_{21}+r_{12} \\ r_{32}+r_{23} \end{bmatrix}$$

$$e_3=\pm\frac{1}{2}\sqrt{1-r_{11}-r_{22}+r_{33}}$$

$$\begin{bmatrix} e_0 \\ e_1 \\ e_2 \end{bmatrix}=\frac{1}{4e_3}\begin{bmatrix} r_{21}-r_{12} \\ r_{31}+r_{13} \\ r_{32}+r_{23} \end{bmatrix}$$

- ✓ The plus and minus sign indicate that $R_{\mathbf{u}}(\varphi)=R_{-\mathbf{u}}(-\varphi)$
- ✓ Numerical inaccuracies can be minimized by using the set with maximum divisor.

EX 2-4-1

A frame ∑1 undergoes three z-x-z Euler rotations (30, 45, 60) deg with respect to ∑0. Find the rotation transformation matrix , associated Euler parameter and axis-angle representation.

$$ {}^{0}R_1 = \begin{bmatrix} c\alpha c\gamma - s\alpha c\beta s\gamma & -c\alpha s\gamma - s\alpha c\beta c\gamma & s\alpha s\beta \\ s\alpha c\gamma + c\alpha c\beta s\gamma & -s\alpha s\gamma + c\alpha c\beta c\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix} = \begin{bmatrix} 0.1268 & -0.9268 & 0.3536 \\ 0.7803 & -0.1268 & -0.6124 \\ 0.6124 & 0.3536 & 0.7071 \end{bmatrix} $$

$$ e_0 = \pm\frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} = \pm 0.6533 $$

$$ \mathbf{e} = \frac{1}{4e_0}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = \pm\begin{bmatrix} 0.3696 \\ -0.0990 \\ 0.6533 \end{bmatrix} $$

$$ \varphi = 2\arccos e_0 = \pm 98.4225\,\text{deg} $$

$$ \mathbf{u} = \mathbf{e} / (\sin\frac{\varphi}{2}) = \pm\begin{bmatrix} 0.4882 \\ -0.1308 \\ 0.8629 \end{bmatrix} $$

Code Session

Ch2_4.m

5. Quaternions

- ✓ A rotation can be characterized by four quantities if we inspect the axis-angle representation: one for angle and three for the axis.
- ✓ The three quantities of the axis vector are not mutually independent because they are subject to the constraint of unity. Therefore, the degrees-of-freedom of a rotation is three, which is the same as a rotation matrix or Euler angles.
- ✓ The axis-angle representation cannot be processed by vector algebra. The problem can be fixed by utilizing quaternions, a hyper-complex number composed of a scalar part and a vector part.

$$q = q_0 + \mathbf{q} = \underbrace{\boxed{q_0}}_{\text{scalar, } s} + \underbrace{\boxed{q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}}}_{\text{vector, } \mathbf{v}} = \langle q_0, \mathbf{q} \rangle = \langle s, \mathbf{v} \rangle \qquad q_0 \in \mathbf{R}, \quad \mathbf{q} \in \mathbf{R}^3$$

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ are orthogonal complex numbers and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\cdot\mathbf{j}\cdot\mathbf{k} = -1$

Properties

1. **Addition** of two quaternions is a quaternion

$$q + p = \langle q_0, \mathbf{q} \rangle + \langle p_0, \mathbf{p} \rangle = \langle q_0 + p_0, \mathbf{q} + \mathbf{p} \rangle$$

2. Multiplication of two quaternions is a quaternion (**Hamiltonian product**)

$$q \circ p = (q_0 + \mathbf{q})(p_0 + \mathbf{p}) = q_0 p_0 + q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{qp}$$

$$\because \mathbf{qp} \triangleq \mathbf{q} \times \mathbf{p} - \mathbf{q} \cdot \mathbf{p}$$

$$q \circ p = \langle q_0 p_0 - \mathbf{q} \cdot \mathbf{p}, \quad q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p} \rangle$$

3. Quaternion multiplication is **not commutative**

$$q \circ p \neq p \circ q$$

4. Quaternion addition is **associative and commutative**

$$(p + q) \circ r = p \circ r + q \circ r = q \circ r + p \circ r = (q + p) \circ r$$

$$(p \circ q) \circ r = p \circ (q \circ r)$$

Properties

5. **Conjugate** of a quaternion

https://blog.csdn.net/weixin_42511255/article/details/112405856

$$q^* = \langle q_0, -\mathbf{q} \rangle = q_0 - \mathbf{q}$$

$$qq^* = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

$$q^{-1} = \frac{1}{q} = \frac{q^*}{|q|^2}$$

$$q = \begin{bmatrix} \cos\frac{\gamma}{2} \\ 0 \\ 0 \\ \sin\frac{\gamma}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\beta}{2} \\ 0 \\ \sin\frac{\beta}{2} \\ 0 \end{bmatrix} \begin{bmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \\ \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} - \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2} \\ \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2} \end{bmatrix}$$

6. A **unit quaternion** corresponds to a set of Euler parameters

7. Properties of a **unit quaternion**

$$e(\varphi, \mathbf{u}) = \langle e_0, \mathbf{e} \rangle = e_0 + \mathbf{e} = \cos\frac{\varphi}{2} + (\sin\frac{\varphi}{2})\mathbf{u} = \langle s, \mathbf{v} \rangle$$

$$|e| = 1 \qquad e^{-1} = e^*$$

$$s = e_0 \qquad \mathbf{v} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}^T$$

Rotation with Unit Quaternions

Theorem: A rotation defined by ${}^0\mathbf{r} = R_{\mathbf{u}}(\varphi)\,\mathbf{r}$ is equivalent to

$$ {}^0\mathbf{r} = e(\varphi,\mathbf{u})\mathbf{r}e^*(\varphi,\mathbf{u}) \qquad \mathbf{r} = e^*(\varphi,\mathbf{u})\,{}^0\mathbf{r}e(\varphi,\mathbf{u}) $$

and two successive rotations $R=R_2R_1$ are defined by

$$ e(\varphi,\mathbf{u}) = e(\varphi_2,\mathbf{u}_2)e(\varphi_1,\mathbf{u}_1) $$

Proof: $e(\varphi,\mathbf{u}) = \langle e_0,\mathbf{e}\rangle =< e_0,(e_1,e_2,e_3)^T >= \langle s,\mathbf{v}\rangle$

$$ \mathbf{r}e^* = \mathbf{r}\langle s,\mathbf{v}\rangle = \mathbf{r}s - (\mathbf{r}\times\mathbf{v} - \mathbf{r}\cdot\mathbf{v}) = \langle \mathbf{r}\cdot\mathbf{v}, \mathbf{r}s - \mathbf{r}\times\mathbf{v}\rangle = \left\langle e_1r_1+e_2r_2+e_3r_3, \begin{bmatrix} e_0r_1+e_2r_3-e_3r_2 \\ e_0r_2-e_1r_3+e_3r_1 \\ e_0r_3+e_1r_2-e_2r_1 \end{bmatrix}\right\rangle = \langle s_1,\mathbf{v}_1\rangle $$

$$ e\mathbf{r}e^* = \langle s,\mathbf{v}\rangle\langle s_1,\mathbf{v}_1\rangle = \langle ss_1 - \mathbf{v}\cdot\mathbf{v}_1, \quad s\mathbf{v}_1 + s_1\mathbf{v} + \mathbf{v}\times\mathbf{v}_1\rangle $$

$$ = \left\langle 0, \begin{bmatrix} 2(e_0^2+e_1^2)-1 & 2(e_1e_2-e_0e_3) & 2(e_0e_2+e_1e_3) \\ 2(e_0e_3+e_1e_2) & 2(e_0^2+e_2^2)-1 & 2(e_2e_3-e_0e_1) \\ 2(e_1e_3-e_0e_2) & 2(e_0e_1+e_2e_3) & 2(e_0^2+e_3^2)-1 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}\right\rangle >= R_{\mathbf{e}}(\varphi)\mathbf{r} $$

EX 2-5-1

Find principle rotation transformation matrices about X, Y, Z axes of a coordinate frame by (α, β, γ)*, respectively, using unit quaternions.*

$$\left.\begin{aligned} e_0 &= \cos\frac{\varphi}{2} \\ \mathbf{e} &= (\sin\frac{\varphi}{2})\mathbf{u} \\ e_0^2 + \mathbf{e}^T\mathbf{e} &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} e_X(\alpha) &= \left\langle \cos\frac{\alpha}{2}, (\sin\frac{\alpha}{2})\mathbf{I} \right\rangle \\ e_Y(\beta) &= \left\langle \cos\frac{\beta}{2}, (\sin\frac{\beta}{2})\mathbf{J} \right\rangle \\ e_Z(\gamma) &= \left\langle \cos\frac{\gamma}{2}, (\sin\frac{\gamma}{2})\mathbf{K} \right\rangle \end{aligned}$$

$$R_{\mathbf{u}}(\varphi) = (e_0^2 - \mathbf{e}^2)\mathbf{I} + 2\mathbf{e}\mathbf{e}^T + 2e_0 S(\mathbf{e})$$

$$\Rightarrow$$

$$R_X(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \quad R_Y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad R_Z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pros & Cons of Representations

By means of **Rotator**, *R*

1. Rotation transformation matrix
2. Euler angles

$$R_{\mathbf{u}}(\varphi) = (\cos\varphi)\mathbf{I} + (1-\cos\varphi)\mathbf{u}\mathbf{u}^T + (\sin\varphi)S(\mathbf{u})$$
$$= (e_0^2 - \mathbf{e}^2)\mathbf{I} + 2\mathbf{e}\mathbf{e}^T + 2e_0 S(\mathbf{e})$$

By means of **Spinor**, *S*

3. Axis-angle
4. Euler parameters
5. Unit quaternions

- ✓ Rotations do not commute
- ✓ Spatial rotations do not topologically allow a smooth mapping in there dimensional Euclidean space
- ✓ *R* is not explicit and not computationally efficient due to too many elements.
- ✓ Angle-axis representation is not efficient to find the composition of two rotations and determine the equivalent angle-axis of rotations. It also suffers from the multiple-solution problem.
- ✓ When Euler angles are employed, there exists the Gimbal Lock problem at singular angles. The advantage is that they use only three quantities which are integrable and can provide good visualization of rotations with no redundancy.
- ✓ Unit quaternions (Euler parameters) are computationally efficient and has no singular problem. But it is not explicit to show how the rotation evolves in Euclidean space.