ARTICULATED ROBOTS

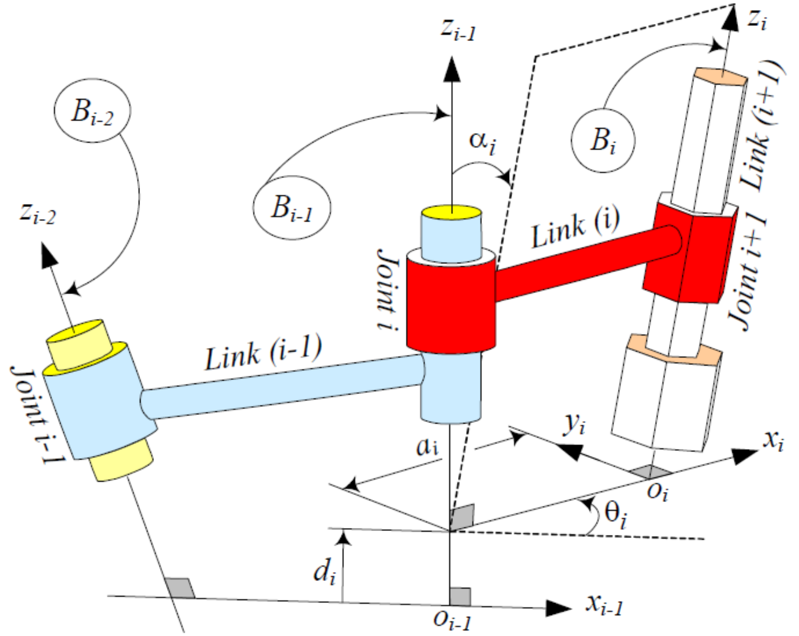
A/P ZHOU, Chunlin (周春琳)

Institute of Cyber-system and Control

College of Control Science and Engineering, Zhejiang University

Email: c_zhou@zju.edu.cn

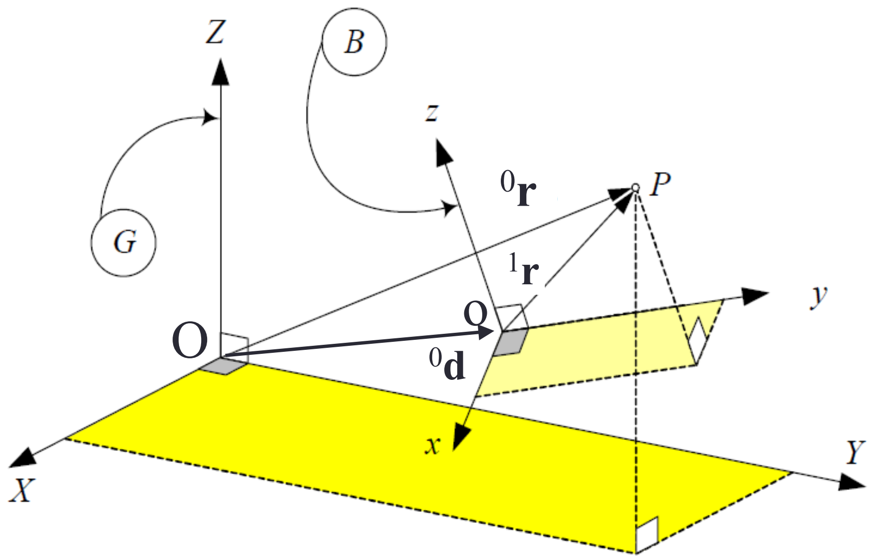
3. FORWARD KINEMATICS I



3.1 Position & Orientation

The position of the reference point P on a end-effector of a manipulator can be represented by a position vector, $\overrightarrow{OP}$.

Position of P : $\overrightarrow{OP} = {}^0\mathbf{r} = {}^0R_1\,{}^1\mathbf{r} + {}^0\mathbf{d}$



$$
{}^0\mathbf{r} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \qquad {}^1\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}
$$

$$
{}^0\mathbf{r} = {}^0R_1\,{}^1\mathbf{r} + {}^0\mathbf{d}
$$

$$
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}
$$

where ${}^0\mathbf{d}$ is a vector in the global frame 0; ${}^0R_1$ is a matrix transforming a vector in the local frame 1 into frame 0.

Homogeneous Transformation

Position of P : $\overrightarrow{OP} = {}^{0}\mathbf{r} = {}^{0}\mathbf{d} + {}^{0}R_1\, {}^{1}\mathbf{r}$

$\Downarrow$

$$\begin{bmatrix} {}^{0}\mathbf{r} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{0}R_1 & {}^{0}\mathbf{d} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} {}^{1}\mathbf{r} \\ 1 \end{bmatrix} = {}^{0}T_1 \begin{bmatrix} {}^{1}\mathbf{r} \\ 1 \end{bmatrix}$$

Homogeneous Transformation Matrix

$${}^{0}T_1 = \left[\begin{array}{ccc|c} r_{11} & r_{12} & r_{13} & d_X \\ r_{21} & r_{22} & r_{23} & d_X \\ r_{31} & r_{32} & r_{33} & d_X \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Note: Representation of an *n*-element vector by an (*n*+1) element vector is called homogeneous representation. The appended element *c* is a scale factor and

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \underbrace{\begin{bmatrix} cr \\ c \end{bmatrix}}_{\text{Homogeneous coordinates}} = \begin{bmatrix} cx \\ cy \\ cz \\ c \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ c \end{bmatrix}$$

If $c \neq 0$, homogeneous coordinates always represent the same vector as *c* varies.

Homogeneous Transformation

Position of P : $\overrightarrow{OP} = {}^0\mathbf{r} = {}^0\mathbf{d} + {}^0R_1\,{}^1\mathbf{r}$

$$\Downarrow$$

$$\begin{bmatrix} {}^0\mathbf{r} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0R_1 & {}^0\mathbf{d} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} {}^1\mathbf{r} \\ 1 \end{bmatrix} = {}^0T_1 \begin{bmatrix} {}^1\mathbf{r} \\ 1 \end{bmatrix}$$

The homogeneous transformation matrix relates coordinates in 1 and 0. It represents both pose and position information by two basic transformations:

Rotation transformation

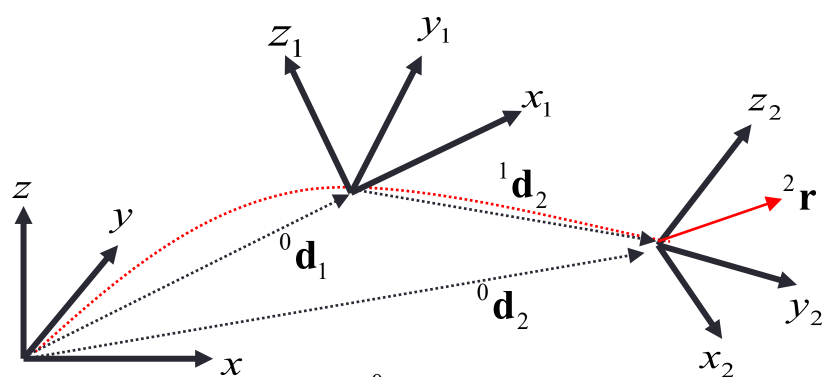
$${}^0T_1 = \begin{bmatrix} {}^0R_1 & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

Translation transformation

$${}^0T_1 = \begin{bmatrix} \mathbf{I}_{3\times 3} & {}^1\mathbf{d} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

Successive Transformation

✓ The final global position of a point P in a rigid body B with position vector **r**, after a sequence of transformation T_1, T_2, T_3, ..., T_n about the global axes can be found by



$${}^0T_2 = ?$$

$$
\begin{aligned}
{}^1\mathbf{r} &= {}^1R_2\,{}^2\mathbf{r} + {}^1\mathbf{d}_2 \\
{}^0\mathbf{r} &= {}^0R_1\,{}^1\mathbf{r} + {}^0\mathbf{d}_1 \\
&= {}^0R_1({}^1R_2\,{}^2\mathbf{r} + {}^1\mathbf{d}_2) + {}^0\mathbf{d}_1 \\
&= ({}^0R_1\,{}^1R_2)\,{}^2\mathbf{r} + ({}^0R_1\,{}^1\mathbf{d}_2 + {}^0\mathbf{d}_1) \\
&= {}^0R_2\,{}^2\mathbf{r} + {}^0\mathbf{d}_2
\end{aligned}
$$

$$
\begin{aligned}
{}^0R_2 &= {}^0R_1\,{}^1R_2 \\
{}^0\mathbf{d}_2 &= {}^0R_1\,{}^1\mathbf{d}_2 + {}^0\mathbf{d}_1
\end{aligned}
\quad \Rightarrow \quad
{}^0T_2 = \begin{bmatrix} {}^0R_1 & {}^0\mathbf{d}_1 \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} {}^1R_2 & {}^1\mathbf{d}_2 \\ \mathbf{0} & 1 \end{bmatrix} = {}^0T_1\,{}^1T_2 \quad \Rightarrow
$$

$${}^0\mathbf{r} = {}^0T_n\,{}^n\mathbf{r}$$

where $${}^0T_n = {}^0T_1\,{}^1T_2 \ldots {}^{n-1}T_n$$

EX 3-1-1

Find the position & pose of a rigid body B in G after B turns α about X-axis, translates a along X-axis, translates d along Z-axis and turns θ about Z-axis.

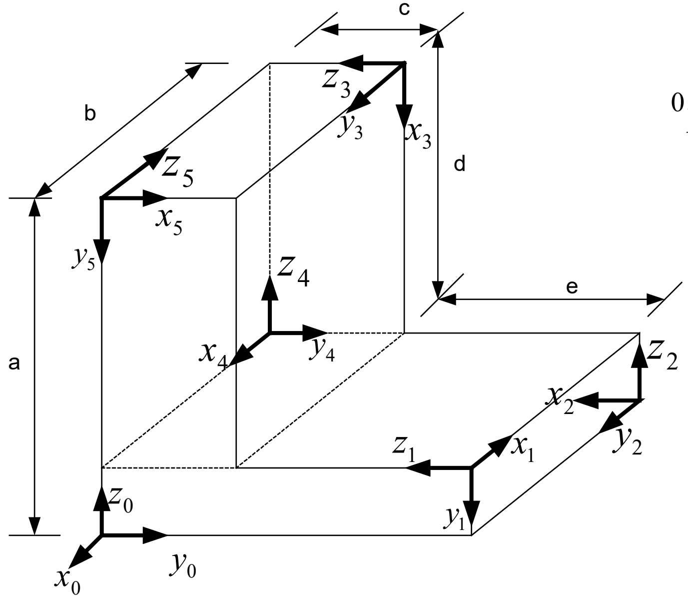
$$T = T_{Z,\theta} T_{Z,d} T_{X,a} T_{X,\alpha} I_{4\times 4}$$

$$= \begin{bmatrix} \mathrm{c}\theta & -\mathrm{s}\theta & 0 & 0 \\ \mathrm{s}\theta & \mathrm{c}\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}\alpha & \mathrm{s} & 0 \\ 0 & \mathrm{s}\alpha & \mathrm{c}\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EX 3-1-2

$$ {}^{0}T_{i} = ? \qquad {}^{i-1}T_{i} = ? \qquad {}^{j}T_{i} = ? $$

Usg properties of transformation matrix

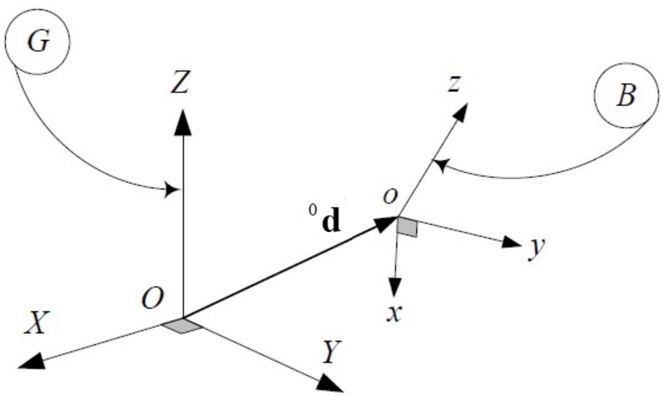


$$ {}^{0}T_{1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & e+c \\ 0 & -1 & 0 & a-d \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}T_{2} = \begin{bmatrix} 0 & -1 & 0 & b \\ 0 & 0 & -1 & a-d \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}T_{2} = \begin{bmatrix} 0 & 1 & 0 & -b \\ -1 & 0 & 0 & e+c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Inverse Homogeneous Transformation

✓ The advantage of simplicity to work with homogeneous transformation matrices come with the penalty of **losg the orthogonality property.**



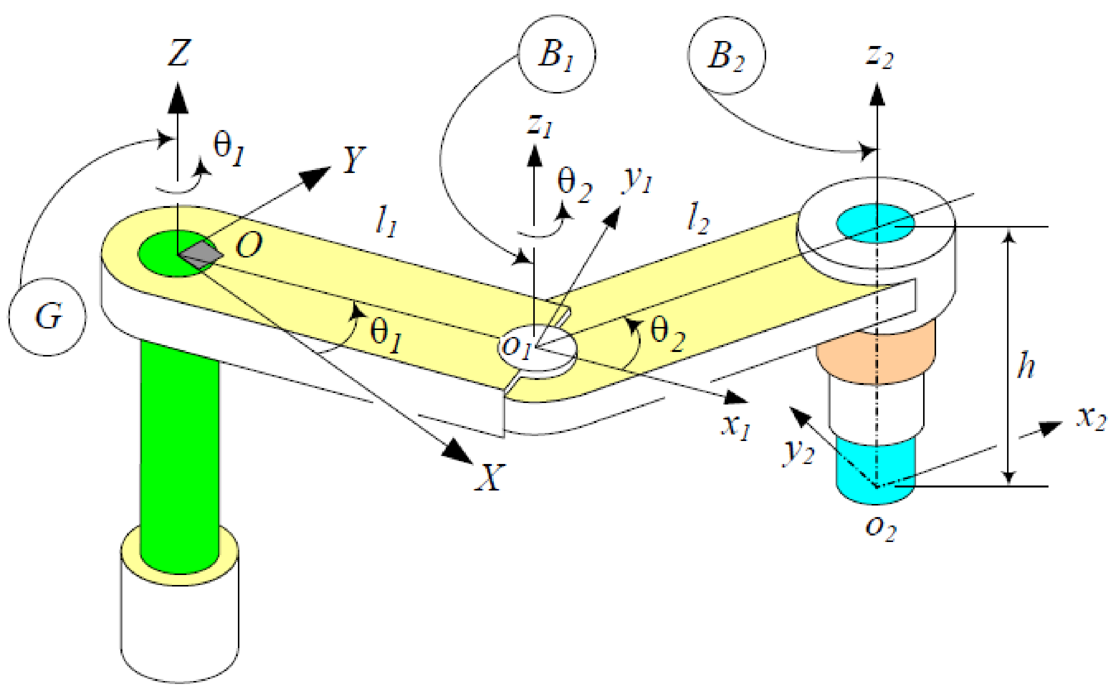
$$^0T_1 = \begin{bmatrix} ^0R_1 & ^0\mathbf{d} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

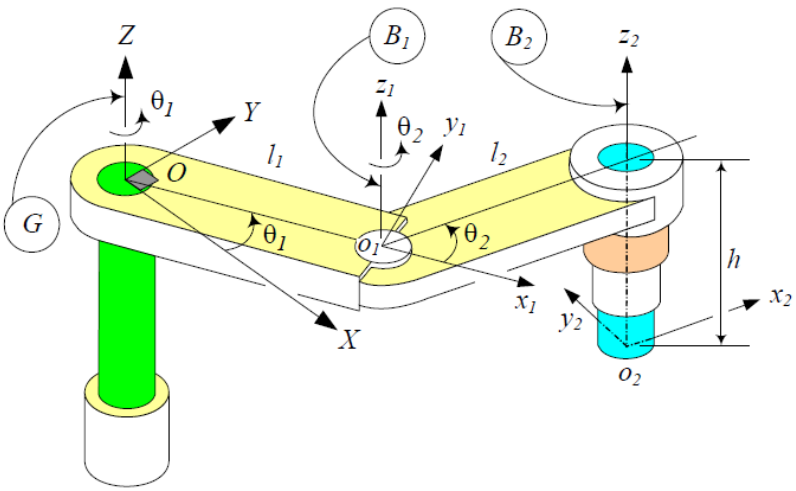
$$^0T_1^{-1} = {}^1T_0 = \begin{bmatrix} ^0R_1^T & -{}^0R_1^T\,{}^0\mathbf{d} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \neq {}^oT_1^T$$

$-{}^0R_1^T\,{}^0\mathbf{d}$ is a vector denoting $\overrightarrow{oO}$ and is represented in B

EX 3-1-3

The figure depicts an R||R||P (SCARA) robot with a global coordinate frame G(OXY Z) attached to the base link along with the coordinate frames $B_1(o_1x_1y_1z_1)$ *and* $B_2(o_2x_2y_2z_2)$ *attached to link (1) and the tip of link (3). Find pose and position of the end-effector.*





1. The T matrix mapping points in $B2$ to $B1$ is

$$ {}^{B_1}T_{B_2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & -h \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

2. The T matrix mapping points in B_1 to G:

$$ {}^{G}T_{B_1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

3. The T matrix mapping points in B_2 to G is

$$ {}^{G}T_{B_2} = {}^{G}T_{B_1}\ {}^{B_1}T_{B_2} $$

$$ = \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & 0 & l_1c\theta_1 + l_2c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & 0 & l_1s\theta_1 + l_2s(\theta_1+\theta_2) \\ 0 & 0 & 1 & -h \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Pose (RPY)

$(0, 0, \theta_1 + \theta_2)$

position

$$ \begin{bmatrix} l_1c\theta_1 + l_2c(\theta_1+\theta_2) \\ l_1s\theta_1 + l_2s(\theta_1+\theta_2) \\ -h \\ 1 \end{bmatrix} $$

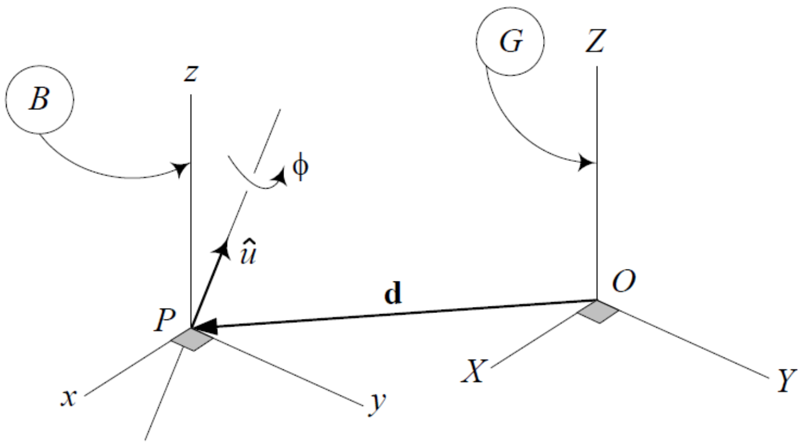
The origin of B2 can also be found by

$$ {}^{G}\mathbf{r}_2 = {}^{G}T_{B_2}\ {}^{B_2}\mathbf{r}_{o_2} = {}^{G}T_{B_2}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} $$

$$ = \begin{bmatrix} l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) \\ -h \\ 1 \end{bmatrix} $$

EX 3-1-4

Find the homogeneous transformation matrix for a rotation about an axis not through origin of coordinate frame.



Rotation about **u** is equivalent to

1. Translate **u** to make it through the origin
2. Rotate about the translated **u**
3. Translate rotated quantities back

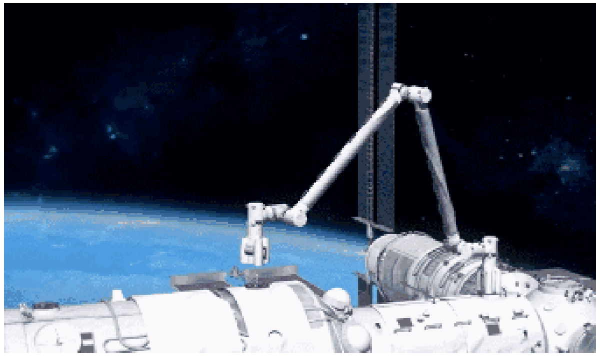
$$
\begin{aligned}
T &= T(\mathbf{d})T_{\mathbf{u}}(\varphi)T(-\mathbf{d}) \\
&= \begin{bmatrix} I & \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} R_{\mathbf{u}}(\varphi) & 0 \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} I & -\mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix} \\
&= \begin{bmatrix} R_{\mathbf{u}}(\varphi) & (I - R_{\mathbf{u}}(\varphi))\mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix}
\end{aligned}
$$

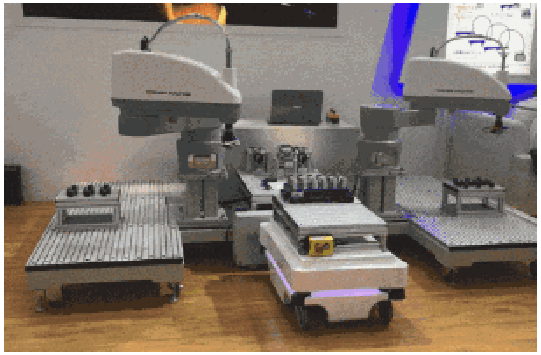
Code Session

Ch3_1.m

3.2 Denavit-Hartenberg Method

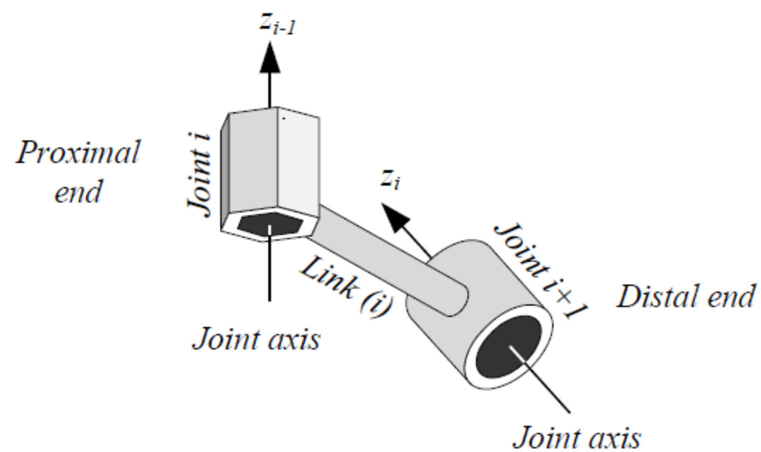
- ✓ Serial articulated robot (a robotic arm) has diversified structure, which brings complexity to the control of robot.
- ✓ It is necessary develop a generic method to define the geometry of a robotic arm in order to control different arms usg the unified method.
- ✓ One of the most useful methods uses the so called Denavit-Hartenberg notations that can describe the structure of a robotic arm in a generic manner.





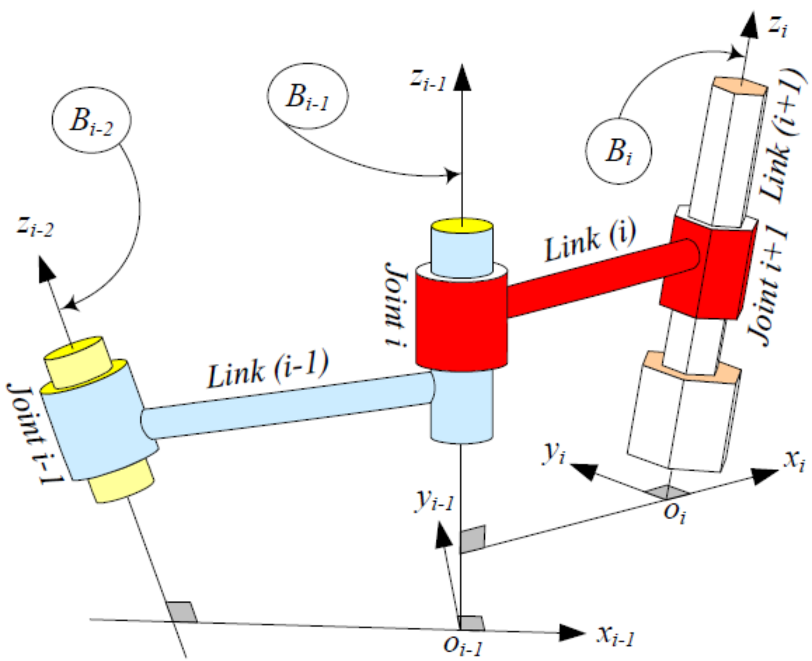
Notations of Links

- ✓ A robot with *n* joints will have *n* movable links and 1 ground fixed link;
- ✓ Numbering of links starts from 0 for the grounded base link to 1 for the first link of the robot, and increases sequentially up to *n* for the end-effector link;
- ✓ The link *i* is connected to its lower link *i*−1 at its proximal end by joint *i* and is connected to its upper link *i*+1 at its distal end by joint *i*+1;



Denavit-Hartenberg Parameters

- ✓ Solving the forward kinematics problem is a process to find *T* between link *i* and link *i*-1 if their relative motions are given;
- ✓ The pose and position of the link *i* with respect to link *i*-1 are decided by two aspects: the **rotation** and the **geometries of mechanical parts**;



- **Rotation**

  1. Relative rotation angle θ

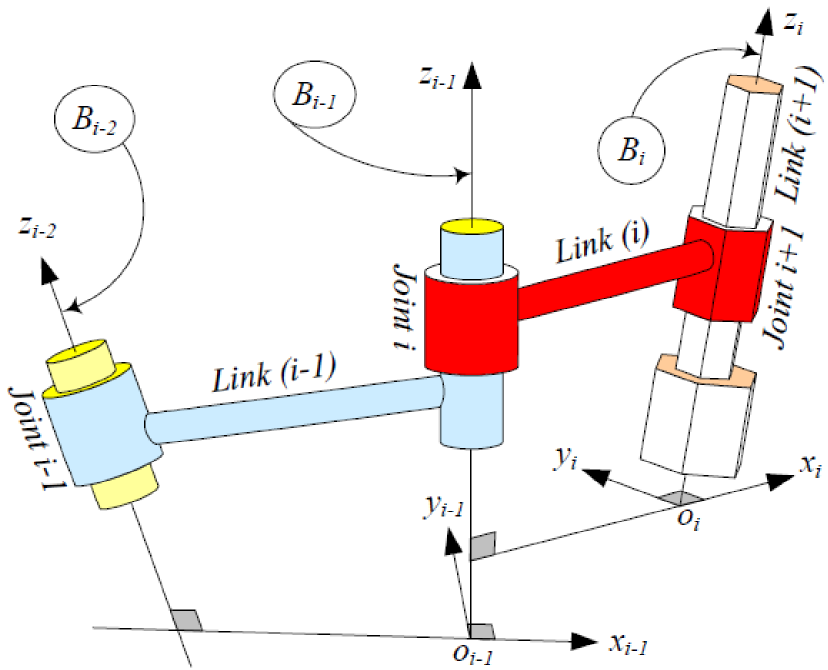
- **Geometries of mechanical parts**

  2. Distance of links
  3. Twist of links
  4. Offset of links

DH Parameters

✓ The process usg the 4 geometric parameters to determine *T* is known as **Denavit-Hartenberg (DH) method**

Step 1: Setup coordinate frames



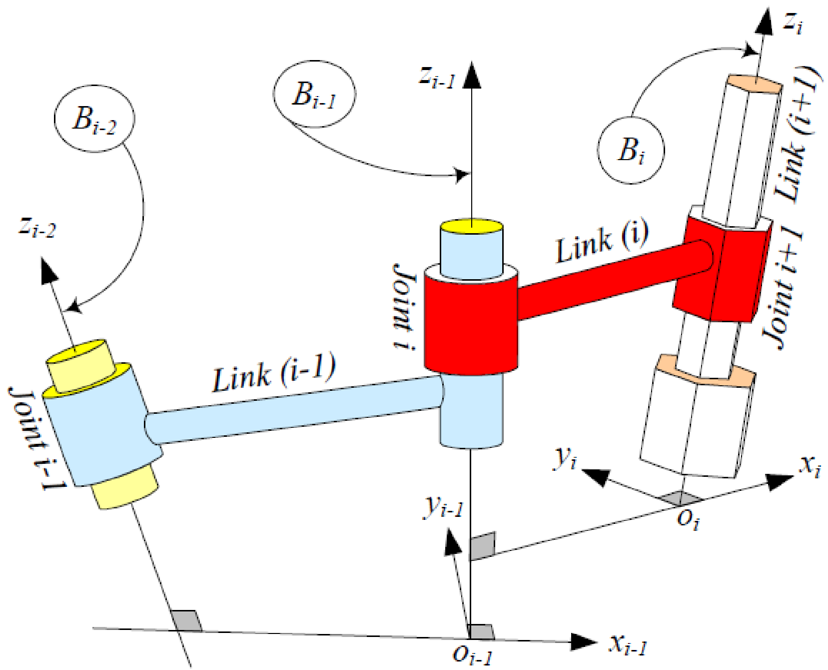
z_i-axis

- aligned with the axis of the distal end joint of the *i*+1 link
- or aligned with the translation direction for a prismatic joint
- both directions are applicable

Step 1: Setup coordinate frames

Origin o_i

- intersect point of the common normal between the $\boldsymbol{z}_{i-1}$ and $\boldsymbol{z}_i$ axes with $\boldsymbol{z}_i$



x_i-axis

- along the common normal between the $\boldsymbol{z}_{i-1}$ and $\boldsymbol{z}_i$ axes, pointing from the $\boldsymbol{z}_{i-1}$ to the $\boldsymbol{z}_i$ –axis

$$x_i = \pm\left(z_{i-1} \times z_i\right) / \left\| z_{i-1} \times z_i \right\|$$

- if two z-axes are parallel, collinear with that of the previous joints

y_i-axis

$$y_i = \left(z_i \times x_i\right) / \left\| z_i \times x_i \right\|$$

Step 2: Identify DH parameters

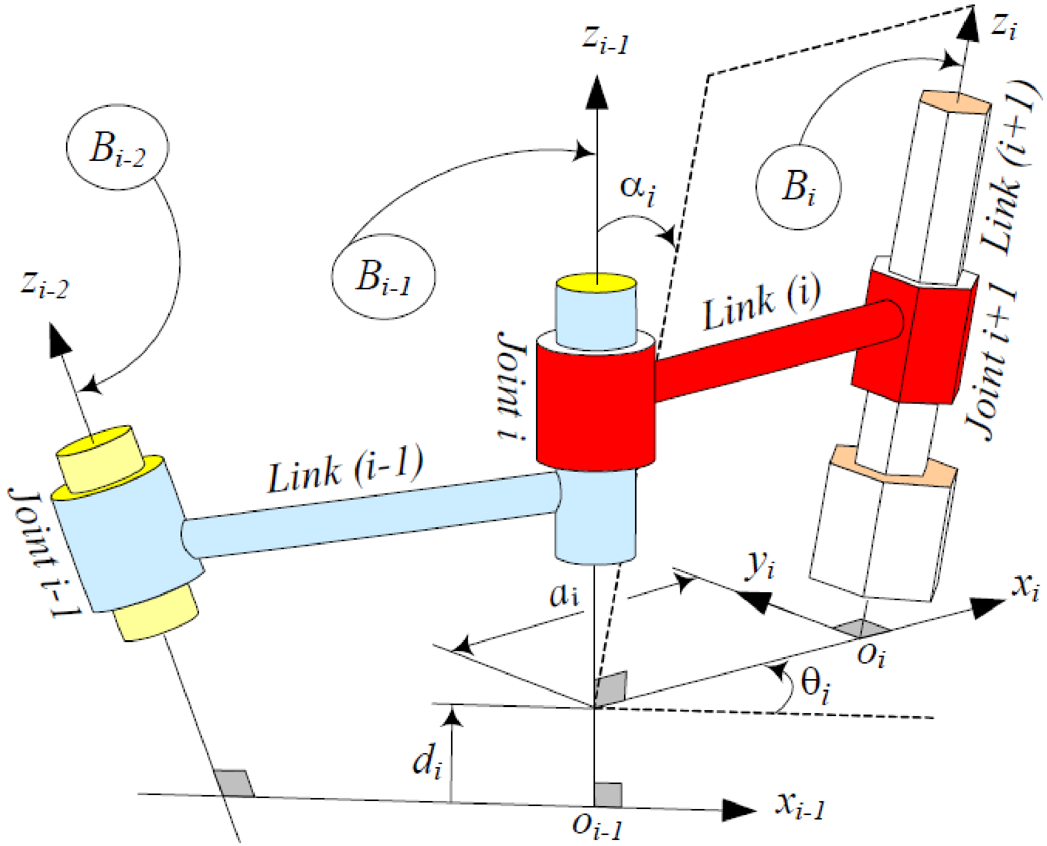
1. **Joint distance d_i :** distance between $x_{i\text{-}1}$ and x_i axes along the z_{i-1} -axis.
2. **Joint angle θ_i :** rotation of the $x_{i\text{-}1}$ -axis about the z_{i-1} -axis to become parallel to the x_i-axis
3. **Link length a_i :** distance between z_{i-1} and z_i axes along the x_i-axis
4. **Link twist α_i :** rotation of z_{i-1} -axis about x_i-axis to be parallel to z_i-axis

Joint parameters

$\theta_i \quad d_i$

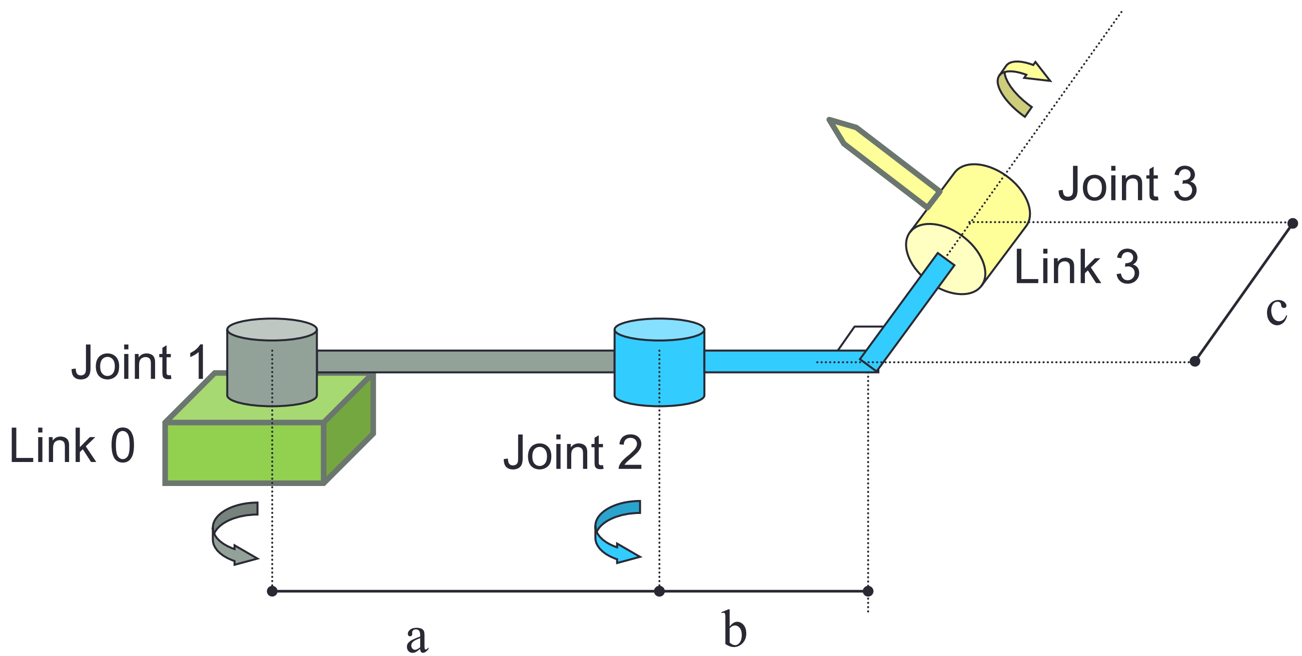
Link parameters

$a_i \quad \alpha_i$



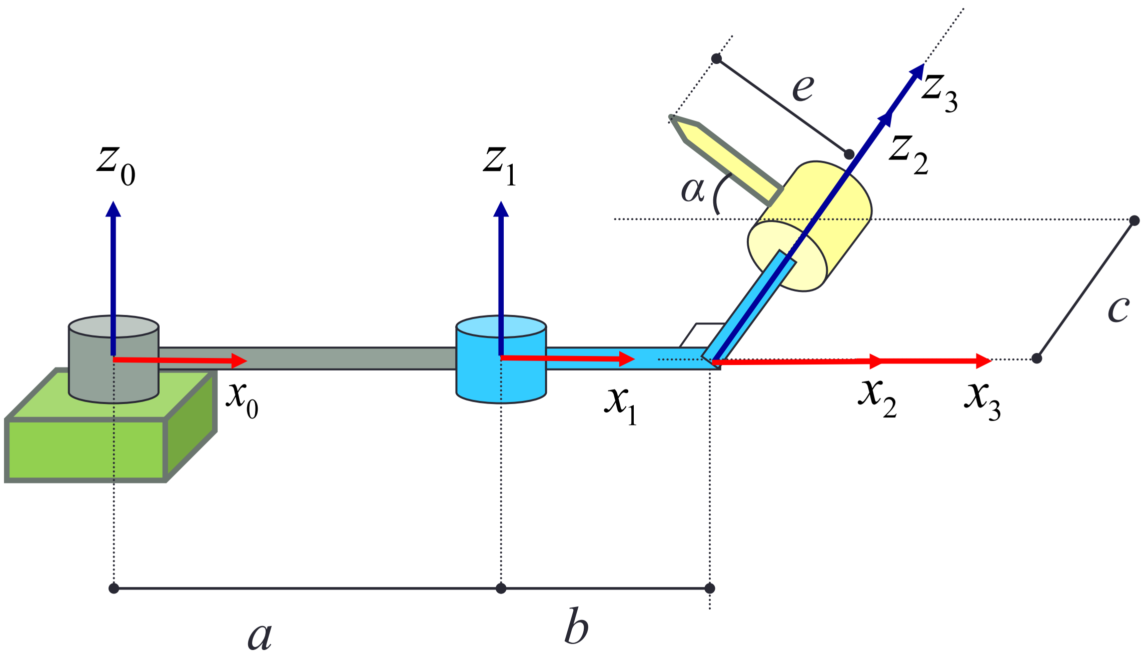
Ex 3-2-1

Fill in the table of DH parameters (type I) for the 3R robotic arm



Ex 3-2-1

Find the forward kinematics solution.

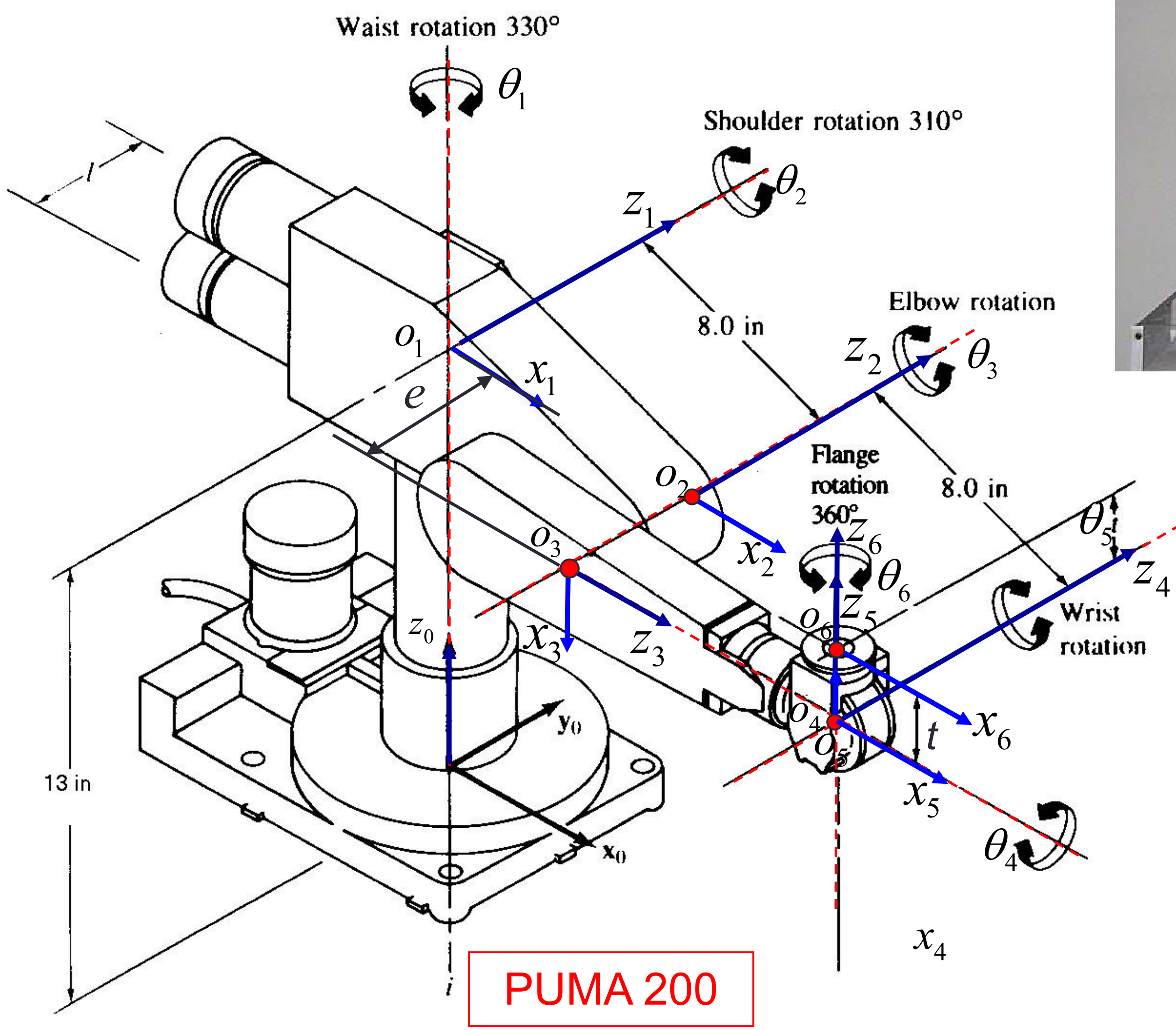


Frame No.	a_i	α_i	d_i	θ_i
1	a	0	0	θ_1
2	b	-90º	0	θ_2
3	0	0	0	θ_3

Notes on DH parameters

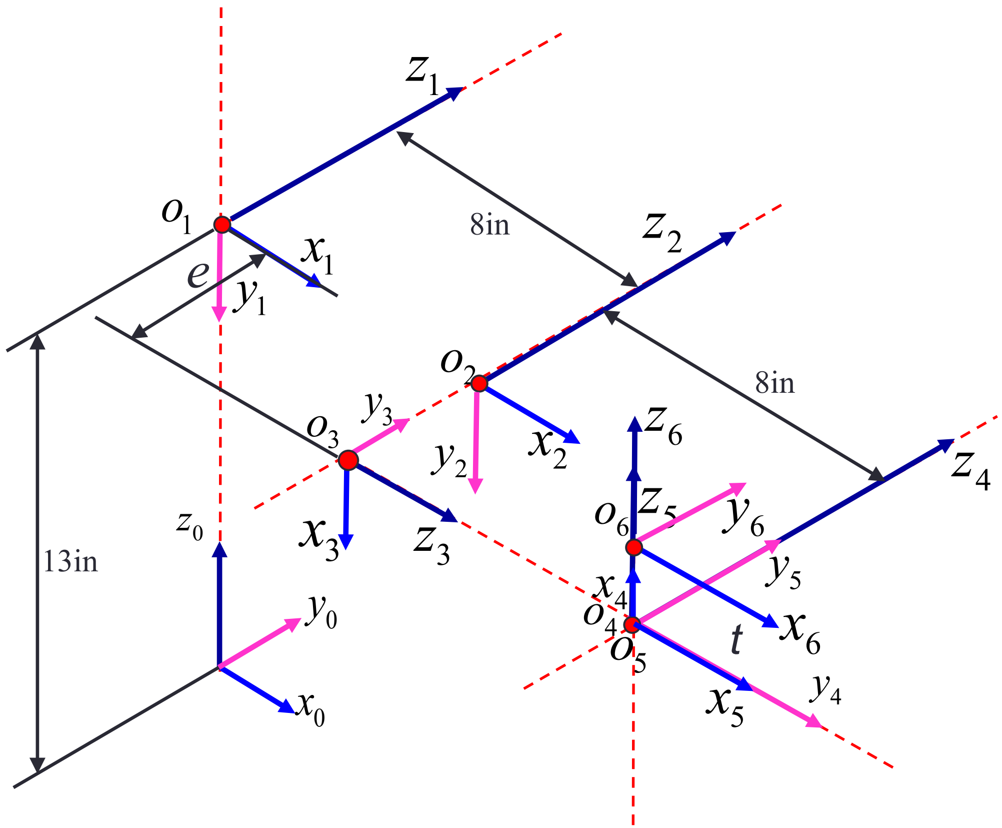
1. The configuration of a robot at which all the joint variables are zero is called the home configuration or rest position, which is the reference for all motions of a robot.
2. The DH coordinate frames are not unique because the direction of z_i-axes are arbitrary, and x-axis will be arbitrary if z_{i-1} and z_i intersects.
3. The direction x_i is to set a more convenient reference frame when most of the joint parameters are zero.
4. The best rest position is where it makes as many axes parallel to each other and coplanar as possible.

Ex 3-2-2 *Fill in the table of DH parameters for the PUMA200 robot*



Ex 3-2-2

Fill in the table of DH parameters for the PUMA200 robot



No.	a_i	α_i	d_i	θ_i
1	0	-90°	13	θ_1
2	8	0	0	θ_2
3	0	90°	$-e$	$\theta_3(90°)$
4	0	90°	8	$\theta_4(180°)$
5	0	90°	0	$\theta_5(90°)$
6	0	0	t	θ_6

✓ The DH coordinate frames are **not unique.**

Code Session

No code