Special Integrals of Gradshteyn and Ryzhik

the Proofs - Volume I

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MONOGRAPHS AND RESEARCH NOTES IN MATHEMATICS

Special Integrals of Gradshteyn and Ryzhik

the Proofs - Volume I

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To my teachers
Maria Pardo and Aida Arriagada

Contents

Introduction

The evaluation of definite integrals is a subject that every student encounters in beginning Calculus courses. It is particularly easy to give examples that produce complicated answers. For instance

$$\int_0^\infty e^{-x}\, dx = 1,$$

is elementary. The value

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

requires a little bit more work. This is a fundamental evaluation and it appears in every Statistics course. On the other hand, the value

$$\int_0^\infty e^{-x^3}\, dx = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

is more advanced and it involves the classical *Gamma function.* It is hard to convince a good student that this last evaluation cannot be written in a simpler form. *This is the best you can do.*

Aside from evaluating integrals just because they are there, the author has discussed in [62] and [68] a variety of mathematical questions coming from these evaluations.

Tables of integrals have been used since the beginning of time. The author recalls his undergraduate days where students carried earlier editions of [84]. Current tables include the encyclopedic treatise by A. P. Prudnikov et al. [73] as well as smaller volumes such as A. Apelblat [8]. There are also large collection of integrals that have appeared as papers. For instance, A. Devoto and D. Duke [34] contains a variety of definite integrals that are useful in the evaluation of Feynman diagrams (more of these appear in the book by V. A. Smirnov [78] and the series by C. C.Grosjean [44, 45, 46, 47]).

A literature search shows that the table of integrals by I. S. Gradshteyn and I. M. Ryzhik is one of the most used by the scientific community. The author became interested in the verification of its entries while trying to verify entry **3.248.5** of [39]

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}[\phi(x)+\sqrt{\phi(x)}]^{1/2}}$$

with

$$\phi(x) = 1 + \frac{4x^2}{3(1+x^2)^2}.$$

The table gave the value $\pi/2\sqrt{6}$. A direct numerical integration shows that this is incorrect. The integral is approximately 0.666377 and the right hand-side is about 0.641275. This error produces two natural questions: what is the value of the integral? and what produced the incorrect answer? To this day, April 2014, the author does not know how to answer either one.

The entry **3.248.5** looked interesting since the author had been involved in the expansion of the double square root [20]

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1} + \sum_{n=0}^{\infty} x_n(a)c^n$$

and had identified the coefficients as

$$x_n(a) = \frac{1}{\pi\sqrt{2}} \frac{(-1)^{n-1}}{n} N_{0,4}(a; n-1)$$

where

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}.$$

The integral $N_{0,4}(a;m)$ is evaluated in the form

$$N_{0,4}(a;m) = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+1/2}},$$

where the coefficients of the polynomial

$$P_m(a) = \sum_{\ell=0}^{m} d_{\ell,m} a^\ell$$

are given by

$$d_{\ell,m} = 2^{-2m} \sum_{k=\ell}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{l}.$$

The formula for $N_{0,4}(a;m)$ was established in [18]. A variety of proofs that have appeared in the literature are reviewed in [5]. Newer proofs continue to appear [6, 7, 48, 52].

The original proof simply showed how the specific form of this integral can be traced to an entry in Gradshteyn and Ryzhik. An earlier elementary proof [23] had produced a more complicated triple sum for the coefficients $d_\ell(m)$. It turns out that this triple sum was crucial in the computer proof by M. Kauers and P. Paule [50] of the logconcavity of these coefficients. The study of the sequence $\{d_\ell(m) : 0 \leq \ell \leq m\}$ has generated some interesting work.

The reader will find in [1, 17] descriptions of their unimodality. In [30, 32] the authors describe further ordering properties and [31] produces a combinatorial interpretation of this family. A summary of properties is presented in [59].

A second entry in Gradshteyn-Ryzhik that has generated interesting results is Entry **6.441.2**

$$\int_0^1 \ln \Gamma(x)\, dx = \ln \sqrt{2\pi}.$$

This is a result due to L. Euler. During the study of integrals containing the Hurwitz zeta function, just by chance, the author found the generalization

$$\int_0^1 \ln^2 \Gamma(x)\, dx = \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{\gamma \ln \sqrt{2\pi}}{3} + \\ \frac{4}{3}\ln^2 \sqrt{2\pi} - (\gamma + 2\ln\sqrt{2\pi})\frac{\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2},$$

presented in [37]. Here $\gamma = -\Gamma'(1)$ is Euler's constant. The corresponding value of the integral of $\ln^3 \Gamma(x)$ has recently been obtained by D. Bailey, D. Borwein and J. Borwein in [11] in terms of the sums

$$\omega_{a,b,c}(r,s,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\log^a m \, \log^b n \, \log^c (m+n)}{n^r m^s (n+m)^t}.$$

The goal of the project is to present self-contained proofs to all the entries in [40]. We have tried to give the most elementary arguments possible. An important step in the verification of these formulas is the use of a symbolic language. The author has used `Mathematica`. A standard reduction technique, that is helpful in the symbolic check of these integrals, is illustrated with entry **4.333**:

$$\int_0^\infty e^{-\mu x^2} \ln x \, dx = -\frac{1}{4}(\gamma + \ln(4\mu))\sqrt{\frac{\pi}{\mu}}.$$

Start with the change of variables $t = \sqrt{\mu} x$. This simplifies the exponent in the exponential to produce

$$\int_0^\infty e^{-\mu x^2} \ln x \, dx = \frac{1}{\sqrt{\mu}} \int_0^\infty e^{-t^2} \ln t \, dt - \frac{\ln \mu}{2\sqrt{\mu}} \int_0^\infty e^{-t^2} \, dt.$$

The first integral is the case $\mu = 1$ of entry **4.333** and second one is the classical normal integral. Both of them can be evaluated symbolically. This provides evidence that the formula *might* be correct. Now one can start looking for a proof. The reader should be warned that, as in every computer code, there are some bugs in the `Integrate` package of `Mathematica`. For instance, version 9.0 gives

$$\int_0^\infty \frac{\sinh(2x)\, dx}{e^{3x} - 1} = \frac{1}{36}(9 + 2\sqrt{3}\pi + \ln 729)$$

while the correct value, coming from entry **3.545.2** in [40]

$$\int_0^\infty \frac{\sinh ax\, dx}{e^{bx}-1} = \frac{1}{2a} - \frac{\pi}{2b}\cot\left(\frac{\pi a}{b}\right),$$

is

$$\int_0^\infty \frac{\sinh(2x)\, dx}{e^{3x}-1} = \frac{1}{36}(9+2\sqrt{3}\pi).$$

In the proofs presented here, we have indicated the entries that cannot be evaluated by `Mathematica`. For the material chosen for this volume, *there are not too many of them*. `Mathematica` has powerful integrating algorithms. A second example where one obtains an *incorrect symbolic answer* comes from entry **3.269.1**:

$$\int_0^1 \frac{x^p - x^{-p}}{1-x^2}\, x\, dx = \frac{\pi}{2}\cot\left(\frac{\pi p}{2}\right) - \frac{1}{p}.$$

`Mathematica` version 9.0 gives the value

$$\int_0^1 \frac{x^p - x^{-p}}{1-x^2}\, x\, dx = \frac{\pi}{2\sin(\pi p)} - \frac{1+p^2}{2p(1-p^2)}.$$

For safety reasons, the author has checked each entry using numerical integration in `Mathematica`.

The message of the previous paragraph is that in the evaluation of definite integrals there is still room for classical human proofs. The author's vision of a perfect entry is one that has a variety of proofs, that can be evaluated as a symbolic language or points to the development of a new algorithm. Moreover, the formula *must have a reason for being.*

The entries are given without indication of the range of parameters for their validity. It is a rewarding exercise for the reader to determine this range. A second ingredient missing from the book is a historical background on these entries. This is an essential component of a table of integrals. The reader should be informed about where the integral came from and a bibliographical reference for it. Some of this information appears in the work of Bierens de Haan [15]. Lack of time (and mostly knowledge) has kept the author from pursuing this goal.

Many of the entries in [40] came from the tables of integrals produced by Bierens de Haan [16]. Information about this author appears in [74]. Corrections to this work were produced by C. F. Lindman [56]. Many other entries have been added through the years. Among the sources cited for the entries in Gradshteyn and Ryzhik [40] the reader will find the classical tables by A. Erdélyi et al. [35, 36] and [12] and also the tables [42] and [43]. The reader will find in

`www.stephenwolfram.com/publications/history-future-special-functions` a lecture by S. Wolfram on Special Functions, where part of the history of this table is described.

The reader of [40] will find in it a section called `Acknowledgments`. This contains a list of all the users of the table that have contributed to the latest edition. It is a way in which the editors thank the community for the continuous effort to make this volume a reliable document. The author wishes to thank Dan Zwillinger for his invitation to participate in the editing process of future editions of this table. The length of this list of users shows that the formulas presented in [40] are the result of a truly collaborative effort. The author wishes to thank, in the same spirit, those who participated in the papers that form the basis for this volume. Other collaborators will be acknowledged in future volumes. Many thanks to Matthew Albano, Jason Rosenberg, and Pat Whitworth, undergraduates at the time when the work was prepared; Luis Medina, Armin Straub, and Erin Beyerstedt, then graduate students; Ronald Posey and Khristo Boyadzhiev, colleagues who were patient with my many emails about integrals. The author wishes to thank Lin Jiu for a careful reading of this manuscript. Finally, particular thanks are given to Tewodros Amdeberhan, who has shown the author the beauty, patterns, and unexpected connections behind these formulas.

List of formulas

The current volume contains the first 15 papers containing proofs of the entries in [40]. The author has looked for collaborators in this task among colleagues and has also included graduate and undergraduate students in it. The first paper in this series [63] appeared in *Revista Scientia*. This is a journal published by the Departamento de Matemáticas of Universidad Técnica Federico Santa Maria in Valparaiso, Chile. The author was an undergraduate student there. Following a suggestion of a referee, the sequence of papers containing proofs of other entries have all appeared in this journal. The author wishes to thank Prof. Iván Szantó, the editor of *Revista Scientia* for his collaboration in this project and for his permission to include these papers in this volume.

The title of the papers starts with *The integrals in Gradshteyn and Ryzhik* followed by an indication of the content of the specific paper. The papers included here are:

Part 1: A family of logarithmic integrals, $\mathbf{14}, 2007, 1-6$.

Part 2: Elementary logarithmic integrals, $\mathbf{14}, 2007, 7-15$.

Part 3: Combinations of logarithms and exponentials, $\mathbf{15}, 2007, 31-36$.

Part 4: The gamma function, $\mathbf{15}, 2007, 37-36$.

Part 5: Some trigonometric integrals, $\mathbf{15}, 2007, 47-60$; with T. Amdeberhan and L. Medina.

Part 6: The beta function, $\mathbf{16}, 2008, 9-24$.

Part 7: Elementary examples, $\mathbf{16}, 2008, 25-39$; with T. Amdeberhan.

Part 8: Combinations of powers, exponentials, and logarithms, $\mathbf{16}, 2008, 41-50$; with J. Rosenberg, A. Straub and P. Whitworth.

Part 9: Combinations of logarithms, rational and trigonometric functions, $\mathbf{17}, 2009, 27-44$; with T. Amdeberhan, J. Rosenberg, A. Straub, and P. Whitworth.

Part 10: The digamma function, $\mathbf{17}, 2009, 45-66$; with L. Medina.

Part 11: The incomplete beta function, $\mathbf{18}, 2009, 61-75$; with K. Boyadzhiev and L. Medina.

Part 12: Some logarithmic integrals, **18**, 2009, 77 – 84;
with R. Posey.

Part 13: Trigonometric forms of the beta function, **19**, 2010, 91 – 96.

Part 14: An elementary evaluation of entry 3.411.5, **19**, 2010, 97 – 103;
with T. Amdeberhan.

Part 15: Frullani integrals, **19**, 2010, 113 – 119;
with M. Albano, T. Amdeberhan, and E. Beyerstedt.

This volume contains the first collection of evaluations from [40] as part of a plan to present proofs of all the entries. These chapters are small modifications of the articles that have appeared in ***Scientia***, so ***the chapters can be read independently of each other***.

The table [40] is divided into definite and indefinite integrals. Starting in Section 2, page 63, the reader will find a list of indefinite integrals. This part is subdivided into sections, with each section named after the type of integrals found in it. For instance, section **2.15** has the title *Forms containing pairs of binomials: $a + bx$ and $\alpha + \beta x$.* Sections are then divided into subsections. The number of entries in each subsection varies a great deal. For instance, subsection 2.152 contains two entries

$$\textbf{2.152.1} \qquad \int \frac{z}{t}\,dx = \frac{bx}{\beta} + \frac{\Delta}{\beta^2}\ln \ell$$

$$\textbf{2.152.2} \qquad \int \frac{t}{z}\,dx = \frac{\beta x}{b} - \frac{\Delta}{b^2}\ln z$$

where $z = a + bx$, $t = \alpha + \beta x$, and $\Delta = a\beta - \alpha b$. The list of indefinite integrals ends on page 245.

The titles of the sections are:

Title		Pages
2.1	Rational Functions	**66** to **82**
2.2	Algebraic Functions	**82** to **105**
2.3	The Exponential Function	**106** to **109**
2.4	Hyperbolic Functions	**110** to **151**
2.5 – 2.6	Trigonometric Functions	**151** to **237**
2.7	Logarithms and Inverse-Hyperbolic Functions	**237** to **241**
2.8	Inverse Trigonometric Functions	**241** to **245**

The part corresponding to indefinite integrals that is considered in the first volume of this series, begins on page 253. The sections are named

Title		**Pages**
3.1 – 3.2	Power and Algebraic Functions	**253** to **333**
3.3 – 3.4	Exponential Functions	**334** to **371**
3.5	Hyperbolic Functions	**371** to **390**
3.6 – 4.1	Trigonometrical Functions	**390** to **527**
4.2	Logarithmic Functions	**527** to **599**
4.2	Inverse Trigonometric Functions	**599** to **607**

Later sections contain integrals of a variety of special functions. This will be discussed in future volumes.

Chapter 1

A family of logarithmic integrals

1.1 Introduction

The values of many definite integrals have been compiled in the classical *Table of Integrals, Series and Products* by I. S. Gradshteyn and I. M. Ryzhik [40]. The table is organized like a phonebook: integrals that *look* similar are placed close together. For example, **4.229.4** gives

$$\int_0^1 \ln\left(\ln\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{\mu-1} dx = \psi(\mu)\Gamma(\mu), \tag{1.1.1}$$

for $\operatorname{Re}\mu > 0$, and **4.229.7** states that

$$\int_{\pi/4}^{\pi/2} \ln\ln\tan x \, dx = \frac{\pi}{2}\ln\left\{\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}\right\}. \tag{1.1.2}$$

In spite of a large amount of work in the development of this table, the latest version of [40] still contains some typos. For example, the exponent u in (1.1.1) should be μ. A list of errors and typos can be found in

http://www.mathtable.com/errata/gr6_errata.pdf

The fact that two integrals are close in the table is not a reflection of the difficulty involved in their evaluation. Indeed, the formula (1.1.1) can be established by the change of variables $v = -\ln x$ followed by differentiating the classical gamma function

$$\Gamma(\mu) := \int_0^\infty t^{\mu-1}e^{-t}\,dt, \quad \operatorname{Re}\mu > 0, \tag{1.1.3}$$

with respect to the parameter μ. The function $\psi(\mu)$ in (1.1.1) is simply the logarithmic derivative of $\Gamma(\mu)$ and the formula has been checked. The situation is quite different for (1.1.2). This formula is the subject of the lovely paper

[82] in which the author uses Analytic Number Theory to check (1.1.2). The ingredients of the proof are quite formidable: the author shows that

$$\int_{\pi/4}^{\pi/2} \ln \ln \tan x \, dx = \frac{d}{ds} \Gamma(s) L(s) \text{ at } s = 1, \tag{1.1.4}$$

where

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots \tag{1.1.5}$$

is the Dirichlet L-function. The computation of (1.1.4) is done in terms of the Hurwitz zeta function

$$\zeta(q,s) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s}, \tag{1.1.6}$$

defined for $0 < q < 1$ and $\text{Re}\, s > 1$. The function $\zeta(q,s)$ can be analytically continued to the whole plane with only a simple pole at $s = 1$ using the integral representation

$$\zeta(q,s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-qt} t^{s-1}}{1 - e^{-t}} \, dt. \tag{1.1.7}$$

The relation with the L-functions is provided by employing

$$L(s) = 2^{-2s} \left(\zeta(s, \tfrac{1}{4}) - \zeta(s, \tfrac{3}{4}) \right). \tag{1.1.8}$$

The functional equation

$$L(1-s) = \left(\frac{2}{\pi} \right)^s \sin \frac{\pi s}{2} \Gamma(s) \, L(s), \tag{1.1.9}$$

and Lerch's identity

$$\zeta'(0,a) = \log \frac{\Gamma(a)}{\sqrt{2\pi}}, \tag{1.1.10}$$

complete the evaluation. More information about these functions can be found in [83].

In the introduction to [22] we expressed the desire to establish *all* the formulas in [40]. This is a *nearly impossible task* as was also noted by a (not so) favorable review given in [76]. This is the first of a series of papers where we present some of these evaluations.

We consider here the family

$$f_n(a) = \int_0^\infty \frac{\ln^{n-1} x \, dx}{(x-1)(x+a)}, \text{ for } n \geq 2 \text{ and } a > 0. \tag{1.1.11}$$

Special examples of f_n appear in [40]. The reader will find

$$f_2(a) = \frac{\pi^2 + \ln^2 a}{2(1+a)} \tag{1.1.12}$$

as formula **4.232.3** and

$$f_3(a) = \frac{\ln a \, (\pi^2 + \ln^2 a)}{3(1+a)} \tag{1.1.13}$$

as formula **4.261.4**. In later sections the persistent reader will find

$$\begin{aligned} f_4(a) &= \frac{(\pi^2 + \ln^2 a)^2}{4(1+a)} \\ f_5(a) &= \frac{\ln a \, (\pi^2 + \ln^2 a)(7\pi^2 + 3\ln^2 a)}{15(1+a)} \\ f_6(a) &= \frac{(\pi^2 + \ln^2 a)^2(3\pi^2 + \ln^2 a)}{6(1+a)} \end{aligned}$$

as **4.262.3**, **4.263.1** and **4.264.3** respectively.

These formulas suggest that

$$h_n(b) := f_n(a) \times (1+a) \tag{1.1.14}$$

is a polynomial in the variable $b = \ln a$. The relatively elementary evaluation of $f_n(a)$ discussed here identifies this polynomial.

There are several classical results that are stated without proof. The reader will find them in [9] and [22].

1.2 The evaluation

The expression (1.1.11) for $f_n(a)$ can be written as

$$f_n(a) = \int_0^1 \frac{\ln^{n-1} x \, dx}{(x-1)(x+a)} + \int_1^\infty \frac{\ln^{n-1} x \, dx}{(x-1)(x+a)},$$

and the transformation $t = 1/x$ in the second integral yields

$$f_n(a) = \int_0^1 \frac{\ln^{n-1} x \, dx}{(x-1)(x+a)} + (-1)^n \int_0^1 \frac{\ln^{n-1} x \, dx}{(x-1)(1+ax)}.$$

The partial fraction decomposition

$$\frac{1}{(x-1)(x+a)} = \frac{1}{1+a}\frac{1}{x-1} - \frac{1}{1+a}\frac{1}{x+a}$$

yields the representation

$$\begin{aligned} f_n(a) \quad = \quad & \frac{1-(-1)^{n-1}}{1+a} \int_0^1 \frac{\ln^{n-1} x \, dx}{x-1} - \frac{1}{1+a} \int_0^1 \frac{\ln^{n-1} x \, dx}{x+a} \\ & + (-1)^{n-1} \frac{a}{1+a} \int_0^1 \frac{\ln^{n-1} x \, dx}{1+ax}. \end{aligned}$$

The evaluation of these integrals require the *polylogarithm* function defined by

$$\mathrm{Li}_n(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^n}. \tag{1.2.1}$$

This function is sometimes denoted by PolyLog$[n, x]$. Detailed information about the polylogarithm functions appears in [54].

Proposition 1.2.1. *For $n \in \mathbb{N}$, $n \geq 2$ and $a > 1$ we have*

$$\begin{aligned}
\int_0^1 \frac{\ln^{n-1} x \, dx}{x-1} &= (-1)^n (n-1)! \, \zeta(n), \\
\int_0^1 \frac{\ln^{n-1} x \, dx}{x+a} &= (-1)^n (n-1)! \, \mathrm{Li_n}(-1/\mathrm{a}), \\
\int_0^1 \frac{\ln^{n-1} x \, dx}{1+ax} &= (-1)^n \frac{(n-1)!}{a} \, \mathrm{Li_n}(-\mathrm{a}).
\end{aligned}$$

Proof. Simply expand the integrand in a geometric series. □

Corollary 1.2.1. *The integral $f_n(a)$ is given by*

$$f_n(a) = \frac{(-1)^n (n-1)!}{1+a} \left\{ \left[(1 - (-1)^{n-1} \right] \zeta(n) - \mathrm{Li_n}\left(-\tfrac{1}{\mathrm{a}}\right) + (-1)^{\mathrm{n}-1} \mathrm{Li_n}(-\mathrm{a}) \right\}.$$

The reduction of the previous expression requires the identity

$$\mathrm{Li}_\nu(\mathrm{z}) = \frac{(2\pi)^\nu}{\Gamma(\nu)} \mathrm{e}^{\pi \mathrm{i} \nu/2} \zeta\left(1-\nu, \frac{\log(-\mathrm{z})}{2\pi \mathrm{i}} + \frac{1}{2}\right) - \mathrm{e}^{\pi \mathrm{i} \nu} \mathrm{Li}_\nu(-1/\mathrm{z}). \tag{1.2.2}$$

This transformation for the polylogarithm function appears in

http://functions.wolfram.com/10.08.17.0007.01

In the special case $z = -a$ and $\nu = n$, with $n \in \mathbb{N}$, $n \geq 2$, we obtain

$$(-1)^{n-1} \mathrm{Li_n}(-\mathrm{a}) - \mathrm{Li_n}(-1/\mathrm{a}) = \frac{(2\pi)^{\mathrm{n}}}{\mathrm{n}! \, \mathrm{i}^{\mathrm{n}}} \mathrm{B_n}\left(\frac{\log \mathrm{a}}{2\pi \mathrm{i}} + \frac{1}{2}\right), \tag{1.2.3}$$

where $B_n(z)$ is the Bernoulli polynomial of order n. This family of polynomials is defined by their exponential generating function

$$\frac{te^{qt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(q) \frac{t^k}{k!}. \tag{1.2.4}$$

The classical identity

$$\zeta(1-k, q) = -\frac{1}{k} B_k(q), \text{ for } k \in \mathbb{N} \tag{1.2.5}$$

is used in (1.2.3). Therefore the result in Corollary 1.2.1 can be written as:

Corollary 1.2.2. *The integral $f_n(a)$ is given by*

$$f_n(a) = \frac{(-1)^n}{1+a}(n-1)!\,[1+(-1)^n]\,\zeta(n) + \frac{(2\pi i)^n}{n(1+a)} B_n\left(\frac{\log a}{2\pi i} + \frac{1}{2}\right).$$

We now proceed to simplify this representation. The Bernoulli polynomials satisfy the addition theorem

$$B_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} B_j(x) y^{n-j}, \tag{1.2.6}$$

and the reflection formula

$$B_n(\tfrac{1}{2} - x) = (-1)^n B_n\left(\tfrac{1}{2} + x\right). \tag{1.2.7}$$

In particular $B_n(\frac{1}{2}) = 0$ if n is odd. For n even, one has

$$B_n(\tfrac{1}{2}) = (2^{1-n} - 1)B_n, \tag{1.2.8}$$

where B_n is the Bernoulli number $B_n(0)$. Thus, the last term in Corollary 1.2.2 becomes

$$B_n\left(\frac{\log a}{2\pi i} + \frac{1}{2}\right) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (2^{1-2j} - 1) B_{2j} \left(\frac{\log a}{2\pi i}\right)^{n-2j}.$$

We have completed the proof of the following closed-form formula for $f_n(a)$:

Theorem 1.2.1. *The integral $f_n(a)$ is given by*

$$\begin{aligned} f_n(a) &= \frac{(-1)^n\,(n-1)!}{1+a}\,[1+(-1)^n]\,\zeta(n) + \\ &+ \frac{1}{n(1+a)} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (2^{2j} - 2)(-1)^{j-1} B_{2j} \pi^{2j} (\log a)^{n-2j}. \end{aligned}$$

Observe that if n is odd, the first term vanishes and there is no contribution of the *odd zeta values*. For n even, the first term provides a rational multiple of π^n in view of Euler's representation of the even zeta values

$$\zeta(2m) = \frac{(-1)^{m+1}\,(2\pi)^{2m} B_{2m}}{2(2m)!}. \tag{1.2.9}$$

The polynomial h_n predicted in (1.1.14) can now be read directly from this expression for the integral f_n. Observe that h_n has positive coefficients because the Bernoulli numbers satisfy $(-1)^{j-1}B_{2j} > 0$.

Note. The change of variables $t = \ln x$ converts $h_n(a)$ into the form

$$h_n(a) = \int_{-\infty}^{\infty} \frac{t^{n-1}\, dt}{(1-e^{-t})(a+e^t)}. \tag{1.2.10}$$

The integrals $h_n(a)$ for $n = 2, \cdots, 5$ appear in [40] as **3.419.2**, $\cdots$, **3.419.6**. The latest edition has an error in the expression for this last value.

Conclusions. We have provided an evaluation of the integral

$$f_n(a) := \int_0^{\infty} \frac{\ln^{n-1} x\, dx}{(x-1)(x+a)}, \tag{1.2.11}$$

given by

$$\begin{aligned} n(1+a)f_n(a) &= (-1)^n n!\,[1+(-1)^n]\,\zeta(n) \\ &+ \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} (2^{2j}-2)(-1)^{j-1} B_{2j} \pi^{2j} (\log a)^{n-2j}. \end{aligned} \tag{1.2.12}$$

Symbolic calculation. At the present time, February 2014, the integrals evaluated in this chapter can be computed with `Mathematica` 9.0 if the parameter n is assigned a numerical (positive integer) value and a is kept as a free parameter. On the other hand, if both n and a are kept in symbolic form, `Mathematica` is unable to give the value of the integral. This also happens if a is assigned a numerical value and n is kept as a parameter.

The integrands described in this chapter have the structral form

$$\text{polynomial in } x \times \text{rational function of } x. \tag{1.2.13}$$

More examples of this class will appear in later chapters.

The indefinite integral

$$\int \frac{\ln^{n-1} x}{x-b}\, dx \tag{1.2.14}$$

that would give the integrals evaluated here after a partial fraction expansion, cannot be computed when n is kept as a parameter. For a given value of n, `Mathematica` gives the indefinite integral in terms of elementary functions and polylogarithms. For example,

$$\begin{aligned} \int \frac{\ln^3 x}{x-b}\, dx = \ln^3 x \ln\left(1-\frac{x}{b}\right) + 3\ln^2 x \text{ PolyLog}\left[2, \frac{x}{b}\right] \\ - 6 \ln x \text{ PolyLog}\left[3, \frac{x}{b}\right] + 6 \text{ PolyLog}\left[3, \frac{x}{b}\right]. \end{aligned}$$

This is discussed in the next chapter.

There are also some issues of producing the simplest expression for these integrals. For instance, for entry **3.419.5**, Version 9.0 of `Mathematica` gives

$$\int_{-\infty}^{\infty} \frac{x^4\,dx}{(\beta+e^x)(1-e^{-x})} = \frac{24}{1+b}\left(\text{PolyLog}\left[5,-\tfrac{1}{b}\right] - \text{PolyLog}\left[5,-b\right]\right).$$

The reader is invited to learn the commands to produce

$$\int_{-\infty}^{\infty} \frac{x^4\,dx}{(\beta+e^x)(1-e^{-x})} = \frac{\ln\beta}{15(\beta+1)}\left(\pi^2+\ln^2\beta\right)\left(7\pi^2+3\ln^2\beta\right).$$

The latest edition [40] has a typo in this entry: the factor $\pi^2+\ln^2\beta$ has an extra square.

Note. On April 2014, the entries **4.229.4** and **4.229.7**, given in (1.1.1) and (1.1.2) respectively, cannot be evaluated by `Mathematica`, version 9.0.

Chapter 2

Elementary logarithmic integrals

2.1 Introduction

The table of integrals by I. S. Gradshteyn and I. M. Ryzhik [40] contains a large selection of definite integrals of the form

$$\int_a^b R(x) \ln^m x \, dx, \tag{2.1.1}$$

where $R(x)$ is a rational function, a, $b \in \mathbb{R}^+$ and $m \in \mathbb{N}$. We call integrals of the form (2.1.1) *elementary logarithmic integrals*. The goal of this note is to present methods to evaluate them. We may assume that $a = 0$ using

$$\int_a^b R(x) \ln^m x \, dx = \int_0^b R(x) \, \ln^m x \, dx - \int_0^a R(x) \, \ln^m x \, dx. \tag{2.1.2}$$

Section 2.2 describes the situation when R is a polynomial. Section 2.3 presents the case in which the rational function has a single simple pole. Finally section 2.4 considers the case of multiple poles.

2.2 Polynomial examples

The first example considered here is

$$I(P; b, m) := \int_0^b P(x) \, \ln^m x \, dx, \tag{2.2.1}$$

where P is a polynomial. This can be evaluated in elementary terms. Indeed, $I(P;b,m)$ is a linear combination of

$$\int_0^b x^j \ln^m x\,dx, \tag{2.2.2}$$

and the change of variables $x = bt$ yields

$$\int_0^b x^j \ln^m x\,dx = b^{j+1}\sum_{k=0}^{m}\binom{m}{k}\ln^{m-k} b\int_0^1 t^j \ln^k t\,dt. \tag{2.2.3}$$

The last integral evaluates to $(-1)^k k!/(j+1)^{k+1}$ either by an easy induction argument or by the change of variables $t = e^{-s}$ that gives it as a value of the gamma function.

Theorem 2.2.1. *Let $P(x)$ be a polynomial given by*

$$P(x) = \sum_{j=0}^{p} a_j x^j. \tag{2.2.4}$$

Then

$$\begin{aligned} I(P;b,m) &:= \int_0^b P(x)\ln^m x\,dx \qquad (2.2.5)\\ &= \sum_{k=0}^{m}(-1)^k k!\binom{m}{k}\ln^{m-k} b\sum_{j=0}^{p} a_j\frac{b^{j+1}}{(j+1)^{k+1}}. \end{aligned}$$

This expression shows that $I(P;b,m)$ is a linear combination of $b^j \ln^k b$, with $1 \leq j \leq 1+p(=1+deg(P))$ and $0 \leq k \leq m$.

2.3 Linear denominators

We now consider the integral

$$f(b;r) := \int_0^b \frac{\ln x\,dx}{x+r} \tag{2.3.1}$$

for $b,\ r > 0$. This corresponds to the case in which the rational function in (2.1.1) has a single simple pole.

The change of variables $x = rt$ produces

$$\int_0^b \frac{\ln x\,dx}{x+r} = \ln r\,\ln(1+b/r) + \int_0^{b/r}\frac{\ln t\,dt}{1+t}. \tag{2.3.2}$$

Therefore, it suffices to consider the function

$$g(b) := \int_0^b \frac{\ln t\, dt}{1+t}, \tag{2.3.3}$$

as we have

$$f(b;r) = \ln r\, \ln\left(1+\frac{b}{r}\right) + g\left(\frac{b}{r}\right). \tag{2.3.4}$$

Before we present a discussion of the function g, we describe some elementary consequences of (2.3.2).

Elementary examples. The special case $r = b$ in (2.3.2) yields

$$\int_0^b \frac{dx}{x+b} = \ln 2\, \ln b + \int_0^1 \frac{\ln t\, dt}{1+t}. \tag{2.3.5}$$

Expanding $1/(1+t)$ as a geometric series, we obtain

$$\int_0^1 \frac{\ln t\, dt}{1+t} = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}. \tag{2.3.6}$$

This appears as **4.231.1** in [40]. Differentiating (2.3.2) with respect to r produces

$$\int_0^b \frac{\ln x\, dx}{(x+r)^2} = -\frac{\ln(b+r)}{r} + \frac{\ln r}{r} + \frac{b\, \ln b}{r(r+b)}. \tag{2.3.7}$$

As b, $r \to 1$ we obtain

$$\int_0^1 \frac{\ln x\, dx}{(1+x)^2} = -\ln 2. \tag{2.3.8}$$

This appears as **4.231.6** in [40]. On the other hand, as $b \to \infty$ we recover **4.231.5** in [40]:

$$\int_0^\infty \frac{\ln x\, dx}{(x+r)^2} = \frac{\ln r}{r}. \tag{2.3.9}$$

The polylogarithm function. The evaluation of the integral

$$g(b) := \int_0^b \frac{\ln t\, dt}{1+t}, \tag{2.3.10}$$

requires the transcendental function

$$\mathrm{Li}_n(x) := \sum_{k=1}^\infty \frac{x^k}{k^n}. \tag{2.3.11}$$

This is the *polylogarithm function* and it has also appeared in [63] in our discussion of the family

$$h_n(a) := \int_0^\infty \frac{\ln^n x\, dx}{(x-1)(x+a)}, \quad n \in \mathbb{R},\, a > 0. \tag{2.3.12}$$

In the current context we have $n = 2$ and we are dealing with the *dilogarithm function*: $\text{Li}_2(x)$.

Lemma 2.3.1. *The function $g(b)$ is given by*

$$g(b) = \ln b \, \ln(1+b) + \text{Li}_2(-\text{b}). \tag{2.3.13}$$

Proof. The change of variables $t = bs$ yields

$$g(b) = \ln b \, \ln(1+b) + \int_0^1 \frac{\ln s \, ds}{1+bs}. \tag{2.3.14}$$

Expanding the integrand in a geometric series yields the final identity. □

Theorem 2.3.1. *Let b, $r > 0$. Then*

$$\int_0^b \frac{\ln x \, dx}{x+r} = \ln b \ln\left(\frac{b+r}{r}\right) + \text{Li}_2\left(-\frac{\text{b}}{\text{r}}\right). \tag{2.3.15}$$

Corollary 2.3.1. *Let $b > 0$. Then*

$$\int_0^b \frac{\ln x \, dx}{x+b} = \ln 2 \, \ln b - \frac{\pi^2}{12}. \tag{2.3.16}$$

Proof. Let $r \to b$ in Theorem 2.3.1 and use

$$\text{Li}_2(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}. \tag{2.3.17}$$

□

The expression in Theorem 2.3.1 and the method of partial fractions gives the explicit evaluation of elementary logarithmic integrals where the rational function has simple poles. For example:

Corollary 2.3.2. *Let $0 < a < b$ and $r_1 \neq r_2 \in \mathbb{R}^+$. Then, with $r = r_2 - r_1$, we have*

$$\int_a^b \frac{\ln x \, dx}{(x+r_1)(x+r_2)} = \frac{1}{r}\left[\ln b \, \ln\left(\frac{r_2(b+r_1)}{r_1(b+r_2)}\right) + \ln a \, \ln\left(\frac{r_1(a+r_2)}{r_2(a+r_1)}\right)\right] + \\ + \frac{1}{r}\left[\text{Li}_2\left(-\frac{\text{b}}{\text{r}_1}\right) - \text{Li}_2\left(-\frac{\text{a}}{\text{r}_1}\right) - \text{Li}_2\left(-\frac{\text{b}}{\text{r}_2}\right) + \text{Li}_2\left(-\frac{\text{a}}{\text{r}_2}\right)\right].$$

The special case $a = r_1$ and $b = r_2$ is of interest:

Corollary 2.3.3. *Let $0 < a < b$. Then*

$$\begin{aligned} \int_a^b \frac{\ln x \, dx}{(x+a)(x+b)} &= \frac{1}{b-a}\left[\ln(ab)\ln(a+b) - \ln 2 \ln(ab) - 2\ln a \, \ln b\right] \\ &+ \frac{1}{b-a}\left[-2\text{Li}_2(-1) + \text{Li}_2\left(-\frac{\text{b}}{\text{a}}\right) + \text{Li}_2\left(-\frac{\text{a}}{\text{b}}\right)\right]. \end{aligned}$$

The integral in Corollary 2.3.3 appears as **4.232.1** in [40]. An interesting problem is to derive **4.232.2**

$$\int_0^\infty \frac{\ln x\,dx}{(x+u)(x+v)} = \frac{\ln^2 u - \ln^2 v}{2(u-v)} \tag{2.3.18}$$

directly from Corollary 2.3.3.

We now present an elementary evaluation of this integral and obtain from it an identity of Euler. We prove that

$$\int_a^b \frac{\ln x\,dx}{(x+a)(x+b)} = \frac{\ln ab}{2(b-a)} \ln \frac{(a+b)^2}{4ab}. \tag{2.3.19}$$

Proof. The partial fraction decomposition

$$\frac{1}{(x+a)(x+b)} \quad = \quad \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b} \right).$$

reduces the problem to the evaluation of

$$I_1 = \int_a^b \frac{\ln x\ dx}{x+a} \text{ and } I_2 = \int_a^b \frac{\ln x\ dx}{x+b}.$$

The change of variables $x = at$ gives, with $c = b/a$,

$$\begin{aligned} I_1 &= \int_1^c \frac{\ln(at)\ dt}{1+t} \\ &= \ln a \int_1^c \frac{dt}{1+t} + \int_1^c \frac{\ln t}{1+t}\,dt \\ &= \ln a \ln(1+c) - \ln a \ln 2 + \int_1^c \frac{\ln t}{1+t}\,dt. \end{aligned}$$

Similarly,

$$I_2 = \ln b \ln 2 - \ln b \ln(1+1/c) + \int_1^{1/c} \frac{\ln t}{1+t}\,dt.$$

Therefore

$$\begin{aligned} I_1 - I_2 &= \ln a \ln(1+c) + \ln b \ln(1+1/c) - \ln 2 \ln a - \ln 2 \ln b + \\ &+ \int_1^c \frac{\ln t}{1+t}\,dt - \int_{1/c}^1 \frac{\ln t}{1+t}\,dt. \end{aligned}$$

Let $s = 1/t$ in the second integral to get

$$\int_{1/c}^1 \frac{\ln t}{1+t}\,dt \quad = \quad \int_c^1 \frac{\ln s}{s(1+s)}\,ds.$$

Replacing in the expression for $I_1 - I_2$ yields

$$\begin{aligned} I_1 - I_2 &= \ln a\,(\ln(a+b) - \ln a - \ln 2) - \ln b\,(\ln 2 - \ln(a+b) + \ln b) + \\ &+ \int_1^c \frac{\ln t}{t}\,dt. \end{aligned}$$

The last integral can now be evaluated by elementary means to produced the result. □

Now comparing the two evaluation of the integral in Corollary 2.3.3 produces an identity for the dilogarithm function.

Corollary 2.3.4. *The dilogarithm function satisfies*

$$\text{Li}_2(-\text{z}) + \text{Li}_2\left(-\frac{1}{\text{z}}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(\text{z}). \tag{2.3.20}$$

In particular,

$$\text{Li}(-1) = -\frac{\pi^2}{12}. \tag{2.3.21}$$

Replacing in Corollary 2.3.3 gives (2.3.19).

This is the first of many interesting functional equations satisfied by the polylogarithm functions. It was established by L. Euler in 1768. The reader will find in [54] a nice description of them.

2.4 A unique multiple pole

In this section we consider the evaluation of

$$f_n(b,r) := \int_0^b \frac{\ln x\,dx}{(x+r)^n}. \tag{2.4.1}$$

This corresponds to the elementary rational integrals with a unique pole (at $x = -r$). The change of variables $x = rt$ yields

$$f_n(b,r) = \frac{\ln r}{(n-1)r^{n-1}}\left[\frac{(b+r)^{n-1} - r^{n-1}}{(b+r)^{n-1}}\right] + \frac{1}{r^{n-1}}h_n(b/r),$$

where

$$h_n(b) := \int_0^b \frac{\ln t\,dt}{(1+t)^n}. \tag{2.4.2}$$

We first establish a recurrence for h_n.

Theorem 2.4.1. *Let $n > 2$ and $b > 0$. Then h_n satisfies the recurrence*

$$\begin{aligned} h_n(b) &= \frac{n-2}{n-1} h_{n-1}(b) + \frac{b \ln b}{(n-1)(1+b)^{n-1}} \\ &+ \frac{1-(1+b)^{n-2}}{(n-1)(n-2)(1+b)^{n-2}}. \end{aligned} \tag{2.4.3}$$

Proof. Start with

$$h_n(b) = \int_0^b \frac{[(1+t)-t] \ln t \, dt}{(1+t)^n} = h_{n-1}(b) - \int_0^b \frac{t \ln t \, dt}{(1+t)^n}.$$

Integrate by parts in the last integral, with $u = t \ln t$ and $dv = dt/(1+t)^n$ to produce the result. □

The initial condition for this recurrence is obtained from the value

$$h_2(b) = \frac{b}{1+b} \ln b - \ln(1+b). \tag{2.4.4}$$

This expression follows by a direct integration by parts in

$$h_2(b) = -\lim_{\epsilon \to 0} \int_\epsilon^b \ln t \, \frac{d}{dt}(1+t)^{-1} \, dt. \tag{2.4.5}$$

The first few values of $h_n(b)$ suggest the introduction of the function

$$q_n(b) := (1+b)^{n-1} h_n(b), \tag{2.4.6}$$

for $n \geq 2$. For example,

$$q_2(b) = b \ln b - (1+b) \ln(1+b). \tag{2.4.7}$$

The recurrence for h_n yields one for q_n.

Corollary 2.4.1. *The recurrence*

$$q_n(b) = \frac{(n-2)}{(n-1)}(1+b) q_{n-1}(b) + \frac{b \ln b}{n-1} - \frac{(1+b)\left[(1+b)^{n-2} - 1\right]}{(n-1)(n-2)}, \tag{2.4.8}$$

holds for $n \geq 3$.

Corollary 2.4.1 establishes the existence of functions $X_n(b)$, $Y_n(b)$ and $Z_n(b)$, such that

$$q_n(b) = X_n(b) \ln b + Y_n(b) \ln(1+b) + Z_n(b). \tag{2.4.9}$$

The recurrence (2.4.8) produces explicit expression for each of these parts.

Proposition 2.4.1. *Let $n \geq 2$ and $b > 0$. Then*

$$X_n(b) = \frac{(1+b)^{n-1} - 1}{n-1}. \tag{2.4.10}$$

Proof. The function X_n satisfies the recurrence

$$X_n(b) = \frac{n-2}{n-1}(1+b)X_{n-1}(b) + \frac{b}{n-1}. \tag{2.4.11}$$

The initial condition is $X_2(b) = b$. The result is now easily established by induction. □

Proposition 2.4.2. *Let $n \geq 2$ and $b > 0$. Then*

$$Y_n(b) = -\frac{(1+b)^{n-1}}{n-1}. \tag{2.4.12}$$

Proof. The function Y_n satisfies the recurrence

$$Y_n(b) = \frac{n-2}{n-1}(1+b)Y_{n-1}(b). \tag{2.4.13}$$

This recurrence and the initial condition $Y_2(b) = -(1+b)$, yield the result. □

It remains to identify the function $Z_n(b)$. It satisfies the recurrence

$$Z_n(b) = \frac{n-2}{n-1}(1+b)Z_{n-1}(b) - \frac{(1+b)\left[(1+b)^{n-2} - 1\right]}{(n-2)(n-1)}. \tag{2.4.14}$$

This recurrence and the initial condition $Z_2(b) = 0$ suggest the definition

$$T_n(b) := -\frac{(n-1)!\, Z_n(b)}{b(1+b)}. \tag{2.4.15}$$

Lemma 2.4.1. *The function $T_n(b)$ is a polynomial of degree $n-3$ with positive integer coefficients.*

Proof. The function $T_n(b)$ satisfies the recurrence

$$T_n(b) = (n-2)(1+b)T_{n-1}(b) + (n-3)!\left[\frac{(1+b)^{n-2} - 1}{b}\right]. \tag{2.4.16}$$

Now simply observe that the right hand side is a polynomial in b. □

Properties of the polynomial $T_n(b)$ will be described in future publications. We now simply observe that its coefficients are *unimodal*. Recall that a polynomial

$$P_n(b) = \sum_{k=0}^{n} c_k b^k \tag{2.4.17}$$

is called *unimodal* if there is an index n^*, such that $c_k \leq c_{k+1}$ for $0 \leq k \leq n^*$ and $c_k \geq c_{k+1}$ for $n^* < k \leq n$. That is, the sequence of coefficients of P_n has a single peak. Unimodal polynomials appear in many different branches of Mathematics. The reader will find in [27] and [80] information about this property. We now use the result of [17] to establish the unimodality of T_n.

Theorem 2.4.2. *Suppose $c_k > 0$ is a nondecreasing sequence. Then $P(x+1)$ is unimodal.*

Therefore we consider the polynomial $S_n(b) := T_n(b-1)$. It satisfies the recurrence

$$S_n(b) = b(n-2)S_{n-1}(b) + (n-3)! \sum_{r=0}^{n-3} b^r. \tag{2.4.18}$$

Now write

$$S_n(b) = \sum_{k=0}^{n-3} c_{k,n} b^k, \tag{2.4.19}$$

and conclude that $c_{0,n} = (n-3)!$ and

$$c_{k,n} = (n-2)c_{k-1,n-1} + (n-3)!, \tag{2.4.20}$$

from which it follows that

$$c_{k+1,n} - c_{k,n} = (n-2)\left[c_{k,n-1} - c_{k-1,n-1}\right]. \tag{2.4.21}$$

We conclude that $c_{k,n}$ is a nondecreasing sequence.

Theorem 2.4.3. *The polynomial $T_n(b)$ is unimodal.*

Conclusions. We have given explicit formulas for integrals of the form

$$\int_a^b R(x)\, \ln x \, dx, \tag{2.4.22}$$

where R is a rational function with real poles. Future reports will describe the case of higher powers

$$\int_a^b R(x)\, \ln^m x \, dx, \tag{2.4.23}$$

as well as the case of complex poles, based on integrals of the form

$$C_n(a,r) := \int_0^b \frac{\ln x \, dx}{(x^2+r^2)^n}. \tag{2.4.24}$$

Note. All the entries discussed in this chapter can be computed using `Mathematica`.

Chapter 3

Combinations of logarithms and exponentials

3.1 Introduction

This is the third in a series of papers dealing with the evaluation of definite integrals in the table of Gradshteyn and Ryzhik [40]. We consider here problems of the form

$$\int_0^\infty e^{-tx}\, P(\ln x)\, dx, \tag{3.1.1}$$

where $t > 0$ is a parameter and P is a polynomial. In future work we deal with the finite interval case

$$\int_a^b e^{-tx}\, P(\ln x)\, dx, \tag{3.1.2}$$

where a, $b \in \mathbb{R}^+$ with $a < b$ and $t \in \mathbb{R}$. The classical example

$$\int_0^\infty e^{-x} \ln x\, dx = -\gamma, \tag{3.1.3}$$

where γ is Euler's constant is part of this family. The integrals of type (3.1.1) are linear combinations of

$$J_n(t) := \int_0^\infty e^{-tx}\, (\ln x)^n\, dx. \tag{3.1.4}$$

The values of these integrals are expressed in terms of the gamma function

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx \tag{3.1.5}$$

and its derivatives.

3.2 The evaluation

In this section we consider the value of $J_n(t)$ defined in (3.1.4). The change of variables $s = tx$ yields

$$J_n(t) = \frac{1}{t} \int_0^\infty e^{-s} \left(\ln s - \ln t \right)^n \, ds. \tag{3.2.1}$$

Expanding the power yields J_n as a linear combination of

$$I_m := \int_0^\infty e^{-x} \left(\ln x \right)^m \, dx, \quad 0 \leq m \leq n. \tag{3.2.2}$$

An analytic expression for these integrals can be obtained directly from the representation of the *gamma function* in (3.1.5).

Proposition 3.2.1. *For $n \in \mathbb{N}$ we have*

$$\int_0^\infty \left(\ln x \right)^n \, x^{s-1} e^{-x} \, dx = \left(\frac{d}{ds} \right)^n \Gamma(s). \tag{3.2.3}$$

In particular

$$I_n := \int_0^\infty \left(\ln x \right)^n \, e^{-x} \, dx = \Gamma^{(n)}(1). \tag{3.2.4}$$

Proof. Differentiate (3.1.5) n-times with respect to the parameter s. □

Example 3.1. Formula **4.331.1** in [40] states that[1]

$$\int_0^\infty e^{-\mu x} \, \ln x \, dx = -\frac{\delta}{\mu} \tag{3.2.5}$$

where $\delta = \gamma + \ln \mu$. This value follows directly by the change of variables $s = \mu x$ and the classical special value $\Gamma'(1) = -\gamma$. The reader will find in Chapter 9 of [22] details on this constant. In particular, if $\mu = 1$, then $\delta = \gamma$ and we obtain (3.1.3):

$$\int_0^\infty e^{-x} \, \ln x \, dx = -\gamma. \tag{3.2.6}$$

The change of variables $x = e^{-t}$ yields the form

$$\int_{-\infty}^\infty t \, e^{-t} \, e^{-e^{-t}} \, dt = \gamma. \tag{3.2.7}$$

[1] The table uses C for the Euler constant.

Many of the evaluations are given in terms of the *polygamma function*

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x). \tag{3.2.8}$$

Properties of ψ are summarized in Chapter 1 of [79]. A simple representation is

$$\psi(x) = \lim_{n \to \infty} \left(\ln n - \sum_{k=0}^{n} \frac{1}{x+k} \right), \tag{3.2.9}$$

from where we conclude that

$$\psi(1) = \lim_{n \to \infty} \left(\ln n - \sum_{k=1}^{n} \frac{1}{k} \right) = -\gamma, \tag{3.2.10}$$

this being the most common definition of the Euler's constant γ. This is precisely the identity $\Gamma'(1) = -\gamma$.

The derivatives of ψ satisfy

$$\psi^{(m)}(x) = (-1)^{m+1} m! \, \zeta(m+1, x), \tag{3.2.11}$$

where

$$\zeta(z, q) := \sum_{n=0}^{\infty} \frac{1}{(n+q)^z} \tag{3.2.12}$$

is the *Hurwitz zeta function*. This function appeared in [63] in the evaluation of some logarithmic integrals.

Example 3.2. Formula **4.335.1** in [40] states that

$$\int_0^\infty e^{-\mu x} (\ln x)^2 \, dx = \frac{1}{\mu} \left[\frac{\pi^2}{6} + \delta^2 \right], \tag{3.2.13}$$

where $\delta = \gamma + \ln \mu$ as before. This can be verified using the procedure described above: the change of variable $s = \mu x$ yields

$$\int_0^\infty e^{-\mu x} (\ln x)^2 \, dx = \frac{1}{\mu} \left(I_2 - 2 I_1 \ln \mu + I_0 \ln^2 \mu \right), \tag{3.2.14}$$

where I_n is defined in (3.2.4). To complete the evaluation we need some special values: $\Gamma(1) = 1$ is elementary, $\Gamma'(1) = \psi(1) = -\gamma$ appeared above and using (3.2.11) we have

$$\psi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \left(\frac{\Gamma'(x)}{\Gamma(x)} \right)^2. \tag{3.2.15}$$

The value

$$\psi'(1) = \zeta(2) = \frac{\pi^2}{6}, \tag{3.2.16}$$

where $\zeta(z) = \zeta(z,1)$ is the Riemann zeta function, comes directly from (3.2.11). Thus

$$\Gamma''(1) = \zeta(2) + \gamma^2. \tag{3.2.17}$$

Let $\mu = 1$ in (3.2.13) to produce

$$\int_0^\infty e^{-x} (\ln x)^2 \, dx = \zeta(2) + \gamma^2. \tag{3.2.18}$$

Similar arguments yields formula **4.335.3** in [40]:

$$\int_0^\infty e^{-\mu x} (\ln x)^3 \, dx = -\frac{1}{\mu} \left[\delta^3 + \tfrac{1}{2}\pi^2 \delta - \psi''(1) \right], \tag{3.2.19}$$

where, as usual, $\delta = \gamma + \ln \mu$. The special case $\mu = 1$ now yields

$$\int_0^\infty e^{-x} (\ln x)^3 \, dx = -\gamma^3 - \tfrac{1}{2}\pi^2\gamma + \psi''(1). \tag{3.2.20}$$

Using the evaluation

$$\psi''(1) = -2\zeta(3) \tag{3.2.21}$$

produces

$$\int_0^\infty e^{-x} (\ln x)^3 \, dx = -\gamma^3 - \tfrac{1}{2}\pi^2\gamma - 2\zeta(3). \tag{3.2.22}$$

Problem 3.2.1. *In [22], page 203, we introduced the notion of* weight *for some real numbers. In particular, we have assigned $\zeta(j)$ the weight j. Differentiation increases the weight by 1, so that $\zeta'(3)$ has weight 4. The task is to check that the integral*

$$I_n := \int_0^\infty e^{-x} (\ln x)^n \, dx \tag{3.2.23}$$

is a homogeneous form of weight n.

3.3 A small variation

Similar arguments are now employed to produce a larger family of integrals. The representation

$$\int_0^\infty x^{s-1} e^{-\mu x} \, dx = \mu^{-s}\Gamma(s), \tag{3.3.1}$$

is differentiated n times with respect to the parameter s to produce

$$\int_0^\infty (\ln x)^n \, x^{s-1} e^{-\mu x} \, dx = \left(\frac{d}{ds} \right)^n \left[\mu^{-s} \Gamma(s) \right]. \tag{3.3.2}$$

The special case $n = 1$ yields

$$\begin{aligned} \int_0^\infty x^{s-1} e^{-\mu x} \, \ln x \, dx &= \frac{d}{ds} \left[\mu^{-s} \Gamma(s) \right] \\ &= \mu^{-s} \left(\Gamma'(s) - \ln \mu \, \Gamma(s) \right) \\ &= \mu^{-s} \Gamma(s) \left(\psi(s) - \ln \mu \right). \end{aligned} \tag{3.3.3}$$

This evaluation appears as **4.352.1** in [40]. The special case $\mu = 1$ yields

$$\int_0^\infty x^{s-1} e^{-x} \, \ln x \, dx = \Gamma'(s), \tag{3.3.4}$$

that is **4.352.4** in [40].

Special values of the gamma function and its derivatives yield more concrete evaluations. For example, the functional equation

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{3.3.5}$$

that is a direct consequence of $\Gamma(x+1) = x\Gamma(x)$, yields

$$\psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k}. \tag{3.3.6}$$

Replacing $s = n+1$ in (3.3.3) we obtain

$$\int_0^\infty x^n e^{-\mu x} \, \ln x \, dx = \frac{n!}{\mu^{n+1}} \left(\sum_{k=1}^n \frac{1}{k} - \gamma - \ln \mu \right), \tag{3.3.7}$$

that is **4.352.2** in [40].

The final formula of Section **4.352** in [40] is **4.352.3**

$$\int_0^\infty x^{n-1/2} e^{-\mu x} \, \ln x \, dx = \frac{\sqrt{\pi} \, (2n-1)!!}{2^n \, \mu^{n+1/2}} \left[2 \sum_{k=1}^n \frac{1}{2k-1} - \gamma - \ln(4\mu) \right].$$

This can also be obtained from (3.3.3) by using the classical values

$$\begin{aligned} \Gamma(n + \tfrac{1}{2}) &= \frac{\sqrt{\pi}}{2^n} (2n-1)!! \\ \psi(n + \tfrac{1}{2}) &= -\gamma + 2 \left(\sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right). \end{aligned}$$

The details are left to the reader.

Section **4.353** of [40] contains three peculiar combinations of integrands. The first two of them can be verified by the methods described above: formula **4.353.1** states

$$\int_0^\infty (x-\nu)x^{\nu-1}e^{-x}\,\ln x\,dx = \Gamma(\nu), \tag{3.3.8}$$

and **4.353.2** is

$$\int_0^\infty (\mu x - n - \tfrac{1}{2})x^{n-\frac{1}{2}}e^{-\mu x}\,\ln x\,dx = \frac{(2n-1)!!}{(2\mu)^n}\sqrt{\frac{\pi}{\mu}}. \tag{3.3.9}$$

Note. All the entries discussed in this chapter can be computed using `Mathematica`.

Chapter 4

The gamma function

4.1 Introduction

The table of integrals [40] contains some evaluations that can be derived by elementary means from the *gamma function*, defined by

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\, dx. \tag{4.1.1}$$

The convergence of the integral in (4.1.1) requires $a > 0$. The goal of this paper is to present some of these evaluations in a systematic manner. The techniques developed here will be employed in future publications. The reader will find in [22] analytic information about this important function.

The gamma function represents the extension of factorials to real parameters. The value

$$\Gamma(n) = (n-1)!, \text{ for } n \in \mathbb{N} \tag{4.1.2}$$

is elementary. On the other hand, the special value

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \tag{4.1.3}$$

is equivalent to the well-known *normal integral*

$$\int_0^\infty \exp(-t^2)\, dt = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right). \tag{4.1.4}$$

The reader will find in [22] proofs of Legendre's duplication formula

$$\Gamma\left(x + \tfrac{1}{2}\right) = \frac{\Gamma(2x)\sqrt{\pi}}{\Gamma(x)\, 2^{2x-1}}, \tag{4.1.5}$$

that produces for $x = m \in \mathbb{N}$ the values

$$\Gamma\left(m+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m}}\,\frac{(2m)!}{m!}. \tag{4.1.6}$$

This appears as **3.371** in [40] in the equivalent form

$$\int_0^\infty x^{n-\frac{1}{2}} e^{-\mu x}\,dx = \frac{\sqrt{\pi}\,(2n-1)!!}{2^n \mu^{n+\frac{1}{2}}}. \tag{4.1.7}$$

4.2 The introduction of a parameter

The presence of a parameter in a definite integral provides great amount of flexibility. The change of variables $x = \mu t$ in (4.1.1) yields

$$\Gamma(a) = \mu^a \int_0^\infty t^{a-1} e^{-\mu t}\,dt. \tag{4.2.1}$$

This appears as **3.381.4** in [40] and the choice $a = n+1$, with $n \in \mathbb{N}$, that reads

$$\int_0^\infty t^n e^{-\mu t}\,dt = n!\,\mu^{-n-1} \tag{4.2.2}$$

appears as **3.351.3**.

The special case $a = m + \frac{1}{2}$, that appears as **3.371** in [40], yields

$$\int_0^\infty t^{m-\frac{1}{2}} e^{-\mu t}\,dt = \frac{\sqrt{\pi}}{2^{2m}}\,\frac{(2m)!}{m!}\mu^{-m-\frac{1}{2}}, \tag{4.2.3}$$

is consistent with (4.1.6).

The combination

$$\int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{x^{\rho+1}}\,dx = \frac{\mu^\rho - \nu^\rho}{\rho}\,\Gamma(1-\rho), \tag{4.2.4}$$

that appears as **3.434.1** in [40] can now be evaluated directly. The parameters are restricted by convergence: μ, $\nu > 0$ and $\rho < 1$. The integral **3.434.2**

$$\int_0^\infty \frac{e^{-\mu x} - e^{-\nu x}}{x}\,dx = \ln\frac{\nu}{\mu}, \tag{4.2.5}$$

is obtained from (4.2.4) by passing to the limit as $\rho \to 0$. This is an example of *Frullani integrals* that will be discussed in Chapter 15.

The reader will be able to check **3.478.1**:

$$\int_0^\infty x^{\nu-1}\exp(-\mu x^p)\,dx = \frac{1}{p}\mu^{-\nu/p}\Gamma\left(\frac{\nu}{p}\right), \tag{4.2.6}$$

and **3.478.2**:

$$\int_0^\infty x^{\nu-1}\left[1-\exp(-\mu x^p)\right]\,dx = -\frac{1}{|p|}\mu^{-\nu/p}\Gamma\left(\frac{\nu}{p}\right) \tag{4.2.7}$$

by introducing appropriate parameter reduction.

The parameters can be used to prove many of the classical identities for $\Gamma(a)$.

Proposition 4.2.1. *The gamma function satisfies*

$$\Gamma(a+1) = a\,\Gamma(a). \tag{4.2.8}$$

Proof. Differentiate (4.2.1) with respect to μ to produce

$$0 = a\mu^{a-1}\int_0^\infty t^{a-1}e^{-\mu t}\,dt - \mu^a\int_0^\infty t^a e^{-\mu t}\,dt. \tag{4.2.9}$$

Now put $\mu = 1$ to obtain the result. □

Differentiating (4.1.1) with respect to the parameter a yields

$$\Gamma'(a) = \int_0^\infty x^{a-1}e^{-x}\,\ln x\,dx. \tag{4.2.10}$$

Further differentiation introduces higher powers of $\ln x$:

$$\Gamma^{(n)}(a) = \int_0^\infty x^{a-1}e^{-x}\,(\ln x)^n\,dx. \tag{4.2.11}$$

In particular, for $a = 1$, we obtain:

$$\int_0^\infty (\ln x)^n\,e^{-x}\,dx = \Gamma^{(n)}(1). \tag{4.2.12}$$

The special case $n = 1$ yields

$$\int_0^\infty e^{-x}\,\ln x\,dx = \Gamma'(1). \tag{4.2.13}$$

The reader will find in [22], page 176, an elementary proof that $\Gamma'(1) = -\gamma$, where

$$\gamma := \lim_{n\to\infty}\sum_{k=1}^n \frac{1}{k} - \ln n \tag{4.2.14}$$

is Euler's constant. This is one of the fundamental numbers of Analysis.

On the other hand, differentiating (4.2.1) produces

$$\int_0^\infty x^{a-1}e^{-\mu x}\left(\ln x\right)^n\,dx = \left(\frac{\partial}{\partial a}\right)^n\left[\mu^{-a}\Gamma(a)\right], \tag{4.2.15}$$

that appears as **4.358.5** in [40]. Using Leibnitz's differentiation formula we obtain

$$\int_0^\infty x^{a-1}e^{-\mu x}\left(\ln x\right)^n\,dx = \mu^{-a}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\ln\mu\right)^k\Gamma^{(n-k)}(a). \tag{4.2.16}$$

In the special case $a=1$ we obtain

$$\int_0^\infty e^{-\mu x}\left(\ln x\right)^n\,dx = \frac{1}{\mu}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\ln\mu\right)^k\Gamma^{(n-k)}(1). \tag{4.2.17}$$

The cases $n=1,\,2,\,3$ appear as **4.331.1**, **4.335.1** and **4.335.3** respectively.

In order to obtain analytic expressions for the terms $\Gamma^{(n)}(1)$, it is convenient to introduce the *polygamma function*

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x). \tag{4.2.18}$$

The derivatives of ψ satisfy

$$\psi^{(n)}(x) = (-1)^{n+1}n!\,\zeta(n+1,x), \tag{4.2.19}$$

where

$$\zeta(z,q) = \sum_{n=0}^{\infty}\frac{1}{(n+q)^z} \tag{4.2.20}$$

is the *Hurwitz zeta function*. In particular this gives

$$\psi^{(n)}(1) = (-1)^{n+1}\,n!\,\zeta(n+1). \tag{4.2.21}$$

The values of $\Gamma^{(n)}(1)$ can now be computed by recurrence via

$$\Gamma^{(n+1)}(1) = \sum_{k=0}^{n}\binom{n}{k}\Gamma^{(k)}(1)\psi^{(n-k)}(1), \tag{4.2.22}$$

obtained by differentiating $\Gamma'(x)=\psi(x)\Gamma(x)$.

Using (4.2.19) the reader will be able to check the first few cases of (4.2.15), with the notation $\delta=\psi(a)-\ln\mu$:

$$\begin{aligned}
\int_0^\infty x^{a-1}e^{-\mu x}\ln^2 x\,dx &= \frac{\Gamma(a)}{\mu^a}\left\{\delta^2+\zeta(2,a)\right\},\\
\int_0^\infty x^{a-1}e^{-\mu x}\ln^3 x\,dx &= \frac{\Gamma(a)}{\mu^a}\left\{\delta^3+3\zeta(2,a)\delta-2\zeta(3,a)\right\},\\
\int_0^\infty x^{a-1}e^{-\mu x}\ln^4 x\,dx &= \frac{\Gamma(a)}{\mu^a}\left\{\delta^4+6\zeta(2,a)\delta^2-8\zeta(3,a)\delta+\right.\\
&\quad\left.+3\zeta^2(2,a)+6\zeta(4,a)\right\}.
\end{aligned}$$

These appear as **4.358.2**, **4.358.3** and **4.358.4**, respectively. `Mathematica` evaluates these entries in terms of polylogarithms.

4.3 Elementary changes of variables

The use of appropriate changes of variables yields, from the basic definition (4.1.1), the evaluation of more complicated definite integrals. For example, let $x = t^b$ to obtain, with $c = ab - 1$,

$$\int_0^\infty t^c \exp(-t^b)\, dt = \frac{1}{b}\Gamma\left(\frac{c+1}{b}\right). \tag{4.3.1}$$

The special case $a = 1/b$, that is $c = 0$, is

$$\int_0^\infty \exp(-t^b)\, dt = \frac{1}{b}\Gamma\left(\frac{1}{b}\right), \tag{4.3.2}$$

that appears as **3.326.1** in [40]. The special case $b = 2$ is the normal integral (4.1.4).

We can now introduce an extra parameter via $t = s^{1/b}x$. This produces

$$\int_0^\infty x^m \exp(-sx^b)\, dx = \frac{\Gamma(a)}{s^a b}, \tag{4.3.3}$$

with $m = ab - 1$. This formula appears (at least) three times in [40]: **3.326.2**, **3.462.9** and **3.478.1**. Moreover, the case $s = 1$, $c = (m + 1/2)n - 1$ and $b = n$ appears as **3.473**:

$$\int_0^\infty \exp(-x^n) x^{\left(m+\frac{1}{2}\right)n-1}\, dx = \frac{(2m-1)!!}{2^m\, n}\sqrt{\pi}. \tag{4.3.4}$$

The form given here can be established using (4.1.6).

Differentiating (4.3.3) with respect to the parameter m (keeping in mind that $a = (m+1)/b$), yields

$$\int_0^\infty x^m e^{-sx^b} \ln x\, dx = \frac{\Gamma(a)}{b^2\, s^a}\left[\psi(a) - \ln s\right]. \tag{4.3.5}$$

In particular, if $b = 1$ we obtain

$$\int_0^\infty x^m e^{-sx} \ln x\, dx = \frac{\Gamma(m+1)}{s^{m+1}}\left[\psi(m+1) - \ln s\right]. \tag{4.3.6}$$

The case $m = 0$ and $b = 2$ gives

$$\int_0^\infty e^{-sx^2} \ln x\, dx = -\frac{\sqrt{\pi}}{4\sqrt{s}}\left(\gamma + \ln 4s\right), \tag{4.3.7}$$

where we have used $\psi(1/2) = -\gamma - 2\ln 2$. This appears as **4.333** in [40].

An interesting example is $b = m = 2$. Using the values

$$\Gamma\left(\tfrac{3}{2}\right) = \sqrt{\pi}/2 \text{ and } \psi\left(\tfrac{3}{2}\right) = 2 - 2\ln 2 - \gamma \tag{4.3.8}$$

the expression (4.3.5) yields

$$\int_0^\infty x^2 e^{-sx^2} \ln x \, dx = \frac{1}{8s}(2 - \ln 4s - \gamma)\sqrt{\frac{\pi}{s}}. \tag{4.3.9}$$

The values of ψ at half-integers follow directly from (4.1.5). Formula (4.3.9) appears as **4.355.1** in [40]. Using (4.3.5) it is easy to verify

$$\int_0^\infty (\mu x^2 - n)x^{2n-1}e^{-\mu x^2} \ln x \, dx = \frac{(n-1)!}{4\mu^n}, \tag{4.3.10}$$

and

$$\int_0^\infty (2\mu x^2 - 2n - 1)x^{2n}e^{-\mu x^2} \ln x \, dx = \frac{(2n-1)!!}{2(2\mu)^n}\sqrt{\frac{\pi}{\mu}}, \tag{4.3.11}$$

for $n \in \mathbb{N}$. These appear as, respectively, **4.355.3** and **4.355.4** in [40]. The term $(2n-1)!!$ is the semi-factorial defined by

$$(2n-1)!! = (2n-1)(2n-3)\cdots 5 \cdot 3 \cdot 1. \tag{4.3.12}$$

Finally, formula **4.369.1** in [40]

$$\int_0^\infty x^{a-1}e^{-\mu x}\left[\psi(a) - \ln x\right] \, dx = \frac{\Gamma(a)\ln\mu}{\mu^a} \tag{4.3.13}$$

can be established by the methods developed here. The more ambitious reader will check that

$$\int_0^\infty x^{n-1}e^{-\mu x}\left\{\left[\ln x - \tfrac{1}{2}\psi(n)\right]^2 - \tfrac{1}{2}\psi'(n)\right\} \, dx = \frac{(n-1)!}{\mu^n}\left\{\left[\ln\mu - \tfrac{1}{2}\psi(n)\right]^2 + \tfrac{1}{2}\psi'(n)\right\}.$$

This is **4.369.2** in [40].

We can also write (4.3.5) in the exponential scale to obtain

$$\int_{-\infty}^\infty te^{mt}\exp\left(-se^{bt}\right) \, dt = \frac{\Gamma(m/b)}{b^2 s^{m/b}}\left(\psi\left(\frac{m}{b}\right) - \ln s\right). \tag{4.3.14}$$

The special case $b = m = 1$ produces

$$\int_{-\infty}^\infty te^t \exp\left(-se^t\right) \, dt = -\frac{(\gamma + \ln s)}{s} \tag{4.3.15}$$

that appears as **3.481.1**. The second special case, appearing as **3.481.2**, is $b = 2$, $m = 1$, that yields

$$\int_{-\infty}^{\infty} te^t \exp\left(-se^{2t}\right)\, dt = -\frac{\sqrt{\pi}\,(\gamma + \ln 4s)}{4\sqrt{s}}. \tag{4.3.16}$$

This uses the value $\psi(1/2) = -(\gamma + 2\ln 2)$.

There are many other possible changes of variables that lead to interesting evaluations. We conclude this section with one more: let $x = e^t$ to convert (4.1.1) into

$$\int_{-\infty}^{\infty} \exp\left(-e^x\right)\, e^{ax}\, dx = \Gamma(a). \tag{4.3.17}$$

This is **3.328** in [40].

As usual one should not prejudge the difficulty of a problem: the example **3.471.3** states that

$$\int_0^a x^{-\mu-1}(a-x)^{\mu-1}e^{-\beta/x}\, dx = \beta^{-\mu}a^{\mu-1}\Gamma(\mu)\exp\left(-\tfrac{\beta}{a}\right). \tag{4.3.18}$$

This can be reduced to the basic formula for the gamma function. Indeed, the change of variables $t = \beta/x$ produces

$$I = \beta^{-\mu}a^{\mu-1}\int_{\beta/a}^{\infty} (t - \beta/a)^{\mu-1}\, e^{-t}\, dt. \tag{4.3.19}$$

Now let $y = t - \beta/a$ to complete the evaluation. The table [40] writes u instead of a: it seems to be a bad idea to have μ and u in the same formula, it leads to typographical errors that should be avoided.

Another simple change of variables gives the evaluation of **3.324.2**:

$$\int_{-\infty}^{\infty} e^{-(x-b/x)^{2n}}\, dx = \frac{1}{n}\Gamma\left(\frac{1}{2n}\right). \tag{4.3.20}$$

The symmetry yields

$$I = 2\int_0^{\infty} e^{-(x-b/x)^{2n}}\, dx. \tag{4.3.21}$$

The change of variables $t = b/x$ yields, using $b > 0$,

$$I = 2b\int_0^{\infty} e^{-(t-b/t)^{2n}}\, \frac{dt}{t^2}. \tag{4.3.22}$$

The average of these forms produces

$$I = \int_0^{\infty} e^{-(x-b/x)^{2n}}\left(1 + \frac{b}{x^2}\right) dx. \tag{4.3.23}$$

Finally, the change of variables $u = x - b/x$ gives the result. Indeed, let $u = x - b/x$ and observe that u is increasing when $b > 0$. This restriction is missing in the table. Then we get

$$I = 2\int_0^\infty e^{-u^{2n}}\, du. \tag{4.3.24}$$

This can now be evaluated via $v = u^{2n}$.

Note. In the case $b < 0$ the change of variables $u = x - b/x$ has an inverse with two branches, splitting at $x = \sqrt{-b}$. Then we write

$$\begin{aligned} I &:= 2\int_0^\infty e^{-(x-b/x)^{2n}}\, dx \qquad (4.3.25)\\ &= 2\int_0^{\sqrt{-b}} e^{-(x-b/x)^{2n}}\, dx + 2\int_{\sqrt{-b}}^\infty e^{-(x-b/x)^{2n}}\, dx. \end{aligned}$$

The change of variables $u = x - b/x$ is now used in each of the integrals to produce

$$I = 2\int_{2\sqrt{-b}}^\infty \frac{u\exp(-u^{2n})\, du}{\sqrt{u^2+4b}}. \tag{4.3.26}$$

The change of variables $z = \sqrt{u^2+4b}$ yields

$$I = 2\int_0^\infty \exp\left(-(z^2-4b)^n\right). \tag{4.3.27}$$

We are unable to simplify it any further.

4.4 The logarithmic scale

Euler prefered the version

$$\Gamma(a) = \int_0^1 \left(\ln\frac{1}{u}\right)^{a-1}\, du. \tag{4.4.1}$$

We will write this as

$$\Gamma(a) = \int_0^1 (-\ln u)^{a-1}\, du, \tag{4.4.2}$$

for better spacing. Many of the evaluations in [40] are special cases of this formula. Section **4.215** in [40] consists of four examples: the first one, **4.215.1** is (4.4.1) itself. The second one, labeled **4.215.2** and written as

$$\int_0^1 \frac{dx}{(-\ln x)^\mu} = \frac{\pi}{\Gamma(\mu)}\text{cosec}\ \mu\pi, \tag{4.4.3}$$

is evaluated as $\Gamma(1-\mu)$ by (4.4.1). The identity

$$\Gamma(\mu)\Gamma(1-\mu) = \frac{\pi}{\sin \pi\mu} \tag{4.4.4}$$

yields the given form. The reader will find in [22] a proof of this identity. The section concludes with the special values

$$\int_0^1 \sqrt{-\ln x}\, dx = \frac{\sqrt{\pi}}{2}, \tag{4.4.5}$$

as **4.215.3** and **4.215.4**:

$$\int_0^1 \frac{dx}{\sqrt{-\ln x}} = \sqrt{\pi}. \tag{4.4.6}$$

Both of them are special cases of (4.4.1).

The reader should check the evaluations **4.269.3**:

$$\int_0^1 x^{p-1} \sqrt{-\ln x}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{p^3}}, \tag{4.4.7}$$

and **4.269.4**:

$$\int_0^1 \frac{x^{p-1}\, dx}{\sqrt{-\ln x}} = \sqrt{\frac{\pi}{p}} \tag{4.4.8}$$

by reducing them to (4.2.1). Also **4.272.5**, **4.272.6** and **4.272.7**

$$\begin{aligned}
\int_1^\infty (\ln x)^p \frac{dx}{x^2} &= \Gamma(1+p), \qquad (4.4.9)\\
\int_0^1 (-\ln x)^{\mu-1}\, x^{\nu-1}\, dx &= \frac{1}{\nu^\mu}\Gamma(\mu),\\
\int_0^1 (-\ln x)^{n-\frac{1}{2}}\, x^{\nu-1}\, dx &= \frac{(2n-1)!!}{(2\nu)^n}\sqrt{\frac{\pi}{\nu}},
\end{aligned}$$

can be evaluated directly in terms of the gamma function.

Differentiating (4.4.1) with respect to a yields **4.229.4** in [40]:

$$\int_0^1 \ln(-\ln x)\, (-\ln x)^{a-1}\, dx = \Gamma'(a) = \psi(a)\Gamma(a), \tag{4.4.10}$$

with $\psi(a)$ defined in (4.2.18). The special case $a=1$ is **4.229.1**:

$$\int_0^1 \ln(-\ln x)\, dx = -\gamma, \tag{4.4.11}$$

and

$$\int_0^1 \ln(-\ln x)\, \frac{dx}{\sqrt{-\ln x}} = -(\gamma + 2\ln 2)\sqrt{\pi}, \tag{4.4.12}$$

that appears as **4.229.3**, is obtained by using the values $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\psi\left(\frac{1}{2}\right) = -(\gamma + 2\ln 2)$.

The same type of arguments confirms **4.325.11**

$$\int_0^1 \ln(-\ln x) \frac{x^{\mu-1}\,dx}{\sqrt{-\ln x}} = -(\gamma + \ln 4\mu)\sqrt{\frac{\pi}{\mu}}, \tag{4.4.13}$$

and **4.325.12**:

$$\int_0^1 \ln(-\ln x)\;(-\ln x)^{\mu-1}\;x^{\nu-1}\,dx = \frac{1}{\nu^\mu}\Gamma(\mu)\left[\psi(\mu) - \ln\nu\right]. \tag{4.4.14}$$

In particular, when $\mu = 1$ we obtain **4.325.8**:

$$\int_0^1 \ln(-\ln x)\,x^{\nu-1}\,dx = -\frac{1}{\nu}\left(\gamma + \ln\nu\right). \tag{4.4.15}$$

4.5 The presence of fake parameters

There are many formulas in [40] that contain parameters. For example, **3.461.2** states that

$$\int_0^\infty x^{2n}e^{-px^2}\,dx = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}} \tag{4.5.1}$$

and **3.461.3** states that

$$\int_0^\infty x^{2n+1}e^{-px^2}\,dx = \frac{n!}{2p^{n+1}}. \tag{4.5.2}$$

The change of variables $t = px^2$ eliminates the *fake* parameter p and reduces **3.461.2** to

$$\int_0^\infty t^{n-\frac{1}{2}}e^{-t}\,dt = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \tag{4.5.3}$$

and **3.461.3** to

$$\int_0^\infty t^n e^{-t}\,dt = n!. \tag{4.5.4}$$

These are now evaluated by identifying them with $\Gamma(n+\frac{1}{2})$ and $\Gamma(n+1)$, respectively.

A second way to introduce fake parameters is to shift the integral (4.2.1) via $s = t + b$ to produce

$$\int_b^\infty (s-b)^a e^{-s\mu}\,ds = \mu^{-a-1}e^{-\mu b}\Gamma(a+1). \tag{4.5.5}$$

This appears as **3.382.2** in [40].

Note. The entries **3.478.2** (in (4.2.7)), **4.358.5** (in (4.2.15)), **3.324.2** (in (4.3.20)), **4.229.4** (in (4.4.10)), and the last four entries in Section 4.4 (these are **4.229.3**, **4.325.11**, **4.325.12**, **4.325.8**) cannot be evaluated using `Mathematica`.

Chapter 5

Some trigonometric integrals

5.1 Introduction

The table of integrals [40] contains a large variety of evaluations of the type

$$I = \int_a^b A(x) R(\sin x, \cos x)\, dx \tag{5.1.1}$$

where A is an algebraic function, R is rational and $-\infty \leq a < b \leq \infty$. We present a systematic discussion of two families of integrals of this type. This paper is part of a general program started in [63, 64, 65, 66] intended to provide proofs and context to the formulas in [40].

The first class considered here corresponds to the complete integrals

$$c(n,p) := \int_0^{\pi/2} x^p \cos^n x\, dx, \tag{5.1.2}$$

and

$$s(n,p) := \int_0^{\pi/2} x^p \sin^n x\, dx, \tag{5.1.3}$$

where n, $p \in \mathbb{N}$. In section 5.2 we present closed-form expressions for these integrals. These expressions involve the sums

$$\sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_j \leq n} \frac{1}{k_1^2 k_2^2 \cdots k_j^2}, \tag{5.1.4}$$

that are closely related to the multiple zeta values

$$\zeta(i_1, i_2, \ldots, i_s) = \sum_{0 < k_1 < k_2 < \cdots < k_s} \frac{1}{k_1^{i_1} k_2^{i_2} \cdots k_s^{i_s}}. \tag{5.1.5}$$

The reader will find in Section 3.4 of [24] an introduction to these sums.

In general, one does not expect such elementary evaluations to extend to $p \notin \mathbb{N}$. For example, the change of variables $x = \pi t^2/2$ produces

$$\int_0^{\pi/2} x^{-1/2} \cos x \, dx = \sqrt{2\pi} \int_0^1 \cos\left(\frac{\pi t^2}{2}\right) dt. \tag{5.1.6}$$

The latter integral is evaluated in terms of the *cosine Fresnel* function

$$\text{FresnelC}[x] := \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt, \tag{5.1.7}$$

which indeed is not an elementary function.

The second class considered here presents generalizations of the formula **3.822.2** in [40] stated as

$$\int_0^\infty \frac{\cos^{2n+1} x}{\sqrt{x}} \, dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{n} \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}, \quad n \in \mathbb{N}. \tag{5.1.8}$$

The integral in (5.1.8) can be transformed via $t = x^2$ to provide the evaluation of

$$\int_0^\infty \cos^{2n+1} t^2 \, dt, \tag{5.1.9}$$

that is given as the case $p = 2$ in Theorem 5.3.1.

Section 5.3 contains analytic expressions for the generalizations

$$C_n(p, b) := \int_0^\infty x^{-p} \cos^{2n+1}(x + b) \, dx, \tag{5.1.10}$$

and

$$S_n(p, b) := \int_0^\infty x^{-p} \sin^{2n+1}(x + b) \, dx. \tag{5.1.11}$$

The last section also contains some evaluations obtained by differentiation with respect to parameters. An illustrative example is

$$\int_0^\infty \int_0^\infty \frac{\log x \, \log y}{\sqrt{xy}} \cos(x + y) \, dx \, dy = (\gamma + 2 \log 2)\pi^2, \tag{5.1.12}$$

that is equivalent to

$$\int_0^\infty \int_0^\infty \log x \, \log y \, \cos(x^2 + y^2) \, dx \, dy = \frac{1}{16}(\gamma + 2 \log 2)\pi^2. \tag{5.1.13}$$

A generalization of this evaluation appears as Example 5.3.

The method described in the present work gives impetus to a class of integrals that are closely related to the particular integral computations addressed in this paper.

5.2 The first example

In this section we present the evaluation in closed-form of the definite integrals

$$c(n,p) := \int_0^{\pi/2} x^p \cos^n x \, dx. \tag{5.2.1}$$

A reduction formula for this integral appears as entry **3.822.1** in [40]. This is established next.

Theorem 5.2.1. *The integral $c(n,p)$ satisfies the recurrence*

$$c(n,p) \quad = \quad \frac{n-1}{n} c(n-2,p) - \frac{p(p-1)}{n^2} c(n,p-2), \tag{5.2.2}$$

for $n \geq 2$, $p \geq 2$.

Proof. The identity $\cos^2 x = 1 - \sin^2 x$ yields

$$c(n,p) \quad = \quad c(n-2,p) - I(n,p) \tag{5.2.3}$$

where

$$I(n,p) \quad := \quad \int_0^{\pi/2} x^p \cos^{n-2} x \, \sin^2 x \, dx.$$

Now

$$\begin{aligned} I(n,p) &= \int_0^{\pi/2} x^p \sin x \times \frac{d}{dx} \left(-\frac{1}{n-1} \cos^{n-1} x \right) dx \\ &= \frac{1}{2n-1} \int_0^{\pi/2} \left(x^p \cos x + p x^{p-1} \sin x \right) \cos^{n-1} x \, dx \\ &= \frac{c(n,p)}{n-1} + \frac{p}{n-1} \int_0^{\pi/2} x^{p-1} \sin x \cos^{n-1} x \; dx. \end{aligned}$$

Moreover

$$\begin{aligned} \int_0^{\pi/2} x^{p-1} \sin x \cos^{n-1} x \; dx &= \int_0^{\pi/2} x^{p-1} \frac{d}{dx} \left(-\frac{1}{n} \cos^n x \right) dx \\ &= \frac{p-1}{n} c(n,p-2). \end{aligned}$$

$\square$

Strategy: According to (5.2.2), the integral $c(n,p)$ can be evaluated in terms of the initial values given in the table. The indices m and q have the same parity as n and p respectively and range over $0 \leq m \leq n$ and $0 \leq q \leq p$.

n modulo 2	p modulo 2	initial conditions
0	0	$c(m,0)\quad c(0,q)$
1	0	$c(m,0)\quad c(1,q)$
0	1	$c(m,1)\quad c(0,q)$
1	1	$c(m,1)\quad c(1,q)$

We now evaluate the initial conditions $c(n,0)$, $c(n,1)$, $c(0,p)$ and $c(1,p)$.

The expression for $c(0,p)$.
The computation of the identity

$$c(0,p) = \frac{1}{p+1}\left(\frac{\pi}{2}\right)^{p+1} \tag{5.2.4}$$

is immediate.

The expression for $c(n,0)$.
This is classical. The result appears as **3.621.3** and **3.621.4** in [40].

Theorem 5.2.2. (Wallis' formula and companion). *Let* $n \in \mathbb{N}_0$. *Then*

$$c(2n,0) = \frac{\pi}{2^{2n+1}}\binom{2n}{n}, \tag{5.2.5}$$

and

$$c(2n+1,0) = \frac{2^{2n}}{(2n+1)\binom{2n}{n}}. \tag{5.2.6}$$

The shortest proof of Theorem 5.2.2 employs the representation

$$c(n,0) = \int_0^{\pi/2} \cos^n x\, dx = 2^{n-1} B\left(\frac{n+1}{2}, \frac{n+1}{2}\right), \tag{5.2.7}$$

that appears as **3.621.1** in [40]. Here B is the *Euler's beta function* defined by the integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt. \tag{5.2.8}$$

The expression (5.2.7) follows from the change of variables $t = \cos u$. To express (5.2.5) and (5.2.6), in terms of the beta function, employ the standard relation

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \tag{5.2.9}$$

and the special values

$$\Gamma(n) = (n-1)! \text{ and } \Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,(2n)!}{2^{2n}\, n!} \tag{5.2.10}$$

that are valid for $n \in \mathbb{N}$.

The identity in Theorem 5.2.2, in the case n is even, that is,

$$c(2n,0) = \int_0^{\pi/2} \cos^{2n}\theta\, d\theta = \frac{\pi}{2^{2n+1}}\binom{2n}{n}, \tag{5.2.11}$$

is Wallis's formula and sometimes found in calculus books (see e.g. [53], page 492). To prove it, first write $\cos^2\theta = 1 - \sin^2\theta$ and use integration by parts to obtain the recursion

$$c(2n,0) \quad = \quad \frac{2n-1}{2n} c(2n-2,0). \tag{5.2.12}$$

Then verify that the right side of (5.2.11) satisfies the same recurrence together with the initial value $\pi/2$ for $n = 0$.

We now present a new proof of Wallis's formula (5.2.11) in the context of rational integrals. Extensions of the ideas in this proof have produced *rational Landen transformations*. The reader will find in [19, 21, 49, 57, 58] details on these transformations.

Start with

$$c(2n,0) \quad = \quad \int_0^{\pi/2} \cos^{2n}\theta\, d\theta = \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^n d\theta.$$

Now introduce $\psi = 2\theta$ and expand and simplify the result by observing that the odd powers of cosine integrate to zero. The inductive proof of (5.2.11) requires

$$c(2n,0) \quad = \quad 2^{-n}\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} c(2i,0). \tag{5.2.13}$$

Note that $c(2n,0)$ is uniquely determined by (5.2.13) along with the initial value $c(0,0) = \pi/2$. Thus (5.2.11) now follows from the identity

$$f(n) := \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{-2i}\binom{n}{2i}\binom{2i}{i} \quad = \quad 2^{-n}\binom{2n}{n}. \tag{5.2.14}$$

We now provide a mechanical proof of (5.2.14) using the theory developed by Wilf and Zeilberger, which is explained in [69, 72]; the sum in (5.2.14) is the example used in [72] (page 113) to illustrate their method. The command

$$ct(\text{binomial}(n,2i)\,\text{binomial}(2i,i)2^{-2i}, 1, i, n, N)$$

produces

$$f(n+1) \quad = \quad \frac{2n+1}{n+1}\, f(n), \tag{5.2.15}$$

and one checks that $2^{-n}\binom{2n}{n}$ satisfies this recursion. Note that (5.2.12) and (5.2.15) are equivalent under

$$c(2n,0) \quad = \quad \frac{\pi}{2^{n+1}} f(n).$$

The proof is complete.

Closed form expression for $c(1,p)$.

We now consider the evaluation of

$$c(1,p) := \int_0^{\pi/2} x^p \cos x \, dx. \tag{5.2.16}$$

The following evaluation appears as **3.761.11** in [40].

Theorem 5.2.3. *Let $p \in \mathbb{N}$ and $\delta_{odd,p}$ be Kronecker's delta function at the odd integers. Then*

$$c(1,p) = \sum_{k=0}^{\xi_p} (-1)^k \frac{p!}{(p-2k)!} \left(\frac{\pi}{2} \right)^{p-2k} - (-1)^{\xi_p} \delta_{odd,p} \, p! \tag{5.2.17}$$

where $\xi_p = \lfloor \frac{p}{2} \rfloor$.

Proof. Both sides of the equation (5.2.17) satisfy the initial value problem

$$u_p - p(p-1)u_{p-2} = \left(\frac{\pi}{2} \right)^p \text{ and } u_0 = 1, \, u_1 = \frac{\pi - 2}{2}. \tag{5.2.18}$$

Actually the recurrence (5.2.18) is obtained using integration by parts in (5.2.16). Iterating this recurrence yields the right hand side of (5.2.17). $\square$

Note 5.2.1. *The result in Theorem 5.2.3 can be expressed in terms of the Taylor polynomial for* $\cos x$*:*

$$f_p(x) = (-1)^{\xi_p} p! \left(-1 + \sum_{k=0}^{\xi_p} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right). \tag{5.2.19}$$

The formula (5.2.17) can be restated

$$c(1,p) = \begin{cases} f_p(\pi/2), & \textit{for } p \textit{ odd}, \\ f_p'(\pi/2), & \textit{for } p \textit{ even}. \end{cases} \tag{5.2.20}$$

Note. `Mathematica` evaluates the integral $c(1,p)$ in terms of a hypergeometric function:

$$\int_0^{\pi/2} x^p \cos x \, dx = \frac{1}{p+1} \left(\frac{\pi}{2} \right)^{p+1} {}_1F_2 \left(\left. \begin{matrix} \frac{p+1}{2} \\ \frac{1}{2}, \frac{p+3}{2} \end{matrix} \right| - \frac{\pi^2}{16} \right). \tag{5.2.21}$$

Closed form expression for $c(n,1)$: in fact, this would be the last initial condition we require to execute the startegy outlined at the beginning of this section.

Theorem 5.2.4. *The integral $c(n,1)$ satisfies the recurrence*

$$c(n,1) \quad = \quad \frac{n-1}{n} c(n-2,1) - \frac{1}{n^2}. \tag{5.2.22}$$

Proof. The identity $\cos^2 x = 1 - \sin^2 x$ yields

$$c(n,1) = c(n-2,1) - J, \tag{5.2.23}$$

where

$$J = \int_0^{\pi/2} x \sin^2 x \, \cos^{n-2} x \, dx. \tag{5.2.24}$$

Integration by parts leads to

$$J = \frac{1}{n-1} \int_0^{\pi/2} (\sin x + x \cos x) \cos^{n-1} x \, dx. \tag{5.2.25}$$

This produces (5.2.22). □

The solution of (5.2.22) yields a closed-form formula for $c(n,1)$.

Theorem 5.2.5. *The integral $c(n,1)$ is given according to the parity of n, by*

$$c(2n,1) = \frac{\binom{2n}{n}}{2^{2n+2}} \left(\frac{\pi^2}{2} - \sum_{k=1}^{n} \frac{2^{2k}}{k^2 \binom{2k}{k}} \right), \tag{5.2.26}$$

for even indices. For odd indices, we have

$$c(2n+1,1) = \frac{2^{2n}}{(2n+1)\binom{2n}{n}} \left(\frac{\pi}{2} - \sum_{k=0}^{n} \frac{\binom{2k}{k}}{2^{2k}(2k+1)} \right). \tag{5.2.27}$$

To establish this result we solve a more general recurrence than (5.2.22).

Lemma 5.2.1. *Let a_n, b_n and r_n be sequences with $a_n, b_n \neq 0$. Assume that z_n satisfies*

$$a_n z_n = b_n z_{n-1} + r_n, \; n \geq 1 \tag{5.2.28}$$

with initial condition z_0. Then

$$z_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \left(z_0 + \sum_{k=1}^{n} \frac{a_1 a_2 \cdots a_{k-1}}{b_1 b_2 \cdots b_k} r_k \right). \tag{5.2.29}$$

Proof. Introduce the integrating factor d_n with the property that $d_n b_n = d_{n-1} a_{n-1}$. The recurrence (5.2.28) becomes

$$D_n - D_{n-1} = d_n r_n, \tag{5.2.30}$$

where $D_n = d_n a_n z_n$. Therefore, by telescoping,

$$D_n = D_0 + \sum_{k=1}^{n} d_k r_k, \tag{5.2.31}$$

with $D_0 = d_0 a_0 z_0$. To find the integrating factor, observe that

$$\frac{d_n}{d_{n-1}} = \frac{a_{n-1}}{b_n}. \tag{5.2.32}$$

Thus

$$d_n = \frac{a_0 a_1 \cdots a_{n-1}}{b_1 b_2 \cdots b_n} d_0. \tag{5.2.33}$$

Replacing in (5.2.31) yields (5.2.29). □

Corollary 5.2.1. *Let $n \in \mathbb{N}$ and assume that z_n satisfies*

$$2nz_n = (2n-1)z_{n-1} + r_n, \; n \geq 1, \tag{5.2.34}$$

with the initial condition z_0. Let $\lambda_j = 2^{2j}\binom{2j}{j}^{-1}$, then

$$z_n = \frac{1}{\lambda_n}\left(z_0 + \sum_{k=1}^{n} \frac{\lambda_k r_k}{2k} \right). \tag{5.2.35}$$

Proof. Use $a_n = 2n$ and $b_n = 2n-1$ in Lemma 5.2.1. □

We now apply Lemma 5.2.1 on the recurrence (5.2.22), repeated here for convenience to the reader,

$$c(n,1) = \frac{n-1}{n} c(n-2,1) - \frac{1}{n^2}.$$

Observe that this recurrence splits naturally into even and odd branches. The value of $c(2n,1)$ is determined completely by $c(0,1)$, and $c(2n+1,1)$ by $c(1,1)$. Hence, there is no computational interaction between $c(2n,1)$ and $c(2n+1,1)$. Let $x_n = c(2n,1)$ so that x_n satisfies

$$2nx_n = (2n-1)x_{n-1} - \frac{1}{4n}, \tag{5.2.36}$$

with the initial condition

$$x_0 = c(0,1) = \frac{\pi^2}{8}. \tag{5.2.37}$$

Similarly, $y_n = c(2n+1,1)$, the odd component of $c(n,1)$, satisfies

$$(2n+1)y_n = 2ny_{n-1} - \frac{1}{2n+1} \tag{5.2.38}$$

and the initial condition

$$y_0 = c(1,1) = \frac{\pi}{2} - 1. \tag{5.2.39}$$

The expressions for z_n in Lemma 5.2.1 yield the formulas for $c(2n,1)$ and also $c(2n+1,1)$ in Theorem 5.2.5. The proof is complete.

Note 5.2.2. *The finite sums in (5.2.26) and (5.2.27) do not have closed-form, but it is a classical result that, in the limit,*

$$\sum_{k=1}^{\infty} \frac{2^{2k}}{k^2 \binom{2k}{k}} = \frac{\pi^2}{2} \tag{5.2.40}$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}(2k+1)} = \frac{\pi}{2}. \tag{5.2.41}$$

Note 5.2.3. *Formula* **3.821.3** *in [40] gives formulas equivalent to (5.2.26) and (5.2.27), respectively.*

Finally, we conclude this section by presenting a closed-form expression for the integral $c(n,p)$, for arbitrary $n,\, p \in \mathbb{N}$. The recurrence (5.2.2), in the case of even indices n, becomes

$$2nX_n(p) = (2n-1)X_{n-1}(p) - \frac{p(p-1)}{2n}X_n(p-2) \tag{5.2.42}$$

where $X_n(p) = c(2n,p)$. The initial value

$$X_0(p) = \frac{1}{(p+1)2^{p+1}}\pi^{p+1} \tag{5.2.43}$$

given in (5.2.4) and the recurrence (5.2.42) show the existence of rational numbers $a_{n,p,p+1-2j}$ such that

$$X_n(p) = \sum_{j=0}^{\xi_p} a_{n,p,p+1-2j}\pi^{p+1-2j}, \tag{5.2.44}$$

with $\xi_p = \lfloor \frac{p}{2} \rfloor$. The recurrence (5.2.42) is now expanded as

$$\begin{aligned} 2n\sum_{j=0}^{\xi_p} a_{n,p,p+1-2j}\pi^{p+1-2j} &= (2n-1)\sum_{j=0}^{\xi_p} a_{n-1,p,p+1-2j}\pi^{p+1-2j} \\ &- \frac{p(p-1)}{2n}\sum_{j=0}^{\xi_{p-1}} a_{n,p-2,p-1-2j}\pi^{p-1-2j}. \end{aligned} \tag{5.2.45}$$

The fact that the coefficients $a_{n,p,j} \in \mathbb{Q}$ allows us to match the corresponding powers of π in (5.2.45). The highest order term is π^{p+1}. Only two of the sums contain this power, therefore

$$2na_{n,p,p+1} = (2n-1)a_{n-1,p,p+1}. \tag{5.2.46}$$

The initial condition

$$a_{0,p,p+1} = \frac{1}{(p+1)2^{p+1}} \tag{5.2.47}$$

comes from (5.2.43). The solution to the initial value problem (5.2.46, 5.2.47) is then found using Corollary 5.2.1 (here $r_n = 0$), namely that

$$a_{n,p,p+1} = \frac{\binom{2n}{n}}{(p+1)2^{2n+p+1}}. \tag{5.2.48}$$

The coefficient of the next highest power π^{p-1}, in (5.2.45), yields the recurrence

$$2na_{n,p,p-1} = (2n-1)a_{n-1,p,p-1} - \frac{p(p-1)}{2n}a_{n,p-2,p-1}. \tag{5.2.49}$$

Observe that the last term in this relation is given by (5.2.48). Moreover, (5.2.43) shows that $a_{0,p,p-1} = 0$. The solution to (5.2.49), following Corollary 5.2.1, is

$$a_{n,p,p-1} = -\frac{p\binom{2n}{n}}{2^{2n+p+1}} \sum_{k_1=1}^{n} \frac{1}{k_1^2}. \tag{5.2.50}$$

The next power of π in (5.2.45) produces

$$2na_{n,p,p-3} = (2n-1)a_{n-1,p,p-3} + \frac{p(p-1)(p-2)}{n2^{2n+p}} \binom{2n}{n} \sum_{k_1=1}^{n} \frac{1}{k_1^2}, \tag{5.2.51}$$

with $a_{0,p,p-3} = 0$. One more use of Corollary 5.2.1 yields

$$a_{n,p,p-3} = \frac{\binom{2n}{n}\, p!}{2^{2n+p+1}\,(p-3)!} \sum_{k_2=1}^{n} \sum_{k_1=1}^{k_2} \frac{1}{k_1^2 k_2^2}. \tag{5.2.52}$$

This procedure can be repeated until all descending powers of π have been exhausted, hence a complete closed form for the integrals $c(n,p)$ will be made possible.

Theorem 5.2.6. *Let $n, p \in \mathbb{N}$ and let $\xi_p = \lfloor \frac{p}{2} \rfloor$. Then the even branches $X_n(p) = c(2n,p)$ of the integral*

$$c(n,p) = \int_0^{\pi/2} x^p \cos^n x \, dx \tag{5.2.53}$$

are given by

$$X_n(p) = \sum_{j=0}^{\xi_p} a_{n,p,p+1-2j} \pi^{p+1-2j} + \delta_{odd,p} \cdot a^*_{n,p}, \tag{5.2.54}$$

and the value of $a_{n,p,p+1-2j}$ *for* $p \geq 2$ *and* $0 \leq j \leq \xi_p$ *is given by*

$$a_{n,p,p+1-2j} = \frac{(-1)^j \binom{2n}{n} p!}{2^{2n+p+1} (p+1-2j)!} \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_j \leq n} \frac{1}{k_1^2 k_2^2 \cdots k_j^2},$$

and

$$a^*_{n,p} = \frac{(-1)^{\xi_p} \binom{2n}{n} p!}{2^{2n}} \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_p \leq n} \frac{1}{k_1^2 k_2^2 \cdots k_p^2} \sum_{j=1}^{k_p} \frac{2^{2j}}{j^2 \binom{2j}{j}}. \tag{5.2.55}$$

Similarly, for the odd branches $Y(n,p) = c(2n+1,p)$ *we have*

$$Y_n(p) = \sum_{j=0}^{\xi_p} b_{n,p,p-2j} \pi^{p-2j} + \delta_{odd,p} \cdot b^*_{n,p}, \tag{5.2.56}$$

with

$$b_{n,p,p-2j} = \frac{(-1)^j \, p! \, 2^{2n+2j-p}}{(2n+1)\binom{2n}{n} (p-2j)!} \sum_{0 \leq k_1 \leq k_2 \leq \cdots \leq k_j \leq n} \frac{1}{(2k_1+1)^2 (2k_2+1)^2 \cdots (2k_j+1)^2},$$

and

$$b^*_{n,p} = \frac{(-1)^{\xi_p} \, p! \, 2^{2n}}{(2n+1)\binom{2n}{n}} \sum_{0 \leq k_1 \leq k_2 \leq \cdots \leq k_p \leq n} \frac{1}{(2k_1+1)^2 (2k_2+1)^2 \cdots (2k_p+1)^2} \sum_{j=0}^{k_p} \frac{\binom{2j}{j}}{2^{2j}(2j+1)}.$$

5.3 Some examples on the halfline

In this section we provide an analytic expression for

$$C_n(p,b) = \int_0^\infty x^{-p} \cos^{2n+1}(x+b) \, dx, \tag{5.3.1}$$

and

$$S_n(p,b) = \int_0^\infty x^{-p} \sin^{2n+1}(x+b)\, dx. \tag{5.3.2}$$

In the table [40] the evaluation of the special case $p = \frac{1}{2}$ and $b = 0$:

$$\int_0^\infty \frac{\cos^{2n+1} x}{\sqrt{x}}\, dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^n \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}, \tag{5.3.3}$$

and

$$\int_0^\infty \frac{\sin^{2n+1} x}{\sqrt{x}}\, dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^n (-1)^k \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}, \tag{5.3.4}$$

as **3.822.2** and **3.821.14**.

Theorem 5.3.1. *Let $0 < p < 1$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then*

$$\int_0^\infty x^{-p} \cos^{2n+1} x\, dx = \frac{\Gamma(1-p)}{2^{2n}} \sin\left(\frac{\pi p}{2}\right) \sum_{k=0}^n \frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}}, \tag{5.3.5}$$

and

$$\int_0^\infty x^{-p} \sin^{2n+1} x\, dx = \frac{\Gamma(1-p)}{2^{2n}} \cos\left(\frac{\pi p}{2}\right) \sum_{k=0}^n (-1)^k \frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}}. \tag{5.3.6}$$

Proof. The identity $2\cos x = e^{ix} + e^{-ix}$ and the binomial theorem yield

$$\int_0^\infty x^{-p} \cos^{2n+1} x\, dx =$$
$$2^{-2n-1} \sum_{k=0}^n \binom{2n+1}{k} \int_0^\infty x^{-p} \left(e^{i(2n+1-2k)x} + e^{-i(2n+1-2k)} \right) dx. \tag{5.3.7}$$

Recall the Heaviside step function defined by $H(x) = 1$, if $x > 0$ and $H(x) = 0$ otherwise. Then, each of the integrals in (5.3.7) is evaluated using the Fourier transform

$$\int_{-\infty}^\infty H(x) x^{-p} e^{-i\omega x}\, dx = \frac{\Gamma(1-p)}{|\omega|^{1-p}} \exp(-i\pi(1-p)\text{sign}(\omega)/2). \tag{5.3.8}$$

□

Corollary 5.3.1. *Let $p > 1$ be real and $n \in \mathbb{N}_0$. Then*

$$\int_0^\infty \cos^{2n+1} x^p\, dx = \frac{1}{2^{2n}} \Gamma\left(\frac{p+1}{p}\right) \cos\left(\frac{\pi}{2p}\right) \sum_{k=0}^n \frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}}, \tag{5.3.9}$$

and

$$\int_0^\infty \sin^{2n+1} x^p\, dx = \frac{1}{2^{2n}} \Gamma\left(\frac{p+1}{p}\right) \sin\left(\frac{\pi}{2p}\right) \sum_{k=0}^n (-1)^k \frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}}. \tag{5.3.10}$$

Proof. The change of variables $x \mapsto x^{1/(1-p)}$ in the results of Theorem 5.3.1 gives the result. $\square$

The last result described here is a further generalization of Theorem 5.3.1.

Theorem 5.3.2. *Assume $b \in \mathbb{R}$, $0 < p < 1$ and $n \in \mathbb{N}_0$. Define*

$$C_n(p,b) = \int_0^\infty x^{-p} \cos^{2n+1}(x+b)\, dx \tag{5.3.11}$$

and

$$S_n(p,b) = \int_0^\infty x^{-p} \sin^{2n+1}(x+b)\, dx. \tag{5.3.12}$$

Then

$$C_n(p,b) = \frac{\Gamma(1-p)}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{\sin(\frac{\pi p}{2} - (2k+1)b)}{(2k+1)^{1-p}}, \tag{5.3.13}$$

and

$$S_n(p,b) = \frac{\Gamma(1-p)}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} \frac{\cos(\frac{\pi p}{2} - (2k+1)b)}{(2k+1)^{1-p}}. \tag{5.3.14}$$

Proof. Denote the left-hand side of (5.3.13) and (5.3.14) by $f_n(b)$ and $g_n(b)$ respectively. Differentiation with respect to the parameter b yields

$$\begin{aligned}
\frac{\partial g_n}{\partial b} - (-1)^n (2n+1) f_n &= (2n+1) \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} f_j(b) \\
\frac{\partial f_n}{\partial b} + (-1)^n (2n+1) g_n &= -(2n+1) \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} g_j(b).
\end{aligned} \tag{5.3.15}$$

Considering b and p fixed, we now show that the right-hand side of (5.3.13) and (5.3.14) satisfy the system (5.3.15) with the same initial conditions. This will establish the result.

In the case of the right-hand side of (5.3.13), it is required to check the identity

$$2^{-2n} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} \frac{\sin(\frac{\pi}{2}p - (2k+1)b)}{(2k+1)^{1-p}} =$$

$$(2n+1) \sum_{j=0}^{n} (-1)^j \binom{n}{j} 2^{-2j} \sum_{k=0}^{j} \binom{2j+1}{j-k} \frac{\sin(\frac{\pi}{2}p - (2k+1)b)}{(2j+1)^{1-p}}.$$

To verify this we compare the coefficients of the transcendental terms

$$\frac{\sin(\frac{\pi}{2}p - (2k+1)b)}{(2k+1)^{1-p}}.$$

It turns out that this question is equivalent to validating the identity

$$(-1)^k 2^{-2n} \binom{2n+1}{n-k} (2k+1) = (2n+1) \sum_{j=k}^{n} (-1)^j 2^{-2j} \binom{n}{j} \binom{2j+1}{j-k}. \quad (5.3.16)$$

To this end, we employ the WZ-technology as explained in [72]. This method produces the recurrence

$$2(n+k+1)(n+1-k)u(n+1-k) - (n+1)(2n+3)u(n,k) = 0. \quad (5.3.17)$$

To prove (5.3.16) simply check that both sides of (5.3.16) satisfy the recurrence (5.3.17) as well as the initial condition $u(0,0) = 1$.

The identities

$$\begin{aligned} \int_0^\infty x^{-p} \cos(x+b)\, dx &= -\Gamma(1-p) \sin(b - \tfrac{p\pi}{2}) \quad (5.3.18) \\ \int_0^\infty x^{-p} \sin(x+b)\, dx &= \Gamma(1-p) \cos(b - \tfrac{p\pi}{2}), \end{aligned}$$

which are special cases of

$$\begin{aligned} \int_0^\infty x^{-p} \cos(ax+b)\, dx &= -a^{p-1}\Gamma(1-p) \sin(b - \tfrac{p\pi}{2}) \quad (5.3.19) \\ \int_0^\infty x^{-p} \sin(ax+b)\, dx &= a^{p-1}\Gamma(1-p) \cos(b - \tfrac{p\pi}{2}), \end{aligned}$$

show that the corresponding initial values in (5.3.13) (respectively 5.3.14) match. The evaluations (5.3.19) appear as **3.764.1** and **3.764.2** in [40]. To establish (5.3.18) expand $\cos(x+b)$ as $\cos x \cos b - \sin x \sin b$, use the change of variables $x \mapsto x^p$, and Theorem 5.3.1. □

We now discuss some definite integrals that follow from Theorem 5.3.2.

Example 5.1. Differentiating (5.3.13) with respect to p and setting $p = \frac{1}{2}$ and $b = 0$ gives, after the change of variables $x \mapsto x^2$,

$$(5.3.20)$$

$$\begin{aligned} \int_0^\infty \log x \, \cos^{2n+1} x^2 \, dx &= -\frac{\sqrt{\pi}}{2^{2n+3}}(\pi + 2\gamma + 4\log 2) \sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{1}{\sqrt{4k+2}} \\ &- \frac{\sqrt{\pi}}{2^{2n+2}} \sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{\log(2k+1)}{\sqrt{4k+2}}, \end{aligned}$$

where we have used the value $\Gamma'(1/2) = -\sqrt{\pi}(\gamma + 2\log 2)$.

Example 5.2. Assume $0 < p,\, q < 1$. Multiplying (5.3.13) by b^{-q} and integrating over the half-line yields (after replacing b by y)

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{\cos^{2n+1}(x+y)}{x^p y^q} dA &= -\Gamma(1-p)\Gamma(1-q)\cos\left(\frac{\pi(p+q)}{2}\right) \\ &\times \sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{(2k+1)^{p+q-2}}{2^{2n}}. \end{aligned}$$

In particular, for $n = 0$,

$$\int_0^\infty \int_0^\infty \frac{\cos(x+y)}{x^p y^q} dA = -\Gamma(1-p)\Gamma(1-q)\cos\left(\frac{\pi(p+q)}{2}\right). \tag{5.3.21}$$

The derivative $\frac{\partial^2}{\partial p \partial q}$ at $p = q = \frac{1}{2}$ produces the evaluation

$$\int_0^\infty \int_0^\infty \frac{\log x \, \log y}{\sqrt{xy}} \cos(x+y)\, dx\, dy = (\gamma + 2\log 2)\pi^2 \tag{5.3.22}$$

that we promised in the Introduction.

Example 5.3. Iterating the method described in the previous example yields

$$\int_{\mathbb{R}_+^n} \left(\cos \|x\|^2\right) \cdot \prod_{j=1}^{n} \log x_j \, dV = \frac{(-1)^{\Delta_n} \pi^{n/2}}{2^{2n}} \begin{cases} \operatorname{Re} \psi_n & \text{if } n \text{ is even,} \\ \operatorname{Im} \psi_n & \text{if } n \text{ is odd,} \end{cases}$$

with

$$\Delta_n = \frac{n(n+1)}{2}, \; \psi_n = \left(\gamma + 2\log 2 + \frac{\pi i}{2}\right)^n e^{\pi i n/4}. \tag{5.3.23}$$

Here $\|x\|^2 = x_1^2 + \cdots + x_n^2$ and γ is Euler's constant. For instance, for $n = 3$ we have

$$\begin{aligned} \int_0^\infty \int_0^\infty \int_0^\infty & \log x \log y \log z \cos(x^2 + y^2 + z^2)\, dx\, dy\, dz \\ &= \frac{\pi^{3/2}}{8}(-16\xi^3 + 12\xi^2\pi + 6\xi\pi^2 - \pi^3), \end{aligned}$$

where $\xi = \gamma + 2\log 2$.

Note. Entry **3.822.2** in (5.1.8) and **3.821.14** in (5.3.4) are the only entries in this chapter that cannot be evaluated by `Mathematica`.

Chapter 6

The beta function

6.1 Introduction

The table of integrals [40] contains some evaluations that can be derived by elementary means from the *beta function*, defined by

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx. \tag{6.1.1}$$

The convergence of the integral in (6.1.1) requires a, $b > 0$. This definition appears as **3.191.3** in [40].

Our goal is to present in a systematic manner, the evaluations appearing in the classical table of Gradshteyn and Ryzhik [40], that involve this function. In this part, we restrict to algebraic integrands leaving the trigonometric forms for a future publication. This paper complements [66] that dealt with the *gamma function* defined by

$$\Gamma(a) := \int_0^\infty x^{a-1}e^{-x}\,dx. \tag{6.1.2}$$

These functions are related by the functional equation

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}. \tag{6.1.3}$$

A proof of this identity can be found in [22].

The special values $\Gamma(n) = (n-1)!$ and

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n}} \frac{(2n)!}{n!} \tag{6.1.4}$$

for $n \in \mathbb{N}$, will be used to simplify the values of the integrals presented here. Proofs of these formulas can be found in [66] as well as in Proposition 6.2.1 below.

The other property that will be employed frequently is

$$\Gamma(a)\,\Gamma(1-a) = \frac{\pi}{\sin \pi a}. \tag{6.1.5}$$

The reader will find in [22] a proof based on the product representation of these functions. A challenging problem is to produce a proof that only employs changes of variables.

The table [40] contains some direct values:

$$\int_0^1 \frac{x^p\,dx}{(1-x)^p} = \frac{p\pi}{\sin p\pi} \tag{6.1.6}$$

is **3.192.1** and is evaluated by identifying it as $B(p+1, 1-p)$. Formula **3.192.2** states that

$$\int_0^1 \frac{x^p\,dx}{(1-x)^{p+1}} = -\frac{\pi}{\sin p\pi}. \tag{6.1.7}$$

The integral is directly evaluated as $B(p+1,-p) = \Gamma(p+1)\Gamma(-p)$. and then simplified to produce the result. Next, **3.192.3** is

$$\int_0^1 \frac{(1-x)^p}{x^{p+1}}\,dx = -\frac{\pi}{\sin p\pi} \tag{6.1.8}$$

and the change of variables $t = 1/x$ in **3.192.4** produces

$$\int_1^\infty (x-1)^{p-1/2}\frac{dx}{x} = \int_0^1 t^{-p-1/2}(1-t)^{p-1/2}\,dt \tag{6.1.9}$$

and this is

$$B\left(\tfrac{1}{2}-p, \tfrac{1}{2}+p\right) = \Gamma\left(\tfrac{1}{2}-p\right)\Gamma\left(\tfrac{1}{2}+p\right) = \frac{\pi}{\cos p\pi}, \tag{6.1.10}$$

as stated in [40].

Let $b = \frac{1}{2}$ in (6.1.1) to obtain

$$\int_0^1 \frac{x^{a-1}\,dx}{\sqrt{1-x}} = B\left(a, \tfrac{1}{2}\right) = \frac{\Gamma(a)\sqrt{\pi}}{\Gamma\left(a+\tfrac{1}{2}\right)}. \tag{6.1.11}$$

The special values $a = n+1$ and $a = n + \frac{1}{2}$ appear as **3.226.1** and **3.226.2**, respectively.

6.2 Elementary properties

Many of the properties of the beta function can be established by simple changes of variables. For example, letting $y = 1 - x$ in (6.1.1) yields the symmetry

$$B(a,b) = B(b,a). \tag{6.2.1}$$

It should not be surprising that a clever change of variables might lead to a beautiful result. This is illustrated following Serret [75]. Start with

$$\begin{aligned} B(a,a) &= \int_0^1 (x - x^2)^{a-1}\,dx \\ &= 2\int_0^{1/2} \left[\tfrac{1}{4} - \left(\tfrac{1}{2} - x\right)^2\right]^{a-1} dx. \end{aligned}$$

The natural change of variables $v = \frac{1}{2} - x$ yields

$$B(a,a) = 2\int_0^{1/2} \left(\tfrac{1}{4} - v^2\right)^{a-1} dv. \tag{6.2.2}$$

The next step is now clear: let $s = 4v^2$ to produce

$$B(a,a) = 2^{1-2a} B\left(a, \tfrac{1}{2}\right). \tag{6.2.3}$$

The functional equation (6.1.3) converts this identity into Legendre's original form:

Proposition 6.2.1. *The gamma function satisfies*

$$\Gamma\left(a + \tfrac{1}{2}\right) = \frac{\Gamma(2a)\,\Gamma(\frac{1}{2})}{\Gamma(a)\,2^{2a-1}}. \tag{6.2.4}$$

In particular, for $a = n \in \mathbb{N}$, this yields (6.1.4).

6.3 Elementary changes of variables

The integral (6.1.1) defining the beta function can be transformed by changes of variables. For example, the new variable $x = t/u$, reduces (6.1.1) to

$$\int_0^u t^{a-1}(u-t)^{b-1}\,dt = u^{a+b-1}B(a,b), \tag{6.3.1}$$

that appears as **3.191.1** in [40]. The effect of this change of variables is to express the beta function as an integral over a finite interval. Observe that the integrand vanishes at both end points. Similarly, the change $t = (v-u)x+u$ maps the interval $[0,1]$ to $[u,v]$. It yields

$$\int_u^v (t-u)^{a-1}(v-t)^{b-1}\,dt = (v-u)^{a+b-1}B(a,b). \tag{6.3.2}$$

This is **3.196.3** in [40]. The special case $u=0$, $v=n$ and $a=\nu$, $b=n+1$ appears as **3.193** in [40] as

$$\int_0^n x^{\nu-1}(n-x)^n\,dx = \frac{n^{\nu+n}\,n!}{\nu(\nu+1)(\nu+2)\cdots(\nu+n)}. \tag{6.3.3}$$

`Mathematica` evaluates this entry as

$$\int_0^n x^{\nu-1}(n-x)^n\,dx = n^{n+\nu}B(\nu,n+1). \tag{6.3.4}$$

Several integrals in [40] can be obtained by a small variation of the definition. For example, the integral

$$\int_0^1 (1-x^a)^{b-1}\,dx = \frac{1}{a}B\left(1/a,b\right) \tag{6.3.5}$$

can be obtained by the change of variables $t=x^a$. This appears as **3.249.7** in [40] and illustrates the fact that it not necessary for the integrand to vanish at *both* end points. The special case $a=2$ appears as **3.249.5**:

$$\int_0^1 (1-x^2)^{b-1}\,dx = \tfrac{1}{2}B\left(\tfrac{1}{2},b\right) = 2^{2b-2}B(b,b), \tag{6.3.6}$$

where the second identity follows from Legendre's duplication formula (6.2.4).

The change of variables $t=cx$ produces a scaled version:

$$\int_0^c (c^a-t^a)^{b-1}\,dt = \frac{1}{a}c^{a(b-1)+1}B\left(1/a,b\right). \tag{6.3.7}$$

The special case $a=2$ yields

$$\int_0^c (c^2-t^2)^{b-1}\,dt = \frac{c^{2b-1}}{2}B\left(1/2,b\right). \tag{6.3.8}$$

The choice $b=n+\frac{1}{2}$ appears as **3.249.2** in [40]:

$$\int_0^c (c^2-t^2)^{n-1/2}\,dt = \frac{\pi c^{2n}}{2^{2n+1}}\binom{2n}{n}. \tag{6.3.9}$$

Similarly **3.251.1** in [40] is

$$\int_0^1 x^{c-1}(1-x^a)^{b-1}\,dx = \frac{1}{a}B\left(\frac{c}{a}, b\right). \tag{6.3.10}$$

The change of variables $t = 1/x$ converts (6.1.1) into

$$\int_1^\infty t^{-a-b}(t-1)^{b-1}\,dt = B(a,b). \tag{6.3.11}$$

Letting $t = x^p$ yields

$$\int_1^\infty x^{p(1-a-b)-1}\left(x^p-1\right)^{b-1}\,dx = \frac{1}{p}B(a,b). \tag{6.3.12}$$

The special case $\nu = b$ and $\mu = p(1-a-b)$ is **3.251.3**:

$$\int_1^\infty x^{\mu-1}\left(x^p-1\right)^{\nu-1}\,dx = \frac{1}{p}B\left(1-\nu-\mu/p, \nu\right). \tag{6.3.13}$$

6.4 Integrals over a halfline

The beta function can also be expressed as an integral over a half-line. The change of variables $t = x/(1-x)$ maps $[0,1]$ onto $[0,\infty)$ and it produces from (6.1.1)

$$B(a,b) = \int_0^\infty \frac{t^{a-1}\,dt}{(1+t)^{a+b}}. \tag{6.4.1}$$

In particular, if $a+b=1$, using (6.1.3) and (6.1.5), we obtain

$$\int_0^\infty \frac{t^{a-1}\,dt}{1+t} = \frac{\pi}{\sin \pi a}. \tag{6.4.2}$$

This can be scaled to produce, for $a > 0$ and $c > 0$,

$$\int_0^\infty \frac{x^{a-1}\,dx}{x+c} = \frac{\pi}{\sin \pi a}c^{a-1} \quad \text{for } c > 0 \tag{6.4.3}$$

that appears as **3.222.2** in [40]. In the case $c < 0$ we have a singular integral. Define $b = -c > 0$ and $s = x/b$, so now we have to evaluate

$$I = -b^{a-1}\int_0^\infty \frac{s^{a-1}\,ds}{1-s}. \tag{6.4.4}$$

The integral is considered as a Cauchy principal value

$$I = \lim_{\epsilon \to 0} \int_0^1 \frac{s^{a-1}\, ds}{(1-s)^{1-\epsilon}} + \int_1^\infty \frac{s^{a-1}\, ds}{(1-s)^{1-\epsilon}}. \tag{6.4.5}$$

Let $y = 1/s$ in the second integral and evaluate them in terms of the beta function to produce

$$I = \lim_{\epsilon \to 0} \epsilon \Gamma(\epsilon) \times \frac{1}{\epsilon} \left(\frac{\Gamma(a)}{\Gamma(a+\epsilon)} - \frac{\Gamma(1-a-\epsilon)}{\Gamma(1-a)} \right). \tag{6.4.6}$$

Use L'Hopital's rule to evaluate and obtain

$$I = -\frac{\Gamma'(a)}{\Gamma(a)} + \frac{\Gamma'(1-a)}{\Gamma(a)}. \tag{6.4.7}$$

Using the relation $\Gamma(a)\Gamma(1-a) = \pi \text{cosec } \pi a$, this reduces to $\pi \cot \pi a$. Therefore we have

$$\int_0^\infty \frac{x^{a-1}\, dx}{x+c} = -\frac{\pi}{\tan \pi a} (-c)^{a-1} \quad \text{ for } c < 0 \tag{6.4.8}$$

The change of variables $x = e^{-t}$ produces, for $c < 0$,

$$\int_{-\infty}^\infty \frac{e^{-\mu t}\, dt}{e^{-t} + c} = -\pi \cot(\mu \pi) \, (-c)^{\mu - 1}. \tag{6.4.9}$$

The special case $c = -1$ appears as **3.313.1**:

$$\int_{-\infty}^\infty \frac{e^{-\mu t}\, dt}{1 - e^{-t}} = \pi \cot(\mu \pi). \tag{6.4.10}$$

Note. This is a singular integral and its value should be computed as a Cauchy principal value; that is,

$$\lim_{a \to 0^-} \int_{-\infty}^a \frac{e^{-\mu t}\, dt}{1 - e^{-t}} + \lim_{b \to 0^+} \int_b^\infty \frac{e^{-\mu t}\, dt}{1 - e^{-t}}. \tag{6.4.11}$$

`Mathematica` states that this integral is divergent.

We now consider several examples in [40] that are direct consequences of (6.4.3) and (6.4.8). In the first example, we combine (6.4.3) with the partial fraction decomposition

$$\frac{1}{(x+a)(x+b)} = \frac{1}{b-a} \left(\frac{1}{x+a} - \frac{1}{x+b} \right) \tag{6.4.12}$$

to produce **3.223.1**:

$$\int_0^\infty \frac{x^{\mu-1}\, dx}{(x+b)(x+a)} = \frac{\pi}{b-a} (a^{\mu-1} - b^{\mu-1}) \text{cosec}(\pi \mu). \tag{6.4.13}$$

Similarly,

$$\frac{1}{x+b} - \frac{1}{x-a} = \frac{a+b}{(a-x)(b+x)} \tag{6.4.14}$$

leads to **3.223.2**:

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{(b+x)(a-x)} = \frac{\pi}{a+b}\left(b^{\mu-1}\,\operatorname{cosec}(\mu\pi) + a^{\mu-1}\cot(\mu\pi)\right), \tag{6.4.15}$$

using (6.4.3) and (6.4.8). The result **3.223.3**:

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{(a-x)(b-x)} = \pi\cot(\mu\pi)\frac{a^{\mu-1}-b^{\mu-1}}{b-a}, \tag{6.4.16}$$

follows from

$$\frac{1}{(a-x)(b-x)} = \frac{1}{a-b}\left(\frac{1}{b-x} - \frac{1}{a-x}\right). \tag{6.4.17}$$

Finally, **3.224**:

$$\int_0^\infty \frac{(x+b)x^{\mu-1}\,dx}{(x+a)(x+c)} = \frac{\pi}{\sin(\mu\pi)}\left(\frac{a-b}{a-c}a^{\mu-1} + \frac{c-b}{c-a}c^{\mu-1}\right), \tag{6.4.18}$$

follows from

$$\frac{x+b}{(x+a)(x+c)} = \frac{b-a}{c-a}\frac{1}{x+a} - \frac{b-c}{c-a}\frac{1}{x+c}. \tag{6.4.19}$$

We can now transform (6.4.1) to the interval $[0,1]$ by splitting $[0,\infty)$ as $[0,1]$ followed by $[1,\infty)$. In the second integral, we let $t = 1/s$. The final result is

$$B(a,b) = \int_0^1 \frac{t^{a-1}+t^{b-1}}{(1+t)^{a+b}}\,dt. \tag{6.4.20}$$

Note. `Mathematica` gives this integral in terms of hypergeometric function:

$$\int_0^1 \frac{t^{a-1}+t^{b-1}}{(1+t)^{a+b}}\,dt = \frac{1}{a}\,{}_2F_1\left(\left.\begin{matrix} a & a+b \\ & a+1 \end{matrix}\right| -1\right) + \frac{1}{b}\,{}_2F_1\left(\left.\begin{matrix} b & a+b \\ & b+1 \end{matrix}\right| -1\right). \tag{6.4.21}$$

The formula (6.4.20), that appears as entry **3.216.1**, makes it apparent that the beta function is symmetric: $B(a,b) = B(b,a)$. The change of variables $s = 1/t$ converts (6.4.20) into **3.216.2**:

$$B(a,b) = \int_1^\infty \frac{s^{a-1}+s^{b-1}}{(1+s)^{a+b}}\,ds. \tag{6.4.22}$$

It is easy to introduce a parameter: let $c > 0$ and consider the change of variables $t = cx$ in (6.4.1) to obtain

$$\int_0^\infty \frac{x^{a-1}\,dx}{(1+cx)^{a+b}} = c^{-a}B(a,b), \tag{6.4.23}$$

that appears as **3.194.3**. We can now shift the lower limit of integration via $t = x + u$ to produce

$$\int_u^\infty (t-u)^{a-1}(t+v)^{-a-b}\,dt = (u+v)^{-b}B(a,b), \tag{6.4.24}$$

where $v = 1/c - u$. This is **3.196.2**, where v is denoted by β. Now let $b = c-a$ in the special case $v = 0$ to obtain

$$\int_u^\infty (t-u)^{a-1}t^{-c}\,dt = u^{a-c}B(a,c-a). \tag{6.4.25}$$

This appears as **3.191.2**.

We now write (6.4.1) using the change of variables $t = x^c$. It produces

$$\int_0^\infty \frac{x^{ac-1}\,dx}{(1+x^c)^{a+b}} = \frac{1}{c}B(a,b). \tag{6.4.26}$$

The special case $c = 2$ and $a = 1+\mu/2$, $b = 1-\mu/2$ produces **3.251.6** in the form

$$\int_0^\infty \frac{x^{\mu+1}\,dx}{(1+x^2)^2} = \frac{\mu\pi}{4\sin\mu\pi/2}. \tag{6.4.27}$$

Now let $b = 1-a$ and choose $a = p/c$ to obtain

$$\int_0^\infty \frac{x^{p-1}\,dx}{1+x^c} = \frac{1}{c}B\left(\frac{p}{c},\frac{c-p}{c}\right) = \frac{\pi}{c}\,\operatorname{cosec}(\pi p/c). \tag{6.4.28}$$

This appears as **3.241.2** in [40].

Similar arguments establish **3.196.4**:

$$\int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = -\frac{\pi}{b}\,\operatorname{cosec}(\nu\pi)\left(\frac{b}{b-a}\right)^\nu. \tag{6.4.29}$$

Indeed, the change of variables $t = x-1$ yields

$$\int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = \int_0^\infty \frac{dt}{[(a-b)-bt]\,t^\nu}, \tag{6.4.30}$$

and scaling via the new variable $z = bt/(b-a)$ gives

$$\int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = -\frac{1}{b}\left(\frac{b}{b-a}\right)^\nu \int_0^\infty \frac{dz}{(1+z)\,z^\nu}. \tag{6.4.31}$$

The result follows from (6.4.1) and the value

$$B(\nu,1-\nu) = \Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin\pi\nu}. \tag{6.4.32}$$

The same argument gives **3.196.5**:

$$\int_{-\infty}^1 \frac{dx}{(a-bx)(1-x)^\nu} = \frac{\pi}{b}\,\operatorname{cosec}(\nu\pi)\left(\frac{b}{a-b}\right)^\nu. \tag{6.4.33}$$

6.5 Some direct evaluations

There are many more integrals in [40] that can be evaluated in terms of the beta function. For example, **3.221.1** states that

$$\int_a^\infty \frac{(x-a)^{p-1}\,dx}{x-b} = \pi(a-b)^{p-1} \operatorname{cosec} \pi p. \tag{6.5.1}$$

To establish these identities, we assume that $a > b$ to avoid the singularities. The change of variables $t = (x-a)/(a-b)$ yields

$$\int_a^\infty \frac{(x-a)^{p-1}\,dx}{x-b} = (a-b)^{p-1} \int_0^\infty \frac{t^{p-1}\,dt}{1+t}, \tag{6.5.2}$$

and this integral appears in (6.4.2).

Similarly, **3.221.2** states that

$$\int_{-\infty}^a \frac{(a-x)^{p-1}\,dx}{x-b} = -\pi(b-a)^{p-1} \operatorname{cosec} \pi p. \tag{6.5.3}$$

This is evaluated by the change of variables $y = (a-x)/(b-a)$.

The table contains several evaluations that are elementary corollaries of (6.4.1). Starting with

$$\int_0^\infty \frac{x^a\,dx}{(1+x)^b} = B(a+1, b-a-1) = \frac{\Gamma(a+1)\,\Gamma(b-a-1)}{\Gamma(b)}, \tag{6.5.4}$$

we find the case $a = p$ and $b = 3$ in **3.225.3**:

$$\int_0^\infty \frac{x^p\,dx}{(1+x)^3} = \frac{\Gamma(p+1)\,\Gamma(2-p)}{\Gamma(3)} = \frac{p(1-p)}{2}\frac{\pi}{\sin(p\pi)}, \tag{6.5.5}$$

using elementary properties of the gamma function.

The change of variables $t = 1 + x$ converts (6.5.4) into

$$\int_1^\infty \frac{(t-1)^a\,dt}{t^b} = B(a+1, b-a-1) = \frac{\Gamma(a+1)\,\Gamma(b-a-1)}{\Gamma(b)}. \tag{6.5.6}$$

The special case $a = p-1$ and $b = 2$ gives

$$\begin{aligned} \int_1^\infty \frac{(t-1)^{p-1}\,dt}{t^2} &= \Gamma(p)\Gamma(2-p) = (1-p)\Gamma(p)\Gamma(1-p) \\ &= \frac{\pi(1-p)}{\sin(p\pi)}. \end{aligned}$$

This appears as **3.225.1**. Similarly, the case $a = 1 - p$ and $b = 3$ produces **3.225.2**:

$$\begin{aligned} \int_1^\infty \frac{(t-1)^{1-p}\,dt}{t^3} &= \frac{\Gamma(2-p)\Gamma(1+p)}{\Gamma(3)} = \frac{1}{2}p(1-p)\Gamma(p)\Gamma(1-p) \\ &= \frac{\pi p(1-p)}{2\sin(p\pi)}. \end{aligned}$$

6.6 Introducing parameters

It is often convenient to introduce free parameters in a definite integral. Starting with (6.4.1), the change of variables $t = \frac{u}{v}x^c$ yields

$$B(a,b) = cu^a v^b \int_0^\infty \frac{t^{ac-1}\,dt}{(v+ut^c)^{a+b}}. \tag{6.6.1}$$

This formula appears as **3.241.4** in [40] with the parameters

$$a = \frac{\mu}{\nu},\ b = n+1-\frac{\mu}{\nu},\ c=\nu,\ u=q,\ \text{and } v=p, \tag{6.6.2}$$

in the form

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{(p+qx^\nu)^{n+1}} = \frac{1}{\nu\, p^{n+1}}\left(\frac{p}{q}\right)^{\mu/\nu} \frac{\Gamma(\mu/\nu)\,\Gamma(n+1-\mu/\nu)}{\Gamma(n+1)}.$$

This is a messy notation and it leaves the wrong impression that n should be an integer.

- The special case $v = c = 1$ and $b = p+1-a$ produces

$$\int_0^\infty \frac{t^{a-1}\,dt}{(1+ut)^{p+1}} = \frac{1}{u^a}B(a,p+1-a). \tag{6.6.3}$$

This appears as **3.194.4** in [40], except that it is written in terms of binomial coefficients as

$$\int_0^\infty \frac{t^{a-1}\,dt}{(1+ut)^{p+1}} = (-1)^p\frac{\pi}{u^a}\binom{a-1}{p}\operatorname{cosec}(\pi a). \tag{6.6.4}$$

We prefer the notation in (6.6.3).

- The special case $v = c = 1$ and $b = 2-a$ produces

$$\int_0^\infty \frac{t^{a-1}\,dt}{(1+ut)^2} = \frac{1}{u^a}B(a,2-a). \tag{6.6.5}$$

Using (6.1.3) and (6.1.5) yields the form

$$\int_0^\infty \frac{t^{a-1}\,dt}{(1+ut)^2} = \frac{(1-a)\,\pi}{u^a\,\sin\pi a}. \tag{6.6.6}$$

This appears as **3.194.6** in [40].

• The special case $u = v = 1$ and $c = q$, and choosing $a = p/q$ and $b = 2 - p/q$ yields **3.241.5** in the form

$$\int_0^\infty \frac{x^{p-1}\,dx}{(1+x^q)^2} = \frac{q-p}{q^2}\,\frac{\pi}{\sin(\pi p/q)}. \tag{6.6.7}$$

• The special case $c = 1$ and $a = m+1$, $b = n - m - \frac{1}{2}$ produces

$$\int_0^\infty \frac{t^m\,dt}{(v+ut)^{n+\frac{1}{2}}} = \frac{1}{u^{m+1}\,v^{n-m-\frac{1}{2}}} B\left(m+1, n-m-\tfrac{1}{2}\right). \tag{6.6.8}$$

Using (6.1.3) and (6.1.4) this reduces to

$$\int_0^\infty \frac{t^m\,dt}{(v+ut)^{n+\frac{1}{2}}} = \frac{m!\,n!\,(2n-2m-2)!}{(n-m-1)!\,(2n)!} 2^{2m+2} \frac{v^{m-n+1/2}}{u^{m+1}}, \tag{6.6.9}$$

for m, $n \in \mathbb{N}$, with $n > m$. This appears as **3.194.7** in [40].

• The special case $u = v = 1$ and $b = \frac{1}{2} - a$ yields

$$\int_0^\infty \frac{t^{ac-1}\,dt}{\sqrt{1+t^c}} = \frac{1}{c} B\left(a, \tfrac{1}{2} - a\right). \tag{6.6.10}$$

Writing $a = p/c$ we recover **3.248.1**:

$$\int_0^\infty \frac{t^{p-1}\,dt}{\sqrt{1+t^c}} = \frac{1}{c} B\left(\tfrac{p}{c}, \tfrac{1}{2} - \tfrac{p}{c}\right). \tag{6.6.11}$$

• Now replace v by v^2 in (6.6.1). Then, with $u = 1$, $a = \frac{1}{2}$, $c = 2$, so that $ac = 1$ and $b = n - \frac{1}{2}$ we obtain

$$\int_0^\infty \frac{dt}{(v^2+t^2)^n} = \frac{1}{2v^{2n-1}} B\left(\tfrac{1}{2}, n - \tfrac{1}{2}\right). \tag{6.6.12}$$

This can be written as

$$\int_0^\infty \frac{dt}{(v^2+t^2)^n} = \frac{\sqrt{\pi}\,\Gamma(n-1/2)}{2\Gamma(n)v^{2n-1}} \tag{6.6.13}$$

that appears as **3.249.1** in [40].

• The special case $v = 1$, $c = 2$ and $b = \frac{n}{2} - a$ in (6.6.1) yields

$$\int_0^\infty \frac{t^{2a-1}\,dt}{(1+ut^2)^{n/2}} = \frac{1}{2u^a} B\left(a, \tfrac{n}{2} - a\right). \tag{6.6.14}$$

Now $a = 1/2$ gives

$$\int_0^\infty (1+ut^2)^{-n/2}\,dt = \frac{1}{2\sqrt{u}} B\left(\tfrac{1}{2}, \tfrac{n-1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{u}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(n/2)}. \tag{6.6.15}$$

It is curious that the table [40] contains **3.249.8** as the special case $u = 1/(n-1)$ of this evaluation.

• We now put $u = v = 1$ and $c = 2$ in (6.6.1). Then, with $b = 1 - \nu - a$ and $a = \mu/2$, we obtain **3.251.2**:

$$\int_0^\infty \frac{t^{\mu-1}\,dt}{(1+t^2)^{1-\nu}} = \frac{1}{2} B\left(\frac{\mu}{2}, 1 - \nu - \frac{\mu}{2}\right). \tag{6.6.16}$$

• We now consider the case $c = 2$ in (6.6.1):

$$\int_0^\infty \frac{t^{2a-1}\,dt}{(v+ut^2)^{a+b}} = \frac{1}{2u^a v^b} B(a,b). \tag{6.6.17}$$

The special case $a = m + \frac{1}{2}$ and $b = n - m + \frac{1}{2}$ yields

$$\int_0^\infty \frac{t^{2m}\,dt}{(v+ut^2)^{n+1}} = \frac{\Gamma(m+1/2)\,\Gamma(n-m+1/2)}{2u^{m+1/2}v^{n-m+1/2}\Gamma(n+1)}, \tag{6.6.18}$$

and using (6.1.4) we obtain **3.251.4**:

$$\int_0^\infty \frac{t^{2m}\,dt}{(v+ut^2)^{n+1}} = \frac{\pi(2m)!(2n-2m)!}{2^{2n+1}m!(n-m)!n!\,u^{m+1/2}v^{n-m+1/2}}, \tag{6.6.19}$$

for $n, m \in \mathbb{N}$ with $n > m$.

On the other hand, if we choose $a = m + 1$ and $b = n - m$ we obtain **3.251.5**:

$$\int_0^\infty \frac{t^{2m+1}\,dt}{(v+ut^2)^{n+1}} = \frac{\Gamma(m+1)\,\Gamma(n-m)}{2u^{m+1}v^{n-m}\Gamma(n+1)} = \frac{m!(n-m-1)!}{2n!u^{m+1}v^{n-m}}. \tag{6.6.20}$$

Several evaluations in [40] come from the form

$$\int_0^1 t^{aq-1}(1-t^q)^{b-1}\,dt = \frac{1}{q} B(a,b), \tag{6.6.21}$$

obtained from (6.1.1) by the change of variables $x = t^q$.

- The choice $a = 1 + p/q$ and $b = 1 - p/q$ produces

$$\int_0^1 t^{p+q-1}(1-t^q)^{-p/q}\,dt = \frac{1}{q}B\left(1+\frac{p}{q}, 1-\frac{p}{q}\right) = \frac{p\pi}{q^2}\,\operatorname{cosec}\left(\frac{p\pi}{q}\right). \tag{6.6.22}$$

This appears as **3.251.8**.

- The choice $a = 1/p$ and $b = 1 - 1/p$ gives

$$\int_0^1 x^{q/p-1}(1-x^q)^{-1/p}\,dx = \frac{1}{q}B\left(\frac{1}{p}, 1-\frac{1}{p}\right) = \frac{\pi}{q}\,\operatorname{cosec}\left(\frac{\pi}{p}\right). \tag{6.6.23}$$

This appears as **3.251.9**.

- The reader can now check that the choice $a = p/q$ and $b = 1-p/q$ yields the evaluation

$$\int_0^1 x^{p-1}(1-x^q)^{-p/q}\,dx = \frac{1}{q}B\left(\frac{p}{q}, 1-\frac{p}{q}\right) = \frac{\pi}{q}\,\operatorname{cosec}\left(\frac{p\pi}{q}\right). \tag{6.6.24}$$

This appears as **3.251.10**.

- Putting $v = 1$ and $b = \nu - a$ in (6.6.1) we get

$$\int_0^\infty \frac{t^{ac-1}\,dt}{(1+ut^c)^\nu} = \frac{1}{cu^a}B(a, \nu - a). \tag{6.6.25}$$

Now let $a = r/c$ to obtain

$$\int_0^\infty \frac{t^{r-1}\,dt}{(1+ut^c)^\nu} = \frac{1}{cu^{r/c}}B\left(\frac{r}{c}, \nu - \frac{r}{c}\right). \tag{6.6.26}$$

This appears as **3.251.11**.

- We now choose $b = 1 - 1/q$ in (6.6.21) to obtain

$$\int_0^1 \frac{t^{aq-1}\,dt}{\sqrt[q]{1-t^q}} = \frac{1}{q}B\left(a, 1-\frac{1}{q}\right). \tag{6.6.27}$$

Finally, writing $a = c - (m-1)/q$ gives the form

$$\int_0^1 \frac{t^{cq-m}\,dt}{\sqrt[q]{1-t^q}} = \frac{1}{q}B\left(c+\frac{1}{q}-\frac{m}{q}, 1-\frac{1}{q}\right). \tag{6.6.28}$$

The special case $q = 2$ produces

$$\int_0^1 \frac{t^{2c-m}\,dt}{\sqrt{1-t^2}} = \tfrac{1}{2}B\left(c+\tfrac{1}{2}-\tfrac{m}{2}, \tfrac{1}{2}\right) = \frac{\Gamma(c+\frac{1}{2}-\frac{m}{2})\sqrt{\pi}}{2\Gamma(c+1-\frac{m}{2})}. \tag{6.6.29}$$

In particular, if $c = n+1$ and $m = 1$ we obtain **3.248.2**:

$$\int_0^1 \frac{t^{2n+1}\,dt}{\sqrt{1-t^2}} = \frac{\sqrt{\pi}\,n!}{2\Gamma(n+3/2)} = \frac{2^{2n}\,n!^2}{(2n+1)!}. \tag{6.6.30}$$

Similarly, $c = n$ and $m = 0$ yield **3.248.3**:

$$\int_0^1 \frac{t^{2n}\,dt}{\sqrt{1-t^2}} = \frac{\pi}{2^{2n+1}}\,\frac{(2n)!}{n!^2} = \frac{\pi}{2^{2n+1}}\binom{2n}{n}. \tag{6.6.31}$$

In the case $q = 3$ we get

$$\int_0^1 \frac{t^{3c-m}\,dt}{\sqrt[3]{1-t^3}} = \frac{1}{3}B\left(c+\frac{1}{3}-\frac{m}{3}, 1-\frac{1}{3}\right). \tag{6.6.32}$$

This includes **3.267.1** and **3.267.2** in [40]:

$$\begin{aligned}
\int_0^1 \frac{t^{3n}\,dt}{\sqrt[3]{1-t^3}} &= \frac{2\pi}{3\sqrt{3}}\frac{\Gamma(n+\frac{1}{3})}{\Gamma(\frac{1}{3})\,\Gamma(n+1)} \\
\int_0^1 \frac{t^{3n-1}\,dt}{\sqrt[3]{1-t^3}} &= \frac{(n-1)!\Gamma(\frac{2}{3})}{3\Gamma(n+\frac{2}{3})}
\end{aligned}$$

The latest edition of [40] has added our suggestion

$$\int_0^1 \frac{t^{3n-2}\,dt}{\sqrt[3]{1-t^3}} = \frac{\Gamma(n-\frac{1}{3})\,\Gamma(\frac{2}{3})}{3\Gamma(n+\frac{1}{3})} \tag{6.6.33}$$

as **3.267.3**.

6.7 The exponential scale

We now present examples of (6.1.1) written in terms of the exponential function. The change of variables $x = e^{-ct}$ in (6.1.1) yields

$$\int_0^\infty e^{-at}(1-e^{-ct})^{b-1}\,dt = \frac{1}{c}B\left(\frac{a}{c}, b\right). \tag{6.7.1}$$

This appears as **3.312.1** in [40]. On the other hand, if we let $x = e^{-ct}$ in (6.4.1) we get

$$\int_{-\infty}^\infty \frac{e^{-act}\,dt}{(1+e^{-ct})^{a+b}} = \frac{1}{c}B(a,b). \tag{6.7.2}$$

This appears as **3.313.2** in [40]. The reader can now use the techniques described above to verify

$$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{(e^{b/a}+e^{-x/a})^\nu} = a\exp\left[b\left(\mu-\frac{\nu}{a}\right)\right]\,B\left(a\mu, \nu-a\mu\right), \tag{6.7.3}$$

that appears as **3.314**. The choice $b=0$, $\nu=1$ and relabelling parameters by $a=1/q$ and $\mu=p$ yield **3.311.3**:

$$\int_{-\infty}^{\infty} \frac{e^{-px}\,dx}{1+e^{-qx}} = \frac{1}{q} B\left(\frac{p}{q}, 1-\frac{p}{q}\right) = \frac{\pi}{q} \text{cosec}\left(\frac{\pi p}{q}\right), \tag{6.7.4}$$

using the identity $B(x, 1-x) = \pi \text{cosec}(\pi x)$ in the last step. This is the form given in the table.

The integral **3.311.9**:

$$\int_{-\infty}^{\infty} \frac{e^{-\mu x}\,dx}{b+e^{-x}} = \pi b^{\mu-1}\ \text{cosec}(\mu\pi) \tag{6.7.5}$$

can be evaluated via the change of variables $t = e^{-x}/b$ and (6.4.2) to produce

$$I = b^{\mu-1} \int_0^{\infty} \frac{t^{\mu-1}\,dt}{1+t}. \tag{6.7.6}$$

6.8 Some logarithmic examples

The beta function appears in the evaluation of definite integrals involving logarithms. For example, **4.273** states that

$$\int_u^v \left(\ln\frac{x}{u}\right)^{p-1} \left(\ln\frac{v}{x}\right)^{q-1} \frac{dx}{x} = B(p,q) \left(\ln\frac{v}{u}\right)^{p+q-1}. \tag{6.8.1}$$

The evaluation is simple: the change of variables $x = ut$ produces, with $c = v/u$,

$$I = \int_1^c \ln^{p-1} t\,(\ln c - \ln t)^{q-1}\,\frac{dt}{t}. \tag{6.8.2}$$

The change of variables $z = \frac{\ln t}{\ln c}$ give the result.

A second example is **4.275.1**:

$$\int_0^1 \left[(-\ln x)^{q-1} - x^{p-1}(1-x)^{q-1}\right]\,dx = \frac{\Gamma(q)}{\Gamma(p+q)}\left[\Gamma(p+q) - \Gamma(p)\right], \tag{6.8.3}$$

that should be written as

$$\int_0^1 \left[(-\ln x)^{q-1} - x^{p-1}(1-x)^{q-1}\right]\,dx = \Gamma(q) - B(p,q). \tag{6.8.4}$$

The evaluation is elementary, using Euler form of the gamma function

$$\Gamma(q) = \int_0^1 (-\ln x)^{q-1}\,dx. \tag{6.8.5}$$

6.9 Examples with a fake parameter

The evaluation **3.217**:

$$\int_0^\infty \left(\frac{b^p x^{p-1}}{(1+bx)^p} - \frac{(1+bx)^{p-1}}{b^{p-1}x^p} \right) dx = \pi \cot \pi p \tag{6.9.1}$$

has the obvious parameter b. We say that this is a *fake parameter* in the sense that a simple scaling shows that the integral is independent of it. Indeed, the change of variables $t = bx$ shows this independence. Therefore the evaluation amounts to showing that

$$\int_0^\infty \left(\frac{t^{p-1}}{(1+t)^p} - \frac{(1+t)^{p-1}}{t^p} \right) dt = \pi \cot \pi p. \tag{6.9.2}$$

To achieve this, we let $y = 1/t$ in the second integral to produce

$$\lim_{\epsilon \to 0} \int_0^\infty \frac{t^{p-1-\epsilon}\, dt}{(1+t)^p} - \int_0^\infty \frac{t^{\epsilon-1}\, dt}{(1+t)^{1-p}}. \tag{6.9.3}$$

The integrals above evaluate to $B(p-\epsilon, \epsilon) - B(\epsilon, 1-p-\epsilon)$. Using

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \text{ and } \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)} \tag{6.9.4}$$

this reduces to

$$I = \lim_{\epsilon \to 0} \epsilon \Gamma(\epsilon) \left(\frac{\Gamma(p-\epsilon)\Gamma(p+\epsilon)\sin(\pi(p+\epsilon)) - \Gamma^2(p)\sin(\pi p)}{\epsilon \Gamma(p)\Gamma(p+\epsilon)\sin(\pi(p+\epsilon))} \right). \tag{6.9.5}$$

Now recall that

$$\lim_{\epsilon \to 0} \epsilon \Gamma(\epsilon) = 1 \tag{6.9.6}$$

and reduce the previous limit to

$$I = \frac{1}{\Gamma^2(p)\, \sin(\pi p)} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\Gamma(p-\epsilon)\Gamma(p+\epsilon)\sin(\pi(p+\epsilon)) - \Gamma^2(p)\sin(\pi p) \right). \tag{6.9.7}$$

Using L'Hopital's rule we find that $I = \pi \cot(\pi p)$ as required.

The example **3.218**

$$\int_0^\infty \frac{x^{2p-1} - (a+x)^{2p-1}}{(a+x)^p\, x^p}\, dx = \pi \cot \pi p \tag{6.9.8}$$

also shows a fake parameter. The change of variable $x = at$ reduces the integral above to

$$\int_0^\infty \frac{t^{2p-1} - (1+t)^{2p-1}}{(1+t)^p\, t^p}\, dt = \pi \cot \pi p. \tag{6.9.9}$$

This can be written as

$$I = \int_0^\infty \left(\frac{t^{p-1}}{(1+t)^p} - \frac{(1+t)^{p-1}}{t^p} \right) dt. \tag{6.9.10}$$

The result now follows from (6.9.2).

6.10 Another type of logarithmic integral

Entry **4.251.1** is

$$\int_0^\infty \frac{x^{a-1}\,\ln x}{x+b}\,dx = \frac{\pi\, b^{a-1}}{\sin \pi a}\left(\ln b - \pi \cot \pi a\right). \tag{6.10.1}$$

To check this evaluation we first scale by $x = bt$ and obtain

$$\int_0^\infty \frac{x^{a-1}\,\ln x}{x+b}\,dx = b^{a-1}\ln b \int_0^\infty \frac{t^{a-1}\,dt}{1+t} + b^{a-1}\int_0^\infty \frac{t^{a-1}\,\ln t}{1+t}\,dt. \tag{6.10.2}$$

The first integral is simply

$$\int_0^\infty \frac{t^{a-1}\,dt}{1+t} = B(a, 1-a) = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}. \tag{6.10.3}$$

The second one is evaluated as

$$\int_0^\infty \frac{t^{a-1}\,\ln t}{1+t}\,dt = -\pi^2 \frac{\cos \pi a}{\sin^2(\pi a)} \tag{6.10.4}$$

by differentiating (6.4.1) with respect to a. The evaluation follows from here.

6.11 A hyperbolic looking integral

The evaluation of **3.457.3**:

$$\int_{-\infty}^\infty \frac{x\,dx}{(a^2e^x + e^{-x})^\mu} = -\frac{1}{2a^\mu} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \ln a, \tag{6.11.1}$$

is done as follows: write

$$I = \frac{1}{a^\mu}\int_{-\infty}^\infty \frac{x\,dx}{(ae^x + a^{-1}e^{-x})^\mu} \tag{6.11.2}$$

and let $t = ae^x$ to produce

$$I = \frac{1}{a^\mu} \int_0^\infty \frac{t^{\mu-1} (\ln t - \ln a)\, dt}{(1+t^2)^\mu}. \tag{6.11.3}$$

The change of variables $s = t^2$ yields

$$I = \frac{1}{4a^\mu} \int_0^\infty \frac{s^{\mu/2-1} \ln s\, ds}{(1+s)^\mu} - \frac{\ln a}{2a^\mu} \int_0^\infty \frac{s^{\mu/2-1}\, ds}{(1+s)^\mu}. \tag{6.11.4}$$

The first integral vanishes. This follows directly from the change $s \mapsto 1/s$. The second integral is the beta value indicated in the formula.

In particular, the value $a = 1$ yields

$$\int_{-\infty}^\infty \frac{x\, dx}{\cosh^\mu x} = 0. \tag{6.11.5}$$

Differentiating with respect to μ produces

$$\int_{-\infty}^\infty x \ln \cosh x\, dx = 0, \tag{6.11.6}$$

that appears as **4.321.1** in [40].

Note. `Mathematica` gives

$$\begin{aligned} \int_{-\infty}^\infty \frac{x\, dx}{(a^2 e^x + e^{-x})^\mu} &= \frac{a^{-2\mu}}{\mu^2} {}_3F_2 \left(\left. \begin{matrix} \frac{\mu}{2}, \frac{\mu}{2}, \mu \\ 1+\frac{\mu}{2}, 1+\frac{\mu}{2} \end{matrix} \right| -\frac{1}{a^2} \right) \\ &- \frac{1}{\mu^2} {}_3F_2 \left(\left. \begin{matrix} \frac{\mu}{2}, \frac{\mu}{2}, \mu \\ 1+\frac{\mu}{2}, 1+\frac{\mu}{2} \end{matrix} \right| -a^2 \right). \end{aligned}$$

Note. The only entries in this chapter that cannot be evaluated by `Mathematica` are **4.273** in (6.8.1) and **3.217** in (6.9.1).

Chapter 7

Elementary examples

7.1 Introduction

Elementary mathematics leaves the impression that there is marked difference between the two branches of calculus. *Differentiation* is a subject that is systematic: every evaluation is a consequence of a number of rules and some basic examples. However, *integration* is a mixture of art and science. The successful evaluation of an integral depends on the right approach, the right change of variables or a patient search in a table of integrals. In fact, the theory of *indefinite* integrals of elementary functions is complete [28, 29]. Risch's algorithm determines whether a given function has an antiderivative within a given class of functions. However, the theory of *definite* integrals is far from complete and there is no general theory available. The level of complexity in the evaluation of a definite integral is hard to predict as can be seen in

$$\int_0^\infty e^{-x}\,dx = 1, \quad \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}, \text{ and } \int_0^\infty e^{-x^3}\,dx = \Gamma\left(\tfrac{4}{3}\right). \quad (7.1.1)$$

The first integrand has an elementary primitive, the second one is the classical Gaussian integral, and the evaluation of the third requires Euler's *gamma* *function* defined by

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\, dx. \tag{7.1.2}$$

The table of integrals [40] contains a large variety of integrals. This paper continues the work initiated in [3, 63, 64, 65, 66, 67] with the objective of providing proofs and context of *all the formulas* in the table [40]. Some of them are truly elementary. In this paper we present a derivation of a small number of them.

7.2 A simple example

The first evaluation considered here is that of **3.249.6**:

$$\int_0^1 (1-\sqrt{x})^{p-1}\, dx = \frac{2}{p(p+1)}. \tag{7.2.1}$$

The evaluation is completely elementary. The change of variables $y = 1-\sqrt{x}$ produces

$$I = -2\int_0^1 y^p\, dy + 2\int_0^1 y^{p-1}\, dy, \tag{7.2.2}$$

and each of these integrals can be evaluated directly to produce the result.

This example can be generalized to consider

$$I(a) = \int_0^1 (1-x^a)^{p-1}\, dx. \tag{7.2.3}$$

The change of variables $t = x^a$ produces

$$I(a) = a^{-1}\int_0^1 t^{1/a-1}(1-t)^{p-1}\, dt. \tag{7.2.4}$$

The integral (7.2.4) appears as **3.251.1** and it can be evaluated in terms of the *beta function*

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx, \tag{7.2.5}$$

as

$$I(a) = a^{-1} B\left(p, a^{-1}\right). \tag{7.2.6}$$

The reader will find in [67] details about this evaluation.

A further generalization is provided in the next lemma.

Lemma 7.2.1. *Let $n \in \mathbb{N}$, a, b, $c \in \mathbb{R}$ with $bc > 0$. Define $u = ac - b^2$. Then*

$$\begin{aligned}\int_0^1 x^{n/2} \frac{a + b\sqrt{x}}{b + c\sqrt{x}}\, dx &= \frac{2u(-b)^{n+1}}{c^{n+3}} \ln\left(1 + \frac{c}{b}\right) + \\ &\quad \frac{2u}{c^2} \sum_{j=0}^{n} \frac{(-1)^j}{n - j + 1} \left(\frac{b}{c}\right)^j + \frac{2b}{(n+2)c}.\end{aligned}$$

Proof. Substitute $y = b + c\sqrt{x}$ and expand the new term $(y - b)^n$. □

7.3 A generalization of an algebraic example

The evaluation

$$\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)\sqrt{4 + 3x^2}} = \frac{\pi}{3} \tag{7.3.1}$$

appears as **3.248.4** in [40]. We consider here the generalization

$$q(a, b) = \int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)\sqrt{b + ax^2}}. \tag{7.3.2}$$

We assume that a, $b > 0$.

The change of variables $x = \sqrt{b}t/\sqrt{a}$ yields

$$q(a, b) = 2\sqrt{a} \int_0^{\infty} \frac{dt}{(a + bt^2)\sqrt{1 + t^2}} \tag{7.3.3}$$

where we have used the symmetry of the integrand to write it over $[0, \infty)$. The standard trigonometric change of variables $t = \tan\varphi$ produces

$$q(a, b) = 2\sqrt{a} \int_0^{\pi/2} \frac{\cos\varphi\, d\varphi}{a\cos^2\varphi + b\sin^2\varphi}. \tag{7.3.4}$$

Finally, $u = \sin\varphi$, produces

$$q(a, b) = 2\sqrt{a} \int_0^1 \frac{du}{a + (b - a)u^2}. \tag{7.3.5}$$

The evaluation of this integral is divided into three cases:

Case 1. $a = b$. Then we simply get $q(a, b) = 2/\sqrt{a}$.

Case 2. $a < b$. The change of variables $s = u\sqrt{b-a}/\sqrt{a}$ produces $(b-a)u^2 = s^2 a$, so that

$$q(a,b) = \frac{2}{\sqrt{b-a}} \int_0^c \frac{ds}{1+s^2} = \frac{2}{\sqrt{b-a}} \tan^{-1} c, \tag{7.3.6}$$

with $c = \sqrt{b-a}/\sqrt{a}$.

Case 3. $a > b$. Then we write

$$q(a,b) = 2\sqrt{a} \int_0^1 \frac{du}{a-(a-b)u^2}. \tag{7.3.7}$$

The change of variables $u = \sqrt{a}\, s/\sqrt{a-b}$ yields

$$q(a,b) = \frac{2}{\sqrt{a-b}} \int_0^c \frac{ds}{1-s^2}, \tag{7.3.8}$$

where $c = \sqrt{a-b}/\sqrt{a}$. The partial fraction decomposition

$$\frac{1}{1-s^2} = \frac{1}{2}\left(\frac{1}{1+s} + \frac{1}{1-s} \right) \tag{7.3.9}$$

now produces

$$q(a,b) = \frac{1}{\sqrt{a-b}} \ln\left(\frac{\sqrt{a}+\sqrt{a-b}}{\sqrt{a}-\sqrt{a-b}} \right). \tag{7.3.10}$$

The special case in **3.248.4** corresponds to $a = 3$ and $b = 4$. The value of the integral is $2\tan^{-1}(1/\sqrt{3}) = \frac{\pi}{3}$, as claimed. This generalization has been included as **3.248.6** in the latest edition of [40].

We now consider a generalization of this integral. The proof requires several elementary steps, given first for the convenience of the reader.

Let a, $b \in \mathbb{R}$ with $a < b$ and $n \in \mathbb{N}$. Introduce the notation

$$I = I_n(a,b) := \int_0^\infty \frac{dt}{(a+bt^2)^n\sqrt{1+t^2}}. \tag{7.3.11}$$

Then we have:

Lemma 7.3.1. *The integral $I_n(a,b)$ is given by*

$$I_n(a,b) = \int_0^1 \frac{(1-v^2)^{n-1}\,dv}{(a+\alpha v^2)^n}, \tag{7.3.12}$$

with $\alpha = b - a$.

Proof. The change of variables $v = t/\sqrt{1+t^2}$ gives the result. $\square$

The identity

$$(1-v^2)^{n-1} = (1-v^2)^n + (1-v^2)^{n-1}\left\{\frac{1}{\alpha}(a+\alpha v^2) - \frac{a}{\alpha}\right\} \qquad (7.3.13)$$

produces

$$I_n(a,b) = \frac{\alpha}{b}\int_0^1 \frac{(1-v^2)^n}{(a+\alpha v^2)^n}\,dv + \frac{1}{b}\int_0^1 \frac{(1-v^2)^{n-1}}{(a+\alpha v^2)^{n-1}}\,dv. \qquad (7.3.14)$$

The evaluation of these integrals requires an intermediate result, that is also of independent interest.

Lemma 7.3.2. *Assume $z \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$. Then*

$$\int_0^1 \frac{dx}{(1+z^2x^2)^{n+1}} = \frac{1}{2^{2n}}\binom{2n}{n}\left(\frac{\tan^{-1} z}{z} + \sum_{k=1}^{n} \frac{2^{2k}}{2k\binom{2k}{k}}\frac{1}{(1+z^2)^k}\right). \qquad (7.3.15)$$

Proof. Define

$$F_n(z) := \int_0^1 \frac{dx}{(1+z^2x^2)^{n+1}} = \frac{1}{z}\int_0^z \frac{dy}{(1+y^2)^{n+1}}. \qquad (7.3.16)$$

Take derivatives with respect to z on both sides of (7.3.16). The outcome is a system of differential-difference equations

$$\begin{aligned} \frac{dF_n(z)}{dz} &= \frac{2(n+1)}{z}F_{n+1}(z) - \frac{2(n+1)}{z}F_n(z) \\ \frac{dF_n}{dz} &= -\frac{1}{z}F_n(z) + \frac{1}{z(1+z^2)^{n+1}}. \end{aligned} \qquad (7.3.17)$$

Solving for a purely recursive relation we obtain (after re-indexing $n \mapsto n-1$):

$$2nF_n(z) = (2n-1)F_{n-1}(z) + \frac{1}{(1+z^2)^n}, \qquad (7.3.18)$$

with the initial condition $F_0(z) = \frac{1}{z}\tan^{-1} z$. This recursion is solved using the procedure described in Lemma 2.7 of [3]. This produces the stated expression for $F_n(z)$. $\square$

The next required evaluation is that of the powers of a simple rational function.

Lemma 7.3.3. *Let a, b, c, d be real numbers such that $cd > 0$. Then*

$$\int_0^1 \left(\frac{ax^2+b}{cx^2+d} \right)^n dx = \\ \frac{a^n}{c^n} + \frac{4a^n}{c^n} \sqrt{\frac{d}{c}} \tan^{-1} \sqrt{c/d} \sum_{k=1}^n \left(\frac{bc-ad}{4ad} \right)^k \binom{n}{k} \binom{2k-2}{k-1} \\ + \frac{4a^n}{c^n} \sum_{k=1}^n \left(\frac{bc-ad}{4ad} \right)^k \binom{n}{k} \binom{2k-2}{k-1} \sum_{j=1}^{k-1} \frac{2^{2j}}{2j\binom{2j}{j}} \frac{d^j}{(c+d)^j}.$$

Proof. Start with the partial fraction expansion

$$G(x) := \frac{ax^2+b}{cx^2+d} = \frac{a}{c} + \frac{bc-ad}{cd} \frac{1}{cx^2/d+1}, \tag{7.3.19}$$

and expand $G(x)^n$ by the binomial theorem. The result follows by using Lemma 7.3.2. $\square$

The next result follows by combining the statements of the previous three lemmas.

Theorem 7.3.1. *Let a, $b \in \mathbb{R}^+$ with $a < b$. Then*

$$\begin{aligned} I_{n+1}(a,b) &:= \int_0^\infty \frac{dt}{(a+bt^2)^{n+1}\sqrt{1+t^2}} \\ &= \frac{1}{a(a-b)^n} \sum_{j=0}^n \binom{n}{j} \left(\frac{-b}{4a} \right)^j \binom{2j}{j} \times \\ &\qquad \left(\frac{\tan^{-1}\sqrt{b/a-1}}{\sqrt{b/a-1}} + \sum_{k=1}^j \frac{2^{2k}}{2k\binom{2k}{k}} \left(\frac{a}{b} \right)^k \right). \end{aligned}$$

7.4 Some integrals involving the exponential function

In [40] we find **3.310**:

$$\int_0^\infty e^{-px}\, dx = \frac{1}{p}, \text{ for } p > 0, \tag{7.4.1}$$

that is probably the most elementary evaluation in the table. The example **3.311.1**

$$\int_0^\infty \frac{dx}{1+e^{px}} = \frac{\ln 2}{p}, \tag{7.4.2}$$

can also be evaluated in elementary terms. Observe first that the change of variables $t = px$, shows that (7.4.2) is equivalent to the case $p = 1$:

$$\int_0^\infty \frac{dx}{1+e^x} = \ln 2. \tag{7.4.3}$$

This can be evaluated by the change of variables $u = e^x$ that yields

$$I = \int_1^\infty \frac{du}{u(1+u)}, \tag{7.4.4}$$

and this can be integrated by partial fractions to produce the result. The parameter in (7.4.2) is what we call *fake*, in the sense that the corresponding integral is independent of it. The advantage of such a parameter is that it provides flexibility to a formula: differentiating (7.4.2) with respect to p produces

$$\int_0^\infty \frac{xe^{px}\,dx}{(1+e^{px})^2} = \frac{\ln 2}{p^2}, \tag{7.4.5}$$

$$\int_0^\infty \frac{x^2e^{px}(e^{px}-1)\,dx}{(1+e^{px})^3} = \frac{2\ln 2}{p^3}, \tag{7.4.6}$$

$$\int_0^\infty \frac{x^3e^{px}(e^{2px}-4e^{px}+1)\,dx}{(1+e^{px})^4} = \frac{6\ln 2}{p^4}. \tag{7.4.7}$$

The general integral formula is obtained by differentiating (7.4.2) n-times with respect to p to produce

$$\int_0^\infty \left(\frac{\partial}{\partial p}\right)^n \frac{dx}{1+e^{px}} = (-1)^n \frac{n!}{p^{n+1}} \ln 2. \tag{7.4.8}$$

The pattern of the integrand is clear:

$$\left(\frac{\partial}{\partial p}\right)^n \frac{1}{1+e^{px}} = \frac{(-1)^n x^n e^{px}}{(1+e^{px})^{n+1}} P_n(e^{px}), \tag{7.4.9}$$

where P_n is a polynomial of degree $n-1$. It follows that

$$\int_0^\infty \frac{x^n e^{px} P_n(e^{px})\,dx}{(1+e^{px})^{n+1}} = \frac{n!\,\ln 2}{p^{n+1}}. \tag{7.4.10}$$

The change of variables $t = px$ shows that p is a fake parameter. The integral is equivalent to

$$\int_0^\infty \frac{x^n e^x P_n(e^x)\,dx}{(1+e^x)^{n+1}} = n!\,\ln 2. \tag{7.4.11}$$

The first few polynomials in the sequence are given by

$$\begin{aligned} P_1(u) &= 1, \\ P_2(u) &= u-1, \\ P_3(u) &= u^2-4u+1, \\ P_4(u) &= u^3-11u^2+11u-1. \end{aligned} \tag{7.4.12}$$

Proposition 7.4.1. *The polynomials $P_n(u)$ satisfy the recurrence*

$$P_{n+1}(u) = (nu-1)P_n(u) - u(1+u)\frac{d}{du}P_n(u). \tag{7.4.13}$$

Proof. The result follows by expanding the relation

$$\frac{(-1)^{n+1}x^{n+1}e^{px}P_{n+1}(e^{px})}{(1+e^{px})^{n+2}} = \frac{\partial}{\partial p}\left(\frac{(-1)^n x^n e^{px} P_n(e^{px})}{(1+e^{px})^{n+1}}\right). \tag{7.4.14}$$

□

Examining the first few values, we observe that

$$Q_n(u) := (-1)^n P_n(-u) \tag{7.4.15}$$

is a polynomial with positive coefficients. This follows directly from the recurrence

$$Q_{n+1}(u) = (nu+1)Q_n(u) + u(1-u)\frac{d}{du}Q_n(u). \tag{7.4.16}$$

This comes directly from (7.4.13). The first few values are

$$\begin{aligned} Q_1(u) &= 1, \\ Q_2(u) &= u+1, \\ Q_3(u) &= u^2+4u+1, \\ Q_4(u) &= u^3+11u^2+11u+1. \end{aligned} \tag{7.4.17}$$

Writing

$$Q_n(u) = \sum_{j=0}^{n-1} E_{j,n}u^j, \tag{7.4.18}$$

the reader will easily verify the recurrence

$$\begin{aligned} E_{0,n+1} &= E_{0,n} \\ E_{j,n+1} &= (n-j+1)E_{j-1,n} + (j+1)E_{j,n} \\ E_{n,n+1} &= E_{n,n}. \end{aligned} \tag{7.4.19}$$

The numbers $E_{j,n}$ are called *Eulerian numbers*. They appear in many situations. For example, they provide the coefficients in the series

$$\sum_{k=1}^{\infty} k^j x^k = \frac{x}{(1-x)^{j+1}} \sum_{n=0}^{m-1} E_{j,n}x^n \tag{7.4.20}$$

and satisfy the simpler recurrence

$$E_{j,n} = nE_{j-1,n} + jE_{j,n-1}, \tag{7.4.21}$$

that can be derived from (7.4.19). These numbers have a combinatorial interpretation: they count the number of permutations of $\{1, 2, \ldots, n\}$ having j permutation ascents. The explicit formula

$$E_{j,n} = \sum_{k=0}^{j+1} (-1)^k \binom{n+1}{k} (j+1-k)^n, \tag{7.4.22}$$

can be checked from the recurrences. The reader will find more information about these numbers in [41].

7.5 A simple change of variables

The table [40] contains the example **3.195**:

$$\int_0^\infty \frac{(1+x)^{p-1}}{(x+a)^{p+1}}\, dx = \frac{1-a^{-p}}{p(a-1)}. \tag{7.5.1}$$

One must include the restrictions $a > 0$, $a \neq 1$, $p \neq 0$. The evaluation is elementary: let

$$u = \frac{1+x}{x+a}, \tag{7.5.2}$$

to obtain

$$I = \frac{1}{a-1} \int_{1/a}^1 u^{p-1}\, du, \tag{7.5.3}$$

that gives the stated value. The formula has been supplemented with the value 1 for $a = 1$ and $\ln a/(a-1)$ when $p = 0$ in the last edition of [40].

Differentiating (7.5.1) n times with respect to the parameter p produces

$$\int_0^\infty \frac{(1+x)^{p-1}}{(x+a)^{p+1}} \ln^n \left(\frac{1+x}{x+a} \right) dx = \frac{(-1)^n a^{-p}}{(a-1)p^{n+1}} \left[n!\,(a^p - 1) - \sum_{k=1}^n \frac{n!(p \ln a)^k}{k!} \right].$$

Naturally, the integral above is just

$$\frac{1}{a-1} \int_{1/a}^1 u^{p-1} \ln^n u\, du \tag{7.5.4}$$

and its value can also be obtained by differentiation of (7.5.3).

The next result presents a generalization of (7.5.1):

Lemma 7.5.1. *Let a, b be free parameters and $n \in \mathbb{N}$. Then*

$$\int_0^\infty \frac{(1+x)^b}{(x+a)^{b+n}}\,dx = (a-1)^{-n} \times \left\{ B(n,b) - \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{a^{-b-k}}{b+k} \right\},$$

where $B(n,b)$ is Euler's beta function.

Proof. Use the change of variables $u = (1+x)/(a+x)$, expand in series and then integrate term by term. □

7.6 Another example

The integral in **3.268.1** states that

$$\int_0^1 \left(\frac{1}{1-x} - \frac{px^{p-1}}{1-x^p} \right) dx = \ln p. \tag{7.6.1}$$

To compute it, and to avoid the singularity at $x=1$, we write

$$I = \lim_{\epsilon \to 0} \int_0^{1-\epsilon} \left(\frac{1}{1-x} - \frac{px^{p-1}}{1-x^p} \right) dx. \tag{7.6.2}$$

This evaluates as

$$I = \lim_{\epsilon \to 0} - \ln \epsilon + \ln(1-(1-\epsilon)^p) = \lim_{\epsilon \to 0} \ln \left(\frac{1-(1-\epsilon)^p}{\epsilon} \right) = \ln p. \tag{7.6.3}$$

Note. `Mathematica` gives 0 as the value of this integral.

7.7 Examples of recurrences

Several definite integrals in [40] can be evaluated by producing a recurrence for them. For example, in **3.622.3** we find

$$\int_0^{\pi/4} \tan^{2n} x\,dx = (-1)^n \left(\frac{\pi}{4} - \sum_{j=0}^{n-1} \frac{(-1)^{j-1}}{2j-1} \right). \tag{7.7.1}$$

To check this identity, define

$$I_n = \int_0^{\pi/4} \tan^{2n} x\,dx \tag{7.7.2}$$

and integrate by parts to produce

$$I_n = -I_{n-1} + \frac{1}{2n-1}. \tag{7.7.3}$$

From here we generate the first few values

$$I_0 = \frac{\pi}{4},\ I_1 = -\frac{\pi}{4} + 1,\ I_2 = \frac{\pi}{4} - 1 + \frac{1}{3},\ \text{and } I_3 = -\frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5},$$

and from here one can *guess* the formula (7.7.1). A proof by induction is easy using (7.7.3).

Note. `Mathematica` expresses this integral in the form

$$\int_0^{\pi/4} \tan^{2n} x\,dx = \frac{1}{4}\left[\psi\left(\frac{n}{2}+\frac{3}{4}\right) - \psi\left(\frac{n}{2}+\frac{1}{4}\right)\right]. \tag{7.7.4}$$

A similar argument produces **3.622.4**:

$$\int_0^{\pi/4} \tan^{2n+1} x\,dx = \frac{(-1)^{n+1}}{2}\left(\ln 2 - \sum_{k=1}^{n} \frac{(-1)^k}{k}\right). \tag{7.7.5}$$

To establish this, define

$$J_n = \int_0^{\pi/4} \tan^{2n+1} x\,dx \tag{7.7.6}$$

and integrate by parts to produce

$$J_n = -J_{n-1} + \frac{1}{2n}. \tag{7.7.7}$$

The value

$$J_0 = \int_0^{\pi/4} \tan x\,dx = \frac{\ln 2}{2}, \tag{7.7.8}$$

and the recurrence (7.7.7) yield the formula.

Note. `Mathematica` gives this integral in the form

$$\int_0^{\pi/4} \tan^{2n+1} x\,dx = \frac{1}{4}\left[\text{HarmonicNumber}\left(\frac{n}{2}\right) - \text{HarmonicNumber}\left(\frac{n-1}{2}\right)\right]. \tag{7.7.9}$$

7.8 A truly elementary example

The evaluation of **3.471.1**

$$\int_0^u \exp\left(-\frac{b}{x}\right)\frac{dx}{x^2} = \frac{1}{b}\exp\left(-\frac{b}{u}\right), \tag{7.8.1}$$

is truly elementary: the change of variables $t = -b/x$ gives the result.

7.9 Combination of polynomials and exponentials

Integration by parts produces

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}. \tag{7.9.1}$$

This appears as **2.321.1** in [40]. Introduce the notation

$$I_n(a) := \int x^n e^{ax}\,dx \tag{7.9.2}$$

so that (7.9.1) states that

$$I_n(a) = \frac{1}{a}x^n e^{ax} - \frac{n}{a}I_{n-1}(a). \tag{7.9.3}$$

This recurrence is now used to prove

$$I_n(a) = n!e^{ax}\sum_{k=0}^{n}\frac{(-1)^k x^{n-k}}{(n-k)!\,a^{k+1}} \tag{7.9.4}$$

by an easy inductive argument. This appears as **2.321.2**. The case $1 \leq n \leq 4$ appear as **2.322.1**, **2.322.2**, **2.322.3**, **2.322.4**, respectively.

Note. `Mathematica` expresses this integral as

$$\int x^n e^{ax}\,dx = (-1)^n a^{-n-1}\Gamma(n+1,-ax), \tag{7.9.5}$$

where $\Gamma(n,x)$ is the incomplete gamma function.

Integrating (7.9.4) between 0 and u produces **3.351.1**:

$$\int_0^u x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}} - e^{-au}\sum_{k=0}^{n}\frac{n!}{k!}\frac{u^k}{a^{n-k+1}}. \tag{7.9.6}$$

This sum can be written in terms of the incomplete gamma function. Details will appear in a future publication. Integrating (7.9.4) from u to ∞ produces

$$\int_u^\infty x^n e^{-ax}dx = e^{-au}\sum_{k=0}^{n}\frac{n!}{k!}\frac{u^k}{a^{n-k+1}}. \tag{7.9.7}$$

This appears as **3.351.2**.

The special case $n = 1$ of **3.351.1** appears as **3.351.7**. The cases $n = 2$ and $n = 3$ appear as **3.351.8** and **3.351.9**, respectively.

7.10 A perfect derivative

In section **4.212** we find a list of examples that can be evaluated in terms of the exponential integral function, defined by

$$\mathrm{Ei}(x) := \int_{-\infty}^{x} \frac{e^t\,dt}{t} \tag{7.10.1}$$

for $x < 0$ and by the Cauchy principal value of (7.10.1) for $x > 0$. An exception is **4.212.7**:

$$\int_1^e \frac{\ln x\,dx}{(1+\ln x)^2} = \frac{e}{2} - 1. \tag{7.10.2}$$

This is an elementary integral: the change of variables $t = 1 + \ln x$ yields

$$I = \frac{1}{e}\int_1^2 \frac{(t-1)}{t^2} e^t\,dt \tag{7.10.3}$$

and to evaluate it, observe that

$$\frac{(t-1)}{t^2} e^t = \frac{d}{dt}\frac{e^t}{t}. \tag{7.10.4}$$

The change of variables $t = \ln x$ in (7.10.2) yields

$$\int_0^1 \frac{t\,e^t\,dt}{(1+t)^2} = \frac{e}{2} - 1. \tag{7.10.5}$$

This is **3.353.4** in [40].

The previous evaluation can be generalized by introducing a parameter.

Lemma 7.10.1. *Let $\alpha \in \mathbb{R}$. Then*

$$\int_1^e \frac{\ln x\,dx}{(\alpha+\ln x)^{\alpha+1}} = \frac{e}{(\alpha+1)^\alpha} - \frac{1}{\alpha^\alpha}. \tag{7.10.6}$$

Proof. Substitute $t = \alpha + \ln x$ and use

$$\frac{d}{dt}\frac{e^t}{t^\alpha} = \frac{t-\alpha}{t^{\alpha+1}} e^t. \tag{7.10.7}$$

The case $\alpha = 1$ corresponds to (7.10.2). □

7.11 Integrals involving quadratic polynomials

There are several evaluation in [40] that involve quadratic polynomials. We assume, for reasons of convergence, that the discriminant $d = b^2 - ac$ is strictly negative.

We start with

$$\int_0^\infty \frac{dx}{ax^2+2bx+c} = \frac{1}{\sqrt{ac-b^2}} \cot^{-1}\left(\frac{b}{\sqrt{ac-b^2}} \right). \tag{7.11.1}$$

This is evaluated by completing the square and a simple trigonometric substitution:

$$\begin{aligned} \int_0^\infty \frac{dx}{ax^2+2bx+c} &= \frac{1}{a}\int_{b/a}^\infty \frac{du}{u^2-d/a^2} \\ &= \frac{1}{\sqrt{-d}}\int_{b/\sqrt{-d}}^\infty \frac{dv}{v^2+1}. \end{aligned}$$

Differentiating (7.11.1) with respect to c produces **3.252.1**:

$$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial c^{n-1}}\left[\frac{\cot^{-1}(b/\sqrt{ac-b^2})}{\sqrt{ac-b^2}} \right]. \tag{7.11.2}$$

We now produce a closed-from expression for this integral.

Lemma 7.11.1. *Let $n \in \mathbb{N}$ and $u := 4(ac-b^2)/ac$. Assume $cu > 0$. Then we have the explicit evaluation*

$$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^n} = \frac{2b}{a(cu)^n}\binom{2n-2}{n-1} \times \left\{ \frac{\sqrt{acu}}{b}\cot^{-1}\left(\frac{2b}{\sqrt{acu}}\right) - \sum_{j=1}^{n-1}\frac{u^j}{j\binom{2j}{j}} \right\}.$$

Proof. The case $n=1$ was described above:

$$h(a,b,c) := \int_0^\infty \frac{dx}{ax^2+2bx+c} = \frac{1}{\sqrt{ac-b^2}}\cot^{-1}\left(\frac{1}{\sqrt{ac-b^2}}\right). \tag{7.11.3}$$

Now observe that $h(a^2, abc, b^2) = h(1,b,1)/ac$. Now change the parameters sequentially as $a \mapsto a^2$; $c \mapsto c^2$; $b \mapsto abc$. In the new format, both sides satisfy the differential-difference equation

$$-2nc(1-b^2)f_{n+1} = \frac{df_n}{dc} + \frac{b}{ac^{2n}}. \tag{7.11.4}$$

The result is obtained by reversing the transformations of paramaters indicated above. □

Corollary 7.11.1. *Using the notations of Lemma 7.11.1 we have*

$$\sum_{j=1}^{\infty} \frac{u^j}{j\binom{2j}{j}} = \frac{\sqrt{acu}}{b} \cot^{-1}\left(\frac{2b}{\sqrt{acu}}\right). \tag{7.11.5}$$

The integral **3.252.2**

$$\int_{-\infty}^{\infty} \frac{dx}{(ax^2+2bx+c)^n} = \frac{(2n-3)!!\pi\, a^{n-1}}{(2n-2)!!\,(ac-b^2)^{n-1/2}} \tag{7.11.6}$$

reduces via $u = a(x+b/a)/\sqrt{ac-b^2}$ to Wallis' integral

$$\int_0^{\infty} \frac{du}{(u^2+1)^n} = \frac{(2n-3)!!}{(2n-2)!!}\,\frac{\pi}{2}, \tag{7.11.7}$$

that appears as **3.249.1**. The reader will find in [22] proofs of Wallis' integral. Observe that the evaluation of **3.252.2** is much simpler than the corresponding half-line example presented in Lemma 7.11.1.

The last example of this type is **3.252.3**:

$$\int_0^{\infty} \frac{dx}{(ax^2+2bx+c)^{n+3/2}} = \frac{(-2)^n}{(2n+1)!!}\frac{\partial^n}{\partial c^n}\left(\frac{1}{\sqrt{c}\,(\sqrt{ac}+b)}\right).$$

A simple trigonometric substitution gives the case $n=0$:

$$\begin{aligned} \int_0^{\infty} \frac{dx}{(ax^2+2bx+c)^{3/2}} &= \frac{\sqrt{a}}{ac-b^2}\int_{b/\sqrt{-d}}^{\infty} \frac{du}{(u^2+1)^{3/2}} \\ &= \frac{1}{\sqrt{c}\,(\sqrt{ac}+b)}. \end{aligned}$$

The general case follows by differentiating with respect to c and observing that

$$\left(\frac{\partial}{\partial c}\right)^j = (-1)^j\frac{(2j+1)!!}{2^j}(ax^2+bx+c)^{-3/2-j}.$$

We now provide a closed-form expression for (7.11.8).

Theorem 7.11.1. *Let a, b, $c \in \mathbb{R}$ and $n \in \mathbb{N}$. Define $u = (ac - b^2)/4ac$ and assume $cu > 0$. Then*

$$\int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^{n+3/2}} = \frac{(cu)^{-n}}{\sqrt{c}\binom{2n}{n}(2n+1)} \left(\frac{1}{\sqrt{ac} + b} - \frac{b}{ac - b^2} \sum_{j=1}^{n} \binom{2j}{j} u^j \right).$$

Proof. Change parameters sequentially as $a \mapsto a^2$; $c \mapsto c^2$; $b \mapsto abc$. Then, in the new format both sides satisfy the differential-difference equation

$$-(2N(1 - b^2)c) f_{N+1} = \frac{df_N}{dc} - \frac{b}{ac^{2N}}, \tag{7.11.8}$$

where $N = n + \frac{3}{2}$. ∎

7.12 An elementary combination of exponentials and rational functions

The table [40] contains two integrals belonging to the family

$$T_j := \int_0^\infty e^{-px}(e^{-x} - 1)^n \frac{dx}{x^j}. \tag{7.12.1}$$

Indeed **3.411.19** gives T_1:

$$\int_0^\infty e^{-px}(e^{-x} - 1)^n \frac{dx}{x} = -\sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(p + n - k), \tag{7.12.2}$$

and **3.411.20** gives T_2:

$$\int_0^\infty e^{-px}(e^{-x} - 1)^n \frac{dx}{x^2} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (p + n - k) \ln(p + n - k), \tag{7.12.3}$$

The next result presents an explicit evaluation of T_j.

Proposition 7.12.1. *Let p be a free parameter, and let n, $j \in \mathbb{N}$ with $n + p > 0$. Then*

$$\int_0^\infty e^{-px}(e^{-x} - 1)^n \frac{dx}{x^j} = \frac{(-1)^j}{(j-1)!} \sum_{k=0}^{n} (-1)^k (p+n-k)^{j-1} \ln(p+n-k). \tag{7.12.4}$$

Proof. Start with the observation that

$$T_j = -\int T_{j-1}(p)dp + C. \tag{7.12.5}$$

Therefore we need to describe the iterative integrals $f_j(p) = \int f_{j-1}(p)\,dp$,with $f_0(p) = \ln(p+\alpha)$. This can be found in page 82 of [22] as

$$f_j(p) = \frac{1}{j!}(p+\alpha)^j \ln(p+\alpha) - \frac{H_j}{j!}(p+\alpha)^j + C, \tag{7.12.6}$$

with $\alpha = p + n - k$ and $H_j = 1 + \frac{1}{2} + \cdots + \frac{1}{j}$ is the harmonic number. To build back the functions T_j employ the fact that, for any polynomial $Q(n,k)$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} Q(n,k) \equiv 0. \tag{7.12.7}$$

Consequently,

$$T_j = C + \frac{(-1)^{j+1}}{j!} \sum_{k=0}^{n} (-1)^k (p+n-k)^j \ln(p+n-k). \tag{7.12.8}$$

The last step is to check that $C = 0$. This follows directly from $T_j \to 0$ as $p \to \infty$. The assertion is now validated. □

7.13 An elementary logarithmic integral

Entry **4.222.1** states that

$$\int_0^\infty \ln\left(\frac{a^2+x^2}{b^2+x^2}\right) dx = (a-b)\pi. \tag{7.13.1}$$

In order to establish this, we consider the finite integral

$$I(m) := \int_0^m \ln\left(\frac{a^2+x^2}{b^2+x^2}\right) dx \tag{7.13.2}$$

and then let $m \to \infty$.

Integration by parts gives

$$\begin{aligned} \int_0^m \ln(a^2+x^2)\,dx &= m\ln(m^2+a^2) - 2\int_0^m \frac{x^2\,dx}{a^2+x^2} \\ &= m\ln(m^2+a^2) - 2m + 2a^2 \int_0^m \frac{dx}{a^2+x^2} \\ &= m\ln(m^2+a^2) - 2m + 2a\tan^{-1}\left(\frac{m}{a}\right). \end{aligned}$$

Therefore

$$I(m) = m \ln \left(\frac{m^2 + a^2}{m^2 + b^2} \right) + 2a \tan^{-1} \left(\frac{m}{a} \right) - 2b \tan^{-1} \left(\frac{m}{b} \right).$$

The limit of the logarithmic part is zero and the arctangent part gives $(a-b)\pi$ as required.

The generalization

$$\int_0^\infty \ln \left(\frac{a^s + x^s}{b^s + x^s} \right) \, dx = (a-b) \frac{\pi}{\sin(\pi/s)} \tag{7.13.3}$$

can be established by elementary methods provided we assume the value

$$\int_0^\infty \frac{dx}{1 + x^s} = \frac{\pi}{s \, \sin(\pi/s)} \tag{7.13.4}$$

as given. This integral is evaluated in terms of Euler's beta function in [67]. Indeed, integration by parts gives

$$\int_0^y \ln(a^s + x^s) \, dx = y \ln(a^s + y^s) - sy + sa^s \int_0^y \frac{dx}{a^s + y^s}, \tag{7.13.5}$$

and similarly for the b-parameter. Combining these evaluations gives

$$\int_0^y \ln \left(\frac{a^s + x^s}{b^s + x^s} \right) \, dx = y \ln \left(\frac{a^s + y^s}{b^s + y^s} \right) + sa^s \int_0^y \frac{dx}{a^s + x^s} - sb^s \int_0^y \frac{dx}{b^s + x^s}.$$

Upon letting $y \to \infty$, we observe that the logarithmic term vanishes and a scaling reduces the remaining integrals to (7.13.4).

Note. `Mathematica` evaluates all the entries in this chapter, with the exception of **3.252.1** in (7.11.2), **3.252.2** in (7.11.6), **3.252.3** in (7.11.8), and also the entries **3.411.19** in (7.12.2) and **3.411.20** in (7.12.3).

Chapter 8

Combinations of powers, exponentials, and logarithms

8.1 Introduction

The uninitiated reader of the table of integrals by I. S. Gradshteyn and I. M. Ryzhik [40] will surely be puzzled by choice of integrands. In this note we provide an elementary proof of the evaluation **4.353.3**

$$\int_0^1 (ax+n+1)x^n e^{ax} \ln x \, dx = e^a \sum_{k=0}^{n} (-1)^{k-1} \frac{n!}{(n-k)!a^{k+1}} + (-1)^n \frac{n!}{a^{n+1}}. \tag{8.1.1}$$

`Mathematica` evaluates this integral in terms of the *Exponential integral function* as

$$\int_0^1 (ax+n+1)x^n e^{ax} \ln x \, dx = \text{ExpIntegral}(-n,-a) + \frac{(-1)^{n+1}}{a^{n+1}} \Gamma(n+1). \tag{8.1.2}$$

We also consider the integrals

$$q_n := \int_0^1 x^n e^{-x} \ln x \, dx \tag{8.1.3}$$

and the companion family

$$p_n := \int_0^1 x^n e^{-x} \, dx. \tag{8.1.4}$$

The integral q_n corresponds to the case $a=-1$ in (8.1.1). Section 8.3 provides closed-form expressions for p_n and q_n. Section 8.4 considers the generalization

$$P_n(a) = \int_0^1 x^n e^{-ax} \, dx \text{ and } Q_n(a) = \int_0^1 x^n e^{-ax} \ln x \, dx. \tag{8.1.5}$$

The main result of this section is the closed-form expressions

$$P_n(a) := \int_0^1 x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}} \left(1 - e^{-a} \sum_{k=0}^{n} \frac{a^k}{k!} \right), \tag{8.1.6}$$

and

$$Q_n(a) := \int_0^1 x^n e^{-ax} \ln x \, dx = \frac{n!}{a^{n+1}} \left[\sum_{k=1}^{n} \frac{1}{k} \left(1 - e^{-a} \sum_{j=0}^{k-1} \frac{a^j}{j!} \right) + aQ_0(a) \right],$$

where

$$Q_0(a) = \int_0^1 e^{-ax} \ln x\,dx = -\frac{1}{a} \left(\gamma + \ln a + \Gamma(0,a) \right), \tag{8.1.7}$$

and $\Gamma(0,a)$ is the incomplete gamma function defined by

$$\Gamma(a,x) := \int_x^\infty t^{a-1} e^{-t}\,dt. \tag{8.1.8}$$

8.2 The evaluation of 4.353.3

The identity

$$\frac{d}{dx} \left(x^{n+1} e^{ax} \right) = (ax+n+1)x^n e^{ax} \tag{8.2.1}$$

and integration by parts yield

$$\int_0^1 (ax+n+1)x^n e^{ax} \ln x\,dx = -\int_0^1 x^n e^{ax}\,dx. \tag{8.2.2}$$

This last integral appears as **3.351.1** in [40]. We have obtained a closed-form expression for it in [4]. This has been shown in Chapter 7. A new proof is presented in Section 8.4.

A closed form expression for the right hand side of (8.2.2) is obtained from

$$\int_0^1 x^n e^{ax}\,dx = \left(\frac{d}{da} \right)^n \frac{e^a - 1}{a}. \tag{8.2.3}$$

The symbolic evaluation of (8.2.3) for small values of $n \in \mathbb{N}$ suggests the existence of a polynomial $P_n(a)$ such that

$$\int_0^1 x^n e^{ax}\,dx = \frac{(-1)^{n+1}\,n!}{a^{n+1}} + \frac{P_n(a)}{a^{n+1}} e^a. \tag{8.2.4}$$

The next lemma confirms the existence of this polynomial.

Lemma 8.2.1. *The function $P_n(a)$ defined by*

$$P_n(a) = a^{n+1}e^{-a}\left(\left(\frac{d}{da}\right)^n \frac{e^a-1}{a} - \frac{(-1)^{n+1}n!}{a^{n+1}}\right) \tag{8.2.5}$$

is a polynomial of degree n.

Proof. Let $D = \frac{d}{da}$. Then $D^{n+1} = D(D^n)$ produces the recurrence

$$P_{n+1}(a) = aP_n'(a) + (a-n-1)P_n(a). \tag{8.2.6}$$

The initial condition $P_0(a) = 1$ and (8.2.6) show that P_n is a polynomial of degree n. □

Theorem 8.2.1. *The polynomial*

$$Q_n(a) := (-1)^n P_n(-a) \tag{8.2.7}$$

has positive integer coefficients, written as

$$Q_n(a) = \sum_{k=0}^n b_{n,k}a^k. \tag{8.2.8}$$

These coefficients satisfy

$$\begin{aligned} b_{n+1,0} &= (n+1)b_{n,0} \\ b_{n+1,k} &= (n+1-k)b_{n,k} + b_{n,k-1}, \quad 1 \leq k \leq n \\ b_{n+1,n+1} &= b_{n,n}. \end{aligned} \tag{8.2.9}$$

Moreover, the polynomial $Q_n(a)$ is given by

$$Q_n(a) = n! \sum_{k=0}^n \frac{a^k}{k!} \tag{8.2.10}$$

Proof. The recurrence (8.2.6) yields

$$Q_{n+1}(a) = -aQ_n'(a) + (a+n+1)Q_n(a). \tag{8.2.11}$$

The recursion for the coefficients $b_{n,k}$ follows directly from here. Morover, it is clear that $b_{n,n} = 1$ and $b_{n,0} = n!$. A little experimentation suggets that $b_{n,k} = n!/k!$, and this can be established from (8.2.9). □

This proposition amounts to the evaluation of **3.351.1** in [40]:

$$\int_0^u x^n e^{ax}\,dx = \frac{(-1)^{n+1}\,n!}{a^{n+1}} + \frac{e^{au}}{a^{n+1}} \sum_{k=0}^n \frac{n!}{k!}(-1)^{n-k}u^k a^k. \tag{8.2.12}$$

The reader will find a proof of this formula in Chapter 7 or in [4].

8.3 A new family of integrals

In this section we consider the family of integrals

$$q_n := \int_0^1 x^n e^{-x} \ln x \, dx, \tag{8.3.1}$$

and its companion

$$p_n := \int_0^1 x^n e^{-x} \, dx. \tag{8.3.2}$$

Lemma 8.3.1. *The integrals p_n, q_n satisfy the recursion*

$$\begin{aligned} p_{n+1} &= (n+1)p_n - e^{-1} && (8.3.3) \\ q_{n+1} &= (n+1)q_n + p_n. && (8.3.4) \end{aligned}$$

Proof. Integrate by parts. □

The initial conditions are

$$p_0 = 1 - e^{-1} \text{ and } q_0 = \int_0^1 e^{-x} \ln x \, dx = \gamma - \text{Ei}(-1). \tag{8.3.5}$$

Here γ is Euler's constant defined by

$$\gamma := \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n \tag{8.3.6}$$

with integral representation

$$\gamma = \int_0^\infty e^{-x} \ln x \, dx \tag{8.3.7}$$

given as **4.331.1**. The reader will find in [22] a proof of this identity. The second term in (8.3.5) is converted into

$$\int_1^\infty e^{-x} \ln x \, dx = \int_1^\infty \frac{e^{-x}}{x} \, dx \tag{8.3.8}$$

and this last form is identified as Ei(−1), where Ei is the exponential integral defined by

$$\text{Ei}(z) = -\int_{-z}^\infty \frac{e^{-x}}{x} \, dx. \tag{8.3.9}$$

In the current context, the value of Ei(−1) will be simply one of the terms in the initial condition q_0.

We determine first an explicit expression for p_n. The recursion (8.3.3) shows the existence of integers a_n, b_n such that

$$p_n = a_n + b_n e^{-1}, \tag{8.3.10}$$

with $a_0 = 1$, $b_0 = -1$. From (8.3.3) we obtain

$$a_{n+1} + b_{n+1}e^{-1} = (n+1)a_n + \left[(n+1)b_n - 1\right] e^{-1}. \tag{8.3.11}$$

The irrationality of e produce the system

$$a_{n+1} = (n+1)a_n, \text{ with } a_0 = 1, \tag{8.3.12}$$
$$b_{n+1} = (n+1)b_n - 1, \text{ with } b_0 = -1. \tag{8.3.13}$$

The expression $a_n = n!$ follows directly from (8.3.12). To solve (8.3.13), define $B_n := b_n/n!$ and observe that

$$B_{n+1} = B_n - \frac{1}{(n+1)!}, \tag{8.3.14}$$

that telescopes to

$$b_n = -n! \sum_{k=0}^{n} \frac{1}{k!}. \tag{8.3.15}$$

We have shown:

Proposition 8.3.1. *The integral p_n in (8.3.2) is given by*

$$p_n = \int_0^1 x^n e^{-x}\, dx = \frac{n!}{e}\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right). \tag{8.3.16}$$

We now determine a similar closed-form for q_n. The recursion (8.3.4) shows the existence of integers c_n, d_n, f_n such that

$$q_n = c_n + d_n e^{-1} + f_n q_0. \tag{8.3.17}$$

In order to produce a system similar to (8.3.12,8.3.13) we will assume that the constants 1, e^{-1} and $q_0 = -(\gamma + \mathrm{Ei}(-1))$ are linearly independent over $\mathbb{Q}$. Under this assumption (8.3.4) produces

$$c_{n+1} = (n+1)c_n + n!, \tag{8.3.18}$$
$$d_{n+1} = (n+1)c_n - n! \sum_{k=0}^{n} \frac{1}{k!}, \tag{8.3.19}$$
$$f_{n+1} = (n+1)f_n, \tag{8.3.20}$$

with the initial conditions $c_0 = 0$, $d_0 = 0$ and $f_0 = 1$.

The expression $f_n = n!$ follows directly from (8.3.20). To solve (8.3.18) and (8.3.19) we employ the following result established in [3].

Lemma 8.3.2. *Let a_n, b_n and r_n be sequences with a_n, $b_n \neq 0$. Assume that z_n satisfies*

$$a_n z_n = b_n z_{n-1} + r_n, \; n \geq 1 \tag{8.3.21}$$

with initial condition z_0. Then

$$z_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \left(z_0 + \sum_{k=1}^{n} \frac{a_1 a_2 \cdots a_{k-1}}{b_1 b_2 \cdots b_k} r_k \right). \tag{8.3.22}$$

We conclude that

$$c_n = n! \sum_{k=1}^{n} \frac{1}{k}, \tag{8.3.23}$$

and

$$d_n = -n! \sum_{k=1}^{n} \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{j!}. \tag{8.3.24}$$

The expression for c_n shows that they coincide with the Stirling numbers of the first kind: $c_n = |s(n+1,2)|$.

We have established

Proposition 8.3.2. *The integral q_n in (8.3.1) is given by*

$$q_n = \int_0^1 x^n e^{-x} \ln x \, dx = n! \left[\frac{1}{e} \sum_{k=1}^{n} \frac{1}{k} \left(e - \sum_{j=0}^{k-1} \frac{1}{j!} \right) + q_0 \right]. \tag{8.3.25}$$

Example 8.1. The expressions for p_n and q_n provide the evaluation of **4.351.1** in [40]

$$\int_0^1 (1-x) e^{-x} \ln x \, dx = \frac{1-e}{e}, \tag{8.3.26}$$

by identifying the integral as $q_0 - q_1$. The recurrence (8.3.4) shows that

$$q_0 - q_1 = -p_0 = e^{-1} - 1, \tag{8.3.27}$$

as claimed.

Example 8.2. The evaluation of **4.362.1** in [40]

$$\int_0^1 x e^x \ln(1-x) \, dx = \int_0^1 (1-t) e^{1-t} \ln t \, dt \tag{8.3.28}$$

is achieved by observing that this integral is $e(q_0 - q_1) = 1 - e$.

8.4 A parametric family

In this section we consider the evaluation of

$$P_n(a) := \int_0^1 x^n e^{-ax}\,dx \tag{8.4.1}$$

$$Q_n(a) := \int_0^1 x^n e^{-ax} \ln x\,dx. \tag{8.4.2}$$

The integrals q_n considered in Section 8.3 corresponds to the special case: $q_n = Q_n(1)$.

We now establish a recursion for Q_n by differentiating (8.4.2).

Lemma 8.4.1. *The integral $Q_n(a)$ satisfies the relation*

$$Q_{n+1}(a) = -\frac{d}{da} Q_n(a). \tag{8.4.3}$$

To obtain a closed-form expression for $Q_n(a)$ we need to determine the initial condition

$$Q_0(a) = \int_0^1 e^{-ax} \ln x\,dx. \tag{8.4.4}$$

This is expressed in terms of the *incomplete gamma function* defined in **8.350.1** by

$$\Gamma(a,x) := \int_x^\infty t^{a-1} e^{-t}\,dt. \tag{8.4.5}$$

Observe that $\Gamma(a,0) = \Gamma(a)$, the usual gamma function.

Lemma 8.4.2. *The initial condition $Q_0(a)$ is given by*

$$Q_0(a) = \int_0^1 e^{-ax} \ln x\,dx = -\frac{1}{a}\left(\gamma + \ln a + \Gamma(0,a)\right). \tag{8.4.6}$$

Proof. The change of variables $t = ax$ yields

$$Q_0(a) = \frac{1}{a}\int_0^a e^{-t} \ln t\,dt - \frac{\ln a}{a}\left(1 - e^{-a}\right). \tag{8.4.7}$$

Then

$$\int_0^a e^{-t} \ln t\,dt = \int_0^\infty e^{-t} \ln t\,dt - \int_a^\infty e^{-t} \ln t\,dt. \tag{8.4.8}$$

The first integral is

$$\int_0^\infty e^{-t} \ln t\,dt = -\gamma, \tag{8.4.9}$$

that simply reflects the fact that $\gamma = -\Gamma'(1)$. Integrating by parts yields

$$\int_a^\infty e^{-t} \ln t \, dt = e^{-a} \ln a + \Gamma(0,a). \tag{8.4.10}$$

The formula for $Q_0(a)$ is established. $\square$

We now determine a closed-form expression for $P_n(a)$ and $Q_n(a)$ following the procedure employed in Section 8.3.

Lemma 8.4.3. *The integrals P_n and $Q_n(a)$ satisfy the recursion*

$$P_{n+1}(a) = \frac{1}{a}\left((n+1)P_n(a) - e^{-a}\right) \tag{8.4.11}$$

$$Q_{n+1}(a) = \frac{1}{a}\left((n+1)Q_n(a) + P_n(a)\right). \tag{8.4.12}$$

The initial conditions are given by

$$P_0(a) = \frac{1}{a}(1-e^{-a}), \text{ and } Q_0(a) = -\frac{1}{a}(\gamma + \Gamma(0,a) + \ln a). \tag{8.4.13}$$

Proof. Integrate by parts. $\square$

We conclude that we can write

$$P_n(a) = A_n(a) - B_n(a)e^{-a}, \tag{8.4.14}$$

and

$$Q_n(a) = C_n(a) - D_n(a)e^{-a} - E_n(a)(\gamma + \Gamma(0,a) + \ln a). \tag{8.4.15}$$

Lemma 8.4.4. *The recursions (8.4.11) and (8.4.12) imply that*

$$\begin{aligned}
A_{n+1}(a) &= \frac{1}{a}(n+1)A_n(a), \qquad (8.4.16)\\
B_{n+1}(a) &= \frac{1}{a}\left[(n+1)B_n(a) + 1\right],\\
C_{n+1}(a) &= \frac{1}{a}\left[(n+1)C_n(a) + A_n(a)\right],\\
D_{n+1}(a) &= \frac{1}{a}\left[(n+1)D_n(a) + B_n(a)\right],\\
E_{n+1}(a) &= \frac{1}{a}(n+1)E_n(a)
\end{aligned}$$

with initial conditions

$$A_0(a) = B_0(a) = E_0(a) = \frac{1}{a} \text{ and } C_0(a) = D_0(a) = 0. \tag{8.4.17}$$

These recursion can now be solved as in Section 8.3 to produce a closed-form expression for the integrals $P_n(a)$ and $Q_n(a)$. We employ the notation

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \tag{8.4.18}$$

for the harmonic numbers and

$$\text{Exp}_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \tag{8.4.19}$$

for the partial sums of the exponential function.

Theorem 8.4.1. *Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then*

$$P_n(a) := \int_0^1 x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}} \left[1 - e^{-a} Exp_n(a)\right], \tag{8.4.20}$$

and

$$Q_n(a) := \int_0^1 x^n e^{-ax} \ln x\, dx = \frac{n!}{a^{n+1}} \left[H_n - G(a) - e^{-a} \sum_{k=1}^{n} \frac{1}{k} Exp_{k-1}(a)\right],$$

where $G(a) = -aQ_0(a) = \gamma + \Gamma(0, a) + \ln a$.

These expressions provide the evaluations of two integrals in [40].

Example 8.3. Formula **4.351.2** states that

$$\int_0^1 e^{-ax}(-ax^2 + 2x) \ln x\, dx = \frac{1}{a^2} \left[-1 + (1 + a)e^{-a}\right]. \tag{8.4.21}$$

In order to verify this, observe that the stated integral is

$$-a \int_0^1 x^2 e^{-ax} \ln x\, dx + 2 \int_0^1 x e^{-ax} \ln x\, dx = -aQ_2(a) + 2Q_1(a). \tag{8.4.22}$$

The expressions in Theorem 8.4.1 now complete the evaluation.

Example 8.4. Formula **4.353.3** in [40] gives the value of

$$I_n(a) := \int_0^1 (-ax + n + 1) x^n e^{-ax} \ln x\, dx. \tag{8.4.23}$$

Observe that

$$I_n(a) = -aQ_{n+1}(a) + (n + 1)Q_n(a), \tag{8.4.24}$$

and using the recursion (8.4.12) we conclude that $I_n(a) = -P_n(a)$. The expression in Theorem 8.4.1 is precisely what appears in [40]. `Mathematica` gives the value

$$I_n(a) = \frac{1}{a^{n+1}} \left(\Gamma(n + 1, a) - \Gamma(n + 1)\right). \tag{8.4.25}$$

We conclude with the evaluation of a series shown to us by Tewodros Amdeberhan. Expand the exponential term in (8.4.21) and integrate term by term to obtain

$$\sum_{k=0}^{\infty} \frac{(-a)^k}{k!\,(n+1+k)^2} = \frac{n!}{a^{n+1}} \left(-\psi(n+1) + \ln a + \Gamma(0,a) + e^{-a} \sum_{k=0}^{n} \frac{1}{k} \text{Exp}_{k-1}(a) \right). \tag{8.4.26}$$

Here

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{8.4.27}$$

is the *digamma* function defined in **8.360.1** of [40]. the identity

$$\psi(n+1) = H_n - \gamma, \tag{8.4.28}$$

that is a direct consequence of the functional equation $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma'(1) = -\gamma$, was used to transform (8.4.26).

The identity (8.4.26) can be used to provide multiple expressions for the incomplete gamma function, such as

$$\int_a^\infty \frac{e^{-x}}{x}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k a^{n+1+k}}{n!\,k!\,(n+1+k)^2} + \psi(n+1) - \ln a - e^{-a} \sum_{k=1}^{n} \frac{\text{Exp}_{k-1}(a)}{k}, \tag{8.4.29}$$

and the special case for $n = 0$:

$$\int_a^\infty \frac{e^{-x}}{x}\,dx = -\gamma - \ln a + \sum_{k=0}^{\infty} \frac{(-1)^k a^{k+1}}{(k+1)!\,(k+1)}. \tag{8.4.30}$$

Note. `Mathematica` evaluates all the entries of [40] discussed in this chapter.

Chapter 9

Combinations of logarithms, rational and trigonometric functions

9.1 Introduction

The table of integrals [40] contains many examples of the form

$$\int_a^b R_1(x)\,\left(\ln R_2(x)\right)^m\,dx, \tag{9.1.1}$$

where R_1 and R_2 are rational functions, $a,\,b \in \mathbb{R}^+$ and $m \in \mathbb{N}$. For example, **4.231.1** states that

$$\int_0^1 \frac{\ln x\,dx}{1+x} = -\frac{\pi^2}{12}. \tag{9.1.2}$$

This result can be established by the elementary methods described here.

Other examples, such as **4.233.1**

$$\int_0^1 \frac{\ln x\,dx}{1+x+x^2} = \frac{2}{9}\left(\frac{2\pi^2}{3} - \psi'(\tfrac{1}{3})\right), \tag{9.1.3}$$

and **4.261.8**

$$\int_0^1 \ln^2 x\,\frac{1-x}{1-x^6}\,dx = \frac{8\sqrt{3}\pi^3 + 351\zeta(3)}{486}, \tag{9.1.4}$$

require more sophisticated special functions. Here ψ is the *digamma function* defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{9.1.5}$$

and $\zeta(s)$ is the classical *Riemann zeta function*. These results will be described in a future publication.

The integrals discussed here can also be framed in the context of trigonometric functions. For example, the change of variables $x = \tan t$ yields the identity

$$\int_0^1 \frac{\ln x\, dx}{1+x^2} = \int_0^{\pi/4} \ln \tan t\, dt = -G. \tag{9.1.6}$$

Here G is the *Catalan's constant* defined by

$$G := \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^2}. \tag{9.1.7}$$

In this paper we concentrate on integrals of the type (9.1.1) where the logarithm appears to the first power and the poles of the rational function are either real or purely imaginary. The method of partial fractions and scaling of the independent variable show that such integrals are linear combinations of

$$h_{n,1}(b) := \int_0^b \frac{\ln t\, dt}{(1+t)^n}, \tag{9.1.8}$$

and

$$h_{n,2}(b) := \int_0^b \frac{\ln t\, dt}{(1+t^2)^n}. \tag{9.1.9}$$

The function $h_{n,1}$ was evaluated in [64], where it was denoted simply by h. We complete this evaluation in Section 9.4, by identifying a polynomial defined in [64]. The closed-form of $h_{n,1}$ involves the *Stirling numbers* of the first kind. The evaluation of $h_{n,2}$ is discussed in Section 9.6. The value of $h_{n,2}$ involves the *tangent integral*

$$\mathrm{Ti}_2(x) := \int_0^x \frac{\tan^{-1} t}{t}. \tag{9.1.10}$$

The case of integrals with more complicated pole structure will be described in a future publication.

9.2 Some elementary examples

We begin our discussion with an elementary example. Entry **4.291.1** states that

$$\int_0^1 \frac{\ln(1+x)}{x}\, dx = \frac{\pi^2}{12}. \tag{9.2.1}$$

To establish this value we consider first a useful series. The result is expressed in terms of the *Riemann zeta function*

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}. \tag{9.2.2}$$

Lemma 9.2.1. *Let $s > 1$. Then*

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = -\frac{2^{s-1}-1}{2^{s-1}}\zeta(s) \tag{9.2.3}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} = \frac{2^s-1}{2^s}\zeta(s) \tag{9.2.4}$$

Proof. The second sum is

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} = \sum_{k=1}^{\infty} \frac{1}{k^s} - \sum_{k=1}^{\infty} \frac{1}{(2k)^s} = (1-2^{-s})\zeta(s). \tag{9.2.5}$$

To evaluate the first sum, split it into even and odd values of the index k:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = \sum_{k=1}^{\infty} \frac{1}{(2k)^s} - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} \tag{9.2.6}$$

and use the value of the first sum. □

To evaluate (9.2.1) we employ the expansion

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \tag{9.2.7}$$

and integrate term by term we obtain

$$\int_0^1 \frac{\ln(1+x)}{x}\, dx = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}. \tag{9.2.8}$$

The result now follows from the lemma and the classical value $\zeta(2) = \pi^2/6$.

A similar calculation yields **4.291.2**:

$$\int_0^1 \frac{\ln(1-x)}{x} dx = -\frac{\pi^2}{6}. \tag{9.2.9}$$

The change of variables $x = e^{-t}$ produces the evaluation of **4.223.1**

$$\int_0^{\infty} \ln(1+e^{-t})\, dt = \frac{\pi^2}{12} \tag{9.2.10}$$

and **4.223.2**:

$$\int_0^{\infty} \ln(1-e^{-t})\, dt = -\frac{\pi^2}{6}. \tag{9.2.11}$$

9.3 More elementary examples

In [64] we analyze the case in which the rational function R_1 has a single multiple pole and $R_2(x) = x$. There are simple examples where the evaluation can be obtained directly. For instance, formula **4.231.1** states that

$$\int_0^1 \frac{\ln x}{1+x}\, dx = -\frac{\pi^2}{12}. \tag{9.3.1}$$

This can be established by simply expanding the term $1/(1+x)$ as a geometric series and integrate term by term. The same is true for **4.231.2**

$$\int_0^1 \frac{\ln x}{1-x}\, dx = -\frac{\pi^2}{6}. \tag{9.3.2}$$

The evaluation of **4.231.3**

$$\int_0^1 \frac{x\,\ln x}{1-x}\, dx = 1 - \frac{\pi^2}{6}, \tag{9.3.3}$$

and **4.231.4**

$$\int_0^1 \frac{1+x}{1-x}\,\ln x\, dx = 1 - \frac{\pi^2}{3}, \tag{9.3.4}$$

follow directly from (9.3.2). Similar elementary algebraic manipulations produce **4.231.19**

$$\int_0^1 \frac{x\,\ln x}{1+x}\, dx = -1 + \frac{\pi^2}{2}, \tag{9.3.5}$$

and **4.231.20**

$$\int_0^1 \frac{(1-x)\,\ln x}{1+x}\, dx = 1 - \frac{\pi^2}{6}. \tag{9.3.6}$$

The same is true for **4.231.14**

$$\int_0^1 \frac{x\,\ln x}{1+x^2}\, dx = -\frac{\pi^2}{48}, \tag{9.3.7}$$

and **4.231.15**

$$\int_0^1 \frac{x\,\ln x}{1-x^2}\, dx = -\frac{\pi^2}{24}, \tag{9.3.8}$$

via the change of variables $t = x^2$.

The evaluation of **4.231.13**:

$$\int_0^1 \frac{\ln x\, dx}{1-x^2} = -\frac{\pi^2}{8} \tag{9.3.9}$$

is a direct consequence of the partial fraction decomposition

$$\frac{1}{1-x^2} = \frac{1}{2}\frac{1}{1-x} + \frac{1}{2}\frac{1}{1+x} \tag{9.3.10}$$

and the identities (9.3.1) and (9.3.2).

It is often the case that a simple change of variables reduces an integral to one that has previously been evaluated. For example, the change of variables $t = 1 - x^2$ produces

$$\int_0^1 \frac{\ln(1-x^2)}{x}\,dx = \frac{1}{2}\int_0^1 \frac{\ln t\,dt}{1-t}. \tag{9.3.11}$$

The right-hand side is given in (9.3.2) and we obtain the value of **4.295.11**:

$$\int_0^1 \frac{\ln(1-x^2)}{x}\,dx = -\frac{\pi^2}{12}. \tag{9.3.12}$$

9.4 A single multiple pole

The situation for a single multiple pole is more delicate. The pole may be placed at $x = -1$ by scaling. The main result established in [64] is:

Theorem 9.4.1. *Let $n \geq 2$ and $b > 0$. Define*

$$h_{n,1}(b) = \int_0^b \frac{\ln t\,dt}{(1+t)^n} \tag{9.4.1}$$

and introduce the function

$$q_n(b) = (1+b)^{n-1}h_{n,1}(b). \tag{9.4.2}$$

Then

$$q_n(b) = X_n(b)\ln b + Y_n(b)\ln(1+b) + Z_n(b), \tag{9.4.3}$$

where

$$X_n(b) = \frac{(1+b)^{n-1}-1}{n-1}, \quad Y_n(b) = -\frac{(1+b)^{n-1}}{n-1}. \tag{9.4.4}$$

Finally, the function

$$T_n(b) := -\frac{(n-1)!\,Z_n(b)}{b(1+b)}, \tag{9.4.5}$$

satisfies $T_2(b) = 0$ and for $n \geq 1$ it satisfies the recurrence

$$T_{n+2}(b) = n(1+b)T_{n+1}(b) + (n-1)!\left(\frac{(1+b)^n - 1}{b}\right). \tag{9.4.6}$$

It follows that $T_n(b)$ is a polynomial in b of degree $n-3$ with positive integer coefficients.

Note 9.4.1. *The case $n = 1$ is expressed in terms of the* polylogarithm function

$$PolyLog[n, z] := \sum_{k=1}^{\infty} \frac{z^k}{k^n}. \tag{9.4.7}$$

Indeed, we have

$$\int_0^b \frac{\ln x}{1+x}\, dx = \ln b \, \ln(1+b) + PolyLog[2, -b]. \tag{9.4.8}$$

We now identify the polynomial T_n in (**??**). The first few are given by

$$\begin{aligned} T_3(b) &= 1, \\ T_4(b) &= 3b+4, \\ T_5(b) &= 11b^2+27b+18, \\ T_6(b) &= 50b^3+176b^2+216b+96. \end{aligned} \tag{9.4.9}$$

For $n \geq 2$, define

$$A_n(b) = \frac{1}{(n-2)!} T_n(b). \tag{9.4.10}$$

Then (9.4.6) becomes

$$A_{n+2}(b) = (1+b)A_{n+1}(b) + \frac{(1+b)^n - 1}{bn}, \tag{9.4.11}$$

with initial condition $A_2(b) = 0$.

The polynomial A_n is written as

$$A_n(b) = \sum_{j=0}^{n-3} a_{n,j} b^j \tag{9.4.12}$$

The recursion (9.4.11) produces

Lemma 9.4.1. *Let $n \geq 4$. Then the coefficients $a_{n,j}$ satisfy*

$$\begin{aligned} a_{n,0} &= a_{n-1,0} + 1, \\ a_{n,j} &= a_{n-1,j} + a_{n-1,j-1} + \frac{(n-3)!}{(j+1)!\,(n-3-j)!}, \text{ for } 1 \leq j \leq n-4, \\ a_{n,n-3} &= a_{n-1,n-4} + \frac{1}{n-2}. \end{aligned} \tag{9.4.13}$$

The expressions $a_{n,0} = n-2$ and $a_{n,n-3} = H_{n-2}$, the harmonic number, are easy to determine from (9.4.13). We now find closed-form expressions for the remaining coefficients. These involve the *Stirling numbers of the first kind* $s(n,j)$ defined by the expansion

$$(x)_n = \sum_{j=1}^{n} s(n,j)x^j, \tag{9.4.14}$$

where $(x)_n := x(x+1)(x+2)\cdots(x+n-1)$ is the Pochhammer symbol (also called rising factorial). The numbers $s(n,1)$ are given by

$$s(n,1) = (-1)^{n-1}(n-1)!, \tag{9.4.15}$$

and the sequence $s(n,j)$ satisfies the recurrence

$$s(n+1,j) = s(n,j-1) - ns(n,j). \tag{9.4.16}$$

Theorem 9.4.2. *The coefficients $a_{n,j}$ are given by*

$$a_{n,j} = \frac{(-1)^j}{(j+1)!}\binom{n-2}{j+1} s(j+2,2). \tag{9.4.17}$$

Proof. Define

$$b_{n,j} := (-1)^j a_{n,j} \times (j+1)!\binom{n-2}{j+1}^{-1}. \tag{9.4.18}$$

The recurrence (9.4.13) shows that $b_{n,j}$ is independent of n and satisfies

$$b_{n,j+1} = -jb_{n,j} + (-1)^{j+1}j! \tag{9.4.19}$$

and this is (9.4.16) in the special case $j=2$. Formula (9.4.17) has been established. □

We now restate the value of $h_{n,1}$. The index n is increased by 1 and the identity $|s(n,k)| = (-1)^{n+k}s(n,k)$ is used in order to make the formula look cleaner.

Corollary 9.4.1. *Assume $b>0$ and $n \in \mathbb{R}$. Then*

$$\begin{aligned}\int_0^b \frac{\ln t\,dt}{(1+t)^{n+1}} &= \frac{1}{n}\left[1-(1+b)^{-n}\right]\ln b - \frac{1}{n}\ln(1+b) \\ &- \frac{1}{n(1+b)^{n-1}}\sum_{j=1}^{n-1}\frac{1}{j!}\binom{n-1}{j}|s(j+1,2)|b^j.\end{aligned} \tag{9.4.20}$$

The special case $b=1$ provides the evaluation

$$\int_0^1 \frac{\ln t\,dt}{(1+t)^{n+1}} = -\frac{\ln 2}{n} - \frac{1}{n2^{n-1}}\sum_{j=1}^{n-1}\frac{1}{j!}\binom{n-1}{j}|s(j+1,2)|.$$

Elementary changes of variables, starting with $t = \tan^2 \varphi$, convert (9.4.20) into

$$\begin{aligned} \int_a^1 s^{n-1} \ln(1-s)\, ds &= \frac{(1-a^n)}{n^2} \left[n \ln(1-a) - 1 \right] \\ &- \frac{1}{n} \sum_{j=1}^{n-1} \frac{1}{j!} \binom{n-1}{j} |s(j+1,2)| a^{n+1-j} (1-a)^j. \end{aligned} \tag{9.4.21}$$

The special case $a = 1/2$ produces

$$\int_{1/2}^1 s^{n-1} \ln(1-s)\, ds = \frac{1}{n2^{n-1}} \sum_{j=1}^{n-1} \frac{\binom{n-1}{j} |s(j+1,2)|}{j!} - \frac{(2^n-1)}{n^2 2^n} (n \ln 2 + 1),$$

and $a = 0$ gives

$$\int_0^1 s^{n-1} \ln(1-s)\, ds = -\left(\frac{1}{n^2} + \frac{|s(n,2)|}{n!} \right). \tag{9.4.22}$$

9.5 Denominators with complex roots

In this section we consider the simplest example of the type (9.1.1), where the rational function $R_1(x)$ has a complex (non-real) pole. This is

$$G := -\int_0^1 \frac{\ln x}{1+x^2}\, dx.$$

The constant G is called *Catalan's constant* and is given by

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}. \tag{9.5.1}$$

Entry **4.231.12** of [40] states

$$\int_0^1 \frac{\ln x}{1+x^2}\, dx = -G. \tag{9.5.2}$$

To verify (9.5.2) simply expand the integrand to produce

$$\begin{aligned} \int_0^1 \frac{\ln x\, dx}{1+x^2} &= -\sum_{k=0}^{\infty} (-1)^k \int_0^1 x^{2k} \ln x\, dx \\ &= -\sum_{k=0}^{\infty} (-1)^k \int_0^{\infty} t e^{-(2k+1)t}\, dt \\ &= -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \int_0^{\infty} \sigma e^{-\sigma}\, d\sigma. \end{aligned}$$

The integral is evaluated by integration by parts or recognizing its value as $\Gamma(2) = 1$.

Integration by parts gives the alternative form

$$\int_0^1 \frac{\tan^{-1} x}{x}\, dx = G, \tag{9.5.3}$$

that appears as **4.531.1**.

There are many definite integrals in [40] that are related to Catalan's constant. For example, the change of variables $t = 1/x$ yields from (9.5.2), the value

$$\int_1^\infty \frac{\ln t\, dt}{1+t^2} = G. \tag{9.5.4}$$

This is the second part of **4.231.12**. Adding these two expressions for G, we conclude that

$$\int_0^\infty \frac{\ln x\, dx}{1+x^2} = 0. \tag{9.5.5}$$

The change of variables $x = at$ in (9.5.2) yields **4.231.11**:

$$\int_0^a \frac{\ln\ dx}{x^2+a^2} = \frac{\pi\ \ln a}{4a} - \frac{G}{a}. \tag{9.5.6}$$

We now introduce material that will provide a genaralization of (9.5.5) to the case of a multiple pole at i. The integral is expressed in terms of the *polygamma function*

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{9.5.7}$$

Lemma 9.5.1. *Let a, $b \in \mathbb{R}$. Then*

$$\int_0^\infty \frac{\ln t\, dt}{(1+t^2)^b} = \frac{\Gamma(\frac{1}{2})\psi(\frac{1}{2}) - \Gamma(b-\frac{1}{2})\psi(b-\frac{1}{2})}{2\Gamma(b)}. \tag{9.5.8}$$

Proof. Define

$$f(a,b) := \int_0^\infty \frac{t^a\, dt}{(1+t^2)^b}. \tag{9.5.9}$$

Then

$$\frac{d}{da} f(a,b) = \int_0^\infty \frac{t^a\ \ln t}{(1+t^2)^b}\, dt. \tag{9.5.10}$$

In particular,

$$\frac{d}{da} f(a,b)\Big|_{a=0} = \int_0^\infty \frac{\ln t\, dt}{(1+t^2)^b}. \tag{9.5.11}$$

The change of variables $s = t^2$ gives

$$f(a,b) = \frac{1}{2}\int_0^\infty \frac{s^{(a-1)/2}\, ds}{(1+s)^b}. \tag{9.5.12}$$

Now use the integral representation

$$B(u,v) = \int_0^\infty \frac{s^{u-1}\,ds}{(1+s)^{u+v}} \tag{9.5.13}$$

(given in **8.380.3** in [40] and proved in [67]) with $u = (a+1)/2$ and $v = b-(a+1)/2$ to obtain

$$f(a,b) = B\left(\frac{a+1}{2}, b - \frac{a+1}{2}\right) = \frac{\Gamma((a+1)/2)\,\Gamma(b-(a+1)/2)}{\Gamma(b)}. \tag{9.5.14}$$

Therefore

$$\begin{aligned} \int_0^\infty \frac{\ln t\,dt}{(1+t^2)^b} &= \frac{d}{da} f(a,b)\Big|_{a=0} \qquad (9.5.15)\\ &= \frac{1}{2\Gamma(b)}\left(\Gamma'((a+1)/2) - \Gamma'(b-(a+1)/2)\right)\Big|_{a=0}. \end{aligned}$$

Now use the relation $\Gamma'(x) = \psi(x)\Gamma(x)$ to obtain the result. □

The case $b = n \in \mathbb{N}$ requires the value

$$\Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n}}\frac{(2n)!}{n!}, \tag{9.5.16}$$

and

$$\psi(n+\tfrac{1}{2}) = -\gamma + 2\ln 2 - 2\sum_{k=1}^{n} \frac{1}{2k-1} = -\gamma - 2\ln 2 + 2H_{2n} - H_n, \tag{9.5.17}$$

that appears in **8.366.3**. Here H_n is the n-th harmonic number. The reader will find a proof of this evaluation in [22], page 212. A proof of (9.5.16) appears as Exercise 10.1.3 on page 190 of [22].

Corollary 9.5.1. *Let* $n \in \mathbb{N}$. *Then*

$$\int_0^\infty \frac{\ln x\,dx}{(1+x^2)^{n+1}} = -\frac{\pi}{2^{2n+1}}\binom{2n}{n}\sum_{k=1}^{n}\frac{1}{2k-1}. \tag{9.5.18}$$

We now provide a proof of Entry **4.231.7** in [40]:

$$\int_0^\infty \frac{\ln x\,dx}{(a^2+b^2x^2)^n} = \frac{\Gamma(n-\frac{1}{2})\sqrt{\pi}}{4(n-1)!\,a^{2n-1}b}\left(2\ln\left(\frac{a}{2b}\right) - \gamma - \psi(n-\tfrac{1}{2})\right).$$

Taking the factor b^2 out of the parenthesis on the left and letting $c = a/b$ yields the equivalent form

$$\int_0^\infty \frac{\ln x\,dx}{(c^2+x^2)^n} = \frac{\Gamma(n-\frac{1}{2})\sqrt{\pi}}{4(n-1)!}\left(2\ln\left(\frac{c}{2}\right) - \gamma - \psi(n-\tfrac{1}{2})\right).$$

It is more convenient to replace n by $n+1$ to obtain

$$\int_0^\infty \frac{\ln x\, dx}{(c^2+x^2)^{n+1}} = \frac{\Gamma(n+\frac{1}{2})\sqrt{\pi}}{4n!}\left(2\ln\left(\frac{c}{2}\right) - \gamma - \psi(n+\tfrac{1}{2})\right).$$

Using (9.5.16) and (9.5.17) the requested evaluation amounts to

$$\int_0^\infty \frac{\ln x\, dx}{(c^2+x^2)^{n+1}} = \frac{\pi}{(2c)^{2n+1}}\binom{2n}{n}\left(\ln c - \sum_{k=1}^{n}\frac{1}{2k-1}\right). \tag{9.5.19}$$

This can be written as

$$\int_0^\infty \frac{\ln x\, dx}{(c^2+x^2)^{n+1}} = \frac{\pi}{(2c)^{2n+1}}\binom{2n}{n}\left(\ln c - H_n + 2H_{2n}\right). \tag{9.5.20}$$

To establish this, make the change of variables $x = ct$ to obtain

$$\int_0^\infty \frac{\ln x\, dx}{(x^2+c^2)^{n+1}} = \frac{\ln c}{c^{2n+1}}\int_0^\infty \frac{dx}{(t^2+1)^{n+1}} + \frac{1}{c^{2n+1}}\int_0^\infty \frac{\ln t\, dt}{(t^2+1)^{n+1}}.$$

Using Wallis' formula

$$\int_0^\infty \frac{dt}{(1+t^2)^{n+1}} = \frac{\pi}{2^{2n+1}}\binom{2n}{n}, \tag{9.5.21}$$

the required evaluation now follows from Corollary 9.5.1.

The special case $n=0$ yields **4.231.8**:

$$\int_0^\infty \frac{\ln x\, dx}{a^2+b^2x^2} = \frac{\pi}{2ab}\ln\left(\frac{a}{b}\right). \tag{9.5.22}$$

This evaluation also appears as **4.231.9** in the form

$$\int_0^\infty \frac{\ln px\, dx}{q^2+x^2} = \frac{\pi}{2q}\ln pq. \tag{9.5.23}$$

9.6 The case of a single purely imaginary pole

In this section we evaluate the integral

$$h_{n,2}(a,b) := \int_0^b \frac{\ln t\, dt}{(t^2+a^2)^{n+1}}, \tag{9.6.1}$$

for a, $b > 0$ and $n \in \mathbb{N}$. This is the generalization of (9.4.1) to the case in which the integrand has a multiple pole at $t = ia$. The change of variables $t = ax$ yields

$$h_{n,2}(a,b) = a^{-2n-1}g_n(b/a), \tag{9.6.2}$$

where

$$g_n(x) := \int_0^x \frac{\ln t \, dt}{(t^2+1)^{n+1}}. \tag{9.6.3}$$

The goal is to produce an analytic expression for $g_n(x)$.

Note. `Mathematica` gives the expression

$$\begin{aligned} h_{n,2}(a,b) &= -\frac{b}{a^{2(n+1)}} \, {}_3F_2 \left(\begin{matrix} \frac{1}{2} & \frac{1}{2} & n+1 \\ & \frac{3}{2} & \frac{3}{2} \end{matrix} \middle| -\frac{b^2}{a^2} \right) \\ & \quad + \frac{b \ln b}{a^{2(n+1)}} \, {}_2F_1 \left(\begin{matrix} \frac{1}{2} & n+1 \\ & \frac{3}{2} \end{matrix} \middle| -\frac{b^2}{a^2} \right). \end{aligned}$$

We produce first a recurrence for the companion integral

$$f_n(x) := \int_0^x \frac{dt}{(t^2+1)^{n+1}}. \tag{9.6.4}$$

Lemma 9.6.1. *Let $n \in \mathbb{N}$ and $x > 0$. Then*

$$2nf_n(x) = (2n-1)f_{n-1}(x) + \frac{x}{(x^2+1)^n}, \tag{9.6.5}$$

and $f_0(x) = \tan^{-1} x$.

Proof. Integrate by parts. □

The recurrence (9.6.5) is now solved using the following result established in [3].

Lemma 9.6.2. *Let $n \in \mathbb{N}$ and define $\lambda_j = 2^{2j}\binom{2j}{j}^{-1}$. Suppose the sequence z_n satisfy the recurrence $2nz_n - (2n-1)z_{n-1} = r_n$, with r_n given. Then*

$$z_n = \frac{1}{\lambda_n}\left(z_0 + \sum_{k=1}^{n} \frac{\lambda_k \, r_k}{2k} \right). \tag{9.6.6}$$

We conclude with an explicit expression for $f_n(x)$.

Proposition 9.6.1. *Let $n \in \mathbb{N}$. Then*

$$\int_0^x \frac{dt}{(1+t^2)^{n+1}} = \frac{\binom{2n}{n}}{2^{2n}} \left(\tan^{-1} x + \sum_{j=1}^{n} \frac{2^{2j}}{2j\binom{2j}{j}} \frac{x}{(x^2+1)^j} \right). \tag{9.6.7}$$

Note. This expression for f_n appears as **2.148.4** of [40] in the alternative form

$$\begin{aligned} \int_0^x \frac{dt}{(1+t^2)^{n+1}} &= \frac{x}{2n+1} \sum_{k=1}^{n} \frac{(2n+1)!!}{(2n-2k+1)!!} \frac{(n-k)!}{2^k n!} \frac{1}{(1+x^2)^{n+1-k}} \\ &+ \frac{(2n-1)!!}{2^n n!} \tan^{-1} x. \end{aligned} \tag{9.6.8}$$

`Mathematica` gives the expression

$$\int_0^x \frac{dt}{(1+t^2)^{n+1}} = x \ {}_2F_1\left(\begin{matrix} \frac{1}{2} & n+1 \\ & \frac{3}{2} \end{matrix} \middle| -x^2 \right). \tag{9.6.9}$$

We now produce a recurrence for the integral $g_n(x)$.

Lemma 9.6.3. *Let $n \in \mathbb{N}$. Then the integrals $g_n(x)$ satisfy*

$$2ng_n(x) - (2n-1)g_{n-1}(x) = 2n \ln x f_n(x) - [(2n-1)\ln x + 1] f_{n-1}(x). \tag{9.6.10}$$

Proof. Integration by parts yields

$$g_n(x) = f_n(x) \ln x - \int_0^x f_n(t) \frac{dt}{t}. \tag{9.6.11}$$

From the recurrence (9.6.5) we obtain

$$2n \int_0^x f_n(t) \frac{dt}{t} - (2n-1) \int_0^x f_{n-1}(t) \frac{dt}{t} = f_{n-1}(x). \tag{9.6.12}$$

Now replace the integral term from (9.6.11) to obtain the result. □

In order to produce a closed-form expression for $g_n(x)$ using Lemma 9.6.2, we need the initial condition

$$g_0(x) = \int_0^x \frac{\ln t \, dt}{1+t^2}. \tag{9.6.13}$$

Lemma 9.6.4. *The function $g_n(x)$ is given by*

$$g_n(x) = \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{k} \frac{x^{2k+1}}{2k+1} \left(\ln x - \frac{1}{2k+1} \right). \tag{9.6.14}$$

Proof. The binomial theorem gives

$$(1+t^2)^{-n-1} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{k} t^{2k}. \tag{9.6.15}$$

The expression for $g_n(x)$ now follows by integrating term by term and the evaluation

$$\int_0^x t^{2k} \ln t \, dt = \left(\ln x - \frac{1}{2k+1} \right) \frac{x^{2k+1}}{2k+1}. \tag{9.6.16}$$

□

In particular, the initial condition $g_0(x)$ of the recurrence (9.6.10) is given by

$$g_0(x) = \ln x \tan^{-1} x - \mathrm{Ti}_2(x), \tag{9.6.17}$$

where

$$\mathrm{Ti}_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)^2} = \int_0^x \frac{\tan^{-1} t}{t}\, dt \tag{9.6.18}$$

is the tangent integral.

The recurrence (9.6.10) is now solved using (9.6.2) to produce

$$\begin{aligned} g_n(x) &= 2^{-2n}\binom{2n}{n} g_0(x) \\ &+ \frac{\binom{2n}{n}}{2^{2n}} \sum_{j=1}^{n} \frac{2^{2j}}{2j\binom{2j}{j}} \left[\{2j f_j(x) - (2j-1) f_{j-1}(x)\} \ln x - f_{j-1}(x) \right]. \end{aligned}$$

Using the recurrence (9.6.5), this can be written as

$$g_n(x) = \frac{\binom{2n}{n}}{2^{2n}} \left(g_0(x) + \sum_{j=1}^{n} \frac{2^{2j}}{2j\binom{2j}{j}} \left[\frac{x \,\ln x}{(x^2+1)^j} - f_{j-1}(x) \right] \right). \tag{9.6.19}$$

Solving the recurrence yields:

Theorem 9.6.1. *Let $n \in \mathbb{N}$. Define the rational function*

$$p_j(x) = \sum_{k=1}^{j} \frac{2^{2k}}{2k\binom{2k}{k}} \frac{x}{(1+x^2)^k}. \tag{9.6.20}$$

Then the integral $g_n(x)$ is given by

$$\int_0^x \frac{\ln t\, dt}{(1+t^2)^{n+1}} = \frac{\binom{2n}{n}}{2^{2n}} \left[g_0(x) + p_n(x) \ln x - \sum_{k=1}^{n} \frac{\tan^{-1} x + p_{k-1}(x)}{2k-1} \right].$$

Note 9.6.1. *The special case $x = 1$ in (9.6.1) produces*

$$\int_0^1 \frac{\ln t\, dt}{(t^2+1)^{n+1}} = 2^{-2n}\binom{2n}{n} \left(G - \sum_{k=1}^{n} \frac{\frac{\pi}{4} + p_{k-1}(1)}{2k-1} \right). \tag{9.6.21}$$

The values

$$p_n(1) = \frac{1}{2} \sum_{j=1}^{n} \frac{2^j}{j\binom{2j}{j}} \tag{9.6.22}$$

do not admit a closed-form, but they do satisfy the three-term recurrence

$$(2n+1)p_{n+1}(1) - (3n+1)p_n(1) + np_{n-1}(1) = 0. \tag{9.6.23}$$

The reader is invited to verify the expansion

$$\sum_{k=1}^{\infty} \frac{x^k}{k\binom{2k}{k}} = \frac{2\sqrt{x}\,\sin^{-1}(\sqrt{x}/2)}{\sqrt{4-x}}, \tag{9.6.24}$$

from which it follows that

$$\sum_{k=1}^{\infty} \frac{2^k}{k\binom{2k}{k}} = \frac{\pi}{2}. \tag{9.6.25}$$

An alternative derivation. Integration by parts produces

$$\int_0^b \frac{\ln s\, ds}{s^2+a} = \frac{1}{\sqrt{a}} \ln b \tan^{-1} \frac{b}{\sqrt{a}} - \frac{1}{\sqrt{a}} \int_0^{b/\sqrt{a}} \frac{\tan^{-1} x}{x}\, dx. \tag{9.6.26}$$

Differentiating n-times with respect to a and using

$$\left(\frac{d}{da}\right)^j \frac{1}{\sqrt{a}} = \frac{(-1)^j\,(2j)!}{j!2^{2j}a^{j+1/2}}, \ \left(\frac{d}{da}\right)^j \frac{1}{b^2+a} = \frac{(-1)^j\,j!}{(b^2+a)^{j+1}},$$

and

$$\left(\frac{d}{da}\right)^j \tan^{-1} \frac{b}{\sqrt{a}} = (-1)^j (j-1)! b \sum_{k=0}^{j-1} \frac{\binom{2k}{k}}{2^{2k+1}a^{k+1/2}(b^2+a)^{j-k}},$$

we obtain

$$\begin{aligned}
\int_0^b \frac{\ln s\, ds}{(s^2+a)^{n+1}} &= \ln b \tan^{-1}\left(\frac{b}{\sqrt{a}}\right) F_n(a) \\
&\quad + \frac{b \ln b}{2} \sum_{k=1}^{n} \frac{F_{n-k}(a)}{k} \sum_{j=0}^{k-1} \frac{F_j(a)}{(b^2+a)^{k-j}} \\
&\quad - F_n(a) \int_0^b \frac{\tan^{-1} x}{x} dx \\
&\quad - \frac{1}{2} \sum_{k=1}^{n} \frac{F_{n-k}(a)}{k} \sum_{j=0}^{j-1} F_j(a) \int_0^b \frac{dt}{(t^2+a)^{k-j}}
\end{aligned}$$

with $F_j(a) = 2^{-2j}a^{-j-1/2}\binom{2j}{j}$. The last integral in this expression can be evaluated using Proposition 9.6.1 to produce a generalization of Theorem 9.6.1. We have replaced the parameter a by a^2 to produce a cleaner formula.

Theorem 9.6.2. *Let $a, b \in \mathbb{R}$ with $b > 0$ and $n \in \mathbb{N}$. Introduce the notation*

$$F_j(a) = \frac{\binom{2j}{j}}{2^{2j} a^{2j+1}}. \tag{9.6.27}$$

Then

$$\begin{aligned}
\int_0^b \frac{\ln s \, ds}{(s^2+a^2)^{n+1}} &= F_n(a) \ln b \tan^{-1}(b/a) \\
&\quad + \frac{b \ln b}{2} \sum_{k=1}^{n} \frac{F_{n-k}(a)}{k} \sum_{j=0}^{k-1} \frac{F_j(a)}{(a^2+b^2)^{k-1}} \\
&\quad - F_n(a) \int_0^{b/a} \frac{\tan^{-1} x}{x} \, dx \\
&\quad - \frac{1}{2} \tan^{-1}(b/a) \left(\sum_{k=1}^{n} \frac{F_{n-k}(a)}{k} \sum_{j=0}^{k-1} F_j(a) F_{k-j-1}(a) \right) \\
&\quad - \frac{b}{4\sqrt{a}} \sum_{k=1}^{n} \frac{F_{n-k}(a)}{k} \sum_{j=0}^{k-1} F_j(a) F_{k-j-1}(a) \sum_{r=1}^{k-j-1} \frac{1}{r F_r(a) (a^2+b^2)^r}.
\end{aligned}$$

9.7 Some trigonometric versions

In this section we provide trigonometric versions of some of the evaluations provided in the previous sections. Many of these integrals correspond to special values of the *Lobachevsky function* defined by

$$L(x) := -\int_0^x \ln \cos t \, dt. \tag{9.7.1}$$

This appears as entry **8.260** in [40] and also as **4.224.4**. The change of variables $t = \pi/2 - x$ provides a proof of **4.224.1**:

$$\int_0^x \ln \sin t \, dt = L(\pi/2 - x) - L(\pi/2). \tag{9.7.2}$$

The special value

$$L(\pi/2) = \int_0^{\pi/2} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2, \tag{9.7.3}$$

appears as **4.224.3**. The change of variables $t = \frac{\pi}{2} - x$ yields **4.224.6**:

$$\int_0^{\pi/2} \ln \cos x \, dx = -\frac{\pi}{2} \ln 2. \tag{9.7.4}$$

To establish these evaluations, observe that, by symmetry,

$$\begin{aligned} 2\int_0^{\pi/2} \ln \sin x\, dx &= \int_0^{\pi/2} \ln \sin x\, dx + \int_0^{\pi/2} \ln \cos x\, dx \\ &= \int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right) dx \\ &= \int_0^{\pi/2} \ln(\sin 2x)\, dx - \frac{\pi}{2}\ln 2. \end{aligned}$$

Now let $t = 2x$ in the last integral to obtain the result.

Combining (9.7.1) and (9.7.2) we obtain **4.227.1**

$$\int_0^u \ln \tan x\, dx = L(u) + L(\pi/2 - u) + \frac{\pi}{2}\ln 2. \tag{9.7.5}$$

The identity (9.1.6) and the evaluation (9.5.2) yield the value of **4.227.2**:

$$\int_0^{\pi/4} \ln \tan t\, dt = -G. \tag{9.7.6}$$

Now observe that

$$\int_0^{\pi/4} \ln \tan t\, dt = \int_0^{\pi/4} \ln \sin t\, dt - \int_0^{\pi/4} \ln \cos t\, dt = -G \tag{9.7.7}$$

and

$$\int_0^{\pi/2} \ln \sin t\, dt = \int_0^{\pi/4} \ln \sin t\, dt + \int_0^{\pi/4} \ln \cos t\, dt = -\frac{\pi}{2}\ln 2. \tag{9.7.8}$$

Solving this system of equations yields

$$\int_0^{\pi/4} \ln \sin t\, dt = -\frac{\pi}{4}\ln 2 - \frac{G}{2} \tag{9.7.9}$$

that appears as **4.224.2** in [40] and

$$\int_0^{\pi/4} \ln \cos t\, dt = -\frac{\pi}{4}\ln 2 + \frac{G}{2} \tag{9.7.10}$$

that appears as **4.224.5**.

We now make use of the identity

$$\cos x - \sin x = \sqrt{2}\cos(x + \pi/4) \tag{9.7.11}$$

to obtain

$$\begin{aligned}\int_0^{\pi/4} \ln(\cos x - \sin x)\, dx &= \frac{\pi}{8} + \int_0^{\pi/2} \ln\cos(x+\pi/4) \\ &= \frac{\pi}{8}\ln 2 + \int_0^{\pi/2} \ln\cos t\, dt - \int_0^{\pi/4} \ln\cos t\, dt.\end{aligned}$$

The first integral is given in (9.7.4) as $-\frac{\pi}{2}\ln 2$ and the second one as $-\frac{\pi}{4}\ln 2 + \frac{G}{2}$ in (9.7.10). We conclude with the evaluation of **4.225.1**

$$\int_0^{\pi/4} \ln(\cos x - \sin x)\, dx = -\frac{\pi}{8}\ln 2 - \frac{G}{2}. \tag{9.7.12}$$

A similar analysis produces **4.225.2**:

$$\int_0^{\pi/4} \ln(\cos x + \sin x)\, dx = -\frac{\pi}{8}\ln 2 + \frac{G}{2}. \tag{9.7.13}$$

These evaluations can be used to produce **4.227.9**:

$$\int_0^{\pi/4} \ln(1 + \tan x)\, dx = \frac{\pi}{8}\ln 2, \tag{9.7.14}$$

and **4.227.11**:

$$\int_0^{\pi/4} \ln(1 - \tan x)\, dx = \frac{\pi}{8}\ln 2 - G. \tag{9.7.15}$$

To prove these formulas, simply write

$$\ln(1 \pm \tan x) = \ln(\cos \pm \sin x) - \ln\cos x. \tag{9.7.16}$$

The same type of calculations provide verification of **4.227.13**

$$\int_0^{\pi/4} \ln(1 + \cot x)\, dx = \frac{\pi}{8}\ln 2 + G \tag{9.7.17}$$

and **4.227.14**

$$\int_0^{\pi/4} \ln(\cot x - 1)\, dx = \frac{\pi}{8}\ln 2. \tag{9.7.18}$$

The next example of this type is **4.227.15**:

$$\int_0^{\pi/4} \ln(\tan x + \cot x)\, dx = \frac{\pi}{2}\ln 2, \tag{9.7.19}$$

This is evaluated by writing the integral as

$$-\int_0^{\pi/4} \ln(\sin x)\, dx - \int_0^{\pi/2} \ln(\cos x)\, dx = \frac{\pi}{2}\ln 2, \tag{9.7.20}$$

using (9.7.9) and (9.7.10).

The evaluation of **4.227.10**

$$\int_0^{\pi/2} \ln(1+\tan x)\,dx = \frac{\pi}{4}\ln 2 + G, \tag{9.7.21}$$

follows from the integrals evaluated here. Indeed,

$$\begin{aligned}\int_0^{\pi/2} \ln(1+\tan x)\,dx &= \int_0^{\pi/2} \ln(\sin x + \cos x)\,dx - \int_0^{\pi/2} \ln(\cos x)\,dx \\ &= 2\int_0^{\pi/4} \ln(\sin x + \cos x)\,dx - \int_0^{\pi/2} \ln(\cos x)\,dx \\ &= 2\left(-\frac{\pi}{8}\ln 2 + \frac{G}{2}\right) + \frac{\pi}{2}\ln 2,\end{aligned}$$

where we have used (9.7.4) and (9.7.13).

The identity (9.5.5) yields

$$\int_0^{\pi/2} \ln\tan t\,dt = 0. \tag{9.7.22}$$

The apparent generalization

$$\int_0^{\pi/2} \ln(a\tan t)\,dt = \frac{\pi}{2}\ln a, \tag{9.7.23}$$

with $a > 0$, appears as **4.227.3**.

The evaluations (9.7.9) and (9.7.10) can be brought back into rational form. The change of variables $t = \tan^{-1} u$ produces from (9.7.10):

$$-\frac{\pi}{4}\ln 2 + \frac{G}{2} = -\frac{1}{2}\int_0^1 \frac{\ln(1+u^2)}{1+u^2}\,du. \tag{9.7.24}$$

We have obtained a proof of **4.295.5**:

$$\int_0^1 \frac{\ln(1+x^2)}{1+x^2}\,dx = \frac{\pi}{2}\ln 2 - G. \tag{9.7.25}$$

The change of variables $t = 1/x$ and (9.5.4) yield **4.295.6**:

$$\int_1^\infty \frac{\ln(1+t^2)}{1+t^2}\,dt = \frac{\pi}{2}\ln 2 + G. \tag{9.7.26}$$

There are many other integrals that may be evaluated by the methods reported here. For instance, integration by parts yields

$$\int_0^x t\,\cot t\,dt = x\,\ln\sin x\Big|_0^x - \int_0^x \ln\sin t\,dt. \tag{9.7.27}$$

Using (9.7.2) we obtain

$$\int_0^x t \cot t \, dt = x \ln \sin x - L\left(\frac{\pi}{2} - x\right) + \frac{\pi}{2} \ln 2. \tag{9.7.28}$$

In particular, from $L(0) = 0$, we obtain **3.747.7**:

$$\int_0^{\pi/2} \frac{t \, dt}{\tan t} = \frac{\pi}{2} \ln 2. \tag{9.7.29}$$

The change of variables $u = \sin x$ produces from here the evaluation of **4.521.1**:

$$\int_0^1 \frac{\text{Arcsin } u}{u} \, du = \frac{\pi}{2} \ln 2. \tag{9.7.30}$$

Note. `Mathematica` evaluates all the entries in this chapter. The Lobachevsky function is expressed in terms of polylogarithms by the identity

$$L(x) = -\frac{i}{24}\left[\pi^2 + 12x^2 + 24ix \ln\left(2e^{ix}\right) + 12\text{PolyLog}\left(2, -e^{2ix}\right)\right].$$

Chapter 10

The digamma function

10.1 Introduction

The table of integrals [40] contains a large variety of definite integrals that involve the *digamma* function

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{10.1.1}$$

Here $\Gamma(x)$ is the gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt. \tag{10.1.2}$$

Many of the analytic properties can be derived from those of $\Gamma(x)$. The next theorem represents a collection of the important properties of $\Gamma(x)$ that are used in the current paper. The reader will find in [22] detailed proofs.

Theorem 10.1.1. *The gamma function satisfies:*

a) the functional equation

$$\Gamma(x+1) = x\Gamma(x). \tag{10.1.3}$$

b) For $n \in \mathbb{N}$, the interpolation formula $\Gamma(n) = (n-1)!$.

c) The Euler constant γ, *defined by*

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n, \tag{10.1.4}$$

is also given by $\gamma = -\Gamma'(1)$. *This appears as the special case* $a=1$ *of formula* **4.331.1***:*

$$\int_0^\infty e^{-ax} \ln x \, dx = -\frac{\gamma + \ln a}{a}. \tag{10.1.5}$$

This was established in [66]. The change of variables $t = ax$ *shows that the case* $a = 1$ *is equivalent to the general case. This is an instance of a* fake parameter*.*

d) The infinite product representation

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{k=1}^{\infty} \left[\left(1 + \frac{x}{k} \right)^{-1} e^{x/k} \right] \tag{10.1.6}$$

is valid for $x \in \mathbb{C}$ *away from the poles at* $x = 0, -1, -2, \ldots$

e) For $n \in \mathbb{N}$ *we have*

$$\Gamma\left(n + \tfrac{1}{2}\right) = \frac{(2n)!}{2^{2n}\, n!} \sqrt{\pi} \tag{10.1.7}$$

and

$$\Gamma\left(\tfrac{1}{2} - n\right) = (-1)^n \frac{2^{2n}\, n!}{(2n)!} \sqrt{\pi}. \tag{10.1.8}$$

f) For $x \in \mathbb{C}$, $x \notin \mathbb{Z}$ *we have the reflection rule*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{10.1.9}$$

Several properties of the digamma function $\psi(x)$ follow directly from the gamma function.

Theorem 10.1.2. *The digamma function* $\psi(x)$ *satisfies*
a) the functional equation

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \tag{10.1.10}$$

b) For $n \in \mathbb{N}$, *we have*

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}. \tag{10.1.11}$$

In particular, $\psi(1) = -\gamma$.

c) For $x \in \mathbb{C}$ away from $x = 0, -1, -2, \cdots$ we have

$$\begin{aligned}\psi(x) &= -\gamma - \frac{1}{x} + x\sum_{k=1}^{\infty}\frac{1}{k(x+k)} \\ &= -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{x+k} - \frac{1}{k+1}\right)\end{aligned} \tag{10.1.12}$$

d) The derivative of ψ is given by

$$\psi'(x) = \sum_{k=0}^{\infty}\frac{1}{(x+k)^2}. \tag{10.1.13}$$

In particular, $\psi'(1) = \pi^2/6$.

e) For $n \in \mathbb{N}$ we have

$$\psi(\tfrac{1}{2} \pm n) = -\gamma - 2\ln 2 + 2\sum_{k=1}^{n}\frac{1}{2k-1}. \tag{10.1.14}$$

In particular,

$$\psi(\tfrac{1}{2}) = -\gamma - 2\ln 2. \tag{10.1.15}$$

f) For $x \in \mathbb{C}$, $x \notin \mathbb{Z}$ we have the reflection rule

$$\psi(1-x) = \psi(x) + \pi\, cot\, \pi x. \tag{10.1.16}$$

10.2 A first integral representation

In this section we establish the integral evaluation **3.429**. Several direct consequences of this formulas are also described.

Proposition 10.2.1. *Assume $a > 0$. Then*

$$\int_0^{\infty}\left[e^{-x} - (1+x)^{-a}\right]\frac{dx}{x} = \psi(a). \tag{10.2.1}$$

Proof. We begin with the double integral

$$\int_0^{\infty}\int_1^{s} e^{-tz}dt\, dz = \int_0^{\infty}\frac{e^{-z} - e^{-sz}}{z}dz. \tag{10.2.2}$$

On the other hand,

$$\int_1^s \int_0^\infty e^{-tz} dz\, dt = \int_1^s \frac{dt}{t} = \ln s. \tag{10.2.3}$$

We conclude that

$$\int_0^\infty \frac{e^{-z} - e^{-sz}}{z}\, dz = \ln s. \tag{10.2.4}$$

This evaluation is equivalent to:

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \ln \frac{b}{a}, \tag{10.2.5}$$

that appears as formula **3.434.2** in [40]. The reader will find a proof in [66].

We now establish the result: start with

$$\begin{aligned}
\Gamma'(a) &= \int_0^\infty e^{-s} s^{a-1} \ln s\, ds \\
&= \int_0^\infty e^{-s} s^{a-1} \int_0^\infty \frac{e^{-z} - e^{-zs}}{z}\, dz\, ds \\
&= \int_0^\infty \left(e^{-z} \int_0^\infty s^{a-1} e^{-s}\, ds - \int_0^\infty s^{a-1} e^{-s(1+z)}\, ds \right) \frac{dz}{z}.
\end{aligned}$$

This formula can be rewritten as

$$\Gamma'(a) = \Gamma(a) \int_0^\infty \left(e^{-z} - (1+z)^{-a} \right) \frac{dz}{z}.$$

This establishes (10.2.1). $\square$

Example 10.1. The special case $a = 1$ yields

$$\int_0^\infty \left(e^{-x} - \frac{1}{1+x} \right) \frac{dx}{x} = -\gamma. \tag{10.2.6}$$

This appears as **3.435.3**.

Example 10.2. The change of variables $w = -\ln x$ gives the value of **4.275.2**:

$$\int_0^1 \left[x - \left(\frac{1}{1 - \ln x} \right)^q \right] \frac{dx}{x \ln x} = - \int_0^\infty \left[e^{-w} - (1+w)^{-q} \right] \frac{dw}{w} = -\psi(q). \tag{10.2.7}$$

Example 10.3. The change of variables $t = 1/(x+1)$ in (10.2.1) yields **3.471.14**:

$$\int_0^1 \frac{e^{(1-1/t)} - t^a}{t(1-t)}\, dt = \psi(a). \tag{10.2.8}$$

Example 10.4. The result of Example 10.1 can be used to prove **3.435.4**:

$$\int_0^\infty \left(e^{-bx} - \frac{1}{1+ax} \right) \frac{dx}{x} = \ln \frac{a}{b} - \gamma. \tag{10.2.9}$$

Indeed, the change of variables $t = bx$ yields from (10.1) the identity

$$\begin{aligned}\int_0^\infty \left(e^{-bx} - \frac{1}{1+ax} \right) \frac{dx}{x} &= \int_0^\infty \left(e^{-t} - \frac{1}{1+at/b} \right) \frac{dt}{t} \\ &= \int_0^\infty \frac{e^{-t} - e^{-at/b}}{t} \, dt + \int_0^\infty \left(e^{-at/b} - \frac{1}{1+at/b} \right) \frac{dt}{t}.\end{aligned}$$

Formula (10.2.5) shows the first integral is $\ln \frac{a}{b}$ and the value of the second one comes from (10.1).

Example 10.5. The evaluation **3.476.2**:

$$\int_0^\infty \left(e^{-x^p} - e^{-x^q} \right) \frac{dx}{x} = \frac{p-q}{pq} \gamma \tag{10.2.10}$$

comes directly from (10.2.1). Indeed, the change of variables $u = x^p$ yields

$$I := \int_0^\infty \left(e^{-x^p} - e^{-x^q} \right) \frac{dx}{x} = \frac{1}{p} \int_0^\infty \left(e^{-u} - e^{-u^{q/p}} \right) \frac{du}{u}.$$

Now write

$$I = \frac{1}{p} \int_0^\infty \left(e^{-u} - \frac{1}{1+u} \right) \frac{du}{u} + \frac{1}{p} \int_0^\infty \left(\frac{1}{1+u} - e^{-u^{q/p}} \right) \frac{du}{u}.$$

The first integral is $-\gamma$ by (10.2.6) and the change of variables $v = u^{q/p}$ gives

$$\begin{aligned} I &= -\frac{\gamma}{p} + \frac{1}{q} \int_0^\infty \left(\frac{1}{1+v^{p/q}} - e^{-v} \right) \frac{dv}{v} \\ &= -\frac{\gamma}{p} + \frac{1}{q} \int_0^\infty \left(\frac{1}{1+v} - e^{-v} \right) \frac{dv}{v} + \frac{1}{q} \int_0^\infty \frac{v - v^{p/q}}{v(1+v)(1+v^{p/q})} \, dv. \end{aligned}$$

Split the last integral from $[0,1]$ to $[1,\infty)$ and use the change of variables $x \mapsto 1/x$ in the second part to check that the whole integral vanishes. Formula (10.2.10) has been established.

Example 10.6. Formula **3.463**:

$$\int_0^\infty \left(e^{-x^2} - e^{-x} \right) \frac{dx}{x} = \frac{\gamma}{2} \tag{10.2.11}$$

corresponds to the choice $p = 2$ and $q = 1$ in (10.2.10).

Example 10.7. Formula **3.469.2**:

$$\int_0^\infty \left(e^{-x^4} - e^{-x} \right) \frac{dx}{x} = \frac{3\gamma}{4} \tag{10.2.12}$$

corresponds to the choice $p = 4$ and $q = 1$ in (10.2.10).

Example 10.8. Formula **3.469.3**:

$$\int_0^\infty \left(e^{-x^4} - e^{-x^2} \right) \frac{dx}{x} = \frac{\gamma}{4} \tag{10.2.13}$$

corresponds to the choice $p = 4$ and $q = 2$ in (10.2.10).

Example 10.9. Formula **3.475.3**:

$$\int_0^\infty \left(e^{-x^{2^n}} - e^{-x} \right) \frac{dx}{x} = (1 - 2^{-n})\gamma \tag{10.2.14}$$

corresponds to the choice $p = 2^n$ and $q = 1$ in (10.2.10).

The case $p = q$ in (10.2.10) is now modified to include a parameter.

Proposition 10.2.2. *Let a, b, $p \in \mathbb{R}^+$. Then* **3.476.1** *in [40] states that*

$$\int_0^\infty \left[e^{-ax^p} - e^{-bx^p} \right] \frac{dx}{x} = \frac{\ln b - \ln a}{p}. \tag{10.2.15}$$

Proof. The change of variables $t = ax^p$ gives

$$\int_0^\infty \left[e^{-ax^p} - e^{-bx^p} \right] \frac{dx}{x} = \frac{1}{p} \int_0^\infty \left(e^{-t} - e^{-bt/a} \right) \frac{dt}{t}.$$

Introduce the term $1/(1+t)$ to obtain

$$\begin{aligned} I &= \frac{1}{p} \int_0^\infty \left(e^{-t} - \frac{1}{1+t} \right) \frac{dt}{t} - \frac{1}{p} \int_0^\infty \left(e^{-bt/a} - \frac{1}{1+t} \right) \frac{dt}{t} \\ &= -\frac{\gamma}{p} - \frac{1}{p} \int_0^\infty \left(e^{-s} - \frac{b}{b+as} \right) \frac{ds}{s}. \end{aligned}$$

Adding and subtracting the term $1/(1+s)$ produces

$$I = \frac{1}{p} \int_0^\infty \left(\frac{b}{b+as} - \frac{1}{1+s} \right) \frac{ds}{s}. \tag{10.2.16}$$

The final result now comes from evaluating the last integral. $\square$

We now present another integral representation of the digamma function.

Proposition 10.2.3. *The digamma function is given by*

$$\psi(a) = \int_0^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-ax}}{1-e^{-x}} \right) dx. \tag{10.2.17}$$

This expression appears as **3.427.1** *in [40].*

Proof. The representation (10.2.1) is written as

$$\psi(a) = \lim_{\delta \to 0} \int_\delta^\infty \frac{e^{-z}}{z}\, dz - \int_\delta^\infty \frac{dz}{z(1+z)^a}, \tag{10.2.18}$$

to avoid the singularity at $z = 0$. The change of variables $z = e^t - 1$ in the second integral gives

$$\psi(a) = \lim_{\delta \to 0} \int_\delta^\infty \frac{e^{-z}}{z}\, dz - \int_{\ln(1+\delta)}^\infty \frac{e^{-at}\, dt}{1-e^{-t}}. \tag{10.2.19}$$

Now observe that

$$\left| \int_\delta^{\ln(1+\delta)} \frac{e^{-t}}{t}\, dt \right| \leq \int_{\ln(1+\delta)}^\delta \frac{dt}{t} \to 0, \tag{10.2.20}$$

as $\delta \to 0$. This completes the proof. $\square$

Example 10.10. The special case $a = 1$ in (10.2.17) gives **3.427.2**:

$$\int_0^\infty \left(\frac{1}{1-e^{-x}} - \frac{1}{x} \right) e^{-x}\, dx = \gamma. \tag{10.2.21}$$

Example 10.11. The change of variables $t = e^{-x}$ in (10.2.17) produces **4.281.4**:

$$\int_0^1 \left(\frac{1}{\ln t} + \frac{t^{a-1}}{1-t} \right) dt = -\psi(a). \tag{10.2.22}$$

Example 10.12. The special case $a = 1$ in (10.2.22) yields **4.281.1**:

$$\int_0^1 \left(\frac{1}{\ln t} + \frac{1}{1-t} \right) dt = \gamma. \tag{10.2.23}$$

Proposition 10.2.4. *Let* $p,\ q \in \mathbb{R}^+$. *Then*

$$\int_0^1 \left(\frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x} \right) dx = \ln p - \psi(q). \tag{10.2.24}$$

This appears as **4.281.5** *in [40].*

Proof. Write

$$\int_0^1 \left(\frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x} \right) dx = \int_0^1 \left(\frac{1}{\ln x} + \frac{x^{q-1}}{1-x} \right) dx + \int_0^1 \frac{x^{p-1}-1}{\ln x} \, dx. \tag{10.2.25}$$

The first integral is $-\psi(q)$ from (10.2.22) and to evaluate the second one, differentiate with respect to p, to produce

$$\frac{d}{dp} \int_0^1 \frac{x^{p-1}-1}{\ln x} \, dx = \int_0^1 x^{p-1} \, dx = \frac{1}{p}. \tag{10.2.26}$$

The value at $p = 1$ shows that the constant of integration vanishes. The formula (10.2.24) has been established. ☐

10.3 The difference of values of the digamma function

In this section we establish an integral representation for the difference of values of the digamma function. The expression appears as **3.231.5** in [40].

Proposition 10.3.1. *Let p, $q \in \mathbb{R}$. Then*

$$\int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x} \, dx = \psi(q) - \psi(p). \tag{10.3.1}$$

Proof. Consider first

$$I(\epsilon) = \int_0^1 x^{p-1}(1-x)^{\epsilon-1} \, dx - \int_0^1 x^{q-1}(1-x)^{\epsilon-1} \, dx, \tag{10.3.2}$$

that avoids the apparent singularity at $x = 1$. The integral $I(\epsilon)$ can be expressed in terms of the beta function

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} \, dx \tag{10.3.3}$$

as $I(\epsilon) = B(p,\epsilon) - B(q,\epsilon)$, and using the relation

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \tag{10.3.4}$$

we obtain

$$I(\epsilon) = \Gamma(\epsilon) \left(\frac{\Gamma(p)}{\Gamma(p+\epsilon)} - \frac{\Gamma(q)}{\Gamma(q+\epsilon)} \right). \tag{10.3.5}$$

Now use $\Gamma(1+\epsilon) = \epsilon\Gamma(\epsilon)$ to write

$$I(\epsilon) = \Gamma(1+\epsilon)\left(\frac{\Gamma(p) - \Gamma(p+\epsilon)}{\epsilon} \frac{1}{\Gamma(p+\epsilon)} - \frac{\Gamma(q) - \Gamma(q+\epsilon)}{\epsilon} \frac{1}{\Gamma(q+\epsilon)} \right), \tag{10.3.6}$$

and obtain (10.3.1) by letting $\epsilon \to 0$. □

Example 10.13. The special value $\psi(1) = -\gamma$ produces

$$\int_0^1 \frac{1 - x^{q-1}}{1-x}\, dx = \gamma + \psi(q). \tag{10.3.7}$$

This appears as **3.265** in [40].

Example 10.14. A second special value appears in **3.268.2**:

$$\int_0^1 \frac{1 - x^a}{1-x}\, x^{b-1}\, dx = \psi(a+b) - \psi(b). \tag{10.3.8}$$

It is obtained from (10.3.1) by choosing $p = b$ and $q = a + b$.

Example 10.15. Now let $q = 1 - p$ in (10.3.1) to produce

$$\int_0^1 \frac{x^{p-1} - x^{-p}}{1-x}\, dx = \psi(1-p) - \psi(p) = \pi \cot \pi p. \tag{10.3.9}$$

This appears as **3.231.1** in [40].

Example 10.16. The special case $p = a + 1$ and $q = 1 - a$ produces

$$\int_0^1 \frac{x^a - x^{-a}}{1-x}\, dx = \psi(1-a) - \psi(1+a) = \pi \cot \pi a - \frac{1}{a}, \tag{10.3.10}$$

where we have used (10.1.10) and (10.1.16) to simplify the result. This is **3.231.3** in [40].

Example 10.17. The change of variables $x = t^a$ in (10.3.1) produces

$$\int_0^1 \frac{t^{ap-1} - t^{aq-1}}{1 - t^a}\, dt = \frac{\psi(q) - \psi(p)}{a}. \tag{10.3.11}$$

Now let $p = 1, a = \nu$ and $q = \frac{\mu}{\nu}$ and the replace μ by p and ν by q to obtain **3.244.3** in [40]:

$$\int_0^1 \frac{t^{q-1} - t^{p-1}}{1 - t^q}\, dt = \frac{1}{q}\left(\gamma + \psi\left(\frac{p}{q} \right) \right). \tag{10.3.12}$$

Example 10.18. The special case $p = b/a$ and $q = 1 - b/a$ in (10.3.11) produces

$$\int_0^1 \frac{x^{b-1} - x^{a-b-1}}{1 - x^a}\, dx = \frac{1}{a} \left(\psi(1 - b/a) - \psi(b/a) \right). \tag{10.3.13}$$

The result is now simplified using (10.1.16) to produce

$$\int_0^1 \frac{x^{b-1} - x^{a-b-1}}{1 - x^a}\, dx = \frac{\pi}{a} \cot \frac{\pi b}{a}. \tag{10.3.14}$$

This is **3.244.2** in [40].

Example 10.19. The special case $a = 2$ in (10.3.11) yields

$$\int_0^1 \frac{t^{2\mu-1} - t^{2\nu-1}}{1 - t^2}\, dt = \frac{1}{2} \left(\psi(\nu) - \psi(\mu) \right). \tag{10.3.15}$$

The choice $\mu = 1 + p/2$ and $\nu = 1 - p/2$:

$$\int_0^1 \frac{x^p - x^{-p}}{1 - x^2}\, x\, dx = \frac{1}{2} \left(\psi(1 + p/2) - \psi(1 - p/2) \right). \tag{10.3.16}$$

The identities $\psi(x+1) = \psi(x) + 1/x$ and $\psi(1-x) - \psi(x) = \pi \cot \pi x$ produce

$$\int_0^1 \frac{x^p - x^{-p}}{1 - x^2}\, x\, dx = \frac{\pi}{2} \cot \left(\frac{p\pi}{2} \right) - \frac{1}{p}. \tag{10.3.17}$$

This appears as **3.269.1** in [40].

Example 10.20. The choice $\mu = \frac{a+1}{2}$ and $\nu = \frac{b+1}{2}$ in (10.3.15) gives **3.269.3**:

$$\int_0^1 \frac{x^a - x^b}{1 - x^2}\, dx = \frac{1}{2} \left(\psi \left(\frac{b+1}{2} \right) - \psi \left(\frac{a+1}{2} \right) \right). \tag{10.3.18}$$

Note. This is the example discussed in the Introduction, where `Mathematica` gives the incorrect value

$$\int_0^1 \frac{x^a - x^b}{1 - x^2}\, dx = \frac{\pi}{2 \sin(\pi p)} - \frac{1 + p^2}{2p(1 - p^2)}. \tag{10.3.19}$$

10.4 Integrals over a halfline

In this section we consider integrals over the halfline $[0, \infty)$ that can be evaluated in terms of the digamma function.

Proposition 10.4.1. *Let $p,\, q \in \mathbb{R}$. Then*

$$\int_0^\infty \left(\frac{t^p}{(1+t)^p} - \frac{t^q}{(1+t)^q} \right) \frac{dt}{t} = \psi(q) - \psi(p). \tag{10.4.1}$$

This is **3.219** *in [40]. Also*

$$\int_0^\infty \left(\frac{1}{(1+t)^p} - \frac{1}{(1+t)^q} \right) \frac{dt}{t} = \psi(q) - \psi(p). \tag{10.4.2}$$

Proof. Let $t = x/(1-x)$ in (10.3.1). The second form comes from the first by the change of variables $x \mapsto 1/x$. □

Example 10.21. The special case $p = 1$ yields

$$\int_0^\infty \left(\frac{1}{1+t} - \frac{1}{(1+t)^q} \right) \frac{dt}{t} = \psi(q) + \gamma. \tag{10.4.3}$$

This appears as **3.233** in [40].

Example 10.22. The evaluation of **3.235**:

$$\int_0^\infty \frac{[(1+x)^a - 1]}{(1+x)^b} \frac{dx}{x} = \psi(b) - \psi(b-a) \tag{10.4.4}$$

can be established directly from (10.4.3). Simply write

$$\begin{aligned} \int_0^\infty \frac{(1+x)^a - 1}{(1+x)^b} \frac{dx}{x} \quad &= \quad \int_0^\infty \left(\frac{1}{1+x} - \frac{1}{(1+x)^b} \right) \frac{dx}{x} \\ &- \quad \int_0^\infty \left(\frac{1}{1+x} - \frac{1}{(1+x)^{b-a}} \right) \frac{dx}{x}, \end{aligned}$$

to obtain the result.

Some examples of integrals over $[0, \infty)$ can be reduced to a pair of integrals over $[0, 1]$.

Proposition 10.4.2. *The formula* **3.231.6** *of [40] states that*

$$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1-x} \, dx = \pi \left(\cot \pi p - \cot \pi q \right). \tag{10.4.5}$$

Proof. To evaluate this, make the change of variables $t = 1/x$ in the part over $[1, \infty)$ to produce

$$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1-x} \, dx = \int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x} \, dx - \int_0^1 \frac{x^{-p} - x^{-q}}{1-x} \, dx. \tag{10.4.6}$$

Now use the result (10.3.1) to write

$$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1-x} \, dx = \psi(q) - \psi(p) - \left[\psi(1-q) - \psi(1-p) \right]. \tag{10.4.7}$$

The relation $\psi(x) - \psi(1-x) = -\pi \, \cot(\pi q)$ yields the result. □

10.5 An exponential scale

In this section we present the evaluation of certain definite integrals involving the exponential function. These are integrals that can be evaluated in terms of the digamma function of the parameters involved.

Example 10.23. The simplest one is **3.317.2**:

$$\int_{-\infty}^{\infty} \left(\frac{1}{(1+e^{-x})^p} - \frac{1}{(1+e^{-x})^q} \right) dx = \psi(q) - \psi(p) \tag{10.5.1}$$

that comes from (10.4.2) via the change of variables $x \mapsto e^{-x}$. `Mathematica` gives

$$\int_{-\infty}^{\infty} \left(\frac{1}{(1+e^{-x})^p} - \frac{1}{(1+e^{-x})^q} \right) dx =$$

$$\frac{1}{p} {}_2F_1 \left(\left. \begin{matrix} p \quad p \\ p+1 \end{matrix} \right| -1 \right) - \frac{1}{q} {}_2F_1 \left(\left. \begin{matrix} q \quad q \\ q+1 \end{matrix} \right| -1 \right) \\ - p \, {}_3F_2 \left(\left. \begin{matrix} 1 \quad 1 \quad 1+p \\ 2 \quad q \end{matrix} \right| -1 \right) + q \, {}_3F_2 \left(\left. \begin{matrix} 1 \quad 1 \quad 1+q \\ 2 \quad q \end{matrix} \right| -1 \right).$$

Example 10.24. The special case $p = 1$ and $\psi(1) = -\gamma$ produces **3.317.1**:

$$\int_{-\infty}^{\infty} \left(\frac{1}{1+e^{-x}} - \frac{1}{(1+e^{-x})^q} \right) dx = \psi(q) + \gamma \tag{10.5.2}$$

Example 10.25. The evaluation of **3.316**:

$$\int_{-\infty}^{\infty} \frac{(1+e^{-x})^p - 1}{(1+e^{-x})^q} \, dx = \psi(q) - \psi(q-p) \tag{10.5.3}$$

comes directly from (10.5.1).

Proposition 10.5.1. *Let p, $q \in \mathbb{R}$. Then*

$$\int_0^{\infty} \frac{e^{-pt} - e^{-qt}}{1 - e^{-t}} \, dt = \psi(q) - \psi(p), \tag{10.5.4}$$

This appears as **3.311.7** *in [40].*

Proof. Make the of variables $x = e^{-t}$ in (10.3.1). ☐

Example 10.26. The evaluation (10.5.4) can also be written as

$$\int_0^{\infty} \frac{e^{t(1-p)} - e^{t(1-q)}}{e^t - 1} \, dt = \psi(q) - \psi(p), \tag{10.5.5}$$

Example 10.27. The special case $p = 1$ and $q = 1 - \nu$ is

$$\int_0^\infty \frac{1 - e^{\nu t}}{e^t - 1}\, dt = \psi(1-\nu) - \psi(1), \tag{10.5.6}$$

and using $\psi(1) = -\gamma$ and $\psi(1-\nu) = \psi(\nu) + \pi \cot \pi\nu$, yields the form

$$\int_0^\infty \frac{1 - e^{\nu t}}{e^t - 1}\, dt = \psi(\nu) + \gamma + \pi \cot \pi\nu, \tag{10.5.7}$$

as it appears in **3.311.5**.

Example 10.28. Another special case of (10.5.4) is **3.311.6**, that corresponds to $p = 1$:

$$\int_0^\infty \frac{e^{-t} - e^{-qt}}{1 - e^{-t}}\, dt = \psi(q) + \gamma. \tag{10.5.8}$$

Example 10.29. The evaluation **3.311.11**:

$$\int_0^\infty \frac{e^{px} - e^{qx}}{e^{rx} - e^{sx}}\, dx = \frac{1}{r-s}\left(\psi\left(\frac{r-q}{r-s}\right) - \psi\left(\frac{r-p}{r-s}\right)\right), \tag{10.5.9}$$

follows directly from (10.5.4) by the change of variables $t = (r-s)x$.

Example 10.30. The evaluation of **3.311.12**:

$$\int_0^\infty \frac{a^x - b^x}{c^x - d^x}\, dx = \frac{1}{\ln c - \ln d}\left(\psi\left(\frac{\ln c - \ln b}{\ln c - \ln d}\right) - \psi\left(\frac{\ln c - \ln a}{\ln c - \ln d}\right)\right), \tag{10.5.10}$$

is proved by simply writing the exponentials in natural base.

Example 10.31. The formula **3.311.10** had a sign error in the sixth edition of [40]. It appears as

$$\int_0^\infty \frac{e^{-px} - e^{-qx}}{1 + e^{-(p+q)x}}\, dx = \frac{\pi}{p+q} \cot\left(\frac{p\pi}{p+q}\right). \tag{10.5.11}$$

It should be

$$\int_0^\infty \frac{e^{-px} - e^{-qx}}{1 - e^{-(p+q)x}}\, dx = \frac{\pi}{p+q} \cot\left(\frac{p\pi}{p+q}\right). \tag{10.5.12}$$

The value (10.5.9) yields

$$\int_0^\infty \frac{e^{-px} - e^{-qx}}{1 - e^{-(p+q)x}}\, dx = \frac{1}{p+q}\left(\psi\left(\frac{q}{p+q}\right) - \psi\left(\frac{p}{p+q}\right)\right), \tag{10.5.13}$$

and the trigonometric answer follows from (10.1.16). This has been corrected in the current edition of [40].

Example 10.32. The evaluation of **3.312.2**:

$$\int_0^\infty \frac{(1-e^{-ax})(1-e^{-bx})e^{-px}}{1-e^{-x}}\,dx = \psi(p+a)+\psi(p+b)-\psi(p+a+b)-\psi(p) \tag{10.5.14}$$

follows directly from (10.3.1). Indeed, the change of variables $t = e^{-x}$ gives

$$I = \int_0^1 \frac{t^{p-1}(1-t^a-t^b+t^{a+b})}{1-t}\,dt \tag{10.5.15}$$

and now split them as

$$I = \int_0^1 \frac{t^{p-1}-t^{p+a-1}}{1-t}\,dt - \int_0^1 \frac{t^{p+b-1}-t^{p+a+b-1}}{1-t}\,dt \tag{10.5.16}$$

and use (10.3.1) to conclude.

10.6 A singular example

The example discussed in this section is

$$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = b^{\mu-1}\pi\,\cot(\pi\mu), \tag{10.6.1}$$

that appears as **3.311.8** in [40]. In the case $b > 0$ this has to be modified in its presentation to avoid the singularity $x = -\ln b$. The case $b < 0$ was discussed in [67]. In order to reduce the integral to a previous example, we let $t = e^{-x}$ to obtain

$$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = \int_0^\infty \frac{t^{\mu-1}\,dt}{b-t}. \tag{10.6.2}$$

The change of variables $t = by$ yields

$$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = b^{\mu-1}\int_0^\infty \frac{y^{\mu-1}\,dy}{1-y}. \tag{10.6.3}$$

Now separate the range of integration into $[0,1]$ and $[1,\infty)$. Then make the change of variables $y = 1/z$ in the second part. This produces

$$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = b^{\mu-1}\int_0^1 \frac{z^{\mu-1}-z^{-\mu}}{1-z}\,dz. \tag{10.6.4}$$

This last integral has been evaluated as $\cot(\pi\mu)$ in (10.3.9).

Note. `Mathematica` gives the value

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-\mu x}\,dx}{b-e^{-x}} &= \frac{1}{b\mu} + \frac{1}{\mu-1}\,{}_2F_1\left(\begin{matrix}1 & 1-\mu \\ & 2-\mu\end{matrix}\Bigg| b\right) \\ &+ \frac{1}{b^2(1+\mu)}\,{}_2F_1\left(\begin{matrix}1 & 1+\mu \\ & 2+\mu\end{matrix}\Bigg| \frac{1}{b}\right).\end{aligned}$$

10.7 An integral with a fake parameter

The example considered in this section is **3.234.1**:

$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx = \frac{\pi}{a^q} \cot \pi q. \tag{10.7.1}$$

We show that the parameter a is *fake*, in the sense that it can be easily scaled out of the formula. The integral is written as $\lim\limits_{\epsilon \to 0} I(\epsilon)$ where

$$\begin{aligned} I(\epsilon) &= \int_0^1 \left(\frac{x^{q-1}}{(1-ax)^{1-\epsilon}} - \frac{x^{-q}}{(a-x)^{1-\epsilon}} \right) dx \\ &= \int_0^1 \frac{x^{q-1}}{(1-ax)^{1-\epsilon}}\,dx - \int_0^1 \frac{x^{-q}}{(a-x)^{1-\epsilon}}\,dx. \end{aligned}$$

Make the change of variables $t = ax$ in the first integral and $x = at$ in the second one to produce

$$I(\epsilon) = a^{-q} \int_0^a \frac{t^{q-1}\,dt}{(1-t)^{1-\epsilon}} - a^{-q+\epsilon} \int_0^{1/a} \frac{t^{-q}\,dt}{(1-t)^{1-\epsilon}},$$

and then let $\epsilon \to 0$ to produce

$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx = a^{-q} \left(\int_0^a \frac{t^{q-1}\,dt}{1-t} - \int_0^{1/a} \frac{t^{-q}\,dt}{1-t} \right).$$

Differentiation with respect to the parameter a, shows that the expression in parenthesis is independent of a. It is now evaluated by using $a = 1$ to obtain

$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx = a^{-q} \left(\int_0^1 \frac{t^{q-1} - t^{-q}}{1-t}\,dt \right).$$

The evaluation (10.3.1) now yields

$$\begin{aligned} \int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx &= a^{-q} \left(\psi(1-q) - \psi(q) \right) \\ &= a^{-q} \pi \cot \pi q. \end{aligned}$$

Formula (10.7.1) has been established.

10.8 The derivative of ψ

In a future publication we will discuss the evaluation of definite integrals in terms of the *polygamma function*

$$\text{PolyGamma}[n, x] := \left(\frac{d}{dx} \right)^n \psi(x). \tag{10.8.1}$$

In this section, we simply describe some integrals in [40] that comes from direct differentiation of the examples described above.

Example 10.33. Differentiating (10.3.1) with respect to the parameter p produces **4.251.4**:

$$\int_0^1 \frac{x^{p-1} \ln x}{1-x} \, dx = -\psi'(p). \tag{10.8.2}$$

Example 10.34. The change of variables $x = t^q$ in (10.8.2), followed by the change of parameter $p \mapsto \frac{p}{q}$ yields **4.254.1**:

$$\int_0^1 \frac{t^{p-1} \ln t}{1-t^q} \, dx = -\frac{1}{q^2} \psi' \left(\frac{p}{q} \right). \tag{10.8.3}$$

Example 10.35. Replace q by $2q$ and p by q in (10.8.3) to produce

$$\int_0^1 \frac{t^{q-1} \ln t}{1-t^{2q}} \, dt = -\frac{1}{4q^2} \psi' \left(\tfrac{1}{2} \right). \tag{10.8.4}$$

To evaluate this last term, differentiate the logarithm of the identity

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \, \Gamma(x + \tfrac{1}{2}), \tag{10.8.5}$$

to obtain

$$2\psi(2x) = 2 \ln 2 + \psi(x) + \psi(x + \tfrac{1}{2}). \tag{10.8.6}$$

One more differentitation produces

$$4\psi'(2x) = \psi'(x) + \psi'(x + \tfrac{1}{2}). \tag{10.8.7}$$

The value $x = \frac{1}{2}$ gives

$$\psi'(\tfrac{1}{2}) = 3\psi'(1) = \frac{\pi^2}{2}. \tag{10.8.8}$$

Therefore we obtain **4.254.6**:

$$\int_0^1 \frac{x^{q-1} \ln x}{1-x^{2q}} \, dx = -\frac{\pi^2}{8q^2}. \tag{10.8.9}$$

Example 10.36. Differentiating (10.3.12) n-times with respect to the parameter p produces **4.271.15**:

$$\int_0^1 \ln^n x \, \frac{x^{p-1} \, dx}{1-x^q} = -\frac{1}{q^{n+1}} \psi^{(n)} \left(\frac{p}{q} \right). \tag{10.8.10}$$

10.9 A family of logarithmic integrals

Several of the integrals appearing in [40] are particular examples of the family evaluated in the next proposition.

Proposition 10.9.1. *Let* $a,\, b \in \mathbb{R}^+$. *Then*

$$\int_0^1 x^{a-1}(1-x)^{b-1} \, \ln x \, dx = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \left(\psi(a) - \psi(a+b)\right). \tag{10.9.1}$$

Proof. Differentiate the identity

$$\int_0^1 x^{a-1}(1-x)^{b-1} \, dx = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \tag{10.9.2}$$

with respect to the parameter a and recall that $\Gamma'(x) = \psi(x)\Gamma(x)$. □

The next corollary appears as **4.253.1** in [40].

Corollary 10.9.1. *Let* $a,\, b,\, c \in \mathbb{R}^+$. *Then*

$$\int_0^1 x^{a-1}(1-x^c)^{b-1} \, \ln x \, dx = \frac{\Gamma(a/c)\,\Gamma(b)}{c^2\,\Gamma(a/c+b)} \left(\psi\left(\frac{a}{c}\right) - \psi\left(\frac{a}{c}+b\right)\right). \tag{10.9.3}$$

Proof. Let $t = x^c$ in the integral (10.9.1). □

Example 10.37. The formula in the previous corollary also appears as **4.256** in the form

$$\int_0^1 \ln\left(\frac{1}{x}\right) \frac{x^{\mu-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}} = \frac{1}{n^2} B\left(\frac{\mu}{n}, \frac{m}{n}\right) \left[\psi\left(\frac{\mu+m}{n}\right) - \psi\left(\frac{\mu}{n}\right)\right]. \tag{10.9.4}$$

Example 10.38. The integral

$$\int_0^1 \frac{x^{2n}\,\ln x}{\sqrt{1-x^2}}\,dx = \int_0^1 x^{2n}(1-x^2)^{-1/2}\,\ln x\,dx \tag{10.9.5}$$

that appears as **4.241.1** in [40], corresponds to $a = 2n+1$, $b = \frac{1}{2}$ and $c = 2$ in (10.9.3). Therefore

$$\int_0^1 \frac{x^{2n}\,\ln x}{\sqrt{1-x^2}}\,dx = \frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{4\Gamma(n+1)} \left[\psi(n+\tfrac{1}{2}) - \psi(n+1)\right]. \tag{10.9.6}$$

Using (10.1.7), (10.1.11) and (10.1.14) yields

$$\int_0^1 \frac{x^{2n}\,\ln x}{\sqrt{1-x^2}}\,dx = \frac{\binom{2n}{n}\pi}{2^{2n+1}} \left(\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \ln 2\right). \tag{10.9.7}$$

This is **4.241.1**.

Example 10.39. The integral in **4.241.2** states that

$$\int_0^1 \frac{x^{2n+1}\,\ln x}{\sqrt{1-x^2}}\,dx = \frac{(2n)!!}{(2n+1)!!}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right). \tag{10.9.8}$$

Writing the integral as

$$I = \int_0^1 x^{2n+1}(1-x^2)^{-1/2}\,\ln x\,dx \tag{10.9.9}$$

we see that it corresponds to the case $a = 2n+2$, $b = \frac{1}{2}$, $c = 2$ in (10.9.3). Therefore

$$I = \frac{\Gamma(n+1)\,\Gamma(\frac{1}{2})}{4\Gamma(n+\frac{3}{2})}\left[\psi(n+1) - \psi(n+\tfrac{3}{2})\right]. \tag{10.9.10}$$

Using (10.1.7), (10.1.11) and (10.1.14) yields

$$\int_0^1 \frac{x^{2n+1}\,\ln x}{\sqrt{1-x^2}}\,dx = \frac{2^{2n}}{(n+1)\binom{2n+1}{n}}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right). \tag{10.9.11}$$

This is equivalent to (10.9.8).

Example 10.40. The integral **4.241.3** in [40] states that

$$\int_0^1 x^{2n}\sqrt{1-x^2}\,\ln x\,dx = \frac{(2n-1)!!}{(2n+2)!!}\cdot\frac{\pi}{2}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k} - \frac{1}{2n+2} - \ln 2\right). \tag{10.9.12}$$

To evaluate the integral, we write it as

$$I = \int_0^1 x^{2n}(1-x^2)^{1/2}\,\ln x\,dx \tag{10.9.13}$$

and we see that it corresponds to the case $a = 2n+1$, $b = \frac{3}{2}$, $c = 2$ in (10.9.3). Therefore

$$I = \frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{3}{2})}{4\Gamma(n+2)}\left[\psi(n+\tfrac{1}{2}) - \psi(n+2)\right]. \tag{10.9.14}$$

Using (10.1.7), (10.1.11) and (10.1.14) yields

$$\int_0^1 x^{2n}\sqrt{1-x^2}\,\ln x\,dx = -\frac{\binom{2n}{n}\,\pi}{2^{2n+2}\,(n+1)}\left(\ln 2 + \frac{1}{2n+2} + \sum_{k=1}^{2n}\frac{(-1)^k}{k}\right). \tag{10.9.15}$$

This is equivalent to (10.9.12).

Example 10.41. The integral **4.241.4** in [40] states that

$$\int_0^1 x^{2n+1}\sqrt{1-x^2}\,\ln x\,dx = \frac{(2n)!!}{(2n+3)!!}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^{k-1}}{k} - \frac{1}{2n+3}\right). \tag{10.9.16}$$

To evaluate the integral, we write it as

$$I = \int_0^1 x^{2n+1}(1-x^2)^{1/2}\,\ln x\,dx \tag{10.9.17}$$

and we see that it corresponds to the case $a = 2n+2$, $b = \frac{3}{2}$, $c = 2$ in (10.9.3). Therefore

$$I = \frac{\Gamma(n+1)\,\Gamma(\frac{3}{2})}{4\Gamma(n+\frac{5}{2})}\left[\psi(n+1) - \psi(n+\tfrac{5}{2})\right]. \tag{10.9.18}$$

Using (10.1.7), (10.1.11) and (10.1.14) yields

$$\int_0^1 x^{2n+1}\sqrt{1-x^2}\,\ln x\,dx = \frac{2^{2n+1}}{(n+1)(n+2)\binom{2n+3}{n+1}}\left(\ln 2 - \frac{1}{2n+3} + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right). \tag{10.9.19}$$

This is equivalent to (10.9.16).

Example 10.42. The integral **4.241.5** in [40] states that

$$\int_0^1 \sqrt{(1-x^2)^{2n-1}}\,\ln x\,dx = -\frac{(2n-1)!!\,\pi}{4\,(2n)!!}\left[\psi(n+1)+\gamma+\ln 4\right]. \tag{10.9.20}$$

To evaluate the integral, we write it as

$$I = \int_0^1 (1-x^2)^{n-\frac{1}{2}}\,\ln x\,dx \tag{10.9.21}$$

and we see that it corresponds to the case $a = 1$, $b = n+\frac{1}{2}$, $c = 2$ in (10.9.3). Therefore

$$I = \frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{4\Gamma(n+1)}\left[\psi(\tfrac{1}{2}) - \psi(n+\tfrac{1}{2})\right]. \tag{10.9.22}$$

Using (10.1.7), (10.1.11) and (10.1.14) yields

$$\int_0^1 (1-x^2)^{n-\frac{1}{2}}\,\ln x\,dx = -\frac{\binom{2n}{n}\pi}{2^{2n+2}}\left(2\ln 2 + \sum_{k=1}^{n}\frac{1}{k}\right). \tag{10.9.23}$$

This is equivalent to (10.9.20). This integral also appears as **4.246**.

Example 10.43. The case $n = 0$ in (10.9.7) yields

$$\int_0^1 \frac{\ln x \, dx}{\sqrt{1-x^2}} = -\frac{\pi}{2} \ln 2. \tag{10.9.24}$$

This appears as **4.241.7** in [40].

Example 10.44. Formula **4.241.8** states that

$$\int_1^\infty \frac{\ln x \, dx}{x^2 \sqrt{x^2-1}} = 1 - \ln 2. \tag{10.9.25}$$

To evaluate this, let $t = 1/x$ to obtain

$$I = -\int_0^1 t(1-t^2)^{-1/2} \, \ln t \, dt. \tag{10.9.26}$$

This corresponds to the case $a = 2$, $b = \frac{1}{2}$, $c = 2$ in (10.9.3). Therefore

$$I = -\frac{\Gamma(1)\,\Gamma(\frac{1}{2})}{4\,\Gamma(\frac{3}{2})} \left[\psi(1) - \psi(\tfrac{3}{2}) \right], \tag{10.9.27}$$

and the value $1 - \ln 2$ comes from (10.1.11) and (10.1.14).

Example 10.45. The case $n = 0$ in (10.9.15) produces

$$\int_0^1 \sqrt{1-x^2} \, \ln x \, dx = -\frac{\pi}{8}(2 \ln 2 + 1). \tag{10.9.28}$$

This appears as **4.241.9** in [40].

Example 10.46. The case $n = 0$ in (10.9.19) produces

$$\int_0^1 x \sqrt{1-x^2} \, \ln x \, dx = \frac{1}{9}(3 \ln 2 - 4). \tag{10.9.29}$$

This appears as **4.241.10** in [40].

Example 10.47. Entry **4.241.11** states that

$$\int_0^1 \frac{\ln x \, dx}{\sqrt{x(1-x^2)}} = -\frac{\sqrt{2\pi}}{8} \Gamma^2\left(\tfrac{1}{4}\right). \tag{10.9.30}$$

To evaluate the integral, write it as

$$I = \int_0^1 x^{-1/2}(1-x^2)^{-1/2} \, \ln x \, dx \tag{10.9.31}$$

and this corresponds to the case $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = 2$ in (10.9.3). Therefore

$$I = \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{4\Gamma(\frac{3}{4})} \left[\psi\left(\tfrac{1}{4}\right) - \psi\left(\tfrac{3}{4}\right) \right]. \tag{10.9.32}$$

The stated form comes from using (10.1.9) and (10.1.16).

Example 10.48. The identity

$$\int_0^1 \frac{x \ln x}{\sqrt{1-x^4}}\,dx = -\frac{\pi}{8}\ln 2 \tag{10.9.33}$$

appears as **4.243** in [40]. To evaluate it, we write it as

$$I = \int_0^1 x(1-x^4)^{-1/2}\,\ln x\,dx \tag{10.9.34}$$

that corresponds to $a=2$, $b=\frac{1}{2}$, $c=4$ in (10.9.3). Therefore,

$$I = \frac{1}{16}\Gamma^2(\tfrac{1}{2})\left[\psi(\tfrac{1}{2}) - \psi(1)\right]. \tag{10.9.35}$$

The values $\psi(1) = -\gamma$ and $\psi(\frac{1}{2}) = -\gamma - 2\ln 2$ give the result.

Example 10.49. The verification of **4.244.1**:

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{x(1-x^2)^2}} = -\frac{1}{8}\Gamma^3\left(\tfrac{1}{3}\right) \tag{10.9.36}$$

is achieved by using (10.9.3) with $a=\frac{2}{3}$, $b=\frac{1}{3}$ and $c=2$ to obtain

$$I = \frac{\Gamma^2\left(\frac{1}{3}\right)}{4\Gamma(\frac{2}{3})}\left[\psi\left(\tfrac{1}{3}\right) - \psi\left(\tfrac{2}{3}\right)\right]. \tag{10.9.37}$$

Using (10.1.9) and (10.1.16) produces the stated result.

Example 10.50. The usual application of (10.9.3) shows that **4.244.2** is

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{9\sqrt{3}}\left(\psi\left(\tfrac{1}{3}\right) + \gamma\right), \tag{10.9.38}$$

where we have used $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = 2\pi/\sqrt{3}$. It remains to evaluate $\psi(\frac{1}{3})$. The identity (10.1.16) gives

$$\psi\left(\tfrac{1}{3}\right) - \psi\left(\tfrac{2}{3}\right) = -\frac{\pi}{\sqrt{3}}. \tag{10.9.39}$$

To obtain a second relation among these quantities, we start from the identity

$$\Gamma(3x) = \frac{3^{3x-1/2}}{2\pi}\Gamma(x)\,\Gamma(x+\tfrac{1}{3})\,\Gamma(x+\tfrac{2}{3}) \tag{10.9.40}$$

that follows directly from (10.1.6), and differentiate logarithmically to obtain

$$\psi(3x) = \ln 3 + \frac{1}{3}\left(\psi(x) + \psi(x+\tfrac{1}{3}) + \psi(x+\tfrac{2}{3})\right). \tag{10.9.41}$$

The special case $x = \frac{1}{3}$ yields

$$\psi\left(\tfrac{1}{3}\right) + \psi\left(\tfrac{2}{3}\right) = -2\gamma - 3\ln 3. \tag{10.9.42}$$

We conclude that

$$\psi(\tfrac{1}{3}) = -\gamma - \frac{3}{2}\ln 3 - \frac{\pi}{2\sqrt{3}} \tag{10.9.43}$$

and

$$\psi(\tfrac{2}{3}) = -\gamma - \frac{3}{2}\ln 3 + \frac{\pi}{2\sqrt{3}}. \tag{10.9.44}$$

This gives

$$\int_0^1 \frac{\ln x\, dx}{\sqrt[3]{1-x^3}} = -\frac{\pi}{3\sqrt{3}}\left(\ln 3 + \frac{\pi}{3\sqrt{3}}\right), \tag{10.9.45}$$

as stated in **4.244.2**.

Example 10.51. The evaluation of **4.244.3**:

$$\int_0^1 \frac{x\ \ln x\, dx}{\sqrt[3]{1-x^3}} = \frac{\pi}{27}(\pi - 3\sqrt{3}\ln 3), \tag{10.9.46}$$

proceeds as in the previous example. The integral is identified as

$$I = \tfrac{1}{9}\Gamma(\tfrac{2}{3})\Gamma(\tfrac{1}{3})\left[\psi\left(\tfrac{2}{3}\right) + \gamma\right]. \tag{10.9.47}$$

The value (10.9.44) gives the rest.

Example 10.52. The change of variables $t = x^4$ yields

$$\int_0^1 \frac{x^p\ \ln x\, dx}{\sqrt{1-x^4}} = \frac{1}{16}\int_0^1 t^{(p-3)/4}(1-t)^{-1/2}\ \ln t\, dt. \tag{10.9.48}$$

The last integral is evaluated using (10.9.3) with $a = \frac{p+1}{4}$, $b = \frac{1}{2}$ and $c = 1$ to obtain

$$\int_0^1 \frac{x^p\ \ln x\, dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{16}\frac{\Gamma(\frac{p+1}{4})}{\Gamma(\frac{p+3}{4})}\left[\psi\left(\frac{p+1}{4}\right) - \psi\left(\frac{p+3}{4}\right)\right]. \tag{10.9.49}$$

The special case $p = 4n+1$ yields

$$\int_0^1 \frac{x^{4n+1}\ \ln x\, dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{16n!}\Gamma(n+\tfrac{1}{2})\left[\psi(n+\tfrac{1}{2}) - \psi(n+1)\right]. \tag{10.9.50}$$

The special case $p = 4n+1$ yields

$$\int_0^1 \frac{x^{4n+1}\ \ln x\, dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}{16n!}\left[\psi(n+\tfrac{1}{2}) - \psi(n+1)\right].$$

Using (10.1.7), (10.1.11) and (10.1.14) yields **4.245.1** in the form

$$\int_0^1 \frac{x^{4n+1}\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{\pi\binom{2n}{n}}{2^{2n+3}}\left(-\ln 2+\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\right). \tag{10.9.51}$$

The special case $p = 4n+3$ yields

$$\int_0^1 \frac{x^{4n+3}\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}\,n!}{16\Gamma(n+\frac{3}{2})}\left[\psi(n+1)-\psi(n+\tfrac{3}{2})\right].$$

Using (10.1.7), (10.1.11) and (10.1.14) yields **4.245.2** in the form

$$\int_0^1 \frac{x^{4n+3}\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{2^{2n-2}}{(2n+1)\binom{2n}{n}}\left(\ln 2+\sum_{k=1}^{2n+1}\frac{(-1)^{k}}{k}\right). \tag{10.9.52}$$

Example 10.53. The change of variables $t = x^{2n}$ produces

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{1-x^{2n}}} = \frac{1}{4n^2}\int_0^1 t^{\frac{1}{2n}-1}(1-t)^{-\frac{1}{n}}\,\ln t\,dt. \tag{10.9.53}$$

Then (10.9.3) with $a = \frac{1}{2n}$, $b = 1-\frac{1}{n}$ and $c = 1$ give the value

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{1-x^{2n}}} = \frac{\Gamma(\frac{1}{2n})\,\Gamma(1-\frac{1}{n})}{4n^2\,\Gamma(1-\frac{1}{2n})}\left[\psi\left(\tfrac{1}{2n}\right)-\psi\left(1-\tfrac{1}{2n}\right)\right]. \tag{10.9.54}$$

Using (10.1.7) and (10.1.14) to obtain

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{1-x^{2n}}} = -\frac{\pi}{8}\frac{B\left(\frac{1}{2n},\frac{1}{2n}\right)}{n^2\,\sin\left(\frac{\pi}{2n}\right)}. \tag{10.9.55}$$

This is **4.247.1** in [40].

Example 10.54. The change of variables $t = x^2$ gives

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{x^{n-1}(1-x^2)}} = \frac{1}{4}\int_0^1 t^{\frac{1}{2n}-1}(1-t)^{-\frac{1}{n}}\,\ln t\,dt.$$

Using (10.9.3) we obtain

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{x^{n-1}(1-x^2)}} = \frac{\Gamma(\frac{1}{2n})\,\Gamma(1-\frac{1}{2n})}{\Gamma(1-\frac{1}{2n})}\left[\psi\left(\tfrac{1}{2n}\right)-\psi\left(1-\tfrac{1}{2n}\right)\right]. \tag{10.9.56}$$

Proceeding as in the previous example, we obtain

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{x^{n-1}(1-x^2)}} = -\frac{\pi}{8}\frac{B\left(\frac{1}{2n},\frac{1}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)}. \tag{10.9.57}$$

This is **4.247.2** in [40].

Some integrals in [40] have the form of the Corollary 10.9.1 after an elementary change of variables.

Example 10.55. Formula **4.293.8** in [40] states that

$$\int_0^1 x^{a-1} \ln(1-x)\,dx = -\frac{1}{a}\left(\psi(a+1)+\gamma\right). \tag{10.9.58}$$

This follows directly from (10.9.3) by the change of variables $x \mapsto 1-x$. The same is true for **4.293.13**:

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,\ln(1-x)\,dx = B(a,b)\left[\psi(b)-\psi(a+b)\right]. \tag{10.9.59}$$

Example 10.56. The change of variables $t = e^{-x}$ gives

$$\int_0^\infty xe^{-x}(1-e^{2x})^{n-\frac{1}{2}}\,dx = -\int_0^1 (1-t^2)^{n-\frac{1}{2}}\,\ln t\,dt. \tag{10.9.60}$$

This latter integral is evaluated using (10.9.3) as

$$I = -\frac{\sqrt{\pi}\Gamma(n+\frac{1}{2})}{4n!}\left(\psi\left(\tfrac{1}{2}\right)-\psi(n+1)\right). \tag{10.9.61}$$

Using (10.1.7) and (10.1.11) we obtain

$$\int_0^\infty xe^{-x}(1-e^{-2x})^{n-\frac{1}{2}}\,dx = \frac{\binom{2n}{n}\pi}{2^{2n+2}}\left(2\ln 2+\sum_{k=1}^n \frac{1}{k}\right). \tag{10.9.62}$$

This appears as **3.457.1** in [40].

10.10 An announcement

There are many integrals in [40] that contain the term $1+x$ in the denominator, instead of the term $1-x$ seen, for instance, in Section 7.2. The evaluation of these integrals can be obtained using the *incomplete beta function*, defined by

$$\beta(a) := \int_0^1 \frac{x^{a-1}\,dx}{1+x} \tag{10.10.1}$$

as it appears in **8.371.1**. This function is related to the digamma function by the identity

$$\beta(a) = \frac{1}{2}\left[\psi\left(\frac{a+1}{2}\right)-\psi\left(\frac{a}{2}\right)\right]. \tag{10.10.2}$$

These evaluations have appeared in [26] and they will be described in Chapter 11.

10.11 One more family

We conclude this collection with a two-parameter family of integrals.

Proposition 10.11.1. *Let* $a, b \in \mathbb{R}^+$. *Then*

$$\int_0^\infty \left(e^{-x^a} - \frac{1}{1+x^b} \right) \frac{dx}{x} = -\frac{\gamma}{a}, \tag{10.11.1}$$

independently of b.

Proof. Write the integral as

$$\int_0^\infty \left(e^{-x^a} - e^{-x^b} \right) \frac{dx}{x} + \int_0^\infty \left(e^{-x^b} - \frac{1}{1+x^b} \right) \frac{dx}{x}. \tag{10.11.2}$$

The first integral is $(a-b)\gamma/ab$ according to (10.2.10). The change of variables $t = x^b$ converts the second one into

$$\frac{1}{b} \int_0^\infty \left(e^{-t} - \frac{1}{1+t} \right) \frac{dt}{t} = -\frac{\gamma}{b}, \tag{10.11.3}$$

according to (10.2.6). The formula has been established. □

Example 10.57. The case $a = 2^n$ and $b = 2^{n+1}$ gives **3.475.1**:

$$\int_0^\infty \left(\exp(-x^{2^n}) - \frac{1}{1+x^{2^{n+1}}} \right) \frac{dx}{x} = -\frac{\gamma}{2^n}. \tag{10.11.4}$$

Example 10.58. The case $a = 2^n$ and $b = 2$ gives **3.475.2**:

$$\int_0^\infty \left(\exp(-x^{2^n}) - \frac{1}{1+x^2} \right) \frac{dx}{x} = -\frac{\gamma}{2^n}. \tag{10.11.5}$$

Example 10.59. The case $a = 2$ and $b = 2$ gives **3.467**:

$$\int_0^\infty \left(e^{-x^2} - \frac{1}{1+x^2} \right) \frac{dx}{x} = -\frac{\gamma}{2}. \tag{10.11.6}$$

Example 10.60. Finally, the change of variables $t = ax$ yields

$$\int_0^\infty \left(e^{-px} - \frac{1}{1+a^2x^2} \right) \frac{dx}{x} = \int_0^\infty \frac{e^{-pt/a} - e^{-t}}{t} \, dt + \int_0^\infty \left(e^{-t} - \frac{1}{1+t^2} \right) \frac{dt}{t}.$$

The first integral is $\ln \frac{a}{p}$ according to (10.2.5), the second one is $-\gamma$. This gives the evaluation of **3.442.3**:

$$\int_0^\infty \left(e^{-px} - \frac{1}{1+a^2x^2} \right) \frac{dx}{x} = \gamma + \ln \frac{a}{p}. \tag{10.11.7}$$

Note. `Mathematica` is unable to evaluate **3.435.4** in (10.2.9), **3.475.1** in (10.11.4), **3.475.2** in (10.11.5), **3.476.1** in (15.2.3), **4.281.5** in (10.2.24), **3.311.11** in (10.5.9), and **3.311.12** in (10.5.10). It also produces a *strange* answer for **3.471.14** in (10.2.8): the correct value $\psi(a)$ has an extra additive term involving the *Meijer G-function*.

Chapter 11

The incomplete beta function

11.1 Introduction

The table of integrals [40] contains a large variety of definite integrals that involve the *incomplete beta* function defined here by the integral

$$\beta(a) = \int_0^1 \frac{x^{a-1}\,dx}{1+x}. \tag{11.1.1}$$

The convergence of the integral requires $a > 0$. Nielsen [70], who used this function extensively, attributed it to Stirling, page 17. The table [40] prefers to introduce first the *digamma function*

$$\psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{11.1.2}$$

and define $\beta(x)$ by the identity

$$\beta(x) = \frac{1}{2}\left(\psi\left(\tfrac{x+1}{2}\right) - \psi\left(\tfrac{x}{2}\right)\right). \tag{11.1.3}$$

This definition appears as **8.370** and (11.1.1) appears as **3.222.1**. Here

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \tag{11.1.4}$$

is the classical gamma function. Naturally, both starting points for β are equivalent, and Corollary 11.2.1 proves (11.1.3). The value

$$\gamma := -\psi(1) = -\Gamma'(1) \tag{11.1.5}$$

is the well-known *Euler's constant.*

In this chapter we will prove elementary properties of this function and use them to evaluate some definite integrals in [40].

11.2 Some elementary properties

The incomplete beta function admits a representation by series.

Proposition 11.2.1. *Let $a \in \mathbb{R}^+$. Then*

$$\beta(a) = \sum_{k=0}^{\infty} \frac{(-1)^k}{a+k}. \tag{11.2.1}$$

Proof. The result follows from the expansion of $1/(1+x)$ in (11.1.1) as a geometric series. ☐

Corollary 11.2.1. *The incomplete beta function is given by*

$$\beta(a) = \frac{1}{2}\left[\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right)\right]. \tag{11.2.2}$$

This is **8.370** *in [40].*

Proof. The expansion for the digamma function ψ

$$\psi(t) = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{t+k} - \frac{1}{k+1}\right) \tag{11.2.3}$$

has been discussed in [61]. Then

$$\psi\left(\frac{a}{2}\right) = -\gamma - \sum_{k=0}^{\infty}\left(\frac{2}{a+2k} - \frac{1}{k+1}\right) \tag{11.2.4}$$

and

$$\psi\left(\frac{a+1}{2}\right) = -\gamma - \sum_{k=0}^{\infty}\left(\frac{2}{a+2k+1} - \frac{1}{k+1}\right). \tag{11.2.5}$$

The identity (11.2.2) comes from adding these two expressions. ☐

These properties are now employed to prove some functional relations of the incomplete beta function. The proofs will employ the identities

$$\psi(x+1) = \frac{1}{x} + \psi(x) \tag{11.2.6}$$

$$\psi(x) - \psi(1-x) = -\pi\cot(\pi x) \tag{11.2.7}$$

$$\psi(x+\tfrac{1}{2}) - \psi(\tfrac{1}{2}-x) = \pi\tan(\pi x) \tag{11.2.8}$$

that were established in [61].

Remark 11.2.1. *Several of the evaluations presented here will employ the special values*

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}, \tag{11.2.9}$$

that appears as **8.365.4**, *and*

$$\psi\left(\tfrac{1}{2} \pm n\right) = -\gamma + 2\left(\sum_{k=1}^{n} \frac{1}{2k-1} - \ln 2\right), \tag{11.2.10}$$

that appears as **8.366.3**.

Many of the proofs of formulas in Section **4.271** *employ the values*

$$\psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}, \tag{11.2.11}$$

that appear as **8.366.11** *and also* **8.366.12/13**:

$$\psi'(\tfrac{1}{2} \pm n) = \frac{\pi^2}{2} \mp 4 \sum_{k=1}^{n} \frac{1}{(2k-1)^2}. \tag{11.2.12}$$

Higher order derivatives are given by

$$\begin{aligned} \psi^{(n)}(1) &= (-1)^{n+1} n! \zeta(n+1) \text{ and} \\ \psi^{(n)}(\tfrac{1}{2}) &= (-1)^{n+1} n! (2^{n+1}-1) \zeta(n+1). \end{aligned}$$

Proposition 11.2.2. *The incomplete beta function satisfies*

$$\begin{aligned} \beta(x+1) &= \frac{1}{x} - \beta(x), && (11.2.13) \\ \beta(1-x) &= \frac{\pi}{\sin \pi x} - \beta(x), && (11.2.14) \\ \beta(x+1) &= \frac{1}{x} - \frac{\pi}{\sin \pi x} + \beta(1-x). && (11.2.15) \end{aligned}$$

Proof. Using (11.2.2) we have

$$\begin{aligned} \beta(x+1) &= \frac{1}{2}\left[\psi\left(\frac{x+2}{2}\right) - \psi\left(\frac{x+1}{2}\right)\right] = \frac{1}{2}\left[\psi\left(\frac{x}{2}+1\right) - \psi\left(\frac{x+1}{2}\right)\right] \\ &= \frac{1}{2}\left[\frac{2}{x} + \psi\left(\frac{x}{2}\right) - \psi\left(\frac{x+1}{2}\right)\right] \\ &= \frac{1}{x} - \beta(x). \end{aligned}$$

This establishes (11.2.13). To prove (11.2.14) we start with

$$\beta(x) + \beta(1-x) = \frac{1}{2}\left[\psi\left(\frac{1}{2}+\frac{x}{2}\right) - \psi\left(\frac{x}{2}\right) + \psi\left(1-\frac{x}{2}\right) - \psi\left(\frac{1}{2}-\frac{x}{2}\right)\right].$$

The formula (11.2.14) now follows from (10.2.17) and (10.2.22). □

11.3 Some elementary changes of variables

The class of integrals evaluated here is obtained from (11.1.1) by some elementary manipulations.

Example 11.1. The change $x = t^p$ in (11.1.1) yields

$$\beta(a) = p\int_0^1 \frac{t^{ap-1}\,dt}{1+t^p}. \tag{11.3.1}$$

Replace a by $\frac{a}{p}$ to obtain **3.241.1**:

$$\int_0^1 \frac{t^{a-1}\,dt}{1+t^p} = \frac{1}{p}\beta\left(\frac{a}{p}\right). \tag{11.3.2}$$

Example 11.2. The special case $p=2$ in Example 11.1 gives

$$\beta(a) = 2\int_0^1 \frac{t^{2a-1}\,dt}{1+t^2}. \tag{11.3.3}$$

Choose $a = \frac{b+1}{2}$, and relabel the variable of integration as x, to obtain **3.249.4**:

$$\int_0^1 \frac{x^b\,dx}{1+x^2} = \frac{1}{2}\beta\left(\frac{b+1}{2}\right). \tag{11.3.4}$$

Example 11.3. The evaluation of **3.251.7**:

$$\int_0^1 \frac{x^a\,dx}{(1+x^2)^2} = -\frac{1}{4} + \frac{a-1}{4}\beta\left(\frac{a-1}{2}\right) \tag{11.3.5}$$

comes from the change of variables $t = x^2$ and integration by parts. Indeed,

$$\begin{aligned}\int_0^1 \frac{x^a\,dx}{(1+x^2)^2} &= \frac{1}{2}\int_0^1 t^{(a-1)/2}\,\frac{d}{dt}\frac{1}{1+t}\,dt \\ &= -\frac{1}{4} + \frac{a-1}{4}\int_0^1 \frac{t^{(a-3)/2}\,dt}{1+t}\,dt,\end{aligned}$$

and (11.3.5) has been established.

Example 11.4. Formula **3.231.2** states that

$$\int_0^1 \frac{x^{p-1} + x^{-p}}{1+x}\, dx = \frac{\pi}{\sin \pi p}. \tag{11.3.6}$$

The integrals is recognized as $\beta(p) + \beta(1-p)$ and its value follows from (11.2.14). Similarly, **3.231.4** is

$$\int_0^1 \frac{x^{p} - x^{-p}}{1+x}\, dx = \frac{1}{p} - \frac{\pi}{\sin \pi p}. \tag{11.3.7}$$

The integral is now recognized as $\beta(1+p) - \beta(1-p)$, and the result follows from (11.2.15).

Example 11.5. The evaluation of **3.244.1**:

$$\int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1+x^q}\, dx = \frac{\pi}{q} \text{cosec} \frac{p\pi}{q} \tag{11.3.8}$$

is

$$I = \frac{1}{q} \left(\beta(p/q) + \beta(1-p/q) \right) \tag{11.3.9}$$

according to (11.3.2). The result now follows from (11.2.14).

Example 11.6. The evaluation of **3.269.2**:

$$\int_0^1 x \frac{x^{p} - x^{-p}}{1+x^2}\, dx = \frac{1}{p} - \frac{\pi}{2\sin(\pi p/2)} \tag{11.3.10}$$

is obtained by the change of variables $t = x^2$, that produces

$$I = \frac{1}{2} \int_0^1 \frac{t^{p/2} - t^{-p/2}}{1+t}\, dt = \frac{1}{2} \left[\beta\left(\frac{p}{2} + 1 \right) - \beta\left(1 - \frac{p}{2} \right) \right]. \tag{11.3.11}$$

The result now follows from (11.2.15).

11.4 Some exponential integrals

In this section we present some exponential integrals that may be evaluated in terms of the β-function.

Example 11.7. The change of variables $x = e^{-t}$ in (11.1.1) gives

$$\beta(a) = \int_0^\infty \frac{e^{-at}\, dt}{1+e^{-t}}. \tag{11.4.1}$$

This appears as **3.311.2** in [40].

Example 11.8. The evaluation of **3.311.13**:

$$\int_0^\infty \frac{e^{-px}+e^{-qx}}{1+e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\text{cosec}\left(\frac{\pi p}{p+q}\right) \tag{11.4.2}$$

is achieved by the change of variables $t=(p+q)x$ that produces

$$\begin{aligned} I &= \frac{1}{p+q}\int_0^\infty \frac{e^{-pt/(p+q)}}{1+e^{-t}}\,dt + \frac{1}{p+q}\int_0^\infty \frac{e^{-qt/(p+q)}}{1+e^{-t}}\,dt \\ &= \frac{1}{p+q}\left[\beta\left(\frac{p}{p+q}\right)+\beta\left(1-\frac{p}{p+q}\right)\right]. \end{aligned}$$

The result now comes from (11.2.15).

11.5 Some trigonometrical integrals

In this section we present the evaluation of some trigonometric integrals using the β-function.

Example 11.9. The change of variables $x=\tan^2 t$ in (11.1.1) gives

$$\beta(a) = 2\int_0^{\pi/4} \tan^{2a-1} t\,dt. \tag{11.5.1}$$

Introduce the new parameter $b=2a-1$ to obtain **3.622.2**:

$$\int_0^{\pi/4} \tan^b t\,dt = \frac{1}{2}\beta\left(\frac{b+1}{2}\right). \tag{11.5.2}$$

Example 11.10. The change of variables $x=\tan t$ in (11.3.5) gives

$$\int_0^{\pi/4} \tan^a t\,\cos^2 t\,dt = -\frac{1}{4}+\frac{a-1}{4}\beta\left(\frac{a-1}{2}\right). \tag{11.5.3}$$

Now use (11.2.13) to obtain

$$\beta\left(\frac{a-1}{2}\right) = \frac{2}{a-1}-\beta\left(\frac{a+1}{2}\right), \tag{11.5.4}$$

that converts (11.5.3) to

$$\int_0^{\pi/4} \tan^a t\,\cos^2 t\,dt = \frac{1}{4}+\frac{1-a}{4}\beta\left(\frac{a+1}{2}\right). \tag{11.5.5}$$

This is the form in which **3.623.3** appears in [40]. `Mathematica` gives

$$\begin{aligned}\int_0^{\pi/4} \tan^a x \cos^2 x\, dx &= \frac{1}{2^{(a-3)/2}(a-3)} {}_2F_1\left(\begin{matrix} \frac{1-a}{2} & \frac{3-a}{2} \\ & \frac{5-a}{2} \end{matrix} \middle| \frac{1}{2} \right) \\ &- -\frac{\pi}{4}(a-1)\sec\left(\frac{\pi a}{2}\right).\end{aligned}$$

Using this form and (11.5.2) we obtain **3.623.2**:

$$\int_0^{\pi/4} \tan^a t \sin^2 t\, dt = -\frac{1}{4} + \frac{1+a}{4}\beta\left(\frac{a+1}{2}\right). \tag{11.5.6}$$

`Mathematica` gives the expression

$$\int_0^{\pi/4} \tan^a x \sin^2 x\, dx = \frac{1}{2^{(a+3)/2}(a+3)} {}_2F_1\left(\begin{matrix} \frac{a+1}{2} & \frac{a+3}{2} \\ & \frac{a+5}{2} \end{matrix} \middle| \frac{1}{2} \right). \tag{11.5.7}$$

Example 11.11. The evaluation of **3.624.1**:

$$\int_0^{\pi/4} \frac{\sin^p x\, dx}{\cos^{p+2} x} = \frac{1}{p+1} \tag{11.5.8}$$

can be done by writing the integral as

$$I = \int_0^{\pi/4} \tan^{p+2} x\, dx + \int_0^{\pi/4} \tan^p x\, dx. \tag{11.5.9}$$

These are evaluated using (11.5.2) to obtain

$$I = \frac{1}{2}\beta\left(\frac{p+3}{2}\right) + \frac{1}{2}\beta\left(\frac{p+1}{2}\right). \tag{11.5.10}$$

The rule (11.2.13) completes the proof.

Example 11.12. The integral **3.651.2**

$$\int_0^{\pi/4} \frac{\tan^\mu x\, dx}{1 - \sin x \cos x} = \frac{1}{3}\left(\beta\left(\frac{\mu+2}{2}\right) + \beta\left(\frac{\mu+1}{2}\right)\right) \tag{11.5.11}$$

can be established directly using the integral definition of β given in (11.1.1). Simply observe that dividing the numerator and denominator of the integrand by $\cos^2 x$ yields, after the change of variables $t = \tan x$, the identity

$$\begin{aligned}\int_0^{\pi/4} \frac{\tan^\mu x\, dx}{1 - \sin x \cos x} &= \int_0^{\pi/4} \frac{\tan^\mu x}{(\sec^2 x - \tan x)} \frac{dx}{\cos^2 x} \\ &= \int_0^1 \frac{t^\mu\, dt}{t^2 - t + 1} \\ &= \int_0^1 \frac{t^{\mu+1} + t^\mu}{t^3 + 1}\, dt.\end{aligned}$$

The change of variables $t = s^{1/3}$ gives the result.

The evaluation of **3.651.1**

$$\int_0^{\pi/4} \frac{\tan^\mu x\,dx}{1+\sin x \cos x} = \frac{1}{3}\left(\psi\left(\frac{\mu+2}{2}\right) - \psi\left(\frac{\mu+1}{2}\right)\right) \tag{11.5.12}$$

can be established along the same lines. This part employs the representation **8.361.7**:

$$\psi(z) = \int_0^1 \frac{x^{z-1}-1}{x-1}\,dx - \gamma \tag{11.5.13}$$

established in [61].

Example 11.13. The elementary identity

$$\frac{1}{1-\sin^2 x\cos^2 x} = \frac{1}{2}\left(\frac{1}{1+\sin x\cos x} + \frac{1}{1-\sin x \cos x}\right) \tag{11.5.14}$$

and the evaluations given in Examples 11.5.12 and 11.5.11 gives a proof of **3.656.1**:

$$\begin{aligned}\int_0^{\pi/4} \frac{\tan^\mu x\,dx}{1-\sin^2 x\cos^2 x} &= \\ &\tfrac{1}{12}\left(-\psi\left(\tfrac{\mu+1}{6}\right) - \psi\left(\tfrac{\mu+2}{6}\right) + \psi\left(\tfrac{\mu+4}{6}\right) + \psi\left(\tfrac{\mu+5}{6}\right)\right) \\ &\qquad + \tfrac{1}{6}\left(\psi\left(\tfrac{\mu+2}{6}\right) - \psi\left(\tfrac{\mu+1}{6}\right)\right).\end{aligned}$$

Example 11.14. The final integral in this section is **3.635.1**:

$$\int_0^{\pi/4} \cos^{\mu-1}(2x)\,\tan x\,dx = \frac{1}{2}\beta(\mu). \tag{11.5.15}$$

This is easy: start with

$$\tan x = \frac{\sin x}{\cos x} = \frac{2\sin x\cos x}{2\cos^2 x} = \frac{\sin 2x}{1+\cos 2x}, \tag{11.5.16}$$

and use the change of variables $t = \cos 2x$ to produce the result.

11.6 Some hyperbolic integrals

This section contains the evaluation of some hyperbolic integrals using the β-function.

Example 11.15. The integral (11.4.1) can be written as

$$\beta(a) = \int_0^\infty \frac{e^{t(1/2-a)}\,dt}{e^{t/2}+e^{-t/2}}, \tag{11.6.1}$$

and with $t=2y$ and $b=2a-1$, we obtain **3.541.6**:

$$\int_0^\infty \frac{e^{-by}\,dy}{\cosh y} = \beta\left(\frac{b+1}{2}\right). \tag{11.6.2}$$

Example 11.16. Integration by parts produces

$$\begin{aligned}\int_0^\infty \frac{e^{-ax}\,dx}{\cosh^2 x} &= 2\int_0^\infty e^{-ax}\frac{d}{dx}\frac{1}{1+e^{-2x}}\,dx \\ &= -1+2a\int_0^\infty \frac{e^{-ax}\,dx}{1+e^{-2x}}.\end{aligned}$$

The change of variables $t=2x$ now gives the evaluation of **3.541.8**:

$$\int_0^\infty \frac{e^{-ax}\,dx}{\cosh^2 x} = a\beta\left(\frac{a}{2}\right)-1. \tag{11.6.3}$$

Example 11.17. The change of variables $t=e^{-x}$ gives

$$\int_0^\infty e^{-ax}\tanh x\,dx = \int_0^1 \frac{t^{a-1}-t^a}{1+t^2}\,dt, \tag{11.6.4}$$

and with $s=t^2$ we get

$$\begin{aligned}I &= \frac{1}{2}\int_0^1 \frac{s^{a/2-1}-s^{(a-1)/2}}{1+s}\,ds \\ &= \frac{1}{2}\left[\beta\left(\frac{a}{2}\right)-\beta\left(\frac{a}{2}+1\right)\right].\end{aligned}$$

The transformation rule (11.2.13) gives the evaluation of **3.541.7**:

$$\int_0^\infty e^{-ax}\tanh x\,dx = \beta\left(\frac{a}{2}\right)-\frac{1}{a}. \tag{11.6.5}$$

11.7 Differentiation formulas

Example 11.18. Differentiating (11.1.1) with respect to the parameter a yields

$$\int_0^1 \frac{x^{a-1}\,\ln x}{1+x}\,dx = \beta'(a), \tag{11.7.1}$$

that appears as **4.251.3** in [40].

Example 11.19. Differentiating (11.3.2) n times with respect to the parameter a produces **4.271.16** written in the form

$$\int_0^1 \frac{x^{a-1}\ln^n x}{1+x^p}\,dx = \frac{1}{p^{n+1}}\beta^{(n)}\left(\frac{a}{p}\right). \tag{11.7.2}$$

The choice $n=1$ now gives formula **4.254.4** in [40]:

$$\int_0^1 \frac{x^{a-1}\ln x}{1+x^p}\,dx = \frac{1}{p^2}\beta'\left(\frac{a}{p}\right). \tag{11.7.3}$$

Example 11.20. The special case $n=1$, $a=1$ and $p=1$ in (11.7.2) produces the elementary integral **4.231.1**:

$$\int_0^1 \frac{\ln x\,dx}{1+x} = -\frac{\pi^2}{12}. \tag{11.7.4}$$

In this evaluation we have employed the values

$$\psi'(1) = \zeta(2) = \frac{\pi^2}{6}, \text{ and } \psi'(1/2) = \frac{\pi^2}{12}, \tag{11.7.5}$$

that appear in (11.2.12).

Example 11.21. Formula **4.231.14**:

$$\int_0^1 \frac{x\,\ln x}{1+x^2}\,dx = -\frac{\pi^2}{48} \tag{11.7.6}$$

comes from (11.7.3) by choosing the parameters $n=1$, $a=2$ and $p=2$. The values of $\psi'(1)$ and $\psi'(1/2)$ are employed again. Naturally, this evaluation also comes from (11.7.4) via the change of variables $x^2 \mapsto x$.

Example 11.22. The choice $n=a=1$ and $p=2$ in (11.7.3) and the values

$$\psi^{(2)}\left(\tfrac{1}{4}\right) = \pi^2 + 8G \text{ and } \psi^{(2)}\left(\tfrac{3}{4}\right) = \pi^2 - 8G, \tag{11.7.7}$$

where G is *Catalan's constant* defined by

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \tag{11.7.8}$$

yields the evaluation of **4.231.12**:

$$\int_0^1 \frac{\ln x\,dx}{1+x^2} = -G. \tag{11.7.9}$$

The change of variables $x = t/a$, with $a>0$, and the elementary integral

$$\int_0^a \frac{dt}{t^2+a^2} = \frac{\pi}{4a}, \tag{11.7.10}$$

give the evaluation of **4.231.11**:

$$\int_0^a \frac{\ln x\,dx}{x^2+a^2} = \frac{\pi\ln a - 4G}{4a}. \tag{11.7.11}$$

Example 11.23. Now choose $n = 1$, $a = 2$ and $p = 1$ in (11.7.3) and use the value $\psi'(3/2) = \pi^2/2 - 4$ given in (11.2.12) to obtain **4.231.19**:

$$\int_0^1 \frac{x \ln x}{1+x}\, dx = \frac{\pi^2}{12} - 1. \tag{11.7.12}$$

Combining this with (11.7.4) gives **4.231.20**:

$$\int_0^1 \frac{1-x}{1+x} \ln x \, dx = 1 - \frac{\pi^2}{6}. \tag{11.7.13}$$

Example 11.24. The values

$$\psi^{(2)}\left(\tfrac{1}{4}\right) = -2\pi^3 - 56\zeta(3) \text{ and } \psi^{(2)}\left(\tfrac{3}{4}\right) = 2\pi^3 - 56\zeta(3), \tag{11.7.14}$$

given in [79], are now used to produce the evaluation of **4.261.6**:

$$\int_0^1 \frac{\ln^2 x \, dx}{1+x^2} = \frac{\pi^3}{16}. \tag{11.7.15}$$

Example 11.25. The relation

$$\psi^{(n)}(1-z) + (-1)^{n+1}\psi^{(n)}(z) = (-1)^n \pi \frac{d^n}{dz^n} \cot \pi z, \tag{11.7.16}$$

and the choice $n = 4$, $a = 1$ and $p = 2$ in (11.7.3) give

$$\begin{aligned} \int_0^1 \frac{\ln^4 x \, dx}{1+x^2} &= \frac{1}{2^5}\beta^{(4)}\left(\tfrac{1}{2}\right) \\ &= \frac{1}{1024}\left(\psi^{(4)}\left(\tfrac{3}{4}\right) - \psi^{(4)}\left(\tfrac{1}{4}\right)\right) \\ &= \frac{1}{1024}\left(-\pi \frac{d^4}{dz^4} \cot \pi z\Big|_{z=3/4}\right). \end{aligned}$$

This yields the evaluation of **4.263.2**:

$$\int_0^1 \frac{\ln^4 x \, dx}{1+x^2} = \frac{5\pi^5}{64}. \tag{11.7.17}$$

The evaluation of **4.265**:

$$\int_0^1 \frac{\ln^6 x \, dx}{1+x^2} = \frac{61\pi^7}{256}, \tag{11.7.18}$$

can be checked by the same method.

Example 11.26. Now choose $n \in \mathbb{N}$ and take $a = n+1$ and $p = 1$ in (11.7.3) to obtain the expression

$$I := \int_0^1 \frac{x^n \ln^2 x}{1+x}\, dx = \frac{1}{8}\beta^{(2)}(n+1). \tag{11.7.19}$$

This is now expressed in terms of the $\psi-$function and then simplified employing the relation

$$\psi^{(m)}(z) = (-1)^{m+1} m! \zeta(m+1, z), \tag{11.7.20}$$

with the *Hurwitz zeta function*

$$\zeta(s, z) := \sum_{k=0}^{\infty} \frac{1}{(z+k)^s}. \tag{11.7.21}$$

We conclude that

$$I = \frac{1}{4} \left(\zeta \left(3, \frac{n+1}{2} \right) - \zeta \left(3, \frac{n+2}{2} \right) \right). \tag{11.7.22}$$

The elementary identity

$$\zeta\left(s, \tfrac{a}{2}\right) - \zeta\left(s, \tfrac{a+1}{2}\right) = 2^s \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^s}, \tag{11.7.23}$$

is now used with $s = 3$ and $a = n+1$ to obtain

$$\int_0^1 \frac{x^n \ln^2 x \, dx}{1+x} = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n+1)^3}. \tag{11.7.24}$$

This is finally transformed to the form

$$\int_0^1 \frac{x^n \ln^2 x \, dx}{1+x} = (-1)^n \left(\frac{3}{2} \zeta(3) + 2 \sum_{k=1}^{n} \frac{(-1)^k}{k^3} \right). \tag{11.7.25}$$

This is **4.261.11** of [40].

The same method produces **4.262.4**:

$$\int_0^1 \frac{x^n \ln^3 x \, dx}{1+x} = (-1)^{n+1} \left(\frac{7\pi^4}{120} - 6 \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^4} \right). \tag{11.7.26}$$

Example 11.27. The method of the previous example yields the value of **4.262.1**:

$$\int_0^1 \frac{\ln^3 x \, dx}{1+x} = -\frac{7\pi^4}{120}. \tag{11.7.27}$$

Here we use $\psi^{(3)}(1) = \pi^4/15$ and $\psi^{(3)}(1/2) = \pi^4$.

Similarly, $\psi^{(5)}(1) = 8\pi^6/63$ and $\psi^{(5)}(1/2) = 8\pi^6$ yield **4.264.1**:

$$\int_0^1 \frac{\ln^5 x \, dx}{1+x} = -\frac{31\pi^6}{252}, \tag{11.7.28}$$

while $\psi^{(7)}(1) = 8\pi^8/15$ and $\psi^{(7)}(1/2) = 136\pi^8$ give **4.266.1**:

$$\int_0^1 \frac{\ln^7 x\, dx}{1+x} = -\frac{127\pi^8}{240}. \tag{11.7.29}$$

Example 11.28. A combination of the evaluations given above produces **4.261.2**:

$$\int_0^1 \frac{\ln^2 x\, dx}{1-x+x^2} = \frac{10\pi^3}{81\sqrt{3}}. \tag{11.7.30}$$

Indeed,

$$\begin{aligned}
\int_0^1 \frac{\ln^2 x\, dx}{1-x+x^2} &= \int_0^1 \frac{1+x}{1+x^3} \ln^2 x\, dx \\
&= \int_0^1 \frac{\ln^2 x\, dx}{1+x^3} + \int_0^1 \frac{x \ln^2 x\, dx}{1+x^3} \\
&= \frac{1}{27} \left(\beta^{(2)}(\tfrac{1}{3}) + \beta^{(2)}(\tfrac{2}{3}) \right) \\
&= \frac{1}{216} \left(\psi^{(2)}\left(\tfrac{2}{3}\right) - \psi^{(2)}\left(\tfrac{1}{3}\right) + \psi^{(2)}\left(\tfrac{5}{6}\right) - \psi^{(2)}\left(\tfrac{1}{6}\right) \right) \\
&= \frac{\pi}{216} \left(\frac{d^2}{dz^2} \cot \pi z \Big|_{z=1/3} + \frac{d^2}{dz^2} \cot \pi z \Big|_{z=1/6} \right) \\
&= \frac{\pi}{216} \left(\frac{8\pi^2}{3\sqrt{3}} + 8\sqrt{3}\pi^2 \right) = \frac{10\pi^3}{81\sqrt{3}}.
\end{aligned}$$

Example 11.29. Replace n by $2n$ in (11.7.2) and set $a = p = 1$ to produce

$$\begin{aligned}
\int_0^1 \frac{\ln^{2n} x\, dx}{1+x} &= \beta^{(2n)}(1) \\
&= \frac{1}{2^{2n+1}} \left(\psi^{(2n)}(1) - \psi^{(2n)}\left(\tfrac{1}{2}\right) \right) \\
&= \frac{2^{2n}-1}{2^{2n}} (2n)! \zeta(2n+1).
\end{aligned}$$

This appears as **4.271.1**.

Example 11.30. The change of variables $t = bx$ in (11.7.2) produces

$$\begin{aligned}
\int_0^b \frac{t^{a-1} \ln t}{b^p + t^p} &= \frac{b^{a-p}}{p^2} \beta'\left(\tfrac{a}{p}\right) + b^{1-a} \ln b \int_0^b \frac{t^{a-1}\, dt}{b^p + t^p} \\
&= \frac{b^{a-b}}{p^2} \beta'\left(\tfrac{a}{p}\right) + \ln b \frac{b^{a-p}}{p} \beta\left(\tfrac{a}{p}\right).
\end{aligned}$$

The last integral was evaluated using (11.3.2).

Differentiate this identity with respect to the parameter b to obtain

$$\begin{aligned} \int_0^b \frac{t^{a-1}\ln t}{(b^p+t^p)^2}\,dt &= \frac{b^{a-2p}\ln b}{2p} + \frac{p-a}{p^3} b^{a-2p}\beta'\left(\tfrac{a}{p}\right) \\ &\quad - \frac{b^{a-2p}}{p^2}(1+(a-p)\ln b)\beta\left(\tfrac{a}{p}\right). \end{aligned} \tag{11.7.31}$$

The special case $a=b=p=1$ yields **4.231.6**:

$$\int_0^1 \frac{\ln x\,dx}{(1+x)^2} = -\beta(1) = -\ln 2. \tag{11.7.32}$$

Similarly, the choice $a=2$, $b=1$ and $p=2$ yields **4.234.2**:

$$\int_0^1 \frac{x\ln x\,dx}{(1+x^2)^2} = -\frac{1}{4}\beta(1) = -\frac{\ln 2}{4}. \tag{11.7.33}$$

Example 11.31. The last example of this section we present an evaluation of **4.234.1**:

$$\int_1^\infty \frac{\ln x\,dx}{(1+x^2)^2} = \frac{G}{2} - \frac{\pi}{8}, \tag{11.7.34}$$

using the methods developed here. We begin with the change of variables $x \mapsto 1/x$ to transform the problem to the interval $[0,1]$. We have

$$\int_1^\infty \frac{\ln x\,dx}{(1+x^2)^2} = -\int_0^1 \frac{x^2\ln x\,dx}{(1+x^2)^2}. \tag{11.7.35}$$

Now choose $a=3$, $b=-1$ and $p=2$ in (11.7.31) to obtain

$$\int_0^1 \frac{x^2\ln x\,dx}{(1+x^2)^2} = -\frac{1}{8}\beta'\left(\frac{3}{2}\right) + \frac{1}{4}\beta\left(\frac{3}{2}\right). \tag{11.7.36}$$

The value of (11.7.34) now follows from

$$\begin{aligned} \tfrac{1}{4}\beta\left(\tfrac{3}{2}\right) &= \tfrac{1}{8}\left(\psi\left(\tfrac{5}{4}\right) - \psi\left(\tfrac{3}{4}\right)\right) \\ &= \tfrac{1}{8}\left(4 + \psi\left(\tfrac{1}{4}\right) - \psi\left(\tfrac{3}{4}\right)\right) \\ &= \tfrac{1}{2} - \tfrac{\pi}{8}, \end{aligned}$$

and

$$\begin{aligned} \tfrac{1}{8}\beta'\left(\tfrac{3}{2}\right) &= \tfrac{1}{32}\left(\psi'\left(\tfrac{5}{4}\right) - \psi'\left(\tfrac{3}{4}\right)\right) \\ &= \tfrac{1}{32}\left(\psi'\left(\tfrac{1}{4}\right) - \psi'\left(\tfrac{3}{4}\right) - 16\right) \\ &= \tfrac{1}{32}\left(\zeta(2,\tfrac{1}{4}) - \zeta(2,\tfrac{3}{4}) - 16\right) \\ &= \tfrac{G}{2} - \tfrac{1}{2}. \end{aligned}$$

11.8 One last example

In this section we discuss the evaluation of **3.522.4**:

$$\int_0^\infty \frac{dx}{(b^2+x^2)\cosh \pi x} = \frac{1}{b}\beta\left(b+\frac{1}{2}\right). \tag{11.8.1}$$

The technique illustrated here will be employed in a future publication to discuss many other evaluations.

To establish (11.8.1), introduce the function

$$h(b,y) := \int_0^\infty e^{-bt}\frac{\cos yt}{\cosh t}\,dt. \tag{11.8.2}$$

This function is harmonic and bounded for $\operatorname{Re} b > 0$. Therefore it admits a Poisson representation

$$h(b,y) = \frac{1}{\pi}\int_{-\infty}^\infty h(0,u)\frac{b}{b^2+(y-u)^2}\,du. \tag{11.8.3}$$

The value $h(0,u)$ is a well-known Fourier transform

$$h(0,u) = \int_0^\infty \frac{\cos yt}{\cosh t}\,dt = \frac{\pi}{2\,\cosh(\pi y/2)}, \tag{11.8.4}$$

that appears as **3.981.3** in [40]. Therefore we have

$$h(b,y) = \frac{b}{2}\int_{-\infty}^\infty \frac{du}{\cosh(\pi u/2)\;[b^2+(y-u)^2]}. \tag{11.8.5}$$

The special value $y=0$ and (11.6.2) give the result (after replacing b by $2b$ and u by $2u$).

We conclude with an interpretation of (11.8.1) in terms of the sine Fourier transform of a function related to $\beta(x)$. The proof is a simple application of the elementary identity

$$\int_0^\infty e^{xt}\sin bt\,dt = \frac{b}{b^2+x^2}. \tag{11.8.6}$$

The details are left to the reader.

Theorem 11.8.1. *Let*

$$\mu(x) := \int_0^\infty \frac{e^{-xt}\,dt}{\cosh t} = \beta\left(\frac{x+1}{2}\right). \tag{11.8.7}$$

Then **3.522.4** *in (11.8.1) is equivalent to the identity*

$$\int_0^\infty \mu(t) \sin xt \, dt = \mu\left(\frac{2x}{\pi}\right). \tag{11.8.8}$$

Note. `Mathematica` evaluates all the entries in this chapter with the single exception of entry **3.522.4** in (11.8.1).

Chapter 12

Some logarithmic integrals

12.1 Introduction

The classical table of integrals by I. Gradshteyn and I. M. Ryzhik [40] contains many entries from the family

$$\int_0^1 R(x) \ln x \, dx \tag{12.1.1}$$

where R is a rational function. For instance, the elementary integral **4.231.1**

$$\int_0^1 \frac{\ln x \, dx}{1+x} = -\frac{\pi^2}{12}, \tag{12.1.2}$$

is evaluated simply by expanding the integrand in a power series. In [2], the first author and collaborators have presented a systematic study of integrals of the form

$$h_{n,1}(b) = \int_0^b \frac{\ln t \, dt}{(1+t)^{n+1}}, \tag{12.1.3}$$

as well as the case in which the integrand has a single purely imaginary pole

$$h_{n,2}(a,b) = \int_0^b \frac{\ln t \, dt}{(t^2+a^2)^{n+1}}. \tag{12.1.4}$$

The work presented here deals with integrals where the rational part of the integrand is allowed to have arbitrary complex poles.

12.2 Evaluations in terms of polylogarithms

In this section we describe the evaluation of

$$g(a) = \int_0^1 \frac{\ln x \, dx}{x^2 - 2ax + 1}, \tag{12.2.1}$$

under the assumption that the denominator has non-real roots, that is, $a^2 < 1$.

The first approach to the evaluation of $g(a)$ is based on the factorization of the quartic as

$$x^2 - 2ax + 1 = (x + r_1)(x + r_2), \tag{12.2.2}$$

where $r_1 = -a + i\sqrt{1 - a^2}$ and $r_2 = -a - i\sqrt{1 - a^2}$. The partial fraction expansion

$$\frac{1}{(x + r_1)(x + r_2)} = \frac{1}{r_2 - r_1} \left(\frac{1}{x + r_1} - \frac{1}{x + r_2} \right), \tag{12.2.3}$$

yields

$$g(a) = \frac{1}{r_2 - r_1} \int_0^1 \frac{\ln x \, dx}{x + r_1} - \frac{1}{r_2 - r_1} \int_0^1 \frac{\ln x \, dx}{x + r_2}. \tag{12.2.4}$$

These integrals are computed in terms of the *dilogarithm* function defined by

$$\text{PolyLog}[2, x] := -\int_0^x \frac{\ln(1 - t)}{t} \, dt. \tag{12.2.5}$$

A direct calculaton shows that

$$\int \frac{\ln x \, dx}{x + a} = \ln x \ln(1 + x/a) + \text{PolyLog}[2, -x/a], \tag{12.2.6}$$

and thus

$$\int_0^1 \frac{\ln x \, dx}{x + a} = \text{PolyLog}\left[2, -\frac{1}{a}\right]. \tag{12.2.7}$$

It follows that

$$g(a) = \frac{1}{r_2 - r_1} \left(\text{PolyLog}\left[2, -\frac{1}{r_1}\right] - \text{PolyLog}\left[2, -\frac{1}{r_2}\right] \right). \tag{12.2.8}$$

Observe that the real integral $g(a)$ appears here expressed in terms of the polylogarithm of complex arguments.

Example 12.1. The case $a = 1/2$ yields

$$\int_0^1 \frac{\ln x \, dx}{x^2 - x + 1} = \frac{i}{\sqrt{3}} \left(\text{PolyLog}\left[2, (1 + i\sqrt{3})/2\right] - \text{PolyLog}\left[2, (1 - i\sqrt{3})/2\right] \right). \tag{12.2.9}$$

The polylogarithm function is evaluated using the representation

$$(1+i\sqrt{3})/2 = e^{i\pi/3}, \tag{12.2.10}$$

to produce

$$\begin{aligned}\text{PolyLog}\left[2,(1+i\sqrt{3})/2\right] &= \sum_{k=1}^{\infty}\frac{\left[\frac{1}{2}(1+i\sqrt{3})\right]^k}{k^2} = \sum_{k=1}^{\infty}\frac{e^{i\pi k/3}}{k^2} \\ &= \sum_{k=1}^{\infty}\frac{\cos\left(\frac{\pi k}{3}\right)+i\sin\left(\frac{\pi k}{3}\right)}{k^2}.\end{aligned}$$

Similarly

$$\text{PolyLog}\left[2,(1-i\sqrt{3})/2\right] = \sum_{k=1}^{\infty}\frac{\cos\left(\frac{\pi k}{3}\right)-i\sin\left(\frac{\pi k}{3}\right)}{k^2},$$

and it follows that

$$\begin{aligned}\int_0^1\frac{\ln x\,dx}{x^2-x+1} &= \frac{i}{\sqrt{3}}\Big(\text{PolyLog}\left[2,(1+i\sqrt{3})/2\right] \\ &\quad -\text{PolyLog}\left[2,(1-i\sqrt{3})/2\right]\Big) = -\frac{2}{\sqrt{3}}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{\pi k}{3}\right)}{k^2}.\end{aligned}$$

The function $\sin(\pi k/3)$ is periodic, with period 6, and repeating values

$$\tfrac{\sqrt{3}}{2}, \tfrac{\sqrt{3}}{2}, 0, -\tfrac{\sqrt{3}}{2}, -\tfrac{\sqrt{3}}{2}, 0.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{\pi k}{3}\right)}{k^2} &= \frac{\sqrt{3}}{2}\Bigg(\sum_{k=0}^{\infty}\frac{1}{(6k+1)^2}+\sum_{k=0}^{\infty}\frac{1}{(6k+2)^2}-\sum_{k=0}^{\infty}\frac{1}{(6k+4)^2} \\ &\quad -\sum_{k=0}^{\infty}\frac{1}{(6k+5)^2}\Bigg).\end{aligned}$$

To evaluate these sums, recall the series representation of the *polygamma* function $\psi(x) = \Gamma'(x)/\Gamma(x)$, given by

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\frac{x}{k(x+k)}. \tag{12.2.11}$$

Differentiation yields

$$\psi'(x) = -\sum_{k=0}^{\infty}\frac{1}{(x+k)^2}, \tag{12.2.12}$$

and we obtain

$$\sum_{k=0}^{\infty} \frac{1}{(6k+j)^2} = \frac{1}{36} \sum_{k=0}^{\infty} \frac{1}{(k+\frac{j}{6})^2}.$$

This provides the expression

$$\sum_{k=1}^{\infty} \frac{\sin\left(\frac{\pi k}{3}\right)}{k^2} = \frac{\sqrt{3}}{72} \left(\psi'\left(\tfrac{1}{6}\right) + \psi'\left(\tfrac{2}{6}\right) - \psi'\left(\tfrac{4}{6}\right) - \psi'\left(\tfrac{5}{6}\right) \right). \tag{12.2.13}$$

The integral (12.2.9) is

$$\int_0^1 \frac{\ln x \, dx}{x^2 - x + 1} = -\frac{1}{36} \left(\psi'\left(\tfrac{1}{6}\right) + \psi'\left(\tfrac{2}{6}\right) - \psi'\left(\tfrac{4}{6}\right) - \psi'\left(\tfrac{5}{6}\right) \right). \tag{12.2.14}$$

The identities

$$\psi(1-x) = \psi(x) + \pi \cot \pi x, \tag{12.2.15}$$

and

$$\psi(2x) = \frac{1}{2} \left(\psi(x) + \psi(x + \tfrac{1}{2}) \right) + \ln 2, \tag{12.2.16}$$

produce

$$\psi'\left(\tfrac{1}{6}\right) = 5\psi'\left(\tfrac{1}{3}\right) - \tfrac{4\pi^2}{3}, \; \psi'\left(\tfrac{2}{3}\right) = -\psi'\left(\tfrac{1}{3}\right) + \tfrac{4\pi^2}{3}, \; \psi'\left(\tfrac{5}{6}\right) = -5\psi'\left(\tfrac{1}{3}\right) + \tfrac{16\pi^2}{3}.$$

Replacing in (12.2.14) yields

$$\int_0^1 \frac{\ln x \, dx}{x^2 - x + 1} = \frac{2\pi^2}{9} - \frac{1}{3}\psi'\left(\frac{1}{3}\right). \tag{12.2.17}$$

This appears as formula **4.233.2** in [40].

Note. The method described in the previous example evaluates logarithmic integrals in terms of the *Clausen function*

$$\text{Cl}_2(x) := \sum_{k=1}^{\infty} \frac{\sin kx}{k^2}. \tag{12.2.18}$$

Note. An identical procedure can be used to evaluate the integrals **4.233.1**, **4.233.3**, **4.233.4** in [40] given by

$$\int_0^1 \frac{\ln x \, dx}{x^2 + x + 1} = \frac{4\pi^2}{27} - \frac{2}{9}\psi'\left(\frac{1}{3}\right), \tag{12.2.19}$$

$$\int_0^1 \frac{x \, \ln x \, dx}{x^2 + x + 1} = -\frac{7\pi^2}{54} + \frac{1}{9}\psi'\left(\frac{1}{3}\right), \tag{12.2.20}$$

and

$$\int_0^1 \frac{x \, \ln x \, dx}{x^2 - x + 1} = \frac{5\pi^2}{36} - \frac{1}{6}\psi'\left(\frac{1}{3}\right), \tag{12.2.21}$$

respectively.

12.3 An alternative approach

In this section we present an alternative evaluation for the integral

$$g(a) = \int_0^1 \frac{\ln x \, dx}{x^2 - 2ax + 1}, \tag{12.3.1}$$

based on the observation that

$$g(a) = \lim_{s \to 0} \frac{d}{ds} \int_0^1 \frac{x^s \, dx}{x^2 - 2ax + 1}. \tag{12.3.2}$$

The proof discussed here is based on the *Chebyshev* polynomials of the second kind $U_n(a)$, defined by

$$U_n(a) = \frac{\sin[(n+1)t]}{\sin t}, \tag{12.3.3}$$

where $a = \cos t$. The relation with the problem at hand comes from the generating function

$$\frac{1}{1 - 2ax + x^2} = \sum_{k=0}^{\infty} U_k(a) x^k. \tag{12.3.4}$$

This appears as **8.945.2** in [40].

Observe that

$$\int_0^1 \frac{x^s \, dx}{x^2 - 2ax + 1} = \sum_{k=0}^{\infty} U_k(a) \int_0^1 x^{k+s} \, dx = \sum_{k=0}^{\infty} \frac{U_k(a)}{k+s+1}.$$

It follows that

$$\int_0^1 \frac{\ln x \, dx}{x^2 - 2ax + 1} = -\sum_{k=0}^{\infty} \frac{U_k(a)}{(k+1)^2}. \tag{12.3.5}$$

Replacing the trigonometric expression (12.3.3) for the *Chebyshev* polynomial, it follows that

$$\int_0^1 \frac{\ln x \, dx}{x^2 - 2ax + 1} = -\frac{1}{\sin t} \sum_{k=0}^{\infty} \frac{\sin kt}{k^2} = -\frac{\text{Cl}_2(t)}{\sin t}. \tag{12.3.6}$$

This reproduces the representation discussed in Section 12.2.

Note. The methods presented here give the value of (12.3.1) in terms of the dilogarithm function. The classical values

$$\text{Cl}_2\left(\tfrac{\pi}{2}\right) = -\text{Cl}_2\left(\tfrac{3\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = G, \tag{12.3.7}$$

are easy to establish. More sophisticated evaluations appear in [79]. These are given in terms of the *Hurwitz zeta function*

$$\zeta(s,q) = \sum_{k=0}^{\infty} \frac{1}{(k+q)^s}. \tag{12.3.8}$$

For instance, the reader will find

$$\text{Cl}_2\left(\tfrac{2\pi}{3}\right) = \sqrt{3}\left(\frac{3^{-s}-1}{2}\zeta(2) + 3^{-s}\zeta(2,\tfrac{1}{3})\right), \tag{12.3.9}$$

and

$$\text{Cl}_2\left(\tfrac{\pi}{3}\right) = \sqrt{3}\left(\frac{3^{-s}-1}{2}\zeta(2) + 6^{-s}\left(\zeta(2,\tfrac{1}{6}) + \zeta(2,\tfrac{1}{3})\right)\right). \tag{12.3.10}$$

Note. Integrals of the form

$$\int_0^1 R(x) \ln\ln\frac{1}{x}\,dx \tag{12.3.11}$$

present new challeges. The reader will find some examples in [60]. The current version of `Mathematica` is able to evaluate

$$\int_0^1 \frac{x\,\ln\ln 1/x}{x^4+x^2+1}\,dx = \frac{\pi}{12\sqrt{3}}\left(6\ln 2 - 3\ln 3 + 8\ln\pi - 12\ln\Gamma(\tfrac{1}{3})\right), \tag{12.3.12}$$

but is unable to evaluate

$$\int_0^1 \frac{x\,\ln\ln 1/x}{x^4-\sqrt{2}x^2+1}\,dx = \frac{\pi}{8\sqrt{2}}\left(7\ln\pi - 4\ln\sin\tfrac{\pi}{8} - 8\ln\Gamma(\tfrac{1}{8})\right). \tag{12.3.13}$$

12.4 Higher powers of logarithms

The method of the previous sections can be used to evaluate integrals of the form

$$\int_0^1 R(x)\ln^p x\,dx, \tag{12.4.1}$$

where R is a rational function. The ideas are illustrated with the verification of formula **4.261.8** in [40]:

$$\int_0^1 \frac{1-x}{1-x^6}\ln^2 x\,dx = \frac{8\sqrt{3}\pi^3 + 351\zeta(3)}{486}. \tag{12.4.2}$$

Define

$$J_1 = \int_0^1 \frac{\ln^2 x \, dx}{1+x}, \qquad J_2 = \int_0^1 \frac{\ln^2 x \, dx}{1-x+x^2},$$
$$J_3 = \int_0^1 \frac{x \ln^2 x \, dx}{1-x+x^2}, \qquad J_4 = \int_0^1 \frac{\ln^2 x \, dx}{1+x+x^2}.$$

The partial fraction decomposition

$$\frac{1-x}{1-x^6} = \frac{1}{3}\frac{1}{1+x} + \frac{1}{6}\frac{1}{1-x+x^2} - \frac{1}{3}\frac{x}{1-x+x^2} + \frac{1}{2}\frac{1}{1+x+x^2},$$

gives

$$\int_0^1 \frac{1-x}{1-x^6} \ln^2 x \, dx = \frac{1}{3}J_1 + \frac{1}{6}J_2 - \frac{1}{3}J_3 + \frac{1}{2}J_4. \tag{12.4.3}$$

Evaluation of J_1. Consider first

$$\int_0^1 \frac{x^s}{1+x}\, dx = \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^1 x^{k+s-1}\, dx = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+s}.$$

Differentiating twice with respect to s gives

$$J_1 = \int_0^1 \frac{\ln^2 x \, dx}{1+x} = 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} = \frac{3}{2}\zeta(3). \tag{12.4.4}$$

Evaluations of J_2. Integrating the expansion

$$\frac{x^s \, dx}{x^2-2ax+1} = \sum_{k=0}^{\infty} \frac{U_k(a)}{s+k+1}, \tag{12.4.5}$$

and differentiating twice with respect to s yields

$$\int_0^1 \frac{x^s \ln^2 x \, dx}{x^2-2ax+1} = 2\sum_{k=0}^{\infty} \frac{U_k(a)}{(s+k+1)^3}. \tag{12.4.6}$$

The value $s=0$ yields

$$\int_0^1 \frac{\ln^2 x \, dx}{x^2-2ax+1} = 2\sum_{k=0}^{\infty} \frac{U_k(a)}{(k+1)^3}. \tag{12.4.7}$$

We conclude that

$$J_2 = 2\sum_{k=0}^{\infty} \frac{U_k(\frac{1}{2})}{(k+1)^3}. \tag{12.4.8}$$

The sequence $U_k(\frac{1}{2})$ is periodic of period 6 and values $1, 0, -1, -1, 0, 1$. Therefore

$$J_2 = 2\sum_{k=1}^{\infty} \frac{1}{(6k+1)^3} - 2\sum_{k=1}^{\infty} \frac{1}{(6k+3)^3} - 2\sum_{k=1}^{\infty} \frac{1}{(6k+4)^3} + 2\sum_{k=1}^{\infty} \frac{1}{(6k+5)^3}. \tag{12.4.9}$$

This can be written as

$$108J_2 = \sum_{k=1}^{\infty} \frac{1}{(k+1/6)^3} - \sum_{k=1}^{\infty} \frac{1}{(k+1/2)^3} - \sum_{k=1}^{\infty} \frac{1}{(k+2/3)^3} + \sum_{k=1}^{\infty} \frac{1}{(k+5/6)^3}.$$

Proceeding along the same lines of the previous argument, employing now the second derivative of the polygamma function yields

$$J_2 = \frac{10\pi^3}{81\sqrt{3}}. \tag{12.4.10}$$

The same type of analysis gives

$$\begin{aligned} J_3 &= \int_0^1 \frac{x \ln^2 x \, dx}{1-x+x^2} = \frac{5\pi^3}{81\sqrt{3}} - \frac{2\zeta(3)}{3}, \\ J_4 &= \int_0^1 \frac{\ln^2 x \, dx}{1+x+x^2} = \frac{81\pi^3}{81\sqrt{3}}. \end{aligned}$$

This completes the proof of (12.4.2).

The reader is invited to use the method developed here to verify

$$\int_0^1 \frac{1-x}{1-x^6} \ln^4 x \, dx = \frac{32\sqrt{3}\pi^5 + 16335\zeta(5)}{1458}, \tag{12.4.11}$$

and

$$\int_0^1 \frac{1-x}{1-x^6} \ln^6 x \, dx = \frac{7(256\sqrt{3}\pi^7 + 1327995\zeta(7))}{26244}. \tag{12.4.12}$$

`Mathematica` evaluates these results.

The methods discussed here constitute the most elementary approach to the evaluations of logarithmic integrals. M. Coffey [33] presents some of the more advanced techniques required for the computation of integrals of the form

$$\int_0^1 R(x) \ln^s x \, dx \tag{12.4.13}$$

for s real and R a rational function.

Chapter 13

Trigonometric forms of the beta function

13.1 Introduction

The table of integrals [40] contains a large variety of definite integrals in trigonometric form that can be evaluated in terms of the *beta function* defined by

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx. \tag{13.1.1}$$

The convergence of the integral requires $a,\ b > 0$.

The change of variables $x = \sin^2 t$ yields the basic representation

$$B(a,b) = 2\int_0^{\pi/2} \sin^{2a-1} t\, \cos^{2b-1} t\, dt, \tag{13.1.2}$$

that, after replacing $(2a, 2b)$ by (a, b), is written as

$$\int_0^{\pi/2} \sin^{a-1} t\, \cos^{b-1} t\, dt = \frac{1}{2} B\left(\frac{a}{2}, \frac{b}{2}\right). \tag{13.1.3}$$

This appears as **3.621.5** in [40].

13.2 Special cases

In this section we present several special cases of formula (13.1.3) that appear in [40].

Example 13.1. The choice $b = 1$ in (13.1.3) gives

$$\int_0^{\pi/2} \sin^{a-1} t\, dt = \frac{1}{2} B\left(\frac{a}{2}, \frac{1}{2}\right). \tag{13.2.1}$$

Legendre's duplication formula

$$\Gamma(2a) = \frac{2^{2a-1}}{\sqrt{\pi}} \Gamma(a)\Gamma(a + \tfrac{1}{2}) \tag{13.2.2}$$

can be used to write (13.2.1) as

$$\int_0^{\pi/2} \sin^{a-1} t\, dt = 2^{a-2} B\left(\frac{a}{2}, \frac{a}{2}\right) = \frac{2^{a-2}\,\Gamma^2(a/2)}{\Gamma(a)}. \tag{13.2.3}$$

This is **3.621.1** in [40]. The dual evaluation

$$\int_0^{\pi/2} \cos^{a-1} t\, dt = 2^{a-2} B\left(\frac{a}{2}, \frac{a}{2}\right) = \frac{2^{a-2}\,\Gamma^2(a/2)}{\Gamma(a)}, \tag{13.2.4}$$

comes from the change of variables $t \mapsto \frac{\pi}{2} - t$. The reader will find a proof of (13.2.2) in [22].

Example 13.2. The special case $a = \frac{1}{2}$ in (13.2.3) gives **3.621.7**:

$$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}}. \tag{13.2.5}$$

Example 13.3. The special case $a = \frac{3}{2}$ in (13.2.3) gives **3.621.6**:

$$\int_0^{\pi/2} \sqrt{\sin x}\, dx = \sqrt{\frac{2}{\pi}} \Gamma^2\left(\tfrac{3}{4}\right). \tag{13.2.6}$$

Example 13.4. The special case $a = \frac{5}{2}$ in (13.2.3) gives **3.621.2**:

$$\int_0^{\pi/2} \sin^{3/2} x\, dx = \frac{1}{6\sqrt{2\pi}} \Gamma^2\left(\tfrac{1}{4}\right). \tag{13.2.7}$$

Example 13.5. The special case $a = 2m + 1$ in (13.2.3) gives

$$\int_0^{\pi/2} \sin^{2m} x\, dx = 2^{2m-1} B\left(m + \tfrac{1}{2}, m + \tfrac{1}{2}\right), \tag{13.2.8}$$

and using the identity

$$\Gamma\left(m + \tfrac{1}{2}\right) = \frac{\pi}{2^{2m}} \frac{(2m)!}{m!} \tag{13.2.9}$$

it yields

$$\int_0^{\pi/2} \sin^{2m} x\, dx = \frac{\binom{2m}{m}\pi}{2^{2m+1}}. \tag{13.2.10}$$

This appears as **3.621.3**. Similarly, $a = 2m+2$ in (13.2.3) gives

$$\int_0^{\pi/2} \sin^{2m+1} x \, dx = 2^{2m} B(m+1, m+1), \tag{13.2.11}$$

that can be written as

$$\int_0^{\pi/2} \sin^{2m+1} x \, dx = \frac{2^{2m}}{(2m+1)} \binom{2m}{m}^{-1}. \tag{13.2.12}$$

This is **3.621.4**.

Example 13.6. The integral **3.622.1** is

$$\begin{aligned}
\int_0^{\pi/2} \tan^{\pm a} x \, dx &= \int_0^{\pi/2} \sin^{\pm a} x \, \cos^{\mp a} x \, dx \\
&= \tfrac{1}{2} B\left(\tfrac{1\pm a}{2}, 1 - \tfrac{1\pm a}{2}\right) \\
&= \tfrac{1}{2} \Gamma\left(\tfrac{1\pm a}{2}\right) \Gamma\left(1 - \tfrac{1\pm a}{2}\right)
\end{aligned}$$

and this reduces to

$$\int_0^{\pi/2} \tan^{\pm a} x \, dx = \frac{\pi}{2 \, \cos(\pi a/2)},$$

as it appears in the table.

Example 13.7. The identity

$$\tan^{a-1} x \, \cos^{2b-2} x = \sin^{a-1} x \, \cos^{2b-a-1} x \tag{13.2.13}$$

shows that

$$\begin{aligned}
\int_0^{\pi/2} \tan^{a-1} x \, \cos^{2b-2} x \, dx &= \int_0^{\pi/2} \sin^{a-1} x \, \cos^{2b-a-1} x \, dx \\
&= \frac{1}{2} B\left(\frac{a}{2}, b - \frac{a}{2}\right).
\end{aligned} \tag{13.2.14}$$

This appears as **3.623.1**.

Example 13.8. The formula **3.624.2** states that

$$\int_0^{\pi/2} \frac{\sin^{a-1/2} x}{\cos^{2a-1} x} \, dx = \frac{\Gamma\left(\frac{a}{2} + \frac{1}{4}\right) \Gamma(1-a)}{2\Gamma\left(\frac{5}{4} - \frac{a}{2}\right)}. \tag{13.2.15}$$

This comes directly from (13.1.3).

Example 13.9. The identity **3.627**:

$$\int_0^{\pi/2} \frac{\tan^a x}{\cos^a x} \, dx = \int_0^{\pi/2} \frac{\cot^a x}{\sin^a x} \, dx = \frac{\Gamma(a)\Gamma(\frac{1}{2} - a)}{2^a \sqrt{\pi}} \sin\left(\frac{\pi a}{2}\right), \tag{13.2.16}$$

can be verified by writing the first integral as

$$I = \int_0^{\pi/2} \sin^a x \, \cos^{1-2a} x \, dx = \frac{1}{2} B\left(\frac{a+1}{2}, \frac{1-2a}{2} \right). \tag{13.2.17}$$

The beta function is

$$\frac{1}{2} B\left(\frac{a+1}{2}, \frac{1-2a}{2} \right) = \frac{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-a\right)}{2\Gamma\left(1-\frac{a}{2}\right)}. \tag{13.2.18}$$

Using $\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin \pi t}$ we can reduce (13.2.18) to the expression in (13.2.16).

Example 13.10. The evaluation of **3.628**

$$\int_0^{\pi/2} \sec^{2p} x \, \sin^{2p-1} x \, dx = \frac{\Gamma(p)\Gamma(\frac{1}{2}-p)}{2\sqrt{\pi}}, \tag{13.2.19}$$

is direct, once we write the integral as

$$\int_0^{\pi/2} \cos^{-2p} x \sin^{2p-1} x \, dx = \frac{1}{2} B\left(\tfrac{1}{2} - p, p \right). \tag{13.2.20}$$

13.3 A family of trigonometric integrals

In this section we present the evaluation of a family of trigonometrical integrals in [40]. Many special cases appear in the table.

Proposition 13.3.1. *Let a, b, $c \in \mathbb{R}$ with the condition*

$$a + b + 2c + 2 = 0. \tag{13.3.1}$$

Then

$$\int_0^{\pi/4} \sin^a x \, \cos^b x \, \cos^c(2x) \, dx = \frac{1}{2} B\left(\frac{a+1}{2}, c+1 \right). \tag{13.3.2}$$

Proof. Let $t = \tan x$ to obtain

$$\int_0^{\pi/4} \sin^a x \, \cos^b x \, \cos^c(2x) \, dx = \int_0^1 t^a (1-t^2)^c (1+t^2)^{-(a+b+2c+2)/2} \, dt \tag{13.3.3}$$

and (13.3.1) yields

$$\int_0^{\pi/4} \sin^a x \, \cos^b x \, \cos^c(2x) \, dx = \int_0^1 t^a (1-t^2)^c \, dt. \tag{13.3.4}$$

The change of variables $s = t^2$ produces

$$\int_0^{\pi/4} \sin^a x \cos^b x \cos^c(2x)\, dx = \frac{1}{2} \int_0^1 s^{(a-1)/2}(1-s)^c\, ds, \tag{13.3.5}$$

and this last integral has the given beta value. □

Example 13.11. The formula (13.3.2), with $a = 2n$, $b = -2p - 2n - 2$, and $c = p$ appears as **3.625.2** in [40]:

$$\int_0^{\pi/4} \frac{\sin^{2n} x \cos^p(2x)}{\cos^{2p+2n+2} x}\, dx = \tfrac{1}{2} B\left(n + \tfrac{1}{2}, p+1\right). \tag{13.3.6}$$

Example 13.12. The formula **3.624.3**

$$\int_0^{\pi/4} \frac{\cos^{n-1/2}(2x)}{\cos^{2n+1} x}\, dx = \frac{\pi}{2^{2n+1}} \binom{2n}{n} \tag{13.3.7}$$

corresponds to the case $a = 0$, $b = -2n - 1$, and $c = n - \frac{1}{2}$.

Example 13.13. Formula **3.624.4** in [40]

$$\int_0^{\pi/4} \frac{\cos^{\mu}(2x)}{\cos^{2(\mu+1)} x}\, dx = 2^{2\mu} B(\mu+1, \mu+1) \tag{13.3.8}$$

corresponds to $a = 0$, $b = -2\mu - 2$, and $c = \mu$. Then (13.3.2) gives

$$\int_0^{\pi/4} \frac{\cos^{\mu}(2x)}{\cos^{2(\mu+1)} x}\, dx = \frac{1}{2} B\left(\frac{1}{2}, \mu+1\right). \tag{13.3.9}$$

The duplication formula

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma(x + \tfrac{1}{2}), \tag{13.3.10}$$

transforms (13.3.9) into (13.3.8).

Example 13.14. The values $a = 2\mu - 2$, $b = 0$, and $c = \mu$ produce **3.624.5**:

$$\int_0^{\pi/4} \frac{\sin^{2\mu-2} x}{\cos^{\mu}(2x)}\, dx = \frac{\Gamma(\mu - \frac{1}{2})\Gamma(1-\mu)}{2\sqrt{\pi}} \tag{13.3.11}$$

directly. Indeed, the answer from (13.3.2) is $B(\mu - 1/2, 1-\mu)/2$. The table also has the alternative answer $2^{1-2\mu} B(2\mu - 1, 1 - \mu)$ that can be obtained using (13.3.10).

Example 13.15. Formula **3.625.1**:

$$\int_0^{\pi/4} \frac{\sin^{2n-1} x \cos^p(2x)}{\cos^{2p+2n+2} x}\, dx = \frac{1}{2} B(n, p+1) \tag{13.3.12}$$

corresponds to $a = 2n - 1$, $b = -2p - 2n - 1$, and $c = p$.

Example 13.16. The choice $a = 2n-1$, $b = -2n-2m$, and $c = m - \frac{1}{2}$ gives **3.625.3**:

$$\int_0^{\pi/4} \frac{\sin^{2n-1} x \, \cos^{m-1/2}(2x)}{\cos^{2n+2m} x} \, dx = \frac{1}{2} B(n, m + \tfrac{1}{2}). \tag{13.3.13}$$

For n, $m \in \mathbb{N}$ we can also write

$$\int_0^{\pi/4} \frac{\sin^{2n-1} x \, \cos^{m-1/2}(2x)}{\cos^{2n+2m} x} \, dx = \frac{2^{2n-1}}{n} \binom{2m}{m} \binom{2n+2m}{n+m}^{-1} \binom{n+m}{n}^{-1}. \tag{13.3.14}$$

Example 13.17. The values $a = 2n$, $b = -2n-2m-1$, and $c = m - \frac{1}{2}$ give **3.625.4**:

$$\int_0^{\pi/4} \frac{\sin^{2n} x \, \cos^{m-1/2}(2x)}{\cos^{2n+2m+1} x} \, dx = \frac{1}{2} B\left(n + \tfrac{1}{2}, m + \tfrac{1}{2}\right). \tag{13.3.15}$$

For n, $m \in \mathbb{N}$ we can also write

$$\int_0^{\pi/4} \frac{\sin^{2n} x \, \cos^{m-1/2}(2x)}{\cos^{2n+2m+1} x} \, dx = \frac{\pi}{2^{2n+2m+1}} \binom{2n}{n} \binom{2m}{m} \binom{n+m}{n}^{-1}. \tag{13.3.16}$$

Example 13.18. Formula **3.626.1**:

$$\int_0^{\pi/4} \frac{\sin^{2n-1} x}{\cos^{2n+2} x} \sqrt{\cos(2x)} \, dx = \frac{1}{2} B(n, 3/2), \tag{13.3.17}$$

comes from (13.3.2) with $a = 2n-1$, $b = -2n-2$, and $c = 1/2$. For $n \in \mathbb{N}$ we have

$$\int_0^{\pi/4} \frac{\sin^{2n-1} x}{\cos^{2n+2} x} \sqrt{\cos(2x)} \, dx = \frac{2^{2n} (n-1)! \, n!}{(2n+1)!}. \tag{13.3.18}$$

Example 13.19. The last example in this section is formula **3.626.2**:

$$\int_0^{\pi/4} \frac{\sin^{2n} x}{\cos^{2n+3} x} \sqrt{\cos(2x)} \, dx = \frac{1}{2} B(n + \tfrac{1}{2}, \tfrac{3}{2}), \tag{13.3.19}$$

comes from (13.3.2) with $a = 2n$, $b = -2n-3$, and $c = 1/2$. For $n \in \mathbb{N}$ we have

$$\int_0^{\pi/4} \frac{\sin^{2n} x}{\cos^{2n+3} x} \sqrt{\cos(2x)} \, dx = \frac{\pi}{2^{2n+2}} \frac{(2n)!}{n! \, (n+1)!}. \tag{13.3.20}$$

Chapter 14

An elementary evaluation of entry 3.411.5

14.1 Introduction

The compilation by I. S. Gradshteyn and I. M. Ryzhik [40] contains about 600 pages of definite integrals. Some of them are quite elementary; for instance, **4.291.1**

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12} \tag{14.1.1}$$

is obtained by expanding the integrand as a power series and using the value

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}. \tag{14.1.2}$$

The latter is reminiscent of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{14.1.3}$$

The reader will find in [22] many proofs of the classical evaluation (14.1.3).

Most entries in [40] appear quite formidable, and their evaluation requires a variety of methods and ingenuity. Entry **4.229.7**

$$\int_{\pi/4}^{\pi/2} \ln\ln\tan x\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\sqrt{2\pi}\right) \tag{14.1.4}$$

illustrates this point. Vardi [82] describes a good amount of mathematics involved in evaluating (14.1.4). The integral is first interpreted in terms of the derivative of the L-function

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} - \cdots \tag{14.1.5}$$

as

$$\int_{\pi/4}^{\pi/2} \ln \ln \tan x \, dx = -\frac{\pi\gamma}{4} + L'(1). \tag{14.1.6}$$

Here γ is *Euler's constant.* Then $L'(1)$ is computed in terms of the *gamma function.* This is an unexpected procedure.

Any treatise such as [40], containing large amount of information is bound to have some errors. Some of them are easy to fix. For instance, formula **3.511.8** in the sixth edition [39] reads

$$\int_0^\infty \frac{dx}{\cosh^2 x} = \sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}. \tag{14.1.7}$$

The source given for this integral is formula **BI**(98)(25) from the table by Bierens de Haan [14], where it appears as

$$\int_0^\infty \frac{1}{e^t + e^{-t}} \frac{dt}{\sqrt{t}} = \sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}. \tag{14.1.8}$$

The change of variable $t = \sqrt{x}$ yields a correct version of (14.1.7):

$$\int_0^\infty \frac{dx}{\cosh(x^2)} = \sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}. \tag{14.1.9}$$

It is now clear what happened: a typo produced (14.1.7). In the latest edition of the integral table [40], the editors have decided to replace this entry with

$$\int_0^\infty \frac{dx}{\cosh^2 x} = 1. \tag{14.1.10}$$

The right-hand side of (14.1.7) has been corrected.

The advent of computer algebra packages has not made these tables obsolete. The latest version of `Mathematica` evaluates (14.1.10) directly, but it is unable to produce (14.1.9).

Most of the errors in [40] are of the type: some parameter has been mistyped, an exponent has been misplaced, parameters are mistaken to be identical (a common mishap is μ and u appearing in the same formula). Despite of this fact, it is a remarkable achievement for such an endeavor. The accuracy of [40] comes from the effort of many generations, beginning with [56] and also including [51, 77].

A different type of error was found by the authors. It turns out that entry **3.248.5** of [39] is incorrect. To explain the reason for looking at any specific entry requires some background. The stated entry **3.248.5** involves the rational function

$$\varphi(x) = 1 + \frac{4x^2}{3(1+x^2)^2} \tag{14.1.11}$$

and the result says

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[\varphi(x)+\sqrt{\varphi(x)}\right]^{1/2}} = \frac{\pi}{2\sqrt{6}}. \tag{14.1.12}$$

The encounter begins with the evaluation of

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \tag{14.1.13}$$

in the form

$$N_{0,4}(a;m) = \frac{\pi}{2}\frac{P_m(a)}{[2(a+1)]^{m+1/2}}, \tag{14.1.14}$$

where $P_m(a)$ is a polynomial of degree m. The reader will find in [5, 59] details about (14.1.13) and properties of the coefficients of P_m. It is rather interesting that $N_{0,4}(a,m)$ appears in the expansion of the double square root function

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}N_{0,4}(a;k-1)c^k. \tag{14.1.15}$$

Browsing [38] on a leisure day, and with double square roots in our mind, formula **3.248.5** caught our attention. After many failed attempts at the proof, a simple numerical integration showed that (14.1.12) is incorrect. In spite of our inability to evaluate this integral, we have produced many equivalent versions. The reader is invited to verify that, if $\sigma(x,p) := \sqrt{x^4+2px^2+1}$ then the integral in (14.1.12) is $I(\frac{5}{3},1)$, where

$$I(p,q) = \int_0^\infty \frac{dx}{\sqrt{\sigma(x,p)}\sqrt{\sigma(x,q)}\sqrt{\sigma(x,p)+\sigma(x,q)}}. \tag{14.1.16}$$

The correct value of (14.1.12) *has eluded us.*

The reader is surely aware that often typos or errors could have profound consequences. In a letter to Larry Glasser, regarding (14.1.12), we mistyped the function $\varphi(x)$ of (14.1.11) as

$$\varphi(x) = 1 + \frac{4x^2}{3(1+x^2)}. \tag{14.1.17}$$

Larry, a consummate integrator, replied with

$$\sqrt{3}\left(\text{Tanh}^{-1}\sqrt{2\omega} - \tfrac{1}{\sqrt{2}}\text{Tanh}^{-1}\sqrt{\omega}\right) \tag{14.1.18}$$

where $\omega = \left(\sqrt{7}-\sqrt{3}\right)/2\sqrt{7}$. Beautiful, but it does not help with (14.1.12). The editors of [38] have found an alternative to this quandry: the latest edition [40] has no entry **3.248.5**.

Another example of errors in [39] has been discussed in the *American Mathematical Monthly* by E. Talvila [81]. Several entries, starting with **3.851.1** [38]

$$\int_0^\infty x \sin(ax^2) \sin(2bx)\, dx = \frac{b}{2a}\sqrt{\frac{\pi}{2a}} \left[\cos \frac{b^2}{a} + \sin \frac{a^2}{b} \right] \tag{14.1.19}$$

are shown to be incorrect. This time, the errors are more dramatic: the integrals are divergent. These entries do not appear in the latest edition [40].

The website http://www.math.tulane.edu/∼vhm/Table.html has the goal *to provide proofs and context to the entries in* [40]. The example chosen for the present article is taken from Section 3.411 consisting of 32 entries. The integrands are combinations of rational functions of powers and exponentials and the domain of integration is the whole real line or the halfline $(0, \infty)$. There is a single exception: entry **3.411.5** states that

$$\int_0^{\ln 2} \frac{x\, dx}{1 - e^{-x}} = \frac{\pi^2}{12}. \tag{14.1.20}$$

The next section presents an elementary proof of (14.1.20).

14.2 A reduction argument

The expansion of the integrand in (14.1.20) as a geometric series yields

$$\frac{x}{1 - e^{-x}} = x + \sum_{k=1}^\infty x e^{-kx}. \tag{14.2.1}$$

Term-by-term integration produces the following expressions

$$\int_0^a \frac{x\, dx}{1 - e^{-x}}\, dx = \frac{1}{2}a^2 - \sum_{k=1}^\infty \frac{e^{-ak}}{k^2} + \sum_{k=1}^\infty \frac{1}{k^2} - a \sum_{k=1}^\infty \frac{e^{-ak}}{k}. \tag{14.2.2}$$

The complexity of these three series decreases as one moves from left to right. We now compute each term in (14.2.2), individually.

The third series. Integrating the geometric series $\sum_{k=0}^\infty x^k = 1/(1-x)$ yields $\sum_{n \geq 1} \frac{x^n}{n} = \ln(1-x)$, which is valid for $|x| < 1$. Evaluating at $x = e^{-a}$ gives

$$\sum_{k=1}^\infty \frac{e^{-ak}}{k} = \ln(1 - e^{-a}). \tag{14.2.3}$$

The second series. The *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{14.2.4}$$

plays a prominent role in the evaluation of the remaining 31 entries in Section 3.411. Indeed, the first of these

$$\int_0^\infty \frac{x^{\nu-1}\,dx}{e^{\mu x}-1} = \frac{1}{\mu^\nu}\Gamma(\nu)\zeta(\nu) \tag{14.2.5}$$

is the classical integral representation for $\zeta(\nu)$. It is becoming that the special value

$$\zeta(2) = \frac{\pi^2}{6} \tag{14.2.6}$$

appears as the second series in (14.2.2).

The first series. The second series in (14.2.2) is the only remaining part, we are alluding to the function $\sum_{k\geq 1} x^k/k^2$ evaluated at $x = e^{-a}$. This is the famous *polylogarithm* studied by Euler. See the introduction to [55] for a historical perspective. Aside from the series representation

$$\text{PolyLog}(2,x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}, \tag{14.2.7}$$

there is a natural integral expression

$$\text{PolyLog}(2,x) = -\int_0^x \frac{\ln(1-t)}{t}\,dt. \tag{14.2.8}$$

Therefore, the identity (14.2.2) reduces to

$$\int_0^a \frac{x\,dx}{1-e^{-x}} = \frac{1}{2}a^2 - \text{PolyLog}[2,e^{-a}] + \frac{\pi^2}{6} - a\ln(1-e^{-a}), \tag{14.2.9}$$

and entry **3.411.5** corresponds to the special value

$$\text{PolyLog}[2,\tfrac{1}{2}] = \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2. \tag{14.2.10}$$

Equivalently,

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}k^2} = \frac{\pi^2}{6} - \ln^2 2. \tag{14.2.11}$$

Euler proved the functional equation

$$\text{PolyLog}[2,x] + \text{PolyLog}[2,1-x] = \frac{\pi^2}{6} - \ln x \ln(1-x) \tag{14.2.12}$$

for the polylogarithm function. In particular, the case $x = \frac{1}{2}$ gives (14.2.11).

14.3 An elementary computation of the first series

A series for $\ln^2 2$ can be obtained by squaring $\ln 2 = -\sum_{n\geq 1} \frac{1}{n2^n}$ so that

$$\begin{aligned} \ln^2 2 &= \left(\sum_{n=1}^{\infty} \frac{1}{n2^n} \right) \times \left(\sum_{m=1}^{\infty} \frac{1}{m2^m} \right) = \sum_{n,m\geq 1} \frac{1}{nm2^{n+m}} \\ &= \sum_{r=1}^{\infty} \left(\sum_{m=1}^{r-1} \frac{1}{(r-m)m} \right) \frac{1}{2^r}. \end{aligned}$$

The partial fraction decomposition $\frac{1}{(r-m)m} = \frac{1}{r}\left(\frac{1}{m} + \frac{1}{r-m} \right)$ gives

$$\ln^2 2 = \sum_{r=1}^{\infty} \frac{H_{r-1}}{r2^{r-1}}, \tag{14.3.1}$$

where $H_{r-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{r-1}$ is the harmonic number. Therefore,

$$\begin{aligned} \ln^2 2 + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}k^2} &= \sum_{r=1}^{\infty} \frac{H_{r-1}}{r2^{r-1}} + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}k^2} \\ &= \sum_{r=1}^{\infty} \frac{1}{r2^{r-1}} \left(H_{r-1} + \frac{1}{r} \right) \\ &= \sum_{r=1}^{\infty} \frac{H_r}{r2^{r-1}}. \end{aligned}$$

It remains to verify that this last series is in fact $\zeta(2)$.

The representation of the harmonic number as

$$H_r = \int_0^1 \frac{1-x^r}{1-x}\, dx \tag{14.3.2}$$

gives the desired step. Indeed, if c_r is a sequence of real numbers and α is fixed, then

$$\sum_{r=1}^{\infty} c_r H_r \alpha^r = \int_0^1 \frac{1}{1-x} \sum_{r=1}^{\infty} (1-x^r) c_r \alpha^r \, dx. \tag{14.3.3}$$

Thus the function

$$f(x) = \sum_{r=1}^{\infty} c_r x^r \tag{14.3.4}$$

appears in the integral representation

$$\sum_{r=1}^{\infty} c_r H_r \alpha^r = \int_0^1 \frac{f(\alpha) - f(\alpha x)}{1-x}\, dx. \tag{14.3.5}$$

In the present case, $c_r = 1/r2^{r-1}$ and $f(x) = -2\ln(1-x/2)$. Therefore,

$$\sum_{r=1}^{\infty} \frac{H_r}{r2^{r-1}} = \int_0^1 \frac{2\ln 2 + 2\ln(1-x/2)}{1-x}\, dx = 2\int_0^1 \frac{\ln(1+y)}{y}\, dy. \tag{14.3.6}$$

This last integral is computable via (14.1.1) and we have come full circle.

The technique described above in exhibiting an elementary proof of (14.2.11) can be applied to

$$\sum_{n=1}^{\infty} \frac{H_{n-1}}{n^2} = \zeta(3). \tag{14.3.7}$$

J. Borwein and D. Bradley [25] have given 32 proofs of this charming identity.

Chapter 15

Frullani integrals

15.1 Introduction

The table of integrals [40] contains many evaluations of the form

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx = [f(0) - f(\infty)] \ln\left(\frac{b}{a}\right). \tag{15.1.1}$$

Expressions of this type are called *Frullani integrals*. Conditions that guarantee the validity of this formula are given in [10] and [71]. In particular, the continuity of f' and the convergence of the integral are sufficient for (15.1.1) to hold.

15.2 A list of examples

Many of the entries in [40] are simply particular cases of (15.1.1).

Example 15.1. The evaluation of **3.434.2** in [40]:

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \ln\left(\frac{b}{a}\right) \tag{15.2.1}$$

corresponds to the function $f(x) = e^{-x}$.

Example 15.2. The change of variables $t = e^{-x}$ in Example 15.1 yields

$$\int_0^1 \frac{t^{b-1} - t^{a-1}}{\ln t}\, dt = \ln\left(\frac{a}{b}\right). \tag{15.2.2}$$

This is **4.267.8** in [40].

Example 15.3. A generalization of the previous example appears as entry **3.476.1** in [40]:

$$\int_0^\infty \left(e^{-vx^p} - e^{-ux^p} \right) \frac{dx}{x} = \frac{1}{p} \ln \left(\frac{u}{v} \right). \tag{15.2.3}$$

This comes from Frullani's result with a simple additional scaling.

Example 15.4. The choice

$$f(x) = \frac{e^{-qx} - e^{-px}}{x}, \tag{15.2.4}$$

with p, $q > 0$ satisfies $f(\infty) = 0$ and

$$f(0) = \lim_{x \to 0} \frac{e^{-qx} - e^{-px}}{x} = p - q. \tag{15.2.5}$$

Then Frullani's theorem yields

$$\int_0^\infty \left(\frac{e^{-aqx} - e^{-apx}}{ax} - \frac{e^{-bqx} - e^{-bpx}}{bx} \right) \frac{dx}{x} = (p-q) \ln \left(\frac{b}{a} \right),$$

that can be written as

$$\int_0^\infty \left(\frac{e^{-aqx} - e^{-apx}}{a} - \frac{e^{-bqx} - e^{-bpx}}{b} \right) \frac{dx}{x^2} = (p-q) \ln \left(\frac{b}{a} \right).$$

This is entry **3.436** in [40].

Example 15.5. Now choose

$$f(x) = \frac{x}{1 - e^{-x}} \exp(-ce^x). \tag{15.2.6}$$

Then Frullani's theorem yields entry **3.329** of [40], in view of $f(0) = e^{-c}$ and $f(\infty) = 0$:

$$\int_0^\infty \left(\frac{a \exp(-ce^{ax})}{1 - e^{-ax}} - \frac{b \exp(-ce^{bx})}{1 - e^{-bx}} \right) dx = e^{-c} \ln \left(\frac{b}{a} \right). \tag{15.2.7}$$

Example 15.6. The next example uses

$$f(x) = (x+c)^{-\mu}, \tag{15.2.8}$$

with c, $\mu > 0$, to produce

$$\int_0^\infty \frac{(ax+c)^{-\mu} - (bx+c)^{-\mu}}{x} \, dx = c^{-\mu} \ln \left(\frac{b}{a} \right). \tag{15.2.9}$$

This is **3.232** in [40].

Example 15.7. Entry **4.536.2** in [40] is

$$\int_0^\infty \frac{\tan^{-1}(px) - \tan^{-1}(qx)}{x}\,dx = \frac{\pi}{2}\ln\left(\frac{p}{q}\right). \tag{15.2.10}$$

This follows directly from (15.1.1) by choosing $f(x) = \tan^{-1} x$.

Example 15.8. The function $f(x) = \ln(a + be^{-x})$ gives the evaluation of entry **4.319.3** of [40]:

$$\int_0^\infty \frac{\ln(a + be^{-px}) - \ln(a + be^{-qx})}{x}\,dx = \ln\left(\frac{a}{a+b}\right)\ln\left(\frac{p}{q}\right). \tag{15.2.11}$$

Example 15.9. The function $f(x) = ab\ln(1+x)/x$ produces entry **4.297.7** of [40]:

$$\int_0^\infty \frac{b\ln(1+ax) - a\ln(1+bx)}{x^2}\,dx = ab\ln\left(\frac{b}{a}\right). \tag{15.2.12}$$

Example 15.10. Entry **3.484** of [40]:

$$\int_0^\infty \left[\left(1 + \frac{a}{qx}\right)^{qx} - \left(1 + \frac{a}{px}\right)^{px}\right]\frac{dx}{x} = (e^a - 1)\ln\left(\frac{q}{p}\right), \tag{15.2.13}$$

is obtained by choosing $f(x) = (1 + a/x)^x$ in (15.1.1).

Example 15.11. The final example in this section corresponds to the function

$$f(x) = \frac{a + be^{-x}}{ce^x + g + he^{-x}} \tag{15.2.14}$$

that produces entry **3.412.1** of [40]:

$$\int_0^\infty \left[\frac{a + be^{-px}}{ce^{px} + g + he^{-px}} - \frac{a + be^{-qx}}{ce^{qx} + g + he^{-qx}}\right]\frac{dx}{x} = \frac{a+b}{c+g+h}\ln\left(\frac{q}{p}\right). \tag{15.2.15}$$

15.3 A separate source of examples

The list presented in this section contains integrals of Frullani type that were found in volume 1 of Ramanujan's Notebooks [13].

Example 15.12.

$$\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x}\,dx = \frac{\pi}{2}\ln\frac{a}{b}$$

Example 15.13.

$$\int_0^\infty \ln \frac{p+qe^{-ax}}{p+qe^{-bx}} \frac{dx}{x} = \ln\left(1+\frac{q}{p}\right) \ln \frac{b}{a}.$$

Example 15.14.

$$\int_0^\infty \left[\left(\frac{ax+p}{ax+q} \right)^n - \left(\frac{bx+p}{bx+q} \right)^n \right] \frac{dx}{x} = \left(1 - \frac{p^n}{q^n} \right) \ln \frac{a}{b}$$

where a, b, p, q are all positive.

Example 15.15.

$$\int_0^\infty \frac{\cos ax - \cos bx}{x} \, dx = \ln \frac{b}{a}.$$

Example 15.16.

$$\int_0^\infty \sin\left(\frac{(b-a)x}{2}\right) \sin\left(\frac{(b+a)x}{2}\right) \frac{dx}{x} = \int_0^\infty \frac{\cos ax - \cos bx}{2x} \, dx = \frac{1}{2} \ln \frac{b}{a}.$$

Example 15.17.

$$\int_0^\infty \sin px \, \sin qx \, \frac{dx}{x} = \int_0^\infty \frac{\cos[(p-q)x] - \cos[(p+q)x]}{2x} \, dx = \frac{1}{2} \ln \frac{p+q}{p-q}.$$

Example 15.18. The evaluation of

$$\int_0^\infty \ln\left(\frac{1+2n\cos ax + n^2}{1+2n\cos bx + n^2}\right) \frac{dx}{x} = \begin{cases} \ln \frac{b}{a} \ln(1+n)^2 & n^2 < 1 \\ \ln \frac{b}{a} \ln\left(1+\frac{1}{n}\right)^2 & n^2 > 1 \end{cases}$$

is more delicate and is given in detail in the next section.

Example 15.19. The value

$$\int_0^\infty \frac{e^{-ax} \sin ax - e^{-bx} \sin bx}{x} \, dx = 0$$

follows directly from (15.1.1) since, in this case $f(x) = e^{-x} \sin x$ satisfies $f(\infty) = f(0) = 0$.

Example 15.20.

$$\int_0^\infty \frac{e^{-ax} \cos ax - e^{-bx} \cos bx}{x} \, dx = \ln \frac{b}{a}.$$

15.4 A more delicate example

Entry **4.324.2** of [40] states that

$$\int_0^\infty \left[\ln(1+2a\cos px + a^2) - \ln(1+2a\cos qx + a^2)\right] \frac{dx}{x} =$$
$$\begin{cases} 2\ln\left(\frac{q}{p}\right)\ln(1+a) & -1 < a \leq 1 \\ 2\ln\left(\frac{q}{p}\right)\ln(1+1/a) & a < -1 \text{ or } a \geq 1. \end{cases} \tag{15.4.1}$$

This requires a different approach since the obvious candidate for a direct application of Frullani's theorem, namely $f(x) = \ln(1+2a\cos x + a^2)$, does not have a limit at infinity.

In order to evaluate this entry, start with

$$\int_0^1 x^y dx = \frac{1}{y+1}, \tag{15.4.2}$$

so

$$\int_0^1 dy \int_0^1 x^y dx = \int_0^1 dx \int_0^1 x^y dy = \int_0^1 \frac{x-1}{\ln x} dx = \int_0^1 \frac{dy}{y+1} = \ln 2. \tag{15.4.3}$$

This is now generalized for arbitrary symbols α and β as

$$\int_0^\infty \frac{e^{\alpha t} - e^{\beta t}}{t} dt = \ln\left(\frac{\beta}{\alpha}\right). \tag{15.4.4}$$

To prove (15.4.4), make the substitution $u = e^{-t}$ that turns the integral into

$$\begin{aligned} \int_0^1 \frac{u^{-1-\beta} - u^{-1-\alpha}}{\ln u} du &= \int_0^1 du \int_{-1-\alpha}^{-1-\beta} u^w dw \\ &= \int_{-1-\alpha}^{-1-\beta} dw \int_0^1 u^w du \\ &= \int_{-1-\alpha}^{-1-\beta} \frac{dw}{w+1} \\ &= \ln\left(\frac{\beta}{\alpha}\right). \end{aligned}$$

Now observe that $\left|\frac{2a\cos(rx)}{1+a^2}\right| \leq 1$; therefore it is legitimate to expand the logarithmic terms as infinite series using $\ln(1+z) = \sum_k (-1)^{k-1}\frac{z^k}{k}$. The outcome reads

$$\int_0^\infty \frac{dx}{x} \sum_{k\geq 1} \frac{(-1)^{k-1}A^k(\cos^k px - \cos^k qx)}{k} =$$
$$\sum_{k\geq 1} \frac{(-1)^{k-1}A^k}{2^k k} \int_0^\infty \frac{(e^{ipx}+e^{-ipx})^k - (e^{iqx}+e^{-iqx})^k}{x} dx;$$

where $A = 2a/(1+a^2)$. The inner integral is evaluated using some binomial expansions. That is,

$$\int_0^\infty \frac{(e^{ipx}+e^{-ipx})^k - (e^{iqx}+e^{-iqx})^k}{x} dx = \sum_{r=0}^{k} \binom{k}{r} \int_0^\infty \frac{e^{(2r-k)ipx} - e^{(2r-k)iqx}}{x} dx. \tag{15.4.5}$$

It is time to employ equation (15.4.4). A closer look at (15.4.5) shows that care must be exercised. The integrals are sensitive to the *parity* of k. More precisely, the quantity $2r-k$ vanishes if and only if k is even and $r = k/2$, in which case there is a zero contribution to summation. Otherwise, the second integral in (15.4.5) is always equal to $\ln(q/p)$. Therefore,

$$\sum_{r=0}^{k} \binom{k}{r} \int_0^\infty \left[e^{(2r-k)ipx} - e^{(2r-k)iqx} \right] \frac{dx}{x} =$$
$$\begin{cases} 2^k \ln\left(\frac{q}{p}\right) & \text{if } k \text{ is odd,} \\ \left(2^k - \binom{k}{k/2}\right) \ln\left(\frac{q}{p}\right) & \text{if } k \text{ is even.} \end{cases}$$

Combining the results obtained thus far yields

$$\begin{aligned} I &= \int_0^\infty \frac{\ln(1+2a\cos(px)+a^2) - \ln(1+2a\cos(qx)+a^2)}{x} dx \\ &= \int_0^\infty \frac{dx}{x} \sum_{k\geq 1} \frac{(-1)^{k-1}A^k(\cos^k px - \cos^k qx)}{k} \\ &= \sum_{k\geq 1} \frac{(-1)^{k-1}A^k}{k2^k} \sum_{r=0}^{k} \binom{k}{r} \int_0^\infty \frac{e^{(2r-k)ipx} - e^{(2r-k)iqx}}{x} dx \\ &= \ln\left(\frac{q}{p}\right) \sum_{k \text{ odd}} \frac{(-1)^{k-1}A^k}{k} + \ln\left(\frac{q}{p}\right) \sum_{k \text{ even}} \frac{(-1)^{k-1}A^k}{k} \left(1 - \frac{1}{2^k}\binom{k}{k/2}\right) \\ &= \ln\left(\frac{q}{p}\right) \sum_{k\geq 1} \frac{(-1)^{k-1}A^k}{k} + \ln\left(\frac{q}{p}\right) \sum_{k\geq 1} \frac{1}{2k} \left(\frac{A}{2}\right)^{2k} \binom{2k}{k} \\ &= \ln\left(\frac{q}{p}\right) \ln(1+A) + \frac{1}{2} \ln\left(\frac{q}{p}\right) \sum_{k\geq 1} \binom{2k}{k} \frac{1}{k} \left(\frac{A^2}{2^2}\right)^k . \end{aligned} \tag{15.4.6}$$

The last step utilizes the Taylor series

$$\sum_{k\geq 1}\binom{2k}{k}\frac{Q^k}{k} = -2\ln\left(\tfrac{1}{2}\left[1+\sqrt{1-4Q}\right]\right). \tag{15.4.7}$$

This follows from the binomial series $\sum_{k\geq 0}\binom{2k}{k}R^k = 1/\sqrt{1-4R}$ after rearranging in the manner

$$\sum_{k\geq 1}\binom{2k}{k}R^{k-1} = \frac{1}{R\sqrt{1-4R}} - \frac{1}{R} = \frac{4}{\sqrt{1-4R}(1+\sqrt{1-4R})},$$

and then integrating from 0 to Q

$$\begin{aligned}\sum_{k\geq 1}\binom{2k}{k}\frac{Q^k}{k} &= \int_0^Q \frac{4\,dR}{\sqrt{1-4R}(1+\sqrt{1-4R})} \\ &= -2\int_1^{\sqrt{1-4Q}}\frac{du}{1+u} \\ &= -2\ln\left(\tfrac{1}{2}\left[1+\sqrt{1-4Q}\right]\right).\end{aligned}$$

Formula (15.4.7) applied to equation (15.4.6) leads to

$$I = \ln\left(\frac{q}{p}\right)\ln(1+A) - \ln\left(\frac{q}{p}\right)\ln\left(\tfrac{1}{2}\left[1+\sqrt{1-4Q}\right]\right).$$

It remains to replace $Q = A^2/2^2 = a^2/(1+a^2)^2$ and use the identity

$$1-4Q = \frac{(a^2-1)^2}{(a^2+1)^2}.$$

Observe that the expression for $\sqrt{1-4Q}$ depends on whether $|a|>1$ or not. The proof is complete.

Note. `Mathematica` is unable to evaluate the entries **3.329** in (15.2.7), **4.536.2** in (15.2.10), **4.319.3** in (15.2.11), **4.297.7** in (15.2.12), **3.484** in (15.2.13), and **3.412.1** in (15.2.15). The same is true for the delicate example in the last section.

Chapter 16

Index of entries

16.1 The list

This index contains the formulas in Gradshteyn and Ryzhik [40] that are established here. The sections are named as in the table. A small collection of formulas, with answers that are too wide to fit in the structure designed here are given at the end.

Section 2.14.
Forms containing the binomial $1 \pm x^n$.

Subsection 2.148

2.148.4 $\displaystyle\int_0^x \frac{dt}{(1+t^2)^{n+1}} = \frac{\binom{2n}{n}}{2^{2n}} \left(\tan^{-1} x + \sum_{j=1}^{n} \frac{2^{2j}}{2j\binom{2j}{j}} \frac{x}{(x^2+1)^j} \right)$ **115**

Section 2.32.
The exponential combined with rational functions of x.

Subsection 2.321

2.321.1 $\displaystyle\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx$ **84**

2.321.2 $\int x^n e^{ax}\, dx = e^{ax} \left(\sum_{k=0}^{n} \frac{(-1)^k k! \binom{n}{k}}{a^{k+1}} x^{n-k} \right)$ **85**

Subsection 2.322

2.322.1 $\int x e^{ax}\,dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right)$ **85**

2.322.2 $\int x^2 e^{ax}\,dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$ **85**

2.322.3 $\int x^3 e^{ax}\,dx = e^{ax}\left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right)$ **85**

2.322.4 $\int x^4 e^{ax}\,dx = e^{ax}\left(\frac{x^4}{a} - \frac{4x^3}{a^2} + \frac{12x^2}{a^3} - \frac{24x}{a^4} + \frac{24}{a^5}\right)$ **85**

Section 3.19 – 3.23
Combinations of powers of x and powers of binomials of the form $(\alpha + \beta x)$

Subsection 3.191

3.191.1 $\int_0^u x^{a-1}(u-x)^{b-1}\,dx = u^{a+b-1}B(a,b)$ **56**

3.191.2 $\int_u^\infty (x-u)^{a-1}x^{-c}\,dx = u^{a-c}B(a,c-a)$ **60**

3.191.3a $\int_0^1 x^{\nu-1}(1-x)^{\mu-1}\,dx = B(\mu,\nu)$ **53**

3.191.3b $\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\,dx = B(\mu,\nu)$ **53**

Subsection 3.192

3.192.1 $\int_0^1 \frac{x^p\,dx}{(1-x)^p} = \frac{\pi p}{\sin \pi p}$ **54**

3.192.2 $\int_0^1 \frac{x^p\,dx}{(1-x)^{p+1}} = -\frac{\pi}{\sin \pi p}$ **54**

3.192.3 $\int_0^1 \frac{(1-x)^p\,dx}{x^{p+1}} = -\frac{\pi}{\sin \pi p}$ **54**

3.192.4 $\int_1^\infty (x-1)^{p-1/2}\frac{dx}{x} = \frac{\pi}{\cos \pi p}$ **54**

Subsection 3.193

3.193 $\int_0^n x^{\nu-1}(n-x)^n\,dx = \frac{n!\,n^{\nu+n}}{\nu(\nu+1)(\nu+2)\cdots(\nu+n)}$ **56**

Subsection 3.194

Subsection 3.195

Subsection 3.196

Subsection 3.216

Subsection 3.217

Subsection 3.218

3.218	$\int_0^\infty \frac{x^{2p-1}-(a+x)^{2p-1}}{(a+x)^p x^p}\,dx$	$=$	$\pi\cot\pi p$	**69**

Subsection 3.219

3.219	$\int_0^\infty \left(\frac{1}{(1+x)^p}-\frac{1}{(1+x)^q}\right)\frac{dx}{x}$	$=$	$\psi(q)-\psi(p)$	**135**

Subsection 3.221

3.221.1	$\int_a^\infty \frac{(x-a)^{p-1}\,dx}{x-b}$	$=$	$\pi(a-b)^{p-1}\operatorname{cosec}(p\pi)$	**61**
3.221.2	$\int_{-\infty}^a \frac{(a-x)^{p-1}\,dx}{x-b}$	$=$	$-\pi(b-a)^{p-1}\operatorname{cosec}(p\pi)$	**61**

Subsection 3.222

3.222.1	$\int_0^1 \frac{x^{\mu-1}\,dx}{1+x}$	$=$	$\beta(\mu)$	**151**
3.222.2a	$\int_0^\infty \frac{x^{\mu-1}\,dx}{x+a}$	$=$	$\pi\operatorname{cosec}(\mu\pi)a^{\mu-1}\quad\text{for } a>0$	**57**
3.222.2b	$\int_0^\infty \frac{x^{\mu-1}\,dx}{x+a}$	$=$	$-\pi\cot(\mu\pi)(-a)^{\mu-1}\quad\text{for } a<0$	**57**

Subsection 3.223

3.223.1	$\int_0^\infty \frac{x^{\mu-1}\,dx}{(x+b)(x+a)}$	$=$	$\frac{\pi}{b-a}\left(a^{\mu-1}-b^{\mu-1}\right)\operatorname{cosec}(\pi\mu)$	**58**
3.223.2	$\int_0^\infty \frac{x^{\mu-1}\,dx}{(x+b)(a-x)}$	$=$	$\frac{\pi}{b-a}\left(b^{\mu-1}\operatorname{cosec}(\mu\pi)+a^{\mu-1}\cot(\mu\pi)\right)$	**59**
3.223.3	$\int_0^\infty \frac{x^{\mu-1}\,dx}{(a-x)(b-x)}$	$=$	$\frac{\pi}{b-a}\left(a^{\mu-1}-b^{\mu-1}\right)\cot(\pi\mu)$	**59**

Subsection 3.224

3.224	$\int_0^\infty \frac{(x+b)x^{\mu-1}\,dx}{(x+a)(x+c)}$	$=$	$\frac{\pi}{\sin(\mu\pi)}\left(\frac{a-b}{a-c}a^{\mu-1}+\frac{c-b}{c-a}c^{\mu-1}\right)$	**59**

Subsection 3.225

3.225.1	$\int_1^\infty \frac{(x-1)^{p-1}\,dx}{x^2}$	$=$	$(1-p)\pi \operatorname{cosec} p\pi$	**62**
3.225.2	$\int_1^\infty \frac{(x-1)^{1-p}\,dx}{x^3}$	$=$	$\frac{\pi}{2}p(1-p)\operatorname{cosec} p\pi$	**62**
3.225.3	$\int_0^\infty \frac{x^p\,dx}{(1+x)^3}$	$=$	$\frac{\pi}{2}p(1-p)\operatorname{cosec} p\pi$	**61**

Subsection 3.226

3.226.1	$\int_0^1 \frac{x^n\,dx}{\sqrt{1-x}}$	$=$	$2\frac{(2n)!!}{(2n+1)!!}$	**54**
3.226.2	$\int_0^1 \frac{x^{n-1/2}\,dx}{\sqrt{1-x}}$	$=$	$\frac{(2n-1)!!}{(2n)!!}\pi$	**54**

Subsection 3.231

3.231.1	$\int_0^1 \frac{x^{p-1}-x^{-p}}{1-x}\,dx$	$=$	$\pi\cot\pi p$	**133**
3.231.2	$\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x}\,dx$	$=$	$\frac{\pi}{\sin\pi p}$	**155**
3.231.3	$\int_0^1 \frac{x^{a}-x^{-a}}{1-x}\,dx$	$=$	$\pi\cot\pi a-\frac{1}{a}$	**133**
3.231.4	$\int_0^1 \frac{x^{p}-x^{-p}}{1+x}\,dx$	$=$	$\frac{1}{p}-\frac{\pi}{\sin\pi p}$	**155**
3.231.5	$\int_0^1 \frac{x^{\mu-1}-x^{\nu-1}}{1-x}\,dx$	$=$	$\psi(\nu)-\psi(\mu)$	**132**
3.231.6	$\int_0^1 \frac{x^{p-1}-x^{q-1}}{1-x}\,dx$	$=$	$\pi(\cot\pi p-\cot\pi q)$	**135**

Subsection 3.232

3.232	$\int_0^\infty \frac{(ax+c)^{-\mu}-(bx+c)^{-\mu}}{x}\,dx$	$=$	$c^{-\mu}\ln\left(\frac{b}{a}\right)$	**192**

Subsection 3.233

3.233 $\displaystyle\int_0^\infty \left(\frac{1}{1+x} - \frac{1}{(1+x)^q} \right) \frac{dx}{x} = \psi(q) + \gamma$ **135**

Subsection 3.234

3.234.1 $\displaystyle\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx = \frac{\pi}{a^q} \cot \pi q$ **139**

Subsection 3.235

3.235 $\displaystyle\int_0^\infty \frac{(1+x)^a - 1}{(1+x)^b} \frac{dx}{x} = \psi(b) - \psi(b-a)$ **135**

Section 3.24 – 3.27.

Powers of x, of binomials of the form $\alpha + \beta x^p$ and of polynomials in x

Subsection 3.241

3.241.1 $\displaystyle\int_0^1 \frac{x^{a-1}\,dx}{1+x^p}\,dx = \frac{1}{p}\beta\left(\frac{a}{p}\right)$ **154**

3.241.2 $\displaystyle\int_0^\infty \frac{x^{\mu-1}\,dx}{1+x^\nu}\,dx = \frac{\pi}{\nu}\mathrm{cosec}\frac{\mu\pi}{\nu} = \frac{1}{\nu}B\left(\frac{\mu}{\nu}, \frac{\nu-\mu}{\nu}\right)$ **60**

3.241.4 $\displaystyle\int_0^\infty \frac{x^{\mu-1}\,dx}{(p+qx^\nu)^{n+1}}\,dx = \left(\frac{p}{q}\right)^{\mu/\nu} \frac{\Gamma(\mu/\nu)\Gamma(n+1-\mu/\nu)}{\nu p^{n+1}\,\Gamma(n+1)}$ **62**

3.241.5 $\displaystyle\int_0^\infty \frac{x^{p-1}\,dx}{(1+x^q)^2}\,dx = \frac{(p-q)\pi}{q^2}\ \mathrm{cosec}\ \frac{(p-q)\pi}{q}$ **63**

Subsection 3.244

3.244.1 $\displaystyle\int_0^1 \frac{x^{p-1}+x^{q-p-1}}{1+x^q}\,dx = \frac{\pi}{q}\mathrm{cosec}\left(\frac{\pi p}{q}\right)$ **155**

3.244.2 $\displaystyle\int_0^1 \frac{x^{b-1}-x^{a-b-1}}{1-x^a}\,dx = \frac{\pi}{a}\cot\ \frac{\pi b}{a}$ **134**

3.244.3 $\displaystyle\int_0^1 \frac{x^{q-1}-x^{p-1}}{1-x^q}\,dx = \frac{1}{q}\left(\gamma + \psi\left(\frac{p}{q}\right)\right)$ **133**

Subsection 3.248

Subsection 3.249

Subsection 3.251

3.251.1 $$\int_0^1 x^{\mu-1}(1-x^\lambda)^{\nu-1}\,dx = \frac{1}{\lambda}B\left(\frac{\mu}{\lambda},\nu\right)$$ **57,74**

3.251.2 $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+x^2)^{1-\nu}} = \frac{1}{2}B\left(\frac{\mu}{2},1-\nu-\frac{\mu}{2}\right)$$ **64,72**

3.251.3 $$\int_1^\infty x^{\mu-1}(x^p-1)^{\nu-1}\,dx = \frac{1}{p}B\left(1-\nu-\frac{\mu}{p},\nu\right)$$ **57**

3.251.4 $$\int_0^\infty \frac{x^{2m}\,dx}{(v+ut^2)^{n+1}} = \frac{\pi(2m)!(2n-2m)!}{2^{2n+1}m!(n-m)!n!u^{m+1/2}v^{n-m+1/2}}$$ **64**

3.251.5 $$\int_0^\infty \frac{x^{2m+1}\,dx}{(v+ut^2)^{n+1}} = \frac{m!(n-m-1)!}{2n!u^{m+1}v^{n-m}}$$ **65**

3.251.6 $$\int_0^\infty \frac{x^{\mu+1}\,dx}{(1+x^2)^2} = \frac{\mu\pi}{4\sin\frac{\mu\pi}{2}}$$ **60**

3.251.7 $$\int_0^1 \frac{x^a\,dx}{(1+x^2)^2} = -\frac{1}{4}+\frac{a-1}{4}\beta\left(\frac{a-1}{2}\right)$$ **154**

3.251.8 $$\int_0^1 x^{p+q-1}(1-x^q)^{-p/q}\,dx = \frac{\pi p}{q^2}\operatorname{cosec}\left(\frac{\pi p}{q}\right)$$ **65**

3.251.9 $$\int_0^1 x^{q/p-1}(1-x^q)^{-1/p}\,dx = \frac{\pi}{q}\operatorname{cosec}\left(\frac{\pi}{p}\right)$$ **65**

3.251.10 $$\int_0^1 x^{p-1}(1-x^q)^{-p/q}\,dx = \frac{\pi}{q}\operatorname{cosec}\left(\frac{p\pi}{q}\right)$$ **65**

3.251.11 $$\int_0^\infty x^{\mu-1}(1+\beta x^p)^{-\nu}\,dx = \frac{1}{p}\beta^{-\frac{\mu}{p}}B\left(\frac{\mu}{p},\nu-\frac{\mu}{p}\right)$$ **65**

Subsection 3.252

3.252.1 $$\int_0^\infty \frac{dx}{(ax^2+bx+c)^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial c^{n-1}}\left[\frac{1}{\sqrt{ac-b^2}}\ \operatorname{arccot}\ \frac{b}{\sqrt{ac-b^2}}\right]$$ **87**

3.252.2 $$\int_{-\infty}^\infty \frac{dx}{(ax^2+bx+c)^n} = \frac{(2n-3)!!\pi a^{n-1}}{(2n-2)!!(ac-b^2)^{n-1/2}}$$ **87**

3.252.3 $$\int_0^\infty \frac{dx}{(ax^2+bx+c)^{n+3/2}} = \frac{(-2)^n}{(2n+1)!!}\frac{\partial^n}{\partial c^n}\left[\frac{1}{\sqrt{c}(\sqrt{ac}+b)}\right]$$ **87**

Subsection 3.265

3.265 $\int_0^1 \frac{1-x^{q-1}}{1-x}\,dx = \psi(q)+\gamma$ **133**

Subsection 3.267

3.267.1 $\int_0^1 \frac{x^{3n}\,dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}}\frac{\Gamma\left(n+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\,\Gamma(n+1)}$ **66**

3.267.2 $\int_0^1 \frac{x^{3n-1}\,dx}{\sqrt[3]{1-x^3}} = \frac{(n-1)!\,\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(n+\frac{2}{3}\right)\,\Gamma(n+1)}$ **66**

3.267.3 $\int_0^1 \frac{x^{3n-2}\,dx}{\sqrt[3]{1-x^3}} = \frac{\Gamma\left(n-\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(n+\frac{1}{3}\right)}$ **66**

Subsection 3.268

3.268.1 $\int_0^1 \left(\frac{1}{1-x}-\frac{px^{p-1}}{1-x^p}\right) = \ln p$ **82**

3.268.2 $\int_0^1 \frac{1-x^a}{1-x}x^{b-1}\,dx = \psi(a+b)-\psi(b)$ **133**

Subsection 3.269

3.269.1 $\int_0^1 x\frac{x^p-x^{-p}}{1-x^2}\,dx = \frac{\pi}{2}\cot\left(\frac{\pi p}{2}\right)-\frac{1}{p}$ **134**

3.269.2 $\int_0^1 x\frac{x^p-x^{-p}}{1+x^2}\,dx = \frac{1}{p}-\frac{\pi}{2\sin(p\pi/2)}$ **155**

3.269.3 $\int_0^1 \frac{x^a-x^b}{1-x^2}\,dx = \frac{1}{2}\left(\psi\left(\frac{b+1}{2}\right)-\psi\left(\frac{a+1}{2}\right)\right)$ **134**

Section 3.31.
Exponential functions

Subsection 3.310

3.310 $\int_0^\infty e^{-px}\,dx = \frac{1}{p}$ **78**

Subsection 3.311

3.311.1 $\displaystyle\int_0^\infty \frac{dx}{1+e^{px}} = \frac{\ln 2}{p}$ **78**

3.311.2 $\displaystyle\int_0^\infty \frac{e^{-ax}\,dx}{1+e^{-x}} = \beta(a)$ **156**

3.311.3 $\displaystyle\int_{-\infty}^\infty \frac{e^{-px}\,dx}{1+e^{-qx}} = \frac{\pi}{|q|}\ \text{cosec}\ \frac{p\pi}{q}$ **67**

3.311.5 $\displaystyle\int_0^\infty \frac{1-e^{\nu x}}{e^x-1}\,dx = \psi(\nu)+\gamma+\pi\cot(\pi\nu)$ **137**

3.311.6 $\displaystyle\int_0^\infty \frac{e^{-x}-e^{-qx}}{1-e^{-x}}\,dx = \psi(q)+\gamma$ **136**

3.311.7 $\displaystyle\int_0^\infty \frac{e^{-px}-e^{-qx}}{1-e^{-x}}\,dx = \psi(q)-\psi(p)$ **136**

3.311.8 $\displaystyle\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = b^{\mu-1}\pi\cot(\pi\mu)$ **138**

3.311.9 $\displaystyle\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b+e^{-x}} = b^{\mu-1}\pi\ \text{cosec}\ (\pi\mu)$ **67**

3.311.10 $\displaystyle\int_0^\infty \frac{e^{-px}-e^{-qx}}{1-e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\cot\left(\frac{p\pi}{p+q}\right)$ **137**

3.311.11 $\displaystyle\int_0^\infty \frac{e^{px}-e^{qx}}{e^{rx}-e^{sx}}\,dx = \frac{1}{r-s}\left(\psi\left(\frac{r-q}{r-s}\right) -\psi\left(\frac{r-p}{r-s}\right)\right)$ **137**

3.311.12 $\displaystyle\int_0^\infty \frac{a^x-b^x}{c^x-d^x}\,dx = \frac{1}{\ln c-\ln d}\left(\psi\left(\frac{\ln c-\ln b}{\ln c-\ln d}\right) - \psi\left(\frac{\ln c-\ln a}{\ln c-\ln d}\right)\right)$ **137**

3.311.13 $\displaystyle\int_0^\infty \frac{e^{-px}+e^{-qx}}{1+e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\text{cosec}\left(\frac{\pi p}{p+q}\right)$ **156**

Subsection 3.312

3.312.1 $\displaystyle\int_0^\infty (1-e^{x/\beta})^{\nu-1}\,e^{-\mu x}\,dx = \beta B(\beta\mu,\nu)$ **67**

3.312.2 $\displaystyle\int_0^\infty \frac{(1-e^{-ax})(1-e^{-bx})e^{-px}}{1-e^{-x}}\,dx = \psi(p+a)+\psi(p+b)-\psi(p+a+b)-\psi(p)$ **138**

Subsection 3.313

3.313.1	$\int_{-\infty}^{\infty} \frac{e^{-\mu x}\,dx}{1-e^{-x}}$	$=$	$\pi \cot(\pi\mu)$	**58**
3.313.2	$\int_{-\infty}^{\infty} \frac{e^{-\mu x}\,dx}{(1+e^{-x})^{\nu}}$	$=$	$B(\mu, \nu-\mu)$	**67**

Subsection 3.314

3.314	$\int_{-\infty}^{\infty} \frac{e^{-\mu x}\,dx}{(e^{\beta/\gamma}+e^{-x/\gamma})^{\nu}}$	$=$	$\gamma \exp\left[\beta(\mu-\nu/\gamma)\right]\, B(\gamma\mu, \nu-\gamma\mu)$	**67**

Subsection 3.316

3.316	$\int_{-\infty}^{\infty} \frac{(1+e^{-x})^{p}-1}{(1+e^{-x})^{q}}\,dx$	$=$	$\psi(q)-\psi(q-p)$	**136**

Subsection 3.317

3.317.1	$\int_{-\infty}^{\infty} \left(\frac{1}{1+e^{-x}} - \frac{1}{(1+e^{-x})^{q}}\right) dx$	$=$	$\psi(q)+\gamma$	**136**
3.317.2	$\int_{-\infty}^{\infty} \left(\frac{1}{(1+e^{-x})^{p}} - \frac{1}{(1+e^{-x})^{q}}\right) dx$	$=$	$\psi(q)-\psi(p)$	**136**

Section 3.32 – 3.34.
Exponentials of more complicated arguments

Subsection 3.324

3.324.2	$\int_{-\infty}^{\infty} e^{-(x-b/x)^{2n}}\,dx$	$=$	$\frac{1}{n}\Gamma\left(\frac{1}{2n}\right)$	**31**

Subsection 3.326

3.326.1	$\int_{0}^{\infty} \exp(-x^{b})\,dx$	$=$	$\frac{1}{b}\Gamma\left(\frac{1}{b}\right)$	**29**
3.326.2	$\int_{0}^{\infty} x^{m}\exp(-\beta x^{n})\,dx$	$=$	$\frac{\Gamma(\gamma)}{n\beta^{\gamma}},\ \gamma = \frac{m+1}{n}$	**29**

Section. Exponentials of exponentials

Subsection 3.328

3.328 $\displaystyle\int_{-\infty}^{\infty} \exp(-e^x) e^{\mu x}\, dx = \Gamma(\mu)$ **31**

3.329 $\displaystyle\int_0^{\infty} \left(\frac{a \exp(-c e^{ax})}{1-e^{-ax}} - \frac{b \exp(-c e^{bx})}{1-e^{-bx}} \right) dx = e^{-c} \ln\left(\frac{b}{a}\right)$ **192**

Section 3.35.
Combinations of exponentials and rational functions

Subsection 3.351

3.351.1 $\displaystyle\int_0^u x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}} - e^{-au} \sum_{k=0}^{n} \frac{n!}{k!} \frac{u^k}{a^{n-k+1}}$ **85**

3.351.2 $\displaystyle\int_u^{\infty} x^n e^{-ax}\, dx = e^{-au} \sum_{k=0}^{n} \frac{n!}{k!} \frac{u^k}{a^{n-k+1}}$ **85**

3.351.3 $\displaystyle\int_0^{\infty} x^n e^{-\mu x}\, dx = n! \mu^{-n-1}$ **26**

3.351.7 $\displaystyle\int_0^u x e^{-ax}\, dx = \frac{1}{a^2} - \frac{1}{a^2} e^{-au}(1+au)$ **85**

3.351.8 $\displaystyle\int_0^u x^2 e^{-ax}\, dx = \frac{2}{a^3} - \frac{1}{a^3} e^{-au}(2+2au+a^2u^2)$ **85**

3.351.9 $\displaystyle\int_0^u x^3 e^{-ax}\, dx = \frac{6}{a^4} - \frac{1}{a^4} e^{-au}(6+6au+3a^2u^2+a^3u^3)$ **85**

Subsection 3.353

3.353.4 $\displaystyle\int_0^1 \frac{x e^x\, dx}{(1+x)^2} = \frac{e}{2} - 1$ **86**

Section 3.36 – 3.37.
Combinations of exponentials and algebraic functions

Subsection 3.371

3.371 $\displaystyle\int_0^{\infty} x^{n-\frac{1}{2}} e^{-\mu x}\, dx = \sqrt{\pi} 2^{-n} \mu^{-n-\frac{1}{2}} (2n-1)!!$ **26**

Section 3.38 – 3.39.
Combinations of exponentials and arbitrary powers

Subsection 3.382

3.382.2 $\int_b^\infty (x-b)^a e^{-\mu x}\,dx \quad = \quad \mu^{-a-1}e^{-\mu b}\Gamma(a+1)$ **29**

Section 3.41 – 3.44.
Combinations of rational functions of powers and exponentials

Subsection 3.411

3.411.5 $\int_0^{\ln 2} \frac{x\,dx}{1-e^{-x}} = \frac{\pi^2}{12}$ **187**

3.411.19 $\int_0^\infty e^{-px}(e^{-x}-1)^n \frac{dx}{x}$

$= -\sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(p+n-k)$ **89**

3.411.20 $\int_0^\infty e^{-px}(e^{-x}-1)^n \frac{dx}{x^2}$

$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (p+n-k)\ln(p+n-k)$ **89**

Subsection 3.412

3.412.1 $\int_0^\infty \left[\frac{a+be^{-px}}{ce^{px}+g+he^{-px}} - \frac{a+be^{-qx}}{ce^{qx}+g+he^{-qx}} \right] \frac{dx}{x}$

$= \quad \frac{a+b}{c+g+h} \ln\left(\frac{q}{p}\right)$ **193**

Subsection 3.419

3.419.2	$\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(\beta+e^x)(1-e^{-x})}$	$=$	$\displaystyle\frac{\pi^2+\ln^2\beta}{2(\beta+1)}$	**6**
3.419.3	$\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(\beta+e^x)(1-e^{-x})}$	$=$	$\displaystyle\frac{(\pi^2+\ln^2\beta)\,\ln\beta}{3(\beta+1)}$	**6**
3.419.4	$\displaystyle\int_{-\infty}^{\infty} \frac{x^3\,dx}{(\beta+e^x)(1-e^{-x})}$	$=$	$\displaystyle\frac{(\pi^2+\ln^2\beta)^2}{4(\beta+1)}$	**6**
3.419.5	$\displaystyle\int_{-\infty}^{\infty} \frac{x^4\,dx}{(\beta+e^x)(1-e^{-x})}$	$=$	$\displaystyle\frac{(\pi^2+\ln^2\beta)(7\pi^2+3\ln^2\beta)\,\ln\beta}{4(\beta+1)}$	**6**
3.419.6	$\displaystyle\int_{-\infty}^{\infty} \frac{x^5\,dx}{(\beta+e^x)(1-e^{-x})}$	$=$	$\displaystyle\frac{(\pi^2+\ln^2\beta)^2(3\pi^2+\ln^2\beta)}{6(\beta+1)}$	**6**

Subsection 3.427

3.427.1	$\displaystyle\int_0^{\infty} \left(\frac{e^{-x}}{x} - \frac{e^{-ax}}{1-e^{-x}}\right) dx$	$=$	$\psi(a)$	**131**
3.427.2	$\displaystyle\int_0^{\infty} \left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right) e^{-x}\,dx$	$=$	γ	**131**

Subsection 3.429

3.429	$\displaystyle\int_0^{\infty} \left[e^{-x} - (1+x)^{-a}\right] \frac{dx}{x}$	$=$	$\psi(a)$	**127**

Subsection 3.434

3.434.1	$\displaystyle\int_0^{\infty} \frac{e^{-\nu x}-e^{-\mu x}}{x^{\rho+1}}\,dx$	$=$	$\displaystyle\frac{\mu^\rho-\nu^\rho}{\rho}\Gamma(1-\rho)$	**26**
3.434.2	$\displaystyle\int_0^{\infty} \frac{e^{-\nu x}-e^{-\mu x}}{x}\,dx$	$=$	$\displaystyle\ln\frac{\nu}{\mu}$	**26, 128,191**

Subsection 3.435

3.435.3	$\displaystyle\int_0^{\infty} \left(e^{-x} - \frac{1}{1+x}\right) \frac{dx}{x}$	$=$	$-\gamma$	**128**
3.435.4	$\displaystyle\int_0^{\infty} \left(e^{-bx} - \frac{1}{1+ax}\right) \frac{dx}{x}$	$=$	$\displaystyle\ln\frac{a}{b}-\gamma$	**129**

Subsection 3.436

3.436	$\int_0^\infty \left(\frac{e^{-aqx} - e^{-apx}}{a} - \frac{e^{-bqx} - e^{-bpx}}{b} \right) \frac{dx}{x^2}$	$=$	$(p-q) \ln \frac{b}{a}$	**192**

Subsection 3.442

3.442.3	$\int_0^\infty \left(e^{-px} - \frac{1}{1+a^2x^2} \right) \frac{dx}{x}$	$=$	$\gamma + \ln \frac{a}{p}$	**150**

Section 3.45.
Combinations of powers and algebraic functions of exponentials

Subsection 3.457

3.457.1	$\int_0^\infty xe^{-x}(1-e^{2x})^{n-\frac{1}{2}}\, dx$	$=$	$\frac{\binom{2n}{n}\pi}{2^{2n+2}} \left(2\ln 2 + \sum_{k=1}^n \frac{1}{k} \right)$	**148**
3.457.3	$\int_{-\infty}^\infty \frac{x\, dx}{(a^2e^x + e^{-x})^\mu}$	$=$	$-\frac{1}{2a^\mu} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \ln a$	**70**

Section 3.3.46 – 3.48.
Combinations of exponentials of more complicated arguments and powers

Subsection 3.461

3.461.2	$\int_0^\infty x^{2n} e^{-px^2}\, dx$	$=$	$\frac{(2n-1)!!}{2(2p)^n} \sqrt{\frac{\pi}{p}}$	**35**
3.461.3	$\int_0^\infty x^{2n+1} e^{-px^2}\, dx$	$=$	$\frac{n!}{2p^{n+1}}$	**35**

Subsection 3.462

3.462.9	$\int_0^\infty e^{-\beta x^n \pm a}\, dx$	$=$	$\frac{e^{\pm a}}{n\beta^{1/n}} \Gamma\left(\frac{1}{n}\right)$	**29**

Subsection 3.463

3.463	$\int_0^\infty \left(e^{-x^2} - e^{-x} \right) \frac{dx}{x}$	$=$	$\frac{\gamma}{2}$	**130**

Subsection 3.467

3.467 $\displaystyle\int_0^\infty \left(e^{-x^2} - \frac{1}{1+x^2} \right) \frac{dx}{x} \quad = \quad -\frac{\gamma}{2}$ **150**

Subsection 3.469

3.469.2 $\displaystyle\int_0^\infty \left(e^{-x^4} - e^{-x} \right) \frac{dx}{x} \quad = \quad \frac{3\gamma}{4}$ **130**

3.469.3 $\displaystyle\int_0^\infty \left(e^{-x^4} - e^{-x^2} \right) \frac{dx}{x} \quad = \quad \frac{\gamma}{4}$ **130**

Subsection 3.471

3.471.1 $\displaystyle\int_0^u \exp\left(-\frac{\beta}{x} \right) \frac{dx}{x^2} \quad = \quad \frac{1}{\beta} \exp\left(-\frac{\beta}{u} \right)$ **84**

3.471.3 $\displaystyle\int_0^a x^{-\mu-1}(a-x)^{\mu-1} e^{-\beta/x}\, dx \quad = \quad \beta^{-\mu} a^{\mu-1} \Gamma(\mu) \exp(-\beta/a)$ **31**

3.471.14 $\displaystyle\int_0^1 \frac{e^{(1-1/x)} - x^a}{x(1-x)}\, dx \quad = \quad \psi(a)$ **129**

Subsection 3.473

3.473 $\displaystyle\int_0^\infty \exp(-x^n) x^{(m+1/2)n-1}\, dx \quad = \quad \frac{(2m-1)!!}{2^m n} \sqrt{\pi}$ **29**

Subsection 3.475

3.475.1 $\displaystyle\int_0^\infty \left(\exp\left(-x^{2^n} \right) - \frac{1}{1+x^{2^{n+1}}} \right) \frac{dx}{x} \quad = \quad -\frac{\gamma}{2^n}$ **150**

3.475.2 $\displaystyle\int_0^\infty \left(\exp\left(-x^{2^n} \right) - \frac{1}{1+x^2} \right) \frac{dx}{x} \quad = \quad -\frac{\gamma}{2^n}$ **150**

3.475.3 $\displaystyle\int_0^\infty \left(e^{-x^{2^n}} - e^{-x} \right) \frac{dx}{x} \quad = \quad (1-2^{-n})\gamma$ **150**

Subsection 3.476

3.476.1 $\displaystyle\int_0^\infty \left(e^{-ax^p} - e^{-bx^p} \right) \frac{dx}{x} \quad = \quad \frac{\ln b - \ln a}{p}$ **130,192**

3.476.2 $\displaystyle\int_0^\infty \left(e^{-x^p} - e^{-x^q} \right) \frac{dx}{x} \quad = \quad \frac{p-q}{pq}\gamma$ **129**

Subsection 3.478

3.478.1	$\int_0^\infty x^{\nu-1}\exp\left(-\mu x^p\right)\,dx$	$=$	$\frac{1}{p}\mu^{-\nu/p}\Gamma\left(\frac{\nu}{p}\right)$	**27, 29**
3.478.2	$\int_0^\infty x^{\nu-1}\left[1-\exp\left(-\mu x^p\right)\right]\,dx$	$=$	$-\frac{1}{\lvert p\rvert}\mu^{-\nu/p}\Gamma\left(\frac{\nu}{p}\right)$	**27**

Subsection 3.481

3.481.1	$\int_{-\infty}^\infty xe^x\exp(-\mu e^x)\,dx$	$=$	$-\frac{1}{\mu}(\gamma+\ln\mu)$	**31**
3.481.2	$\int_{-\infty}^\infty xe^x\exp(-\mu e^{2x})\,dx$	$=$	$-\frac{1}{4}\left[\gamma+\ln(4\mu)\right]\sqrt{\frac{\pi}{\mu}}$	**31**

Subsection 3.484

3.484	$\int_0^\infty \left[\left(1+\frac{a}{qx}\right)^{qx}-\left(1+\frac{a}{px}\right)^{px}\right]\frac{dx}{x}$	$=$	$(e^a-1)\ln\left(\frac{q}{p}\right)$	**193**

Section 3.52 – 3.53.
Combinations of hyperbolic functions and algebraic functions

Subsection 3.522

3.522.4	$\int_0^\infty \frac{dx}{\left(b^2+x^2\right)\cosh\pi x}$	$=$	$\frac{1}{b}\beta\left(b+\frac{1}{2}\right)$	**165**

Section 3.54.
Combinations of hyperbolic functions and exponentials

Section 3.541

3.541.6	$\int_0^\infty \frac{e^{-\mu x}}{\cosh x}\,dx$	$=$	$\beta\left(\frac{\mu+1}{2}\right)$	**159**
3.541.7	$\int_0^\infty e^{-\mu x}\tanh x\,dx$	$=$	$\beta\left(\frac{\mu}{2}\right)-\frac{1}{\mu}$	**159**
3.541.8	$\int_0^\infty \frac{e^{-\mu x}}{\cosh^2 x}\,dx$	$=$	$\mu\beta\left(\frac{\mu}{2}\right)-1$	**159**

Section 3.62.
Powers of trigonometric functions

Subsection 3.621

3.621.1a	$\int_0^{\pi/2} \sin^{\mu-1} x\, dx$	$=$	$2^{\mu-2} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right)$	**40,178**
3.621.1b	$\int_0^{\pi/2} \cos^{\mu-1} x\, dx$	$=$	$2^{\mu-2} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right)$	**40,178**
3.621.2	$\int_0^{\pi/2} \sin^{3/2} x\, dx$	$=$	$\frac{1}{6\sqrt{2\pi}} \Gamma^2\left(\frac{1}{4}\right)$	**40,179**
3.621.3a	$\int_0^{\pi/2} \sin^{2m} x\, dx$	$=$	$\frac{(2m-1)!!}{(2m)!!}\frac{\pi}{2}$	**40,179**
3.621.3b	$\int_0^{\pi/2} \cos^{2m} x\, dx$	$=$	$\frac{(2m-1)!!}{(2m)!!}\frac{\pi}{2}$	**40,178**
3.621.4a	$\int_0^{\pi/2} \sin^{2m+1} x\, dx$	$=$	$\frac{(2m)!!}{(2m+1)!!}$	**179**
3.621.4b	$\int_0^{\pi/2} \cos^{2m+1} x\, dx$	$=$	$\frac{(2m)!!}{(2m+1)!!}\frac{\pi}{2}$	**178**
3.621.5	$\int_0^{\pi/2} \sin^{a-1} x\, \cos^{b-1} x\, dx$	$=$	$\frac{1}{2} B\left(\frac{a}{2}, \frac{b}{2}\right)$	**177**
3.621.6	$\int_0^{\pi/2} \sqrt{\sin x}\, dx$	$=$	$\sqrt{\frac{2}{\pi}} \Gamma^2\left(\frac{1}{4}\right)$	**178**
3.621.7	$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}}$	$=$	$\frac{1}{2\sqrt{2\pi}} \Gamma^2\left(\frac{1}{4}\right)$	**178**

Subsection 3.622

3.622.1	$\int_0^{\pi/2} \tan^{\pm a} x\, dx$	$=$	$\frac{\pi}{2\cos(\pi a/2)}$	**179**
3.622.2	$\int_0^{\pi/4} \tan^{\mu} x\, dx$	$=$	$\frac{1}{2}\beta\left(\frac{\mu+1}{2}\right)$	**156**
3.622.3	$\int_0^{\pi/4} \tan^{2n} x\, dx$	$=$	$(-1)^n \left(\frac{\pi}{4} - \sum_{j=0}^{n-1} \frac{(-1)^{j-1}}{2j-1}\right)$	**83**
3.622.4	$\int_0^{\pi/4} \tan^{2n+1} x\, dx$	$=$	$\frac{(-1)^{n+1}}{2}\left(\ln 2 - \sum_{k=1}^{n} \frac{(-1)^k}{k}\right)$	**83**

Subsection 3.623

3.623.1a	$\int_0^{\pi/2} \tan^{a-1} x \, \cos^{2b-2} x \, dx$	$=$	$\frac{1}{2} B\left(\frac{a}{2}, b - \frac{a}{2}\right)$	**179**
3.623.1b	$\int_0^{\pi/2} \operatorname{cotg}^{a-1} x \, \sin^{2b-2} x \, dx$	$=$	$\frac{1}{2} B\left(\frac{a}{2}, b - \frac{a}{2}\right)$	**179**
3.623.2	$\int_0^{\pi/4} \tan^{\mu} x \sin^2 x \, dx$	$=$	$-\frac{1}{4} + \frac{1+\mu}{4} \beta\left(\frac{1+\mu}{2}\right)$	**157**
3.623.3	$\int_0^{\pi/4} \tan^{\mu} x \sin^2 x \, dx$	$=$	$\frac{1}{4} + \frac{1-\mu}{4} \beta\left(\frac{1+\mu}{2}\right)$	**157**

Subsection 3.624

3.624.1	$\int_0^{\pi/4} \frac{\sin^p x \, dx}{\cos^{p+2} x}$	$=$	$\frac{1}{p+1}$	**157**
3.624.2	$\int_0^{\pi/2} \frac{\sin^{a-1/2} x \, dx}{\cos^{2a-1} x}$	$=$	$\frac{\Gamma\left(\frac{a}{2} + \frac{1}{4}\right) \Gamma\left(1-a\right)}{2\Gamma\left(\frac{5}{4} - \frac{a}{2}\right)}$	**179**
3.624.3	$\int_0^{\pi/4} \frac{\cos^{n-1/2}(2x) \, dx}{\cos^{2n+1} x}$	$=$	$\frac{\pi}{2^{2n+1}} \binom{2n}{n}$	**181**
3.624.4	$\int_0^{\pi/4} \frac{\cos^{\mu}(2x) \, dx}{\cos^{2(\mu+1)} x}$	$=$	$2^{2\mu} B(\mu+1, \mu+1)$	**181**
3.624.5	$\int_0^{\pi/4} \frac{\sin^{2\mu-2} x \, dx}{\cos^{\mu}(2x)}$	$=$	$\frac{\Gamma\left(\mu - \frac{1}{2}, 1-\mu\right)}{2\sqrt{\pi}}$	**181**

Subsection 3.625

3.625.1	$\int_0^{\pi/4} \frac{\sin^{2n-1} x \, \cos^p(2x)}{\cos^{2p+2n+2} x} \, dx$	$=$	$\frac{1}{2} B\left(n, p+1\right)$	**182**
3.625.2	$\int_0^{\pi/4} \frac{\sin^{2n} x \, \cos^p(2x)}{\cos^{2p+2n+2} x} \, dx$	$=$	$\frac{1}{2} B\left(n + \frac{1}{2}, p+1\right)$	
	$\sin\left(\frac{\pi a}{2}\right)$			**181**
3.625.3	$\int_0^{\pi/4} \frac{\sin^{2n-1} x \, \cos^{m-1/2}(2x)}{\cos^{2n+2m} x} \, dx$	$=$	$\frac{1}{2} B\left(n, m + \frac{1}{2}\right)$	**182**
3.625.4	$\int_0^{\pi/4} \frac{\sin^{2n} x \, \cos^{m-1/2}(2x)}{\cos^{2n+2m+1} x} \, dx$	$=$	$\frac{1}{2} B\left(n + \frac{1}{2}, m + \frac{1}{2}\right)$	**182**

Subsection 3.626

3.626.1 $$\int_0^{\pi/4} \frac{\sin^{2n-1} x}{\cos^{2n+2} x} \sqrt{\cos(2x)}\, dx = \frac{1}{2} B\left(n+\tfrac{1}{2}, \tfrac{3}{2}\right)$$ **182**

3.626.2 $$\int_0^{\pi/4} \frac{\sin^{2n} x}{\cos^{2n+3} x} \sqrt{\cos(2x)}\, dx = \frac{1}{2} B\left(n, \tfrac{3}{2}\right)$$ **182**

Subsection 3.627

3.627a $$\int_0^{\pi/2} \frac{\tan^a x\, dx}{\cos^a x} = \frac{\Gamma(a)\Gamma\left(\frac{1}{2}-a\right)}{2^a\sqrt{\pi}} \sin\left(\frac{\pi a}{2}\right)$$ **180**

3.627b $$\int_0^{\pi/2} \frac{\cot^a x\, dx}{\sin^a x} = \frac{\Gamma(a)\Gamma\left(\frac{1}{2}-a\right)}{2^a\sqrt{\pi}} \sin\left(\frac{\pi a}{2}\right)$$ **180**

Subsection 3.628

3.628 $$\int_0^{\pi/2} \sec^{2p} x\, \sin^{2p-1} x\, dx = \frac{\Gamma(p)\Gamma\left(\frac{1}{2}-p\right)}{2\sqrt{\pi}}$$ **180**

Section 3.63.
Powers of trigonometric functions and trigonometric functions of linear functions

Subsection 3.635

3.635.1 $$\int_0^{\pi/4} \cos^{\mu-1}(2x)\, \tan x\, dx = \frac{\beta(\mu)}{2}$$ **158**

Section 3.64 – 3.65.
Powers and rational functions of trigonometric functions

Subsection 3.651

3.651.1 $$\int_0^{\pi/4} \frac{\tan^\mu x\, dx}{1+\sin x \cos x} = \frac{1}{3}\left(\psi\left(\frac{\mu+2}{2}\right) - \psi\left(\frac{\mu+1}{2}\right)\right)$$ **158**

3.651.2 $$\int_0^{\pi/4} \frac{\tan^\mu x\, dx}{1-\sin x \cos x} = \frac{1}{3}\left(\beta\left(\frac{\mu+2}{2}\right) + \beta\left(\frac{\mu+1}{2}\right)\right)$$ **158**

Subsection 3.656

$$\textbf{3.656.1} \quad \int_0^{\pi/4} \frac{\tan^\mu x\,dx}{1-\sin^2 x\cos^2 x} = \tfrac{1}{12}\Big(-\psi\left(\tfrac{\mu+1}{6}\right) - \psi\left(\tfrac{\mu+2}{6}\right) + \psi\left(\tfrac{\mu+4}{6}\right)\Big) + \tfrac{1}{12}\Big(\psi\left(\tfrac{\mu+5}{6}\right) + 2\psi\left(\tfrac{\mu+2}{6}\right) - 2\psi\left(\tfrac{\mu+1}{6}\right)\Big). \qquad \textbf{158}$$

Section 3.72 – 3.74.
Combinations of trigonometric and rational functions

Subsection 3.747

$$\textbf{3.747.7} \quad \int_0^{\pi/2} x\,\cot x\,dx = \frac{\pi}{2}\ln 2 \qquad \textbf{123}$$

Section 3.76 – 3.77.
Combinations of trigonometric functions and powers

Subsection 3.761

$$\textbf{3.761.11} \quad \int_0^{\pi/2} x^m\,\cos x\,dx = \sum_{k=0}^{\lfloor m/2\rfloor} (-1)^k \frac{m!}{(m-2k)!}\left(\frac{\pi}{2}\right)^{m-2k} + (-1)^{\lfloor m/2\rfloor}\left(2\lfloor\frac{m}{2}\rfloor - m\right) m! \qquad \textbf{42}$$

Subsection 3.764

$$\textbf{3.764.1} \quad \int_0^{\infty} x^p \sin(ax+b)\,dx = \frac{1}{a^{p+1}}\Gamma(1+p)\cos\left(b+\frac{p\pi}{2}\right) \qquad \textbf{50}$$

$$\textbf{3.764.2} \quad \int_0^{\infty} x^p \cos(ax+b)\,dx = -\frac{1}{a^{p+1}}\Gamma(1+p)\sin\left(b+\frac{p\pi}{2}\right) \qquad \textbf{50}$$

Section 3.82 – 3.83.
Powers of trigonometric functions combined with other powers

Subsection 3.821

3.821.3a $$\int_0^{\pi/2} x\cos^{2n} x\,dx = \frac{\binom{2n}{n}}{2^{2n+2}}\left(\frac{\pi^2}{2} - \sum_{k=1}^{n} \frac{2^{2k}}{k^2\binom{2k}{k}}\right)$$ **45**

3.821.3b $$\int_0^{\pi/2} x\cos^{2n+1} x\,dx = \frac{2^{2n}}{(2n+1)\binom{2n}{n}} \left(\frac{\pi}{2} - \sum_{k=0}^{n} \frac{\binom{2k}{k}}{2^{2k}(2k+1)}\right)$$ **45**

3.821.14 $$\int_0^{\infty} x^{-1/2}\sin^{2n+1}(px)\,dx = \frac{1}{2^{2n}}\sqrt{\frac{\pi}{2p}}\sum_{k=0}^{n}(-1)^k \binom{2n+1}{n+k+1}\frac{1}{\sqrt{2k+1}}$$ **48**

Subsection 3.822

3.822.1 $$\int_0^{\pi/2} x^p\cos^m x\,dx = -\frac{p(p-1)}{m^2}\int_0^{\pi/2} x^{p-2}\cos^m x\,dx + \frac{m-1}{m}\int_0^{\pi/2} x^p\cos^{m-2} x\,dx$$ **39**

3.822.2 $$\int_0^{\infty} x^{-1/2}\cos^{2n+1}(px)\,dx = \frac{1}{2^{2n}}\sqrt{\frac{\pi}{2p}}\sum_{k=0}^{n}\binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}$$ **38,48**

Section 4.21.
Logarithmic functions

Subsection 4.212

4.212.7 $$\int_1^{e} \frac{\ln x\,dx}{(1+\ln x)^2}\,dx = \frac{e}{2} - 1$$ **85**

Subsection 4.215

4.215.1	$\int_0^1 \left(\ln \frac{1}{x}\right)^{\mu-1} dx$	$=$	$\Gamma(\mu)$	**33**
4.215.2	$\int_0^1 \frac{dx}{(-\ln x)^{\mu-1}}$	$=$	$\frac{\pi}{\Gamma(\mu)} \text{ cosec } \mu\pi$	**33**
4.215.3	$\int_0^1 \sqrt{-\ln x}\, dx$	$=$	$\frac{\sqrt{\pi}}{2}$	**33**
4.215.4	$\int_0^1 \frac{dx}{\sqrt{-\ln x}}$	$=$	$\sqrt{\pi}$	**33**

Section 4.22.
Logarithms of more complicated functions

Subsection 4.222

4.222.1	$\int_0^\infty \ln\left(\frac{a^2+x^2}{b^2+x^2}\right) dx$	$=$	$(a-b)\pi$	**90**

Subsection 4.223

4.223.1	$\int_0^\infty \ln\left(1+e^{-x}\right) dx$	$=$	$\frac{\pi^2}{12}$	**106**
4.223.2	$\int_0^\infty \ln\left(1-e^{-x}\right) dx$	$=$	$-\frac{\pi^2}{6}$	**106**

Subsection 4.224

4.224.1	$\int_0^x \ln \sin t\, dt$	$=$	$L(\pi/2-x)-L(\pi/2)$	**119**
4.224.2	$\int_0^{\pi/4} \ln \sin x\, dx$	$=$	$-\frac{\pi}{4}\ln 2-\frac{1}{2}G$	**120**
4.224.3	$\int_0^{\pi/2} \ln \sin x\, dx$	$=$	$-\frac{\pi}{2}\ln 2$	**119**
4.224.4	$\int_0^x \ln \cos t\, dt$	$=$	$-L(x)$	**119**
4.224.5	$\int_0^{\pi/4} \ln \cos x\, dx$	$=$	$-\frac{\pi}{4}\ln 2+\frac{1}{2}G$	**120**
4.224.6	$\int_0^{\pi/2} \ln \cos x\, dx$	$=$	$-\frac{\pi}{2}\ln 2$	**119**

Subsection 4.225

4.225.1	$\int_0^{\pi/4} \ln(\cos x - \sin x)\, dx$	$=$	$-\frac{\pi}{8}\ln 2 - \frac{G}{2}$	**121**
4.225.2	$\int_0^{\pi/4} \ln(\cos x + \sin x)\, dx$	$=$	$-\frac{\pi}{8}\ln 2 + \frac{G}{2}$	**121**

Subsection 4.227

4.227.1	$\int_0^{u} \ln\tan x\, dx$	$=$	$L(u) + L(\pi/2 - u) + \frac{\pi}{2}\ln 2$	**120**
4.227.2	$\int_0^{\pi/4} \ln\tan x\, dx$	$=$	$-G$	**120**
4.227.3	$\int_0^{\pi/2} \ln(a\tan x)\, dx$	$=$	$\frac{\pi}{2}\ln a$	**122**
4.227.9	$\int_0^{\pi/4} \ln(1+\tan x)\, dx$	$=$	$\frac{\pi}{8}\ln 2$	**121**
4.227.10	$\int_0^{\pi/4} \ln(1+\tan x)\, dx$	$=$	$\frac{\pi}{4}\ln 2 + G$	**122**
4.227.11	$\int_0^{\pi/4} \ln(1-\tan x)\, dx$	$=$	$\frac{\pi}{8}\ln 2 - G$	**121**
4.227.13	$\int_0^{\pi/4} \ln(1+\cot x)\, dx$	$=$	$\frac{\pi}{8}\ln 2 + G$	**121**
4.227.14	$\int_0^{\pi/4} \ln(\cot x - 1)\, dx$	$=$	$\frac{\pi}{8}\ln 2$	**121**
4.227.15	$\int_0^{\pi/4} \ln(\tan x + \cot x)\, dx$	$=$	$\frac{\pi}{2}\ln 2$	**121**

Subsection 4.229

4.229.1	$\int_0^{1} \ln\left(\ln\frac{1}{x}\right)\, dx$	$=$	$-\gamma$	**34**
4.229.3	$\int_0^{1} \ln\left(\ln\frac{1}{x}\right) \frac{dx}{\sqrt{-\ln x}}$	$=$	$-(\gamma + 2\ln 2)\sqrt{\pi}$	**34**
4.229.4	$\int_0^{1} \ln\left(\ln\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{\mu-1}\, dx$	$=$	$\psi(\mu)\Gamma(\mu)$	**1,34**
4.229.7	$\int_{\pi/4}^{\pi/2} \ln\ln\tan x\, dx$	$=$	$\frac{\pi}{2}\ln\left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}\right)$	**1,183**

Section 4.23.
Combinations of logarithms and rational functions

Subsection 4.231

4.231.1	$\int_0^1 \frac{\ln x\,dx}{1+x}$	$=$	$-\frac{\pi^2}{12}$	**11, 99,103**
4.231.2	$\int_0^1 \frac{\ln x\,dx}{1-x}$	$=$	$-\frac{\pi^2}{6}$	**106**
4.231.3	$\int_0^1 \frac{x\,\ln x\,dx}{1-x}$	$=$	$1-\frac{\pi^2}{6}$	**106**
4.231.4	$\int_0^1 \frac{1+x}{1-x}\,\ln x\,dx$	$=$	$1-\frac{\pi^2}{3}$	**106**
4.231.5	$\int_0^\infty \frac{\ln x\,dx}{(x+a)^2}$	$=$	$\frac{\ln a}{a}$	**11**
4.231.6	$\int_0^1 \frac{\ln x\,dx}{(1+x)^2}$	$=$	$-\ln 2$	**11,164**
4.231.8	$\int_0^\infty \frac{\ln x\,dx}{a^2+b^2x^2}$	$=$	$\frac{\pi}{2ab}\ln\left(\frac{a}{b}\right)$	**114**
4.231.9	$\int_0^\infty \frac{\ln px\,dx}{q^2+x^2}$	$=$	$\frac{\pi}{2q}\ln pq$	**114**
4.231.11	$\int_0^a \frac{\ln x\,dx}{x^2+a^2}$	$=$	$\frac{\pi\ln a}{4a}-\frac{G}{a}$	**112,155**
4.231.12a	$\int_0^1 \frac{\ln x\,dx}{1+x^2}$	$=$	$-G$	**111,155**
4.231.12b	$\int_1^\infty \frac{\ln x\,dx}{1+x^2}$	$=$	G	**107,155**
4.231.13	$\int_0^1 \frac{\ln x\,dx}{1-x^2}$	$=$	$-\frac{\pi^2}{8}$	**107**
4.231.14	$\int_0^1 \frac{x\,\ln x\,dx}{1+x^2}$	$=$	$-\frac{\pi^2}{48}$	**160**
4.231.15	$\int_0^1 \frac{x\,\ln x\,dx}{1-x^2}$	$=$	$-\frac{\pi^2}{24}$	**107**
4.231.19	$\int_0^1 \frac{x\,\ln x\,dx}{1+x}$	$=$	$-1+\frac{\pi^2}{2}$	**106,161**
4.231.20	$\int_0^1 \frac{(1-x)\,\ln x\,dx}{1+x}$	$=$	$1-\frac{\pi^2}{6}$	**107,161**

Subsection 4.232

4.232.1	$\int_u^v \frac{\ln x\,dx}{(x+u)(x+v)}$	$=$	$\frac{\ln uv}{2(v-u)} \ln \frac{(u+v)^2}{4uv}$	**13**
4.232.2	$\int_0^\infty \frac{\ln x\,dx}{(x+\beta)(x+\gamma)}$	$=$	$\frac{\ln^2\beta - \ln^2\gamma}{2(\beta-\gamma)}$	**13**
4.232.3	$\int_0^\infty \frac{\ln x\,dx}{(x+a)(x-1)}$	$=$	$\frac{\pi^2+\ln^2 a}{2(a+1)}$	**3**

Subsection 4.233

4.233.1	$\int_0^1 \frac{\ln x\,dx}{1+x+x^2}$	$=$	$\frac{2}{9}\left(\frac{2\pi^2}{3} - \psi'\left(\frac{1}{3}\right)\right)$	**103,170**
4.233.2	$\int_0^1 \frac{\ln x\,dx}{1-x+x^2}$	$=$	$\frac{2\pi^2}{9} - \frac{1}{3}\psi'\left(\frac{1}{3}\right)$	**170**
4.233.3	$\int_0^1 \frac{x\,\ln x\,dx}{1+x+x^2}$	$=$	$-\frac{7\pi^2}{54} + \frac{1}{9}\psi'\left(\frac{1}{3}\right)$	**170**
4.233.4	$\int_0^1 \frac{x\,\ln x\,dx}{1-x+x^2}$	$=$	$\frac{5\pi^2}{36} - \frac{1}{6}\psi'\left(\frac{1}{3}\right)$	**170**

Subsection 4.234

4.234.1	$\int_1^\infty \frac{\ln x\,dx}{(1+x^2)^2}$	$=$	$\frac{G}{2} - \frac{\pi}{8}$	**164**
4.234.2	$\int_0^1 \frac{x\,\ln x\,dx}{(1+x^2)^2}$	$=$	$-\frac{\ln 2}{4}$	**164**

Section 4.24.
Combinations of logarithms and algebraic functions

Subsection 4.241

4.241.1	$\int_0^1 \frac{x^{2n}\,\ln x\,dx}{\sqrt{1-x^2}}$	$=$	$\frac{\binom{2n}{n}\pi}{2^{2n+1}}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k} - \ln 2\right)$	**142**
4.241.2	$\int_0^1 \frac{x^{2n+1}\,\ln x\,dx}{\sqrt{1-x^2}}$	$=$	$\frac{(2n)!!}{(2n+1)!!}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right)$	**142**

Subsection 4.245

4.245.1	$\int_0^1 \frac{x^{4n+1}\ \ln x\,dx}{\sqrt{1-x^4}}$	$=$	$\frac{2^{2n-2}}{(2n+1)\binom{2n}{n}}\left(\ln 2+\sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right)$	**147**
4.245.2	$\int_0^1 \frac{x^{4n+3}\ \ln x\,dx}{\sqrt{1-x^4}}$	$=$	$\frac{\pi\binom{2n}{n}}{2^{2n+3}}\left(-\ln 2+\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\right)$	**147**

Subsection 4.246

4.246	$\int_0^1 (1-x^2)^{n-1/2}\ \ln x\,dx$	$=$	$-\frac{\binom{2n}{n}\pi}{2^{2n+2}}\left(2\ln 2+\sum_{k=1}^{n}\frac{1}{k}\right)$	**144**

Subsection 4.247

4.247.1	$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{1-x^{2n}}}$	$=$	$-\frac{\pi}{8}\frac{B(\frac{1}{2n},\frac{1}{2n})}{n^2\sin\left(\frac{\pi}{2n}\right)}$	**147**
4.247.2	$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{x^{n-1}(1-x^2)}}$	$=$	$-\frac{\pi}{8}\frac{B(\frac{1}{2n},\frac{1}{2n})}{\sin\left(\frac{\pi}{2n}\right)}$	**148**

Subsection 4.251

4.251.1	$\int_0^\infty \frac{x^{a-1}\ \ln x\,dx}{x+b}$	$=\frac{\pi\,b^{a-1}}{\sin\pi a}(\ln b-\pi\cot\pi a)$	**69**
4.251.3	$\int_0^\infty \frac{x^{\mu-1}\ \ln x}{1+x}\,dx$	$=\beta'(\mu)$	**160**
4.251.4	$\int_0^1 \frac{x^{p-1}\ \ln x\,dx}{1-x}$	$=-\psi'(p)$	**140**
4.253.1	$\int_0^1 x^{a-1}(1-x^c)^{b-1}\ \ln x\,dx$	$=\frac{\Gamma(a/c)\Gamma(b)}{c^2\Gamma(a/c+b)}\left(\psi\left(\frac{a}{c}\right)-\psi\left(\frac{a}{c}+b\right)\right)$	**141**

Subsection 4.254

4.254.1	$\int_0^1 \frac{x^{p-1}\ \ln x\,dx}{1-x^q}$	$=$	$-\frac{1}{q^2}\psi'\left(\frac{p}{q}\right)$	**140**
4.254.4	$\int_0^1 \frac{x^{a-1}\ \ln x\,dx}{1+x^p}$	$=$	$\frac{1}{p^2}\beta'\left(\frac{a}{p}\right)$	**160**
4.254.6	$\int_0^1 \frac{x^{q-1}\ \ln x\,dx}{1-x^{2q}}$	$=$	$-\frac{\pi^2}{8q^2}$	**141**

Subsection 4.256

4.256	$\int_0^1 \ln\left(\frac{1}{x}\right) \frac{x^{\mu-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}}$	$=$	$\frac{1}{n^2} B\left(\frac{\mu}{n}, \frac{m}{n}\right) \left[\psi\left(\frac{\mu+m}{n}\right) - \psi\left(\frac{\mu}{n}\right)\right]$	**141**

Subsection 4.26 – 4.27.
Combinations involving powers of the logarithm and other powers

Subsection 4.261

4.261.2	$\int_0^1 \frac{\ln^2 x\,dx}{1-x+x^2}$	$=$	$\frac{10\pi^3}{81\sqrt{3}}$	**163**
4.261.4	$\int_0^\infty \frac{\ln^2 x\,dx}{(x+a)(x-1)}$	$=$	$\frac{(\pi^2+\ln^2 a)\,\ln a}{3(a+1)}$	**3**
4.261.6	$\int_0^1 \frac{\ln^2 x\,dx}{1+x}$	$=$	$\frac{\pi^3}{16}$	**161**
4.261.8	$\int_0^1 \frac{1-x}{1-x^6}\,\ln^2 x\,dx$	$=$	$\frac{8\sqrt{3}\pi^3+351\zeta(3)}{486}$	**103,173**
4.261.11	$\int_0^1 \frac{x^n \ln^2 x\,dx}{1+x}$	$=$	$(-1)^n\left(\frac{3}{2}\zeta(3)+2\sum_{k=1}^{n}\frac{(-1)^k}{k^3}\right)$	**163**

Subsection 4.262

4.262.1	$\int_0^1 \frac{\ln^3 x\,dx}{1+x}$	$=$	$-\frac{7\pi^4}{120}$	**163**
4.262.3	$\int_0^\infty \frac{\ln^3 x\,dx}{(x+a)(x-1)}$	$=$	$\frac{(\pi^2+\ln^2 a)^2}{4(a+1)}$	**3**
4.262.4	$\int_0^1 \frac{x^n \ln^3 x\,dx}{1+x}$	$=$	$(-1)^{n+1}\left(\frac{7\pi^4}{120}-6\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)^4}\right)$	**163**

Subsection 4.263

4.263.1	$\int_0^\infty \frac{\ln^4 x\,dx}{(x+a)(x-1)}$	$=$	$\frac{(\pi^2+\ln^2 a)(7\pi^2+3\ln^2 a)\ln a}{15(a+1)}$	**3**
4.263.2	$\int_0^1 \frac{\ln^4 x\,dx}{1+x^2}$	$=$	$\frac{5\pi^5}{64}$	**162**

Subsection 4.264

4.264.1	$\int_0^1 \frac{\ln^5 x\,dx}{1+x^2}$	$=$	$-\frac{31\pi^6}{252}$	**163**
4.264.3	$\int_0^\infty \frac{\ln^5 x\,dx}{(x+a)(x-1)}$	$=$	$\frac{(\pi^2+\ln^2 a)^2(3\pi^2+\ln^2 a)}{6(a+1)}$	**3**

Subsection 4.265

4.265	$\int_0^1 \frac{\ln^6 x\,dx}{1+x^2}$	$=$	$\frac{61\pi^7}{256}$	**162**

Subsection 4.266

4.266.1	$\int_0^1 \frac{\ln^7 x\,dx}{1+x}$	$=$	$-\frac{127\pi^8}{240}$	**163**

Subsection 4.267

4.267.8	$\int_0^1 \frac{x^{b-1}-x^{a-1}}{\ln x}$	$=$	$\ln\left(\frac{a}{b}\right)$	**191**

Subsection 4.269

4.269.3	$\int_0^1 x^{p-1}\sqrt{-\ln x}\,dx$	$=$	$\frac{1}{2}\sqrt{\frac{\pi}{p^3}}$	**33**
4.269.4	$\int_0^1 \frac{x^{p-1}\,dx}{\sqrt{-\ln x}}$	$=$	$\sqrt{\frac{\pi}{p}}$	**33**

Subsection 4.271

4.271.1	$\int_0^1 \frac{\ln^{2n} x\,dx}{1+x}$	$=$	$\frac{2^{2n}-1}{2^{2n}}(2n)!\zeta(2n+1)$	**164**
4.271.15	$\int_0^1 \frac{x^{p-1}\ \ln^n x\,dx}{1-x^q}$	$=$	$-\frac{1}{q^{n+1}}\psi^{(n)}\left(\frac{p}{q}\right)$	**141**
4.271.16	$\int_0^1 \frac{x^{p-1}\ \ln^n x\,dx}{1+x^q}$	$=$	$\frac{1}{q^{n+1}}\beta^{(n)}\left(\frac{p}{q}\right)$	**160**

Subsection 4.272

4.272.5	$\int_1^\infty (\ln x)^p \frac{dx}{x^2}$	$=$	$\Gamma(1+p)$	**34**
4.272.6	$\int_0^1 (-\ln x)^{\mu-1} x^{\nu-1}\, dx$	$=$	$\frac{1}{\nu^\mu}\Gamma(\mu)$	**34**
4.272.7	$\int_0^1 (-\ln x)^{n-1/2} x^{\nu-1}\, dx$	$=$	$\frac{(2n-1)!!}{(2\nu)^n}\sqrt{\frac{\pi}{\nu}}$	**34**

Subsection 4.273

4.273	$\int_u^v \left(\ln \frac{x}{u}\right)^{p-1} \left(\ln \frac{v}{x}\right)^{q-1} \frac{dx}{x}$	$=$	$B(p,q)\left(\ln \frac{v}{u}\right)^{p+q-1}$	**67**

Subsection 4.275

4.275.1	$\int_0^1 \left[(-\ln x)^{q-1} - x^{p-1}(1-x)^{q-1}\right]\, dx$	$=$	$\Gamma(q) - B(p,q)$	**68**
4.275.2	$\int_0^1 \left[x - \left(\frac{1}{1-\ln x}\right)^q\right] \frac{dx}{x\,\ln x}$	$=$	$-\psi(q)$	**128**

Section 4.28.
Combinations of rational functions of $\ln x$ and powers

Subsection 4.281

4.281.1	$\int_0^1 \left(\frac{1}{\ln x} + \frac{1}{1-x}\right) dx$	$=$	γ	**131**
4.281.4	$\int_0^1 \left(\frac{1}{\ln x} + \frac{x^{a-1}}{1-x}\right) dx$	$=$	$-\psi(a)$	**131**
4.281.5	$\int_0^1 \left(\frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x}\right) dx$	$=$	$\ln p - \psi(q)$	**131**

Section 4.29 – 4.32. Combinations of logarithmic functions of more complicated arguments and powers

Subsection 4.291

4.291.1 $$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}$$ **105**

4.291.2 $$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}$$ **106**

Subsection 4.293

4.293.8 $$\int_0^1 x^{a-1}\ln(1-x)\,dx = -\tfrac{1}{a}\left(\psi(a+1)+\gamma\right)$$ **148**

4.293.13 $$\int_0^1 x^{a-1}(1-x)^{b-1}\ln(1-x)\,dx = B(a,b)\Big[\psi(b) - \psi(a+b)\Big]$$ **148**

Subsection 4.295

4.295.5 $$\int_0^1 \frac{\ln(1+x^2)}{1+x^2}\,dx = \frac{\pi}{2}\ln 2 - G$$ **122**

4.295.6 $$\int_1^\infty \frac{\ln(1+x^2)}{1+x^2}\,dx = \frac{\pi}{2}\ln 2 + G$$ **122**

4.295.11 $$\int_0^1 \frac{\ln(1-x^2)}{x}\,dx = -\frac{\pi^2}{12}$$ **107**

Subsection 4.297

4.297.7 $$\int_0^\infty \frac{b\,\ln(1+ax)-a\,\ln(1+bx)}{x^2}\,dx = ab\ln\left(\frac{b}{a}\right)$$ **193**

Subsection 4.319

4.319.3 $$\int_0^\infty \frac{\ln(a+be^{-px})-\ln(a+be^{-qx})}{x} = \ln\left(\frac{a}{a+b}\right)\ln\left(\frac{p}{q}\right)$$ **193**

Subsection 4.321

4.321.1 $\displaystyle\int_{-\infty}^{\infty} x \, \ln\cosh x \, dx \quad = \quad 0$ **71**

Subsection 4.324

4.324.2a $\displaystyle\int_0^{\infty} \left[\ln\left(\frac{1+2a\cos px + a^2}{1+2a\cos qx+a^2} \right) \right] \frac{dx}{x} \quad = \quad 2\ln\left(\frac{q}{p}\right)\ln(1+a)$

if $a \in (-1,1]$ **195**

4.324.2b $\displaystyle\int_0^{\infty} \left[\ln\left(\frac{1+2a\cos px + a^2}{1+2a\cos qx+a^2} \right) \right] \frac{dx}{x} \quad = \quad 2\ln\left(\frac{q}{p}\right)\ln(1+1/a)$

if $a \notin (-1,1]$ **195**

Subsection 4.325

4.325.8 $\displaystyle\int_0^1 \ln(-\ln x)\, x^{\mu-1}\, dx \quad = \quad -\frac{1}{\mu}(\gamma+\ln\mu)$ **34**

4.325.11 $\displaystyle\int_0^1 \ln(-\ln x) \frac{x^{\mu-1}\,dx}{\sqrt{-\ln x}} \quad = \quad -(\gamma+\ln 4\mu)\sqrt{\frac{\pi}{\mu}}$ **34**

4.325.12 $\displaystyle\int_0^1 \ln(-\ln x)\,(-\ln x)^{\mu-1}\, x^{\mu-1}\, dx \quad = \quad \frac{1}{\nu^{\mu}}\Gamma(\mu)\left[\psi(\mu)-\ln\nu\right]$ **34**

Section 4.33 – 4.34.
Combinations of logarithms and exponentials

Subsection 4.331

4.331.1 $\displaystyle\int_0^{\infty} e^{-x}\ln x\, dx \quad = \quad -\gamma$ **20,28,96**

Subsection 4.333

4.333 $\displaystyle\int_0^{\infty} e^{-\mu x^2}\ln x\, dx \quad = \quad -\frac{1}{4}(\gamma+\ln 4\mu)\sqrt{\frac{\pi}{\mu}}$ **30**

Subsection 4.335

4.335.1 $$\int_0^\infty e^{-\mu x} \ln^2 x \, dx = \frac{1}{\mu}\left[\frac{\pi^2}{6} + (\gamma + \ln \mu)^2\right]$$ **21,28**

4.335.3 $$\int_0^\infty e^{-\mu x} \ln^3 x \, dx = \frac{1}{\mu}\left[(\gamma + \ln \mu)^3 + \frac{\pi^2}{2}(\gamma + \ln \mu) - \psi''(1)\right]$$ **22,28**

Section 4.35 – 4.36.
Combinations of logarithms, exponentials, and powers

Subsection 4.351

4.351.1 $$\int_0^\infty (1-x)e^{-x} \ln x \, dx = \frac{1-e}{e}$$ **98**

4.351.2 $$\int_0^1 e^{-ax}(-ax^2 + 2x) \ln x \, dx = \frac{1}{a^2}\left[-1 + (1+a)e^{-a}\right]$$ **101**

Subsection 4.352

4.352.1 $$\int_0^\infty x^{\nu-1} e^{-\mu x} \ln x \, dx = \frac{1}{\mu^\nu}\Gamma(\nu)\left[\psi(\nu) - \ln \mu\right]$$ **23**

4.352.2 $$\int_0^\infty x^n e^{-\mu x} \ln x \, dx = \frac{n!}{\mu^{n+1}}\left[\sum_{j=1}^n \frac{1}{j} - \gamma - \ln \mu\right]$$ **23**

4.352.3 $$\int_0^\infty x^{n-\frac{1}{2}} e^{-\mu x} \ln x \, dx = \sqrt{\pi}\frac{(2n-1)!!}{2^n \mu^{n+\frac{1}{2}}} \left[2\left(\sum_{j=1}^n \frac{1}{2j-1}\right) - \gamma - \ln 4\mu\right]$$ **23**

4.352.4 $$\int_0^\infty x^{\mu-1} e^{-x} \ln x \, dx = \Gamma'(\mu)$$ **23**

Subsection 4.353

4.353.1 $$\int_0^\infty (x-\nu)^{\mu-1} e^{-x} \ln x \, dx = \Gamma'(\mu)$$ **24**

4.353.2 $$\int_0^\infty (\mu x - \nu - \tfrac{1}{2}) x^{n-\frac{1}{2}} e^{-\mu x} \ln x \, dx = \frac{(2n-1)!!}{(2\mu)^n}\sqrt{\frac{\pi}{\mu}}$$ **24**

4.353.3 $\int_0^1 (ax+n+1)x^n e^{ax} \ln x \, dx = e^a \sum_{k=0}^{n} (-1)^{k-1} \frac{n!}{(n-k)!a^{k+1}} + (-1)^n \frac{n!}{a^{n+1}} + (-1)^n \frac{n!}{a^{n+1}}$ **101**

Subsection 4.355

4.355.1 $\int_0^\infty x^2 e^{-\mu x^2} \ln x \, dx = \frac{1}{8\mu}(2 - \ln 4\mu - \gamma)\sqrt{\frac{\pi}{\mu}}$ **30**

4.355.3 $\int_0^\infty (\mu x^2 - n) x^{2n-1} e^{-\mu x^2} \ln x \, dx = \frac{(n-1)!}{4\mu^n}$ **30**

4.355.4 $\int_0^\infty (2\mu x^2 - 2n - 1) x^{2n} e^{-\mu x^2} \ln x \, dx = \frac{(2n-1)!!}{2(2\mu)^n}\sqrt{\frac{\pi}{\mu}}$ **30**

Subsection 4.358

4.358.2 $\int_0^\infty x^{a-1} e^{-\mu x} \ln^2 x \, dx = \frac{\Gamma(a)}{\mu^a}\left[\delta^2 + \zeta(2,a)\right], \quad \delta = \psi(a) - \ln \mu$ **29**

4.358.3 $\int_0^\infty x^{a-1} e^{-\mu x} \ln^3 x \, dx = \frac{\Gamma(a)}{\mu^a}\left[\delta^3 + 3\zeta(2,a)\delta - 2\zeta(3,a)\right], \quad \delta = \psi(a) - \ln \mu$ **29**

4.358.4 $\int_0^\infty x^{a-1} e^{-\mu x} \ln^4 x \, dx = \frac{\Gamma(a)}{\mu^a}\Big[\delta^4 + 6\zeta(2,a)\delta^2 - 8\zeta(3,a)\delta + 3\zeta^2(2,a) + 6\zeta(4,a)\Big]$ **29**

4.358.5 $\int_0^\infty x^{a-1} e^{-\mu x} (\ln x)^n \, dx = \left(\frac{\partial}{\partial a}\right)^n \left[\mu^{-a}\Gamma(a)\right]$ **28**

Subsection 4.362

4.362.1 $\int_0^1 x e^x \ln(1-x) \, dx = 1 - e$ **98**

Subsection 4.369

4.369.1 $\displaystyle\int_0^\infty x^{\nu-1}e^{-\mu x}\left[\psi(\nu)-\ln x\right]\,dx = \frac{\Gamma(\nu)\ln\mu}{\mu^\nu}$ **30**

4.369.2 $\displaystyle\int_0^\infty x^{n}e^{-\mu x}\left[\ln x - a_n - b_n\right]\,dx = \frac{n!}{\mu^{n+1}}\left\{[\ln\mu - a_n]^2 + b_n\right\}$ **31**

$$a_n = \tfrac{1}{2}\psi(n+1),$$
$$b_n = \tfrac{1}{2}\psi'(n+1)$$

Section 4.52.
Combinations of arcsines, arccosines, and powers

Subsection 4.521

4.521.1 $\displaystyle\int_0^1 \frac{\text{Arcsin } x}{x}\,dx = \frac{\pi}{2}\ln 2$ **123**

Section 4.53 – 4.54.
Combinations of arctangents, arccotangents, and powers

Subsection 4.531

4.531.1 $\displaystyle\int_0^1 \frac{\tan^{-1} x}{x}\,dx = G$ **111**

Subsection 4.536

4.536.2 $\displaystyle\int_0^\infty \frac{\tan^{-1}(px)-\tan^{-1}(qx)}{x} = \frac{\pi}{2}\ln\left(\frac{p}{q}\right)$ **193**

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