简　介

这是本人收集的大学数学习题集, 都是我自己或别人做过的题目, 当然很多方法都是别人的, 在这里非常感谢高等数学贴吧的吧友提供的题目和方法, 我相当于当了一回搬运工, 把他们的方法整理了一遍. 都是些难度比较高的题目, 题目会比较杂, 涉及到数学分析, 高等代数等, 主要题型是积分、极限、级数, 以及比较难的高等代数题, 题目偏向竞赛性质甚至高于竞赛. 纯粹 LaTeX 手打, 错误在所难免, 请读者见谅.

数学难题

向　　禹

*若有错漏, 欢迎批评指正. 739049687@qq.com

1. 设 $f(x)$ 在 $(0,\pi)$ 连续, $\int_0^\pi f(x)\cos kx\mathrm{d}x=\int_0^\pi f(x)\sin kx\mathrm{d}x=0$ 对 $1\leqslant k\leqslant n$ 都成立. 证明: $f(x)$ 在 $(0,\pi)$ 至少有 $2n$ 个零点.

证明 由于 $f(x)$ 连续, 我们只需要证明 $f(x)$ 在 $(0,\pi)$ 至少改变 $2n$ 次符号. 为此, 只要说明对任何给定 $0<x_1<x_2<\cdots<x_m<\pi, m\leqslant 2n-1$, 存在形如

$$f(x)=\sum_{k=1}^n(a_k\cos kx+b_k\sin kx)$$

的函数恰好在上述点的附近改变符号: 比如在 $(0,x_1)$ 为正, 在 (x_1,x_2) 为负, 在 (x_2,x_3) 为正, 在 (x_3,x_4) 为负,$\cdots$, 若 $m<2n-1$, 则记 $x_{m+1}=\cdots=x_{2n-1}=0$. 令

$$g(x)=\prod_{k=1}^{2n-1}\sin\frac{x-x_k}{2}$$

我们有

$$g(x)=\prod_{k=1}^{2n-1}\left(\sin\frac{x}{2}\cos\frac{x_k}{2}-\cos\frac{x}{2}\sin\frac{x_k}{2}\right)=C\prod_{k=1}^{2n-1}\left(\sin\frac{x}{2}-c_k\cos\frac{x}{2}\right)$$

其中 $C>0, c_1,c_2,\cdots,c_{2n-1}\leqslant 0$. 我们要证明有 $\alpha\in[0,\pi]$ 使得 $f(x)=g(x)\cos\frac{x-\alpha}{2}$ 具有形式

$$\sum_{k=1}^n(a_k\cos kx+b_k\sin kx)$$

此时 $f(x)$ 满足要求.

用和差化积公式可知 $g(x)\cos\frac{x-\alpha}{2}$ 是一个 n 阶的三角多项式. 所以只要证明存在 $\alpha\in[0,\pi]$ 使得

$$\int_0^{2\pi}g(x)\cos\frac{x-\alpha}{2}\mathrm{d}x=0$$

这只要证明

$$\int_0^{2\pi}g(x)\cos\frac{x}{2}\mathrm{d}x=2\int_0^\pi g(2x)\cos x\mathrm{d}x$$

与

$$\int_0^{2\pi}g(x)\sin\frac{x}{2}\mathrm{d}x=2\int_0^\pi g(2x)\sin x\mathrm{d}x$$

异号（只要其中有一个为零也认为是异号）.

$$g(2x)=C\prod_{k=1}^{2n-1}(\sin x-c_k\cos x)=\sum_{k=1}^{2n-1}(-1)^k\alpha_k\sin^{2n-1-k}\cos^k x$$

其中 α_k 非负. 由此立即得到 $\int_0^\pi g(2x)\sin x\mathrm{d}x$ 非负, $\int_0^\pi g(2x)\cos x\mathrm{d}x$ 非正. 综上所述, 结论成立. □

2. 求极限

$$\lim_{n\to\infty}n^3\left(\tan\int_0^\pi\sqrt[n]{\sin x}\mathrm{d}x+\sin\int_0^\pi\sqrt[n]{\sin x}\mathrm{d}x\right).$$

解 当 $x \to 0$ 时, $\tan x - \sin x \sim \dfrac{x^3}{2}$,
于是

$$\begin{aligned}
&\lim_{n\to\infty} n^3\left(\tan\int_0^{\pi}\sqrt[n]{\sin x}\mathrm{d}x + \sin\int_0^{\pi}\sqrt[n]{\sin x}\mathrm{d}x\right)\\
=&\lim_{n\to\infty} n^3\left(\tan\int_0^{\pi}(\sqrt[n]{\sin x}-1)\mathrm{d}x - \sin\int_0^{\pi}(\sqrt[n]{\sin x}-1)\mathrm{d}x\right)\\
=&\lim_{n\to\infty}\frac{\left(n\int_0^{\pi}(\sqrt[n]{\sin x}-1)\mathrm{d}x\right)^3}{2}\\
=&\frac{\left(\int_0^{\pi}\ln\sin x\mathrm{d}x\right)^3}{2}\\
=&-\frac{(\pi\ln 2)^3}{2}
\end{aligned}$$

其中

$$\lim_{n\to\infty} n\int_0^{\pi}\left(\sqrt[n]{\sin x}-1\right)\mathrm{d}x = \lim_{n\to\infty}\int_0^{\pi}\frac{\sqrt[n]{\sin x}-1}{1/n}\mathrm{d}x = \int_0^{\pi}\ln(\sin x)\,\mathrm{d}x = -\pi\ln 2$$

□

3. 把 $f(x)=\cos ax (a\notin\mathbb{Z})$ 在 $[-\pi,\pi]$ 上展开为 Fourier 级数.

解 把 f 延拓为整个数轴上的以 2π 为周期的函数. 记延拓以后的函数为 $\tilde{f}$, 那么 $\tilde{f}$ 是 $(-\infty,+\infty)$ 上的周期为 2π 的连续偶函数. 因此

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi}\cos ax\cos nx\mathrm{d}x\\
&= \frac{1}{\pi}\int_0^{\pi}[\cos(a-n)x+\cos(a+n)x]\mathrm{d}x\\
&= \frac{1}{\pi}\left[\frac{\sin(a-n)\pi}{a-n}+\frac{\sin(a+n)\pi}{a+n}\right]\\
&= \frac{(-1)^n}{\pi}\frac{2a\sin a\pi}{a^2-n^2}\quad (n=0,1,2,\cdots),\\
b_n &= 0\quad (n=1,2,\cdots).
\end{aligned}$$

由 Dini 判别法, 即得 $\tilde{f}$ 的 Fourier 展开式为

$$\tilde{f}(x)=\frac{\sin a\pi}{\pi}\left[\frac{1}{a}+\sum_{n=1}^{\infty}(-1)^n\frac{2a}{a^2-n^2}\cos nx\right] \tag{1}$$

限制在 $[-\pi,\pi]$ 上, 就得到

$$\cos ax=\frac{\sin a\pi}{\pi}\left[\frac{1}{a}+\sum_{n=1}^{\infty}(-1)^n\frac{2a}{a^2-n^2}\cos nx\right] \tag{2}$$

如果在 (2) 式中取 $x=0$ 可得

$$\frac{\pi}{\sin a\pi}=\frac{1}{a}+\sum_{n=1}^{\infty}(-1)^n\frac{2a}{a^2-n^2}\quad (a\notin\mathbb{Z}) \tag{3}$$

在 (2) 式中取 $x=\pi$ 得

$$\cos a\pi=\frac{\sin a\pi}{\pi}\left[\frac{1}{a}+\sum_{n=1}^{\infty}\frac{2a}{a^2-n^2}\right] \tag{4}$$

再令 $a\pi=t$, 得到

$$\cot t=\frac{1}{t}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2t}{t^2-n^2\pi^2}, t\neq 0,\pm\pi,\pm 2\pi,\cdots \tag{5}$$

在 (4) 式中令 $a\pi=t$ 还可以得到

$$\frac{1}{\sin t}=\frac{1}{t}+\sum_{n=1}^{\infty}(-1)^n\frac{2t}{t^2-n^2\pi^2}, t\neq 0,\pm\pi,\pm 2\pi,\cdots \tag{6}$$

□

4. 计算积分

$$\int_0^{\infty}\left(\frac{a}{\sinh ax}-\frac{b}{\sinh bx}\right)\frac{\mathrm{d}x}{x}.$$

解 首先由例 4 的 (6) 式，令 $t=\mathrm{i}ax$, 可得

$$\frac{a}{\sinh ax}=\frac{1}{x}+2\sum_{n=1}^{\infty}(-1)^n\frac{ax}{a^2x^2+n^2\pi^2}$$

这里运用了公式 $\sinh x=\dfrac{\sin(\mathrm{i}x)}{\mathrm{i}}$. 于是

$$\begin{aligned}
I_1=\int_0^{\infty}\left(\frac{a}{\sinh ax}-\frac{1}{x}\right)\frac{\mathrm{d}x}{x}&=2\sum_{n=1}^{\infty}\int_0^{\infty}(-1)^n\frac{a^2x}{a^2x^2+n^2\pi^2}\mathrm{d}x\\
&=2\sum_{n=1}^{\infty}\frac{(-1)^na}{n\pi}\arctan\left(\frac{ax}{n\pi}\right)\Big|_0^{\infty}\\
&=2\cdot\frac{\pi a}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\\
&=-a\ln 2.
\end{aligned}$$

同理有 $I_2=\displaystyle\int_0^{\infty}\left(\frac{b}{\sinh bx}-\frac{1}{x}\right)\mathrm{d}x=-b\ln 2$, 因此

$$\int_0^{\infty}\left(\frac{a}{\sinh ax}-\frac{b}{\sinh bx}\right)\mathrm{d}x=(b-a)\ln 2.$$

□

5. 计算积分

$$I=\int_0^{\infty}\frac{\ln x\ln(1+x)}{1+x^2}\mathrm{d}x.$$

解 令 $x=\frac{1}{t}$ 得

$$I=\int_0^{\infty}\frac{\ln\left(\frac{1}{t}\right)\ln\left(1+\frac{1}{t}\right)}{1+t^2}\mathrm{d}t=\int_0^{\infty}\frac{\ln^2 t}{1+t^2}-\int_0^{\infty}\frac{\ln t\ln(1+t)}{1+t^2}\mathrm{d}t.$$

于是

$$
\begin{aligned}
I &= \frac{1}{2}\int_0^{\infty}\frac{\ln^2 x}{1+x^2}\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}}\ln^2\tan x\,\mathrm{d}x\\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}(\ln\sin x-\ln\cos x)^2\,\mathrm{d}x\\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\ln^2\sin x+\ln^2\cos x-2\ln\sin x\ln\cos x\right)\mathrm{d}x\\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(2\ln^2\sin x-2\ln\sin x\ln\cos x\right)\mathrm{d}x\\
&= \int_0^{\frac{\pi}{2}}\left(\left(-\ln 2-\sum_{n=1}^{\infty}\frac{\cos 2nx}{n}\right)^2-\left(-\ln 2-\sum_{n=1}^{\infty}\frac{\cos 2nx}{n}\right)\left(-\ln 2+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos 2nx}{n}\right)\right)\mathrm{d}x\\
&= \int_0^{\frac{\pi}{2}}\sum_{n=1}^{\infty}\left(\frac{\cos^2 2nx}{n^2}+(-1)^{n-1}\frac{\cos^2 2nx}{n^2}\right)\mathrm{d}x\\
&= \int_0^{\frac{\pi}{2}}\sum_{n=1}^{\infty}\left(\frac{1+\cos 4nx}{2n^2}+(-1)^{n-1}\frac{1+\cos 4nx}{2n^2}\right)\mathrm{d}x\\
&= \frac{\pi}{4}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\right)\\
&= \frac{\pi}{4}\left(\frac{\pi^2}{6}+\frac{\pi^2}{12}\right)=\frac{\pi^3}{16}.
\end{aligned}
$$

□

6. 计算积分

$$
I=\int_0^{\infty}\frac{1}{x^4-x^2+1}\ln^2\frac{x^2}{x^4-x^2+1}\mathrm{d}x.
$$

解 作变换 $x\to\frac{1}{x}$ 可得

$$
I=\int_0^{\infty}\frac{x^2}{x^4-x^2+1}\ln^2\frac{x^2}{x^4-x^2+1}\mathrm{d}x.
$$

于是

$$
\begin{aligned}
I &= \frac{1}{2}\int_0^{\infty}\frac{x^2+1}{x^4-x^2+1}\ln^2\frac{x^2}{x^4-x^2+1}\mathrm{d}x\\
&= \frac{1}{2}\int_0^{\infty}\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1}\ln^2\left(x^2+\frac{1}{x^2}-1\right)\mathrm{d}x \xlongequal{t=x-\frac{1}{x}} \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\ln^2(t^2+1)}{t^2+1}\mathrm{d}t\\
&= \int_0^{\frac{\pi}{2}}\ln^2\cos^2 u\,\mathrm{d}u = 4\int_0^{\frac{\pi}{2}}\ln^2\sin u\,\mathrm{d}u\\
&= \frac{\pi^3}{6}+2\pi\ln^2 2.
\end{aligned}
$$

□

7. 计算主值积分

$$
\int_0^{\infty}\frac{\sin(\tan x)}{x}\mathrm{d}x.
$$

解 方法一

$$
\begin{aligned}
\int_0^{\infty} \frac{\sin(\tan x)}{x} \mathrm{d}x &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sin(\tan x)}{x} \mathrm{d}x \\
&= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\tan x) \sum_{k=-\infty}^{+\infty} \frac{1}{x+k\pi} \mathrm{d}x \\
&= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\tan x) \left(\frac{1}{x} + 2x \sum_{k=1}^{+\infty} \frac{1}{x^2 - k^2\pi^2} \right) \mathrm{d}x \\
&= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(\tan x)}{\tan x} \mathrm{d}x \\
&= \frac{1}{2} \Im \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x(x^2+1)} \mathrm{d}x \\
&= \frac{1}{2} \Im \left[2\pi \mathrm{i} \left(-\frac{1}{2} \operatorname{Res} f(0) + \operatorname{Res} f(\mathrm{i}) \right) \right] \\
&= -\frac{\pi}{2} \frac{\mathrm{e}+1}{\mathrm{e}}.
\end{aligned}
$$

方法二 考虑围道积分 $\int_C \frac{\mathrm{e}^{\mathrm{i}\tan z}}{z} \mathrm{d}z$, 其中 C 是上半平面内高为 R 的矩形围道, 则

$$
\mathrm{V.P.} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\tan z}}{z} \mathrm{d}z - \mathrm{V.P.} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\tan(z+\mathrm{i}R)}}{z+\mathrm{i}R} \mathrm{d}z = \pi \mathrm{i} \operatorname{Res}(f(z), z=0)
$$

令 $R \to \infty$, $\tan(z+\mathrm{i}R) \to \mathrm{i}$, 于是

$$
\mathrm{V.P.} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\tan z}}{z} = \pi \mathrm{i}(1 - \mathrm{e}^{-1}).
$$

最后我们得到

$$
\int_0^{\infty} \frac{\sin(\tan x)}{x} = \frac{\pi}{2}(1 - \mathrm{e}^{-1})
$$

□

8. 计算积分

$$
\int_0^1 \frac{\sinh(a \ln x) \ln(1+x) \ln x}{x} \mathrm{d}x
$$

解

$$
\begin{aligned}
I(a) &= \int_0^1 \frac{\cosh(a \ln x) \ln(1+x)}{x} \mathrm{d}x \\
&= \int_0^{\infty} \cosh(ax) \ln(1 + \mathrm{e}^{-x}) \mathrm{d}x \\
&= \frac{1}{2} \int_0^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\mathrm{e}^{-(k+a)x} + \mathrm{e}^{-(k-a)x} \right) \mathrm{d}x \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2 - a^2} = \frac{1}{2a} \left(\frac{\pi}{\sin \pi a} - \frac{1}{a} \right)
\end{aligned}
$$

这里我们利用了公式 $\dfrac{1}{\sin x}=\dfrac{1}{x}+\sum\limits_{k=1}^{\infty}\dfrac{2x}{x^2-k^2\pi^2}$.

因此

$$I'(a)=\int_0^1\sinh(a\ln x)\frac{\ln(1+x)\ln x}{x}\mathrm{d}x=\frac{1}{a^3}-\frac{\pi(1+a\pi\cot a\pi)}{2a^3\sin\pi a}$$

□

9. 计算积分

$$\int_0^{\infty}\frac{\cos x-\mathrm{e}^{-x^n}}{x}\mathrm{d}x$$

解 首先有

$$F(s)=\int_0^{\infty}x^{s-1}(\cos x-\mathrm{e}^{-x^n})\mathrm{d}x=\Gamma(s)\cos\left(\frac{\pi s}{2}\right)-\frac{1}{n}\Gamma\left(\frac{s}{n}\right).$$

其中

$$\int_0^{\infty}x^{s-1}\cos x\mathrm{d}x=\Re\int_0^{\infty}x^{s-1}\mathrm{e}^{\mathrm{i}x}\mathrm{d}x=\Re\left[\mathrm{i}^s\cdot\Gamma(s)\right]=\Gamma(s)\cos\left(\frac{\pi s}{2}\right)$$

$$\int_0^{\infty}x^{s-1}\mathrm{e}^{-x^n}\mathrm{d}x=\frac{1}{n}\int_0^{\infty}x^{\frac{s}{n}-1}\mathrm{e}^{-x}\mathrm{d}x=\frac{1}{n}\Gamma\left(\frac{s}{n}\right)$$

于是

$$\begin{aligned}\int_0^{\infty}\frac{\cos x-\mathrm{e}^{-x^n}}{x}\mathrm{d}x&=\lim_{s\to0}\left(\Gamma(s)\cos\left(\frac{\pi s}{2}\right)-\frac{1}{n}\Gamma\left(\frac{s}{n}\right)\right)\\&=\lim_{s\to0}\frac{\Gamma(s+1)\cos\left(\frac{\pi s}{2}\right)-\Gamma\left(\frac{s}{n}+1\right)}{s}\\&=\lim_{s\to0}\Gamma'(s+1)\cos\left(\frac{\pi s}{2}\right)-\frac{\pi}{2}\Gamma(s+1)\sin\left(\frac{\pi s}{2}\right)-\frac{1}{n}\Gamma'\left(\frac{s}{n}+1\right)\\&=\left(1-\frac{1}{n}\right)\Gamma'(1)\\&=-\left(1-\frac{1}{n}\right)\gamma\end{aligned}$$

这里 $\Gamma'(1)=-\gamma,\Gamma(z)=\dfrac{1}{z}-\gamma+\dfrac{1}{6}\left(3\gamma^2+\dfrac{\pi^2}{2}\right)+O(z^2)$.

□

10. 计算积分

$$\int_0^{\infty}\mathrm{e}^{-ax^2-\frac{b}{x^2}}\mathrm{d}x$$

解

$$\begin{aligned}\int_0^{\infty}\mathrm{e}^{-ax^2-\frac{b}{x^2}}\mathrm{d}x&=\int_0^{\infty}\mathrm{e}^{-\left(ax^2+\frac{b}{x^2}\right)}\mathrm{d}x\\&=\int_0^{\infty}\mathrm{e}^{-\sqrt{ab}\left(\sqrt{\frac{a}{b}}x^2+\sqrt{\frac{b}{a}}\frac{1}{x^2}\right)}\mathrm{d}x\\&=\sqrt[4]{\frac{b}{a}}\int_0^{\infty}\mathrm{e}^{-\sqrt{ab}\left[\left(\sqrt[4]{\frac{a}{b}}x\right)^2+\left(\frac{1}{x}\sqrt[4]{\frac{b}{a}}\right)^2\right]}\mathrm{d}\left(\sqrt[4]{\frac{a}{b}}x\right)\\&=\sqrt[4]{\frac{b}{a}}\left(\int_0^1+\int_1^{\infty}\right)\mathrm{e}^{-\sqrt{ab}\left(x^2+\frac{1}{x^2}\right)}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \sqrt[4]{\frac{b}{a}}\left(\int_1^{\infty} \mathrm{e}^{-\sqrt{ab}\left(x^2+\frac{1}{x^2}\right)}\frac{1}{x^2}\mathrm{d}x + \int_1^{\infty} \mathrm{e}^{-\sqrt{ab}\left(x^2+\frac{1}{x^2}\right)}\mathrm{d}x\right)\\
&= \sqrt[4]{\frac{b}{a}}\int_1^{\infty} \mathrm{e}^{-\sqrt{ab}\left(x^2+\frac{1}{x^2}\right)}\left(1+\frac{1}{x^2}\right)\mathrm{d}x\\
&= \sqrt[4]{\frac{b}{a}}\int_1^{\infty} \mathrm{e}^{-\sqrt{ab}\left[\left(x-\frac{1}{x}\right)^2+2\right]}\mathrm{d}\left(x-\frac{1}{x}\right)\\
&= \frac{1}{2}\sqrt[4]{\frac{b}{a}}\mathrm{e}^{-2\sqrt{ab}}\int_0^{\infty} \mathrm{e}^{-\sqrt{ab}x^2}\mathrm{d}x\\
&= \frac{1}{2}\sqrt{\frac{\pi}{a}}\mathrm{e}^{-2\sqrt{ab}}
\end{aligned}
$$

□

11. 计算积分

$$\int_0^{\infty} \mathrm{e}^{-ax}\sin^{2n} x\mathrm{d}x$$

解 由两次分部积分可得

$$\int_0^{\infty} \mathrm{e}^{-ax}\sin^{2n} x\mathrm{d}x = \frac{2n(2n-1)}{a^2+4n^2}\int_0^{\infty} \mathrm{e}^{-ax}\sin^{2n-2} x\mathrm{d}x$$

而 $\int_0^{\infty} \mathrm{e}^{-ax}\mathrm{d}x = \frac{1}{a}$, 于是

$$\int_0^{\infty} \mathrm{e}^{-ax}\sin^{2n} x\mathrm{d}x = \frac{(2n)!}{a(a^2+2^2)\cdots(a^2+4n^2)}$$

□

12. 计算积分

$$\int_0^{\infty}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!!}\right)\left(\sum_{n=0}^{\infty}\frac{x^{2n}}{((2n)!!)^2}\right)\mathrm{d}x$$

解 因为

$$
\begin{aligned}
\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!!}\right)\mathrm{d}x &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{x^2}{2}\right)^n\mathrm{d}x^2\\
&= \frac{1}{2}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x^2
\end{aligned}
$$

所以原积分

$$
\begin{aligned}
I &= \frac{1}{2}\int_0^{\infty}\mathrm{e}^{-\frac{x^2}{2}}\sum_{n=0}^{\infty}\frac{(x^2)^n}{(2^2)^n(n!)^2}\mathrm{d}x^2 = \int_0^{\infty}\mathrm{e}^{-t}\sum_{n=0}^{\infty}\frac{t^n}{2^n(n!)^2}\mathrm{d}t\\
&= \sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{2^n(n!)^2} = \sum_{n=0}^{\infty}\frac{1}{2^n n!} = \mathrm{e}^{\frac{1}{2}}
\end{aligned}
$$

□

13. 计算积分

$$\int_0^{2\pi} \mathrm{e}^{\cos x}\cos(\sin x - x)\mathrm{d}x$$

解

$$\begin{aligned}\int_0^{2\pi} \mathrm{e}^{\cos x}\cos(\sin x - x)\mathrm{d}x &= \int_0^{2\pi} \mathrm{e}^{\cos x}\cos(x - \sin x)\mathrm{d}x \\ &= \Re \int_0^{2\pi} \mathrm{e}^{\cos x}\mathrm{e}^{\mathrm{i}(x-\sin x)}\mathrm{d}x \\ &= \Re \int_0^{2\pi} \mathrm{e}^{\mathrm{e}^{-\mathrm{i}x}}\mathrm{e}^{\mathrm{i}x}\mathrm{d}x \\ &= \Re\left(\frac{1}{\mathrm{i}}\int_{|z|=1} \mathrm{e}^{\frac{1}{z}}\mathrm{d}z\right) \\ &= \Re\left(2\pi\mathrm{i}\cdot\frac{1}{\mathrm{i}}\mathrm{Res}\left(\mathrm{e}^{\frac{1}{z}}, z=0\right)\right) \\ &= 2\pi\end{aligned}$$

□

14. 计算积分

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos(\tan x - x)}{\cos x}\mathrm{d}x$$

解

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{\cos(\tan x - x)}{\cos x}\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} [\cos(\tan x) + \tan x\sin(\tan x)]\mathrm{d}x \\ &= \int_0^{\infty} \frac{\cos t + t\sin t}{1+t^2}\mathrm{d}t \qquad (t = \tan x) \\ &= 2\pi\mathrm{i}\cdot\mathrm{Res}\left(\frac{\cos t + t\sin t}{1+t^2}, t = \mathrm{i}\right) \\ &= \frac{\pi}{\mathrm{e}}\end{aligned}$$

□

15. 计算积分

$$I(a) = \int_0^{\infty} \frac{\sin x}{\cosh ax + \cos x}\frac{x}{x^2-\pi^2}\mathrm{d}x$$

解 根据

$$\sum_{k=1}^{\infty}(-1)^{k+1}\mathrm{e}^{-akx}\sin kx = \frac{1}{2}\frac{\sin x}{\cosh ax + \cos x}$$

得

$$I(a) = \int_0^{\infty} \frac{\sin x}{\cosh ax + \cos x}\frac{x}{x^2-\pi^2}\mathrm{d}x$$

$$
\begin{aligned}
&= 2\sum_{k=1}^{\infty}(-1)^{k+1}\left(\int_0^{\infty}\frac{x}{x^2-\pi^2}\mathrm{e}^{-akx}\sin kx\mathrm{d}x\right)\\
&= 2\sum_{k=1}^{\infty}(-1)^{k+1}\left(\int_0^{\infty}\frac{t}{t^2-k^2\pi^2}\mathrm{e}^{-at}\sin t\mathrm{d}t\right)\\
&= \int_0^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2t}{t^2-k^2\pi^2}\right)\mathrm{e}^{-at}\sin t\mathrm{d}t\\
&= \int_0^{\infty}\left(\frac{1}{t}-\csc t\right)\mathrm{e}^{-at}\sin t\mathrm{d}t\\
&= \arctan\left(\frac{1}{a}\right)-\frac{1}{a}
\end{aligned}
$$

□

16. 计算积分

$$\int_0^{\infty}\mathrm{e}^{-x^2}\ln x\mathrm{d}x$$

解

$$
\begin{aligned}
\int_0^{\infty}\mathrm{e}^{-x^2}\ln x\mathrm{d}x &= \frac{1}{4}\int_0^{\infty}\mathrm{e}^{-t}t^{-\frac{1}{2}}\ln t\mathrm{d}t\\
&= \frac{1}{4}\lim_{s\to 1}\partial_s\int_0^{\infty}\mathrm{e}^{-t}t^{s-1-\frac{1}{2}}\mathrm{d}t\\
&= \frac{1}{4}\lim_{s\to 1}\partial_s\Gamma\left(s-\frac{1}{2}\right)\\
&= \frac{1}{4}\lim_{s\to 1}\Gamma\left(s-\frac{1}{2}\right)\psi\left(s-\frac{1}{2}\right)\\
&= \frac{1}{4}\Gamma\left(\frac{1}{2}\right)\psi\left(\frac{1}{2}\right)\\
&= \frac{\sqrt{\pi}}{4}(\gamma+2\ln 2)
\end{aligned}
$$

□

17. 计算积分

$$I(a)=\int_0^{\infty}\frac{\cos x}{\tanh ax}\mathrm{d}x$$

解 首先有

$$I'(a)=\int_0^{\infty}\frac{\cos x}{\cosh^2 ax}\mathrm{d}x$$

考查围道积分

$$\int_C\frac{\mathrm{e}^{\mathrm{i}z}}{\cosh^2 az}\mathrm{d}z$$

其中 C 为上半平面内高为 $\frac{2\pi\mathrm{i}}{a}$ 的围道, 那么有

$$\left(1-\mathrm{e}^{-\frac{2\pi}{a}}\right)\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}z}}{\cosh^2 az}\mathrm{d}z=2\pi\mathrm{i}\left(\mathrm{Res}\left(f(z),z=\frac{\pi\mathrm{i}}{2a}\right)+\mathrm{Res}\left(f(z),z=\frac{3\pi\mathrm{i}}{2a}\right)\right)$$

于是

$$I'(a)=\frac{1}{2}\Re\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}z}}{\cosh^2 az}\mathrm{d}z=\frac{\pi}{2n^2\sinh\left(\frac{\pi}{2n}\right)}$$

最后得到

$$I(a)=\ln\coth\left(\frac{\pi}{2n}\right)$$

□

18. 设 $a>0,b>0$, 计算积分

$$\int_0^{\infty}\frac{\cos ax}{x^2+b^2}\left(\frac{\sin x}{x}\right)^n\mathrm{d}x$$

解 根据留数定理

$$\begin{aligned}\int_0^{\infty}\frac{\cos ax}{x^2+b^2}\left(\frac{\sin x}{x}\right)^n\mathrm{d}x&=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos ax}{x^2+b^2}\left(\frac{\sin x}{x}\right)^n\mathrm{d}x\\&=\frac{1}{2}\Re\left(2\pi\mathrm{i}\mathrm{Res}\left(\frac{\mathrm{e}^{\mathrm{i}az}}{z^2+b^2}\left(\frac{\sin z}{z}\right)^n,b\mathrm{i}\right)\right)\\&=\Re\left(\pi\mathrm{i}\lim_{z\to b\mathrm{i}}(z-b\mathrm{i})\frac{\mathrm{e}^{\mathrm{i}az}}{z^2+b^2}\left(\frac{\sin z}{z}\right)^n\right)\\&=\frac{\pi\mathrm{e}^{-ab}}{2b^{n+1}}\sinh^n b\quad\left(\sin b\mathrm{i}=\frac{\mathrm{e}^{\mathrm{i}b\mathrm{i}}-\mathrm{e}^{-\mathrm{i}b\mathrm{i}}}{2\mathrm{i}}=-\frac{\sinh b}{\mathrm{i}}\right)\end{aligned}$$

□

19. 计算积分

$$\int_0^1\frac{\ln\left(x+\sqrt{1-x^2}\right)}{x}\mathrm{d}x$$

解 考虑积分

$$I(t)=\int_0^1\frac{\ln\left(tx+\sqrt{1-x^2}\right)}{x}\mathrm{d}x$$

那么

$$\begin{aligned}I(0)&=\int_0^1\frac{\ln\left(\sqrt{1-x^2}\right)}{x}\mathrm{d}x\\&=\frac{1}{2}\left(\int_0^1\frac{\ln(1+x)}{x}\mathrm{d}x+\int_0^1\frac{\ln(1-x)}{x}\mathrm{d}x\right)\\&=\frac{1}{2}\left(\frac{\pi^2}{12}-\frac{\pi^2}{6}\right)=-\frac{\pi^2}{24}\end{aligned}$$

而

$$I'(t)=\int_0^1\frac{1}{tx+\sqrt{1-x^2}}\mathrm{d}\theta=\int_0^{\infty}\frac{\cos\theta}{t\sin\theta+\cos\theta}\mathrm{d}\theta=\frac{\pi}{2}\frac{1}{1+t^2}+\frac{t\ln t}{1+t^2}$$

上式对 t 积分得

$$I(t)=\frac{\pi}{2}\arctan t+\frac{1}{2}\ln(1+t^2)\ln t-\frac{1}{2}\int_0^t\frac{\ln(1+x^2)}{x}\mathrm{d}x+C$$

其中

$$C = I(0) = -\frac{\pi^2}{24}, I = I(1) = \frac{\pi^2}{8} + 0 - \frac{1}{2}\cdot\frac{\pi^2}{24} - \frac{\pi^2}{24} = \frac{\pi^2}{16}$$

□

20. 计算不定积分

$$\int \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\mathrm{d}x$$

解

$$\begin{aligned}
&\int \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\mathrm{d}x = \int \frac{\sqrt{\tan x}}{\sqrt{\tan x}+1}\mathrm{d}x \\
=& \int \frac{2u^2}{(1+u)(1+u^4)}\mathrm{d}u \quad \left(u=\sqrt{\tan x}\right) \\
=& \int \left(\frac{1}{1+u} + \frac{-u^3+u^2+u-1}{1+u^4}\right)\mathrm{d}x \\
=& \ln(1+u) - \frac{1}{4}\ln(1+u^4) + \int \frac{\mathrm{d}\left(u+\frac{1}{u}\right)}{\left(u+\frac{1}{u}\right)^2-1} + \frac{1}{2}\int \frac{\mathrm{d}(u^2)}{1+(u^2)^2} \\
=& \ln(1+u) - \frac{1}{4}\ln(1+u^4) + \frac{1}{2}\ln\left(\frac{u^2-u+1}{u^2+u+1}\right) + \frac{1}{2}\arctan u^2 + C \\
=& \ln(1+\sqrt{\tan x}) - \frac{1}{4}\ln(1+\tan^2 x) + \frac{1}{2}\ln\left(\frac{\tan x - \sqrt{\tan x}+1}{\tan^2 x+\sqrt{\tan x}+1}\right) + \frac{1}{2} + C
\end{aligned}$$

□

21. 计算积分

$$\iint_{x,y\geqslant 0} y\mathrm{e}^{-(x+y)}\sin\left(\ln\frac{x}{y}\right)\mathrm{d}x\mathrm{d}y$$

解

$$\begin{aligned}
\iint_{x,y\geqslant 0} y\mathrm{e}^{-(x+y)}\sin\left(\ln\frac{x}{y}\right)\mathrm{d}x\mathrm{d}y &= \Im\iint_{x,y\geqslant 0} y\mathrm{e}^{-(x+y)}\mathrm{e}^{\mathrm{i}\ln\left(\frac{x}{y}\right)}\mathrm{d}x\mathrm{d}y \\
&= \Im\iint_{x,y\geqslant 0} y\mathrm{e}^{-(x+y)}\left(\frac{x}{y}\right)^{\mathrm{i}}\mathrm{d}x\mathrm{d}y \\
&= \Im\int_0^\infty x^{\mathrm{i}}\mathrm{e}^{-x}\mathrm{d}x\int_0^\infty y^{1-\mathrm{i}}\mathrm{e}^{-y}\mathrm{d}y \\
&= \Im\left(\Gamma(1+\mathrm{i})\Gamma(2-\mathrm{i})\right) \\
&= \Im\left((1-\mathrm{i})\mathrm{i}\Gamma(\mathrm{i})\Gamma(1-\mathrm{i})\right) \\
&= \Im\left((1-\mathrm{i})\mathrm{i}\frac{\pi}{\sin(\pi\mathrm{i})}\right) \\
&= -\frac{\pi}{\sinh\pi}
\end{aligned}$$

□

22. 计算积分

$$I(a) = \int_0^\infty \frac{\arctan x}{\mathrm{e}^{ax}-1}\mathrm{d}x$$

解

$$\begin{aligned}
I(a) &= \int_0^\infty \frac{\arctan x}{\mathrm{e}^{ax}-1}\mathrm{d}x \\
&= \int_0^\infty \arctan x \sum_{n=0}^\infty \mathrm{e}^{-(n+1)ax}\mathrm{d}x \\
&= \int_0^\infty \sum_{n=1}^\infty \mathrm{e}^{-nax}\arctan x\,\mathrm{d}x \\
&= \frac{1}{a}\int_0^\infty \sum_{n=1}^\infty \frac{1}{n}\frac{\mathrm{e}^{-nax}}{1+x^2}\mathrm{d}x \\
&= \frac{1}{a}\int_0^\infty \mathrm{e}^{-au}\sum_{n=1}^\infty \frac{1}{n^2+u^2}\mathrm{d}x \\
&= \frac{1}{a}\int_0^\infty \mathrm{e}^{-au}\frac{\coth(\pi u)-\frac{1}{\pi u}}{2u/\pi}\mathrm{d}u \\
&= \frac{\pi}{2a}\int_0^\infty \left(\frac{1}{u}\mathrm{e}^{-\frac{au}{\pi}}\right)\left(\coth u-\frac{1}{u}\right)\mathrm{d}u \\
&= \frac{\pi}{2a}\int_0^\infty \left(\int_{\frac{a}{\pi}}^\infty \mathrm{e}^{-us}\mathrm{d}s\right)\left(\coth u-\frac{1}{u}\right)\mathrm{d}u \\
&= \frac{\pi}{2a}\int_{\frac{a}{\pi}}^\infty \left[\int_0^\infty \mathrm{e}^{-us}\left(\coth u-\frac{1}{u}\right)\mathrm{d}u\right]\mathrm{d}s
\end{aligned}$$

而

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-us}\left(\coth u-\frac{1}{u}\right)\mathrm{d}u \\
&= \int_0^\infty \mathrm{e}^{-us}\left(\frac{1}{u}-\frac{1+\mathrm{e}^{-2u}}{1-\mathrm{e}^{-2u}}\right)\mathrm{d}u \\
&= \int_0^\infty \mathrm{e}^{-\frac{sw}{2}}\left(\frac{1}{w}-\frac{1+\mathrm{e}^{-w}}{1-\mathrm{e}^{-w}}\right)\mathrm{d}w \\
&= \int_0^\infty \left[\frac{\mathrm{e}^{-w}-(\mathrm{e}^{-w}-\mathrm{e}^{-sw/2})}{w}-\frac{1}{2}\frac{\mathrm{e}^{-sw/2}+\mathrm{e}^{-sw/2}-\mathrm{e}^{-sw/2}+\mathrm{e}^{-(1+s/2)w}}{1-\mathrm{e}^{-w}}\right]\mathrm{d}w \\
&= \int_0^\infty \left(\frac{\mathrm{e}^{-w}}{w}-\frac{\mathrm{e}^{-ws/2}}{1-\mathrm{e}^{-w}}\right)\mathrm{d}w+\int_0^\infty \frac{\mathrm{e}^{-ws/2}}{2}\mathrm{d}w+\int_0^\infty \frac{\mathrm{e}^{-ws/2}-\mathrm{e}^{-w}}{w}\mathrm{d}w \\
&= \psi\left(\frac{s}{2}\right)+\frac{1}{s}-\ln\left(\frac{s}{2}\right)
\end{aligned}$$

因此

$$\begin{aligned}
I(a) &= -\frac{\pi}{2a}\int_{\frac{a}{\pi}}^\infty \left[\psi\left(\frac{s}{2}\right)+\frac{1}{s}-\ln\left(\frac{s}{2}\right)\right]\mathrm{d}s \\
&= \frac{\pi}{2a}\left[2\ln\Gamma\left(\frac{s}{2}\right)+\ln(s)-s\ln\left(\frac{s}{2}\right)+s\right]_\infty^{\frac{a}{\pi}} \\
&= \frac{\pi}{a}\ln\Gamma\left(\frac{a}{2\pi}\right)+\frac{\pi}{2a}\ln\left(\frac{a}{\pi}\right)-\frac{1}{2}\ln\left(\frac{a}{2\pi}\right)+\frac{1}{2}-\ln(4\pi)
\end{aligned}$$

□

23. 计算不定积分

$$\int\left(\frac{\arctan x}{x-\arctan x}\right)^2\mathrm{d}x$$

解

$$\begin{aligned}
\int\left(\frac{\arctan x}{x-\arctan x}\right)^2\mathrm{d}x &= \int\frac{t^2}{(\tan t-t)^2}\sec^2 t\,\mathrm{d}t\\
&= \int\frac{t^2}{(\sin t-t\cos t)^2}\mathrm{d}t\\
&= \int\left(-\frac{t}{\sin t}\right)\left(-\frac{t\sin t}{(\sin t-t\cos t)^2}\right)\mathrm{d}t\\
&= -\frac{t}{\sin t}\frac{1}{\sin t-t\cos t}+\int\frac{\mathrm{d}t}{\sin^2 t}\\
&= -\frac{(1+\tan^2 t)t}{\tan t(\tan t-t)^2}-\frac{1}{\tan t}+C\\
&= -\frac{(1+x^2)\arctan x}{x(x-\arctan x)}-\frac{1}{x}+C\\
&= -\frac{1+x\arctan x}{x-\arctan x}+C
\end{aligned}$$

□

24. 计算积分

$$I=\int_0^{\infty}\frac{x^n\sin x}{\cosh x-\cos x}\mathrm{d}x$$

解 根据

$$\frac{\sin x}{\cosh x-\cos x}=2\sum_{k=1}^{\infty}\mathrm{e}^{-kx}\sin kx$$

可得

$$\begin{aligned}
I &= \int_0^{\infty}\sum_{k=1}^{\infty}x^n\mathrm{e}^{-kx}\sin kx\,\mathrm{d}x\\
&= \frac{\sin\left(\frac{(n+1)\pi}{4}\right)\Gamma(n+1)}{2^{\frac{n+1}{2}}}\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}\\
&= \frac{1}{2^{\frac{n+1}{2}}}\sin\left(\frac{(n+1)\pi}{4}\right)\Gamma(n+1)\zeta(n+1)
\end{aligned}$$

□

25. 首先有恒等式

$$\begin{aligned}
\sum_{k=1}^{\infty}a^k\cos(kx) &= \Re\sum_{k=1}^{\infty}a^k\mathrm{e}^{\mathrm{i}kx}\\
&= \Re\frac{\mathrm{e}^{\mathrm{i}x+\ln a}}{1-\mathrm{e}^{\mathrm{i}x+\ln a}}\\
&= \Re\frac{a\mathrm{e}^{\mathrm{i}x}(1-a\mathrm{e}^{-\mathrm{i}x})}{(1-a\mathrm{e}^{\mathrm{i}x})(1-a\mathrm{e}^{-\mathrm{i}x})}
\end{aligned}$$

$$
\begin{aligned}
&= \Re \frac{a e^{ix-a^2}}{1-2a\cos x+a^2} \\
&= \frac{a\cos x - a^2}{1-2a\cos x+a^2}
\end{aligned}
$$

于是

$$
\int_0^{\infty} \frac{a-\cos x}{(1-2a\cos x+a^2)(1+x^2)} = -\frac{1}{a}\int_0^{\infty}\sum_{k=1}^{\infty} a^k \frac{\cos(kx)}{1+x^2} = -\frac{\pi}{2a}\sum_{k=1}^{\infty}\frac{a^k}{e^k} = \frac{e\pi}{2(a-e)}
$$

26. 设 $0<a<1$, 计算积分

$$
I = \int_{-\infty}^{\infty} \frac{\sinh^2 ax}{\sinh^2 x} dx
$$

解 首先有

$$
\int_{-\infty}^{\infty} \frac{\sinh^2 ax}{\sinh^2 x} dx = 2\int_{-\infty}^{0} \frac{(e^{ax}-e^{-ax})^2 e^{2x}}{(e^{2x}-1)^2} dt = \int_0^1 \frac{t^a - t^{-a} - 2}{(1-t)^2} dt
$$

考虑积分

$$
\begin{aligned}
I(s) &= \int_0^1 (t^a - t^{-a} - 2)(1-t)^{s-2} dt \\
&= B(a+1, s-1) + B(-a+1, s-1) - 2B(1, s-1) \\
&= \frac{\Gamma(a+1)\Gamma(s-1)}{\Gamma(a+s)} + \frac{\Gamma(-a+1)\Gamma(s-1)}{\Gamma(-a+s)} - \frac{2\Gamma(s-1)}{\Gamma(s)} \\
&= \frac{\Gamma(s+1)}{s-1}\left(\frac{\Gamma(a+1)-\Gamma(a+s)}{s\Gamma(a+s)} + \frac{\Gamma(-a+1)-\Gamma(-a+s)}{s\Gamma(-a+s)}\right)
\end{aligned}
$$

其中 $\Gamma(s+1) = s\Gamma(s) = s(s-1)\Gamma(s-1)$.
令 $s \to 0$ 得

$$
\begin{aligned}
I = I(0) &= -\left(\frac{\Gamma'(a+1)}{\Gamma(a)} + \frac{\Gamma'(-a+1)}{\Gamma(-a)}\right) \\
&= a(\psi(-a+1) - \psi(a+1)) \\
&= a(\psi(a) - \psi(a+1) - \pi\cot a\pi) \\
&= 1 - a\pi\cot a\pi
\end{aligned}
$$

□

27. 计算积分

$$
I = \int_0^{\infty} \frac{e^{-t}\cosh(a\sqrt{t})}{\sqrt{t}} dt
$$

解

$$
\begin{aligned}
I &= \int_0^{\infty} \frac{e^{-t}\cosh(a\sqrt{t})}{\sqrt{t}} dt \\
&= 2\int_0^{\infty} e^{-t^2}\cosh(at)\, dt = \int_0^{\infty} e^{-t^2}\left(e^{at} + e^{-at}\right) dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\infty} \left(\mathrm{e}^{-t^2+at} + \mathrm{e}^{-t^2-at}\right) \mathrm{d}t \\
&= \int_0^{\infty} \mathrm{e}^{\frac{a^2}{4}-\left(t-\frac{a}{2}\right)^2} \mathrm{d}t + \int_0^{\infty} \mathrm{e}^{\frac{a^2}{4}-\left(t+\frac{a}{2}\right)^2} \mathrm{d}t \\
&= \mathrm{e}^{\frac{a^2}{4}} \left(\int_0^{\infty} \mathrm{e}^{-\left(t-\frac{a}{2}\right)^2} \mathrm{d}t + \int_0^{\infty} \mathrm{e}^{-\left(t+\frac{a}{2}\right)^2} \mathrm{d}t\right) \\
&= \mathrm{e}^{\frac{a^2}{4}} \left(\int_{-\frac{a}{2}}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x + \int_{\frac{a}{2}}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x\right) \\
&= \mathrm{e}^{\frac{a^2}{4}} \left(\int_{-\infty}^{\frac{a}{2}} \mathrm{e}^{-x^2} \mathrm{d}x + \int_{\frac{a}{2}}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x\right) \\
&= \mathrm{e}^{\frac{a^2}{4}} \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x = \sqrt{\pi}\mathrm{e}^{\frac{a^2}{4}}
\end{aligned}
$$

□

28. 计算积分

$$
I = \int_0^{\infty} \mathrm{e}^{-ax} \left(\frac{1}{x} - \coth x\right) \mathrm{d}x
$$

解

$$
\begin{aligned}
I &= \int_0^{\infty} \mathrm{e}^{-ax} \left(\frac{1}{x} - \coth x\right) \mathrm{d}x \\
&= a\int_0^{\infty} \mathrm{e}^{-ax} \ln x \mathrm{d}x - a\int_0^{\infty} \mathrm{e}^{-ax} \ln(\sinh x) \mathrm{d}x \\
&= -\gamma - \ln a - a\int_0^{\infty} \mathrm{e}^{-ax} \ln\left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right) \mathrm{d}x \\
&= -\gamma - \ln a - a\int_0^{\infty} x\mathrm{e}^{-ax} \mathrm{d}x - a\ln 2\int_0^{\infty} \mathrm{e}^{-ax} \mathrm{d}x - a\int_0^{\infty} \mathrm{e}^{-ax} \ln(1-\mathrm{e}^{-2x}) \mathrm{d}x \\
&= -\gamma - \frac{1}{a} + \ln\left(\frac{2}{a}\right) - \sum_{k=1}^{\infty} \frac{a}{k}\int_0^{\infty} \mathrm{e}^{-2kx-ax} \mathrm{d}x \\
&= -\gamma - \frac{1}{a} + \ln\left(\frac{2}{a}\right) - \sum_{k=0}^{\infty} \frac{a}{k(a+2k)} \\
&= \psi\left(1+\frac{a}{2}\right) - \frac{1}{a} + \ln\left(\frac{2}{a}\right) \\
&= \psi\left(\frac{a}{2}\right) + \ln\left(\frac{2}{a}\right) + \frac{1}{a}
\end{aligned}
$$

□

29. 计算积分

$$
I = \int_0^1 \frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)(-\ln x)} \mathrm{d}x
$$

解

$$
\begin{aligned}
I &= \int_0^1 \frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)(-\ln x)} \mathrm{d}x \\
&= \int_0^1 \frac{(1-x^a)(1-x^b)}{1-x} \mathrm{d}x \int_0^c x^y \mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^c \mathrm{d}y \int_0^1 \frac{x^y(1-x^a-x^b+x^{a+b})}{1-x}\mathrm{d}x \\
&= \int_0^c \mathrm{d}y \left(\int_0^1 \frac{x^y}{1-x}\mathrm{d}x - \int_0^1 \frac{a+y}{1-x}\mathrm{d}x - \int_0^1 \frac{b+y}{1-x}\mathrm{d}x + \int_0^1 \frac{a+b+y}{1-x}\mathrm{d}x\right) \\
&= \int_0^c \mathrm{d}y \int_0^1 \sum_{n=0}^{\infty} \left(x^{y+n} - x^{a+y+n} - x^{b+y+n} + x^{a+b+y+n}\right) \mathrm{d}x \\
&= \int_0^c \sum_{n=0}^{\infty} \left(\frac{1}{y+n+1} - \frac{1}{a+y+n+1} - \frac{1}{b+y+n+1} + \frac{1}{a+b+y+n+1}\right) \mathrm{d}y \\
&= \int_0^c \sum_{n=0}^{\infty} [\psi(a+y+1)+\psi(b+y+1)-\psi(y+1)-\psi(a+b+y+1)]\,\mathrm{d}y \\
&= \ln \frac{\Gamma(b+c+1)\Gamma(a+c+1)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)\Gamma(a+b+c+1)}
\end{aligned}
$$

□

30. 计算积分

$$
I = \int_{\frac{1}{2}}^1 \frac{\ln(2x-1)}{\sqrt[6]{x(1-x)(1-2x)^4}}\mathrm{d}x
$$

解

$$
\begin{aligned}
I &= \int_{\frac{1}{2}}^1 \frac{\ln(2x-1)}{\sqrt[6]{x(1-x)(1-2x)^4}}\mathrm{d}x \xlongequal{t=(2x-1)^2} 4^{-\frac{4}{3}} \int_0^1 (1-t)^{-\frac{1}{6}} t^{-\frac{5}{6}} \ln t \,\mathrm{d}t \\
&= 4^{-\frac{4}{3}} \lim_{s\to 0} \partial_s \int_0^1 (1-t)^{-\frac{1}{6}} t^{s-\frac{5}{6}} \mathrm{d}t \\
&= 4^{-\frac{4}{3}} \lim_{s\to 0} \partial_s B\left(\frac{5}{6}, s+\frac{1}{6}\right) \\
&= 4^{-\frac{4}{3}} \lim_{s\to 0} \partial_s \frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(s+\frac{1}{6}\right)}{\Gamma(1+s)} \\
&= 4^{-\frac{4}{3}} \Gamma\left(\frac{5}{6}\right) \lim_{s\to 0} \frac{\Gamma\left(s+\frac{1}{6}\right)}{\Gamma(1+s)} \left[\psi\left(s+\frac{1}{6}\right) - \psi(1+s)\right] \\
&= 4^{-\frac{4}{3}} \Gamma\left(\frac{5}{6}\right) \Gamma\left(\frac{1}{6}\right) \left[\psi\left(\frac{1}{6}\right) - \psi(1)\right] \\
&= 4^{-\frac{4}{3}} \pi \csc\left(\frac{\pi}{6}\right) \left[\psi\left(\frac{1}{6}\right) + \gamma\right] \\
&= \frac{\pi}{\sqrt[3]{256}} \left[2\psi\left(\frac{1}{6}\right) + 2\gamma\right]
\end{aligned}
$$

□

31. 求极限

$$
\lim_{n\to\infty} \sqrt{n} \int_0^{\infty} \cos^{2n-1} x \mathrm{e}^{-\pi x} \mathrm{d}x
$$

解 首先有

$$
\int_0^{\infty} \cos(nx)\mathrm{e}^{-\pi x}\mathrm{d}x = \frac{\pi}{n^2+\pi^2}
$$

接下来求出 $\cos^{2n-1} x$ 的 Fourier 余弦级数即可.

$$\cos^{2n-1} x = \frac{2}{4^n}(\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x})^{2n-1} = \frac{1}{4^{n-1}} \sum_{k=0}^{n} \binom{2n-1}{k} \cos((2n-2k-1)x)$$

于是

$$\int_0^{\infty} \cos^{2n-1} x \mathrm{e}^{-\pi x} \mathrm{d}x = \frac{1}{4^{n-1}} \sum_{k=1}^{n} \binom{2n-1}{k} \frac{\pi}{(2k-1)^2 + \pi^2}$$

对于每个固定的 $0 \leqslant k \leqslant n$, 根据 Stirling 公式我们有

$$\lim_{n\to\infty} \frac{\sqrt{n}}{4^{n-1}} \binom{2n-1}{k} = \frac{2}{\sqrt{\pi}}$$

因此原来的极限为

$$L = 2\sqrt{\pi} \sum_{k=1}^{\infty} \frac{1}{\pi^2 + (2k-1)^2} = \frac{\sqrt{\pi}}{2} \tanh\left(\frac{\pi^2}{2}\right)$$

□

32. 对固定的正整数 n，计算积分

$$I_n = \int_1^{\infty} \frac{\mathrm{d}x}{x(x+1)(x+2)\cdots(x+n)}$$

解 用有理函数的部分分式法, 令

$$\frac{1}{x(x+1)(x+2)\cdots(x+n)} = \sum_{k=0}^{n} \frac{A_k}{x+k}$$

于是得到

$$1 = \sum_{k=0}^{n} A_k x(x+1)\cdots(x+k-1)(x+k+1)\cdots(x+n)$$

令 $x = -k$ 得到

$$1 = A_k[(-k)(-k+1)\cdots(-k+k+1)][(-k+k+1)\cdots(-k+k+n)] = A_k[(-1)^k k!][(n-k)!]$$

于是 $A_k = \dfrac{(-1)^k}{n!}\dbinom{n}{k}$, 原积分

$$\begin{aligned}
I_n &= \int_1^{\infty} \frac{\mathrm{d}x}{x(x+1)(x+2)\cdots(x+n)} \\
&= \int_1^{\infty} \sum_{k=0}^{n} \frac{A_k}{x+k} \mathrm{d}x \\
&= \sum_{k=0}^{n} A_k \ln(x+k)\Big|_1^{\infty} \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{n!} \ln(x+k)\Big|_1^{\infty}
\end{aligned}$$

而 $\displaystyle\lim_{x\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{n!} \ln|x+k| = 0$, 因此 $\displaystyle I_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{n!} \ln|k+1|$.

□

33. 设 $a > b > 0$, 计算积分

$$\int_0^{\pi} \ln(a + b\cos x)\mathrm{d}x$$

解 记 $I(b) = \int_0^{\pi} \ln(a + b\cos x)\mathrm{d}x$, 那么

$$\begin{aligned}
I'(b) &= \int_0^{\pi} \frac{\cos x}{a + b\cos x}\mathrm{d}x \\
&= \frac{1}{b} - \frac{a}{b}\int_0^{\pi} \frac{\mathrm{d}x}{a + b\cos x} \\
&= \frac{\pi}{b} - \frac{2a}{b}\int_0^{\infty} \frac{\mathrm{d}t}{(a+b) + (a-b)t^2} \quad (t = \tan(x/2)) \\
&= \frac{\pi}{b} - \frac{2a}{b} \cdot \frac{1}{\sqrt{a^2 - b^2}} \arctan\left(\sqrt{\frac{a-b}{a+b}}u\right)\Bigg|_0^{\infty} \\
&= \frac{\pi}{b} - \frac{2a}{b} \cdot \frac{1}{\sqrt{a^2 - b^2}} \cdot \frac{\pi}{2} = \frac{\pi}{2} \\
&= \frac{\pi}{b} - \frac{\pi a}{b\sqrt{a^2 - b^2}}
\end{aligned}$$

□

34. 计算积分

$$I = \int_0^1 \frac{\ln 2 - \ln(1 + \sqrt{1 - x^2})}{x}\mathrm{d}x$$

解

$$\begin{aligned}
I &= \int_0^1 \frac{\ln 2 - \ln(1 + \sqrt{1 - x^2})}{x}\mathrm{d}x \\
&= \int_0^1 \frac{1}{x}\ln\left(\frac{2}{1 + \sqrt{1 - x^2}}\right)\mathrm{d}x \quad \left(u = \frac{2}{1 + \sqrt{1 - x^2}}\right) \\
&= \int_1^2 \frac{u}{2\sqrt{u-1}}\ln u \cdot \frac{2-u}{u^2\sqrt{u-1}}\mathrm{d}u \\
&= \frac{1}{2}\int_1^2 \frac{2-u}{u(u-1)}\ln u\mathrm{d}u \\
&= \frac{1}{2}\int_1^2 \frac{\ln u}{u-1}\mathrm{d}u - \int_1^2 \frac{\ln u}{u}\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 \frac{\ln(1+t)}{t}\mathrm{d}t - \frac{\ln^2 2}{2} \\
&= \frac{\pi^2}{24} - \frac{\ln^2 2}{2}
\end{aligned}$$

□

35. 计算积分

$$\int_0^1 \ln\Gamma(x)\cos(2n\pi x)\mathrm{d}x$$

解

$$
\begin{aligned}
\int_0^1 \ln\Gamma(x)\cos(2n\pi x)\mathrm{d}x &= \int_0^1 \ln\Gamma(1-x)\cos(2n\pi x)\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 [\ln\Gamma(1-x)+\ln\Gamma(x)]\cos(2n\pi x)\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \ln\left(\frac{\pi}{\sin(\pi x)}\right)\cos(2nx)\mathrm{d}x \\
&= \frac{1}{4n\pi}\int_0^\pi (\ln\pi - \ln\sin x)\mathrm{d}\left(\sin(2nx)\right) \\
&= -\frac{\ln\sin x}{4n\pi}\sin(2n\pi)\Big|_0^\pi + \frac{1}{4n\pi}\int_0^\pi \frac{\sin(2nx)\cos x}{\sin x}\mathrm{d}x \\
&= \frac{1}{8n\pi}\int_0^\pi \frac{\sin(2n+1)x+\sin(2n-1)x}{\sin x}\mathrm{d}x = \frac{1}{4n}
\end{aligned}
$$

其中 $\displaystyle\int \frac{\sin(2n+1)x}{\sin x}\mathrm{d}x = x + 2\sum_{k=1}^{n}\frac{\sin 2kx}{2k} + C$, $\displaystyle\int_0^\pi \frac{\sin(2n+1)x}{\sin x}\mathrm{d}x = \pi$.

□

36. 计算积分

$$
I = \int_0^\infty \frac{\sin x - x - x^3}{x^3(x^2+1)}\mathrm{d}x
$$

解 首先我们有

$$
I = \int_0^\infty \frac{\sin x - x - x^3}{x^3(x^2+1)}\mathrm{d}x = \int_0^\infty \left(\frac{\sin x}{x^3} - \frac{\sin x}{x} + \frac{x\sin x}{1+x^2} - \frac{1}{x^2}\right)\mathrm{d}x
$$

其中

$$
\begin{aligned}
\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}, &\quad \int_0^\infty \frac{x\sin x}{1+x^2}\mathrm{d}x = \frac{\pi}{2\mathrm{e}}, \\
\int_0^\infty \frac{\sin x - x}{x^3}\mathrm{d}x &= -\frac{1}{2}\int_0^\infty (\sin x - x)\mathrm{d}\left(\frac{1}{x^2}\right) \\
&= \frac{1}{2}\int_0^\infty \frac{\cos x - 1}{x^2}\mathrm{d}x \\
&= -\frac{1}{2}\int_0^\infty \frac{\sin x}{x}\mathrm{d}x \\
&= -\frac{\pi}{4}
\end{aligned}
$$

故 $I = \dfrac{\pi}{2\mathrm{e}} - \dfrac{3\pi}{4}$.

□

37. 计算积分

$$
I = \int_0^\infty \frac{\arctan^2 x}{x(1+x^2)^2}\mathrm{d}x
$$

解

$$
I = \int_0^\infty \frac{\arctan^2 x}{x(1+x^2)^2}\mathrm{d}x
$$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \frac{x^2\cos^3 x}{\sin x}\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} x^2 \mathrm{d}\left(\frac{1}{4}\cos(2x)+\ln(\sin x)\right) \\
&= -\frac{\pi^2}{16} - \int_0^{\frac{\pi}{2}} x\left(\frac{1}{2}\cos(2x)+2\ln(\sin x)\right)\mathrm{d}x \\
&= -\frac{\pi^2}{16} + \frac{1}{4} - 2\int_0^{\frac{\pi}{2}} x\ln(\sin x)\mathrm{d}x \\
&= -\frac{\pi^2}{16} + \frac{1}{4} - 2\int_0^{\frac{\pi}{2}} x\left(-\ln 2 - \sum_{k=1}^{\infty}\frac{\cos 2kx}{k}\right)\mathrm{d}x \\
&= -\frac{\pi^2}{16} + \frac{1}{4} + \frac{\pi^2\ln 2}{4} \\
&= -\frac{\pi^2}{16} + \frac{1}{4} + \frac{\pi^2\ln 2}{4} - \sum_{k=0}^{\infty}\frac{1}{(2k-1)^3} \\
&= -\frac{\pi^2}{16} + \frac{1}{4} + \frac{\pi^2\ln 2}{4} - \frac{7}{8}\zeta(3)
\end{aligned}
$$

□

38. 计算积分

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{e}^x\mathrm{d}x}{(\mathrm{e}^{2x}+\mathrm{e}^{2t})(x^2+\pi^2)}$$

解 矩形围道 $C: -\infty - \mathrm{i}\pi \to \infty - \mathrm{i}\pi \to \infty + \mathrm{i}\pi \to -\infty + \mathrm{i}\pi$

于是由留数定理可得

$$
\begin{aligned}
&\oint_C \frac{\mathrm{e}^z}{z(\mathrm{e}^{2t}+\mathrm{e}^{2z})}\mathrm{d}z \\
&= \int_{-\infty-\mathrm{i}\pi}^{\infty-\mathrm{i}\pi} \frac{\mathrm{e}^x}{x(\mathrm{e}^{2t}+\mathrm{e}^{2x})}\mathrm{d}x - \int_{-\infty+\mathrm{i}\pi}^{\infty+\mathrm{i}\pi} \frac{\mathrm{e}^x}{x(\mathrm{e}^{2t}+\mathrm{e}^{2x})}\mathrm{d}x \\
&= -\int_{-\infty}^{\infty} \frac{2\pi\mathrm{i}\mathrm{e}^x}{(x^2+\pi^2)(\mathrm{e}^{2t}+\mathrm{e}^{2x})}\mathrm{d}x \\
&= 2\pi\mathrm{i}\left[\sum \mathrm{Res}\left(\frac{\mathrm{e}^x}{x(\mathrm{e}^{2t}+\mathrm{e}^{2x})}, \left\{x = t+\frac{\pi\mathrm{i}}{2}, t-\frac{\pi\mathrm{i}}{2}, 0\right\}\right)\right] \\
&= 2\pi\mathrm{i}\left(-\frac{2\pi\mathrm{e}^{-t}}{4t^2+\pi^2} + \frac{1}{\mathrm{e}^{2t}+1}\right),
\end{aligned}
$$

所以原积分 $I = \int_{-\infty}^{\infty} \frac{\mathrm{e}^x\mathrm{d}x}{(\mathrm{e}^{2x}+\mathrm{e}^{2t})(x^2+\pi^2)} = \left(\frac{2\pi\mathrm{e}^{-t}}{4t^2+\pi^2} - \frac{1}{\mathrm{e}^{2t}+1}\right)$.

用同样的方法还可以求出另一个积分 $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(\mathrm{e}^x+\mathrm{e}^{-x})(x^2+\pi^2)} = \frac{2}{\pi} - \frac{1}{2}$.

□

39. 计算积分

$$I = \int_1^{\infty} \frac{1}{x}\ln\left(\frac{x}{x-1}\right)\ln\left[\ln\left(\frac{x}{x-1}\right)\right]\mathrm{d}x$$

解 令 $t = \ln\left(\frac{x}{x-1}\right)$, 得 $I = \int_0^{\infty} \frac{t\ln t}{\mathrm{e}^t - 1}\mathrm{d}t$.

$$f(p) = \int_0^{\infty} \frac{t^p}{\mathrm{e}^t-1}\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^\infty t^p \sum_{n=1}^\infty \mathrm{e}^{-nt}\mathrm{d}t = \sum_{n=1}^\infty \int_0^\infty \mathrm{e}^{-nt} t^p \mathrm{d}t \\
&= \sum_{n=1}^\infty \frac{\Gamma(p+1)}{n^{p+1}} = \zeta(p+1)\Gamma(p+1),
\end{aligned}
$$

于是 $I = f'(1) = \zeta(2)\psi(2) - \zeta'(2) = \frac{1}{6}(1-\gamma)\pi^2 - \zeta'(2)$.

□

40. 计算积分

$$I = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+\sin^2(\tan x)}$$

解

$$I = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+\sin^2(\tan x)} = \int_0^\infty \frac{\mathrm{d}x}{(1+x^2)(1+\sin^2 x)} = \int_0^\infty \frac{\mathrm{d}x}{(1+x^2)(2-\cos^2 x)}$$

考虑 Poisson 积分

$$
\begin{aligned}
I(\eta) &= \int_0^\infty \frac{\mathrm{d}x}{(1+x^2)(\eta^2 - 2\eta\cos 2x + 1)} \\
&= \frac{1}{1-\eta^2}\int_0^\infty \left(2\sum_{k=0}^\infty \eta^k \cos 2kx - 1\right)\frac{\mathrm{d}x}{1+x^2} \\
&= \frac{\pi}{2}\frac{1}{1-\eta^2}\frac{\mathrm{e}^2+\eta}{\mathrm{e}^2-\eta}.
\end{aligned}
$$

同时有 $I(\eta) = \frac{1}{1+\eta^2}\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)\left(1-\frac{2\eta}{1+\eta^2}\cos 2x\right)}$, 令 $\frac{2\eta}{1+\eta^2} = \frac{1}{3}$, 解得 $\eta = 3-2\sqrt{2}$, 于是最后得到

$$I = \frac{2}{3}I(3-2\sqrt{2}) = \frac{\pi}{2\sqrt{2}}\frac{\mathrm{e}^2+3-2\sqrt{2}}{\mathrm{e}^2-3+2\sqrt{2}}.$$

□

41. 计算积分

$$\int_0^1 \frac{\ln x \ln(1+x^2)}{x}\mathrm{d}x$$

解

$$
\begin{aligned}
I &= \int_0^1 \frac{\ln x \ln(1+x^2)}{x}\mathrm{d}x \\
&= -\int_0^\infty t\ln(\mathrm{e}^{-2t}+1)\mathrm{d}t \\
&= -\int_0^\infty t\sum_{n=1}^\infty \frac{(-1)^{n+1}\mathrm{e}^{-2nt}}{n}\mathrm{d}t \\
&= -\sum_{n=1}^\infty \frac{(-1)^{n+1}\int_0^\infty t\mathrm{e}^{-2nt}\mathrm{d}t}{n}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3} \\
&= -\frac{3}{16}\zeta(3).
\end{aligned}
$$

□

42. 计算积分

$$
I = \int_0^{\frac{\pi}{2}} \ln\left(x^2 + \ln^2(\cos x)\right) \mathrm{d}x
$$

解 考虑参数积分

$$
\begin{aligned}
J(a) &= \int_{-\pi}^{\pi} \ln\left(\mathrm{i}x + \ln(a\cos x)\right) \mathrm{d}x \\
&= \oint_{|z|=1} \ln\left(\ln z + \ln\left(a\frac{z+z^{-1}}{2}\right)\right)\frac{-\mathrm{i}\mathrm{d}z}{z} \qquad z = \mathrm{e}^{\mathrm{i}x} \\
&= \oint_{|z|=1} \ln\left(\ln\left(a\frac{1+z^2}{2}\right)\right)\frac{-\mathrm{i}\mathrm{d}z}{z} \\
&= 2\pi\mathrm{Res}\left[\ln\left(\ln\left(a\frac{1+z^2}{2}\right)\right), z = 0\right] \\
&= 2\pi\ln\left(\ln\left(\frac{a}{2}\right)\right).
\end{aligned}
$$

于是得到

$$
\begin{aligned}
I(a) &= \int_0^{\frac{\pi}{2}} \ln\left(x^2 + \ln^2(a\cos x)\right) \mathrm{d}x \\
&= 2\Re\int_0^{\frac{\pi}{2}} \ln\left(\mathrm{i}x + \ln(a\cos x)\right) \mathrm{d}x \\
&= \frac{1}{2}\Re\int_{-\pi}^{\pi} \ln\left(\mathrm{i}x + \ln(a\cos x)\right) \mathrm{d}x \\
&= \frac{1}{2}\Re J(a).
\end{aligned}
$$

故原积分 $I = \int_0^{\frac{\pi}{2}} \ln\left(x^2 + \ln^2(\cos x)\right) \mathrm{d}x = \frac{1}{2}\Re J(1) = \pi\ln(\ln 2)$.

□

43. 计算积分

$$
I = \int_0^{\pi} \sin(xt)(2\sin t)^y \mathrm{d}t
$$

解 考查围道积分 $\oint_C z^{p-q-1}(z-z^{-1})^{p+q-2}\mathrm{d}z$, 这里 C 是单位圆在上半平面的部分 C_R 与从 -1 到 1 的实轴 L 构成的围道, 那么有

$$
\begin{aligned}
I_1 &= \int_{C_R} z^{p-q-1}(z-z^{-1})^{p+q-2}\mathrm{d}z \xrightarrow{z=\mathrm{e}^{\mathrm{i}t}} \mathrm{e}^{\mathrm{i}\pi\frac{p+q-1}{2}}\int_0^{\pi}\mathrm{e}^{\mathrm{i}(p-q)t}(2\sin t)^{p+q-2}\mathrm{d}t \\
I_2 &= \int_L z^{p-q-1}(z-z^{-1})^{p+q-2}\mathrm{d}z = \int_{-1}^{1} x^{p-q-1}(x-x^{-1})^{p+q-2}\mathrm{d}x \\
&= \int_0^1 x^{p-q-1}(x-x^{-1})^{p+q-2}\mathrm{d}x + \int_{-1}^{0} x^{p-q-1}(x-x^{-1})^{p+q-2}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi(p+q)}B(1-q,p+q-1) - \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi(p-q)}B(1-q,p+q-1) \\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}p\pi}\sin(q\pi)B(1-q,p+q-1).
\end{aligned}
$$

由留数定理有 $I_1+I_2=0$, 再令 $x=p-q, y=p+q-2$, 比较虚部可得

$$
I = \int_0^{\pi} \sin(xt)(2\sin t)^y \mathrm{d}t = \frac{\pi \sin\left(\frac{x\pi}{2}\right)\Gamma(y+1)}{\Gamma\left(\frac{y+x}{2}+1\right)\Gamma\left(\frac{y-x}{2}+1\right)}.
$$

同理还可以得到

$$
I = \int_0^{\pi} \cos(xt)(2\sin t)^y \mathrm{d}t = \frac{\pi \cos\left(\frac{x\pi}{2}\right)\Gamma(y+1)}{\Gamma\left(\frac{y+x}{2}+1\right)\Gamma\left(\frac{y-x}{2}+1\right)}.
$$

□

44. 计算积分

$$
I = \int_0^{\pi} \frac{\ln(\sin x)}{x^2+\ln^2(\sin x)}\mathrm{d}x
$$

解 根据 $\dfrac{a}{a^2+b^2} = \displaystyle\int_0^{\infty} \mathrm{e}^{-ab}\cos by\mathrm{d}y$ 可得

$$
\begin{aligned}
-I &= \int_0^{\pi}\left(\int_0^{\infty} \mathrm{e}^{y\ln(\sin x)}\cos xy\mathrm{d}y\right)\mathrm{d}x \\
&= \int_0^{\infty}\mathrm{d}y\int_0^{\pi}\sin^y x\cos xy\mathrm{d}x \\
&= \int_0^{\infty}2^{-y}\mathrm{d}y\int_0^{\pi}(2\sin x)^y\cos xy\mathrm{d}x
\end{aligned}
$$

由上一题可得 $\displaystyle\int_0^{\pi}(2\sin x)^y\cos xy\mathrm{d}x = \pi\cos\left(\frac{y\pi}{2}\right)$, 再根据上一题有

$$
I = -\pi\int_0^{\infty}2^{-y}\cos\left(\frac{y\pi}{2}\right)\mathrm{d}y = -\frac{4\pi\ln 2}{\pi^2+4\ln^2 2}.
$$

用类似的方法可以得到

$$
\int_0^{\pi}\frac{x}{x^2+\ln^2(\sin x)}\mathrm{d}x = \frac{2\pi^2}{\pi^2+4\ln^2 2}.
$$

□

45. 设 $b>0$, 计算积分

$$
A = \int_0^{\infty}\frac{\cos bx}{x^s}\mathrm{d}x \quad (0<s<1), \qquad B = \int_0^{\infty}\frac{\sin bx}{x^s}\mathrm{d}x \quad (0<s<2).
$$

解 首先容易得到 $\dfrac{1}{x^s} = \dfrac{1}{\gamma(s)}\displaystyle\int_0^{\infty} z^{s-1}\mathrm{e}^{-xz}\mathrm{d}z$, 所以

$$
\begin{aligned}
A &= \frac{1}{\Gamma(s)}\int_0^{\infty}\cos bx\mathrm{d}x\int_0^{\infty}z^{s-1}\mathrm{e}^{-xz}\mathrm{d}z \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty}z^{s-1}\mathrm{d}z\int_0^{\infty}\mathrm{e}^{-xz}\cos bx\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{z^s \mathrm{d}z}{z^2+b^2} \\
&= \frac{b^{s-1}}{\Gamma(s)} \int_0^{\frac{\pi}{2}} \tan^s t \mathrm{d}t \quad (z = b\tan t) \\
&= \frac{b^{s-1}}{2\Gamma(s)} B\left(\frac{s+1}{2}, \frac{1-s}{2}\right) \\
&= \frac{b^{s-1}}{2\Gamma(s)} \cdot \frac{\pi}{\sin\left(\frac{s+1}{2}\pi\right)} \\
&= \frac{\pi b^{s-1}}{2\Gamma(s)} \cos\left(\frac{s\pi}{2}\right)
\end{aligned}
$$

类似地, 可以得到 $B = \dfrac{\pi b^{s-1}}{2\Gamma(s)\sin\left(\frac{s\pi}{2}\right)}$ （本题的另解见例 10）.

□

46. 计算积分

$$
I = \int_0^{\infty} (1+x^n)\ln\left(1+\mathrm{e}^{-x}\right) \mathrm{d}x
$$

解

$$
\begin{aligned}
I &= \int_0^{\infty} (1+x^n)\ln\left(1+\mathrm{e}^{-x}\right) \mathrm{d}x \\
&= \int_0^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\mathrm{e}^{-kx}}{k} \mathrm{d}x + \int_0^{\infty} x^n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\mathrm{e}^{-kx}}{k} \mathrm{d}x \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} + \Gamma(n+1)\sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k^{n+1}} \\
&= \frac{\pi^2}{12} + (1-2^{-n})\zeta(n+1)\Gamma(n+1)
\end{aligned}
$$

□

47. 计算积分

$$
I = \int_0^{\infty} \frac{\ln(x^3+1)}{x^3+1} \mathrm{d}x
$$

解 令 $t = x^3$ 得 $I = \dfrac{1}{3}\displaystyle\int_0^{\infty} \frac{\ln(1+t)}{(1+t)t^{\frac{2}{3}}} \mathrm{d}t$, 考虑

$$
\begin{aligned}
I(a) &= \int_0^{\infty} \frac{(1+t)^a}{(1+t)t^{\frac{2}{3}}} \mathrm{d}t \\
&= \int_0^1 y^{-\frac{1}{3}-a}(1-y)^{-\frac{2}{3}} \mathrm{d}y \quad \left(1+t = \frac{1}{y}\right) \\
&= B\left(\frac{2}{3}-a, \frac{1}{3}\right) = \frac{\Gamma\left(\frac{2}{3}-a\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma(1-a)}
\end{aligned}
$$

于是 $\ln I(a) = \ln\Gamma\left(\dfrac{2}{3}-a\right) + \ln\Gamma\left(\dfrac{1}{3}\right) - \ln\Gamma(1-a)$, 那么原积分

$$
I = I'(0) = \frac{1}{3} I(0)\left[-\psi\left(\frac{2}{3}\right) + \psi(1)\right] = \frac{\pi\ln 3}{\sqrt{3}} - \frac{\pi^2}{9}
$$

其中 $I(0)=\dfrac{2\pi}{\sqrt{3}}, \psi(1)=-\gamma, \psi\left(\dfrac{2}{3}\right)=-\gamma-\dfrac{3}{2}\ln 3+\dfrac{\pi}{2\sqrt{3}}, \psi\left(\dfrac{2}{3}\right)$ 的值可以借助 $\psi(1-x)=\psi(x)+\pi\cot\pi x$ 以及对公式 $\Gamma(3x)=\dfrac{3^{3x-\frac{1}{2}}}{2\pi}\Gamma(x)\Gamma\left(x+\dfrac{1}{3}\right)\Gamma\left(x+\dfrac{2}{3}\right)$ 求导然后取 $x=\dfrac{1}{3}$ 得到.

□

48. 计算积分

$$I=\int_0^{\frac{\pi}{4}}\ln(1-\tan^2 x)\mathrm{d}x$$

解 $I=\displaystyle\int_0^{\frac{\pi}{4}}\ln(1-\tan^2 x)\mathrm{d}x=\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\mathrm{d}x+\int_0^{\frac{\pi}{4}}\ln(1-\tan x)\mathrm{d}x$, 其中

$$\begin{aligned}
I_1&=\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\mathrm{d}x=\int_0^{\frac{\pi}{4}}\ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)\mathrm{d}x\\
&=\int_0^{\frac{\pi}{4}}\ln\left(1+\frac{1-\tan x}{1+\tan x}\right)\mathrm{d}x=\int_0^{\frac{\pi}{4}}\ln\left(\frac{2}{1+\tan x}\right)\mathrm{d}x\\
&=\frac{\pi}{4}\ln 2-I_1=\frac{\pi}{8}\ln 2\\
I_2&=\int_0^{\frac{\pi}{4}}\ln(1-\tan x)\mathrm{d}x=\int_0^{\frac{\pi}{4}}\ln\left(1-\tan\left(\frac{\pi}{4}-x\right)\right)\mathrm{d}x\\
&=\int_0^{\frac{\pi}{4}}\ln\left(1-\frac{1-\tan x}{1+\tan x}\right)\mathrm{d}x=\int_0^{\frac{\pi}{4}}\ln\left(\frac{2\tan x}{1+\tan x}\right)\mathrm{d}x\\
&=\frac{\pi}{4}\ln 2+\int_0^{\frac{\pi}{4}}\ln(\tan x)\mathrm{d}x-I_2=\frac{\pi}{8}\ln 2-G
\end{aligned}$$

因此 $I=I_1+I_2=\dfrac{\pi}{4}\ln 2-G$.

□

49. 计算积分

$$I=\int_0^{\infty}\frac{(1-x^2)\arctan(x^2)}{x^4+4x^2+1}\mathrm{d}x$$

解 设 $y\geqslant 0$, 考虑含参积分

$$\begin{aligned}
F(y)&=\int_0^{\infty}\frac{\arctan(yx^2)}{1+x^2}\mathrm{d}x\\
&=\int_0^{\infty}\int_0^{y}\frac{x^2}{(1+t^2x^4)(1+x^2)}\mathrm{d}t\mathrm{d}x\\
&=\int_0^{y}\mathrm{d}t\int_0^{\infty}\left[\frac{1+t^2x^2}{(1+t^2x^4)(1+t^2)}-\frac{1}{(1+t^2)(1+x^2)}\right]\mathrm{d}x\\
&=\int_0^{y}\mathrm{d}t\int_0^{\infty}\frac{1+tx^2}{(1+x^4)(1+t^2)\sqrt{t}}\mathrm{d}x-\int_0^{y}\mathrm{d}t\int_0^{\infty}\frac{1}{(1+t^2)(1+x^2)}\mathrm{d}x\\
&=\frac{\sqrt{2}\pi}{4}\int_0^{y}\frac{1+t}{(1+t^2)\sqrt{t}}\mathrm{d}t-\frac{\pi}{2}\int_0^{y}\frac{1}{1+t^2}\mathrm{d}t\\
&=\frac{\pi}{2}\left[\arctan\left(\sqrt{2y}+1\right)+\arctan\left(\sqrt{2y}-1\right)-\arctan y\right].
\end{aligned}$$

所以

$$I=\int_0^{\infty}\frac{(1-x^2)\arctan(x^2)}{x^4+4x^2+1}\mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^\infty \frac{\left(\sqrt{3}-1\right)\arctan(x^2)}{2\left(x^2+2-\sqrt{3}\right)}\mathrm{d}x - \int_0^\infty \frac{\left(\sqrt{3}+1\right)\arctan(x^2)}{2\left(x^2+2+\sqrt{3}\right)}\mathrm{d}x \\
&= \frac{\sqrt{2}}{2}\left[F\left(2-\sqrt{3}\right) - F\left(2+\sqrt{3}\right)\right] \\
&= -\frac{\sqrt{2}\pi}{4}\left[\arctan\left(\sqrt{2y+1}\right) + \arctan\left(\sqrt{2y-1}\right) - \arctan y\right]\Big|_{y=2-\sqrt{3}}^{y=2+\sqrt{3}} \\
&= -\frac{\sqrt{2}\pi}{4}\left[-\arctan\left(\sqrt{3}-2\right) + \arctan\left(2-\sqrt{3}\right)\right] \\
&= -\frac{\sqrt{2}\pi^2}{24}
\end{aligned}
$$

□

50. 计算积分

$$
I = \int_0^\infty \frac{(1-\mathrm{e}^{-6x})\mathrm{e}^{-x}}{x(1+\mathrm{e}^{-2x}+\mathrm{e}^{-4x}+\mathrm{e}^{-6x}+\mathrm{e}^{-8x})}\mathrm{d}x
$$

解

$$
\begin{aligned}
I &= \int_0^\infty \frac{(1-\mathrm{e}^{-6x})\mathrm{e}^{-x}}{x(1+\mathrm{e}^{-2x}+\mathrm{e}^{-4x}+\mathrm{e}^{-6x}+\mathrm{e}^{-8x})}\mathrm{d}x \\
&= \int_0^\infty \frac{(1-\mathrm{e}^{-6x})\mathrm{e}^{-x}(1-\mathrm{e}^{-2x})}{x(1-\mathrm{e}^{-10x})}\mathrm{d}x \\
&= \int_0^\infty \sum_{n=0}^\infty \frac{(1-\mathrm{e}^{-6x})\mathrm{e}^{-(10n+1)x}(1-\mathrm{e}^{-2x})}{x} \\
&= \int_0^\infty \sum_{n=0}^\infty \frac{\left(\mathrm{e}^{-(10n+1)x}-\mathrm{e}^{-(10n+7)x}\right)-\left(\mathrm{e}^{-(10n+3)x}-\mathrm{e}^{-(10n+9)x}\right)}{x}\mathrm{d}x \\
&= \sum_{n=0}^\infty \ln\left[\frac{(10n+7)(10n+3)}{(10n+1)(10n+9)}\right] \\
&= \sum_{n=0}^\infty \ln\left[\frac{\left(\frac{7}{10}+n\right)\left(\frac{3}{10}+n\right)}{\left(\frac{1}{10}+n\right)\left(\frac{9}{10}+n\right)}\right] \\
&= \lim_{n\to\infty} \ln\left[\frac{\Gamma\left(\frac{1}{10}\right)\Gamma\left(\frac{9}{10}\right)\Gamma\left(\frac{3}{10}+n\right)\Gamma\left(\frac{7}{10}+n\right)}{\Gamma\left(\frac{3}{10}\right)\Gamma\left(\frac{7}{10}\right)\Gamma\left(\frac{1}{10}+n\right)\Gamma\left(\frac{9}{10}+n\right)}\right] \\
&== \ln\left(\frac{\sin\frac{3\pi}{10}}{\sin\frac{\pi}{10}}\right) = \ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)
\end{aligned}
$$

□

51. 设 $a_0 = 1, a_1 = \dfrac{2}{3}$,

$$
(n+1)a_{n+1} - (n-1)a_{n-1} = \frac{2}{3}a_n.
$$

求极限 $\lim\limits_{n\to\infty} n^{\frac{2}{3}}a_n$.

解 设 $S(x) = \sum\limits_{n=1}^\infty n a_n x^{n-1}$, 由递推关系可得

$$
[(1-x^2)S(x)]' = \frac{2}{3}S(x)
$$

解得
$$S(x)=\frac{2}{3}(1+x)^{-\frac{2}{3}}(1-x)^{-\frac{4}{3}}.$$
利用 Cauchy 乘积并比较系数得到当 $n\geqslant 1$ 时有
$$\begin{aligned}
a_n&=\frac{2}{3n\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}\sum_{\substack{k+j=n-1\\k,j\geqslant 0}}(-1)^k\frac{\Gamma\left(k+\frac{2}{3}\right)\Gamma\left(j+\frac{4}{3}\right)}{k!j!}\\
&=\frac{2}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}\sum_{\substack{k+j=n-1\\k,j\geqslant 0}}(-1)^k\binom{n-1}{k}\int_0^1 t^{k-\frac{1}{3}}(1-t)^{j+\frac{1}{3}}\mathrm{d}t\\
&=\frac{2}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}\int_0^1(1-2t)^{n-1}\frac{\sqrt[3]{1-t}}{\sqrt[3]{t}}\mathrm{d}t\\
&=\frac{1}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}\int_{-1}^1\frac{\sqrt[3]{1+s}}{\sqrt[3]{1-s}}s^{n-1}\mathrm{d}s\\
&=\frac{1}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}\left(\int_0^1\frac{\sqrt[3]{2}}{\sqrt[3]{1-s}}s^{n-1}\mathrm{d}s+\int_0^1\frac{\sqrt[3]{1+s}-\sqrt[3]{2}}{\sqrt[3]{1-s}}s^{n-1}\mathrm{d}s+\int_{-1}^0\frac{\sqrt[3]{1+s}}{\sqrt[3]{1-s}}s^{n-1}\mathrm{d}s\right)\\
&\triangleq F_n+G_n+H_n
\end{aligned}$$

容易说明 nG_n,nH_n 有界, $\lim\limits_{n\to\infty}n^{\frac{2}{3}}a_n=\lim\limits_{n\to\infty}n^{\frac{2}{3}}F_n=\dfrac{\sqrt[3]{2}}{\Gamma\left(\frac{1}{3}\right)}$.

□

52. 计算三重积分
$$I=\int_{-\infty}^{\infty}\mathrm{d}t\int_t^{\infty}\mathrm{d}w\int_w^{\infty}\mathrm{e}^{-\left(u^2+t^2+\frac{1}{2}w^2\right)}\mathrm{d}u$$

解 记 $\Omega=(u,w,t)|-\infty<t\leqslant w\leqslant u<\infty$, 做广义球坐标变换
$$\begin{cases}u=r\cos\theta\sin\varphi\\ w=\sqrt{2}r\sin\theta\sin\varphi\\ t=r\cos\varphi\\ r\geqslant 0,\theta\in[-\pi,\pi],\varphi\in[0,\pi]\end{cases}$$

设 $\alpha\in\left(0,\frac{\pi}{2}\right)$ 满足 $\sin\alpha=\dfrac{1}{\sqrt{3}}$, $\varphi(\theta)\in(0,\pi)$ 满足 $\cot\varphi(\theta)=\sqrt{2}\sin\theta$, 则原积分
$$\begin{aligned}
I&=\iiint\limits_{\Omega}\mathrm{e}^{-\left(u^2+t^2+\frac{1}{2}w^2\right)}\mathrm{d}u\mathrm{d}w\mathrm{d}t\\
&=\int_0^{\infty}\mathrm{d}r\int_{-\pi+\alpha}^{\alpha}\mathrm{d}\theta\int_{\varphi(\theta)}^{\pi}\sqrt{2}\mathrm{e}^{-r^2}r^2\sin\varphi\mathrm{d}\varphi\\
&=\frac{\sqrt{2\pi}}{4}\int_{-\pi+\alpha}^{\alpha}\mathrm{d}\theta\int_{\varphi(\theta)}^{\pi}\sin\varphi\mathrm{d}\varphi\\
&=\frac{\sqrt{2\pi}}{4}\int_{-\pi+\alpha}^{\alpha}[1+\cos\varphi(\theta)]\,\mathrm{d}\theta\\
&=\frac{\sqrt{2\pi}}{4}\int_{-\pi+\alpha}^{\alpha}\left(1+\frac{\sqrt{2}\sin\theta}{\sqrt{1+2\sin^2\theta}}\right)\mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{2\pi}}{4}\left(\pi - 2\int_0^{\frac{2}{3}} \frac{\mathrm{d}s}{\sqrt{1-s^2}}\right) \\
&= \frac{\sqrt{2\pi}}{4}\left[\pi - 2\arcsin\left(\frac{2}{3}\right)\right] \\
&= \sqrt{2\pi}\arcsin\left(\frac{1}{\sqrt{6}}\right)
\end{aligned}
$$

□

53. 计算积分

$$
I = \int_0^1 \frac{\sqrt[n]{x^m(1-x)^{n-m}}}{(1+x)^3}\mathrm{d}x
$$

解

$$
\begin{aligned}
I &= \int_0^1 \frac{\sqrt[n]{x^m(1-x)^{n-m}}}{(1+x)^3}\mathrm{d}x \\
&= \int_0^1 \left(\frac{x}{1+x}\right)^{\frac{m}{n}}\left(\frac{1-x}{1+x}\right)^{\frac{n-m}{n}}\frac{\mathrm{d}x}{(1+x)^2} \\
&= 2^{-\frac{n+m}{n}}\int_0^1 t^{\frac{m}{n}}(1-t)^{\frac{n-m}{n}}\mathrm{d}t \quad \left(t = \frac{x}{1+x}\right) \\
&= \frac{2^{-\frac{n+m}{n}}}{\Gamma(3)}\Gamma\left(\frac{m+n}{n}\right)\Gamma\left(\frac{2n-m}{n}\right) \\
&= 2^{-\frac{n+m}{n}}\cdot\frac{m}{n}\frac{n-m}{n}\cdot\Gamma\left(\frac{m}{n}\right)\cdot\Gamma\left(1-\frac{m}{n}\right) \\
&= 2^{-\frac{n+m}{n}}\cdot\frac{m(n-m)}{n^2}\cdot\frac{\pi}{\sin\left(\frac{m\pi}{n}\right)}
\end{aligned}
$$

□

54. 计算积分

$$
I = \int_0^1 \sin(\pi x)\ln\Gamma(x)\mathrm{d}x
$$

解

$$
\begin{aligned}
I &= \int_0^1 \sin(\pi x)\ln\Gamma(x)\mathrm{d}x \\
&= \int_0^1 \sin\pi(1-x)\ln\Gamma(1-x)\mathrm{d}x \\
&= \int_0^1 \sin(\pi x)\ln\Gamma(1-x)\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \sin(\pi x)\ln[\Gamma(x)\Gamma(1-x)]\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \sin(\pi x)\ln\left(\frac{\pi}{\sin(\pi x)}\right)\mathrm{d}x \\
&= \ln\pi\int_0^1 \sin(\pi x)\mathrm{d}x - \int_0^1 \sin(\pi x)\ln(\sin\pi x)\mathrm{d}x \\
&= \frac{\ln\pi}{\pi} - \frac{1}{2\pi}\int_0^\pi \sin t\ln(\sin t)\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\ln \pi}{\pi} - \frac{1}{2\pi}\left(\cos t - \ln\left(\cos\frac{t}{2}\right) + \ln\left(\sin\frac{t}{2}\right) - \cos t \ln(\sin t)\right)\Big|_0^{\pi} \\
&= \frac{\ln \pi}{\pi} - \frac{2\ln 2 - 2}{2\pi} = \frac{\ln\left(\frac{\pi}{2}\right) + 1}{\pi}
\end{aligned}
$$

□

55. 计算积分

$$
I = \int_0^{\frac{\pi}{4}} \frac{\ln(\sin x)\ln(\cos x)}{\sin 2x}\mathrm{d}x
$$

解

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} \frac{\ln(\sin x)\ln(\cos x)}{\sin 2x}\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\ln(\sin x)\ln(\cos x)}{\sin 2x}\mathrm{d}x \\
&= \frac{1}{16}\int_0^{\frac{\pi}{2}} \frac{\cos x \ln(\sin^2 x)\ln(\cos^2 x)}{\sin x \cos^2 x}\mathrm{d}x \\
&= \frac{1}{32}\int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)}\mathrm{d}x \\
&= \frac{1}{32}\int_0^1 \ln x \ln(1-x)\left(\frac{1}{x} + \frac{1}{1-x}\right)\mathrm{d}x \\
&= \frac{1}{16}\int_0^1 \frac{\ln x \ln(1-x)}{x}\mathrm{d}x \\
&= -\frac{1}{16}\int_0^1 \left(\ln x \sum_{k=1}^{\infty} \frac{x^{k-1}}{k}\right)\mathrm{d}x \\
&= \frac{1}{16}\zeta(3)
\end{aligned}
$$

□

56. 计算积分

$$
\int_0^{\frac{\pi}{2}} \frac{\arctan\left(\sqrt{2}\tan x\right)}{\tan x}\mathrm{d}x
$$

解 令 $I(a) = \int_0^{\frac{\pi}{2}} \frac{\arctan(a\tan x)}{\tan x}\mathrm{d}x$, 则 $I(0) = 0$,

$$
\begin{aligned}
I'(a) &= \int_0^{\frac{\pi}{2}} \frac{\tan x}{(1 + a^2\tan^2 x)\tan x}\mathrm{d}x \\
&= \frac{1}{1-a^2}\int_0^{\infty} \frac{(1 + a^2u^2) - a^2(1+u^2)}{(1+a^2u^2)(1+u^2)}\mathrm{d}u \\
&= \frac{1}{1-a^2}\int_0^{\infty} \frac{1}{1+u^2}\mathrm{d}u - \frac{a^2}{1-a^2}\int_0^{\infty} \frac{1}{1+a^2u^2}\mathrm{d}u \\
&= \frac{\pi}{2}\left(\frac{1}{1-a^2} - \frac{a}{1-a^2}\right) \\
&= \frac{\pi}{2}\frac{1}{1+a}
\end{aligned}
$$

于是原积分 $I=I\left(\sqrt{2}\right)=\int_0^{\sqrt{2}} I'(a)\mathrm{d}a=\frac{\pi}{2}\ln\left(1+\sqrt{2}\right)$.

□

57. 计算积分

$$\int_{-\infty}^{\infty}\frac{x^2}{(1+\mathrm{e}^x)(1+\mathrm{e}^{-x})}\mathrm{d}x$$

解

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{x^2}{(1+\mathrm{e}^x)(1+\mathrm{e}^{-x})}\mathrm{d}x &= 2\int_0^{\infty}\frac{x^2\mathrm{e}^x}{(1+\mathrm{e}^x)^2}\mathrm{d}x\\
&= \left.\frac{-2x^2}{1+\mathrm{e}^x}\right|_0^{\infty}+4\int_0^{\infty}\frac{x}{1+\mathrm{e}^x}\mathrm{d}x\\
&= 4\int_0^{\infty}\frac{x\mathrm{e}^{-x}}{1+\mathrm{e}^x}\mathrm{d}x\\
&= 4\int_0^{\infty}x\mathrm{e}^{-x}\sum_{k=0}^{\infty}(-\mathrm{e})^{-kx}\mathrm{d}x\\
&= 4\sum_{k=1}^{\infty}\int_0^{\infty}(-1)^{k-1}x\mathrm{e}^{-kx}\mathrm{d}x\\
&= 4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}=\frac{\pi^2}{3}
\end{aligned}$$

□

58. 计算积分

$$\int_0^{\frac{\pi}{2}}\frac{x\mathrm{e}^{-\tan^2 x}\sin(4x)}{\cos^8 x}\mathrm{d}x$$

解

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}}\frac{x\mathrm{e}^{-\tan^2 x}\sin(4x)}{\cos^8 x}\mathrm{d}x\\
&=\int_0^{\infty}\arctan t\cdot\mathrm{e}^{-t^2}2\frac{2t}{1+t^2}\frac{1-t^2}{1+t^2}\mathrm{d}(\arctan t)\\
&=2\int_0^{\infty}\mathrm{e}^{-t^2}(1-t^4)\arctan t\,\mathrm{d}(t^2)\\
&=-2\int_0^{\infty}(1-t^4)\arctan t\,\mathrm{d}\left(\mathrm{e}^{-t^2}\right)\\
&=-2\left.(1-t^4)\arctan t\right|_0^{\infty}+2\int_0^{\infty}\mathrm{e}^{-t^2}\left[(1-t^2)-4t^3\arctan t\right]\mathrm{d}t\\
&=2\int_0^{\infty}\mathrm{e}^{-t^2}(1-t^2)\mathrm{d}t-4\int_0^{\infty}\mathrm{e}^{-t^2}\arctan t\,\mathrm{d}(t^2)\\
&=2\int_0^{\infty}\mathrm{e}^{-t^2}(1-t^2)\mathrm{d}t+4\int_0^{\infty}t^2\arctan t\,\mathrm{d}\left(\mathrm{e}^{-t^2}\right)\\
&=2\int_0^{\infty}\mathrm{e}^{-t^2}(1-t^2)\mathrm{d}t-4\int_0^{\infty}\mathrm{e}^{-t^2}\left[\frac{t^2}{1+t^2}+2t\arctan t\right]\mathrm{d}t\\
&=2\int_0^{\infty}\mathrm{e}^{-t^2}(1-t^2)\mathrm{d}t-4\int_0^{\infty}\mathrm{e}^{-t^2}\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= -2\int_0^{\infty} \mathrm{e}^{-t^2}(1+t^2)\mathrm{d}t \\
&= -\frac{3}{2}\sqrt{\pi}
\end{aligned}
$$

□

59. 计算积分

$$
\int_0^{\pi} \mathrm{e}^{p\cos x}\cos(p\sin x)\cos qx\mathrm{d}x
$$

解

$$
\begin{aligned}
\int_0^{\pi} \mathrm{e}^{p\cos x}\cos(p\sin x)\cos qx\mathrm{d}x &= \Re\int_0^{\pi} \mathrm{e}^{p\cos x}\mathrm{e}^{\mathrm{i}p\sin x}\cos qx\mathrm{d}x \\
&= \Re\int_0^{\pi} \mathrm{e}^{\mathrm{i}p\sin x+p\cos x}\cos qx\mathrm{d}x \\
&= \Re\int_0^{\pi} \mathrm{e}^{p\mathrm{e}^{\mathrm{i}x}}\cos qx\mathrm{d}x \\
&= \Re\int_0^{\pi}\sum_{k=0}^{\infty}\left(\frac{p^k\mathrm{e}^{\mathrm{i}kx}}{k!}\right)\cos qx\mathrm{d}x \\
&= \sum_{k=0}^{\infty}\frac{p^k}{k!}\int_0^{\pi}\cos qx\cos kx\mathrm{d}x \\
&= \sum_{k=0}^{\infty}\frac{p^k}{2k!}\left[\frac{\sin(k-q)x}{k-q}+\frac{\sin(k+q)x}{k+q}\right]\Bigg|_0^{\pi} \\
&= \frac{\pi}{2}\frac{p^q}{q!}
\end{aligned}
$$

□

60. 求极限

$$
\lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{1}{3k+1}-\frac{1}{3}\ln n\right)
$$

解　首先有

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{1}{3k+1} &= 1+\frac{1}{3}\left(\sum_{k=1}^{n}\left(\frac{1}{k+1/3}-\frac{1}{k}\right)\right) \\
&= 1+\frac{1}{3}\sum_{k=1}^{n}\left(\frac{1}{k+1/3}-\frac{1}{k}\right)+\frac{1}{p}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)+\frac{1}{3}\ln n
\end{aligned}
$$

于是

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{1}{3k+1}-\frac{1}{3}\ln n &= 1+\frac{1}{3}\sum_{k=1}^{n}\left(\frac{1}{k+1/3}-\frac{1}{k}\right)+\frac{1}{p}\left(\sum_{k=1}^{n}\frac{1}{k}\right) \\
&= 1+\frac{1}{3}\sum_{k=1}^{n}\left(\int_0^1 x^{k+1/3-1}\mathrm{d}x-\int_0^1 x^{k-1}\mathrm{d}x\right)+\frac{1}{3}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right) \\
&= 1+\frac{1}{3}\left(\int_0^1\frac{x^{1/3}-1}{1-x}\mathrm{d}x\right)+\frac{1}{3}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + \int_0^1 \frac{x^{1/3}-1}{1-x}\mathrm{d}x + \frac{1}{3}\left(\sum_{k=1}^n \frac{1}{k} - \ln n\right) \\
&= 1 - \int_0^1 \frac{x^2}{x^2+x+1}\mathrm{d}x + \frac{1}{3}\left(\sum_{k=1}^n \frac{1}{k} - \ln n\right) \\
&= \frac{\pi\sqrt{3}}{18} + \frac{1}{2}\ln 3 + \frac{1}{3}\left(\sum_{k=1}^n \frac{1}{k} - \ln n\right)
\end{aligned}
$$

因此 $\lim\limits_{n\to\infty}\left(\sum\limits_{k=0}^n \frac{1}{3k+1} - \frac{1}{3}\ln n\right) = \frac{1}{3}\gamma + \frac{\pi\sqrt{3}}{18} + \frac{1}{2}\ln 3.$

□

61. 设 $a_0, a_1 > 0, a_{n+2} = \frac{1}{a_n} + \frac{1}{a_{n+1}}$, 证明: $\lim\limits_{n\to\infty} a_n = \sqrt{2}$.

解 令

$$M = \max\left\{a_0, a_1, \frac{2}{a_0}, \frac{2}{a_1}\right\}$$

则利用递推公式可以归纳证明 $\frac{2}{M} \leqslant a_n \leqslant M$. 设 a_n 的上下极限为 L, ℓ. 则

$$\frac{2}{M} \leqslant \ell \leqslant M.$$

且

$$\ell = \varliminf_{n\to\infty} a_{n+2} \geqslant \varliminf_{n\to\infty} \frac{1}{a_{n+1}} + \varliminf_{n\to\infty} \frac{1}{a_n} = \frac{2}{L} \tag{1}$$

$$L = \varlimsup_{n\to\infty} a_{n+2} \leqslant \varlimsup_{n\to\infty} \frac{1}{a_{n+1}} + \varlimsup_{n\to\infty} \frac{1}{a_n} = \frac{2}{\ell} \tag{2}$$

因此 $L\ell = 2$.

由上极限的性质, 存在子列 a_{m_k+3} 使得 $\lim\limits_{k\to\infty} a_{m_k+3} = L$. 进一步抽取子列 (为方便起见, 子列记号不变), 可以使 $a_{m_k+2}, a_{m_k+1}, a_{m_k}$ 都收敛, 设极限依次为 x, y, z, 则 $\ell \leqslant x, y, z \leqslant L$. 于是由递推公式可得

$$L = \frac{1}{x} + \frac{1}{y}, \quad x = \frac{1}{y} + \frac{1}{z}$$

由 (1) 式结合 $\ell \leqslant x, y \leqslant L$ 以及 $L = \frac{2}{\ell}$ 得到 $x = y = \ell$. 类似地, 可由第二式得到 $y = z = L$. 所以 $L = \ell$, 从而 $L = \ell = \lim\limits_{n\to\infty} a_n = \sqrt{2}$.

□

62. 计算积分

$$\int_0^1 \frac{\left(\frac{1}{2}-x\right)\ln(1-x)}{x^2-x+1}\mathrm{d}x$$

解 首先有 $I = \int_0^1 \frac{\left(\frac{1}{2}-x\right)\ln(1-x)}{x^2-x+1}\mathrm{d}x = \int_0^1 \frac{\left(x-\frac{1}{2}\right)\ln x}{x^2-x+1}\mathrm{d}x$, 利用

$$\sum_{n=1}^\infty x^{n-1}\sin(n a\pi) = \frac{\sin(a\pi)}{x^2 - 2\cos(a\pi) + 1}$$

令 $a=\frac{1}{3}$ 可得

$$
\begin{aligned}
I &= \frac{2}{\sqrt{3}}\int_0^1 \sum_{n=1}^{\infty} x^{n-1}\sin\left(\frac{n\pi}{3}\right)\left(x-\frac{1}{2}\right)\ln x\mathrm{d}x \\
&= \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{3}\right)\int_0^1 x^{n-1}\left(x-\frac{1}{2}\right)\ln x\mathrm{d}x \\
&= \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{3}\right)\left[\frac{1}{2n^2}-\frac{1}{(n+1)^2}\right] \\
&= \frac{\pi^2}{36}
\end{aligned}
$$

接下来计算 $J=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{3}\right)\left[\frac{1}{2n^2}-\frac{1}{(n+1)^2}\right]$.

$$
\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{3}\right)}{(n+1)^2}=\sum_{n=2}^{\infty}\frac{\sin\left(\frac{n\pi}{3}-\frac{\pi}{3}\right)}{n^2}=\sum_{n=2}^{\infty}\frac{\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)-\cos\left(\frac{n\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)}{n^2}
$$

于是

$$
J=\frac{\sin\left(\frac{\pi}{3}\right)}{2}+\sum_{n=2}^{\infty}\frac{\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)}{n^2}=\frac{\sqrt{3}}{2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{3}\right)}{n^2}
$$

利用 x^2 的 Fourier 展开式 $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}=\frac{3x^2-6\pi x+2\pi^2}{12}$, 令 $x=\frac{\pi}{3}$ 得 $I=\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{3}\right)}{n^2}=\frac{\pi^2}{36}$. □

63. 计算积分

$$
\int_0^{\pi}\sin(a\sin x)\csc x\mathrm{d}x
$$

解

$$
\begin{aligned}
\int_0^{\pi}\sin(a\sin x)\csc x\mathrm{d}x &= \Im\int_0^{\pi}\mathrm{e}^{a\cos x}\mathrm{e}^{\mathrm{i}a\sin x}\csc x\mathrm{d}x \\
&= \Im\int_0^{\pi}\frac{\mathrm{e}^{\mathrm{i}a\sin x+a\cos x}}{\sin x}\mathrm{d}x \\
&= \Im\int_0^{\pi}\frac{\mathrm{e}^{a\mathrm{e}^{\mathrm{i}x}}}{\sin x}\mathrm{d}x \\
&= \Im\int_0^{\pi}\frac{\sum_{k=0}^{\infty}\frac{a^k\mathrm{e}^{\mathrm{i}kx}}{k!}}{\sin x}\mathrm{d}x \\
&= \sum_{k=0}^{\infty}\frac{a^k}{k!}\int_0^{\pi}\frac{\sin kx}{\sin x}\mathrm{d}x \\
&= \pi\sum_{k=0}^{\infty}\frac{a^{2k+1}}{(2k+1)!}=\pi\sinh(a)
\end{aligned}
$$

其中

$$
\int_0^{\pi}\frac{\sin nx}{\sin x}\mathrm{d}x=\begin{cases}\pi, n=2k+1\\ 0, n=2k\end{cases}
$$

□

64. 求极限

$$\lim_{n\to\infty}\frac{1}{n^2}\int_0^{\frac{\pi}{2}} x\left(\frac{\sin nx}{\sin x}\right)^4 \mathrm{d}x$$

解 由

$$\lim_{x\to 0} x^2\left(\frac{1}{\sin^4 x}-\frac{1}{x^4}\right)=\frac{2}{3}$$

知存在常数 $C>0$ 使得

$$\left|\frac{x\sin^4 nx}{\sin^4 x}-\frac{\sin^4 nx}{x^3}\right|\leqslant \frac{C\sin^4 nx}{x}\leqslant Cn,\quad \forall x\in\left(0,\frac{\pi}{2}\right).$$

由此得到

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n^2}\int_0^{\frac{\pi}{2}} x\left(\frac{\sin nx}{\sin x}\right)^4 \mathrm{d}x &= \lim_{n\to\infty}\frac{1}{n^2}\int_0^{\frac{\pi}{2}}\frac{\sin^4 nx}{x^3}\mathrm{d}x\\
&=\lim_{n\to\infty}\int_0^{\frac{n\pi}{2}}\frac{\sin^4 x}{x^3}\mathrm{d}x=\int_0^{\infty}\frac{\sin^4 x}{x^3}\mathrm{d}x\\
&=\int_0^{\infty}\frac{2\sin^3 x\cos x}{x^2}\mathrm{d}x\\
&=\int_0^{\infty}\frac{6\sin^2 x\cos^2 x-2\sin^4 x}{x}\mathrm{d}x\\
&=\int_0^{\infty}\frac{\cos 2x-\cos 4x}{x}\mathrm{d}x=\ln 2.
\end{aligned}$$

□

65. 求极限

$$\lim_{n\to\infty}\left(\int_0^{\pi}\frac{\sin^2 nx}{\sin x}\mathrm{d}x-\sum_{k=1}^{n}\frac{1}{k}\right)$$

解 先考虑积分 $I(n)=\displaystyle\int_0^{\pi}\frac{\sin^2 nx}{\sin x}\mathrm{d}x$, 有

$$I(n)-I(n-1)=\int_0^{\pi}\frac{\sin^2 nx}{\sin x}\mathrm{d}x=\frac{2}{2n-1}$$

于是

$$I(n)=2\sum_{k=1}^{n}\frac{1}{2k-1}\sim\ln 2+\ln(2n)+\gamma=\ln(4n)+\gamma,\quad n\to\infty$$

故 $\displaystyle\lim_{n\to\infty}\left(\int_0^{\pi}\frac{\sin^2 nx}{\sin x}\mathrm{d}x-\sum_{k=1}^{n}\frac{1}{k}\right)=\ln 4.$

□

66. 把方程 $\tan x=x$ 的正根按从小到大顺序排成数列 x_n, 求极限

$$\lim_{n\to\infty} x_n^2\sin(x_{n+1}-x_n)$$

解 首先容易得到 $x_n\in\left((n-1)\pi,(n-1)\pi+\dfrac{\pi}{2}\right)$, 于是 $x_n-(n-1)\pi\in\left(0,\dfrac{\pi}{2}\right)$, 故

$$x_n=\tan x_n=\tan(x_n-(n-1)\pi)$$

所以 $\arctan x_n = x_n - (n-1)\pi$, 且 $x_n - (n-1)\pi \to \dfrac{\pi}{2}, n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} x_n^2 \sin(x_{n+1} - x_n) &= \lim_{n\to\infty} x_n^2 \sin(\arctan x_{n+1} - \arctan x_n + \pi) \\
&= -\lim_{n\to\infty} n^2\pi^2 \sin\left[\arctan\left(\frac{x_{n+1} - x_n}{1 + x_n x_{n+1}}\right)\right] \\
&= -\lim_{n\to\infty} n^2\pi^2 \frac{x_{n+1} - x_n}{1 + x_n x_{n+1}} = -\lim_{n\to\infty} (x_{n+1} - x_n) \\
&= -\lim_{n\to\infty} [x_{n+1} - n\pi - (x_n - (n-1)\pi)] - \pi \\
&= -\pi.
\end{aligned}$$

□

67. 求极限

$$\lim_{x\to 0} \frac{\Gamma(x+1) - \sin x\,\Gamma(\sin x)}{x^4\Gamma(\sin x)}$$

解

$$\begin{aligned}
\lim_{x\to 0} \frac{\Gamma(x+1) - \sin x\,\Gamma(\sin x)}{x^4\Gamma(\sin x)} &= \lim_{x\to 0} \frac{\Gamma(x - \sin x + \sin x + 1) - \Gamma(\sin x + 1)}{x^3\Gamma(\sin x + 1)} \\
&= \lim_{x\to 0} \frac{\Gamma(x - \sin x + \sin x + 1) - \Gamma(\sin x + 1)}{6(x - \sin x)\Gamma(\sin x + 1)} \\
&= \frac{\psi(1)}{6} = -\frac{\gamma}{6}.
\end{aligned}$$

□

68. 计算主值积分

$$\int_0^\infty \sin(x)\sin(\sqrt{x})\mathrm{d}x$$

解

$$\begin{aligned}
&\int_0^\infty \sin(x)\sin(\sqrt{x})\mathrm{d}x \\
&= \frac{1}{2}\int_0^\infty \left[\cos(x - \sqrt{x}) - \cos(x + \sqrt{x})\right]\mathrm{d}x \\
&= \frac{1}{2}\int_0^\infty \left[\cos\left(\left(\sqrt{x} - \frac{1}{2}\right)^2 - \frac{1}{4}\right) - \cos\left(\left(\sqrt{x} + \frac{1}{2}\right)^2 - \frac{1}{4}\right)\right]\mathrm{d}x \\
&= \frac{1}{2}\int_{-\frac{1}{2}}^\infty (2t+1)\cos\left(t^2 - \frac{1}{4}\right)\mathrm{d}t - \frac{1}{2}\int_{\frac{1}{2}}^\infty (2t-1)\cos\left(t^2 - \frac{1}{4}\right)\mathrm{d}t \\
&= \frac{1}{2}\int_{-\frac{1}{2}}^\infty (2t+1)\cos\left(t^2 - \frac{1}{4}\right)\mathrm{d}t + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}} (2t+1)\cos\left(t^2 - \frac{1}{4}\right)\mathrm{d}t \\
&= \frac{1}{2}\int_{-\infty}^\infty \cos\left(t^2 - \frac{1}{4}\right)\mathrm{d}t \\
&= \frac{1}{2}\int_{-\infty}^\infty \left[\cos\left(\frac{1}{4}\right)\cos t^2 + \sin\left(\frac{1}{4}\right)\sin t^2\right]\mathrm{d}t \\
&= \frac{1}{2}\sqrt{\frac{\pi}{2}}\left[\cos\left(\frac{1}{4}\right) + \sin\left(\frac{1}{4}\right)\right]
\end{aligned}$$

$$= \frac{\sqrt{\pi}}{2} \cos\left(\frac{\pi - 1}{4}\right)$$

□

69. 计算积分

$$\int_0^\infty \frac{1 - \cos x}{x(e^x - 1)} dx$$

解 考虑积分 $I(a) = \int_0^\infty \frac{1 - \cos(ax)}{x(e^x - 1)} dx$, 则

$$\begin{aligned} I'(a) &= \int_0^\infty \frac{\sin(ax)}{e^x - 1} dx \\ &= \int_0^\infty \sin(ax) \sum_{n=1}^\infty e^{-nx} dx \\ &= \sum_{n=1}^\infty \sin(ax) e^{-nx} dx \\ &= \sum_{n=1}^\infty \frac{a}{n^2 + a^2} \end{aligned}$$

因此

$$I = I(1) = \int_0^1 \sum_{n=1}^\infty \frac{a}{n^2 + a^2} da = \frac{1}{2} \sum_{n=1}^\infty \ln\left(1 + \frac{1}{n^2}\right) = \frac{1}{2} \ln\left(\frac{\sinh(\pi)}{\pi}\right)$$

□

70. 求和

$$\sum_{n=1}^\infty \arctan\left(\frac{1}{1+n^2}\right)$$

解

$$\begin{aligned} \sum_{n=1}^\infty \arctan\left(\frac{1}{1+n^2}\right) &= \sum_{n=1}^\infty \arg\left(1 + \frac{i}{1+n^2}\right) \\ &= \arg \prod_{n=1}^\infty \left(\frac{1 + \frac{1+i}{n^2}}{1 + \frac{1}{n^2}}\right) \\ &= \arg \frac{\prod_{n=1}^\infty \left(1 + \frac{1+i}{n^2}\right)}{\prod_{n=1}^\infty \left(1 + \frac{1}{n^2}\right)} \\ &= \arg \frac{\prod_{n=1}^\infty \left(1 + \frac{1+i}{n^2}\right)}{\frac{\sinh(\pi)}{\pi}}. \end{aligned}$$

令 $(1+i)\pi^2 = (u + iv)^2, u = \sqrt{\frac{1+\sqrt{2}}{2}}\pi, v = \sqrt{\frac{\sqrt{2}-1}{2}}\pi$, 则

$$\sum_{n=1}^\infty \arctan\left(\frac{1}{1+n^2}\right) = \arg\left[\frac{\pi}{\sinh(\pi)} \frac{\sinh[\pi(u + iv)]}{u + iv}\right]$$

$$
\begin{aligned}
&= \arg\left(\frac{\pi\sinh(u+\mathrm{i}v)}{(u+\mathrm{i}v)\sinh(\pi)}\right) \\
&= \arg(\tanh u + \mathrm{i}\tan v) - \frac{1}{2}\arg(1+\mathrm{i}) \\
&= \arctan\left(\frac{\tan v}{\tanh u}\right) - \frac{\pi}{8} \\
&= \arctan\left(\frac{\tan\sqrt{\frac{1+\sqrt{2}}{2}}\pi}{\tan\sqrt{\frac{\sqrt{2}-1}{2}}\pi}\right) - \frac{\pi}{8}.
\end{aligned}
$$

利用公式 $\arctan a - \arctan b = \arctan\left(\dfrac{a-b}{1+ab}\right)$, $(a, b > 0)$ 以及下面几个等式

$$
\begin{aligned}
\frac{2}{n^2} &= \frac{2}{1+(n-1)(n+1)} \\
\frac{1}{2n^2} &= \frac{2}{1+(2n-1)(2n+1)} \\
\frac{2n}{n^4+n^2+2} &= \frac{2n}{1+(n^2-n+1)(n^2+n+1)}
\end{aligned}
$$

可求出下面的反正切和

$$
\begin{aligned}
\sum_{n=1}^{\infty}\arctan\left(\frac{2}{n^2}\right) &= \sum_{n=1}^{\infty}[\arctan(n+1)-\arctan(n-1)] = \frac{\pi}{2} \\
\sum_{n=1}^{\infty}\arctan\left(\frac{1}{2n^2}\right) &= \sum_{n=1}^{\infty}[\arctan(2n+1)-\arctan(2n-1)] = \frac{\pi}{4} \\
\sum_{n=1}^{\infty}\arctan\left(\frac{2n}{n^4+n^2+2}\right) &= \sum_{n=1}^{\infty}[\arctan(n^2+n+1)-\arctan(n^2-n+1)] = \frac{\pi}{4}
\end{aligned}
$$

□

71. 数列 $\{a_n\}$ 定义为 $a_1 = 2, a_2 = 8, a_n = 4a_{n-1} - a_{n-2} (n = 2, 3, \cdots)$, 求和 $\sum\limits_{n=1}^{\infty}\operatorname{arccot}(a_n^2)$.

解 利用递推式可得

$$
\begin{aligned}
&a_n(4a_{n-1}) = a_{n-1}a_n \\
\Rightarrow\ &a_n(a_n+a_{n-2}) = a_{n-1}(a_{n+1}+a_{n-1}) \\
\Rightarrow\ &a_n^2 - a_{n-1}a_{n+1} = a_{n-1}^2 - a_n a_{n-2}
\end{aligned}
$$

根据上述递推关系可得, 对 $\forall n \geqslant 2$,

$$
a_n^2 - a_{n-1}a_{n+1} = a_{n-1}^2 - a_n a_{n-2} = \cdots = a_2^2 - a_1 a_3 = 4.
$$

根据反余切公式 $\operatorname{arccot} a - \operatorname{arccot} b = \operatorname{arccot}\left(\dfrac{1+ab}{b-a}\right)$ 可得

$$
\operatorname{arccot}\left(\frac{a_{n+1}}{a_n}\right) - \operatorname{arccot}\left(\frac{a_n}{a_{n-1}}\right) = \operatorname{arccot}\left(\frac{1+\frac{a_{n+1}}{a_n}\cdot\frac{a_n}{a_{n-1}}}{\frac{a_n}{a_{n-1}}-\frac{a_{n+1}}{a_n}}\right)
$$

$$
\begin{aligned}
&= \operatorname{arccot}\left[\frac{a_n(a_{n-1}+a_{n+1})}{a_n^2-a_{n-1}a_{n+1}}\right] \\
&= \operatorname{arccot}\left[\frac{a_n(4a_n)}{4}\right] \\
&= \operatorname{arccot} a_n^2.
\end{aligned}
$$

由特征根方法可得 $\{a_n\}$ 的通项公式为 $a_n=\dfrac{1}{\sqrt{3}}\left[(2+\sqrt{3})^n-(2-\sqrt{3})^n\right]$, 于是

$$
\begin{aligned}
\sum_{n=1}^{\infty}\operatorname{arccot}(a_n^2) &= \lim_{n\to\infty}\sum_{k=1}^{n}\operatorname{arccot}(a_n^2) \\
&= \operatorname{arccot} a_1^2+\lim_{n\to\infty}\sum_{k=2}^{n}\left[\operatorname{arccot}\left(\frac{a_{k+1}}{a_k}\right)-\operatorname{arccot}\left(\frac{a_k}{a_{k-1}}\right)\right] \\
&= \operatorname{arccot} a_1^2+\lim_{n\to\infty}\left[\operatorname{acrcot}\left(\frac{a_{n+1}}{a_n}\right)-\operatorname{acrcot}\left(\frac{a_2}{a_1}\right)\right] \\
&= \lim_{n\to\infty}\operatorname{arccot}\left(\frac{a_{n+1}}{a_n}\right)=\operatorname{arccot}(2+\sqrt{3})=\frac{\pi}{12}.
\end{aligned}
$$

□

72. 计算积分

$$
\int_0^{\infty}\frac{(\arctan x)^3}{x^{\frac{3}{2}}}\mathrm{d}x.
$$

解 考虑积分 $I(a)=\displaystyle\int_0^{\pi}\frac{\cos(ax)}{\sin^b x}\mathrm{d}x=2\int_0^{\frac{\pi}{2}}\frac{\cos(2ax)}{\sin^b(2x)}\mathrm{d}x.$
那么

$$
\begin{aligned}
I''(a) &= -8\int_0^{\frac{\pi}{2}}x^2\frac{\cos(2ax)}{\sin^b(2x)}\mathrm{d}x, \\
I''\left(\frac{1}{2}\right)\bigg|_{b=\frac{1}{2}} &= -\frac{8}{\sqrt{2}}\int_0^{\frac{\pi}{2}}x^2\sqrt{\cot x}\mathrm{d}x=-4\sqrt{2}A.
\end{aligned}
$$

回到原积分

$$
\begin{aligned}
\int_0^{\infty}\frac{(\arctan x)^3}{x^{\frac{3}{2}}}\mathrm{d}x &= 6\int_0^{\infty}\frac{(\arctan x)^2}{\sqrt{x}(1+x^2)}\mathrm{d}x \\
&= 6\int_0^{\frac{\pi}{2}}x^2\sqrt{\cot x}\mathrm{d}x \\
&= 6A.
\end{aligned}
$$

由例 42 得 $I(a)=\dfrac{\pi\cdot 2^b\cdot\cos\left(\frac{\pi a}{2}\right)\Gamma(1-b)}{\Gamma\left(\frac{a}{2}-\frac{b}{2}+1\right)\Gamma\left(-\frac{a}{2}-\frac{b}{2}+1\right)}$, 且

$$
\begin{aligned}
I''\left(\frac{1}{2}\right)\bigg|_{b=\frac{1}{2}} &= -\frac{5\pi^3}{2}+\pi\ln^2 2+\pi^2\ln 2, \\
A &= \frac{\sqrt{2}}{8}\left(\frac{5\pi^3}{12}-\pi\ln^2 2-\pi^2\ln 2\right).
\end{aligned}
$$

最后得到

$$\int_0^{\infty}\frac{(\arctan x)^3}{x^{\frac{3}{2}}}\mathrm{d}x=6A=\frac{5\pi^3}{8\sqrt{2}}-\frac{3\pi\ln^2 2}{2\sqrt{2}}-\frac{3\pi^2\ln 2}{2\sqrt{2}}.$$

□

73. 计算积分

$$\int_0^{\infty}\frac{1}{\sinh^{\frac{1}{4}}x\cosh x}\mathrm{d}x.$$

解

$$\begin{aligned}
\int_0^{\infty}\frac{1}{\sinh^{\frac{1}{4}}x\cosh x}\mathrm{d}x&=\int_0^{\infty}\frac{1}{\sinh^{x}\frac{1}{4}(1+\sinh^2 x)}\mathrm{d}(\sinh x)\\
&=\int_0^{\infty}\frac{1}{x^{\frac{1}{4}}(1+x^2)}\mathrm{d}x\xrightarrow{t=\frac{1}{1+x^2}}\frac{1}{2}\int_0^1 t^{\frac{3}{8}}(1-t)^{\frac{1}{8}}\mathrm{d}x\\
&=\frac{1}{2}B\left(\frac{5}{8},\frac{3}{8}\right)=\frac{\pi}{2\sin\left(\frac{3}{8}\pi\right)}\\
&=\frac{\pi}{\sqrt{2+\sqrt{2}}}.
\end{aligned}$$

□

74. 计算积分

$$\int_0^{\infty}\frac{\ln x}{\sqrt{x}\sqrt{x+1}\sqrt{2x+1}}\mathrm{d}x.$$

解

$$\begin{aligned}
I&=-\int_0^{\infty}\frac{\ln(2u)}{\sqrt{u}\sqrt{u+1}\sqrt{2u+1}}\mathrm{d}u\\
&=-\int_0^{\infty}\frac{\ln(2)}{\sqrt{u}\sqrt{u+1}\sqrt{2u+1}}\mathrm{d}u-\int_0^{\infty}\frac{\ln(u)}{\sqrt{u}\sqrt{u+1}\sqrt{2u+1}}\mathrm{d}u\\
&=-\int_0^{\infty}\frac{\ln(2)}{\sqrt{u}\sqrt{u+1}\sqrt{2u+1}}\mathrm{d}u-I\\
\Rightarrow I&=-\frac{\ln 2}{2}\int_0^{\infty}\frac{\mathrm{d}x}{\sqrt{x}\sqrt{x+1}\sqrt{2x+1}}\\
&=-\frac{\ln 2}{2}\int_1^{\infty}\frac{\mathrm{d}u}{\sqrt{u-1}\sqrt{u+1}\sqrt{u}}\\
&=-\frac{\ln 2}{2}\int_1^{\infty}\frac{\mathrm{d}u}{\sqrt{u^2-1}\sqrt{u}}\\
&=-\frac{\ln 2}{2}\int_1^{0}\frac{t^{3/2}}{1-t^2}\frac{(-1)}{t^2}\mathrm{d}t\\
&=-\frac{\ln 2}{2}\int_0^{1}\frac{\mathrm{d}t}{\sqrt{t}\sqrt{1-t^2}}\\
&=-\frac{\ln 2}{4}\int_0^{1}\frac{\mathrm{d}w}{w^{3/4}\sqrt{1-w}}\\
&=-\frac{\ln 2}{4}B\left(\frac{1}{4},\frac{1}{2}\right)=\frac{\ln 2}{4}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{}\Gamma\left(\frac{3}{4}\right)
\end{aligned}$$

$$=-\frac{\ln 2}{2\sqrt{2}}\frac{\pi^{3/2}}{\Gamma^2\left(\frac{3}{4}\right)}.$$

□

75. 计算积分

$$\int_0^{\infty}\frac{\sin\left(\frac{\pi}{4}-x\right)}{\sqrt{x}(x^2+1)}\mathrm{d}x$$

解

$$\begin{aligned}
&\int_0^{\infty}\frac{\sin\left(\frac{\pi}{4}-x\right)}{\sqrt{x}(x^2+1)}\mathrm{d}x\\
&=\frac{\sqrt{2}}{2}\int_0^{\infty}\frac{\cos x-\sin x}{\sqrt{x}(x^2+1)}\mathrm{d}x=\frac{\sqrt{2}}{2}\int_{-\infty}^{\infty}\frac{\cos(x^2)-\sin(x^2)}{x^4+1}\mathrm{d}x\\
&=2\pi\mathrm{i}\cdot\frac{\sqrt{2}}{2}\left[\operatorname{Res}\left(\frac{\cos(x^2)-\sin(x^2)}{x^4+1},\frac{1+\mathrm{i}}{\sqrt{2}}\right)+\operatorname{Res}\left(\frac{\cos(x^2)-\sin(x^2)}{x^4+1},\frac{-1+\mathrm{i}}{\sqrt{2}}\right)\right]\\
&=\sqrt{2}\pi\mathrm{i}\cdot\left[-\frac{1+\mathrm{i}}{8}\left(\sqrt{2}\cosh(1)-\sqrt{2}\sinh(1)\right)+\frac{1+\mathrm{i}}{8}\left(\sqrt{2}\sinh(1)-\sqrt{2}\cosh(1)\right)\right]\\
&=\frac{\pi}{2\mathrm{e}}.
\end{aligned}$$

□

76. 计算积分

$$\int_0^1\frac{\ln(1+x)}{x(x^2+1)}\mathrm{d}x$$

解 令 $f(a)=\displaystyle\int_0^1\frac{\ln(1+ax)}{x(x^2+1)}\mathrm{d}x$, 则

$$f'(a)=\int_0^1\frac{\mathrm{d}x}{(1+ax)(1+x^2)}=\frac{\pi}{4(1+a^2)}-\frac{a\ln 2}{2(1+a^2)}+\frac{a\ln(1+a)}{1+a^2}.$$

于是

$$\begin{aligned}
I&=\int_0^1\frac{\ln(1+x)}{x(x^2+1)}\mathrm{d}x=f(1)=\int_0^1 f'(a)\mathrm{d}a\\
&=\frac{\pi^2}{16}-\frac{1}{4}\ln^2 2+\int_0^1\frac{a\ln(1+a)}{1+a^2}\mathrm{d}a
\end{aligned}$$

令 $g(b)=\displaystyle\int_0^1\frac{a\ln(1+ba)}{1+a^2}\mathrm{d}a$, 则

$$g'(b)=\int_0^1\frac{a^2}{(1+ba)(1+a^2)}\mathrm{d}a=\frac{\ln(1+b)}{b}-f'(b)$$

因此

$$g(1)=\int_0^1\frac{\ln(1+b)}{b}\mathrm{d}b-I=\frac{\pi^2}{12}-I$$

所以

$$I=\frac{\pi^2}{16}-\frac{1}{4}\ln^2 2+\frac{\pi^2}{12}-I\Rightarrow I=\frac{7\pi^2}{96}-\frac{\ln^2 2}{8}.$$

□

77. 计算积分

$$\int_0^\infty \frac{x^2}{2\mathrm{e}^x-1}\mathrm{d}x.$$

解

$$\begin{aligned}
\int_0^\infty \frac{x^2}{2\mathrm{e}^x-1}\mathrm{d}x &= \frac{1}{2}\int_0^\infty \frac{x^2\mathrm{e}^{-x}}{1-\frac{1}{2}\mathrm{e}^{-x}}\mathrm{d}x \\
&= \frac{1}{2}\sum_{n=0}^\infty \int_0^\infty x^2\mathrm{e}^{-x}\left(\frac{1}{2}\mathrm{e}^{-x}\right)\mathrm{d}x \\
&= \frac{1}{2}\sum_{n=0}^\infty \frac{1}{2^n}\int_0^\infty x^2\mathrm{e}^{-(n+1)x}\mathrm{d}x \\
&= \sum_{n=0}^\infty \frac{1}{2^n(n+1)^3} \\
&= \lim_{x\to\frac{1}{2}}\frac{1}{x}\sum_{n=0}^\infty \frac{x^{n+1}}{(n+1)^3} = 2\mathrm{Li}_3\left(\frac{1}{2}\right) \\
&= 2\cdot\frac{21\zeta(3)+4\ln^3 2-2\pi^2\ln 2}{24} \\
&= \frac{7}{4}\zeta(3)+\frac{1}{3}\ln^3 2-\frac{\pi^2}{6}\ln 2.
\end{aligned}$$

□

78. 计算积分

$$\int_0^{\frac{\pi}{2}} \ln(1+\sin x)\ln(1+\cos x)\tan x\mathrm{d}x.$$

解

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \ln(1+\sin x)\ln(1+\cos x)\tan x\mathrm{d}x \\
=& \int_0^1 \ln\left(1+\sqrt{1-x^2}\right)\frac{\ln(1+x)}{x}\mathrm{d}x \\
=& -\mathrm{Li}_2(-x)\ln\left(1+\sqrt{1-x^2}\right)\Big|_0^1+\int_0^1 \mathrm{Li}(-x)\left(\frac{1}{x}-\frac{1}{x\sqrt{1-x^2}}\right)\mathrm{d}x \\
=& \mathrm{Li}_3(-1)-\int_0^1\frac{\mathrm{Li}_2(-x)}{x\sqrt{1-x^2}}\mathrm{d}x \\
=& -\frac{3}{4}\zeta(3)+\frac{1}{2}\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^2}B\left(\frac{k}{2},\frac{1}{2}\right)
\end{aligned}$$

利用 $\dfrac{(-1)^{k-1}}{k^2}B\left(\dfrac{k}{2},\dfrac{1}{2}\right)x^{k-1}=\dfrac{\pi-2\arcsin x}{\sqrt{1-x^2}}$ 可得

$$\begin{aligned}
\frac{(-1)^{k-1}}{k^2}B\left(\frac{k}{2},\frac{1}{2}\right) &= \int_0^{\frac{\pi}{2}}\frac{\pi x\cos x-x^2\cos x}{\sin x}\mathrm{d}x \\
&= \frac{\pi^2}{2}\ln 2-\frac{\pi^2}{4}\ln 2+\frac{7}{8}\zeta(3)
\end{aligned}$$

$$= \frac{\pi^2}{4}\ln 2 + \frac{7}{8}\zeta(3)$$

因此

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \ln(1+\sin x)\ln(1+\cos x)\tan x\,\mathrm{d}x \\
=& -\frac{3}{4}\zeta(3) + \frac{1}{2}\left(\frac{\pi^2}{4}\ln 2 + \frac{7}{8}\zeta(3)\right) \\
=& \frac{\pi^2}{8}\ln 2 - \frac{5}{16}\zeta(3).
\end{aligned}$$

□

79. 计算积分

$$\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4}\arccos\left(\frac{2x}{1+x^2}\right)\mathrm{d}x.$$

解

$$\begin{aligned}
&\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4}\arccos\left(\frac{2x}{1+x^2}\right)\mathrm{d}x \\
=& \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\tan^4 t}{1-\tan^2 t}\left(\frac{\pi}{2}-t\right)\mathrm{d}t \\
=& \pi\int_0^{\frac{\pi}{6}} \frac{\tan^4 t}{1-\tan^2 t}\mathrm{d}t \\
=& -\pi\int_0^{\frac{\pi}{6}}(1+\tan^2 t)\mathrm{d}t + \pi\int_0^{\frac{\pi}{6}}\frac{1}{1-\tan^2 t}\mathrm{d}t \\
=& -\frac{\pi}{\sqrt{3}} + \pi\int_0^{\frac{\pi}{6}}\frac{1+\cos 2t}{2\cos 2t}\mathrm{d}t \\
=& -\frac{\pi}{\sqrt{3}} + \frac{\pi^2}{12} + \frac{\pi}{4}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right).
\end{aligned}$$

□

80. 计算积分

$$\int_0^{\infty} \mathrm{e}^{-at}\sin(bt)\frac{\ln t}{t}\mathrm{d}t.$$

解 考虑积分

$$\begin{aligned}
I(s) =& \int_0^{\infty} t^{s-1}\mathrm{e}^{-at}\sin(bt)\mathrm{d}t \\
=& \Im\int_0^{\infty}\mathrm{e}^{-at+\mathrm{i}bt}t^{s-1}\mathrm{d}t \\
=& \Im\left[\frac{1}{(a-\mathrm{i}b)^s}\int_0^{\infty}x^{s-1}\mathrm{e}^{-x}\mathrm{d}x\right] \quad (x=(a-\mathrm{i}b)t) \\
=& \Im\left[\frac{\Gamma(s)(a+\mathrm{i}b)^s}{(a^2+b^2)^s}\right] \\
=& \frac{\Gamma(s)}{(a^2+b^2)^s}\Im\left[\mathrm{e}^{\frac{s}{2}\ln(a^2+b^2)+\mathrm{i}s\arctan\left(\frac{b}{a}\right)}\right]
\end{aligned}$$

$$= \frac{\Gamma(s)}{(a^2+b^2)^{\frac{s}{2}}}\sin\left[s\arctan\left(\frac{b}{a}\right)\right]$$

那么

$$\int_0^{\infty} \mathrm{e}^{-at}\sin(bt)\frac{\ln t}{t}\mathrm{d}t = I'(0) = -\left(\frac{\ln(a^2+b^2)}{2}+\gamma\right)\arctan\left(\frac{b}{a}\right).$$

□

81. 设 $a,b>0$, 计算积分

$$\int_0^{\infty} \mathrm{e}^{-x^2}\left[\cos\left(\frac{a^2}{x^2}\right)+\sin\left(\frac{a^2}{x^2}\right)\right]\mathrm{d}x$$

解 由例 11 的结论得

$$\begin{aligned}\int_0^{\infty} \mathrm{e}^{-x^2+\mathrm{i}\frac{a^2}{x^2}}\mathrm{d}x &= \frac{\sqrt{\pi}}{2}\mathrm{e}^{-2a\sqrt{\mathrm{i}}}\\ &= \frac{\sqrt{\pi}}{2}\mathrm{e}^{-\sqrt{2}a(1-\mathrm{i})}\\ &= \frac{\sqrt{\pi}}{2}\mathrm{e}^{-\sqrt{2}a}\mathrm{e}^{\mathrm{i}\sqrt{2}a}\\ &= \frac{\sqrt{\pi}}{2}\mathrm{e}^{-\sqrt{2}a}\left[\cos\left(\sqrt{2}a\right)+\mathrm{i}\sin\left(\sqrt{2}a\right)\right].\end{aligned}$$

因此

$$\int_0^{\infty} \mathrm{e}^{-x^2}\left[\cos\left(\frac{a^2}{x^2}\right)+\sin\left(\frac{a^2}{x^2}\right)\right]\mathrm{d}x = \frac{\sqrt{\pi}}{2}\mathrm{e}^{-\sqrt{2}a}\left[\cos\left(\sqrt{2}a\right)+\sin\left(\sqrt{2}a\right)\right].$$

□

82. 计算积分

$$\int_0^{\infty} \sin(x^2)\operatorname{erf}(x)\mathrm{d}x.$$

解 考虑积分 $f(a) = \int_0^{\infty} \sin(x^2)\operatorname{erf}(ax)\mathrm{d}x$, 则

$$\begin{aligned}f'(a) &= \frac{2}{\sqrt{\pi}}\int_0^{\infty} x\mathrm{e}^{-a^2x^2}\sin(x^2)\mathrm{d}x\\ &= \frac{1}{\sqrt{\pi}}\int_0^{\infty} t\mathrm{e}^{-a^2t}\sin t\,\mathrm{d}t\\ &= \frac{1}{\sqrt{\pi}(1+a^4)}.\end{aligned}$$

$$\begin{aligned}\int_0^{\infty} \sin(x^2)\operatorname{erf}(x)\mathrm{d}x = I(1) &= \int_0^1 \frac{1}{\sqrt{\pi}(1+a^4)}\mathrm{d}a + I(0)\\ &= \frac{\pi+2\coth^{-1}\left(\sqrt{2}\right)}{4\sqrt{2\pi}} + \frac{1}{2}\int_0^1 \sin(x^2)\mathrm{d}x\\ &= \frac{\pi+\coth^{-1}\left(\sqrt{2}\right)}{2\sqrt{2\pi}}.\end{aligned}$$

□

83. 计算积分

$$\int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{2x}{3}\right)}{\tan x}\mathrm{d}x.$$

解

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{2x}{3}\right)}{\tan x}\mathrm{d}x &= \frac{3}{2}\int_0^{\frac{\pi}{3}} \frac{\sin x}{\tan\left(\frac{3x}{2}\right)}\mathrm{d}x \\
&= \frac{3}{2}\int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{2u}{1+u^2}}{\frac{3u-u^3}{1-3u^2}}\frac{2\mathrm{d}u}{1+u^2} \\
&= 6\int_0^{\frac{1}{\sqrt{3}}} \frac{1-3u^2}{(3-u^2)(1+u^2)^2}\mathrm{d}u \\
&= \left[\left(\frac{6u}{u^2+1} + \sqrt{3}\ln\left(\sqrt{3}-u\right) - \sqrt{3}\ln\left(u+\sqrt{3}\right)\right)\right]\Bigg|_0^{\frac{1}{\sqrt{3}}} \\
&= \frac{\sqrt{3}}{4}(3-2\ln 2).
\end{aligned}$$

□

84. 已知 $K(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}$, 计算积分

$$\int_0^1 \frac{K(k)}{1+k}\mathrm{d}k$$

解

$$\begin{aligned}
I &= \int_0^1 \frac{K(k)}{1+k}\mathrm{d}k \\
&= \int_0^1 \frac{1}{\sqrt{1-t^2}}\mathrm{d}t \int_0^1 \frac{1}{(1+k)\sqrt{1-k^2t^2}}\mathrm{d}k \\
&= \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{\ln\left(1+\sqrt{1-t^2}\right)}{\sqrt{1-t^2}}\mathrm{d}t \\
&= \sum_{n=0}^{\infty}\int_0^1 t^{2n}\ln\left(1+\sqrt{1-t^2}\right)\mathrm{d}t \\
&= \sum_{n=0}^{\infty} \frac{\sqrt{\pi}\left(n+\frac{1}{2}\right)! - n!}{(2n+1)^2 n!} \\
&= \frac{\pi^2}{8}.
\end{aligned}$$

□

85. 求和

$$\sum_{n=1}^{\infty} \frac{1}{2^n\left(1+\sqrt[2^n]{2}\right)}.$$

解 首先注意到

$$\frac{1}{2^n\left(\sqrt[2^n]{2}-1\right)} - \frac{1}{2^n\left(\sqrt[2^n]{2}+1\right)} = \frac{1}{2^{n-1}\left(\sqrt[2^{n-1}]{2}-1\right)}.$$

于是得到

$$\frac{1}{2^n\left(\sqrt[2^n]{2}+1\right)}=\left[\frac{1}{2^n\left(\sqrt[2^n]{2}-1\right)}-1\right]-\left[\frac{1}{2^{n-1}\left(\sqrt[2^{n-1}]{2}-1\right)}-1\right]$$

且当 $n=1$ 时,

$$\frac{1}{2^{n-1}\left(\sqrt[2^{n-1}]{2}-1\right)}-1=0.$$

因此可求得部分和

$$\sum_{n=1}^{m}\frac{1}{2^n\left(1+\sqrt[2^n]{2}\right)}=\frac{1}{2^m\left(\sqrt[2^m]{2}-1\right)}-1.$$

令 $m\to\infty$ 可得

$$\sum_{n=1}^{\infty}\frac{1}{2^n\left(1+\sqrt[2^n]{2}\right)}=\frac{1}{\ln 2}-1.$$

□

86. 计算积分

$$\int_0^1\frac{x^{p-1}-x^{q-1}}{1-x}\mathrm{d}x.$$

解 首先考虑积分

$$\begin{aligned}I(s)&=\int_0^1 x^{p-1}(1-x)^{1-s}\mathrm{d}x-\int_0^1 x^{q-1}(1-x)^{1-s}\mathrm{d}x\\&=B(p,s)-B(q,s)\\&=\Gamma(s)\left[\frac{\Gamma(p)}{\Gamma(p+s)}-\frac{\Gamma(q)}{\Gamma(q+s)}\right]\\&=\Gamma(1+s)\left[\frac{\Gamma(p)-\Gamma(p+s)}{s}-\frac{\Gamma(q)-\Gamma(q+s)}{s}\right].\end{aligned}$$

于是

$$\int_0^1\frac{x^{p-1}-x^{q-1}}{1-x}\mathrm{d}x=\lim_{s\to0}I(s)=\psi(q)-\psi(p).$$

特别地,

$$\int_0^1\frac{1-x^{q-1}}{1-x}\mathrm{d}x=\gamma+\psi(q),$$

$$\int_0^1\frac{x^{p-1}-x^{-p}}{1-x}\mathrm{d}x=\psi(1-p)-\psi(p)=\pi\cot\pi p.$$

□

87. 求和

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(k+1)(2k+1)}.$$

解 根据 $\cot x$ 的幂级数展开得到

$$\sum_{k=1}^{\infty}\zeta(2k)\,x^{2k}=\frac{1-\pi x\cot\pi x}{2}.$$

于是逐项积分得

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)x^{2k+2}}{(2k+1)(2k+2)}=\int_0^x \mathrm{d}v\int_0^v \frac{1-\pi u\cot\pi u}{2}\mathrm{d}u.$$

注意到

$$\begin{aligned}
\int_0^1 \mathrm{d}v\int_0^v \frac{1-\pi u\cot\pi u}{2}\mathrm{d}u &= -\int_0^1 \mathrm{d}v\int_0^v \frac{\pi u\cot\pi u}{2}\mathrm{d}u+\int_0^1 \mathrm{d}v\int_0^v \frac{\mathrm{d}u}{2}\\
&=-\frac{1}{2}\int_0^1 \pi u\cot\pi u\mathrm{d}u\int_u^1 \mathrm{d}v+\frac{1}{4}\\
&=\frac{1}{4}-\frac{\pi}{2}\int_0^1 u(1-u)\cot\pi u\mathrm{d}u\\
&=\frac{1}{4}+\frac{\pi}{2}\int_0^1 u(1-u)\cot\pi u\mathrm{d}u\\
&=\frac{1}{4}.
\end{aligned}$$

因此 $\displaystyle\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(k+1)(2k+1)}=\frac{1}{2}$.

□

88. 求和

$$\sum_{k=1}^{\infty}\frac{1}{k^4+k^2+1}.$$

解 令 $\omega=\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$, 则

$$n^4+n^2+1=(n^2-\omega)(n^2-\omega^2),$$

于是

$$\frac{1}{1+n^2+n^4}=\frac{1}{\mathrm{i}\sqrt{3}}\left(\frac{1}{n^2-\omega}-\frac{1}{n^2-\omega^2}\right).$$

利用

$$\sum_{n=1}^{\infty}\frac{1}{n^2+a}=\frac{-1+\pi\sqrt{a}\coth\left(\pi\sqrt{a}\right)}{2a}$$

可得

$$\sum_{k=1}^{\infty}\frac{1}{k^4+k^2+1}=\frac{1}{\sqrt{3}}\Im\left(\sum_{n=1}^{\infty}\frac{1}{n^2-\omega}\right)=\frac{1}{6}\left[-3+\pi\sqrt{3}\tanh\left(\frac{\pi\sqrt{3}}{2}\right)\right].$$

□

89. 求和

$$\sum_{n=1}^{\infty}\frac{\sinh\pi}{\cosh(2n\pi)-\cosh\pi}.$$

解 在等式

$$\sum_{n=1}^{\infty}\mathrm{e}^{-nt}\sin nx=\frac{1}{2}\frac{\sin x}{\cosh t-\cos x}$$

中令 $x=\pi\mathrm{i}, t=2k\pi$ 得

$$\frac{\sinh\pi}{\cosh(2k\pi)-\cosh\pi}=2\sum_{n=1}^{\infty}\mathrm{e}^{-2nk\pi}\sinh(n\pi).$$

于是

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sinh\pi}{\cosh(2n\pi)-\cosh\pi}&=2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mathrm{e}^{-2nk\pi}\sinh(n\pi)\\&=2\sum_{n=1}^{\infty}\sinh(n\pi)\sum_{k=1}^{\infty}\mathrm{e}^{-2nk\pi}\\&=2\sum_{n=1}^{\infty}\frac{\sinh(n\pi)}{\mathrm{e}^{2n\pi}-1}\\&=\sum_{n=1}^{\infty}\mathrm{e}^{-n\pi}=\frac{1}{\mathrm{e}^{\pi}-1}.\end{aligned}$$

□

90. 求和

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)-\zeta(3n)}{n}.$$

解

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\zeta(2n)-\zeta(3n)}{n}&=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{\infty}\left(\frac{1}{k^{2n}}-\frac{1}{k^{3n}}\right)\\&=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=2}^{\infty}\left(\frac{1}{k^{2n}}-\frac{1}{k^{3n}}\right)\\&=\sum_{k=2}^{\infty}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{k^2}\right)^n-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{k^3}\right)^n\right]\\&=\sum_{k=2}^{\infty}\left[-\ln\left(1-\frac{1}{k^2}\right)+\ln\left(1-\frac{1}{k^3}\right)\right]\\&=-\ln\left[\prod_{k=2}^{\infty}\ln\left(1-\frac{1}{k^2}\right)\right]+\ln\left[\prod_{k=2}^{\infty}\ln\left(1-\frac{1}{k^3}\right)\right]\\&=-\ln\left(\frac{1}{2}\right)+\ln\left[\frac{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{3\pi}\right]\\&=\ln\left[\frac{2\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{3\pi}\right].\end{aligned}$$

□

91. 求和

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{n^3+1}.$$

解

$$
\begin{aligned}
S&=\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{n^3+1}\\
&=\sum_{n=1}^{\infty}(-1)^n\frac{1}{3}\left(\frac{2n-1}{n^2-n+1}+\frac{1}{n+1}\right)\\
&=\frac{1}{3}\sum_{n=1}^{\infty}(-1)^n\frac{2n-1}{n^2+n+1}+\frac{1}{3}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n+1}\\
&=\frac{1}{3}N+\frac{1}{3}(\ln 2-1).
\end{aligned}
$$

其中 $N=\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{2n-1}{n^2-n+1}$, 考虑函数 $f(z)=\pi\csc(\pi z)\dfrac{2z-1}{z^2-z+1}$,

$$
\begin{aligned}
\sum_{n=1}^{\infty}\operatorname{Res}(f(z),z=n)&=\sum_{n=1}^{\infty}\frac{2n-1}{n^2-n+1}\lim_{z\to n}\frac{\pi}{\sin(\pi z)}(z-n)\\
&=\sum_{n=1}^{\infty}\frac{2n-1}{n^2-n+1}\lim_{z\to n}\frac{\pi}{\pi\cos(\pi z)}\\
&=\sum_{n=1}^{\infty}(-1)^n\frac{2n-1}{n^2-n+1}=N.
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Res}\left(f(z),z=\frac{1+\sqrt{3}\mathrm{i}}{2}\right)&=\frac{\pi\left(2\frac{1+\sqrt{3}\mathrm{i}}{2}-1\right)}{\sin\left(\pi\frac{1+\sqrt{3}\mathrm{i}}{2}\right)}\lim_{z\to\frac{1+\sqrt{3}\mathrm{i}}{2}}\frac{z-\frac{1+\sqrt{3}\mathrm{i}}{2}}{z^2-z+1}\\
&=\frac{\sqrt{3}\pi\mathrm{i}}{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}\cdot\frac{-\mathrm{i}}{\sqrt{3}}=\frac{\pi}{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}\\
&\Rightarrow N+\frac{\pi}{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}=0.
\end{aligned}
$$

于是

$$
S=\frac{1}{3}\left(-\frac{\pi}{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}\right)+\frac{1}{3}(\ln 2-1)=\frac{1}{3}\left(\ln 2-1-\frac{\pi}{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}\right).
$$

□

92. 计算积分

$$
\int_0^{\infty}\frac{x\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}-1}\mathrm{d}x.
$$

解

$$
\begin{aligned}
\int_0^{\infty}\frac{x\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}-1}\mathrm{d}x&=\int_0^{\infty}\frac{x\mathrm{e}^{-2x}}{\mathrm{e}^{-2x}-\mathrm{e}^{-x}+1}\mathrm{d}x\\
&=\int_0^{\infty}\frac{x\mathrm{e}^{-2x}(1+\mathrm{e}^{-x})}{1+\mathrm{e}^{-3x}}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty \sum_{n=0}^\infty (-1)^n \mathrm{e}^{-(2n+1)x}(1+\mathrm{e}^{-x})x\mathrm{d}x \\
&= \sum_{n=0}^\infty (-1)^n \int_0^\infty \left[x\mathrm{e}^{-(3n+2)x} + x\mathrm{e}^{-(3n+3)x}\right]\mathrm{d}x \\
&= \sum_{n=0}^\infty (-1)^n \left[\frac{1}{(3n+2)^2} + \frac{1}{(3n+3)^2}\right] \\
&= \frac{1}{36}\left[\psi'\left(\frac{1}{3}\right) - \psi'\left(\frac{5}{6}\right)\right] + \frac{\pi^2}{108}.
\end{aligned}
$$

□

93. 设 $a > 0$, 计算积分

$$\int_0^\infty \left[\mathrm{e}^{-x} - (1+x)^{-a}\right]\frac{\mathrm{d}x}{x}.$$

解 首先根据 Frullani 积分可得

$$\int_0^\infty \int_1^s \mathrm{e}^{-tz}\mathrm{d}t\mathrm{d}z = \int_0^\infty \frac{\mathrm{e}^{-z} - \mathrm{e}^{-sz}}{z}\mathrm{d}z = \ln s.$$

于是可得

$$
\begin{aligned}
\Gamma'(a) &= \int_0^\infty \mathrm{e}^{-s}s^{a-1}\ln s\mathrm{d}s \\
&= \int_0^\infty \mathrm{e}^{-s}s^{a-1}\int_0^\infty \frac{\mathrm{e}^{-z} - \mathrm{e}^{-zs}}{z}\mathrm{d}z\mathrm{d}s \\
&= \int_0^\infty \left(\mathrm{e}^{-z}\int_0^\infty s^{a-1}\mathrm{e}^{-s}\mathrm{d}s - \int_0^\infty s^{a-1}\mathrm{e}^{-s(1+z)}\mathrm{d}s\right)\frac{\mathrm{d}z}{z} \\
&= \Gamma(a)\int_0^\infty \left[\mathrm{e}^{-x} - (1+x)^{-a}\right]\frac{\mathrm{d}x}{x}
\end{aligned}
$$

因此

$$\int_0^\infty \left[\mathrm{e}^{-x} - (1+x)^{-a}\right]\frac{\mathrm{d}x}{x} = \frac{\Gamma'(a)}{\Gamma(a)} = \psi(a).$$

□

94. 计算积分

$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x}\right)\mathrm{d}x.$$

解 首先考虑积分

$$
\begin{aligned}
I(s) &= \int_0^1 \left(\frac{x^{q-1}}{(1-ax)^{1-s}} - \frac{x^{-q}}{(a-x)^{1-s}}\right)\mathrm{d}x \\
&= \int_0^1 \frac{x^{q-1}}{(1-ax)^{1-s}}\mathrm{d}x - \int_0^1 \frac{x^{-q}}{(a-x)^{1-s}}\mathrm{d}x \\
&= a^{-q}\int_0^a \frac{t^{q-1}}{(1-t)^{1-s}}\mathrm{d}t - a^{-q+s}\int_0^{\frac{1}{a}} \frac{t^{-q}}{(1-t)^{1-s}}\mathrm{d}t.
\end{aligned}
$$

令 $s \to 0$ 得到

$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x}\right)\mathrm{d}x = a^{-q}\left(\int_0^a \frac{t^{q-1}}{1-t}\mathrm{d}t - \int_0^{\frac{1}{a}} \frac{t^{-q}}{1-t}\mathrm{d}t\right).$$

上式右端括号的部分对参数 a 求导为零, 因此是一个独立于 a 的常数, 于是

$$\begin{aligned}\int_0^1\left(\frac{x^{q-1}}{1-ax}-\frac{x^{-q}}{a-x}\right)\mathrm{d}x&=a^{-q}\left(\int_0^1\frac{t^{q-1}}{1-t}\mathrm{d}t-\int_0^1\frac{t^{-q}}{1-t}\mathrm{d}t\right)\\&=a^{-q}\int_0^1\frac{t^{q-1}-t^{-q}}{1-t}\mathrm{d}t\\&=a^{-q}\left[\psi(1-q)-\psi(q)\right]\\&=a^{-q}\pi\cot\pi q.\end{aligned}$$

□

95. 计算积分

$$\int_0^\infty\frac{\ln^2x}{a^2+x^2}\mathrm{d}x.$$

解 方法一

$$\begin{aligned}\int_0^\infty\frac{\ln^2x}{a^2+x^2}\mathrm{d}x&=\lim_{s\to0}\partial_s^2\int_0^\infty\frac{x^s}{a^2+x^2}\mathrm{d}x\\&=\frac{1}{a}\lim_{s\to0}\partial_s^2\left(a^s\int_0^\infty\frac{x^s}{1+x^2}\mathrm{d}x\right)\\&=\frac{1}{2a}\lim_{s\to0}\partial_s^2\left[a^s\int_0^1t^{-\frac{s+1}{2}}(1-t)^{\frac{s+1}{2}-1}\mathrm{d}t\right]\quad\left(t=\frac{1}{1+x^2}\right)\\&=\frac{1}{2a}\lim_{s\to0}\partial_s^2\left[a^sB\left(\frac{s+1}{2},1-\frac{s+1}{2}\right)\right]\\&=\frac{1}{2a}\lim_{s\to0}\partial_s^2\left[a^s\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)\right]\\&=\frac{\pi}{2a}\lim_{s\to0}\partial_s^2\left[a^s\sec\left(\frac{\pi s}{2}\right)\right]\\&=\frac{\pi}{8a}\left(\pi^2+4\ln^2a\right).\end{aligned}$$

方法二

$$\begin{aligned}\int_0^\infty\frac{\ln^2x}{a^2+x^2}\mathrm{d}x&=\frac{1}{a}\int_0^{\frac{\pi}{2}}\ln^2(a\tan t)\mathrm{d}t\quad(x=a\tan t)\\&=\frac{1}{a}\int_0^{\frac{\pi}{2}}\left[\ln^2a+\ln^2(\tan t)+2\ln a\ln(\tan t)\right]\mathrm{d}t\\&=\frac{1}{a}\left(\frac{\pi^3}{8}+\frac{\pi}{2}\ln^2a\right).\end{aligned}$$

其中 $\displaystyle\int_0^{\frac{\pi}{2}}\ln^2(\tan t)\mathrm{d}t=\frac{\pi^3}{8}$ 参见例 6.

□

96. 计算积分

$$\int_0^\infty\frac{\sinh x}{x\cosh^3x}\mathrm{d}x.$$

解

$$\int_0^\infty\frac{\sinh x}{x\cosh^3x}\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sinh x}{x\cosh^3 x}\mathrm{d}x \\
&= \pi\mathrm{i}\sum_{k=0}^{\infty}\operatorname{Res}\left(\frac{\sinh x}{x\cosh^3 x}, x=\frac{\pi\mathrm{i}}{2}+2k\pi\mathrm{i}\right)+\pi\mathrm{i}\sum_{k=1}^{\infty}\operatorname{Res}\left(\frac{\sinh x}{x\cosh^3 x}, x=-\frac{\pi\mathrm{i}}{2}+2k\pi\mathrm{i}\right) \\
&= \pi\mathrm{i}\left(-\frac{8\mathrm{i}}{\pi^3}\right)\pi^3\left(\sum_{k=0}^{\infty}\frac{1}{(4k+1)^3}+\sum_{k=0}^{\infty}\frac{1}{(4k+3)^3}\right) \\
&= \frac{7}{\pi^2}\zeta(3).
\end{aligned}
$$

□

97. 计算积分

$$
\int_0^{\infty} \mathrm{e}^{-x^2}\ln x\,\mathrm{d}x.
$$

解

$$
\begin{aligned}
\int_0^{\infty} \mathrm{e}^{-x^2}\ln x\,\mathrm{d}x &= \frac{1}{4}\int_0^{\infty}\mathrm{e}^{-t}t^{-\frac{1}{2}}\ln t\,\mathrm{d}t \\
&= \frac{1}{4}\lim_{s\to 1}\partial_s\int_0^{\infty}\mathrm{e}^{-t}t^{s-1-\frac{1}{2}}\mathrm{d}t \\
&= \frac{1}{4}\lim_{s\to 1}\partial_s\left[\Gamma\left(s-\frac{1}{2}\right)\right] \\
&= \frac{1}{4}\lim_{s\to 1}\Gamma\left(s-\frac{1}{2}\right)\psi\left(s-\frac{1}{2}\right) \\
&= \frac{1}{4}\Gamma\left(\frac{1}{2}\right)\psi\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{\pi}}{4}(\gamma+2\ln 2).
\end{aligned}
$$

□

98. 计算积分

$$
\int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x}\mathrm{d}x.
$$

解 方法一首先有

$$
\frac{1}{\sin x} = \frac{2\mathrm{i}\mathrm{e}^{-\mathrm{i}x}}{1-\mathrm{e}^{-2\mathrm{i}x}} = 2\mathrm{i}\mathrm{e}^{-\mathrm{i}x}\sum_{k=0}^{\infty}\mathrm{e}^{-2k\mathrm{i}x} = 2\mathrm{i}\sum_{k=0}^{\infty}\mathrm{e}^{-(2k+1)\mathrm{i}x}.
$$

于是

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x}\mathrm{d}x &= 2\mathrm{i}\sum_{k=0}^{\infty}\int_0^{\frac{\pi}{2}} x^2\mathrm{e}^{-(2k+1)\mathrm{i}x}\mathrm{d}x \\
&= 2\mathrm{i}\sum_{k=0}^{\infty}(-1)^k\frac{\left[(2\pi k+\pi)^2+8\mathrm{i}(-1)^k-4\mathrm{i}\pi(2k+1)-8\right]}{4(2k+1)^2} \\
&= 2\pi G-\frac{7}{2}\zeta(3).
\end{aligned}
$$

方法二

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{x^2}{\sin x}\mathrm{d}x&=\int_0^{\frac{\pi}{2}}x^2\mathrm{d}\left[\ln\left(\tan\frac{x}{2}\right)\right]\\&=-2\int_0^{\frac{\pi}{2}}x\ln\left(\tan\frac{x}{2}\right)\mathrm{d}x\\&=4\int_0^{\frac{\pi}{2}}x\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{2k-1}\mathrm{d}x\\&=4\left[\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}-\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\right]\\&=2\pi G-\frac{7}{2}\zeta(3).\end{aligned}$$

□

99. 计算积分

$$\int_0^1\frac{1}{\left(1+x^{\sqrt{2}}\right)^{\sqrt{2}}}\mathrm{d}x.$$

解

$$\begin{aligned}\int_0^1\frac{1}{\left(1+x^{\sqrt{2}}\right)^{\sqrt{2}}}\mathrm{d}x&=\frac{1}{\sqrt{2}}\int_0^1\frac{t^{\frac{\sqrt{2}}{2}-1}}{(1+t)^{\sqrt{2}}}\mathrm{d}t\quad\left(t=x^{\sqrt{2}}\right)\\&=\frac{1}{\sqrt{2}}\int_0^{\frac{1}{2}}\frac{u^{\frac{\sqrt{2}}{2}-1}(1-u)^{\sqrt{2}}}{(1-u)^{\frac{\sqrt{2}}{2}-1}(1-u)^2}\mathrm{d}u\quad\left(u=\frac{t}{1+t}\right)\\&=\frac{1}{\sqrt{2}}\int_{\frac{1}{2}}^1\frac{v^{\frac{\sqrt{2}}{2}-1}(1-v)^{\sqrt{2}}}{(1-v)^{\frac{\sqrt{2}}{2}-1}(1-v)^2}\mathrm{d}v\quad(u=1-v)\\&=\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\int_0^1u^{\frac{\sqrt{2}}{2}-1}(1-u)^{\frac{\sqrt{2}}{2}-1}\mathrm{d}u\\&=\frac{1}{2\sqrt{2}}B\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\\&=\frac{1}{2\sqrt{2}}\frac{\Gamma\left(\frac{1}{\sqrt{2}}\right)\Gamma\left(\frac{1}{\sqrt{2}}\right)}{\Gamma(\sqrt{2})}\\&=\frac{1}{2\sqrt{2}}\frac{\Gamma^2\left(\frac{1}{\sqrt{2}}\right)}{\Gamma(\sqrt{2})}.\end{aligned}$$

□

100. 计算积分

$$\int_0^{\infty}\frac{\ln^2 x}{1+x^4}\mathrm{d}x.$$

解

$$\int_0^{\infty}\frac{\ln^2 x}{1+x^4}\mathrm{d}x=\lim_{s\to 0}\partial_s^2\int_0^{\infty}\frac{x^s}{1+x^4}\mathrm{d}x$$

$$
\begin{aligned}
&\frac{1}{4}\lim_{s\to 0}\partial_s^2\int_0^{\infty} t^{\frac{s+1}{4}-1}(1-t)^{-\frac{s+1}{4}}\,\mathrm{d}t \\
&=\frac{1}{4}\lim_{s\to 0}\partial_s^2\left[B\left(\frac{s+1}{4},1-\frac{s+1}{4}\right)\right] \\
&=\frac{1}{4}\lim_{s\to 0}\partial_s^2\left[\Gamma\left(\frac{s+1}{4}\right)\Gamma\left(1-\frac{s+1}{4}\right)\right] \\
&=\frac{\pi}{4}\lim_{s\to 0}\partial_s^2\left[\csc\left(\frac{s+1}{4}\pi\right)\right] \\
&=\frac{\pi^3}{64}\lim_{s\to 0}\left[1+2\cot^2\left(\frac{s+1}{4}\pi\right)\csc\left(\frac{s+1}{4}\pi\right)\right] \\
&=\frac{\pi^3}{64}\left[1+2\cot^2\left(\frac{\pi}{4}\right)\csc\left(\frac{\pi}{4}\right)\right] \\
&=\frac{3\sqrt{2}\pi^3}{64}.
\end{aligned}
$$

□

101. 计算积分

$$\int_0^{\infty}\frac{\mathrm{e}^{a\cos x}\sin(a\sin x)}{x}\mathrm{d}x.$$

解

$$
\begin{aligned}
\int_0^{\infty}\frac{\mathrm{e}^{a\cos x}\sin(a\sin x)}{x}\mathrm{d}x &= \Im\int_0^{\infty}\frac{\mathrm{e}^{a\mathrm{e}^{\mathrm{i}x}}}{x}\mathrm{d}x \\
&= \Im\int_0^{\infty}\frac{1}{x}\sum_{k=0}^{\infty}\frac{a^k\mathrm{e}^{\mathrm{i}kx}}{k!}\mathrm{d}x \\
&= \Im\int_0^{\infty}\frac{1}{x}\sum_{k=0}^{\infty}\frac{a^k\sin(kx)}{k!}\mathrm{d}x \\
&= \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{a^k}{k!} \\
&= \frac{\pi}{2}(\mathrm{e}^a-1).
\end{aligned}
$$

□

102. 计算积分

$$\int_0^{\infty}\frac{\sin x}{1+\cosh x-\sinh x}\mathrm{d}x.$$

解

$$
\begin{aligned}
\int_0^{\infty}\frac{\sin x}{1+\cosh x-\sinh x}\mathrm{d}x &= \int_0^{\infty}\frac{\sin x}{1+\mathrm{e}^{-x}}\mathrm{d}x \\
&= \sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}\mathrm{e}^{-nx}\sin x\,\mathrm{d}x = \sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1} \\
&= \sum_{n=0}^{\infty}\frac{1}{1+(2n)^2}-\sum_{n=0}^{\infty}\frac{1}{1+(2n+1)^2} \\
&= \frac{1}{2}+\frac{\pi}{4}\coth\left(\frac{\pi}{2}\right)-\frac{\pi}{4}\tanh\left(\frac{\pi}{2}\right)
\end{aligned}
$$

$$= \frac{1}{2} + \frac{\pi}{2}\operatorname{csch}\pi.$$

□

103. 计算积分

$$\int_0^\infty \frac{\sin(x^n)\ln x}{x}\mathrm{d}x.$$

解 令 $u = x^n$, 则原积分 $I = \displaystyle\int_0^\infty \frac{\sin u \ln u}{u}\mathrm{d}u$, 考虑积分

$$\begin{aligned} f(t) &= \int_0^\infty \frac{\sin x}{x^t}\mathrm{d}x \\ &= \frac{1}{\Gamma(t)}\int_0^\infty \left(\int_0^\infty \mathrm{e}^{-xu}\sin x\mathrm{d}x\right)u^{t-1}\mathrm{d}u \\ &= \frac{1}{\Gamma(t)}\int_0^\infty \frac{u^{t-1}}{1+u^2}\mathrm{d}u \\ &= \frac{1}{\Gamma(t)}\int_0^{\frac{\pi}{2}} \tan^{t-1}\theta\mathrm{d}\theta \\ &= \frac{1}{2\Gamma(t)}\frac{\pi}{\sin\left(\frac{\pi t}{2}\right)}. \end{aligned}$$

于是

$$\begin{aligned} I = -f'(1) &= \frac{\pi}{2}\frac{\psi(t)\Gamma(t)\sin\left(\frac{\pi t}{2}\right) - \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)\Gamma(t)}{\left[\Gamma(t)\sin\left(\frac{\pi t}{2}\right)\right]^2}\Bigg|_{t=1} \\ &= \frac{\pi}{2}\psi(1) = -\frac{\gamma\pi}{2}. \end{aligned}$$

□

104. 设 $f:[0,1]\to\mathbb{R}$ 是连续函数, 且 $\displaystyle\int_0^1 f^3(x)\mathrm{d}x = 0$. 求证:

$$\int_0^1 f^4(x)\mathrm{d}x \geqslant \frac{27}{4}\left(\int_0^1 f(x)\mathrm{d}x\right)^4.$$

解 令

$$I_n = \int_0^1 f^n(x)\mathrm{d}x$$

由 Cauchy 不等式得

$$I_2 \geqslant I_1^2$$

再由 Cauchy 不等式得

$$\left(\int_0^1 \left(r + f^2(x)\right)f(x)\mathrm{d}x\right)^2 \leqslant \int_0^1 \left(r+f^2(x)\right)^2\mathrm{d}x\int_0^1 f^2(x)\mathrm{d}x$$

展开得到

$$r^2I_1^2 \leqslant r^2I_2 + 2rI_2^2 + I_2I_4$$

也即

$$(I_1^2-I_2)r^2-2I_2^2r-I_2I_4\leqslant 0$$

于是上式左边的最大值也小于等于 0, 最大值在 $r=\dfrac{I_2^2}{I_1^2-I_2}$ 取到, 即满足

$$\frac{I_4^4}{I_1^2-I_2}-\frac{2I_2^4}{I_1^2-I_2}-I_2I_4\leqslant 0$$

即

$$I_4\geqslant\frac{I_2^3}{I_2-I_1^2}$$

所以只要证明

$$\frac{I_2^3}{I_2-I_1^2}\geqslant\frac{27}{4}I_1^4$$

注意到

$$(I_2-I_1^2)I_1^4=\frac{1}{2}(2I_2-2I_1^2)I_1^2\cdot I_1^2\leqslant\frac{4}{27}I_2^3$$

即

$$\frac{I_2^3}{I_2-I_1^2}\geqslant\frac{27}{4}I_1^4$$

故有

$$\int_0^1 f^4(x)\mathrm{d}x\geqslant\frac{27}{4}\left(\int_0^1 f(x)\mathrm{d}x\right)^4.$$

□

105. 设 f 是在 $[0,1]$ 上非负的连续的凹函数, 且 $f(0)=1$, 求证:

$$2\int_0^1 x^2f(x)\mathrm{d}x+\frac{1}{12}\leqslant\left(\int_0^1 f(x)\mathrm{d}x\right)^2$$

证明 设

$$F(x)=\int_0^x f(t)\mathrm{d}t, I=\int_0^1 x^2f(x)\mathrm{d}x, U=\int_0^1 f(x)\mathrm{d}x$$

由于

$$f(ax)=f(ax+(1-a)\cdot 0)\geqslant af(x)+(1-a)f(0)=af(x)+1-a$$

上式对 a 从 0 到 1 积分得

$$\int_0^1 f(tx)\mathrm{d}t\geqslant\frac{1}{2}f(x)+\frac{1}{2}$$

换元即得

$$2F(x)\geqslant xf(x)+x$$

另外我们有

$$I=\int_0^1 x^2f(x)\mathrm{d}x=x^2F(x)|_0^1-2\int_0^1 xF(x)\mathrm{d}x\leqslant F(1)-2\int_0^1 x(xf(x)+x)\mathrm{d}x$$

即

$$2I\leqslant F(1)-\frac{1}{3}=U-\frac{1}{3}$$

所以

$$U^2-2I-\frac{1}{12}\geqslant\left(2I+\frac{1}{3}\right)^2-2I-\frac{1}{12}=\left(2I-\frac{1}{6}\right)^2\geqslant 0$$

□

106. 设 $f:[0,1]\to\mathbb{R}$ 是可积函数, 且 $|f(x)|\leqslant 1$, $\int_0^1 xf(x)\mathrm{d}x=0$. 令 $F(x)=\int_0^x f(y)\mathrm{d}y\geqslant 0$, 求证:

$$\int_0^1 f^2(x)\mathrm{d}x+5\int_0^1 F^2(x)\mathrm{d}x\geqslant 10\int_0^1 f(x)F(x)\mathrm{d}x.$$

证明 由于

$$\int_0^1\left(\int_0^x f(y)\mathrm{d}y\right)\mathrm{d}x=\int_0^1\left(\int_y^1 f(y)\mathrm{d}x\right)\mathrm{d}y$$
$$\int_0^1\left(\int_0^x f(x)f(y)\mathrm{d}y\right)\mathrm{d}x=\int_0^1\left(\int_y^1 f(x)f(y)\mathrm{d}x\right)\mathrm{d}x$$

故有

$$\int_0^1 F(x)\mathrm{d}x=\int_0^1(1-y)f(y)\mathrm{d}y$$
$$\int_0^1 f(x)F(x)\mathrm{d}x=F^2(1)-\int_0^1 f(y)F(y)\mathrm{d}y$$

利用 $\int_0^1 xf(x)\mathrm{d}x=0$ 得

$$\int_0^1 F(x)\mathrm{d}x=\int_0^1 f(x)\mathrm{d}x=F(1),\ \int_0^1 f(x)F(x)\mathrm{d}x=\frac{1}{2}F^2(1)\geqslant 0$$

利用 $A^2+B^2\geqslant 0$ 则有

$$\int_0^1 f^2(x)\mathrm{d}x+\int_0^1 F^2(x)\mathrm{d}x\geqslant 2\int_0^1 f(x)F(x)\mathrm{d}x$$

另外由 Cauchy 不等式得

$$4\int_0^1 F^2(x)\mathrm{d}x\geqslant\left(\int_0^1 2F(x)\mathrm{d}x\right)^2=4F^2(1)=8\int_0^1 f(x)F(x)\mathrm{d}x$$

相加即证得原式.

注意到有更一般的式子

$$A\int_0^1 f^2(x)\mathrm{d}x+B\int_0^1 F^2(x)\mathrm{d}x\geqslant 2(A+B)\int_0^1 f(x)F(x)\mathrm{d}x$$

等号成立当且仅当 $A=B=0$ 或 $f(x)=F(x)=0$.

注意到

$$\int_0^1 f(x)\mathrm{d}x=F(1)=\int_0^1\frac{\mathrm{d}(xF(x))}{\mathrm{d}x}\mathrm{d}x=\int_0^1 xf(x)\mathrm{d}x+\int_0^1 F(x)\mathrm{d}x=\int_0^1 F(x)\mathrm{d}x$$

$$F^2(1)=\int_0^1\frac{\mathrm{d}F^2(x)}{\mathrm{d}x}\mathrm{d}x=2\int_0^1 f(x)F(x)\mathrm{d}x$$

由 Cauchy 不等式得

$$\int_0^1 f^2(x)\mathrm{d}x \geqslant \left(\int_0^1 f(x)\mathrm{d}x\right)^2 = F^2(1) = 2\int_0^1 f(x)F(x)\mathrm{d}x$$

$$\int_0^1 F^2(x)\mathrm{d}x \geqslant \left(\int_0^1 f(x)\mathrm{d}x\right)^2 = F^2(1) = 2\int_0^1 f(x)F(x)\mathrm{d}x$$

相加即可.

□

107. 设函数 $f \in C(a,b)$ 不恒为零, 满足 $0 \leqslant f(x) \leqslant M$, 试证明:

$$\left(\int_a^b f(x)\mathrm{d}x\right)^2 \leqslant \left(\int_a^b f(x)\sin x\mathrm{d}x\right)^2 + \left(\int_a^b f(x)\cos x\mathrm{d}x\right)^2 + \frac{M^2(b-a)^4}{12}$$

证明 令

$$A = \left(\int_a^b f(x)\mathrm{d}x\right)^2 = \iint\limits_D f(x)f(y)\mathrm{d}x\mathrm{d}y$$

$$B = \left(\int_a^b f(x)\sin x\mathrm{d}x\right)^2 = \iint\limits_D f(x)f(y)\sin x\sin y\mathrm{d}x\mathrm{d}y$$

$$C = \left(\int_a^b f(x)\cos x\mathrm{d}x\right)^2 = \iint\limits_D f(x)f(y)\cos x\cos y\mathrm{d}x\mathrm{d}y$$

这里区域 $D = \{(x,y)|a \leqslant x \leqslant b, a \leqslant y \leqslant b\}$.
则有

$$B + C = \iint\limits_D f(x)f(y)(\sin x\sin y + \cos x\cos y)\mathrm{d}x\mathrm{d}y = \iint\limits_D f(x)f(y)\cos(x-y)\mathrm{d}x\mathrm{d}y$$

$$\begin{aligned}
A - (B+C) &= \iint\limits_D f(x)f(y)[1-\cos(x-y)]\mathrm{d}x\mathrm{d}y \\
&= 2\iint\limits_D f(x)f(y)\sin^2\left(\frac{x-y}{2}\right)\mathrm{d}x\mathrm{d}y \\
&\leqslant \frac{M^2}{2}\iint\limits_D (x-y)^2\mathrm{d}x\mathrm{d}y \\
&= \frac{M^2}{2}\int_a^b \mathrm{d}y\int_a^b (x-y)^2\mathrm{d}y \\
&= \frac{M^2(b-a)^4}{12}
\end{aligned}$$

□

108. 求极限

$$\lim_{x\to+\infty}\sqrt{x}\int_0^{\frac{\pi}{4}}\mathrm{e}^{x(\cos t-1)}\cos t\,\mathrm{d}t$$

解 一方面有

$$\cos t-1=-\frac{1}{2}t^2+o(t^2)\quad t\to 0$$

于是

$$\mathrm{e}^{x(\cos t-1)}=\mathrm{e}^{-\frac{1}{2}xt^2+o(xt^2)}$$

且注意到

$$\lim_{x\to 0}\cos x=1$$

于是对任意 $\varepsilon>0$, 存在 $\delta>0$, 当 $0<x<\delta$ 时

$$\cos t>1-\varepsilon$$

$$\sqrt{x}\int_0^{\delta}\mathrm{e}^{x(\cos t-1)}\cos t\,\mathrm{d}t\leqslant\sqrt{x}\int_0^{\delta}\mathrm{e}^{-\frac{1}{2}xt^2+o(xt^2)}\mathrm{d}t=\sqrt{2}\int_0^{\frac{\delta}{\sqrt{2}}\sqrt{x}}\mathrm{e}^{-y^2+o(y^2)}\mathrm{d}y\to\sqrt{\frac{\pi}{2}}$$

另外

$$\sqrt{x}\int_0^{\delta}\mathrm{e}^{x(\cos t-1)}\cos t\,\mathrm{d}t\geqslant(1-\varepsilon)\sqrt{x}\int_0^{\delta}\mathrm{e}^{-\frac{1}{2}xt^2+o(xt^2)}\mathrm{d}t\to\sqrt{\frac{\pi}{2}}(1-\varepsilon)$$

不难得到

$$\sqrt{x}\int_{\delta}^{\frac{\pi}{4}}\mathrm{e}^{x(\cos t-1)}\cos t\,\mathrm{d}t=0$$

这里只需要用

$$\cos t-1=-2\sin^2\left(\frac{t}{2}\right)\leqslant-2\left(\frac{x}{\pi}\right)^2$$

最后, 由 ε 的任意性知

$$\lim_{x\to+\infty}\sqrt{x}\int_0^{\frac{\pi}{4}}\mathrm{e}^{x(\cos t-1)}\cos t\,\mathrm{d}t=\sqrt{\frac{\pi}{2}}$$

□

109. 求极限

$$\lim_{n\to\infty}\frac{n!}{n^n}\left(\sum_{k=0}^{n}\frac{n^k}{k!}-\sum_{k=n+1}^{\infty}\frac{n^k}{k!}\right)$$

解 我们有

$$\mathrm{e}^n=\sum_{k=0}^{n}\frac{n^k}{k!}+\sum_{k=n+1}^{\infty}\frac{n^k}{k!}=\sum_{k=0}^{n}\frac{n^k}{k!}+\frac{1}{n!}\int_0^n\mathrm{e}^t(n-t)^n\mathrm{d}t$$

所以

$$\sum_{k=n+1}^{\infty}\frac{n^k}{k!}=\frac{1}{n!}\int_0^n\mathrm{e}^t(n-t)^n\mathrm{d}t$$

$$\sum_{k=0}^{n}\frac{n^k}{k!}=\mathrm{e}^n-\frac{1}{n!}\int_0^n\mathrm{e}^t(n-t)^n\mathrm{d}t$$

因此只要计算

$$\lim_{n\to\infty}\frac{n!}{n^n}\left(\mathrm{e}^n-\frac{2}{n!}\int_0^n \mathrm{e}^t(n-t)^n\mathrm{d}t\right)$$

我们有

$$\begin{aligned}\int_0^n \mathrm{e}^t(n-t)^n\mathrm{d}t &= n^{n+1}\int_0^1 \mathrm{e}^{nz}(1-z)^n\mathrm{d}z\\ &= n^{n+1}\int_0^1 \mathrm{e}^{n(z+\ln(1-z))}\mathrm{d}z\\ &= n^{n+1}\int_0^1 \mathrm{e}^{-\frac{1}{2}nz^2-\frac{1}{3}nz^3+o(nz^3)}\mathrm{d}z\\ &= n^{n+1}\int_0^1 \mathrm{e}^{-\frac{1}{2}nz^2}\left(1-\frac{1}{3}nz^3+o(nz^3)\right)\mathrm{d}z\end{aligned}$$

$$\begin{aligned}&\lim_{n\to\infty}\frac{n!}{n^n}\left(\mathrm{e}^n-\frac{2}{n!}\int_0^n \mathrm{e}^t(n-t)^n\mathrm{d}t\right)\\ =&\frac{n!\mathrm{e}^n}{n^n}-2n\left[\int_0^1 \mathrm{e}^{-\frac{1}{2}nz^2}\left(1-\frac{1}{3}nz^2+o(nz^2)\right)\mathrm{d}z\right]\\ =&\left(\sqrt{2\pi n}\mathrm{e}^{\frac{\theta_n}{12n}}-2n\int_0^1 \mathrm{e}^{-\frac{1}{2}nz^2}\mathrm{d}z\right)+2n\int_0^1 \mathrm{e}^{-\frac{1}{2}nz^2}\left(\frac{1}{3}nz^2+o(nz^3)\right)\mathrm{d}z\quad \theta_n\in(0,1)\end{aligned}$$

显然有

$$\lim_{n\to 0}\left(\sqrt{2\pi n}\mathrm{e}^{\frac{\theta_n}{12n}}-2n\int_0^1 \mathrm{e}^{-\frac{1}{2}nz^2}\mathrm{d}z\right)=0$$

$$\lim_{n\to\infty}2n\int_0^1 \mathrm{e}^{-\frac{1}{2}nz^2}\left(\frac{1}{3}nz^3+o(nz^3)\right)\mathrm{d}z=\lim_{n\to\infty}\frac{4}{3}\left(\int_0^{\frac{n}{2}}\mathrm{e}^{-z}z\mathrm{d}z+o\left(\frac{1}{n}\right)\right)=\frac{4}{3}$$

所以

$$\lim_{n\to\infty}\frac{n!}{n^n}\left(\sum_{k=0}^{n}\frac{n^k}{k!}-\sum_{k=n+1}^{\infty}\frac{n^k}{k!}\right)=\frac{4}{3}$$

□

110. 计算积分

$$\int_0^1 \frac{\arctan\sqrt{x^2+2}}{(x^2+1)\sqrt{x^2+2}}\mathrm{d}x$$

解

$$\begin{aligned}\frac{\pi^2}{16}&=\int_0^1\int_0^1\frac{\mathrm{d}x\mathrm{d}y}{(1+x^2)(1+y^2)}\\ &=\int_0^1\int_0^1\left[\frac{1}{(1+x^2)(2+x^2+y^2)}+\frac{1}{(1+y^2)(2+x^2+y^2)}\right]\mathrm{d}x\mathrm{d}y\\ &=2\int_0^1\int_0^1\frac{1}{(1+x^2)(2+x^2+y^2)}\mathrm{d}y\mathrm{d}x\\ &=2\int_0^1\frac{1}{(1+x^2)\sqrt{2+x^2}}\arctan\frac{1}{\sqrt{2+x^2}}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^1 \left[\frac{\pi}{2(1+x^2)\sqrt{2+x^2}} - \frac{\arctan\sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}}\right]\mathrm{d}x \\
&= \frac{\pi^2}{6} - 2\int_0^1 \frac{\arctan\sqrt{x^2+2}}{(x^2+1)\sqrt{x^2+2}}\mathrm{d}x \\
&\Rightarrow \int_0^1 \frac{\arctan\sqrt{x^2+2}}{(x^2+1)\sqrt{x^2+2}}\mathrm{d}x = \frac{5}{96}\pi^2
\end{aligned}
$$

□

111. 设 $f(x) \in C^2(0,1)$ 且 $\lim\limits_{x\to 1^-} f(x) = 0$. 若存在 $M > 0$, 使得 $(1-x)^2|f''(x)| \leqslant M(0<x<1)$, 证明

$$\lim_{x\to 1^-}(1-x)f'(x) = 0$$

证明 对 $t, x \in (0,1), t > x$, 由 Taylor 公式得

$$f(t) = f(x) + f'(x)(t-x) + f''(\xi)\frac{(t-x)^2}{2}, x<\xi<t$$

并取 $t = x + (1-x)\delta, 0 < \delta < \dfrac{1}{2}$, 我们有

$$
\begin{aligned}
f(t) - f(x) &= \delta(1-x)f'(x) + \frac{\delta^2}{2}f''(\xi)(1-x)^2 \\
\Leftrightarrow (1-x)f'(x) &= \frac{f(t)-f(x)}{8} - \frac{\delta}{2}f''(\xi)(1-x)^2 \\
|f'(x)(1-x)| &\leqslant \frac{|f(t)-f(x)|}{\delta} + \frac{\delta}{2}|f''(\xi)(1-x)^2
\end{aligned}
$$

注意到

$$
\begin{aligned}
&\xi = x + (t-x)\theta, 0<\theta<1 \\
\Rightarrow &(1-\xi)^2 = (1-x)^2(1-\delta\theta)^2 > \frac{1}{4}(1-x)^2
\end{aligned}
$$

这里是由于 $0 < \delta\theta < \dfrac{1}{2}$. 结合条件 $(1-x)^2|f''(x)| \leqslant M(0<x<1)$ 得

$$
\begin{aligned}
&\frac{\delta}{2}|f''(\xi)|(1-\xi)^2 \cdot \frac{(1-x)^2}{(1-\xi)^2} \cdot \frac{\delta}{2} < 2M\delta \\
\Rightarrow &|f'(x)(1-x)| \leqslant \frac{|f(t)-f(x)|}{\delta} + 2M\delta
\end{aligned}
$$

现在, 对 $\forall\varepsilon$, 取 $\delta = \dfrac{\varepsilon}{4M}$, 对上述 $\delta\varepsilon$, 存在 $\eta > 0$, 对 $\forall 0 < 1-x < \eta$ 有

$$|f(t)-f(x)| < \frac{\delta\varepsilon}{2}$$

这样, 对 $\forall 0 < 1-x < \eta$, 就有

$$|f'(x)(1-x)| < \varepsilon$$

□

112. 计算积分

$$\int_0^{\frac{\pi}{2}} x\ln(\sin x)\ln(\cos x)\mathrm{d}x$$

解 首先设

$$I=\int_0^{\frac{\pi}{2}} x\ln(\sin x)\ln(\cos x)\mathrm{d}x$$

显然有

$$I=\frac{\pi}{4}\int_0^{\frac{\pi}{2}} \ln(\sin x)\ln(\cos x)\mathrm{d}x$$

根据 Fourier 级数

$$\ln(2\cos x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos 2nx}{n},\qquad -\frac{\pi}{2}<x<\frac{\pi}{2}$$

而另一方面

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\cos 2nx\ln(\sin x)\mathrm{d}x &= \int_0^{\frac{\pi}{2}}\ln(\sin x)\mathrm{d}\left(\frac{\sin 2nx}{2n}\right)\\
&= \frac{1}{2n}\sin 2nx\cdot\ln(\sin x)\Big|_0^{\frac{\pi}{2}}-\frac{1}{2n}\int_0^{\frac{\pi}{2}}\frac{\cos x\cdot\sin 2nx}{\sin x}\mathrm{d}x\\
&= -\frac{1}{4n}\int_0^{\frac{\pi}{2}}\frac{\sin(2n+1)x+\sin(2n-1)x}{\sin x}\mathrm{d}x\\
&= -\frac{\pi}{4n}
\end{aligned}$$

所以

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\ln(\sin x)\ln(\cos x)\mathrm{d}x &= \int_0^{\frac{\pi}{2}}\ln(2\cos x)\ln(\sin x)\mathrm{d}x-\ln 2\cdot\int_0^{\frac{\pi}{2}}\ln(\sin x)\mathrm{d}x\\
&= \sum_{n=1}^{\infty}(-1)^n\frac{\pi}{4n^2}+\frac{\pi}{2}\ln^2 2\\
&= \frac{\pi}{2}\ln^2 2-\frac{1}{48}\pi^2
\end{aligned}$$

于是

$$I=\frac{(\pi\ln 2)^2}{8}-\frac{\pi^4}{192}$$

□

113. 计算积分

$$\int_0^{\infty}\frac{1}{(x^4+(1+2\sqrt{2})x^2+1)(x^{100}-x^{98}+\cdots+1)}\mathrm{d}x$$

解 记

$$I=\int_0^{\infty}\frac{1}{(x^4+(1+2\sqrt{2})x^2+1)(x^{100}-x^{98}+\cdots+1)}\mathrm{d}x$$

把 x 换成 $\frac{1}{x}$ 得

$$I=\int_0^{\infty}\frac{x^{102}}{(x^4+(1+2\sqrt{2})x^2+1)(x^{100}-x^{98}+\cdots+1)}\mathrm{d}x$$

注意到

$$x^{100}-x^{98}+\cdots+1=\frac{1+x^{102}}{1+x^2}$$

于是

$$\begin{aligned} I &= \frac{1}{2}\int_0^{\infty}\frac{1+x^2}{x^4+(1+2\sqrt{2})x^2+1}\mathrm{d}x \\ &= \frac{1}{2}\int_0^{\infty}\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1+2\sqrt{2}}\mathrm{d}x \\ &= \frac{\pi}{2(1+\sqrt{2})} \end{aligned}$$

□

114. 求极限

$$\lim_{n\to\infty}\frac{1}{(2n-1)^{2017}}\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^{2016}\sin^3 x\cos^2 x\mathrm{d}x$$

解 由推广的积分第一中值定理, 对每个正整数 n, $\exists\theta_n\in(0,1)$ 使得

$$\int_{2n\pi}^{(2n+1)\pi}x^{2016}\sin^3 x\cos^2 x\mathrm{d}x=((2n+\theta_n)\pi)^{2016}\int_{2n\pi}^{(2n+1)\pi}\sin^3 x\cos^2 x\mathrm{d}x$$

由此得

$$\begin{aligned} &\int_{2n\pi}^{(2n+1)\pi}x^{2016}\sin^3 x\cos^2 x\mathrm{d}x \\ =& \left((2n\pi)^{2016}+o(n^{2016})\right)\int_{2n\pi}^{(2n+1)\pi}\sin^3 x\cos^2 x\mathrm{d}x \\ =& \left((2n\pi)^{2016}+o(n^{2016})\right)\left(\frac{\cos 5x}{80}-\frac{\cos 3x}{48}-\frac{\cos x}{8}\right)\Bigg|_{2n\pi}^{(2n+1)\pi} \\ =& \frac{4}{15}\left((2n\pi)^{2016}+o(n^{2016})\right)\quad n\to\infty \end{aligned}$$

另外

$$(2n+1)^{2017}-(2n-1)^{2017}=4034(2n)^{2016}+o(n^{2016})\quad n\to\infty$$

于是由 Stolz 定理得

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{(2n-1)^{2017}}\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^{2016}\sin^3 x\cos^2 x\mathrm{d}x \\ =& \lim_{n\to\infty}\frac{\int_{2n\pi}^{(2n+1)\pi}x^{2016}\sin^3 x\cos^2 x\mathrm{d}x}{(2n+1)^{2017}-(2n-1)^{2017}} \\ =& \frac{2}{30510}\lim_{n\to\infty}\frac{(2n\pi)^{2016}+o(n^{2016})}{(2n)^{2016}+o(n^{2016})} \\ =& \frac{2\pi^{2016}}{30510} \end{aligned}$$

更一般的结果是

$$\lim_{n\to\infty}\frac{1}{(2n-1)^{p+1}}\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^p\sin^3 x\cos^2 x\mathrm{d}x=\frac{2\pi^p}{15(p+1)}.$$

□

115. 求极限

$$\lim_{y\to+\infty}\left(\ln^2 y-2\int_0^y \frac{\ln x}{\sqrt{x^2+1}}\mathrm{d}x\right)$$

解 首先有

$$\ln^2 y-2\int_0^y \frac{\ln x}{\sqrt{x^2+1}}\mathrm{d}x=\ln^2 y-\frac{y}{\sqrt{y^2+1}}\cdot\ln^2 y+\int_0^y \frac{\ln^2 x}{(x^2+1)^{\frac{3}{2}}}\mathrm{d}x$$

注意到

$$\lim_{y\to+\infty}\left(\ln^2 y-\frac{y}{\sqrt{y^2+1}}\cdot\ln^2 y\right)=0$$

于是我们只需要计算积分

$$\int_0^\infty \frac{\ln^2 x}{(x^2+1)^{\frac{3}{2}}}\mathrm{d}x$$

记

$$\begin{aligned}
I(p,q)&=\int_0^\infty \frac{x^p}{(1+x^2)^q}\mathrm{d}x\\
&=\int_0^{\frac{\pi}{2}}\frac{\tan^p\theta}{\sec^{2q}\theta}\sec^2\theta\,\mathrm{d}\theta\\
&=\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^{2q-p-2}\theta\,\mathrm{d}\theta\\
&=\frac{1}{2}B\left(\frac{p+1}{2},\frac{2q-p-1}{2}\right)\\
&=\frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{2q-p-1}{2}\right)}{\Gamma(q)}
\end{aligned}$$

于是

$$\int_0^\infty \frac{\ln^2 x}{(x^2+1)^{\frac{3}{2}}}\mathrm{d}x=\frac{\partial^2 I}{\partial p^2}\left(0,\frac{3}{2}\right)$$

注意到

$$I\left(p,\frac{3}{2}\right)=\frac{\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(1-\frac{q}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

求导可得

$$\frac{\partial I}{\partial p}\left(p,\frac{3}{2}\right)=\frac{\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(1-\frac{p}{2}\right)}{2\Gamma\left(\frac{1}{2}\right)}\left[\psi\left(\frac{1+p}{2}\right)-\psi\left(1-\frac{p}{2}\right)\right]$$

再求导

$$\frac{\partial^2 I}{\partial p^2}\left(p,\frac{3}{2}\right)=\frac{\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(1-\frac{p}{2}\right)}{4\Gamma\left(\frac{1}{2}\right)}\left[\left\{\psi\left(\frac{1+p}{2}\right)-\psi\left(1-\frac{p}{2}\right)\right\}^2+\left\{\psi'\left(\frac{1+p}{2}\right)+\psi'\left(1-\frac{p}{2}\right)\right\}\right]$$

代入 $p=0$ 可得

$$\frac{\partial^2 I}{\partial p^2}\left(0,\frac{3}{2}\right)=\frac{1}{4}\left[\left\{\psi\left(\frac{1}{2}\right)-\psi(1)\right\}^2+\left\{\psi'\left(\frac{1}{2}\right)+\psi'(1)\right\}\right]$$

根据公式

$$\psi(s) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+s}\right)$$

可得

$$\begin{aligned}\psi\left(\frac{1}{2}\right) - \psi(1) &= \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+\frac{1}{2}}\right)\\ &= -2\sum_{n=0}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right)\\ &= -2\ln 2\end{aligned}$$

类似的有

$$\begin{aligned}\psi'\left(\frac{1}{2}\right) + \psi'(1) &= \sum_{n=0}^{\infty}\left[\frac{1}{(n+\frac{1}{2})^2} + \frac{1}{(n+1)^2}\right]\\ &= 4\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)^2} + \frac{1}{(2n+2)^2}\right]\\ &= 4\zeta(2)\end{aligned}$$

因此最后得到

$$\lim_{y\to+\infty}\left(\ln^2 y - 2\int_0^y \frac{\ln x}{\sqrt{x^2+1}}\mathrm{d}x\right) = \zeta(2) + \ln^2 2 = \frac{\pi^2}{6} + \ln^2 2$$

□

116. 设 $-1 < a < 1$, 计算积分

$$\int_0^1 \frac{\cosh(a\ln x)\ln(1+x)}{x}\mathrm{d}x$$

解

$$\begin{aligned}\int_0^1 \frac{\cosh(a\ln x)\ln(1+x)}{x}\mathrm{d}x &= \int_{-\infty}^0 \cosh(at)\ln\left(1+\mathrm{e}^t\right)\mathrm{d}t\\ &= \int_{-\infty}^0 \frac{\mathrm{e}^{at}+\mathrm{e}^{-at}}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\mathrm{e}^{kt}\\ &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_{-\infty}^0\left[\mathrm{e}^{(k+a)t} + \mathrm{e}^{(k-a)t}\right]\mathrm{d}t\\ &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{1}{k+a} + \frac{1}{k-a}\right)\\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2-a^2}\\ &= \frac{1}{2a}\left(\frac{\pi}{\sin(\pi a)} - \frac{1}{a}\right)\end{aligned}$$

□

117. 计算积分

$$\int_0^\infty \mathrm{si}(x)\tan^{-1}\left(\frac{\sin x}{2-\cos x}\right)\mathrm{d}x$$

其中 $\mathrm{si}(x)=-\int_x^\infty \frac{\sin t}{t}\mathrm{d}t$.

解 考虑积分

$$I(a)=\int_0^\infty \mathrm{si}(x)\tan^{-1}\left(\frac{\sin x}{a-\cos x}\right)\mathrm{d}x$$

则

$$\begin{aligned}
I'(a) &= -\int_0^\infty \mathrm{si}(x)\frac{\sin x}{a^2-2a\cos x+1}\mathrm{d}x\\
&= -\sum_{n=1}^\infty \frac{1}{a^{n+1}}\int_0^\infty \mathrm{si}(x)\sin(nx)\mathrm{d}x\\
&= -\sum_{n=1}^\infty \frac{1}{a^{n+1}}\left[\mathrm{si}(x)\frac{-\cos(nx)}{n}\Big|_0^\infty+\frac{1}{n}\int_0^\infty \frac{\sin x\cos(nx)}{x}\mathrm{d}x\right]\\
&= -\sum_{n=1}^\infty \frac{1}{a^{n+1}}\left[-\frac{\pi}{2n}+\frac{1}{2n}\int_0^\infty \frac{\sin(n+1)x-\sin(n-1)x}{x}\mathrm{d}x\right]\\
&= -\sum_{n=1}^\infty \frac{1}{a^{n+1}}\left[-\frac{\pi}{2n}+\frac{\pi}{4n}(\mathrm{sgn}(1-n)+\mathrm{sgn}(1+n))\right]\\
&= \frac{\pi}{2}\sum_{n=2}^\infty \frac{1}{na^{n+1}}+\frac{\pi}{4a^2}\\
\Rightarrow I(a) &= -\frac{\pi}{2}\sum_{n=2}^\infty \frac{1}{n^2a^n}-\frac{\pi}{4}+C
\end{aligned}$$

注意到 $a\to\infty$ 时,$I(a)\to 0$, 因此 $C=0$.

$$\begin{aligned}
I(2) &= \int_0^\infty \mathrm{si}(x)\tan^{-1}\left(\frac{\sin x}{2-\cos x}\right)\mathrm{d}x\\
&= \frac{\pi}{2}\left(-\frac{1}{2}+\sum_{n=1}^\infty \frac{1}{n^2a^n}\right)-\frac{\pi}{8}\\
&= \frac{\pi}{8}-\frac{\pi}{2}\sum_{n=1}^\infty \frac{1}{n^2\cdot 2^n}\\
&= -\frac{\pi}{2}\cdot\frac{1}{12}\left(\pi^2-6\ln^2 2\right)+\frac{\pi}{8}\\
&= -\frac{\pi^3}{24}+\frac{\pi}{4}\ln^2 2+\frac{\pi}{8}
\end{aligned}$$

□

118. 求和

$$\sum_{n=0}^\infty \left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right)^2$$

解 首先注意到

$$\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots=\int_0^1(x^n-x^{n+1}+x^{n+2}-\cdots)\mathrm{d}x=\int_0^1\frac{x^n}{1+x}\mathrm{d}x$$

于是可得

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right)^2&=\sum_{n=0}^{\infty}\left(\int_0^1\frac{x^n}{x+1}\mathrm{d}x\right)\left(\int_0^1\frac{y^n}{y+1}\mathrm{d}y\right)\\&=\int_0^1\int_0^1\frac{1}{(1+x)(1+y)}\left(\sum_{n=0}^{\infty}(xy)^n\right)\mathrm{d}x\mathrm{d}y\\&=\int_0^1\int_0^1\frac{1}{(1+x)(1+y)(1-xy)}\mathrm{d}x\mathrm{d}y\\&=\int_0^1\frac{1}{1+x}\left(\int_0^1\frac{1}{(1+y)(1-xy)}\mathrm{d}y\right)\mathrm{d}x\\&=\int_0^1\frac{1}{1+x}\left(\frac{\ln 2-\ln(1-x)}{1+x}\right)\mathrm{d}x\\&=\left(\frac{(1-x)\ln(1-x)}{2(1+x)}+\frac{\ln(1+x)}{2}-\frac{\ln 2}{1+x}\right)\Bigg|_0^1\\&=\ln 2\end{aligned}$$

□

119. 计算积分

$$\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\frac{ax}{1+x^2}}\sin\left(\frac{a}{1+x^2}\right)}{1+x^2}\mathrm{d}x$$

解

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\frac{ax}{1+x^2}}\sin\left(\frac{a}{1+x^2}\right)}{1+x^2}\mathrm{d}x&=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\frac{-ax}{1+x^2}}\sin\left(\frac{a}{1+x^2}\right)}{1+x^2}\mathrm{d}x\\&=\Im\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\frac{a(\mathrm{i}-x)}{1+x^2}}}{1+x^2}\mathrm{d}x\\&=\Im\left[2\pi\mathrm{i}\mathrm{Res}\left(\frac{\mathrm{e}^{\frac{a(\mathrm{i}-x)}{1+x^2}}}{1+x^2},x=\mathrm{i}\right)\right]\\&=\Im(\pi\mathrm{e}^{\frac{\mathrm{i}a}{2}})\\&=\pi\sin\left(\frac{a}{2}\right)\end{aligned}$$

□

120. 计算积分

$$\int_0^1\frac{t^2-1}{(t^2+1)\ln t}\mathrm{d}t$$

解

$$\int_0^1\frac{t^2-1}{(t^2+1)\ln t}\mathrm{d}t=\int_0^{\infty}\frac{1-\mathrm{e}^{-2x}}{x(1+\mathrm{e}^{-2x})}\mathrm{e}^{-x}\mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(-1)^n \int_0^{\infty} \frac{1-\mathrm{e}^{-2x}}{x}\mathrm{e}^{-(2n+1)x}\mathrm{d}x \\
&= \sum_{n=0}^{\infty}(-1)^n \ln\left(\frac{2n+3}{2n+1}\right) \quad \text{Frullani积分} \\
&= \sum_{n=0}^{\infty}\ln\frac{(4n+3)(4n+3)}{(4n+1)(4n+5)} \\
&= \sum_{n=0}^{\infty}\ln\frac{\left(n+\frac{3}{4}\right)^2}{\left(n+\frac{1}{4}\right)\left(n+\frac{5}{4}\right)} \\
&= \lim_{n\to\infty}\ln\left[\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\Gamma^2\left(n+\frac{3}{4}\right)}{\Gamma^2\left(\frac{3}{4}\right)\Gamma\left(n+\frac{1}{4}\right)\Gamma\left(n+\frac{5}{4}\right)}\right] \\
&= 2\ln\left[\frac{\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)}\right]
\end{aligned}
$$

□

121. 计算积分

$$
\int_0^{\infty}\frac{\ln x}{\cosh^2 x}\mathrm{d}x
$$

解 设 $a>0$, 考虑积分

$$
\begin{aligned}
I(a) &= \int_0^{\infty}\frac{x^a}{\cosh^2 x}\mathrm{d}x \\
&= 4\int_0^{\infty}\frac{x^a}{(\mathrm{e}^x+\mathrm{e}^{-x})^2}\mathrm{d}x \\
&= 4\int_0^{\infty}x^a\frac{\mathrm{e}^{-2x}}{\left(1+\mathrm{e}^{-2x}\right)^2}\mathrm{d}x \\
&= 4\int_0^{\infty}x^a\sum_{n=1}^{\infty}(-1)^{n-1}n\mathrm{e}^{-2nx}\mathrm{d}x \\
&= 4\sum_{n=1}^{\infty}(-1)^{n-1}n\int_0^{\infty}x^a\mathrm{e}^{-2nx}\mathrm{d}x \\
&= 4\sum_{n=1}^{\infty}(-1)^{n-1}n\frac{\Gamma(a+1)}{(2n)^{a+1}} \\
&= \frac{2\Gamma(a+1)}{2^a}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^a} \\
&= \frac{2\Gamma(a+1)\eta(a)}{2^a}
\end{aligned}
$$

积分求导得

$$
\begin{aligned}
I'(a) &= \int_0^{\infty}\frac{x^a\ln x}{\cosh^2 x}\mathrm{d}x \\
&= 2\frac{\left[\Gamma'(a+1)\eta(a)+\Gamma(a+1)\eta'(a)\right]\cdot 2^a-\Gamma(a+1)\eta(a)\ln 2\cdot 2^a}{2^{2a}}
\end{aligned}
$$

因此

$$\int_0^\infty \frac{\ln x}{\cosh^2 x}\mathrm{d}x = I'(0) = 2\left(\Gamma'(0)\,\eta(0) + \eta'(0) - \eta(0)\ln 2\right)$$

利用关系式

$$\eta(s) = \left(1 - 2^{1-s}\right)\zeta(s)$$

以及

$$\zeta(0) = -\frac{1}{2}\text{和}\zeta'(0) = -\frac{1}{2}\ln(2\pi)$$

可得

$$\int_0^\infty \frac{\ln x}{\cosh^2 x}\mathrm{d}x = 2\left[-\gamma\left(\frac{1}{2}\right) + \frac{1}{2}\ln\left(\frac{\pi}{2}\right) - \frac{1}{2}\ln 2\right] = \ln\left(\frac{\pi}{4}\right) - \gamma$$

□

122. 设 $f(x)$ 是连续实值函数, 且满足

$$\int_0^1 f(x)\mathrm{d}x = \int_0^1 xf(x)\mathrm{d}x = \cdots = \int_0^1 x^{n-1}f(x)\mathrm{d}x = 1$$

证明:

$$\int_0^1 f^2(x)\mathrm{d}x \geqslant n^2$$

证明 考虑多项式

$$P(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$$

如果多项式 $P(x)$ 也满足上面的条件, 那么

$$\int_0^1 P^2(x)\mathrm{d}x = a_0 + a_1 + \cdots + a_{n-1}$$

为了求出系数 a_i, 再次利用条件

$$\int_0^1 x^kP(x)\mathrm{d}x = 1 \quad k = 0, 1, \cdots, n-1$$

$$\Rightarrow \frac{a_0}{k+1} + \frac{a_1}{k+2} + \cdots + \frac{a_{n-1}}{k+n} = 1 \quad k = 0, 1, \cdots, n-1$$

设

$$H(x) = \frac{a_0}{x+1} + \frac{a_1}{x+2} + \cdots + \frac{a_{n-1}}{x+n}$$

则显然有

$$H(0) = H(1) = \cdots = H(n-1) = 0$$

于是

$$H(x) = \frac{Ax(x-1)(x-2)\cdots(x-n+1)}{(x+1)(x+2)\cdots(x+n)}$$

对比系数可得 $A = -1$ 以及

$$a_k = (-1)^{n-k+1}\frac{(n+k)!}{(k!)^2(n-k+1)!} \quad k = 0, 1, \cdots, n-1$$

用数学归纳法可以证明

$$\sum_{k=0}^{n-1} a_k = n^2$$

所以, 多项式 $P(x)$ 满足上面的性质, 则

$$\int_0^1 P^2(x)\mathrm{d}x = a_0 + a_1 + \cdots + a_{n-1} = n^2$$

取满足以上条件的多项式 $P(x)$, 由 Cauchy 不等式得

$$\int_0^1 P^2(x)\mathrm{d}x \int_0^1 f^2(x)\mathrm{d}x \geqslant \left(\int_0^1 P(x)f(x)\mathrm{d}x\right)^2 = n^4$$

$$\Rightarrow \int_0^1 f^2(x)\mathrm{d}x \geqslant n^2$$

□

123. 计算积分

$$\int_0^{\pi} \left(\frac{\mathrm{e} + \cos x}{1 + 2\mathrm{e}\cdot\cos x + \mathrm{e}^2}\right)^2 \mathrm{d}x$$

解

$$\begin{aligned}
&\int_0^{\pi} \left(\frac{\mathrm{e} + \cos x}{1 + 2\mathrm{e}\cdot\cos x + \mathrm{e}^2}\right)^2 \mathrm{d}x \\
=& \frac{1}{2}\oint_{|z|=1} \left(\frac{\mathrm{e} + \frac{z^2+1}{2z}}{1 + 2\mathrm{e}\frac{z^2+1}{2z} + \mathrm{e}^2}\right)^2 \frac{\mathrm{d}z}{\mathrm{i}z} \\
=& \frac{1}{2}\cdot 2\pi\mathrm{i}\cdot\mathrm{Res}\left[\frac{1}{\mathrm{i}z}\left(\frac{\mathrm{e} + \frac{z^2+1}{2z}}{1 + 2\mathrm{e}\frac{z^2+1}{2z} + \mathrm{e}^2}\right)^2, |z| < 1\right] \quad \left(z = -\frac{1}{\mathrm{e}}, 0\right) \\
=& \frac{\pi}{4}\left(\frac{3\mathrm{e}^2 - 1}{\mathrm{e}^2\left(\mathrm{e}^2 - 1\right)} + \frac{1}{\mathrm{e}^2}\right) \\
=& \frac{\pi}{2}\left(\frac{2\mathrm{e}^2 - 1}{\mathrm{e}^2 - 1}\right)
\end{aligned}$$

□

124. 求和

$$\sum_{n=1}^{\infty} \frac{1}{\sinh\left(2^n\right)}$$

解

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{\sinh\left(2^n\right)} &= \sum_{n=1}^{\infty} \frac{2}{\mathrm{e}^{2^n} - \mathrm{e}^{-2^n}} \\
&= \sum_{n=1}^{\infty} \frac{2}{\mathrm{e}^{2^n}\left(1 - \mathrm{e}^{-2\cdot 2^n}\right)} \\
&= 2\sum_{n=1}^{\infty} \mathrm{e}^{-2^n} \sum_{k=0}^{\infty} \mathrm{e}^{-2\cdot 2^n\cdot k}
\end{aligned}$$

$$= 2\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \mathrm{e}^{-(2k+1)\cdot 2^n}$$
$$= 2\sum_{m=1}^{\infty} \mathrm{e}^{-2m} = \frac{2}{\mathrm{e}^2-1}$$

□

125. 证明: 对 $n \geqslant 1$,

$$\frac{n(n+1)(n+2)}{3} < \sum_{k=1}^{n} \frac{1}{\ln^2\left(1+\frac{1}{k}\right)} < \frac{n}{4} + \frac{n(n+1)(n+2)}{3}$$

证明 求导容易证明对 $x \in (0,1]$ 有

$$\frac{2x}{2+x} < \ln(1+x) < \frac{x}{\sqrt{1+x}}$$

因此对 $1 \leqslant k \leqslant n$

$$k(k+1) = \frac{1+\frac{1}{k}}{\left(\frac{1}{k}\right)^2} < \frac{1}{\ln^2\left(1+\frac{1}{k}\right)} < \left(\frac{2+\frac{1}{k}}{\frac{2}{k}}\right)^2 = k(k+1) + \frac{1}{4}$$

求和即得

$$\frac{n(n+1)(n+2)}{3} < \sum_{k=1}^{n} \frac{1}{\ln^2\left(1+\frac{1}{k}\right)} < \frac{n}{4} + \frac{n(n+1)(n+2)}{3}$$

□

126. 求极限

$$\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{\zeta(k)}{\Gamma(n-k)}, \lim_{n\to\infty}\sum_{k=1}^{n-2}(-1)^{k-1}\frac{\zeta(n-k)}{\Gamma(n-k)}$$

解 首先注意到

$$\sum_{k=2}^{\infty}(\zeta(k)-1) = \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{n^k} = \sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{n^k}$$
$$= \sum_{n=2}^{\infty}\frac{\frac{1}{n^2}}{1-\frac{1}{n}} = \sum_{n=2}^{\infty}\frac{1}{n(n-1)}$$
$$= \sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right) = 1$$

而且对任意 $z \in \mathbb{C}$,

$$\sum_{k=1}^{\infty}\frac{z^{k-1}}{\Gamma(k)} = \sum_{k=0}^{\infty}\frac{z^{k-1}}{(k-1)!} = \mathrm{e}^z$$

这两个绝对收敛级数的 Cauchy 乘积为

$$\sum_{n=3}^{\infty}\left(\sum_{k=2}^{n-1}\frac{(\zeta(k)-1)z^{n-k+1}}{\Gamma(n-k)}\right) = \mathrm{e}^z$$

根据 Mertens 定理知该级数是收敛的, 因此通项趋于 0, 即

$$\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{(\zeta(k)-1)z^{n-k+1}}{\Gamma(n-k)}=0$$

因此

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{\zeta(k)z^{n-k+1}}{\Gamma(n-k)}&=\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{z^{n-k+1}}{\Gamma(n-k)}+\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{(\zeta(k)-1)z^{n-k+1}}{\Gamma(n-k)}\\&=\lim_{n\to\infty}\sum_{k=1}^{n-2}\frac{z^{k-1}}{\Gamma(k)}=\mathrm{e}^{z}\end{aligned}$$

在这个结果中取 $z=1$ 和 $z=-1$ 可得

$$\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{\zeta(k)}{\Gamma(n-k)}=\mathrm{e},\ \lim_{n\to\infty}\sum_{k=1}^{n-2}\frac{(-1)^{k-1}\zeta(n-k)}{\Gamma(k)}=\lim_{n\to\infty}\sum_{k=1}^{n-2}\frac{(-1)^{n-k-1}}{\Gamma(n-k)}=\mathrm{e}^{-1}$$

□

127. 设 k 是一个整数,$k\geqslant 2$, 计算积分

$$I_k=\int_0^{+\infty}\frac{\ln|1-x|}{x^{1+\frac{1}{k}}}\mathrm{d}x$$

解 首先有

$$\begin{aligned}\int_1^{+\infty}\frac{\ln|1-x|}{x^{1+\frac{1}{k}}}\mathrm{d}x&=\int_1^{+\infty}\frac{\ln(x-1)}{x^{1+\frac{1}{k}}}\mathrm{d}x\\&=\int_0^1\frac{\ln\left(\frac{1}{t}-1\right)}{\left(\frac{1}{t}\right)^{1+\frac{1}{k}}}\mathrm{d}\left(\frac{1}{t}\right)\\&=\int_0^1\frac{\ln(1-t)}{t^{1-\frac{1}{k}}}\mathrm{d}t-\int_0^1\frac{\ln t}{t^{1-\frac{1}{k}}}\mathrm{d}t\end{aligned}$$

而

$$\int_0^1\frac{\ln t}{t^{1-\frac{1}{k}}}\mathrm{d}t=\left[-k^2t^{\frac{1}{k}}+kt^{\frac{1}{k}}\ln t\right]\Big|_{t=0^+}^{1}=-k^2$$

进一步, 如果 $\alpha\in(0,2)$, 则

$$-\int_0^1\frac{\ln(1-t)}{t^{\alpha}}\mathrm{d}t=\int_0^1\frac{1}{t^{\alpha}}\sum_{j=1}^{\infty}\frac{t^j}{j}\mathrm{d}t=\sum_{j=1}^{\infty}\frac{1}{j}\int_0^1 t^{j-\alpha}\mathrm{d}t=\sum_{j=1}^{\infty}\frac{1}{j(j+1-\alpha)}$$

因此

$$\begin{aligned}I_k&=k^2+\int_0^1\frac{\ln(1-t)}{t^{1-\frac{1}{k}}}\mathrm{d}t+\int_0^1\frac{\ln(1-t)}{t^{1+\frac{1}{k}}}\mathrm{d}t\\&=k^2-\sum_{j=1}^{\infty}\frac{1}{j\left(j+\frac{1}{k}\right)}-\sum_{j=1}^{\infty}\frac{1}{j\left(j-\frac{1}{k}\right)}\\&=k^2+2\sum_{j=1}^{\infty}\frac{1}{\frac{1}{k^2}-j^2}=k\pi\cot\left(\frac{\pi}{k}\right)\end{aligned}$$

最后一步我们利用了例 4 的等式 (5).

□

128. 设 $f(x)$ 是 $[0,1]$ 上的 n 阶连续可微函数, 满足 $f\left(\frac{1}{2}\right)=f^{(i)}\left(\frac{1}{2}\right)=0$, 其中 i 是不超过 n 的偶数, 证明

$$\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2 \leqslant \frac{1}{(2n+1)\,4^n\,(n!)^2}\int_0^1\left(f^{(n)}(x)\right)^2\mathrm{d}x$$

证明 如果 $g\in C^n([0,1])$, 则对任意 $a\in(0,1)$, 由分部积分可得

$$\int_0^a g(x)\,\mathrm{d}x=\sum_{i=0}^{n-1}\frac{(-1)^i a^{i+1}g^{(i)}(a)}{(i+1)!}+\frac{(-1)^n}{n!}\int_0^a x^n g^{(n)}(x)\,\mathrm{d}x$$

因此

$$\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x=\sum_{i=0}^{n-1}\frac{(-1)^i f^{(i)}\left(\frac{1}{2}\right)}{2^{i+1}(i+1)!}+\frac{(-1)^n}{n!}\int_0^{\frac{1}{2}} x^n f^{(n)}(x)\,\mathrm{d}x$$

以及

$$\int_{\frac{1}{2}}^1 f(x)\,\mathrm{d}x=\int_0^{\frac{1}{2}} f(1-x)\,\mathrm{d}x\sum_{i=0}^{n-1}\frac{(-1)^i f^{(i)}\left(\frac{1}{2}\right)}{2^{i+1}(i+1)!}+\frac{1}{n!}\int_0^{\frac{1}{2}} x^n f^{(n)}(1-x)\,\mathrm{d}x$$

由于 $f^{(i)}\left(\frac{1}{2}\right)=0$, 其中 i 是小于 n 的偶数, 于是

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x&=\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x+\int_{\frac{1}{2}}^1 f(x)\,\mathrm{d}x\\&=\frac{1}{n!}\left(\int_0^{\frac{1}{2}} x^n f^{(n)}(x)\,\mathrm{d}x+\int_0^{\frac{1}{2}} x^n f^{(n)}(1-x)\,\mathrm{d}x\right)\end{aligned}$$

最后由 Cauchy 不等式得

$$\begin{aligned}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2&\leqslant\frac{2}{(n!)^2}\left[\left(\int_0^{\frac{1}{2}} x^n f^{(n)}(x)\,\mathrm{d}x\right)^2+\left(\int_0^{\frac{1}{2}} x^n f^{(n)}(1-x)\,\mathrm{d}x\right)^2\right]\\&\leqslant\left[\int_0^{\frac{1}{2}} x^{2n}\mathrm{d}x\int_0^{\frac{1}{2}}\left(f^{(n)}(x)\right)^2\mathrm{d}x+\int_0^{\frac{1}{2}} x^{2n}\mathrm{d}x\int_0^{\frac{1}{2}}\left(f^{(n)}(1-x)\right)^2\mathrm{d}x\right]\\&\leqslant\frac{1}{(2n+1)\,4^n\,(n!)^2}\int_0^1\left(f^{(n)}(x)\right)^2\mathrm{d}x\end{aligned}$$

□

129. 求和

$$\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n)^4+((m+n)(m+p))^2}$$

解 令上述和为 $S, a=m+n, b=m+p$, 由对称性

$$\begin{aligned}S&=\sum_{m=1}^{\infty}\sum_{a>m}\sum_{b>m}\frac{1}{a^2\left(a^2+b^2\right)}\\&=\sum_{m=1}^{\infty}\sum_{a>m}\sum_{b>m}\left(\frac{1}{a^2b^2}-\frac{1}{b^2\left(a^2+b^2\right)}\right)\end{aligned}$$

$$= \sum_{m=1}^{\infty} \sum_{a>m} \sum_{b>m} \frac{1}{a^2 b^2} - S$$

因此

$$\begin{aligned}
S &= \frac{1}{2} \sum_{m=1}^{\infty} \sum_{a>m} \sum_{b>m} \frac{1}{a^2 b^2} \\
&= \sum_{m=1}^{\infty} \sum_{a>b>m} \frac{1}{a^2 b^2} + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{a>m} \frac{1}{a^4} \\
&= \sum_{a>b\geqslant 1} \frac{1}{a^2 b^2} \sum_{m=1}^{b-1} 1 + \frac{1}{2} \sum_{a\geqslant 1} \frac{1}{a^4} \sum_{m=1}^{a-1} 1 \\
&= \sum_{a>b\geqslant 1} \frac{1}{a^2 b} - \sum_{a>b\geqslant 1} \frac{1}{a^2 b^2} + \frac{1}{2} \sum_{a\geqslant 1} \frac{1}{a^3} - \frac{1}{2} \sum_{a\geqslant 1} \frac{1}{a^4} \\
&= \zeta(2,1) - \zeta(2,2) + \frac{1}{2}\zeta(3) - \frac{1}{2}\zeta(4) \\
&= \frac{3}{2}\zeta(3) - \frac{5}{4}\zeta(4)
\end{aligned}$$

这里我们运用了基本结论 $\zeta(2,1) = \zeta(3), \zeta(2,2) = \frac{3}{4}\zeta(4)$.

□

130. 设 f 是 $[0,1]$ 上二阶连续可导的实值函数, 满足 $f\left(\frac{1}{2}\right) = 0$, 证明

$$\int_0^1 \left(f''(x)\right)^2 \mathrm{d}x \geqslant 320 \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

证明 利用 Taylor 公式可得

$$f(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{x} f''(t)(x-t)\,\mathrm{d}t$$

由于 $f\left(\frac{1}{2}\right) = 0$, 于是有

$$\begin{aligned}
\int_0^1 f(x)\,\mathrm{d}x &= \int_0^1 \left(\int_{\frac{1}{2}}^{x} f''(t)(x-t)\,\mathrm{d}t\right)\mathrm{d}x \\
&= \int_{x=0}^{\frac{1}{2}} \int_{t=x}^{\frac{1}{2}} f''(t)(t-x)\,\mathrm{d}t\,\mathrm{d}x + \int_{x=\frac{1}{2}}^{1} \int_{t=\frac{1}{2}}^{x} f''(t)(x-t)\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_{t=0}^{\frac{1}{2}} \int_{x=0}^{t} f''(t)(t-x)\,\mathrm{d}x\,\mathrm{d}t + \int_{t=\frac{1}{2}}^{1} \int_{x=t}^{1} f''(t)(x-t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{t=0}^{\frac{1}{2}} f''(t) \left[-\frac{(t-x)^2}{2}\right]\Bigg|_{x=0}^{t} \mathrm{d}t + \int_{t=\frac{1}{2}}^{1} f''(t) \left[\frac{(x-t)^2}{2}\right]\Bigg|_{x=t}^{1} \mathrm{d}t \\
&= \frac{1}{2} \int_{t=0}^{\frac{1}{2}} f''(t)\,t^2\mathrm{d}t + \frac{1}{2} \int_{t=\frac{1}{2}}^{1} f''(t)(1-t)^2\,\mathrm{d}t \\
&= \frac{1}{2} \int_{t=0}^{1} f''(t)\,h(t)\,\mathrm{d}t
\end{aligned}$$

其中

$$h(t)=\begin{cases}t^2, & t\in\left[0,\frac{1}{2}\right]\\(1-t)^2, & t\in\left[\frac{1}{2},1\right]\end{cases}$$

因此由 Cauchy 不等式得

$$\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2\leqslant\frac{1}{4}\int_0^1(h(t))^2\,\mathrm{d}t\cdot\int_0^1\left(f''(t)\right)^2\mathrm{d}t=\frac{1}{320}\int_0^1\left(f''(t)\right)^2\mathrm{d}t$$

□

131. $\{x\}$ 表示 x 的小数部分, 计算积分

$$\int_0^{\frac{\pi}{2}}\frac{\{\tan x\}}{\tan x}\mathrm{d}x$$

解 令 $t_n=\arctan(n)$, 则

$$\begin{aligned}I&=\int_0^{\frac{\pi}{2}}\frac{\{\tan x\}}{\tan x}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\left(1-\frac{[\tan x]}{\tan x}\right)\mathrm{d}x\\&=\frac{\pi}{2}-\sum_{n=1}^{\infty}n\int_{t_n}^{t_{n+1}}\frac{\mathrm{d}x}{\tan x}\\&=\frac{\pi}{2}-\sum_{n=1}^{\infty}n\left(\ln(\sin t_{n+1})-\ln(\sin t_n)\right)\end{aligned}$$

由于 $t>0$ 时,$\sin(\arctan t)=\dfrac{1}{\sqrt{1+\frac{1}{t^2}}}$, 那么对 $N\geqslant 2$ 有

$$\begin{aligned}&\sum_{n=1}^{N-1}n\left(\ln(\sin t_{n+1})-\ln(\sin t_n)\right)\\=&\sum_{n=1}^{N-1}\left[(n+1)\ln(\sin t_{n+1})-n\ln(\sin t_n)\right]+\sum_{n=1}^{N-1}\ln(\sin t_{n+1})\\=&N\ln(\sin t_N)-\sum_{n=0}^{N-1}\ln(\sin t_{n+1})\\=&N\ln\left(\frac{1}{\sqrt{1+\frac{1}{N^2}}}\right)-\ln\left(\prod_{n=1}^{N}\frac{1}{\sqrt{1+\frac{1}{n^2}}}\right)\end{aligned}$$

令 $N\to\infty$, 可得

$$\begin{aligned}I&=\frac{\pi}{2}-\frac{1}{2}\ln\left(\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)\right)\\&=\frac{\pi}{2}-\frac{1}{2}\ln\left(\frac{\sin(\pi\mathrm{i})}{\pi\mathrm{i}}\right)\\&=\frac{\pi}{2}-\frac{1}{2}\ln\left(\frac{\sinh\pi}{\pi}\right)\\&=\frac{1}{2}\ln\left(\frac{2\pi}{1-\mathrm{e}^{-2\pi}}\right)\end{aligned}$$

□

132. 对实数 a 和 b 以及整数 $n \geqslant 1$, 令 $\gamma_n(a,b) = -\ln(n+a) + \sum\limits_{k=1}^{n} \frac{1}{k+b}$, 证明极限 $\gamma(a,b) = \lim\limits_{n\to\infty} \gamma_n(a,b)$ 存在且有限, 并求极限

$$\lim_{n\to\infty}\left(\ln\left(\frac{\mathrm{e}}{n+a}\right)+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\right)^{n}$$

解 如果 $x > 0$, 我们有

$$\psi(x) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{x-1+k}\right) = \ln x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right)$$

这就意味着 (其中 b 不能是负整数)

$$\begin{aligned}
\gamma_n(a,b)+\psi(b+1) &= -\ln(n+a)+\sum_{k=1}^{n}\frac{1}{k+b}+\psi(b+1)\\
&= -\ln(n+a)+\psi(b+n+1)\\
&= -\ln(n+a)+\ln(b+n+1)-\frac{1}{2(b+n+1)}+O\left(\frac{1}{n^2}\right)\\
&= \ln\left(\frac{b+n+1}{n+a}\right)-\frac{1}{2(b+n+1)}+O\left(\frac{1}{n^2}\right)\\
&= \frac{b-a+\frac{1}{2}}{n}+O\left(\frac{1}{n^2}\right)
\end{aligned}$$

因此 $\gamma(a,b) = \lim\limits_{n\to\infty}\gamma_n(a,b) = -\psi(b+1)$, 进一步有

$$\begin{aligned}
\ln\left(\frac{\mathrm{e}}{n+a}\right)+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b) &= 1-\ln(n+a)+\sum_{k=1}^{n}\frac{1}{k+b}+\psi(b+1)\\
&= 1+\frac{b-a+\frac{1}{2}}{n}+O\left(\frac{1}{n^2}\right)
\end{aligned}$$

于是最后得到

$$\lim_{n\to\infty}\left(\ln\left(\frac{\mathrm{e}}{n+a}\right)+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\right)^{n} = \mathrm{e}^{b-a+\frac{1}{2}}$$

□

133. 设 f 是 $[0,1]$ 上的连续非负函数, 证明

$$\int_0^1 f^3(x)\,\mathrm{d}x \geqslant 4\left(\int_0^1 x^2 f(x)\,\mathrm{d}x\right)\left(\int_0^1 x f^2(x)\,\mathrm{d}x\right)$$

证明 这里我们证明一个更一般的结论: 设 f, g 是 $[0,1]$ 上的连续非负函数,a 和 b 是非负实数, 则

$$\int_0^1 f^{a+b}(x)\,\mathrm{d}x\int_0^1 g^{a+b}(x)\,\mathrm{d}x \geqslant \left(\int_0^1 f^a(x)g^b(x)\,\mathrm{d}x\right)\left(\int_0^1 f^b(x)g^a(x)\,\mathrm{d}x\right)$$

设 A, B 是非负实数, 则

$$(A^a-B^a)(A^b-B^b)\geqslant 0$$

这就意味着

$$A^{a+b}+B^{a+b}\geqslant A^aB^b+A^bB^a$$

令 $A=f(x)g(y), B=f(y)g(x)$, 并在 $[0,1]\times[0,1]$ 上积分, 我们有

$$\begin{aligned}&\int_0^1\left(\int_0^1[f(x)g(y)]^{a+b}\,\mathrm{d}x\right)\mathrm{d}y+\int_0^1\left(\int_0^1[f(y)g(x)]^{a+b}\,\mathrm{d}x\right)\mathrm{d}y\\ \geqslant&\int_0^1\left(\int_0^1(f(x)g(y))^a(f(y)g(x))^b\,\mathrm{d}x\right)\mathrm{d}y+\int_0^1\left(\int_0^1(f(x)g(y))^b(f(y)g(x))^a\,\mathrm{d}x\right)\mathrm{d}y\end{aligned}$$

也就是

$$\begin{aligned}&\left(\int_0^1 f^{a+b}(x)\,\mathrm{d}x\right)\left(\int_0^1 g^{a+b}(y)\,\mathrm{d}y\right)+\left(\int_0^1 f^{a+b}(y)\,\mathrm{d}y\right)\left(\int_0^1 g^{a+b}(x)\,\mathrm{d}x\right)\\ \geqslant&\left(\int_0^1 f^a(x)g^b(x)\,\mathrm{d}x\right)\left(\int_0^1 f^a(y)g^b(y)\,\mathrm{d}y\right)+\left(\int_0^1 f^a(y)g^b(y)\,\mathrm{d}y\right)\left(\int_0^1 f^a(x)g^b(x)\,\mathrm{d}x\right)\end{aligned}$$

得证, 那么在待证式中取 $g(x)=x, a=2, b=1$ 即可.

□

134. 设 $H_n=\sum_{k=1}^{n}\frac{1}{k}$, 求和

$$\sum_{n=1}^{\infty}H_n\left(\zeta(3)-\sum_{k=1}^{n}\frac{1}{k^3}\right)$$

解

$$\begin{aligned}\sum_{n=1}^{\infty}H_n\left(\zeta(3)-\sum_{k=1}^{n}\frac{1}{k^3}\right)&=\sum_{n=1}^{\infty}H_n\sum_{k=n+1}^{\infty}\frac{1}{k^3}=\sum_{k=2}^{\infty}\frac{1}{k^3}\sum_{n=1}^{k-1}H_n\\ &=\sum_{k=2}^{\infty}\frac{1}{k^3}\sum_{n=1}^{k-1}\sum_{j=1}^{n}\frac{1}{j}=\sum_{k=2}^{\infty}\frac{1}{k^3}\sum_{j=1}^{k-1}\frac{1}{j}\sum_{n=j}^{k-1}1\\ &=\sum_{k=2}^{\infty}\frac{1}{k^3}\sum_{j=1}^{k-1}\frac{k-j}{j}=\sum_{k=2}^{\infty}\frac{1}{k^2}\sum_{j=1}^{k-1}\frac{1}{j}-\sum_{k=2}^{\infty}\frac{k-1}{k^3}\\ &=\zeta(2,1)-\zeta(2)+\zeta(3)=2\zeta(3)-\zeta(2)\end{aligned}$$

□

135. 设 f 是 $[0,1]$ 上的非负函数, 证明

$$\frac{3}{4}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2\leqslant\frac{1}{16}+\int_0^1 f^3(x)\,\mathrm{d}x$$

证明 首先注意到对 $t\geqslant 0$ 有

$$t^3-\frac{3}{4}t^2+\frac{1}{6}=\frac{(4t+1)(2t-1)^2}{16}\geqslant 0$$

由于 f 是非负函数, 则

$$\int_0^1\left(f^3(x)-\frac{3}{4}f^2(x)+\frac{1}{16}\right)\mathrm{d}x\geqslant 0$$

那么由 Cauchy 不等式得

$$\int_0^1 f^3(x)\,\mathrm{d}x + \frac{1}{6} \geqslant \frac{3}{4}\int_0^1 f^2(x)\,\mathrm{d}x \geqslant \frac{3}{4}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

□

136. 求极限

$$\lim_{n\to\infty} n^2 \int_{((n+1)!)^{-1/(n+1)}}^{((n)!)^{-1/n}} \Gamma(nx)\,\mathrm{d}x$$

解 我们将证明如果 f 是 (a,b) 上的实值连续函数且 $\mathrm{e}\in(a,b)$, 则

$$\lim_{n\to\infty} n^2 \int_{((n+1)!)^{-1/(n+1)}}^{((n)!)^{-1/n}} f(nx)\,\mathrm{d}x = \mathrm{e}f(\mathrm{e})$$

令 $b_n = n(n!)^{-1/n}, a_n = n((n+1)!)^{-1/(n+1)}$, 那么由积分平均值定理可得

$$\lim_{n\to\infty} n^2 \int_{((n+1)!)^{-1/(n+1)}}^{((n)!)^{-1/n}} f(nx)\,\mathrm{d}x = n\int_{a_n}^{b_n} f(t)\,\mathrm{d}t = n(b_n-a_n)f(t_n)$$

对某个 $t_n\in(a_n,b_n)$ 成立. 再由 Stirling 公式得

$$\ln(n!) = n\ln n - n + \frac{1}{2}\ln n + \ln\sqrt{2\pi} + O\left(\frac{1}{n}\right)$$

因此

$$b_n = n\mathrm{e}^{-\frac{\ln(n!)}{n}} = \mathrm{e} - \frac{\mathrm{e}\ln n}{2n} - \frac{\mathrm{e}\ln\sqrt{2\pi}}{n} + O\left(\frac{\ln^2 n}{n^2}\right)$$

$$b_n - a_n = b_n - \frac{nb_{n+1}}{n+1} = \frac{\mathrm{e}}{n} + O\left(\frac{\ln n}{n^2}\right) = \mathrm{e}$$

也就意味着

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} t_n = \mathrm{e}$$

再由 f 在 e 处的连续性

$$\lim_{n\to\infty} n(b_n-a_n)f(t_n) = \mathrm{e}f(\mathrm{e})$$

而这里的话,Γ 函数是 $(0,+\infty)$ 上的实值连续函数, 因而极限是 $\mathrm{e}\Gamma(\mathrm{e})$.

□

137. 求和

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{2k-1}$$

解 首先有熟知的等式

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1} = \frac{\pi}{4} \quad \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{8}$$

因此

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{2k-1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\left(\frac{\pi}{4}+\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}\right)$$

$$
\begin{aligned}
&= -\frac{\pi^2}{16} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sum_{k=1}^{n} \frac{(-1)^k}{2k-1} \\
&= -\frac{\pi^2}{16} + \sum_{1 \leqslant k \leqslant n} \frac{(-1)^{n+k}}{(2k-1)(2n-1)} \\
&= -\frac{\pi^2}{16} + \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \right) \\
&= -\frac{\pi^2}{16} + \frac{1}{2} \left(\frac{\pi^2}{16} + \frac{\pi^2}{8} \right) = \frac{\pi^2}{32}
\end{aligned}
$$

□

138. 设 f 是 $[0,+\infty)$ 上的有界非负的连续函数, 求极限

$$
\lim_{n\to\infty} n \left(\sqrt[n]{\int_0^{\infty} f^{n+1}(x)\,\mathrm{e}^{-x}\mathrm{d}x} - \sqrt[n]{\int_0^{\infty} f^{n}(x)\,\mathrm{e}^{-x}\mathrm{d}x} \right)
$$

解 令 $h(t) = f(-\ln t)$, 则 h 是 $(0,1]$ 上有界非负的连续函数, 令 $a_n = \int_0^1 h^n(t)\mathrm{d}t$, 则

$$
L = \lim_{n\to\infty} n \left(\sqrt[n]{\int_0^{\infty} f^{n+1}(x)\,\mathrm{e}^{-x}\mathrm{d}x} - \sqrt[n]{\int_0^{\infty} f^{n}(x)\,\mathrm{e}^{-x}\mathrm{d}x} \right) = \lim_{n\to\infty} n \left(\sqrt[n]{a_{n+1}} - \sqrt[n]{a_n} \right)
$$

我们将证明 $L = M\ln M$, 这里 $M = \sup\limits_{x\in[0,+\infty)} f(x) = \sup\limits_{t\in(0,1]} h(t) \geqslant 0$, 如果 $M=0$, 则显然 $f = h = 0$, 则 $L = 0$, 下设 $M > 0$. 对 $0 < \varepsilon < M$, 存在 $(0,1]$ 内的一个非空区间 I 使得对任意 $t \in I, h(t) \geqslant M - \varepsilon > 0$, 因此

$$
(M-\varepsilon)|I|^{\frac{1}{n}} = \left((M-\varepsilon)^n |I|\right)^{\frac{1}{n}} \leqslant \sqrt[n]{a_n} \leqslant \left(M^n |(0,1]|\right)^{\frac{1}{n}} = M
$$

且 $|I|^{\frac{1}{n}} \to 1(|I|>0)$ 意味着 $\sqrt[n]{a_n} \to M$. 现在我们考虑数列 $\left\{\frac{a_{n+1}}{a_n}\right\}$, 它是有界的, 因为

$$
\frac{a_{n+1}}{a_n} = \frac{1}{a_n}\int_0^1 h^{n+1}(t)\,\mathrm{d}t \leqslant \frac{M}{a_n}\int_0^1 h^n(t)\,\mathrm{d}t + M
$$

它还是单调递增的, 因为利用 Cauchy 不等式得

$$
a_{n+1}^2 = \left(\int_0^1 h^{\frac{n+2}{2}}(t)\,h^{\frac{n}{2}}(t)\,\mathrm{d}t\right) \leqslant \int_0^1 h^{n+2}(t)\,\mathrm{d}t \int_0^1 h^n(t)\,\mathrm{d}t = a_{n+2}a_n
$$

所以 $\left\{\frac{a_{n+1}}{a_n}\right\}$ 存在极限 M', 由 Stolz-Cesàro 定理,

$$
\ln M = \lim_{n\to\infty} \ln\left(\sqrt[n]{a_n}\right) = \lim_{n\to\infty}\frac{\ln a_n}{n} = \lim_{n\to\infty}(\ln a_{n+1} - \ln a_n) = \lim_{n\to\infty}\ln\left(\frac{a_{n+1}}{a_n}\right) = \ln M'
$$

最后得到

$$
\begin{aligned}
&\lim_{n\to\infty} n\left(\sqrt[n]{a_{n+1}} - \sqrt[n]{a_n}\right) \\
&= \lim_{n\to\infty} n\sqrt[n]{a_n}\left(\exp\left(\frac{1}{n}\ln\left(\frac{a_{n+1}}{a_n}\right)\right) - 1\right) \\
&= \lim_{n\to\infty} \sqrt[n]{a_n}\ln\left(\frac{a_{n+1}}{a_n}\right) = M\ln M
\end{aligned}
$$

□

139. 计算二重积分

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1-\cos(x+y)}{2-\cos x-\cos y}\mathrm{d}x\mathrm{d}y$$

解 首先有

$$\frac{1-\cos(x+y)}{2-\cos x-\cos y}=\frac{1-\cos(x+y)}{2-2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}$$

作二重积分换元 $x=u+v, y=u-v$, 则 $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=2$, 于是积分域变为正方形 $(u,v):-\pi\leqslant u\pm v\leqslant\pi$, 由对称性

$$\begin{aligned}
I&=4\iint\limits_{0\leqslant u+v\leqslant\pi}\frac{1-\cos 2u}{1-\cos u\cos v}\mathrm{d}u\mathrm{d}v\\
&=4\int_0^{\pi}\left(\frac{1-\cos 2u}{\cos u}\int_0^{\pi-u}\frac{\mathrm{d}v}{\sec u-\cos v}\right)\mathrm{d}u\\
&=4\int_0^{\pi}\left(\frac{1-\cos 2u}{\cos u}\frac{2}{\sqrt{\sec^2 u-1}}\arctan\left(\sqrt{\frac{\sec u+1}{\sec u-1}}\tan\frac{v}{2}\right)\Bigg|_{v=0}^{\pi-u}\right)\mathrm{d}u\\
&=16\int_0^{\pi}\sin u\arctan\left(\cot^2\left(\frac{u}{2}\right)\right)\mathrm{d}u\\
&=64\int_0^{\infty}\frac{w}{(1+w^2)^2}\arctan(w^2)\mathrm{d}w\quad\left(w=\cot\left(\frac{u}{2}\right)\right)\\
&=32\int_0^{\infty}\frac{\arctan t}{(1+t)^2}\mathrm{d}t\\
&=8\pi
\end{aligned}$$

□

140. 记 $f(n)=\sum\limits_{k=1}^{n}k^k$, 令 $g(n)=\sum\limits_{k=1}^{n}f(k)$, 求极限

$$\lim_{n\to\infty}\left(\frac{g(n+2)}{g(n+1)}-\frac{g(n+1)}{g(n)}\right)$$

解 容易得到

$$g(n)=\sum_{k=1}^{n}(n+1-k)k^k=\sum_{k=1}^{n}(n+1-k)k^k=\sum_{k=1}^{n}k(n+1-k)^{n+1-k}$$

于是我们有

$$\frac{g(n)}{n}=\sum_{k=1}^{n}\frac{k}{n^{k-1}}\left(1-\frac{k-1}{n}\right)^{n+1-k}=1+\frac{2}{n}\left(1-\frac{1}{n}\right)^{n-1}+h(n)+o\left(\frac{1}{n}\right)$$

因为

$$0\leqslant h(n)=\sum_{k=3}^{n}\frac{k}{n^{k-1}}\left(1-\frac{k-1}{n}\right)^{n+1-k}\leqslant\frac{3}{n^2}+\frac{4}{n^3}+\frac{1}{n^4}\sum_{k=5}^{n}k\leqslant\frac{3}{n^2}+\frac{4}{n^3}+\frac{n^2+n}{2n^4}$$

因此

$$\begin{aligned}\frac{g(n+1)}{n^{n+1}} &= \left(1+\frac{1}{n}\right)^{n+1}\cdot\frac{g(n+1)}{(n+1)^{n+1}}\\&= \mathrm{e}\left(1+\frac{2}{n}+o\left(\frac{1}{n}\right)\right)\left(1+\frac{2\mathrm{e}^{-1}}{n}+o\left(\frac{1}{n}\right)\right)\\&= \mathrm{e}+\frac{\mathrm{e}+4}{2n}+o\left(\frac{1}{n}\right)\end{aligned}$$

且

$$\begin{aligned}\frac{g(n+2)}{n^{n+2}} &= \left(1+\frac{2}{n}\right)^{n+2}\cdot\frac{g(n+2)}{(n+2)^{n+2}}\\&= \mathrm{e}\left(1+\frac{2}{n}+o\left(\frac{1}{n}\right)\right)\left(1+\frac{2\mathrm{e}^{-1}}{n}+o\left(\frac{1}{n}\right)\right)\\&= \mathrm{e}^2+\frac{2\mathrm{e}^2+2\mathrm{e}}{n}+o\left(\frac{1}{n}\right)\end{aligned}$$

最后得到

$$\begin{aligned}&\frac{g(n+2)}{g(n+1)}-\frac{g(n+1)}{g(n)}\\=&\frac{g(n+2)g(n)-g^2(n+1)}{g(n+1)g(n)}\\=&\left(\frac{n}{\mathrm{e}}+o(1)\right)\left[\left(\mathrm{e}^2+\frac{2\mathrm{e}^2+2\mathrm{e}}{n}+o\left(\frac{1}{n}\right)\right)\left(1-\frac{2\mathrm{e}^{-1}}{n}+o\left(\frac{1}{n}\right)\right)-\left(\mathrm{e}+\frac{\mathrm{e}+4}{2n}+o\left(\frac{1}{n}\right)\right)^2\right]\\=&\left(\frac{n}{\mathrm{e}}+o(1)\right)\left(\frac{\mathrm{e}^2}{n}+o\left(\frac{1}{n}\right)\right)=\mathrm{e}+o\left(\frac{1}{n}\right)\to\mathrm{e}\end{aligned}$$

□

141. 计算二重积分

$$I=\int_0^{\infty}\left(\int_x^{\infty}\mathrm{e}^{-(x-y)^2}\sin^2\left(x^2+y^2\right)\frac{x^2-y^2}{\left(x^2+y^2\right)^2}\mathrm{d}y\right)\mathrm{d}x$$

解 转化为极坐标可得

$$\begin{aligned}I &= \int_0^{\infty}\left(\int_{\pi/4}^{\frac{\pi}{2}}\mathrm{e}^{-\rho^2(1-\sin 2\theta)}\sin^2\rho^2\frac{\rho^2\cos 2\theta}{\rho^4}\mathrm{d}\theta\right)\mathrm{d}\rho\\&= \int_0^{\infty}\frac{\mathrm{e}^{-\rho^2}\sin^2\rho^2}{2\rho^3}\left(\int_{\pi/4}^{\frac{\pi}{2}}\mathrm{e}^{\rho^2\sin 2\theta}\mathrm{d}\left(\rho^2\sin 2\theta\right)\right)\mathrm{d}\rho\\&= \int_0^{\infty}\frac{\mathrm{e}^{-\rho^2}\sin^2\rho^2}{2\rho^3}\left(\mathrm{e}^{\rho^2\sin 2\theta}\right)\Big|_{\theta=\frac{\pi}{4}}^{\pi}\mathrm{d}\rho\\&= \int_0^{\infty}\frac{\left(\mathrm{e}^{-\rho^2}-1\right)\sin^2\rho^2}{2\rho^3}\mathrm{d}\rho\\&= \frac{1}{4}\int_0^{\infty}\frac{\left(\mathrm{e}^{-t}-1\right)\sin^2 t}{t^2}\mathrm{d}t\end{aligned}$$

注意到

$$\int_0^{\infty}\frac{\sin^2 t}{t^2}\mathrm{d}t=-\int_0^{\infty}\sin^2 t\,\mathrm{d}\left(\frac{1}{t}\right)=\int_0^{\infty}\frac{\sin 2t}{t}\mathrm{d}t=\frac{\pi}{2}$$

对积分 $\int_0^{\infty}\frac{\mathrm{e}^{-t}\sin^2 t}{t^2}\mathrm{d}t$, 考虑参数积分

$$I\left(a\right)=\int_0^{\infty}\frac{\mathrm{e}^{-t}\sin^2 at}{t^2}\mathrm{d}t=\int_0^{\infty}\frac{\mathrm{e}^{-t}\left(1-\cos 2at\right)}{2t^2}\mathrm{d}t$$

$$I'\left(a\right)=\int_0^{\infty}\frac{\mathrm{e}^{-t}\sin 2at}{t}\mathrm{d}t,\ I''\left(a\right)=2\int_0^{\infty}\mathrm{e}^{-t}\cos 2at\mathrm{d}t=\frac{2}{1+4a^2}$$

注意到 $I(0)=I'(0)=0$, 于是

$$I'\left(a\right)=\arctan 2a,\ I''\left(a\right)=a\arctan 2a-\frac{1}{4}\ln\left(1+4a^2\right)$$

原积分为

$$I=\frac{1}{4}\left(I\left(1\right)-\frac{\pi}{2}\right)=\frac{1}{4}\arctan 2-\frac{1}{16}\ln 5-\frac{\pi}{8}$$

□

142. 求和

$$S=\sum_{n=1}^{\infty}\frac{2^{2n-1}}{2n+1}\left(\frac{(n-1)!}{(2n-1)!!}\right)^2$$

解 首先我们有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2^{2n-1}}{2n+1}\left(\frac{(n-1)!}{(2n-1)!!}\right)^2&=\sum_{n=1}^{\infty}\frac{2}{2n+1}\left(\frac{(2n-2)!!}{(2n-1)!!}\right)^2\\&=\sum_{n=1}^{\infty}\frac{2}{2n+1}\int_0^{\frac{\pi}{2}}\sin^{2n-1}x\mathrm{d}x\int_0^{\frac{\pi}{2}}\sin^{2n-1}y\mathrm{d}y\\&=\sum_{n=1}^{\infty}\frac{2}{2n+1}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^{2n-1}x\sin^{2n-1}y\mathrm{d}x\mathrm{d}y\end{aligned}$$

利用对数函数的幂级数公式不难得到

$$\sum_{n=1}^{\infty}\frac{2\sin^{2n-1}x\sin^{2n-1}y}{2n+1}=\frac{1}{\sin^2 x\sin^2 y}\left(\ln\frac{1+\sin x\sin y}{1-\sin x\sin y}-2\sin x\sin y\right)$$

考虑参变量积分

$$I\left(a\right)=\int_0^{\frac{\pi}{2}}\frac{1}{\sin^2 y}\left(\ln\frac{1+a\sin y}{1-a\sin y}-2a\sin y\right)\mathrm{d}y\quad |a|<1$$

则可得

$$\begin{aligned}&I(0)=0\\&I'\left(a\right)=\int_0^{\frac{\pi}{2}}\frac{1}{\sin y}\left(\frac{1}{1+a\sin y}+\frac{1}{1-a\sin y}-2\right)\mathrm{d}y\\&\quad=2a^2\int_0^{\frac{\pi}{2}}\frac{\sin y}{1-a^2\sin^2 y}\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= 2a^2\int_0^1 \frac{\mathrm{d}t}{1-a^2\left(1-t^2\right)} \quad (t=\cos y)\\
&= 2\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{t^2+\left(1-a^2\right)/a^2}\\
&= \frac{2a}{\sqrt{1-a^2}}\arctan\frac{a}{\sqrt{1-a^2}}
\end{aligned}
$$

那么

$$
\begin{aligned}
I(\sin x) &= \int_0^{\sin x} \frac{2a}{\sqrt{1-a^2}}\arctan\frac{a}{\sqrt{1-a^2}}\mathrm{d}a\\
&= 2\int_0^x u\sin u\mathrm{d}u \quad (a=\sin u)\\
&= 2(\sin x - x\cos x)
\end{aligned}
$$

于是

$$
\begin{aligned}
S &= \int_0^{\frac{\pi}{2}}\left[\int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x\sin^2 y}\left(\ln\frac{1+\sin x\sin y}{1-\sin x\sin y}-2\sin x\sin y\right)\mathrm{d}y\right]\mathrm{d}x\\
&= \int_0^{\frac{\pi}{2}} \frac{I(\sin x)}{\sin^2 x}\mathrm{d}x = 2\int_0^{\frac{\pi}{2}}\frac{\sin x - x\cos x}{\sin^2 x}\mathrm{d}x\\
&= -2(\sin x - x\cos)\cot x\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} x\cos x\mathrm{d}x\\
&= 2\int_0^{\frac{\pi}{2}} x\mathrm{d}(\sin x) = \pi - 2
\end{aligned}
$$

□

143. 计算二重积分

$$
\int_0^{\infty}\left(\int_y^{\infty}\frac{(x-y)^2\ln\left(\frac{x+y}{x-y}\right)}{xy\sinh(x+y)}\mathrm{d}x\right)\mathrm{d}y
$$

解 作积分换元 $u=x+y, v=\dfrac{x-y}{x+y}$, 积分化为

$$
\begin{aligned}
I &= \int_0^1\left(\int_0^{\infty}\frac{-(uv)^2\ln v}{\left(u^2\left(1-v^2\right)/4\right)\sinh u}\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\right)\mathrm{d}v\\
&= \int_0^1\frac{-2v^2\ln v}{1-v^2}\mathrm{d}v\int_0^{\infty}\frac{u}{\sinh u}\mathrm{d}u\\
&= \frac{\pi^4}{16}-\frac{\pi^2}{2}
\end{aligned}
$$

其中

$$
\begin{aligned}
\int_0^1\frac{-2v^2\ln v}{1-v^2}\mathrm{d}v &= 2\int_0^1\sum_{k=1}^{\infty}v^{2k}\ln v\mathrm{d}v\\
&= 2\sum_{k=1}^{\infty}\int_0^{\infty}\mathrm{e}^{-(2k+1)t}t\mathrm{d}t \quad (t=-\ln v)
\end{aligned}
$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{(2k+1)^2} = 2\left(\frac{\pi^2}{8}-1\right)$$

$$\int_0^{\infty}\frac{u}{\sinh u}\mathrm{d}u = \int_0^{\infty}\frac{2u\mathrm{e}^{-u}}{1-\mathrm{e}^{-2u}}\mathrm{d}u$$

$$= 2\sum_{k=0}^{\infty}\int_0^{\infty}u\mathrm{e}^{-(2k+1)u}\mathrm{d}u = \frac{\pi^2}{4}$$

□

144. 求极限

$$\lim_{n\to\infty}\frac{1}{n}\int_0^n\left(\frac{x\ln(1+x/n)}{1+x}\right)\mathrm{d}x$$

解 令 $t=\dfrac{x}{n}$, 则

$$\frac{1}{n}\int_0^n\left(\frac{x\ln(1+x/n)}{1+x}\right)\mathrm{d}x = \int_0^1\frac{nt}{1+nt}\ln(1+t)\,\mathrm{d}t$$

$$= \int_0^1\ln(1+t)\,\mathrm{d}t - \int_0^1\frac{t}{1+nt}\frac{\ln(1+t)}{t}\mathrm{d}t$$

$$= 2\ln 2 - 1 - \int_0^1\frac{t}{1+nt}\frac{\ln(1+t)}{t}\mathrm{d}t$$

由于

$$0 \leqslant \int_0^1\frac{t}{1+nt}\frac{\ln(1+t)}{t}\mathrm{d}t \leqslant \frac{1}{n}\int_0^1\frac{\ln(1+t)}{t}\mathrm{d}t < \frac{1}{n}$$

因此

$$\lim_{n\to\infty}\frac{1}{n}\int_0^n\left(\frac{x\ln(1+x/n)}{1+x}\right)\mathrm{d}x = 2\ln 2 - 1$$

□

145. 计算二重积分

$$I = \int_0^{\infty}\frac{1}{x}\int_0^x\frac{\cos(x-y)-\cos x}{y}\mathrm{d}y\mathrm{d}x$$

解 考虑参变量积分

$$I(t) = \int_0^{\infty}\frac{1}{x}\int_0^x\frac{\cos(x-ty)-\cos x}{y}\mathrm{d}y\mathrm{d}x$$

则

$$I(0) = 0$$

$$I'(t) = \int_0^{\infty}\frac{1}{x}\int_0^x\sin(x-ty)\,\mathrm{d}y\mathrm{d}x$$

$$= \int_0^{\infty}\frac{1}{x}\left(\frac{1}{t}\cos(x-ty)\bigg|_{y=0}^{y=x}\right)\mathrm{d}x$$

$$= \int_0^{\infty}\frac{\cos(1-t)x-\cos x}{tx}\mathrm{d}x$$

$$= -\frac{\ln(1-t)}{t}$$

上面最后一步我们利用了 Frullani 积分, 于是

$$
\begin{aligned}
I &= \int_0^\infty \frac{1}{x}\int_0^x \frac{\cos(x-y)-\cos x}{y}\mathrm{d}y\mathrm{d}x \\
&= -\int_0^1 \frac{\ln(1-t)}{t}\mathrm{d}t = \int_0^1 \sum_{k=1}^\infty \frac{t^{k-1}}{k}\mathrm{d}t = \sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}
\end{aligned}
$$

□

146. 设函数 $f:[0,1]\to\mathbb{R}$ 是连续可微函数, 证明不等式

$$\int_0^1 [f'(x)]^2\mathrm{d}x \geqslant 12\left(\int_0^1 f(x)\,\mathrm{d}x - 2\int_0^{1/2} f(x)\,\mathrm{d}x\right)^2$$

证明 利用 Cauchy 不等式得

$$\int_0^{1/2}[f'(x)]^2\mathrm{d}x\int_0^{1/2}x^2\mathrm{d}x \geqslant \left(\int_0^{1/2} xf'(x)\,\mathrm{d}x\right)^2 = \left[\frac{1}{2}f\left(\frac{1}{2}\right) - \int_0^{1/2} f(x)\,\mathrm{d}x\right]^2$$

$$\Rightarrow \int_0^{1/2}[f'(x)]^2\,\mathrm{d}x \geqslant 24\left[\frac{1}{2}f\left(\frac{1}{2}\right) - \int_0^{1/2} f(x)\,\mathrm{d}x\right]^2$$

再利用 Cauchy 不等式得

$$\int_{1/2}^1[f'(x)]^2\,\mathrm{d}x\int_{1/2}^1(1-x)^2\,\mathrm{d}x \geqslant \left[-\frac{1}{2}f\left(\frac{1}{2}\right) + \int_{1/2}^1 f(x)\,\mathrm{d}x\right]^2$$

$$\Rightarrow \int_{1/2}^1[f'(x)]^2\,\mathrm{d}x \geqslant 24\left[-\frac{1}{2}f\left(\frac{1}{2}\right) + \int_{1/2}^1 f(x)\,\mathrm{d}x\right]^2$$

两式相加, 利用不等式 $2(a^2+b^2)\geqslant(a+b)^2$ 得

$$
\begin{aligned}
\int_{10}^1[f'(x)]^2\,\mathrm{d}x &\geqslant 24\left[\left(\frac{1}{2}f\left(\frac{1}{2}\right) - \int_0^{1/2} f(x)\,\mathrm{d}x\right)^2 + \left(-\frac{1}{2}f\left(\frac{1}{2}\right) + \int_{1/2}^1 f(x)\,\mathrm{d}x\right)^2\right] \\
&\geqslant 12\left(\int_0^1 f(x)\,\mathrm{d}x - 2\int_0^{1/2} f(x)\,\mathrm{d}x\right)^2
\end{aligned}
$$

特别地, 当 $\int_0^{1/2} f(x)\,\mathrm{d}x = 0$ 时, 我们有

$$\int_0^1[f'(x)]^2\,\mathrm{d}x \geqslant 12\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

□

147. 求和

$$\sum_{n=1}^\infty \frac{H_{n+2}}{n(n+2)}$$

解 首先注意到

$$H_{n+2}=\sum_{k=1}^{n+2}\frac{1}{k}=\int_0^1\sum_{k=0}^{n}x^k\mathrm{d}x=\int_0^1\frac{1-x^{n+2}}{1-x}\mathrm{d}x$$

于是

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_{n+2}}{n\left(n+2\right)}&=\int_0^1\frac{1}{1-x}\sum_{n=1}^{\infty}\frac{1-x^{n+2}}{n\left(n+2\right)}\mathrm{d}x\\&=\int_0^1\frac{1}{1-x}\left(\frac{3}{4}-\frac{x}{2}-\frac{x^2}{4}-2\left(1-x^2\right)\ln\left(1-x\right)\right)\mathrm{d}x\\&=\int_0^1\left(\frac{x+3}{4}+\frac{1}{2}\left(1+x\right)\ln\left(1-x\right)\right)\mathrm{d}x\\&=\frac{7}{4}\end{aligned}$$

□

148. 求和

$$\sum_{n=1}^{\infty}\arctan\left(\sinh n\right)\cdot\arctan\left(\frac{\sinh 1}{\cosh n}\right)$$

解 注意到

$$\begin{aligned}\arctan\left(\sinh n\right)&=\arctan\left(\frac{\mathrm{e}^n-\mathrm{e}^{-n}}{2}\right)=\arctan\left(\frac{\mathrm{e}^n-\mathrm{e}^{-n}}{1+\mathrm{e}^n\cdot\mathrm{e}^{-n}}\right)\\&=\arctan\left(\mathrm{e}^n\right)-\arctan\left(\mathrm{e}^{-n}\right)=2\arctan\left(\mathrm{e}^n\right)-\frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\arctan\left(\frac{\sinh 1}{\cosh n}\right)&=\arctan\left(\frac{\mathrm{e}-\mathrm{e}^{-1}}{\mathrm{e}^n+\mathrm{e}^{-n}}\right)=\arctan\left(\frac{\mathrm{e}^{n+1}-\mathrm{e}^{n-1}}{1+\mathrm{e}^{n+1}\cdot\mathrm{e}^{n-1}}\right)\\&=\arctan\left(\mathrm{e}^{n+1}\right)-\arctan\left(\mathrm{e}^{n-1}\right)\end{aligned}$$

因此

$$\begin{aligned}&\sum_{n=1}^{\infty}\arctan\left(\sinh n\right)\cdot\arctan\left(\frac{\sinh 1}{\cosh n}\right)\\=&\sum_{n=1}^{\infty}\left[2\arctan\left(\mathrm{e}^n\right)-\frac{\pi}{2}\right]\left[\arctan\left(\mathrm{e}^{n+1}\right)-\arctan\left(\mathrm{e}^{n-1}\right)\right]\\=&2\left[\lim_{n\to\infty}\arctan\left(\mathrm{e}^n\right)\arctan\left(\mathrm{e}^{n+1}\right)-\frac{\pi}{4}\arctan\left(\mathrm{e}\right)\right]\\&-\frac{\pi}{2}\left[\lim_{n\to\infty}\left(\arctan\left(\mathrm{e}^n\right)+\arctan\left(\mathrm{e}^{n+1}\right)\right)-\frac{\pi}{4}-\arctan\left(\mathrm{e}\right)\right]\\=&2\left(\frac{\pi^2}{4}-\frac{\pi}{4}\arctan\left(\mathrm{e}\right)\right)-\frac{\pi}{2}\left(\frac{3}{4}\pi-\arctan\left(\mathrm{e}\right)\right)=\frac{\pi^2}{8}\end{aligned}$$

□

149. 求和

$$\sum_{n=1}^{\infty}\arctan\left(\frac{1}{n^2}\right)$$

解

$$
\begin{aligned}
\tan\left[\sum_{n=1}^{\infty}\arctan\left(\frac{1}{n^2}\right)\right] &= \tan\left[\sum_{n=1}^{\infty}\arg\left(1+\frac{\mathrm{i}}{n^2}\right)\right]\\
&= \tan\left[\arg\left(\prod_{n=1}^{\infty}\left(1+\frac{\left(\pi(1+\mathrm{i})/\sqrt{2}\right)^2}{n^2\pi^2}\right)\right)\right]\\
&= \tan\left[\arg\left(\frac{\sinh\left(\pi(1+\mathrm{i})/\sqrt{2}\right)}{\pi(1+\mathrm{i})/\sqrt{2}}\right)\right]\\
&= \tan\left[\arctan\left(\frac{\tan\left(\pi/\sqrt{2}\right)-\tanh\left(\pi/\sqrt{2}\right)}{\tan\left(\pi/\sqrt{2}\right)+\tanh\left(\pi/\sqrt{2}\right)}\right)\right]\\
&= \frac{\tan\left(\pi/\sqrt{2}\right)-\tanh\left(\pi/\sqrt{2}\right)}{\tan\left(\pi/\sqrt{2}\right)+\tanh\left(\pi/\sqrt{2}\right)}
\end{aligned}
$$

因此

$$
\sum_{n=1}^{\infty}\arctan\left(\frac{1}{n^2}\right)=\arctan\left(\frac{\tan\left(\pi/\sqrt{2}\right)-\tanh\left(\pi/\sqrt{2}\right)}{\tan\left(\pi/\sqrt{2}\right)+\tanh\left(\pi/\sqrt{2}\right)}\right)
$$

□

150. 计算积分

$$
\int_0^1\frac{\arctan x}{\sqrt{1-x^2}}\mathrm{d}x
$$

解 考虑参变量积分 $I(a)=\int_0^1\frac{\arctan(ax)}{\sqrt{1-x^2}}\mathrm{d}x$, 则

$$
I(0)=0,\quad I(\infty)=\frac{\pi}{2}\int_0^1\frac{\mathrm{d}x}{\sqrt{1-x^2}}=\frac{\pi}{2}\arcsin 1=\frac{\pi^2}{4}
$$

$$
\begin{aligned}
I'(a) &= \int_0^1\frac{x\mathrm{d}x}{(1+a^2x^2)\sqrt{1-x^2}}\\
&= \int_0^1\frac{\mathrm{d}t}{1+a^2-a^2t^2}\qquad\left(t=\sqrt{1-x^2}\right)\\
&= \frac{1}{a\sqrt{1+a^2}}\operatorname{arctanh}\left(\frac{at}{\sqrt{1+a^2}}\right)\Bigg|_0^1\\
&= \frac{1}{a\sqrt{1+a^2}}\operatorname{arctanh}\left(\frac{a}{\sqrt{1+a^2}}\right)=\frac{\operatorname{arcsinh} a}{a\sqrt{1+a^2}}
\end{aligned}
$$

于是

$$
\begin{aligned}
\int_0^1\frac{\arctan x}{\sqrt{1-x^2}}\mathrm{d}x &= \int_0^1 I'(a)\mathrm{d}a=-\int_0^1\frac{\operatorname{arcsinh} a}{\sqrt{1+1/a^2}}\mathrm{d}\left(\frac{1}{a}\right)\\
&= -\int_0^1\operatorname{arcsinh} a\,\mathrm{d}\left(\operatorname{arcsinh}\left(\frac{1}{a}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\operatorname{arcsinh}^2 1 + \int_0^1 \frac{1}{\sqrt{1+a^2}} \operatorname{arcsinh}\left(\frac{1}{a}\right) \mathrm{d}a \\
&= \int_1^\infty \frac{\operatorname{arcsinh} u}{u\sqrt{1+u^2}} \mathrm{d}u - \operatorname{arcsinh}^2 1 = \int_1^\infty I'(u)\,\mathrm{d}u - \operatorname{arcsinh}^2 1 \\
&= I(\infty) - I(1) - \operatorname{arcsinh}^2 1 = \frac{1}{2}\left[I(\infty) - \operatorname{arcsinh}^2 1\right] \\
&= \frac{1}{2}\left[\frac{\pi^2}{4} - \ln^2\left(1+\sqrt{2}\right)\right] = \frac{\pi^2}{8} - \frac{1}{2}\ln^2\left(1+\sqrt{2}\right)
\end{aligned}
$$

□

151. 求和

$$
\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2}
$$

解 首先注意到

$$
\begin{aligned}
\frac{1}{\binom{2n}{n}(2n+1)^2} &= \frac{(n!)^2}{(2n+1)!\,(2n+1)} \\
&= \frac{1}{2n+1} \cdot \frac{n!}{(2n+1)!! \cdot 2^n} \\
&= \frac{1}{2^{2n}(2n+1)} \cdot \frac{(2n)!!}{(2n+1)!!} \\
&= \frac{1}{2^{2n}(2n+1)} \int_0^{\frac{\pi}{2}} \cos^{2n+1} x \mathrm{d}x
\end{aligned}
$$

于是

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2} &= \sum_{n=0}^{\infty} \frac{1}{2n+1} \int_0^{\frac{\pi}{2}} \left(\frac{\cos x}{2}\right)^{2n+1} \mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\frac{2+\cos x}{2-\cos x}\right) \mathrm{d}x
\end{aligned}
$$

考虑参数积分

$$
\begin{aligned}
I(t) &= \int_0^{\frac{\pi}{2}} \ln\left(\frac{2+t\cos x}{2-t\cos x}\right) \mathrm{d}x \\
I'(t) &= \int_0^{\frac{\pi}{2}} \frac{\cos x}{2+t\cos x} \mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{\cos x}{2-t\cos x} \mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{4\cos x}{4-t^2\cos^2 x} \mathrm{d}x = 4\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}(\sin x)}{(4-t^2)+t^2\sin^2 x} \\
&= \frac{4}{t\sqrt{4-t^2}} \arctan \frac{t}{\sqrt{4-t^2}}
\end{aligned}
$$

因此

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2} &= I(1) = \int_0^1 \frac{4}{t\sqrt{4-t^2}} \arctan \frac{t}{\sqrt{4-t^2}} \mathrm{d}t + I(0) \\
&= 2\int_0^{\frac{\pi}{6}} \frac{u}{\sin u} \mathrm{d}u \quad (t = 2\sin u)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_0^{\frac{\pi}{6}} u\mathrm{d}\left(\ln\left(\tan\frac{u}{2}\right)\right) = -\frac{\pi}{3}\ln\left(2+\sqrt{3}\right) - 2\int_0^{\frac{\pi}{6}}\ln\left(\tan\frac{u}{2}\right)\mathrm{d}u \\
&= 4\int_0^{\frac{\pi}{6}}\sum_{k=1}^{\infty}\frac{\cos(2k-1)u}{2k-1}\mathrm{d}u - \frac{\pi}{3}\ln\left(2+\sqrt{3}\right) \\
&= \frac{8}{3}G - \frac{\pi}{3}\ln\left(2+\sqrt{3}\right)
\end{aligned}
$$

□

152. 设 H_n 表示第 n 个调和数, 即 $H_n = \sum_{k=1}^{n}\frac{1}{k}$, $E_n = H_n^2 - \sum_{k=1}^{n}\frac{H_{\max(k,n-k)}}{k}$, 求 $\lim_{n\to\infty} E_n$.

解 令 $H_n^{(2)} = \sum_{k=1}^{n}\frac{1}{k^2}$, 我们将要证明的是

$$E_n = \frac{1}{2}H_{\left[\frac{n}{2}\right]}^{(2)}$$

那么就有

$$\lim_{n\to\infty} E_n = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{12}$$

由于 $E_1 = \frac{1}{2}H_0^{(2)} = 0$, 且对 $n > 0$ 有

$$\frac{1}{2}H_{\left[\frac{n+1}{2}\right]}^{(2)} - \frac{1}{2}H_{\left[\frac{n}{2}\right]}^{(2)} = \begin{cases} 0, & n\text{是偶数} \\ \dfrac{2}{(n+1)^2}, & n\text{是奇数} \end{cases}$$

只需要证明差分 $E_{n+1} - E_n$ 满足相同的性质即可, 注意到

$$E_n = H_n^2 - \sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{H_{n-k}}{k} - \sum_{k=\left[\frac{n}{2}\right]+1}^{n}\frac{H_k}{k}$$

先假定 n 是正整数, 则 $\left[\frac{n}{2}\right] = \left[\frac{n+1}{2}\right] = \frac{n}{2}$, 且

$$
\begin{aligned}
E_{n+1} - E_n &= H_{n+1}^2 - H_n^2 - \sum_{k=1}^{\frac{n}{2}}\frac{H_{n+1-k}}{k} - \frac{H_{n+1}}{n+1} \\
&= \frac{H_{n+1}+H_n}{n+1} - \sum_{k=1}^{\frac{n}{2}}\frac{1}{k(n+1-k)} - \frac{H_{n+1}}{n+1} \\
&= \frac{H_n}{n+1} - \frac{1}{n+1}\sum_{k=1}^{\frac{n}{2}}\left(\frac{1}{n+1-k} + \frac{1}{k}\right) = 0
\end{aligned}
$$

再假定 n 是奇数, 则 $\left[\frac{n}{2}\right] = \frac{n-1}{2}$, $\left[\frac{n+1}{2}\right] = \frac{n+1}{2}$, 且

$$E_{n+1} - E_n = H_{n+1}^2 - H_n^2 - \sum_{k=1}^{\frac{n-1}{2}}\frac{H_{n+1-k} - H_{n-k}}{k} - \frac{H_{n+1}}{n+1} + \frac{2H_{\frac{n+1}{2}}}{n+1}$$

$$
\begin{aligned}
&= \frac{H_{n+1}+H_n}{n+1} - \frac{1}{n+1}\sum_{k=1}^{\frac{n-1}{2}}\left(\frac{1}{n+1-k}+\frac{1}{k}\right) - \frac{H_{n+1}}{n+1} \\
&= \frac{H_n}{n+1} - \frac{1}{n+1}\left(H_n - \frac{2}{n+1}\right) = \frac{2}{(n+1)^2}
\end{aligned}
$$

□

153. 求和

$$
\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(n+1)}
$$

解 注意到对 $0 \leqslant x < 1$ 有

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n}x^n &= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{\infty}\left(\frac{\sqrt{x}}{k}\right)^{2n} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\sqrt{x}}{k}\right)^{2n} \\
&= -\ln\prod_{k=1}^{\infty}\left(1-\left(\frac{\sqrt{x}}{k}\right)^2\right) = -\ln\left(\frac{\sin\left(\pi\sqrt{x}\right)}{\pi\sqrt{x}}\right)
\end{aligned}
$$

于是我们有

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(n+1)} &= \int_0^1\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n}x^n\,\mathrm{d}x = -\int_0^1 \ln\left(\frac{\sin\left(\pi\sqrt{x}\right)}{\pi\sqrt{x}}\right)\mathrm{d}x \\
&= -\int_0^1 \ln\left(\sin\pi\sqrt{x}\right)\mathrm{d}x + \int_0^1 \ln\pi\,\mathrm{d}x + \frac{1}{2}\int_0^1 \ln x\,\mathrm{d}x \\
&= -\frac{2}{\pi^2}\int_0^{\pi} t\ln(\sin t)\,\mathrm{d}t + \ln\pi - \frac{1}{2} = \ln(2\pi) - \frac{1}{2}
\end{aligned}
$$

□

154. 设 r 是一个整数, 求和

$$
\sum_{n=-\infty}^{\infty}\arctan\left(\frac{\sinh r}{\cosh n}\right)
$$

解 首先有

$$
\begin{aligned}
\arctan\left(\frac{\sinh r}{\cosh n}\right) &= \arctan\left(\frac{\mathrm{e}^r-\mathrm{e}^{-r}}{\mathrm{e}^n+\mathrm{e}^{-n}}\right) = \arctan\left(\frac{\mathrm{e}^{-(n-r)}-\mathrm{e}^{-(n+r)}}{1+\mathrm{e}^{-2n}}\right) \\
&= \arctan\left(\mathrm{e}^{-(n-r)}\right) - \arctan\left(\mathrm{e}^{-(n+r)}\right)
\end{aligned}
$$

不失一般性, 不妨设 $r \geqslant 0$, 我们有

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}\arctan\left(\frac{\sinh r}{\cos hn}\right) \\
&= 2\sum_{n=1}^{\infty}\arctan\left(\frac{\sinh r}{\cosh n}\right) + \arctan(\sinh r) \\
&= 2\sum_{n=1}^{\infty}\left(\arctan\left(\mathrm{e}^{-(n-r)}\right) - \arctan\left(\mathrm{e}^{-(n+r)}\right)\right) + \arctan\left(\mathrm{e}^r\right) - \arctan\left(\mathrm{e}^{-r}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{m\geqslant 1-r}\arctan\left(\mathrm{e}^{-m}\right) - 2\sum_{m\geqslant 1+r}\arctan\left(\mathrm{e}^{-m}\right) + \arctan\left(\mathrm{e}^{r}\right) - \arctan\left(\mathrm{e}^{-r}\right) \\
&= 2\sum_{1-r\leqslant m\leqslant r}\arctan\left(\mathrm{e}^{-m}\right) + \arctan\left(\mathrm{e}^{r}\right) - \arctan\left(\mathrm{e}^{-r}\right) \\
&= 2\sum_{-r\leqslant m\leqslant r}\arctan\left(\mathrm{e}^{-m}\right) - \arctan\left(\mathrm{e}^{r}\right) - \arctan\left(\mathrm{e}^{-r}\right) \\
&= 2\sum_{1\leqslant m\leqslant r}\left[\arctan\left(\mathrm{e}^{m}\right) + \arctan\left(\mathrm{e}^{-m}\right)\right] + 2\arctan(1) - \arctan\left(\mathrm{e}^{r}\right) - \arctan\left(\mathrm{e}^{-r}\right) \\
&= 2\sum_{1\leqslant m\leqslant r}\frac{\pi}{2} + 2\cdot\frac{\pi}{4} - \frac{\pi}{2} = \pi r
\end{aligned}
$$

□

155. 求和

$$\sum_{n=1}^{\infty}\operatorname{arcsinh}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right)$$

解 记

$$a_n = \frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}, \quad b_n = \frac{\sqrt{2^b+1}-\sqrt{3}}{2^{\frac{n+1}{2}}}$$

不难得到

$$b_{n+1}\sqrt{1+b_n^2} - b_n\sqrt{1+b_{n+1}^2} = a_n$$

根据基本性质

$$\operatorname{arcsinh}\left(x\sqrt{1+y^2} - y\sqrt{1+x^2}\right) = \operatorname{arcsinh}(x) - \operatorname{arcsinh}(y)$$

我们得到

$$\sum_{n=1}^{N}\operatorname{arcsinh}(a_n) = \sum_{n=1}^{N}\left(\operatorname{arcsinh}(b_{n+1}) - \operatorname{arcsinh}(b_n)\right) = \operatorname{arcsinh}(b_{N+1}) - \operatorname{arcsinh}(b_1)$$

现在 $b_1 = 0, b_{N+1} \to \frac{1}{\sqrt{2}}$, 因此

$$
\begin{aligned}
\sum_{n=1}^{\infty}\operatorname{arcsinh}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right) &= \lim_{N\to\infty}\operatorname{arcsinh}(b_{N+1}) \\
&= \operatorname{arcsinh}\left(\frac{1}{\sqrt{2}}\right) = \frac{\ln\left(2+\sqrt{3}\right)}{2}
\end{aligned}
$$

□

156. 设 F_k 表示第 k 个 Fibonacci 数, 求和

$$\sum_{n=1}^{\infty}\left(\arctan\frac{1}{F_{4n-3}} + \arctan\frac{1}{F_{4n-2}} + \arctan\frac{1}{F_{4n-1}} - \arctan\frac{1}{F_{4n}}\right)$$

解 令 $\varphi = \frac{\sqrt{5}+1}{2}$, 对 $n > 0$ 有

$$\arctan\left(\varphi^{-(2n+1)}\right) + \arctan\left(\varphi^{-(2n-1)}\right) = \arctan\left(\frac{\varphi^{-(2n+1)}+\varphi^{-(2n-1)}}{1-\varphi^{-4n}}\right)$$

$$
\begin{aligned}
&= \arctan\left(\frac{\varphi^{-1}+\varphi}{\varphi^{2n}-\varphi^{-2n}}\right) = \arctan\left(\frac{\sqrt{5}}{\varphi^{2n}-\varphi^{-2n}}\right) \\
&= \arctan\left(\frac{1}{F_{2n}}\right)
\end{aligned}
$$

进一步, 对 $n>1$,

$$
\begin{aligned}
\arctan\frac{1}{F_{2n}} + \arctan\frac{1}{F_{2n-1}} &= \arctan\left(\frac{F_{2n}^{-1}+F_{2n-1}^{-1}}{1-F_{2n}^{-1}F_{2n-1}^{-1}}\right) \\
&= \arctan\left(\frac{F_{2n}+F_{2n-1}}{F_{2n}F_{2n-1}-1}\right) = \arctan\left(\frac{F_{2n+1}}{F_{2n}F_{2n-1}-1}\right) \\
&= \arctan\frac{1}{F_{2n-2}}
\end{aligned}
$$

因此

$$
\begin{aligned}
&\sum_{n=1}^{N}\left(\arctan\frac{1}{F_{4n-2}} - \arctan\frac{1}{F_{4n}}\right) \\
&= \sum_{n=1}^{2N}(-1)^{n+1}\arctan\frac{1}{F_{2n}} \\
&= \sum_{n=1}^{2N}(-1)^{n+1}\left(\arctan\left(\varphi^{-(2n+1)}\right)+\arctan\left(\varphi^{-(2n-1)}\right)\right) \\
&= \sum_{n=2}^{2N+1}(-1)^{n+1}\arctan\left(\varphi^{-(2n+1)}\right) + \sum_{n=1}^{2N}(-1)^{n+1}\arctan\left(\varphi^{-(2n-1)}\right) \\
&= \arctan\left(\varphi^{-1}\right) - \arctan\left(\varphi^{-(4N+1)}\right) \\
&= \frac{\pi}{2} - \arctan(\varphi) - \arctan\left(\varphi^{-(4N+1)}\right)
\end{aligned}
$$

并且

$$
\begin{aligned}
\sum_{n=1}^{N}\left(\arctan\frac{1}{F_{4n-3}} + \arctan\frac{1}{F_{4n-1}}\right) &= \sum_{n=1}^{2N}\arctan\frac{1}{F_{2n-1}} \\
&= \arctan\frac{1}{F_1} + \sum_{n=2}^{2N}\left(\arctan\frac{1}{F_{2n-2}} - \arctan\frac{1}{F_{2n}}\right) \\
&= \arctan\frac{1}{F_1} + \arctan\frac{1}{F_2} - \arctan\frac{1}{F_{4N}} \\
&= \frac{\pi}{2} - \arctan\frac{1}{F_{4N}}
\end{aligned}
$$

最后得到

$$
\sum_{n=1}^{\infty}\left(\arctan\frac{1}{F_{4n-3}} + \arctan\frac{1}{F_{4n-2}} + \arctan\frac{1}{F_{4n-1}} - \arctan\frac{1}{F_{4n}}\right) = \pi - \arctan(\varphi)
$$

□

157. 记 C_n 是第 n 个 Catalan 数, 即 $C_n=\dfrac{1}{n+1}\dbinom{2n}{n}$, 证明:

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{C_n}=5+\frac{3\pi}{2}$; (b) $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{C_n}=22+8\sqrt{3}\pi$.

解 首先有展开式

$$f(z)=\left(\arcsin\left(\frac{z}{2}\right)\right)^2=\frac{1}{2}\sum_{n=1}^{\infty}\frac{z^{2n}}{n^2\binom{2n}{n}}$$

因此

$$\begin{aligned}D_z\left(z^3D_z\left(zD_zf\right)\right)&=D_z\left(z^3D_z\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{z^{2n}}{n^2}\binom{2n}{n}\right)\right)\\&=D_z\left(z^3D_z\left(zD_zf\right)\right)=D_z\left(z^3D_z\left(\sum_{n=1}^{\infty}\frac{z^{2n}}{n\binom{2n}{n}}\right)\right)\\&=2D_z\left(\sum_{n=1}^{\infty}\frac{z^{2n+2}}{\binom{2n}{n}}\right)=4z\sum_{n=1}^{\infty}\frac{z^{2n}}{C_n}\end{aligned}$$

因此

$$\sum_{n=1}^{\infty}\frac{z^{2n}}{C_n}=1+\frac{1}{4z}D_z\left(z^3D_z\left(zD_zf\right)\right)=g(z)=\frac{2\left(8+z^2\right)}{\left(4-z^2\right)^2}+\frac{24z\arcsin\left(\frac{z}{2}\right)}{\left(4-z^2\right)^{\frac{5}{2}}}$$

意味着 (a) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{C_n}=5+\frac{3\pi}{2}$; (b) $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{C_n}=22+8\sqrt{3}\pi$.

□

158. 求极限

$$\lim_{n\to\infty}n\prod_{m=1}^{n}\left(1-\frac{1}{m}+\frac{5}{4m^2}\right)$$

解 令 $z=-\dfrac{1}{2}+\mathrm{i}$, 则

$$\begin{aligned}\mathrm{e}^{\gamma z}\prod_{m=1}^{n}\left(1+\frac{z}{m}\right)\mathrm{e}^{-\frac{z}{m}}\cdot\mathrm{e}^{\gamma\bar{z}}\prod_{m=1}^{n}\left(1+\frac{\bar{z}}{m}\right)\mathrm{e}^{-\frac{\bar{z}}{m}}&=\mathrm{e}^{H_n-\gamma}\prod_{m=1}^{n}\left|1+\frac{z}{m}\right|^2\\&=\frac{\mathrm{e}^{H_n-\gamma}}{n}n\prod_{m=1}^{n}\left(1-\frac{1}{m}+\frac{5}{4m^2}\right)\end{aligned}$$

其中 $H_n=\displaystyle\sum_{k=1}^{n}\frac{1}{k}$, 由于 $H_n=\ln n+\gamma+o(1)$, 则 $\dfrac{\mathrm{e}^{H_n-\gamma}}{n}\to 1$.

进一步, 由 Γ 函数的定义

$$\frac{1}{z\Gamma(z)}=\mathrm{e}^{\gamma z}\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)\mathrm{e}^{-\frac{z}{m}}$$

因此

$$\lim_{n\to\infty}\prod_{m=1}^{n}\left(1-\frac{1}{m}+\frac{5}{4m^2}\right)=\frac{1}{z\Gamma(z)}\cdot\frac{1}{\bar{z}\Gamma(\bar{z})}=\frac{1}{\Gamma(z+1)}\cdot\frac{1}{\Gamma(\bar{z}+1)}$$

由于 $z+1=\frac{1}{2}+\mathrm{i},\bar{z}+1=\frac{1}{2}-\mathrm{i}=1-(z+1)$, 由余元公式

$$\lim_{n\to\infty}\prod_{m=1}^{n}\left(1-\frac{1}{m}+\frac{5}{4m^2}\right)=\frac{\sin\left(\pi\left(z+1\right)\right)}{\pi}=\frac{\cos\left(\pi\mathrm{i}\right)}{\pi}=\frac{\cosh\left(\pi\right)}{\pi}$$

□

159. ψ 表示 Digamma 函数,$\varphi=\frac{1+\sqrt{5}}{2}$, 证明

$$\sum_{n=1}^{\infty}\frac{\psi\left(n+\varphi\right)-\psi\left(n-\frac{1}{\varphi}\right)}{n^2+n-1}=\frac{\pi^2}{2\sqrt{5}}+\frac{\pi^2\tan^2\left(\frac{\sqrt{5}\pi}{2}\right)}{\sqrt{5}}+\frac{4\pi\tan\left(\frac{\sqrt{5}\pi}{2}\right)}{5}$$

证明 由于 $\psi(n+z)=\psi(z)+\sum_{k=0}^{n-1}\frac{1}{k+z}$, 那么如果 a 和 b 不是非正整数, 以及任意 $N\geqslant 0$ 有

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\psi\left(n+a\right)-\psi\left(n+b\right)}{\left(n+a\right)\left(n+b\right)}\\&=\left(\psi\left(a\right)-\psi\left(b\right)\right)\sum_{n=0}^{N}\frac{1}{\left(n+a\right)\left(n+b\right)}\\&\quad+\frac{b-a}{2}\left(\left(\sum_{n=0}^{N}\frac{1}{\left(n+a\right)\left(n+b\right)}\right)^2-\sum_{n=0}^{N}\frac{1}{\left(\left(n+a\right)\left(n+b\right)\right)^2}\right)\end{aligned}$$

现在 $a=\varphi,b=-\frac{1}{\varphi}$, 令 $N\to\infty$, 我们得到

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\psi\left(n+\varphi\right)-\psi\left(n-\frac{1}{\varphi}\right)}{n^2+n-1}\\&=\left(\psi\left(\varphi\right)-\psi\left(-\frac{1}{\varphi}\right)\right)\left(1+\sum_{n=0}^{\infty}\frac{1}{n^2+n-1}\right)\\&\quad-\frac{\sqrt{5}}{2}\left(\left(\sum_{n=0}^{N}\frac{1}{n^2+n-1}\right)^2-\sum_{n=0}^{N}\frac{1}{\left(n^2+n-1\right)^2}\right)\end{aligned}$$

(1) 利用余元公式可以得出

$$\psi\left(\varphi\right)-\psi\left(-\frac{1}{\varphi}\right)=\psi\left(\varphi\right)-\psi\left(1-\varphi\right)=\frac{\pi}{\tan\left(\pi\varphi\right)}=\pi\tan\left(\frac{\sqrt{5}\pi}{2}\right)$$

(2) 利用留数定理

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n^2+n-1}&=\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2+n-1}\\&=\frac{\pi}{2}\sum\mathrm{Res}\left(\frac{\cot\left(\pi z\right)}{z^2+z-1},z\right)\\&=-\frac{\pi}{2}\left(\mathrm{Res}\left(\frac{\cot\left(\pi z\right)}{z^2+z-1},-\varphi\right)+\mathrm{Res}\left(\frac{\cot\left(\pi z\right)}{z^2+z-1},\frac{1}{\varphi}\right)\right)\end{aligned}$$

$$= -\frac{\pi\cot(\pi\varphi)}{\sqrt{5}} = \frac{\pi\tan\left(\frac{\sqrt{5}\pi}{2}\right)}{\sqrt{5}}$$

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{\left(n^2+n-1\right)^2} &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{\left(n^2+n-1\right)^2} \\
&= \frac{\pi}{2}\sum\mathrm{Res}\left(\frac{\cot(\pi z)}{\left(z^2+z-1\right)^2}, z\right) \\
&= -\frac{\pi}{2}\left(\mathrm{Res}\left(\frac{\cot(\pi z)}{\left(z^2+z-1\right)^2}, -\varphi\right) + \mathrm{Res}\left(\frac{\cot(\pi z)}{\left(z^2+z-1\right)^2}, \frac{1}{\varphi}\right)\right) \\
&= -\frac{\pi}{5}\left(-\frac{\pi}{\sin^2(\pi\varphi)} - \frac{2\cot(\pi\varphi)}{\sqrt{5}}\right) \\
&= \frac{\pi^2}{5} + \frac{\pi^2\tan^2\left(\frac{\sqrt{5}\pi}{2}\right)}{5} - \frac{2\pi\tan\left(\frac{\sqrt{5}\pi}{2}\right)}{5\sqrt{5}}
\end{aligned}$$

上述三个式子相加证得原式.

□

160. 计算积分

$$\int_0^1\left[\frac{1+\sqrt{1-x}}{x} + \frac{2}{\ln(1-x)}\right]\mathrm{d}x$$

解 方法一 令 $\ln(1-x) = -t$, 有

$$\begin{aligned}
\int_0^1\left[\frac{1+\sqrt{1-x}}{x} + \frac{2}{\ln(1-x)}\right]\mathrm{d}x &= \int_0^{\infty}\mathrm{e}^{-t}\left(\frac{1}{1-\mathrm{e}^{-\frac{t}{2}}} - \frac{2}{t}\right)\mathrm{d}t \\
&= \int_0^{\infty}\left[-2\left(\frac{\mathrm{e}^{-t}}{t} - \frac{1}{\mathrm{e}^t-1}\right) - \frac{\mathrm{e}^{-\frac{t}{2}}}{1+\mathrm{e}^{\frac{t}{2}}}\right]\mathrm{d}t \\
&= -2\int_0^{\infty}\left(\frac{\mathrm{e}^{-t}}{t} - \frac{1}{\mathrm{e}^t-1}\right)\mathrm{d}t - \int_0^{\infty}\frac{\mathrm{e}^{-\frac{t}{2}}}{1+\mathrm{e}^{\frac{t}{2}}}\mathrm{d}t \\
&= 2\gamma - \int_0^{\infty}\left(\mathrm{e}^{-\frac{t}{2}} - \frac{\mathrm{e}^{-\frac{t}{2}}}{1+\mathrm{e}^{-\frac{t}{2}}}\right)\mathrm{d}t \\
&= 2\gamma - 2 + 2\ln 2
\end{aligned}$$

方法二 令 $1-x \to x$, 有

$$\begin{aligned}
\int_0^1\left[\frac{1+\sqrt{1-x}}{x} + \frac{2}{\ln(1-x)}\right]\mathrm{d}x &= \int_0^1\left(\frac{1+\sqrt{x}}{1-x} - \frac{2}{\ln x}\right)\mathrm{d}x \\
&= \int_{\infty}^0\int_0^1\left(\frac{x^t\ln x}{1-x} + \frac{x^{t+\frac{1}{2}}\ln x}{1-x} + 2x^t\right)\mathrm{d}x\mathrm{d}t \\
&= -\int_0^{\infty}\left[\psi'(t+1) + \psi'\left(t+\frac{2}{3}\right) - \frac{2}{t+1}\right]\mathrm{d}t \\
&= -\psi(1) - \psi\left(\frac{3}{2}\right) = 2\gamma - 2 + 2\ln 2
\end{aligned}$$

□

161. 计算积分

$$\int_0^{\pi} \arctan\left(\frac{\ln(\sin x)}{x}\right) \mathrm{d}x$$

解 考虑参数积分

$$I(s) = \int_0^{\pi} \arctan\left(\frac{\ln(s \sin x)}{x}\right) \mathrm{d}x$$

那么在积分下求导有

$$\begin{aligned}
I'(s) &= \frac{1}{s}\int_0^{\pi} \frac{x}{x^2 + \ln^2(s\sin x)} \mathrm{d}x = -\frac{1}{s}\int_0^{\pi} \frac{1}{\ln\left(\frac{s e^{\mathrm{i}x} - s}{2\mathrm{i}}\right)} \mathrm{d}x \\
&= -\frac{1}{s}\Im \int_{|z|=1} \frac{1}{\ln\left(\frac{sz-s}{2\mathrm{i}}\right)} \frac{\mathrm{d}z}{2\mathrm{i}z} = -\frac{1}{s}\Im \frac{\pi}{\ln\left(-\frac{s}{2\mathrm{i}}\right)} \\
&= -\frac{1}{s}\Im\left(\frac{\pi}{\ln\left(\frac{s}{2}\right) + \frac{\pi \mathrm{i}}{2}} \cdot \frac{\ln\left(\frac{s}{2}\right) - \frac{\pi\mathrm{i}}{2}}{\ln\left(\frac{s}{2}\right) - \frac{\pi\mathrm{i}}{2}}\right) \\
&= \frac{1}{2s} \frac{\pi^2}{\ln^2\left(\frac{s}{2}\right) + \frac{\pi^2}{4}}
\end{aligned}$$

因此

$$\begin{aligned}
I(1) &= I(\infty) + \frac{\pi^2}{2}\int_{s=\infty}^{s=1} \frac{1}{\ln^2\left(\frac{s}{2}\right) + \frac{\pi i}{2}} \mathrm{d}\left(\ln\left(\frac{s}{2}\right)\right) \\
&= \frac{\pi^2}{2} + \frac{\pi^2}{2} \cdot \frac{2}{\pi} \arctan\left(\frac{2\ln\left(\frac{s}{2}\right)}{\pi}\right)\Bigg|_{\infty}^{1} \\
&= -\pi \arctan\left(\frac{2\ln 2}{\pi}\right)
\end{aligned}$$

□

162. 证明

$$G = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)\cdot 4^n}\left(1 - \frac{2}{4^n}\right)$$

其中 G 是 Catalan 常数, 对 $s > 1, \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \zeta(0) = -\frac{1}{2}$.

证明 根据 $\cot x$ 的幂级数展开得

$$\sum_{n=1}^{\infty} \zeta(2n) x^{2n} = \frac{1 - \pi x \cot \pi x}{2}$$

于是

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\zeta(2n) x^{2n}}{2n+1} &= \frac{1}{x}\int_0^{x} \frac{1 - \pi t \cot \pi t}{2} \mathrm{d}t \\
&= \frac{1}{2} - \frac{1}{2\pi x}\int_0^{\pi x} u \cot u \mathrm{d}u \\
&= \frac{1}{2} - \frac{1}{2\pi x}\int_0^{\pi x} u \mathrm{d}(\ln(\sin u))
\end{aligned}$$

$$= \frac{1}{2} - \frac{1}{2}\ln(\sin \pi x) + \frac{1}{2\pi x}\int_0^{\pi x} \ln(\sin u)\,\mathrm{d}u$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)\cdot 4^n} &= \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n+1}\left(\frac{1}{2}\right)^{2n} \\ &= \frac{1}{2} - \frac{1}{2}\ln\left(\sin\frac{\pi}{2}\right) + \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \ln(\sin u)\,\mathrm{d}u \\ &= \frac{1}{2} - \frac{1}{2}\ln 2\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)\cdot 4^n}\frac{2}{4^n} &= 2\sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n+1}\left(\frac{1}{4}\right)^{2n} \\ &= 2\left[\frac{1}{2} - \frac{1}{2}\ln\left(\sin\frac{\pi}{4}\right) + \frac{2}{\pi}\int_0^{\frac{\pi}{4}} \ln(\sin u)\,\mathrm{d}u\right] \\ &= 1 + \frac{1}{2}\ln 2 + \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \ln(\sin u)\,\mathrm{d}u \\ &= 1 + \frac{1}{2}\ln 2 + \frac{4}{\pi}\left(-\frac{\pi}{4}\ln 2 - \frac{G}{2}\right) \\ &= 1 - \frac{1}{2}\ln 2 - \frac{2}{\pi}G\end{aligned}$$

最后得到

$$\begin{aligned}\frac{\pi}{2}\sum_{n=1}^{\infty} \zeta(2n)\,x^{2n} &= \frac{\pi}{2}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)\,4^n} - \frac{2\zeta(2n)}{(2n+1)\,4^{2n}}\right] \\ &= \frac{\pi}{2}\left(\frac{1}{2} + \frac{1}{2} - \frac{1}{2}\ln 2 - 1 + \frac{1}{2}\ln 2 + \frac{2}{\pi}G\right) \\ &= G\end{aligned}$$

其中利用 $\ln(\sin u)$ 的 Fourier 展开式有

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \ln(\sin u)\,\mathrm{d}u &= \int_0^{\frac{\pi}{4}} \left(-\ln 2 - \sum_{n=1}^{\infty} \frac{\cos 2nu}{n}\right)\mathrm{d}u \\ &= -\frac{\pi}{4}\ln 2 - \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{2n^2} \\ &= -\frac{\pi}{4}\ln 2 - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \\ &= -\frac{\pi}{4}\ln 2 - \frac{G}{2}.\end{aligned}$$

□

163. 计算积分

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^3 \frac{1}{3+\cos 2x}\mathrm{d}x$$

解

$$
\begin{aligned}
&\int_0^{\infty}\left(\frac{\sin x}{x}\right)^3\frac{1}{3+\cos 2x}\mathrm{d}x\\
=&\sum_{n=0}^{\infty}\int_{\frac{n\pi}{2}}^{\frac{n+1}{2}\pi}\left(\frac{\sin x}{x}\right)^3\frac{1}{3+\cos 2x}\mathrm{d}x\\
=&\sum_{n=0}^{\infty}\int_{n\pi}^{\frac{2n+1}{2}\pi}\left(\frac{\sin x}{x}\right)^3\frac{1}{3+\cos 2x}\mathrm{d}x+\sum_{n=1}^{\infty}\int_{\frac{2n-1}{2}\pi}^{n\pi}\left(\frac{\sin x}{x}\right)^3\frac{1}{3+\cos 2x}\mathrm{d}x\\
=&\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}}\left(\frac{\sin(x+n\pi)}{x+n\pi}\right)^3\frac{1}{3+\cos 2(x+n\pi)}\mathrm{d}x\\
&+\sum_{n=1}^{\infty}\int_{-\frac{\pi}{2}}^{0}\left(\frac{\sin(x+n\pi)}{x+n\pi}\right)^3\frac{1}{3+\cos 2(x+n\pi)}\mathrm{d}x\\
=&\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}}\left(\frac{\sin x}{x+n\pi}\right)^3\frac{(-1)^n}{3+\cos 2x}\mathrm{d}x+\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}}\left(\frac{\sin x}{x-n\pi}\right)^3\frac{(-1)^n}{3+\cos 2x}\mathrm{d}x\\
=&\int_0^{\frac{\pi}{2}}\frac{\sin^3 x}{3+\cos 2x}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(x+n\pi)^3}\mathrm{d}x
\end{aligned}
$$

由已知等式 $\dfrac{1}{\sin x}=\displaystyle\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{x+n\pi}$ 得

$$
\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(x+n\pi)^3}=\frac{(-1)^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{1}{\sin x}\right)=\frac{1+\cos^2 x}{2\sin^3 x}
$$

因此

$$
\int_0^{\infty}\left(\frac{\sin x}{x}\right)^3\frac{1}{3+\cos 2x}\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1+\cos^2 x}{3+\cos 2x}\mathrm{d}x=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{3+\cos 2x}{3+\cos 2x}\mathrm{d}x=\frac{\pi}{8}.
$$

□

164. 计算积分

$$
\int_0^{\pi}\frac{x^2\sin^2 x}{(1+|\cos x|)^2}\mathrm{d}x
$$

解

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}}\frac{x^2\sin^2 x}{(1+|\cos x|)^2}\mathrm{d}x&=\int_0^{\frac{\pi}{2}}\frac{4x^2\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{4\cos^4\frac{x}{2}}\mathrm{d}x\\
&=\int_0^{\frac{\pi}{2}}x^2\left(\sec^2\frac{x}{2}-1\right)\mathrm{d}x=2\int_0^{\frac{\pi}{2}}x^2\mathrm{d}\left(\tan\frac{x}{2}\right)-\frac{\pi^3}{24}\\
&=\frac{\pi^2}{2}-\frac{\pi^3}{24}-4\int_0^{\frac{\pi}{2}}x\tan\frac{x}{2}\mathrm{d}x\\
&=\frac{\pi^2}{2}-\frac{\pi^3}{24}+8\int_0^{\frac{\pi}{2}}x\mathrm{d}\left(\ln\left(\cos\frac{x}{2}\right)\right)\\
&=\frac{\pi^2}{2}-\frac{\pi^3}{24}-2\pi\ln 2-8\int_0^{\frac{\pi}{2}}\ln\left(\cos\frac{x}{2}\right)\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\pi^2}{2} - \frac{\pi^3}{24} - 2\pi\ln 2 - 8\int_0^{\frac{\pi}{2}} \left(-\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos nx}{n}\right) \mathrm{d}x \\
&= \frac{\pi^2}{2} - \frac{\pi^3}{24} + 2\pi\ln 2 + 8\sum_{n=1}^{\infty} \frac{(-1)^n \sin\frac{n\pi}{2}}{n^2} \\
&= \frac{\pi^2}{2} - \frac{\pi^3}{24} + 2\pi\ln 2 - 8G
\end{aligned}
$$

$$
\begin{aligned}
\int_{\frac{\pi}{2}}^{\pi} \frac{x^2\sin^2 x}{(1+|\cos x|)^2}\mathrm{d}x &= \int_{\frac{\pi}{2}}^{\pi} \frac{4x^2\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{4\sin^4\frac{x}{2}}\mathrm{d}x \\
&= \int_{\frac{\pi}{2}}^{\pi} x^2\left(\csc^2\frac{x}{2} - 1\right)\mathrm{d}x = -2\int_{\frac{\pi}{2}}^{\pi} x^2 \mathrm{d}\left(\cot\frac{x}{2}\right) - \frac{\pi^3}{24} \\
&= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 4\int_{\frac{\pi}{2}}^{\pi} x\cot\frac{x}{2}\mathrm{d}x \\
&= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 8\int_{\frac{\pi}{2}}^{\pi} x\mathrm{d}\left(\ln\left(\sin\frac{x}{2}\right)\right) \\
&= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 2\pi\ln 2 - 8\int_{\frac{\pi}{2}}^{\pi} \ln\left(\sin\frac{x}{2}\right)\mathrm{d}x \\
&= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 2\pi\ln 2 - 8\int_{\frac{\pi}{2}}^{\pi} \left(-\ln 2 - \sum_{n=1}^{\infty}\frac{\cos nx}{n}\right)\mathrm{d}x \\
&= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 6\pi\ln 2 - 8\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2} \\
&= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 6\pi\ln 2 - 8G
\end{aligned}
$$

因此

$$
\int_0^{\pi} \frac{x^2\sin^2 x}{(1+|\cos x|)^2}\mathrm{d}x = \pi^2 - \frac{\pi^3}{3} + 8\pi\ln 2 - 16G
$$

□

165. 设 $a > 0$, 计算积分

$$
\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)!}{(n+2)!n!4^{2n+3}}
$$

解

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)!}{(n+2)!n!4^{2n+3}} &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^{2n+3}} \frac{(2n+2)!}{[(n+1)!]^2(n+2)} \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{4^{2n+1}} \frac{(2n)!}{[(n)!]^2(n+1)} \\
&= -2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^{2n+2}} \frac{(2n)!}{[(n)!]^2(n+1)} \\
&= -2\sum_{n=0}^{\infty} \binom{2n}{n} \frac{\left(-\frac{1}{16}\right)^{n+1}}{(n+1)} - \frac{1}{8}
\end{aligned}
$$

令 $y^2 - y + x = 0$, 应用 Lagrange 反演公式得

$$y = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{n+1}}{n+1}$$

再令 $x = -\frac{1}{16}$ 得 $y = \frac{2-\sqrt{5}}{4}$, 故

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)!}{(n+2)!n!4^{2n+3}} = -2 \cdot \frac{2-\sqrt{5}}{4} - \frac{1}{8} = \frac{4\sqrt{5}-9}{8}$$

□

166. 求和

$$\int_0^{\frac{\pi}{2}} \frac{\sin ax}{\sin x} \mathrm{d}x$$

解

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin ax}{\sin x} \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{\mathrm{e}^{\mathrm{i}(a-1)x} - \mathrm{e}^{-\mathrm{i}(a-1)x}}{1-\mathrm{e}^{-2\mathrm{i}x}} \mathrm{d}x = \frac{\mathrm{i}}{2} \int_{-1}^{1} \frac{(-t)^{-\frac{a-1}{2}-1} - (-t)^{\frac{a+1}{2}-1}}{1+t} \mathrm{d}t \\
&= \frac{\mathrm{i}}{2} \left[\int_{-1}^{0} \frac{(-t)^{-\frac{a-1}{2}-1} - (-t)^{\frac{a+1}{2}-1}}{1+t} \mathrm{d}t + \int_0^1 \frac{(-t)^{-\frac{a-1}{2}-1} - (-t)^{\frac{a+1}{2}-1}}{1+t} \mathrm{d}t \right] \\
&= \frac{\mathrm{i}}{2} \int_0^1 \frac{s^{-\frac{a-1}{2}-1} - s^{\frac{a+1}{2}-1}}{1-s} \mathrm{d}s + \frac{1}{2} \int_0^1 \frac{\mathrm{e}^{\mathrm{i}\frac{a\pi}{2}} t^{-\frac{a-1}{2}-1} - \mathrm{e}^{-\mathrm{i}\frac{a\pi}{2}} t^{\frac{a+1}{2}-1}}{1+t} \mathrm{d}t \\
&= \frac{\mathrm{i}}{2} \left[\psi\left(\frac{1+a}{2}\right) - \psi\left(\frac{1-a}{2}\right) \right] - \frac{1}{2} \left[\mathrm{e}^{\mathrm{i}\frac{a\pi}{2}} \beta\left(\frac{1+a}{2}\right) - \mathrm{e}^{-\mathrm{i}\frac{a\pi}{2}} \beta\left(\frac{1-a}{2}\right) \right] \\
&= \frac{\mathrm{i}\pi}{2} \tan\left(\frac{a\pi}{2}\right) - \frac{1}{2} \left[\left(\mathrm{e}^{\mathrm{i}\frac{a\pi}{2}} + \mathrm{e}^{-\mathrm{i}\frac{a\pi}{2}}\right) \beta\left(\frac{1+a}{2}\right) - \mathrm{e}^{-\mathrm{i}\frac{a\pi}{2}} \pi \sec\left(\frac{a\pi}{2}\right) \right] \\
&= \frac{\pi}{2} - \cos\left(\frac{a\pi}{2}\right) \beta\left(\frac{1+a}{2}\right)
\end{aligned}$$

□

167. 计算积分

$$\int_0^1 \left(\frac{\arctan x}{x}\right)^3 \mathrm{d}x$$

解

$$\begin{aligned}
\int_0^1 \left(\frac{\arctan x}{x}\right)^3 \mathrm{d}x &= \int_0^{\frac{\pi}{4}} \frac{t^3 \mathrm{d}(\sin t)}{\sin^3 t} = -\frac{1}{2} \int_0^{\frac{\pi}{4}} t^3 \mathrm{d}\left(\frac{1}{\sin^2 t}\right) \\
&= -\frac{1}{2} \left(\frac{2\pi^3}{4^3} - 3 \int_0^{\frac{\pi}{4}} \frac{t^2}{\sin^2 t} \mathrm{d}t \right) = -\frac{\pi^3}{64} - \frac{3}{2} \int_0^{\frac{\pi}{4}} t^2 \mathrm{d}(\cot t) \\
&= -\frac{\pi^3}{64} - \frac{3}{2} \left(\frac{\pi^2}{16} - 2 \int_0^{\frac{\pi}{4}} t \cot t \, \mathrm{d}t \right) \\
&= -\frac{\pi^3}{64} - \frac{3\pi^2}{32} + 3 \left(-\frac{\pi}{8} \ln 2 - \int_0^{\frac{\pi}{4}} \ln(\sin t) \, \mathrm{d}t \right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\pi^3}{64} - \frac{3\pi^2}{32} - \frac{3\pi}{8}\ln 2 - 3\int_0^{\frac{\pi}{4}} \ln(\sin t)\,\mathrm{d}t \\
&= -\frac{\pi^3}{64} - \frac{3\pi^2}{32} + \frac{3\pi}{8}\ln 2 + \frac{3}{2}G
\end{aligned}
$$

□

168. 求和

$$
S = \sum_{n=1}^{\infty} \frac{16^n}{(2n+1)^2 n^2 \binom{2n}{n}^2}
$$

解 首先有

$$
\begin{aligned}
\frac{16^n}{(2n+1)^2 n^2 \binom{2n}{n}^2} &= \frac{16^n}{(2n+1)^2 n^2}\left[\frac{(n!)^2}{(2n)!}\right]^2 \\
&= \frac{16^n}{(2n+1)^2 n^2}\left[\frac{n!}{(2n-1)!!\cdot 2^n}\right]^2 \\
&= \frac{2}{n(2n+1)}\cdot\frac{(2n)!!}{(2n+1)!!}\cdot\frac{(2n-2)!!}{(2n-1)!!} \\
&= \frac{2}{n(2n+1)}\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,\mathrm{d}x \int_0^{\frac{\pi}{2}} \sin^{2n-1} y\,\mathrm{d}y
\end{aligned}
$$

记

$$
I(t) = \sum_{n=1}^{\infty} \frac{t^{2n+1}}{n(2n+1)} \int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,\mathrm{d}x \int_0^{\frac{\pi}{2}} \sin^{2n-1} y\,\mathrm{d}y
$$

则

$$
\begin{aligned}
I'(t) &= \sum_{n=1}^{\infty} \frac{t^{2n}}{n} \int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,\mathrm{d}x \int_0^{\frac{\pi}{2}} \sin^{2n-1} y\,\mathrm{d}y \\
&= -\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin y} \ln\left(1 - t^2\sin^2 x\sin^2 y\right)\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

于是

$$
S = -2\int_0^1\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin y}\ln\left(1 - t^2\sin^2 x\sin^2 y\right)\mathrm{d}y\mathrm{d}x\mathrm{d}t
$$

考虑

$$
f(u) = \int_0^{\frac{\pi}{2}} \frac{1}{\sin y}\ln\left(1 - u\sin^2 y\right)\mathrm{d}y
$$

则

$$
f'(u) = -\int_0^{\frac{\pi}{2}} \frac{\sin y}{1 - u\sin^2 y}\mathrm{d}y = -\frac{1}{\sqrt{u-u^2}}\arctan\sqrt{\frac{u}{1-u}}
$$

于是

$$
\begin{aligned}
S &= 2\int_0^1\int_0^{\frac{\pi}{2}}\int_0^{t^2\sin^2 x} \frac{\sin x}{\sqrt{u-u^2}}\arctan\sqrt{\frac{u}{1-u}}\,\mathrm{d}u\mathrm{d}x\mathrm{d}t \\
&= 2\int_0^1\int_0^{\frac{\pi}{2}} \sin x\arctan^2\left(\sqrt{\frac{u}{1-u}}\right)\Bigg|_{u=0}^{t^2\sin^2 x}\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$\begin{aligned}
&= 2\int_0^1\int_0^{\frac{\pi}{2}} \sin x \arctan^2\left(\frac{t\sin x}{\sqrt{1-t^2\sin^2 x}}\right)\mathrm{d}x\mathrm{d}t \\
&= 2\int_0^{\frac{\pi}{2}}\int_0^{x} z^2\cos z\mathrm{d}z\mathrm{d}x \quad \left(t=\frac{\sin z}{\sin x}\right) \\
&= 2\int_0^{\frac{\pi}{2}} \left(2x\cos x + x^2\sin x - 2\sin x\right)\mathrm{d}x \\
&= 4\pi - 12
\end{aligned}$$

□

169. 计算积分

$$I=\int_0^1 \exp\left(4\sqrt{\frac{t-t^2}{1+8t}}\right)\cdot\sqrt{\frac{1-8t+16t^2}{t+7t^2-8t^3}}\mathrm{d}t$$

解 设 $t=x^2$, 那么有

$$\begin{aligned}
I &= 2\int_0^1 \exp\left(4\sqrt{\frac{t-t^2}{1+8t}}\right)\cdot\sqrt{\frac{1-8t+16t^2}{t+7t^2-8t^3}}\mathrm{d}t \\
&= \int_0^1 \mathrm{e}^{4xy}\frac{\left|1-4x^2\right|}{\sqrt{\left(1-x^2\right)\left(1+8x^2\right)}}\mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \mathrm{e}^{4xy}\frac{1-4x^2}{\sqrt{\left(1-x^2\right)\left(1+8x^2\right)}}\mathrm{d}x + \int_{\frac{1}{2}}^1 \mathrm{e}^{4xy}\frac{4x^2-1}{\sqrt{\left(1-x^2\right)\left(1+8x^2\right)}}\mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \mathrm{e}^{4xy}\frac{1-4x^2}{\sqrt{\left(1-x^2\right)\left(1+8x^2\right)}}\mathrm{d}x + \int_0^{\frac{1}{2}} \mathrm{e}^{4xy}\frac{3\left(1-4y^2\right)}{\left(1+8y^2\right)\sqrt{\left(1-y^2\right)\left(1+8y^2\right)}}\mathrm{d}y \\
&= \int_0^{\frac{1}{2}} \mathrm{e}^{4xy}\frac{1-4x^2}{\sqrt{\left(1-x^2\right)\left(1+8x^2\right)}}\mathrm{d}x + \int_0^{\frac{1}{2}} \mathrm{e}^{4xy}\frac{3\left(1-4x^2\right)}{\left(1+8x^2\right)\sqrt{\left(1-x^2\right)\left(1+8x^2\right)}}\mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \mathrm{e}^{4xy}\frac{4\left(1-4x^2\right)\left(1+2x^2\right)}{\left(1+8x^2\right)\sqrt{\left(1-x^2\right)\left(1+8x^2\right)}}\mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \mathrm{e}^{4xy}\left(4xy\right)_x'\,\mathrm{d}x = \left[\exp\left(4x\sqrt{\frac{1-x^2}{1+8x^2}}\right)\right]\Bigg|_0^{\frac{1}{2}} = 1
\end{aligned}$$

□

计算积分

$$\int_0^\infty \frac{x^2}{\left(1+x^2\right)^4}\ln^2\left(\frac{2x}{1+x^2}\right)\mathrm{d}x$$

解

$$\begin{aligned}
\int_0^\infty \frac{x^2}{\left(1+x^2\right)^4}\ln^2\left(\frac{2x}{1+x^2}\right)\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \sin^2 t\cos^4 t\ln^2\left(\sin 2t\right)\mathrm{d}t \quad (x=\tan t) \\
&= \int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\ln^2\left(\sin 2t\right)\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\sin^2 t + \cos^2 t\right) \sin^2 t \cos^2 t \ln^2 (\sin 2t)\, \mathrm{d}t \\
&= \frac{1}{8}\int_0^{\frac{\pi}{2}} \sin^2 2t \ln^2 (\sin 2t)\, \mathrm{d}t = \frac{1}{8}\int_0^{\frac{\pi}{2}} \sin^2 t \ln^2 (\sin t)\, \mathrm{d}t \\
&= \frac{1}{16}\int_0^{\frac{\pi}{2}} \ln^2 (\sin t)\, \mathrm{d}t - \frac{1}{6}\int_0^{\frac{\pi}{2}} \cos 2t \ln^2 (\sin t)\, \mathrm{d}t \\
&= \frac{1}{16}\left(\frac{\pi}{2}\ln^2 2 + \frac{\pi^3}{24}\right) - \frac{1}{32}\int_0^{\frac{\pi}{2}} \ln^2 (\sin t)\, \mathrm{d}(\sin 2t) \\
&= \frac{\pi}{32}\ln^2 2 + \frac{\pi^3}{384} + \frac{1}{32}\int_0^{\frac{\pi}{2}} \sin 2t \frac{2\ln(\sin t)}{\sin t} \cos t\, \mathrm{d}t \\
&= \frac{\pi}{32}\ln^2 2 + \frac{\pi^3}{384} + \frac{1}{8}\int_0^{\frac{\pi}{2}} \cos^2 t \ln (\sin t)\, \mathrm{d}t \\
&= \frac{\pi}{32}\ln^2 2 + \frac{\pi^3}{384} - \frac{\pi}{32}\ln 2 - \frac{\pi}{64}
\end{aligned}
$$

□

170. 计算积分

$$
I = \frac{1}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \frac{\mathrm{d}u\mathrm{d}v\mathrm{d}w}{1 - \cos u \cos v \cos w}
$$

解 首先作换元

$$
\tan\frac{u}{2} = x, \qquad \tan\frac{v}{2} = y, \qquad \tan\frac{w}{2} = z
$$

再转化成极坐标

$$
x = r\sin\theta\cos\phi, \qquad y = t\sin\theta\sin\phi, \qquad z = r\cos\theta
$$

然后令 $\varphi = 2\phi$, 于是按照上面的换元我们得到

$$
\begin{aligned}
I &= \frac{8}{\pi^3}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(1+x^2\right)\left(1+y^2\right)\left(1+z^2\right) - \left(1-x^2\right)\left(1-y^2\right)\left(1-z^2\right)} \\
&= \frac{4}{\pi^3}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{x^2+y^2+z^2+x^2y^2z^2} \\
&= \frac{4}{\pi^3}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi}{1 + r^2\sin^4\theta\cos^2\theta\sin^2\phi\cos^2\phi} \\
&= \frac{4}{\pi^3}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi}{1 + \frac{1}{4}r^2\sin^4\theta\cos^2\theta\sin^2\varphi}
\end{aligned}
$$

令 $t = r\sin\theta\frac{1}{2}\cos\theta\sin\varphi$, 则

$$
\begin{aligned}
I &= \frac{4}{\pi^3}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi}{\left(1+t^4\right)\sqrt{\frac{1}{2}\cos\theta\sin\varphi}} \\
&= \frac{4\sqrt{2}}{\pi^3}\int_0^{\infty}\frac{\mathrm{d}t}{1+t^4}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\sin\varphi}} \\
&= \frac{4\sqrt{2}}{\pi^3}\cdot\frac{\pi}{2\sqrt{2}}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} \\
&= \frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^3}
\end{aligned}
$$

□

171. 计算积分

$$\int_0^1 \frac{\ln^2(1-x)\ln x}{x}\mathrm{d}x$$

解

$$\begin{aligned}
\int_0^1 \frac{\ln^2(1-x)\ln x}{x}\mathrm{d}x &= -\int_0^1 \ln(1-x)\ln(x)\,\mathrm{d}\left(\mathrm{Li}_2(x)\right)\\
&= \int_0^1 \mathrm{Li}_2(x)\frac{\ln(1-x)}{x}\mathrm{d}x - \int_0^1 \mathrm{Li}_2(x)\frac{\ln x}{1-x}\mathrm{d}x\\
&= -\frac{1}{2}\mathrm{Li}_2^2(1) - \int_0^1 \frac{\ln x}{1-x}\sum_{n=1}^{\infty}\frac{x^n}{n^2}\mathrm{d}x\\
&= -\frac{1}{2}\mathrm{Li}_2^2(1) + \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=n+1}^{\infty}\frac{1}{k^2}\\
&= -\frac{\pi^4}{72} + \frac{\pi^4}{120} = -\frac{\pi^4}{180}
\end{aligned}$$

□

172. 计算积分

$$\int_0^{\infty}\int_0^{\infty}\frac{\sin x\sin y\sin(x+y)}{xy(x+y)}\mathrm{d}x\mathrm{d}y$$

解 考虑参数积分

$$I(t) = \int_0^{\infty}\int_0^{\infty}\frac{\sin x\sin y\sin t(x+y)}{xy(x+y)}\mathrm{d}x\mathrm{d}y,\quad 0<t<1$$

则

$$\begin{aligned}
I'(t) &= \int_0^{\infty}\int_0^{\infty}\frac{\sin x\sin y\cos t(x+y)}{xy}\mathrm{d}x\mathrm{d}y\\
&= \int_0^{\infty}\int_0^{\infty}\frac{\sin x\sin y\left[\cos(tx)\cos(ty)-\sin(tx)\sin(ty)\right]}{xy}\mathrm{d}x\mathrm{d}y
\end{aligned}$$

其中

$$\begin{aligned}
\int_0^{\infty}\int_0^{\infty}\frac{\sin x\sin y\cos(tx)\cos(ty)}{xy}\mathrm{d}x\mathrm{d}y &= \int_0^{\infty}\frac{\sin x\cos(tx)}{x}\mathrm{d}x\int_0^{\infty}\frac{\sin y\cos(ty)}{y}\mathrm{d}x\\
&= \left(\int_0^{\infty}\frac{\sin x\cos(tx)}{x}\mathrm{d}x\right)^2\\
&= \left(\frac{1}{2}\int_0^{\infty}\frac{\sin(1+t)x+\sin(1-t)x}{x}\mathrm{d}x\right)^2\\
&= \left(\frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right)^2 = \frac{\pi^2}{4}\quad \text{(Dirichlet Integral)}
\end{aligned}$$

$$\int_0^{\infty}\int_0^{\infty}\frac{\sin x\sin y\sin(tx)\sin(ty)}{xy}\mathrm{d}x\mathrm{d}y = \int_0^{\infty}\frac{\sin x\sin(tx)}{x}\mathrm{d}x\int_0^{\infty}\frac{\sin y\sin(ty)}{y}\mathrm{d}y$$

$$
\begin{aligned}
&= \left(\int_0^{\infty} \frac{\sin x \sin(tx)}{x} \mathrm{d}x\right)^2 \\
&= \left(\frac{1}{2}\int_0^{\infty} \frac{\cos(1-t)x - \cos(1+t)x}{x} \mathrm{d}x\right)^2 \\
&= \frac{1}{4}\ln^2\left(\frac{1-t}{1+t}\right) \quad \text{(Frullani Integral)}
\end{aligned}
$$

于是

$$
\begin{aligned}
I &= I(0) + \int_0^1 I'(t)\,\mathrm{d}t = \frac{\pi^2}{4} - \frac{1}{4}\int_0^1 \ln^2\left(\frac{1-t}{1+t}\right)\mathrm{d}t \\
&= \frac{\pi^2}{4} - 2\int_0^1 \frac{\ln^2 u}{(1+u)^2}\mathrm{d}u = \frac{\pi^2}{6}
\end{aligned}
$$

□

173. 计算积分

$$
\int_0^1 \frac{\arccos(\sqrt{x})\,\mathrm{Li}_2(x)}{x}\mathrm{d}x
$$

解

$$
\begin{aligned}
\int_0^1 \frac{\arccos(\sqrt{x})\,\mathrm{Li}_2(x)}{x}\mathrm{d}x &= 2\int_0^1 \frac{\arccos(x)\,\mathrm{Li}_2(x^2)}{x}\mathrm{d}x \\
&= 2\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{2n-1}\arccos(x)\,\mathrm{d}x \\
&= \sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\mathrm{d}x = \sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^{\frac{\pi}{2}} \sin^{2n} x\,\mathrm{d}x \\
&= \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} \sin^{2n} x\,\mathrm{d}x \int_0^{\infty} \mathrm{e}^{-ny}\mathrm{d}y \int_0^{\infty} \mathrm{e}^{-nz}\mathrm{d}z \\
&= -\int_0^{\infty}\int_0^{\infty}\int_0^{\frac{\pi}{2}} \ln\left(1-\mathrm{e}^{-(y+z)}\sin^2 x\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \pi\int_0^{\infty}\int_0^{\infty} \ln\left(\frac{2}{1+\sqrt{1-\mathrm{e}^{-(y+z)}}}\right)\mathrm{d}y\,\mathrm{d}z \\
&= -\pi\int_0^1\int_0^1 \frac{1}{uv}\ln\left(\frac{1+\sqrt{1-uv}}{2}\right)\mathrm{d}u\,\mathrm{d}v \\
&= -\frac{\pi}{2}\int_0^1 \frac{\mathrm{d}v}{v}\int_0^1 \frac{\ln u}{\sqrt{1-uv}+1-uv}\mathrm{d}u \\
&= -\pi\int_0^1 \frac{\ln v}{v}\left(\ln 2 - \ln\left(1+\sqrt{1-v}\right)\right)\mathrm{d}v \\
&= -\frac{\pi}{2}\int_0^1 \left(\ln 2 - \ln\left(1+\sqrt{1-v}\right)\right)\mathrm{d}\left(\ln^2 v\right) \\
&= \frac{\pi}{4}\int_0^1 \frac{\ln^2 v}{\sqrt{1-v}+1-v}\mathrm{d}v \\
&= \frac{\pi}{2}\int_0^1 \frac{\ln^2\left(1-t^2\right)}{1+t}\mathrm{d}t = \frac{\pi}{2}\left(I_1 + 2I_2 + I_3\right)
\end{aligned}
$$

其中

$$I_1=\int_0^1\frac{\ln^2(1+t)}{1+t}\mathrm{d}t=\frac{1}{3}\ln^3 2$$

$$I_2=\int_0^1\frac{\ln(1+t)\ln(1-t)}{1+t}\mathrm{d}t=\frac{1}{3}\ln^3 2-\frac{\pi^2}{12}\ln 2+\frac{1}{8}\zeta(3)$$

$$I_3=\int_0^1\frac{\ln^2(1-t)}{1+t}\mathrm{d}t=\frac{1}{3}\ln^3 2-\frac{\pi^2}{6}\ln 2+\frac{7}{4}\zeta(3)$$

因此原积分

$$I=\frac{\pi}{2}(I_1+2I_2+I_3)=\frac{2\pi}{3}\ln^3 2-\frac{\pi^3}{6}\ln 2+\pi\zeta(3)$$

□

174. 计算积分

$$\int_0^{\frac{1}{2}}\frac{x\ln\left(\frac{\ln 2-\ln(1+2x)}{\ln 2-\ln(1-2x)}\right)}{3+4x^2}\mathrm{d}x$$

解 首先有

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\frac{x\ln\left(\frac{\ln 2-\ln(1+2x)}{\ln 2-\ln(1-2x)}\right)}{3+4x^2}\mathrm{d}x=\frac{1}{4}\int_0^1\frac{x\ln\left(\frac{\ln 2-\ln(1+x)}{\ln 2-\ln(1-x)}\right)}{3+x^2}\mathrm{d}x\\
=&\frac{1}{4}\int_0^1\frac{x}{3+x^2}\ln\left(\frac{\ln\frac{1+x}{2}}{\ln\frac{1-x}{2}}\right)\mathrm{d}x=\frac{1}{4}\int_{-1}^0\frac{x}{3+x^2}\ln\left(\frac{\ln\frac{1+x}{2}}{\ln\frac{1-x}{2}}\right)\mathrm{d}x\\
=&\frac{1}{8}\int_{-1}^1\frac{x}{3+x^2}\ln\left(\frac{\ln\frac{1+x}{2}}{\ln\frac{1-x}{2}}\right)\mathrm{d}x=\frac{1}{4}\left[\int_{-1}^1\frac{x}{3+x^2}\ln\left(\left|\ln\frac{1+x}{2}\right|\right)\mathrm{d}x\right]\\
=&\frac{1}{2}\Re\left[\int_0^1\frac{2t-1}{3+(2t-1)^2}\ln(\ln t)\,\mathrm{d}x\right]=\frac{1}{8}\int_0^1\frac{(2t-1)\ln(-\ln t)}{t^2-t+1}\mathrm{d}t\\
=&\frac{1}{8}\int_0^1\ln(-\ln t)\,\mathrm{d}\left(\ln\left(t^2-t+1\right)\right)\quad(x=2t-1)\\
=&-\frac{1}{8}\int_0^1\frac{\ln\left(t^2-t+1\right)}{t\ln t}\mathrm{d}t=\frac{1}{8}\int_0^\infty\frac{\ln\left(\mathrm{e}^{-2s}-\mathrm{e}^{-s}+1\right)}{s}\mathrm{d}s\quad\left(t=\mathrm{e}^{-s}\right)\\
=&\frac{1}{8}\int_0^\infty\frac{\ln\left(1+\mathrm{e}^{-3s}\right)-\ln\left(1+\mathrm{e}^{-s}\right)}{s}\mathrm{d}s
\end{aligned}$$

考虑参数积分 $I(a,b)=\int_0^\infty\frac{\ln(1+\mathrm{e}^{-as})-\ln\left(1+\mathrm{e}^{-bs}\right)}{s}\mathrm{d}s$, 则 $I(b,b)=0$,

$$I_a'(a,b)=-\int_0^\infty\frac{\mathrm{e}^{-as}}{1+\mathrm{e}^{-as}}\mathrm{d}s=-\frac{1}{a}\ln 2$$

于是

$$I(a,b)=-\ln 2\int_b^a\frac{1}{u}\mathrm{d}u=-\ln 2\ln\frac{a}{b}$$

原积分 $I=\frac{1}{8}I(3,1)=-\frac{1}{8}\ln 2\ln 3$.

□

175. 计算积分

$$\int_0^1 (1+\ln x)\arctan(x)\ln\left(\ln\frac{1}{x}\right)\mathrm{d}x$$

解 首先有

$$\begin{aligned}\int_0^1 (1+\ln x)\arctan(x)\ln\left(\ln\frac{1}{x}\right)\mathrm{d}x &= \int_0^\infty (1-t)\arctan\left(\mathrm{e}^{-t}\right)\ln(t)\,\mathrm{e}^{-t}\mathrm{d}t \\ &= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1}\int_0^\infty (1-t)\,\mathrm{e}^{-2nt}\ln t\,\mathrm{d}t\end{aligned}$$

考虑积分

$$\begin{aligned}I(a) &= \int_0^\infty t^a\mathrm{e}^{-2nt}\mathrm{d}t = \frac{\Gamma(a+1)}{(2n)^{a+1}}, \\ I'(a) &= \frac{\Gamma'(a+1)}{(2n)^{a+1}} - \frac{\Gamma(a+1)}{(2n)^{a+1}}\ln(2n).\end{aligned}$$

于是

$$\begin{aligned}&\sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1}\int_0^\infty (1-t)\,\mathrm{e}^{-2nt}\ln t\,\mathrm{d}t \\ =& \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1}\left(I'(0)-I'(1)\right) \\ =& \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1}\left(\frac{-\gamma}{2n}-\frac{\ln(2n)}{2n}-\frac{1-\gamma}{(2n)^2}+\frac{\ln(2n)}{(2n)^2}\right) \\ =& \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1}\left(-\gamma\frac{2n-1}{(2n)^2}-\frac{(2n-1)\ln(2n)}{(2n)^2}-\frac{1}{(2n)^2}\right) \\ =& \gamma\sum_{n=1}^\infty \frac{(-1)^n}{(2n)^2}+\sum_{n=1}^\infty \frac{(-1)^n\ln(2n)}{(2n)^2}+\sum_{n=1}^\infty \frac{(-1)^n}{(2n-1)(2n)^2} \\ =& -\frac{\pi^2}{48}\gamma+\frac{1}{8}\zeta'(2)+\sum_{n=1}^\infty \frac{(-1)^{n-1}}{4n^2}+\sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n}-\sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1} \\ =& -\frac{\pi^2}{48}\gamma+\frac{1}{8}\zeta'(2)+\frac{\pi^2}{48}+\frac{1}{2}\ln 2-\frac{\pi}{4}\end{aligned}$$

□

176. 计算积分

$$\int_0^\infty x\sin x\ln\left(1-\mathrm{e}^{-x}\right)\mathrm{d}x$$

解

$$\begin{aligned}\int_0^\infty x\sin x\ln\left(1-\mathrm{e}^{-x}\right)\mathrm{d}x &= -\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty x\mathrm{e}^{-nx}\sin x\,\mathrm{d}x \\ &= -\sum_{n=1}^\infty \frac{1}{n}\Im\int_0^\infty x\mathrm{e}^{-(n-\mathrm{i})x}\mathrm{d}x \\ &= -\Im\sum_{n=1}^\infty \frac{1}{n(n-\mathrm{i})^2} = -\Im\sum_{n=1}^\infty \frac{(n+\mathrm{i})^2}{n\left(n^2+1\right)^2}\end{aligned}$$

$$= -\sum_{n=1}^{\infty}\frac{2}{\left(n^2+1\right)^2} = 1-\frac{\pi}{2\tanh(\pi)}-\frac{\pi^2}{2\sinh^2(\pi)}$$

□

177. 设 $f(x):(1,+\infty)\to\mathbb{R}$, 且是连续可导的函数, 满足

$$f(x)\leqslant x^2\ln x,\quad f'(x)>0, x\in(1,+\infty).$$

证明：积分 $\displaystyle\int_1^{+\infty}\frac{1}{f'(x)}\mathrm{d}x$ 发散.

证明 如果 $f'(x)$ 有界, 结论显然成立, 不妨设 $f'(x)$ 无界, 这时 $f(x)$ 单调趋于 $+\infty$. 对 $\forall A>0$, 由 Cauchy 不等式得

$$\left(\int_{\mathrm{e}^{A/2}}^{\mathrm{e}^A}\frac{\mathrm{d}x}{f'(x)}\right)\left(\int_{\mathrm{e}^{A/2}}^{\mathrm{e}^A}\frac{f'(x)}{x^2\ln^2 x}\mathrm{d}x\right)\geqslant\left(\int_{\mathrm{e}^{A/2}}^{\mathrm{e}^A}\frac{\mathrm{d}x}{x\ln x}\right)^2=\ln^2 2$$

由 $f(x)\leqslant x^2\ln x$ 得 $f(\mathrm{e}^x)\leqslant x\mathrm{e}^{2x}$, 因此

$$\begin{aligned}
\int_{\mathrm{e}^{\frac{A}{2}}}^{\mathrm{e}^A}\frac{f'(x)}{x^2\ln^2 x}\mathrm{d}x &= \int_{\frac{A}{2}}^{A}\frac{f'\left(\mathrm{e}^t\right)\mathrm{e}^t}{t^2\mathrm{e}^{2t}}\mathrm{d}t=\int_{\frac{A}{2}}^{A}\frac{\mathrm{d}\left[f\left(\mathrm{e}^t\right)\right]}{t^2\mathrm{e}^{2t}}\\
&= \left.\frac{f\left(\mathrm{e}^t\right)}{t^2\mathrm{e}^{2t}}\right|_{\frac{A}{2}}^{A}+\int_{\frac{A}{2}}^{A}\frac{2t^2\mathrm{e}^{-2t}+2t\mathrm{e}^{-2t}}{t^4}f\left(\mathrm{e}^t\right)\mathrm{d}t\\
&\leqslant \frac{f\left(\mathrm{e}^A\right)}{A^2\mathrm{e}^{2A}}+\int_{\frac{A}{2}}^{A}\frac{2t^2\mathrm{e}^{-2t}+2t\mathrm{e}^{-2t}}{t^4}t\mathrm{e}^{2t}\mathrm{d}t\\
&\leqslant \frac{1}{A}+2\left(\ln 2+\frac{1}{A}\right)=2\ln 2+\frac{3}{A}.
\end{aligned}$$

取 A 充分大, 则 $\displaystyle\int_{\mathrm{e}^{\frac{A}{2}}}^{\mathrm{e}^A}\frac{f'(x)}{x^2\ln^2 x}\mathrm{d}x\leqslant 2$, 因此

$$\int_{\mathrm{e}^{A/2}}^{\mathrm{e}^A}\frac{\mathrm{d}x}{f'(x)}\geqslant\frac{\ln^2 2}{2}$$

对任意充分大的 A 都成立, 于是积分 $\displaystyle\int_1^{+\infty}\frac{1}{f'(x)}\mathrm{d}x$ 发散.

□

178. 计算积分

$$\int_0^1\sinh\left(\ln\sqrt{x}\right)\frac{\ln x\ln(1+x)}{x}\mathrm{d}x$$

解

$$\begin{aligned}
&\int_0^1\sinh\left(\ln\sqrt{x}\right)\frac{\ln x\ln(1+x)}{x}\mathrm{d}x\\
=&\,4\int_0^{\infty}x\sinh(x)\ln\left(1+\mathrm{e}^{-2x}\right)\mathrm{d}x\quad x(\to\mathrm{e}^{-2x})
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^{\infty} x\left(\mathrm{e}^x-\mathrm{e}^{-x}\right)\ln\left(1+\mathrm{e}^{-2x}\right)\mathrm{d}x \\
&= -2\int_0^{\infty}\left(\frac{1}{x}-x\right)\ln(x)\ln\left(1+x^2\right)\frac{\mathrm{d}x}{x} \quad (x\to -\ln x) \\
&= -2\int_0^{\infty}\left(1-x^2\right)\ln(x)\ln\left(1+x^2\right)\frac{\mathrm{d}x}{x^2} \\
&= 2\int_0^1 \ln(x)\ln\left(1+x^2\right)\mathrm{d}x - 2\int_0^1 \ln(x)\ln\left(1+x^2\right)\frac{\mathrm{d}x}{x^2} \\
&= 2\int_0^1 \ln(x)\ln\left(1+x^2\right)\mathrm{d}\left(\frac{1}{x}\right) - 2\int_0^1 \ln\left(1+x^2\right)\mathrm{d}x - 4\int_0^1 \ln(x)\frac{x^2}{1+x^2}\mathrm{d}x \\
&= -2\int_0^1 \ln\left(1+x^2\right)\frac{\mathrm{d}x}{x^2} - 4\int_0^1 \frac{\ln(x)}{1+x^2}\mathrm{d}x - 2\ln(2) + 4\int_0^1 \frac{x^2}{1+x^2}\mathrm{d}x - 4\int_0^1 \ln(x)\frac{x^2}{1+x^2}\mathrm{d}x \\
&= 2\int_0^1 \ln\left(1+x^2\right)\mathrm{d}\left(\frac{1}{x}\right) - 4\int_0^1 \ln(x)\,\mathrm{d}x - 2\ln(2) + 4 - \pi \\
&= 2\ln(2) - 4\int_0^1 \frac{\mathrm{d}x}{1+x^2} + 4 - 2\ln(2) + 4 - \pi = 8 - 2\pi
\end{aligned}
$$

□

179. 计算积分

$$
I=\int_0^1 \frac{\ln(1+x)}{\left(1+x^2\right)\left(1+x^3\right)}\mathrm{d}x
$$

解

$$
\begin{aligned}
I &= \int_0^1 \frac{\ln(1+x)}{\left(1+x^2\right)\left(1+x^3\right)}\mathrm{d}x \\
&= \int_0^1 \frac{x^2\left[\ln(1+x)-\ln x\right]}{\left(1+x^2\right)\left(1+x^3\right)}\mathrm{d}x \\
&= \frac{1}{2}\left(\int_0^{\infty}\frac{\ln(1+x)}{1+x^2}\mathrm{d}x - \int_0^{\infty}\frac{\left(x^2+1-1\right)\ln x}{\left(1+x^2\right)\left(1+x^3\right)}\mathrm{d}x\right) \\
&= \int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x - \int_0^{\infty}\frac{\ln x}{1+x^2}\mathrm{d}x - \int_0^{\infty}\frac{\ln x}{\left(1+x^2\right)\left(1+x^3\right)}\mathrm{d}x
\end{aligned}
$$

注意到 $\dfrac{1}{\left(1+x^2\right)\left(1+x^3\right)}=\dfrac{1+x}{1+x^2}-\dfrac{x^2+x-1}{1+x^3}$, 考虑

$$
\begin{aligned}
J(a) &= \int_0^{\infty}\frac{x^a}{\left(1+x^2\right)\left(1+x^3\right)}\mathrm{d}x \\
&= \frac{1}{2}\left(\int_0^{\infty}\frac{x^{a+1}+x^a}{1+x^2}\mathrm{d}x - \int_0^{\infty}\frac{x^{a+2}+x^{a+1}-x^a}{1+x^3}\mathrm{d}x\right) \\
&= \frac{1}{4}\int_0^{\infty}\frac{x^{\frac{a}{2}}+x^{\frac{a-1}{2}}}{1+x}\mathrm{d}x - \frac{1}{6}\int_0^{\infty}\frac{x^{\frac{a}{3}}+x^{\frac{a-1}{3}}-x^{\frac{a-2}{3}}}{1+x}\mathrm{d}x \\
&= \frac{B\left(1+\frac{a}{2},-\frac{a}{2}\right)+B\left(\frac{1+a}{2},\frac{1-a}{2}\right)}{4} - \frac{B\left(1+\frac{a}{3},-\frac{a}{3}\right)+B\left(\frac{2+a}{3},\frac{1-a}{3}\right)-B\left(\frac{1+a}{3},\frac{2-a}{3}\right)}{6}
\end{aligned}
$$

那么

$$
J'(0)=\int_0^{\infty}\frac{\ln x}{\left(1+x^2\right)\left(1+x^3\right)}\mathrm{d}x=-\frac{37\pi^2}{432}
$$

又

$$\begin{aligned}\int_0^\infty \frac{\ln(1+x)}{1+x^2}\mathrm{d}x &= 2\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x - \int_0^1 \frac{\ln x}{1+x^2}\mathrm{d}x \\ &= \frac{\pi\ln 2}{4} + \int_0^1 \frac{\arctan x}{x}\mathrm{d}x = \frac{\pi\ln 2}{4} + \gamma\end{aligned}$$

$$I = \int_0^\infty \frac{\ln(1+x)}{(1+x^2)(1+x^3)}\mathrm{d}x = \frac{1}{2}\left(\int_0^\infty \frac{\ln(1+x)}{1+x^2}\mathrm{d}x - \frac{37\pi^2}{432}\right) = \frac{\pi\ln 2}{8} + \frac{\gamma}{2} - \frac{37\pi^2}{864}$$

□

180. 计算积分

$$\int_0^1 \frac{\ln^2 x}{1-x+x^2}\mathrm{d}x$$

解

$$\begin{aligned}\int_0^1 \frac{\ln^2 x}{1-x+x^2}\mathrm{d}x &= \int_0^1 \frac{1+x}{1+x^3}\ln^2 x\mathrm{d}x \\ &= \int_0^1 \frac{\ln^2 x}{1+x^3}\mathrm{d}x + \int_0^1 \frac{x\ln^2 x}{1+x^3}\mathrm{d}x \\ &= \frac{1}{27}\left(\beta''\left(\frac{1}{3}\right) + \beta''\left(\frac{2}{3}\right)\right) \\ &= \frac{1}{216}\left(\psi''\left(\frac{2}{3}\right) - \psi''\left(\frac{1}{3}\right) + \psi''\left(\frac{5}{6}\right) - \psi''\left(\frac{1}{6}\right)\right) \\ &= \frac{\pi}{216}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\cos\pi z)\,\Big|_{z=\frac{1}{3}} + \frac{\mathrm{d}^2}{\mathrm{d}z^2}(\cos\pi z)\,\Big|_{z=\frac{1}{6}}\right) \\ &= \frac{\pi}{216}\left(\frac{8\pi^2}{3\sqrt{3}} + 8\sqrt{3}\pi^2\right) = \frac{10\pi^3}{81\sqrt{3}}\end{aligned}$$

□

181. 计算积分

$$\int_0^\infty\int_0^\infty \frac{\ln(x)\ln(y)}{\sqrt{xy}}\cos(x+y)\,\mathrm{d}x\mathrm{d}y$$

解

$$\begin{aligned}I &= \int_0^\infty\int_0^\infty \frac{\ln(x)\ln(y)}{\sqrt{xy}}\cos(x+y)\,\mathrm{d}x\mathrm{d}y \\ &= \Re\int_0^\infty\int_0^\infty \frac{\ln(x)\ln(y)}{\sqrt{xy}}\mathrm{e}^{-\mathrm{i}x}\mathrm{e}^{-\mathrm{i}y}\mathrm{d}x\mathrm{d}y \\ &= \Re\left(\int_0^\infty \frac{\ln x}{\sqrt{x}}\mathrm{e}^{-\mathrm{i}x}\mathrm{d}x\right)^2\end{aligned}$$

$$\begin{aligned}\int_0^\infty \frac{\ln x}{\sqrt{x}}\mathrm{e}^{-\mathrm{i}x}\mathrm{d}x &= \lim_{s\to1}\partial_s\int_0^\infty \mathrm{e}^{-\mathrm{i}x}x^{s-1-\frac{1}{2}}\mathrm{d}x \\ &= \lim_{s\to1}\partial_s\left(\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\left(s-\frac{1}{2}\right)}\Gamma\left(s-\frac{1}{2}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{s\to 1} \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\left(s-\frac{1}{2}\right)} \Gamma\left(s-\frac{1}{2}\right)\left[-\mathrm{i}\frac{\pi}{2}+\psi\left(s-\frac{1}{2}\right)\right] \\
&= \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\sqrt{\pi}\left[-\mathrm{i}\frac{\pi}{2}-\gamma-2\ln 2\right]
\end{aligned}
$$

于是

$$
I = \Re\left[\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\pi\left(-\mathrm{i}\frac{\pi}{2}-\gamma-2\ln 2\right)^2\right] = \pi^2\left(\gamma+2\ln 2\right)
$$

□

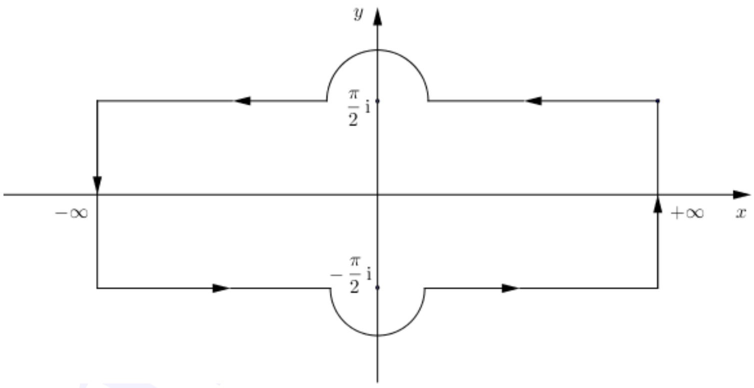
182. 计算积分

$$
\int_0^1 \frac{1}{1+a^2x^2}\left[\left(1-\frac{x}{2}\right)\ln\frac{1+x}{1-x}+\frac{\pi^2x^2}{4}\right]^{-1}\mathrm{d}x
$$

解 先作换元 $x\to\tanh x$, 可得

$$
\begin{aligned}
&\int_0^1 \frac{1}{1+a^2x^2}\left[\left(1-\frac{x}{2}\right)\ln\frac{1+x}{1-x}+\frac{\pi^2x^2}{4}\right]^{-1}\mathrm{d}x \\
=& \int_0^\infty \frac{1}{1+a^2\tanh^2 x}\left(\frac{\coth^2 x-1}{(\coth x - x)^2+\frac{\pi^2x^2}{4}}\right)\mathrm{d}x \\
=& \frac{1}{2}\int_{-\infty}^\infty \frac{1}{1+a^2\tanh^2 x}\left(\frac{\coth^2 x-1}{(\coth x - x)^2+\frac{\pi^2x^2}{4}}\right)\mathrm{d}x
\end{aligned}
$$

于是我们考虑函数 $f(z)=\dfrac{1}{1+a^2\coth^2 z}\cdot\dfrac{\tanh^2 z-1}{\tanh z - z}$ 的如下围道积分



注意到 $f(z)=\dfrac{1}{1+a^2\coth^2 z}\cdot\dfrac{\tanh^2 z-1}{\tanh z-z}$ 的极点为 $z=0, z=\pm\dfrac{\pi}{2}\mathrm{i}$（这两个极点在围道边界上）, 以及 $1+a^2\coth^2 z=0$ 的根 $z=\pm\mathrm{i}\cdot\operatorname{arccoth}\left(\dfrac{\mathrm{i}}{a}\right)=\pm\mathrm{i}\cdot\arctan(a)$, 因此根据留数定理有

$$
\begin{aligned}
&\int_{-\infty-\frac{\pi}{2}\mathrm{i}}^{\infty-\frac{\pi}{2}\mathrm{i}} f(z)\,\mathrm{d}z - \int_{-\infty+\frac{\pi}{2}\mathrm{i}}^{\infty+\frac{\pi}{2}\mathrm{i}} f(z)\,\mathrm{d}z \\
=& 2\pi\mathrm{i}\left(\operatorname{Res}\left[f(z), z=0\right]+\left(\operatorname{Res}\left[f(z),\mathrm{i}\cdot\arctan(a)\right]+\operatorname{Res}\left[f(z),z=-\mathrm{i}\cdot\arctan(a)\right]\right)\right) \\
&+\pi\mathrm{i}\left(\operatorname{Res}\left[f(z),z=\frac{\pi}{2}\mathrm{i}\right]+\operatorname{Res}\left[f(z),z=-\frac{\pi}{2}\mathrm{i}\right]\right) \\
=& 2\pi\mathrm{i}\left(\frac{3}{a^2}-\frac{a}{2\left(a-\arctan(a)\right)}-\frac{a}{2\left(a-\arctan(a)\right)}\right)+\pi\mathrm{i}\left(1+1\right)
\end{aligned}
$$

$$= 2\pi\mathrm{i}\left(\frac{3}{a^2} - \frac{\arctan(a)}{a - \arctan(a)}\right)$$

注意到 $\tanh\left(z \pm \frac{\pi}{2}\mathrm{i}\right) = \coth z, \coth\left(z \pm \frac{\pi}{2}\mathrm{i}\right) = \tanh z$, 于是

$$\int_{-\infty-\frac{\pi}{2}\mathrm{i}}^{\infty-\frac{\pi}{2}\mathrm{i}} f(z)\,\mathrm{d}z - \int_{-\infty+\frac{\pi}{2}\mathrm{i}}^{\infty+\frac{\pi}{2}\mathrm{i}} f(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} f\left(x - \frac{\pi}{2}\mathrm{i}\right)\mathrm{d}x - \int_{-\infty}^{\infty} f\left(x + \frac{\pi}{2}\mathrm{i}\right)\mathrm{d}x$$
$$= \int_{-\infty}^{\infty} \frac{1}{1 + a^2\tanh^2 x} \cdot \frac{\coth^2 x - 1}{\coth x - \left(x - \frac{\pi}{2}\mathrm{i}\right)^2}\mathrm{d}x - \int_{-\infty}^{\infty} \frac{1}{1 + a^2\tanh^2 x} \cdot \frac{\coth^2 x - 1}{\coth x - \left(x + \frac{\pi}{2}\mathrm{i}\right)^2}\mathrm{d}x$$
$$= -\pi\mathrm{i}\int_{-\infty}^{\infty} \frac{1}{1 + a^2\tanh^2 x} \cdot \frac{\coth^2 x - 1}{(\coth x - x)^2 + \frac{\pi^2}{4}}\mathrm{d}x = 2\pi\mathrm{i}\left(\frac{3}{a^2} - \frac{\arctan(a)}{a - \arctan(a)}\right)$$

因此

$$\int_0^{\infty} \frac{1}{1 + a^2\tanh^2 x} \cdot \frac{\coth^2 x - 1}{(\coth x - x)^2 + \frac{\pi^2}{4}}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1 + a^2\tanh^2 x} \cdot \frac{\coth^2 x - 1}{(\coth x - x)^2 + \frac{\pi^2}{4}}\mathrm{d}x$$
$$= \frac{\arctan(a)}{a - \arctan(a)} - \frac{3}{a^2}$$

□

183. 计算积分

$$\int_0^{\infty} \frac{\ln(x)}{1 + \mathrm{e}^x}\mathrm{d}x$$

解

$$\int_0^{\infty} \frac{\ln(x)}{1 + \mathrm{e}^x}\mathrm{d}x = \int_0^1 \frac{\ln(x)}{1 + \mathrm{e}^x}\mathrm{d}x + \int_1^{\infty} \frac{\ln(x)}{1 + \mathrm{e}^x}\mathrm{d}x$$
$$= -\ln(x)\ln\left(\frac{1 + \mathrm{e}^{-x}}{2}\right)\Big|_0^1 + \int_0^1 \ln\left(\frac{1 + \mathrm{e}^{-x}}{2}\right)\frac{\mathrm{d}x}{x} - \ln(x)\ln\left(1 + \mathrm{e}^{-x}\right)\Big|_1^{\infty} + \int_1^{\infty} \ln\left(1 + \mathrm{e}^{-x}\right)\frac{\mathrm{d}x}{x}$$
$$= \int_0^1 \ln\left(\frac{1 - \mathrm{e}^{-xy}}{y}\right)\Big|_{y=1}^{y=2}\frac{\mathrm{d}x}{x} + \int_1^{\infty} \ln\left(1 - \mathrm{e}^{-xy}\right)\Big|_{y=1}^{y=2}\frac{\mathrm{d}x}{x}$$
$$= \int_0^1\int_1^2 \left(\frac{1}{\mathrm{e}^{xy} - 1} - \frac{1}{xy}\right)\mathrm{d}y\,\mathrm{d}x + \int_1^{\infty}\int_1^2 \frac{\mathrm{d}x\,\mathrm{d}y}{\mathrm{e}^{xy} - 1}$$
$$= \int_1^2 \frac{\mathrm{d}y}{y}\left[\ln\left(\frac{1 - \mathrm{e}^{-xy}}{x}\right)\Big|_{x=0}^{x=1} + \ln\left(1 - \mathrm{e}^{-xy}\right)\Big|_{x=1}^{x=\infty}\right]$$
$$= -\int_1^2 \frac{\ln(y)}{y}\mathrm{d}y = -\frac{\ln^2 2}{2}$$

□

184. 计算积分

$$\int_0^{\frac{\pi}{2}} \frac{x\left(1 + \sin^2 x\right)\cos x}{\left(1 + 3\sin^2 x\right)\left(\sin^2 x + 3\right)}\mathrm{d}x$$

解

$$I = \int_0^{\frac{\pi}{2}} \frac{x\left(1 + \sin^2 x\right)\cos x}{\left(1 + 3\sin^2 x\right)\left(\sin^2 x + 3\right)}\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(\frac{x}{1+3\sin^2 x}+\frac{x}{\sin^2 x+3}\right)\mathrm{d}(\sin x)\\
&= \frac{1}{4\sqrt{3}}\left[\int_0^{\frac{\pi}{2}} x\mathrm{d}\left[\arctan\left(\sqrt{3}\sin x\right)\right]+\int_0^{\frac{\pi}{2}} x\mathrm{d}\left[\arctan\left(\frac{\sin x}{\sqrt{3}}\right)\right]\right]
\end{aligned}
$$

记

$$
J=\int_0^{\frac{\pi}{2}} x\mathrm{d}\left[\arctan\left(\sqrt{3}\sin x\right)\right]=x\arctan(a\sin x)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\arctan(a\sin x)\,\mathrm{d}x
$$

$$
\begin{aligned}
K(a) &= \int_0^{\frac{\pi}{2}}\arctan(a\sin x)\,\mathrm{d}x=\int_0^1\frac{\arctan(ax)}{\sqrt{1-x^2}}\mathrm{d}x,\\
K'(a) &= \int_0^1\frac{x}{(1+a^2x^2)\sqrt{1-x^2}}\mathrm{d}x=\frac{\operatorname{arcsinh}(a)}{a\sqrt{1+a^2}}
\end{aligned}
$$

$$
\begin{aligned}
K(a) &= \int_0^a\frac{\operatorname{arcsinh}(t)}{t\sqrt{1+t^2}}\mathrm{d}t=-\int_0^a\operatorname{arcsinh}(t)\,\mathrm{d}\left[\operatorname{arcsinh}\left(\frac{1}{t}\right)\right]+\int_0^a\frac{\operatorname{arcsinh}\left(\frac{1}{t}\right)}{\sqrt{1+t^2}}\mathrm{d}t\\
&= -\operatorname{arcsinh}(a)\operatorname{arcsinh}\left(\frac{1}{a}\right)+\int_{\frac{1}{a}}^{\infty}\frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}}\mathrm{d}u\\
&= -\operatorname{arcsinh}(a)\operatorname{arcsinh}\left(\frac{1}{a}\right)+\int_0^{\infty}\frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}}\mathrm{d}u-\int_0^{\frac{1}{a}}\frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}}\mathrm{d}u
\end{aligned}
$$

$$
\int_0^a\frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}}\mathrm{d}u+\int_0^{\frac{1}{a}}\frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}}\mathrm{d}u=-\operatorname{arcsinh}(a)\operatorname{arcsinh}\left(\frac{1}{a}\right)+\int_0^{\infty}\frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}}\mathrm{d}u
$$

故

$$
\begin{aligned}
I &= \frac{1}{4\sqrt{3}}\left[x\arctan(a\sin x)\Big|_0^{\frac{\pi}{2}}+x\arctan\left(\frac{\sin x}{a}\right)\Big|_0^{\frac{\pi}{2}}\right]-\left(\int_0^a\frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}}\mathrm{d}u+\int_0^{\frac{1}{a}}\frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}}\mathrm{d}u\right)\\
&= \frac{1}{4\sqrt{3}}\left[\frac{\pi^2}{4}+\operatorname{arcsinh}(a)\operatorname{arcsinh}\left(\frac{1}{a}\right)\Big|_{a=\sqrt{3}}-\int_0^{\infty}\frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}}\mathrm{d}u\right]\\
&= \frac{1}{8\sqrt{3}}\ln(3)\ln\left(2+\sqrt{3}\right)
\end{aligned}
$$

其中

$$
\int_0^{\infty}\frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}}\mathrm{d}u=\int_0^{\infty}\frac{t}{\sinh(t)}\mathrm{d}t=\int_0^{\infty}\sum_{k=0}^{\infty}2t\mathrm{e}^{-(2k+1)t}\mathrm{d}t=\sum_{k=0}^{\infty}\frac{2}{(2k+1)^2}=\frac{\pi^2}{4}
$$

□

185. 设 $a,b>0$, 计算积分

$$
\int_0^{\frac{\pi}{4}}\frac{x\ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)}{\cos x(\cos x+\sin x)}\mathrm{d}x
$$

解 对 $x\in\left[0,\frac{\pi}{2}\right]$,

$$
\frac{x\ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)}{\cos x(\cos x+\sin x)}=\frac{x\ln\left(\frac{1+\tan x}{1-\tan x}\right)}{\cos^2 x(1+\tan x)}=\frac{x\left(1+\tan^2 x\right)\ln\left(\frac{1+\tan x}{1-\tan x}\right)}{1+\tan x}
$$

于是

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} \frac{x\left(1+\tan^2 x\right)\ln\left(\frac{1+\tan x}{1-\tan x}\right)}{1+\tan x}\mathrm{d}x \\
&= \int_0^1 \frac{\arctan x \ln\left(\frac{1+x}{1-x}\right)}{1+x}\mathrm{d}x \\
&= \int_0^1 \frac{\arctan\left(\frac{x-1}{1+x}\right)\ln x}{1+x}\mathrm{d}x \\
&= \int_0^1 \frac{\arctan(x)\ln x}{1+x}\mathrm{d}x - \frac{\pi}{4}\int_0^1 \frac{\ln x}{1+x}\mathrm{d}x
\end{aligned}
$$

其中 $\int_0^1 \frac{\ln x}{1+x}\mathrm{d}x = -\frac{\pi^2}{12}$, 记 $F(x) = \int_0^x \frac{\ln t}{1+t}\mathrm{d}t = \int_0^1 \frac{x\ln(xy)}{1+xy}\mathrm{d}y$, 则 $F(1) = -\frac{\pi^2}{12}$,

$$
\begin{aligned}
\int_0^1 \frac{\arctan(x)\ln x}{1+x}\mathrm{d}x &= \left[F(x)\arctan(x)\right]\Big|_0^1 - \int_0^1 \frac{F(x)}{1+x}\mathrm{d}x \\
&= -\frac{\pi^3}{48} - \int_0^1\int_0^1 \frac{x\ln(xy)}{(1+xy)\left(1+x^2\right)}\mathrm{d}x\mathrm{d}y \\
&= -\frac{\pi^3}{48} - \int_0^1\int_0^1 \frac{x\ln(x)}{(1+xy)\left(1+x^2\right)}\mathrm{d}x\mathrm{d}y - \int_0^1\int_0^1 \frac{x\ln(y)}{(1+xy)\left(1+x^2\right)}\mathrm{d}x\mathrm{d}y \\
&= -\frac{\pi^3}{48} - \int_0^1 \left[\frac{\ln(x)\ln(1+xy)}{1+x^2}\right]\Bigg|_{y=0}^{y=1}\mathrm{d}x - \\
&\quad \int_0^1 \left[-\frac{\ln(y)\ln(1+xy)}{1+y^2} + \frac{\ln(y)\ln\left(1+x^2\right)}{2\left(1+y^2\right)} + \frac{y\ln(y)\arctan x}{1+y^2}\right]\Bigg|_{x=0}^{x=1}\mathrm{d}y \\
&= -\frac{\pi^3}{48} - \int_0^1 \frac{\ln(x)\ln(1+x)}{1+x^2}\mathrm{d}x + \int_0^1 \frac{\ln(y)\ln(1+y)}{1+y^2}\mathrm{d}y \\
&\quad - \frac{\ln(2)}{2}\int_0^1 \frac{\ln y}{1+y^2}\mathrm{d}y - \frac{\pi}{4}\int_0^1 \frac{y\ln y}{1+y^2}\mathrm{d}y \\
&= \frac{G}{2} - \frac{\pi^3}{64}
\end{aligned}
$$

于是 $I = \frac{G\ln 2}{2} + \frac{\pi^3}{192}$.

□

186. 设 $0 < a < \frac{\pi}{2}$ 计算积分

$$
\int_0^\infty \frac{x}{1+x^2}\frac{\cos(ax)}{\sinh(\pi x)}\mathrm{d}x
$$

解

$$
\begin{aligned}
\int_0^\infty \frac{x}{1+x^2}\frac{\cos(ax)}{\sinh(\pi x)}\mathrm{d}x &= \int_0^\infty \frac{\cos(ax)}{\sinh(\pi x)}\int_0^\infty \mathrm{e}^{-t}\sin(xt)\,\mathrm{d}t\mathrm{d}x \\
&= \int_0^\infty \mathrm{e}^{-t}\mathrm{d}t\int_0^\infty \frac{\sin(tx)\cos(ax)}{\sinh(\pi x)}\mathrm{d}x \\
&= \frac{1}{2}\int_0^\infty \mathrm{e}^{-t}\mathrm{d}t\int_0^\infty \frac{\sin(t+a)x+\sin(t-a)x}{\sinh(\pi x)}\mathrm{d}x \\
&= \frac{1}{2}\int_0^\infty \mathrm{e}^{-t}\left(\tanh\frac{t+a}{2} + \tanh\frac{t-a}{2}\right)\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{1}{4}\left(\int_0^1 \frac{\mathrm{e}^a-s}{\mathrm{e}^a+s}\mathrm{d}s+\int_0^1 \frac{\mathrm{e}^{-a}-s}{\mathrm{e}^{-a}+s}\mathrm{d}s\right)\\
&=-\frac{1}{2}\left(1-\mathrm{e}^a\ln\left(\frac{1+\mathrm{e}^a}{\mathrm{e}^a}\right)-\mathrm{e}^{-a}\ln\left(\frac{1+\mathrm{e}^{-a}}{\mathrm{e}^{-a}}\right)\right)\\
&=-\frac{1}{2}-\frac{a\mathrm{e}^a}{2}+\cosh(a)\ln\left(1+\mathrm{e}^a\right)
\end{aligned}
$$

其中

$$
\int_0^\infty \frac{\sin(vx)}{\sinh(\mu x)}\mathrm{d}x=2\sum_{k=0}^\infty\int_0^\infty \mathrm{e}^{-(2k+1)\mu x}\sin(vx)\,\mathrm{d}x=2\sum_{k=0}^\infty\frac{v}{v^2+(2k+1)^2\mu^2}=\frac{\pi}{2\mu}\tanh\left(\frac{v\pi}{2\mu}\right)
$$

□

187. 计算积分

$$
\int_0^1\int_0^1\frac{\left(1-x^2\right)\mathrm{d}x\mathrm{d}y}{\left(1+x^2y^2\right)\ln^2(xy)}
$$

解

$$
\begin{aligned}
I&=\int_0^1\int_0^1\int_0^\infty\int_0^\infty\frac{1-x^2}{1+x^2y^2}(xy)^{w+z}\,\mathrm{d}w\mathrm{d}z\mathrm{d}x\mathrm{d}y\\
&=\int_0^1\int_0^1\int_0^\infty\int_0^\infty\sum_{k=0}^\infty\left(-x^2y^2\right)^k\left(1-x^2\right)(xy)^{w+z}\mathrm{d}w\mathrm{d}z\mathrm{d}x\mathrm{d}y\\
&=\sum_{k=0}^\infty(-1)^k\int_0^\infty\int_0^\infty\int_0^1\int_0^1\left(x^{2k+w+z}y^{2k+w+z}-x^{2k+w+z+2}y^{2k+w+z+2}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}z\\
&=\sum_{k=0}^\infty(-1)^k\int_0^\infty\int_0^\infty\frac{2}{(2k+w+z+1)^2(2k+2+z+3)}\mathrm{d}w\mathrm{d}z\\
&=\sum_{k=0}^\infty(-1)^k\frac{1}{2}\ln\frac{k+\frac{3}{2}}{k+\frac{1}{2}}+\sum_{k=1}^\infty(-1)^k k\left(\ln\frac{2k-1}{2k+1}-1\right)
\end{aligned}
$$

$$
\frac{1}{2}\sum_{k=0}^\infty(-1)^k\ln\frac{k+\frac{3}{2}}{k+\frac{1}{2}}=\frac{1}{2}\prod_{k=0}^\infty\frac{2k+\frac{3}{2}}{2k+\frac{1}{2}}\cdot\frac{2k+\frac{3}{2}}{2k+\frac{5}{2}}=\frac{1}{2}\ln\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{3}{4}\right)}=\ln\frac{\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)}
$$

因为

$$
\lim_{n\to\infty}\prod_{k=1}^{2n}\left(1-\frac{2}{2k+1}\right)^{k\cdot(-1)^k}=\exp\left(\frac{2G}{\pi}-\frac{1}{2}\right)\quad\lim_{n\to\infty}\prod_{k=1}^{2n+1}\left(1-\frac{2}{2k+1}\right)^{k\cdot(-1)^k}=\exp\left(\frac{2G}{\pi}+\frac{1}{2}\right)
$$

所以

$$
\sum_{k=1}^\infty(-1)^k k\left(\ln\frac{2k-1}{2k+1}-1\right)=\frac{2G}{\pi}-\frac{1}{2}
$$

$$
I=\ln\frac{\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)}+\frac{2G}{\pi}-\frac{1}{2}
$$

□

188. 计算积分

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\ln(1+\cos x)-\ln(1+\cos y)}{\cos x-\cos y}\mathrm{d}x\mathrm{d}y$$

解 考虑参数积分

$$I(a)=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\ln(1+a\cos x)-\ln(1+a\cos y)}{\cos x-\cos y}\mathrm{d}x\mathrm{d}y,\ I(0)=0$$

则

$$\begin{aligned}I'(a)&=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{1}{(1+a\cos x)(1+a\cos y)}\mathrm{d}x\mathrm{d}y\\&=\left(\int_0^{\frac{\pi}{2}}\frac{1}{1+a\cos x}\mathrm{d}x\right)^2\\&=\frac{4}{1-a^2}\arctan^2\left(\sqrt{\frac{1-a}{1+a}}\right)\end{aligned}$$

于是

$$\begin{aligned}I=I(1)&=\int_0^1\frac{4}{1-a^2}\arctan^2\left(\sqrt{\frac{1-a}{1+a}}\right)\mathrm{d}a\\&=\int_0^{\frac{\pi}{2}}\frac{t^2}{\sin t}\mathrm{d}t=-2\int_0^{\frac{\pi}{2}}t\ln\left(\tan\frac{t}{2}\right)\mathrm{d}t\quad a=\cos t\\&=4\sum_{k=1}^{\infty}\int_0^{\frac{\pi}{2}}\frac{t\cos(2k-1)t}{2k-1}\mathrm{d}t\\&=4\sum_{k=1}^{\infty}\frac{\pi}{2}\frac{(-1)^{k-1}}{(2k-1)^2}-4\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\\&=2\pi G-\frac{7}{2}\zeta(3)\end{aligned}$$

□

189. 设 $a>0,b>0$, 计算积分

$$\int_0^{\infty}\frac{\cos(x^a)-\cos(x^b)}{x}\mathrm{d}x$$

解 对任意 $0<s<1$,

$$\int_0^{\infty}\frac{\cos x}{x^{1-s}}\mathrm{d}x=\Re\left(\int_0^{\infty}x^{s-1}\mathrm{e}^{\mathrm{i}x}\mathrm{d}x\right)=\Re\left[\mathrm{i}^s\cdot\Gamma(s)\right]=\Gamma(s)\cos\left(\frac{\pi s}{2}\right)$$

于是

$$\int_0^{\infty}\frac{\cos(x^a)}{x^{1-s}}\mathrm{d}x=\frac{1}{a}\int_0^{\infty}t^{\frac{s}{a}-1}\cos t\,\mathrm{d}t=\frac{1}{a}\Gamma\left(\frac{s}{a}\right)\cos\left(\frac{\pi s}{2a}\right)$$

$$\int_0^{\infty}\frac{\cos\left(x^b\right)}{x^{1-s}}\mathrm{d}x=\frac{1}{b}\Gamma\left(\frac{s}{b}\right)\cos\left(\frac{\pi s}{2b}\right)$$

于是

$$\int_0^{\infty}\frac{\cos(x^a)-\cos(x^b)}{x}\mathrm{d}x=\lim_{s\to0}\left[\frac{1}{a}\Gamma\left(\frac{s}{a}\right)\cos\left(\frac{\pi s}{2a}\right)-\frac{1}{b}\Gamma\left(\frac{s}{b}\right)\cos\left(\frac{\pi s}{2b}\right)\right]$$

$$
\begin{aligned}
&= \lim_{s\to 0}\left[\frac{1}{a}\left(\frac{a}{s}-\gamma+o\left(1\right)\right)-\frac{1}{b}\left(\frac{b}{s}-\gamma+o\left(1\right)\right)\right] \\
&= \left(\frac{1}{b}-\frac{1}{a}\right)\gamma
\end{aligned}
$$

□

190. 求极限

$$
\lim_{n\to\infty} n\left[\left(\int_0^1 \frac{1}{1+x^n}\mathrm{d}x\right)^n-\frac{1}{2}\right]
$$

解 首先有

$$
\begin{aligned}
I_n &= \int_0^1 \frac{1}{1+x^n}\mathrm{d}x = \frac{1}{n}\int_0^1 \frac{t^{\frac{1}{n}-1}}{1+t}\mathrm{d}t \\
&= \frac{1}{n}\int_0^1 t^{\frac{1}{n}}\left(\frac{1}{t}-\frac{1}{1+t}\right)\mathrm{d}t = 1-\frac{1}{n}\int_0^1 \frac{t^{\frac{1}{n}}}{1+t}\mathrm{d}t \\
&= 1-\sum_{k=0}^{\infty}\frac{1}{n^{k+1}k!}\int_0^1 \frac{\ln^k x}{1+x}\mathrm{d}x
\end{aligned}
$$

因此不难得到

$$
I\left(n\right) = 1-\frac{\ln 2}{n}+\frac{\pi^2}{12n^2}+o\left(\frac{1}{n^2}\right)
$$

故

$$
\begin{aligned}
I^n\left(n\right) &= \mathrm{e}^{n\ln\left[1-\frac{\ln 2}{n}+\frac{\pi^2}{12n^2}+o\left(\frac{1}{n^2}\right)\right]} = \mathrm{e}^{n\left[-\frac{\ln 2}{n}+\frac{\pi^2}{12n^2}-\frac{\ln^2 2}{2n^2}+o\left(\frac{1}{n^2}\right)\right]} \\
&= \frac{1}{2}\left[1+\left(\frac{\pi^2}{12}-\frac{1}{2}\ln^2 2\right)\frac{1}{n}+o\left(\frac{1}{n}\right)\right]
\end{aligned}
$$

于是最后得到

$$
\lim_{n\to\infty} n\left[I^n\left(n\right)-\frac{1}{2}\right] = \frac{\pi^2}{24}-\frac{1}{4}\ln^2 2
$$

□

191. 计算积分

$$
\int_0^1 \frac{1-x}{1+x}\cdot\frac{2k+3+x^2}{1+x^2}\cdot\frac{\mathrm{d}x}{\ln x}
$$

解 记

$$
I\left(k\right) = \int_0^1 \frac{1-x}{1+x}\cdot\frac{2k+3+x^2}{1+x^2}\cdot\frac{\mathrm{d}x}{\ln x}
$$

则

$$
\begin{aligned}
I\left(-1\right) &= \int_0^1 \frac{1-x}{1+x}\frac{\mathrm{d}x}{\ln x} = -\int_0^\infty \frac{1-\mathrm{e}^{-t}}{1+\mathrm{e}^{-t}}\frac{\mathrm{e}^{-t}}{t}\mathrm{d}t \\
&= \int_0^\infty \frac{1-\mathrm{e}^{-t}}{t}\sum_{k=1}^{\infty}\left(-\mathrm{e}^{-t}\right)^k\mathrm{d}t = \sum_{k=1}^{\infty}\left(-1\right)^k\int_0^\infty \frac{\mathrm{e}^{-kt}-\mathrm{e}^{-(k+1)t}}{t}\mathrm{d}t \\
&= \sum_{k=1}^{\infty}\left(-1\right)^k\ln\frac{k+1}{k} = \sum_{k=1}^{\infty}\ln\left(\frac{2k+1}{2k}\cdot\frac{2k-1}{2k}\right)
\end{aligned}
$$

$$= \ln\left(\prod_{k=1}^{\infty}\frac{2k+1}{2k}\cdot\frac{2k-1}{2k}\right) = -\ln\left(\frac{\pi}{2}\right)$$

$$\begin{aligned}
I'(k) &= \int_0^1 \frac{1-x}{1+x}\cdot\frac{2}{1+x^2}\cdot\frac{\mathrm{d}x}{\ln x} = \int_1^{\infty}\frac{1-x}{1+x}\cdot\frac{2}{1+x^2}\cdot\frac{\mathrm{d}x}{\ln x}\\
&= \int_0^{\infty}\frac{1-x}{1+x}\cdot\frac{1}{1+x^2}\cdot\frac{\mathrm{d}x}{\ln x} = -\int_0^{\infty}\frac{\mathrm{d}x}{(1+x)(1+x^2)}\int_0^1 x^y\mathrm{d}y\\
&= \frac{1}{2}\int_0^1\int_0^{\infty}\left(\frac{x^{y+1}}{1+x^2}-\frac{x^y}{1+x^2}-\frac{x^y}{1+x}\right)\mathrm{d}x\mathrm{d}y\\
&= -\frac{\pi}{4}\int_0^1\frac{\mathrm{d}y}{\cos\frac{\pi y}{4}\left(\cos\frac{\pi y}{4}+\sin\frac{\pi y}{4}\right)} = -\ln 2
\end{aligned}$$

于是

$$I(k) = I(-1) + \int_{-1}^{k}(-\ln 2)\,\mathrm{d}t = -\ln\left(\frac{\pi}{2}\right) - (k+1)\ln 2 = -\ln\left(2^k\pi\right)$$

□

192. 求和

$$\sum_{n=1}^{\infty}\arctan\frac{10n}{(3n^2+2)(9n^2-1)}$$

解

$$\begin{aligned}
S &= \sum_{n=1}^{\infty}\arctan\frac{10n}{(3n^2+2)(9n^2-1)}\\
&= \sum_{n=1}^{\infty}\arg\left(1+\frac{10\mathrm{i}n}{(3n^2+2)(9n^2-1)}\right)\\
&= \arg\prod_{n=1}^{\infty}\left(1+\frac{10\mathrm{i}n}{(3n^2+2)(9n^2-1)}\right)\\
&= \arg\prod_{n=1}^{\infty}\left(\frac{\left(1-\frac{\mathrm{i}}{n}\right)\left(1+\frac{\mathrm{i}}{3n-1}\right)\left(1+\frac{\mathrm{i}}{3n+1}\right)\left(1+\frac{\mathrm{i}}{3n}\right)}{1+\frac{2}{3n^2}}\right)\\
&= \lim_{m\to\infty}\sum_{n=1}^{m}\left[\arctan\left(\frac{1}{3n-1}\right)+\arctan\left(\frac{1}{3n}\right)+\arctan\left(\frac{1}{3n+1}\right)-\arctan\left(\frac{1}{n}\right)\right]\\
&= -\arctan(1) + \lim_{m\to\infty}\sum_{n=m+1}^{3m+1}\left[\frac{1}{n}+O\left(\frac{1}{n^3}\right)\right] = \ln 3 - \frac{\pi}{4}
\end{aligned}$$

□

193. 设 $f(x)$ 是 $[0,+\infty)$ 上正的连续函数, 且 $\int_0^{\infty}\frac{\mathrm{d}x}{f(x)}$ 收敛. 记 $F(x)=\int_0^x f(t)\,\mathrm{d}t$, 求证

$$\int_0^{\infty}\frac{x}{F(x)}\mathrm{d}x < 2\int_0^{\infty}\frac{\mathrm{d}x}{f(x)}$$

证明 由 Cauchy 不等式得

$$\left(\int_0^x f(t)\,\mathrm{d}t\right)\left(\int_0^x\frac{t^2}{f(t)}\mathrm{d}t\right) \geqslant \left(\int_0^x t\,\mathrm{d}t\right)^2 = \frac{1}{4}x^4$$

所以

$$\int_0^\infty \frac{x}{F(x)}\mathrm{d}x \leqslant \int_0^\infty \frac{4}{x^3}\int_0^x \frac{t^2}{f(t)}\mathrm{d}t\,\mathrm{d}x$$

注意到

$$\lim_{x\to 0+}\int_0^x \frac{t^2}{f(t)}\mathrm{d}t = \int_0^\xi \frac{1}{f(t)}\mathrm{d}t < \int_0^\infty \frac{\mathrm{d}t}{f(t)}$$

故

$$\begin{aligned}\int_0^A \frac{4}{x^3}\int_0^x \frac{t^2}{f(t)}\mathrm{d}t &= \int_0^A \left(\int_0^x \frac{t^2}{f(t)}\mathrm{d}t\right)\mathrm{d}\left(-\frac{2}{x^2}\right)\\ &= \int_0^A \frac{2}{f(x)}\mathrm{d}x - \frac{2}{A^2}\int_0^A \frac{t^2}{f(t)}\mathrm{d}t < \int_0^A \frac{2}{f(x)}\mathrm{d}x - 2\int_\xi^A \frac{t^2}{f(t)}\mathrm{d}t\end{aligned}$$

令 $A\to+\infty$, 得到

$$\int_0^\infty \frac{4}{x^3}\int_0^x \frac{t^2}{f(t)}\mathrm{d}t\,\mathrm{d}x \leqslant \int_0^\infty \frac{2}{f(x)}\mathrm{d}x - 2\int_\xi^\infty \frac{t^2}{f(t)}\mathrm{d}t < \int_0^\infty \frac{2}{f(x)}\mathrm{d}x$$

另外, 当我们取 $f(x)=x^a+1(a>1)$ 时, 有 $\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{x^a+1}=\frac{\pi}{a\sin\frac{\pi}{a}}$ 收敛. 此时有

$$\lim_{a\to 1}\frac{\int_0^\infty \frac{x}{F(x)}\mathrm{d}x}{\int_0^\infty \frac{\mathrm{d}x}{f(x)}} = \lim_{a\to 1}\frac{\int_0^\infty \frac{x}{x^{a+1}/(a+1)+x}\mathrm{d}x}{\int_0^\infty \frac{\mathrm{d}x}{x^a+1}} = \lim_{a\to 1}\frac{\int_0^\infty \frac{1}{x^a/(a+1)+1}\mathrm{d}x}{\int_0^\infty \frac{\mathrm{d}x}{x^a+1}} = \lim_{a\to 1}(a+1)^{\frac{1}{a}} = 2$$

□

194. 设 $f(x)$ 是 $[0,+\infty)$ 上周期为 T 的局部可积函数, 且 $\displaystyle\int_0^a \frac{f(x)}{x}\mathrm{d}x$ 收敛, 其中 $0<a<\pi$, 证明

$$\lim_{n\to\infty}\frac{1}{\ln n}\int_0^a \frac{f(nx)}{\sin x}\mathrm{d}x = \frac{1}{T}\int_0^T f(x)\,\mathrm{d}x$$

证明 由于 $f(x)$ 局部可积故有界,$\exists M>0$, 使得 $|f(x)|<M$, 而 $\displaystyle\int_0^a \frac{f(nx)}{x} = \int_0^{na}\frac{f(t)}{t}\mathrm{d}t\ (n\in\mathbb{N}_+)$.
由于 $\displaystyle\int_0^a \frac{f(x)}{x}\mathrm{d}x$ 收敛, 故 $\displaystyle\int_0^{na}\frac{f(t)}{t}\mathrm{d}t = \int_0^a \frac{f(nx)}{x}\mathrm{d}x$ 存在, 而

$$\left|\int_0^a \frac{f(nx)}{\sin x}\mathrm{d}x - \int_0^a \frac{f(nx)}{x}\mathrm{d}x\right| = \left|\int_0^a f(nx)\left(\frac{1}{\sin x}-\frac{1}{x}\right)\mathrm{d}x\right| \le M\int_0^a \frac{x-\sin x}{x\sin x}\mathrm{d}x$$

由于 $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=0$, 故 $\displaystyle\int_0^a \frac{x-\sin x}{x\sin x}\mathrm{d}x$ 存在且为有限数, 从而

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\ln n}\int_0^a \frac{f(nx)}{\sin x}\mathrm{d}x &= \lim_{n\to\infty}\frac{1}{\ln n}\int_0^a \frac{f(nx)}{x}\mathrm{d}x = \lim_{n\to\infty}\frac{1}{\ln n}\int_0^{na}\frac{f(t)}{t}\mathrm{d}t\\ &= \lim_{n\to\infty}\frac{1}{\ln(na)-\ln a}\int_0^{na}\frac{f(t)}{t}\mathrm{d}t = \lim_{x\to+\infty}\frac{1}{\ln x}\int_0^x \frac{f(t)}{t}\mathrm{d}t\\ &= \frac{1}{T}\int_0^T f(x)\,\mathrm{d}x\end{aligned}$$

□

195. 计算积分

$$\int_0^\infty [\operatorname{si}(x)\cos x-\operatorname{ci}(x)\sin x]^2\,\mathrm{d}x$$

其中 $\operatorname{si}(x)=-\int_x^\infty \frac{\sin t}{t}\mathrm{d}t,\ \operatorname{ci}(x)=-\int_x^\infty \frac{\cos t}{t}\mathrm{d}t$.

解

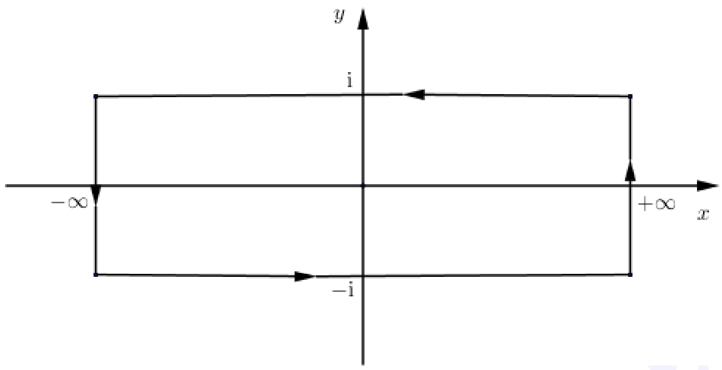
$$\begin{aligned}
\int_0^\infty [\operatorname{si}(x)\cos x-\operatorname{ci}(x)\sin x]^2\,\mathrm{d}x &= \int_0^\infty \left[\sin x\int_x^\infty \frac{\cos t}{t}\mathrm{d}t-\cos x\int_x^\infty \frac{\sin t}{t}\mathrm{d}t\right]^2\mathrm{d}x\\
&=\int_0^\infty\left[\int_x^\infty \frac{\sin(x-t)}{t}\mathrm{d}t\right]^2\mathrm{d}x=\int_0^\infty\left(\int_0^\infty\frac{\sin\theta}{x+\theta}\mathrm{d}\theta\right)^2\mathrm{d}x\\
&=\int_0^\infty\left(\int_0^\infty \sin\theta\mathrm{d}\theta\int_0^\infty \mathrm{e}^{-(x+\theta)y}\mathrm{d}y\right)^2\mathrm{d}x\\
&=\int_0^\infty\left(\int_0^\infty \mathrm{e}^{-xy}\mathrm{d}y\int_0^\infty \mathrm{e}^{-y\theta}\sin\theta\mathrm{d}\theta\right)^2\mathrm{d}x\\
&=\int_0^\infty\left(\int_0^\infty \frac{\mathrm{e}^{-xy}}{1+y^2}\mathrm{d}y\right)^2\mathrm{d}x\\
&=\int_0^\infty\int_0^\infty\int_0^\infty\frac{\mathrm{e}^{-xy}}{1+y^2}\cdot\frac{\mathrm{e}^{-xz}}{1+z^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&=\int_0^\infty\int_0^\infty\frac{\mathrm{d}y\mathrm{d}z}{(1+y^2)(1+z^2)}\int_0^\infty \mathrm{e}^{-(y+z)x}\mathrm{d}x\\
&=\int_0^\infty\int_0^\infty\frac{\mathrm{d}y\mathrm{d}z}{(1+y^2)(1+z^2)(y+z)}\\
&=\int_0^\infty\frac{\mathrm{d}z}{(1+z^2)}\left(z\int_0^\infty\frac{\mathrm{d}y}{1+y^2}+\int_0^\infty\left(\frac{1}{y+z}-\frac{y}{1+y^2}\right)\mathrm{d}y\right)\\
&=\int_0^\infty\frac{\mathrm{d}z}{(1+z^2)}\left(\frac{\pi}{2}z-\ln z\right)\\
&=\frac{\pi}{2}\int_0^\infty\frac{z\mathrm{d}z}{(1+z^2)^2}-\int_0^\infty\frac{\ln z}{(1+x^2)^2}\mathrm{d}z\\
&=\frac{\pi}{4}-\left(\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty\frac{z^{s-1}}{(1+z^2)^2}\mathrm{d}z\right)_{s=1}\\
&=\frac{\pi}{4}-\frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}s}B\left(2-\frac{s}{2},\frac{s}{2}\right)\right]_{s=1}\\
&=\frac{\pi}{4}-\frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\left(1-\frac{s}{2}\right)\frac{\pi}{\sin\frac{\pi s}{2}}\right)\right]_{s=1}=\frac{\pi}{2}
\end{aligned}$$

□

196. 计算积分

$$\int_{-\infty}^\infty\frac{\mathrm{e}^{7\pi x}}{\left(\mathrm{e}^{3\pi x}+\mathrm{e}^{-3\pi x}\right)^3(1+x^2)}\mathrm{d}x$$

解 考虑函数 $f(z)=\dfrac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z}$ 的如下围道积分, 以 $y=\pm\mathrm{i}$ 和 $x=\pm\infty$ 为边的矩形围道（实质是令长方形的长趋于 ∞）.



易知在 $x=\pm\infty$ 两条围道的积分是 0, 在矩形内部的所有极点为 $z=0,\pm\frac{\mathrm{i}}{6},\pm\frac{\mathrm{i}}{2},\frac{5}{6}\mathrm{i}$, 那么由留数定理有

$$
\begin{aligned}
&\int_{-\infty-\mathrm{i}}^{+\infty-\mathrm{i}}\frac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z}\mathrm{d}z-\int_{-\infty+\mathrm{i}}^{+\infty+\mathrm{i}}\frac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z}\mathrm{d}z\\
=&\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{7\pi(x-\mathrm{i})}}{\left(\mathrm{e}^{3\pi(x-\mathrm{i})}+\mathrm{e}^{-3\pi(x-\mathrm{i})}\right)^3(x-\mathrm{i})}\mathrm{d}x-\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{7\pi(x+\mathrm{i})}}{\left(\mathrm{e}^{3\pi(x+\mathrm{i})}+\mathrm{e}^{-3\pi(x+\mathrm{i})}\right)^3(x+\mathrm{i})}\mathrm{d}x\\
=&2\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{7\pi x}}{\left(\mathrm{e}^{3\pi x}+\mathrm{e}^{-3\pi x}\right)^3(1+x^2)}\mathrm{d}x\\
=&2\pi\mathrm{i}\sum\operatorname{res}\left(\frac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z},z=0,\pm\frac{\mathrm{i}}{6},\pm\frac{\mathrm{i}}{2},\pm\frac{5}{6}\mathrm{i}\right)
\end{aligned}
$$

其中

$$
\begin{aligned}
\operatorname{res}\left(\frac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z},z=0\right)&=\frac{1}{8}\\
\operatorname{res}\left(\frac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z},z=\frac{\mathrm{i}}{6}\right)&=-\frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{6}}\left(-18+21\pi\mathrm{i}+10\pi^2\right)}{18\pi^3}\\
\operatorname{res}\left(\frac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z},z=-\frac{\mathrm{i}}{6}\right)&=\frac{\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\left(-18+21\pi\mathrm{i}+10\pi^2\right)}{18\pi^3}\\
\operatorname{res}\left(\frac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z},z=\frac{\mathrm{i}}{2}\right)&=\frac{\mathrm{i}\left(-2+7\pi\mathrm{i}+10\pi^2\right)}{54\pi^3}\\
\operatorname{res}\left(\frac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z},z=-\frac{\mathrm{i}}{2}\right)&=-\frac{\mathrm{i}\left(-2-7\pi\mathrm{i}+10\pi^2\right)}{54\pi^3}\\
\operatorname{res}\left(\frac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z},z=\frac{5\mathrm{i}}{6}\right)&=-\frac{\mathrm{e}^{\frac{5}{6}\pi\mathrm{i}}\left(-18+105\pi\mathrm{i}+250\pi^2\right)}{2250\pi^3}\\
\operatorname{res}\left(\frac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z},z=-\frac{5\mathrm{i}}{6}\right)&=\frac{\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}\left(-18-205\pi\mathrm{i}+250\pi^2\right)}{2250\pi^3}
\end{aligned}
$$

于是最后得到

$$
\int_{-\infty}^{\infty}\frac{\mathrm{e}^{7\pi x}}{\left(\mathrm{e}^{3\pi x}+\mathrm{e}^{-3\pi x}\right)^3(1+x^2)}\mathrm{d}x=\pi\sum\operatorname{res}\left(\frac{\mathrm{e}^{7\pi z}}{\left(\mathrm{e}^{3\pi z}+\mathrm{e}^{-3\pi z}\right)^3 z},z=0,\pm\frac{\mathrm{i}}{6},\pm\frac{\mathrm{i}}{2},\pm\frac{5}{6}\mathrm{i}\right)
$$

$$= \frac{\pi}{8} + \frac{4\left(837+5\pi\left(161-75\sqrt{3}\pi\right)\right)}{3375\pi^2}$$

□

197. 设 $s>0$,n 是正整数, 计算积分

$$\int_0^1\int_0^1 \frac{(xy)^{s-1}-y^n}{(1-xy)\ln(xy)}\mathrm{d}x\mathrm{d}y$$

解 令 $u=xy, x=\frac{u}{y}, \mathrm{d}x=\frac{\mathrm{d}u}{y}$,

$$\begin{aligned}
\int_0^1\int_0^1 \frac{(xy)^{s-1}-y^n}{(1-xy)\ln(xy)}\mathrm{d}x\mathrm{d}y &= \int_0^1\int_u^1 \frac{u^{s-1}-y^n}{(1-u)\ln u}\frac{\mathrm{d}u\mathrm{d}y}{y}\\
&= \int_0^1 \frac{1}{(1-u)\ln u}\left(\int_u^1\left(\frac{u^{s-1}}{y}-y^{n-1}\right)\mathrm{d}y\right)\mathrm{d}u\\
&= \int_0^1 \frac{1}{(1-u)\ln u}\left(u^{s-1}\ln y-\frac{y^n}{n}\right)\Big|_{y=u}^1\mathrm{d}u\\
&-\int_0^1\left(\frac{u^{s-1}}{1-u}+\frac{1}{\ln u}\right)\mathrm{d}u+\int_0^1\left(\frac{1-u^n}{n(1-u)}-1\right)\frac{\mathrm{d}u}{\ln u}\\
&= -\int_0^1\left(\frac{u^{s-1}}{1-u}+\frac{1}{\ln u}\right)\mathrm{d}u-\frac{1}{n}\sum_{k=1}^{n-1}\int_0^1\frac{u^k-1}{\ln u}\mathrm{d}u\\
&= \frac{\Gamma'(s)}{\Gamma(s)}-\frac{\ln(n!)}{n}
\end{aligned}$$

□

198. 求和

$$\sum_{k=1}^{\infty}\left(H_k-\ln k-\gamma-\frac{1}{2k}+\frac{1}{12k^2}\right)$$

解 首先有

$$\sum_{k=1}^n H_k = \sum_{k=1}^n\sum_{j=1}^k\frac{1}{j} = \sum_{j=1}^n\frac{1}{j}\sum_{k=j}^n 1 = \sum_{j=1}^n\frac{n+1-j}{j} = (n+1)H_n-n$$

因此

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left(H_k-\ln k-\gamma-\frac{1}{2k}+\frac{1}{12k^2}\right) = (n+1)H_n-n-\ln(n!)-n\gamma-\frac{H_n}{2}\\
&= \left(n+\frac{1}{2}\right)\left(\ln n+\gamma+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)\right)-n-n\gamma-\left(n\ln n-n+\frac{\ln(2\pi)}{2}+\frac{\ln n}{2}+O\left(\frac{1}{n}\right)\right)\\
&= \frac{1+\gamma}{2}-\frac{\ln(2\pi)}{2}+O\left(\frac{1}{n}\right)
\end{aligned}$$

其中

$$H_n = \ln n+\gamma+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)$$

$$\ln(n!) = n\ln n-n+\frac{\ln(2\pi)}{2}+\frac{\ln n}{2}+O\left(\frac{1}{n}\right)$$

且

$$\sum_{k=1}^{\infty}\frac{1}{12k^2}=\frac{1}{12}\zeta(2)=\frac{\pi^2}{72}$$

因此

$$\sum_{k=1}^{\infty}\left(H_k-\ln k-\gamma-\frac{1}{2k}+\frac{1}{12k^2}\right)=\frac{1+\gamma}{2}-\frac{\ln(2\pi)}{2}+\frac{\pi^2}{72}$$

□

199. 求和

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{H_{n+m}}{n+m}$$

解 首先有

$$H_n=\int_0^1\frac{1-x^n}{1-x}\mathrm{d}x\quad\frac{1}{n}=\int_0^{\infty}\mathrm{e}^{-ny}\mathrm{d}y$$

于是

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{H_{n+m}}{n+m}&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^1\frac{1-x^{n+m}}{1-x}\mathrm{d}x\int_0^{\infty}(-\mathrm{e}^{y})^{m+n}\mathrm{d}y\\&=\int_0^1\int_0^{\infty}\left[\frac{1}{(1-x)(1+\mathrm{e}^y)^2}-\frac{x^2}{(1-x)(x+\mathrm{e}^y)^2}\right]\mathrm{d}y\mathrm{d}x\\&=\int_0^1\left(\frac{\ln 2-\frac{1}{2}}{1-x}-\frac{1}{1-x}\left(\ln(1+x)-1+\frac{1}{1+x}\right)\right)\mathrm{d}x\\&=\int_0^1\frac{\ln 2-\ln(1+x)}{1-x}\mathrm{d}x-\int_0^1\frac{1}{2(1+x)}\mathrm{d}x\\&=\frac{\pi^2}{12}-\frac{\ln^2 2}{2}-\frac{\ln 2}{2}\end{aligned}$$

□

200. 求和

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2$$

解 首先有

$$\begin{aligned}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2&=\left((-1)^n\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{k}\right)^2\\&=\left(\sum_{k=n+1}^{\infty}\int_0^1(-x)^{k-1}\mathrm{d}x\right)^2=\int_0^1\frac{x^n}{1+x}\mathrm{d}x\int_0^1\frac{y^n}{1+y}\mathrm{d}y\end{aligned}$$

于是

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2=\sum_{n=0}^{\infty}(-1)^n\int_0^1\frac{x^n}{1+x}\mathrm{d}x\int_0^1\frac{y^n}{1+y}\mathrm{d}y$$

$$
\begin{aligned}
&=\int_0^1\int_0^1\frac{1}{(1+x)(1+y)(1+xy)}\mathrm{d}x\mathrm{d}y\\
&=\int_0^1\frac{1}{1+x}\left(\int_0^1\frac{1}{(1+y)(1+xy)}\mathrm{d}y\right)\mathrm{d}x\\
&=\int_0^1\frac{1}{1+x}\frac{\ln 2-\ln(1+x)}{1-x}\mathrm{d}x\\
&=\frac{1}{2}\int_0^1\frac{\ln(1+x)-\ln(1-x)}{1+x}\mathrm{d}x\\
&=\frac{\pi^2}{24}
\end{aligned}
$$

□

201. 设 n 和 q 都是正整数, 满足 $2n>q\geqslant 1$, 令

$$
f(t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-t\left(x_1^{2n}+\cdots+x_q^{2n}\right)}}{1+x_1^{2n}+\cdots+x_q^{2n}}\mathrm{d}x_1\cdots\mathrm{d}x_n
$$

证明

$$
\lim_{t\to+\infty}t^{\frac{q}{2n}}f(t)=n^{-q}\Gamma^q\left(\frac{1}{2n}\right)
$$

证明 令 $x_i^{2n}=u_i, i=1,\cdots,q$, 则

$$
\begin{aligned}
f(t)&=2^q\int_{[0,\infty)^q}\frac{\mathrm{e}^{-t\left(x_1^{2n}+\cdots+x_q^{2n}\right)}}{1+x_1^{2n}+\cdots+x_q^{2n}}\mathrm{d}x_1\cdots\mathrm{d}x_q\\
&=n^{-q}\int_{[0,\infty)^q}\frac{\prod_{i=1}^q u_i^{\frac{1}{2n}-1}\mathrm{e}^{-tu_i}}{1+u_1+\cdots+u_n}\mathrm{d}u_1\cdots\mathrm{d}u_q\\
&=n^{-q}\int_{[0,\infty)^q}\prod_{i=1}^q u_i^{\frac{1}{2n}-1}\mathrm{e}^{-tu_i}\left(\int_0^\infty \mathrm{e}^{-s(1+u_1+\cdots+u_n)}\mathrm{d}s\right)\mathrm{d}u_1\cdots\mathrm{d}u_q\\
&=n^{-q}\int_0^\infty \mathrm{e}^{-s}\left(\int_0^\infty u^{\frac{1}{2n}-1}\mathrm{e}^{-(s+t)u}\mathrm{d}u\right)^q\mathrm{d}s\\
&=n^{-q}\int_0^\infty \mathrm{e}^{-s}\left(\frac{1}{(s+t)^{\frac{1}{2n}}}\int_0^\infty r^{\frac{1}{2n}-1}\mathrm{e}^{-r}\mathrm{d}r\right)^q\mathrm{d}s\\
&=n^{-q}\int_0^\infty \mathrm{e}^{-s}\left(\frac{\Gamma\left(\frac{1}{2n}\right)}{(s+t)^{\frac{1}{2n}}}\right)^q\mathrm{d}s\\
&=n^{-q}\Gamma^q\left(\frac{1}{2n}\right)\int_0^\infty\frac{\mathrm{e}^{-s}}{(s+t)^{\frac{q}{2n}}}\mathrm{d}s
\end{aligned}
$$

因此

$$
\lim_{t\to+\infty}t^{\frac{q}{2n}}f(t)=n^{-q}\Gamma^q\left(\frac{1}{2n}\right)\lim_{t\to+\infty}\int_0^\infty \mathrm{e}^{-s}\left(\frac{t}{s+t}\right)^{\frac{q}{2n}}\mathrm{d}s=n^{-q}\Gamma^q\left(\frac{1}{2n}\right)
$$

□

202. 求和

$$
\sum_{n=1}^{\infty}4^n\sin^4\left(2^{-n}\theta\right)
$$

解 我们归纳证明对非负整数 N, 有

$$\sum_{n=1}^{N} 4^n \sin^4\left(2^{-n}\theta\right) = 4^N \sin^2\left(2^{-N}\theta\right) - \sin^2\theta$$

等式对 $N = 0$ 显然, 假定对 $N \geqslant 0$ 成立, 则

$$\begin{aligned}\sum_{n=1}^{N+1} 4^n \sin^4\left(2^{-n}\theta\right) &= 4^N \sin^2\left(2^{-N}\theta\right) - \sin^2\theta + 4^{N+1}\sin^4\left(2^{-(N+1)}\theta\right)\\ &= 4^N \sin^2\left(2^{-N}\theta\right) - \sin^2\theta + 4^N\left(1-\cos\left(2^{-N}\theta\right)\right)^2\\ &= 4^N\left(2-2\cos\left(2^{-N}\theta\right) - \sin^2\theta\right)\\ &= 4^{N+1}\sin^2\left(2^{-(N+1)}\theta\right) - \sin^2\theta\end{aligned}$$

因此令 $N \to \infty$ 得

$$\sum_{n=1}^{\infty} 4^n \sin^4\left(2^{-n}\theta\right) = \theta^2 - \sin^2\theta$$

□

203. 证明

$$\prod_{n=2}^{\infty}\left(\left(\frac{n^2-1}{n^2}\right)^{2(n^2-1)}\left(\frac{n+1}{n-1}\right)^n\right) = \pi$$

解 先考虑有限乘积

$$\begin{aligned}\prod_{n=2}^{N}\left(\frac{n+1}{n-1}\right)^n &= \frac{3^2}{1^2}\cdot\frac{4^3}{2^3}\cdot\frac{5^4}{3^4}\cdots\frac{N^{N-1}}{(N-2)^{N-1}}\cdot\frac{(N+1)^N}{(N-1)^N}\\ &= \frac{N^{N-1}(N+1)^N}{2\left((N-1)!\right)^2} = \frac{N^{2N+1}}{2(N!)^2}\left(1+\frac{1}{N}\right)^N\\ &= \frac{\mathrm{e}^{2N+1}}{4\pi}(1+o(1))\end{aligned}$$

再考虑另一个有限乘积

$$\begin{aligned}\prod_{n=2}^{N}\left(\frac{n^2-1}{n^2}\right)^{n^2-1} &= \prod_{n=2}^{N}\frac{(n-1)^{n^2-1}(n+1)^{n^2-1}}{n^{2(n^2-1)}}\\ &= \frac{1^3\cdot 3^3}{2^6}\cdot\frac{2^8\cdot 4^8}{3^{16}}\cdot\frac{3^{15}\cdot 5^{15}}{4^{30}}\cdots\frac{(N-1)^{N^2-1}(N+1)^{N^2-1}}{N^{2(N^2-1)}}\\ &= \frac{\left((N-1)!\right)^2(N+1)^{N^2-1}}{N^{N^2+2N+2}} = \frac{(N!)^2}{N^{2N+1}}\left(1+\frac{1}{N}\right)^{N^2-1}\\ &= \frac{2\pi}{\mathrm{e}^{2N}}\mathrm{e}^{N-\frac{1}{2}}(1+o(1)) = \frac{2\pi}{\mathrm{e}^{N+\frac{1}{2}}}(1+o(1))\end{aligned}$$

因此最后得到

$$\prod_{n=2}^{N}\left(\left(\frac{n^2-1}{n^2}\right)^{2(n^2-1)}\left(\frac{n+1}{n-1}\right)^n\right) = \left(\prod_{n=2}^{N}\left(\frac{n^2-1}{n^2}\right)^{n^2-1}\right)^2\prod_{n=2}^{N}\left(\frac{n+1}{n-1}\right)^n$$

$$= \frac{4\pi^2}{e^{2N+1}}(1+o(1))\frac{e^{2N+1}}{4\pi}(1+o(1)) \to \pi$$

□

204. 证明

$$\prod_{n=2}^{\infty}\left(\frac{1}{e}\left(\frac{n^2}{n^2-1}\right)^{n^2-1}\right) = \frac{e\sqrt{e}}{2\pi}$$

证明 取对数后得

$$\begin{aligned}\sum_{n=2}^{\infty}\left(-1+\left(n^2-1\right)\ln\left(\frac{1}{1-\frac{1}{n^2}}\right)\right) &= \sum_{n=2}^{\infty}\left(-1+\left(n^2-1\right)\sum_{k=1}^{\infty}\frac{1}{kn^{2k}}\right)\\ &= \sum_{n=2}^{\infty}\left(-1+\sum_{k=1}^{\infty}\frac{1}{kn^{2k-2}}-\sum_{k=1}^{\infty}\frac{1}{kn^{2k}}\right)\\ &= \sum_{k=2}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{(k+1)n^{2k}}-\sum_{k=1}^{5}\frac{1}{kn^{2k}}\right)\\ &= \sum_{k=2}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{(k+1)n^{2k}}-\sum_{k=1}^{5}\frac{1}{kn^{2k}}\right)\end{aligned}$$

其中

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{k} = \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{kn^{2k}} = \sum_{n=2}^{\infty}\ln\left(\frac{n^2}{n^2-1}\right) = \ln 2$$

□

205. 设

$$L = \lim_{n\to\infty}\int_0^1 \sqrt[n]{x^n+(1-x)^n}\,dx.$$

(a) 求 L.

(b) 求极限

$$\lim_{n\to\infty} n^2\left(\int_0^1 \sqrt[n]{x^n+(1-x)^n}\,dx - L\right)$$

解

(a) 令 $t = \frac{x}{1-x}, x = \frac{t}{1+t}, dx = \frac{dt}{(1+t)^2}$, 且

$$\int_0^1 \sqrt[n]{x^n+(1-x)^n}\,dx = 2\int_0^{\frac{1}{2}}(1-x)\sqrt[n]{1+\left(\frac{x}{1-x}\right)^n}\,dx = 2\int_0^1\frac{\sqrt[n]{1+t^n}}{(1+t)^3}dt$$

对任意 $t\in[0,1]$, 且

$$\frac{\sqrt[n]{1+t^n}}{(1+t)^3} \to \frac{1}{(1+t)^3} \qquad 0 \leqslant \frac{\sqrt[n]{1+t^n}}{(1+t)^3} \leqslant \frac{2}{(1+t)^3}$$

因此, 由 Lebesgue 控制收敛定理,

$$L = \lim_{n\to\infty}\int_0^1 \sqrt[n]{x^n+(1-x)^n}\,dx = 2\int_0^1\frac{dt}{(1+t)^3} = \frac{3}{4}$$

(b) 令 $s=t^n$, 则 $t=s^{\frac{1}{n}}, \mathrm{d}t=\frac{1}{n}s^{\frac{1}{n}-1}$, 且

$$\begin{aligned}
n^2\left(\int_0^1 \sqrt[n]{x^n+(1-x)^n}\mathrm{d}x-L\right) &= 2n^2\int_0^1 \frac{\sqrt[n]{1+t^n}-1}{(1+t)^3}\mathrm{d}t \\
&= 2n\int_0^1 \frac{(1+s)^{\frac{1}{n}}-1}{\left(1+s^{\frac{1}{n}}\right)^3}s^{\frac{1}{n}-1}\mathrm{d}s \\
&= 2n\int_0^1 \frac{\exp\left(\frac{1}{n}\ln(1+s)\right)-1}{\left(1+s^{\frac{1}{n}}\right)^3}s^{\frac{1}{n}-1}\mathrm{d}s \\
&= 2\int_0^1 \frac{\exp(h_n(s))\ln(1+s)}{\left(1+s^{\frac{1}{n}}\right)^3}s^{\frac{1}{n}-1}\mathrm{d}s
\end{aligned}$$

其中根据中值定理,$0<h_n(s)<\frac{1}{n}\ln(1+s)$, 对 $s\in(0,1]$,

$$\frac{\exp(h_n(s))\ln(1+s)}{\left(1+s^{\frac{1}{n}}\right)^3}s^{\frac{1}{n}-1}\to\frac{\ln(1+s)}{8s} \quad 0\leqslant\frac{\exp(h_n(s))\ln(1+s)}{\left(1+s^{\frac{1}{n}}\right)^3}\leqslant\frac{2\ln(1+s)}{s}$$

再由 Lebesgue 控制收敛定理得

$$\lim_{n\to\infty}n^2\left(\int_0^1 \sqrt[n]{x^n+(1-x)^n}\mathrm{d}x-L\right)=\frac{1}{4}\int_0^1\frac{\ln(1+s)}{s}\mathrm{d}s=\frac{\pi^2}{48}$$

□

206. 计算积分

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\ln(2-\sin\theta\cos\varphi)\sin\theta}{2-2\sin\theta\cos\varphi+\sin^2\theta\cos^2\varphi}\mathrm{d}\theta\mathrm{d}\varphi$$

解

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\ln(2-\sin\theta\cos\varphi)\sin\theta}{1+(1-\sin\theta\cos\varphi)^2}\mathrm{d}\theta\mathrm{d}\varphi \\
&= \int_0^1\int_0^{\sqrt{1-x^2}}\frac{\ln(2-x)}{1+(1-x)^2}\frac{1}{\sqrt{1-x^2-y^2}}\mathrm{d}y\mathrm{d}x \\
&= \int_0^1\frac{\ln(2-x)}{1+(1-x)^2}\left[\arctan\left(\frac{y}{\sqrt{1-x^2-y^2}}\right)\right]\Bigg|_{y=0}^{\sqrt{1-x^2}}\mathrm{d}x \\
&= \frac{\pi}{2}\int_0^1\frac{\ln(2-x)}{1+(1-x)^2}\mathrm{d}x=\frac{\pi}{2}\int_0^1\frac{\ln(1+x)}{1+x^2}\mathrm{d}x \\
&= \frac{\pi^2}{16}\ln 2
\end{aligned}$$

□

207. 计算主值积分

$$\int_0^{\infty}\frac{\ln(1+\cos x)}{\mathrm{e}^x+1}\mathrm{d}x$$

解 首先有

$$\begin{aligned}\int_0^{\infty}\frac{\ln(1+\cos x)}{\mathrm{e}^x+1}\mathrm{d}x &= -\ln^2 2-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\infty}\frac{\cos(nx)}{\mathrm{e}^x+1}\mathrm{d}x\\ &= -\ln^2 2+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}m}{n\left(m^2+n^2\right)}\end{aligned}$$

于是

$$\begin{aligned}2I &= -2\ln^2 2+2\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}m}{n\left(m^2+n^2\right)}+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}n}{m\left(m^2+n^2\right)}\right]\\ &= -2\ln^2 2+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}}{m^2+n^2}\left(\frac{n}{m}+\frac{m}{n}\right)\\ &= -2\ln^2 2+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}}{mn}\\ &= -2\ln^2 2+2\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\right)^2=0\end{aligned}$$

□

208. 计算积分

$$\int_0^1\cos(\pi x)\ln\left(\frac{\Gamma(x)}{\Gamma^2\left(\frac{x}{2}\right)}\right)\mathrm{d}x$$

解 利用傅里叶展开公式

$$\begin{aligned}\ln\left(\frac{\Gamma(x)}{\Gamma^2\left(\frac{x}{2}\right)}\right) &= \sum_{n=1}^{\infty}\frac{\ln(2n\pi)+\gamma}{n\pi}\sin(2n\pi x)+\sum_{n=1}^{\infty}\frac{\cos(2n\pi x)}{2n}+\frac{\ln(2\pi)}{2}\\ &\quad -2\left[\sum_{n=1}^{\infty}\frac{\ln(2n\pi)+\gamma}{n\pi}\sin(n\pi x)+\sum_{n=1}^{\infty}\frac{\cos(n\pi x)}{2n}+\frac{\ln(2\pi)}{2}\right]\end{aligned}$$

再利用三角积分的正交性有

$$\int_0^1\cos(\pi x)\sin(2n-1)\pi x\mathrm{d}x=0,\int_0^1\cos(\pi x)\cos(n\pi x)\mathrm{d}x=0\ (n\neq 1)$$

于是

$$\begin{aligned}I &= \int_0^1\cos(\pi x)\left(-\frac{\ln 2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n\pi x)}{n}-\cos(\pi x)\right)\mathrm{d}x\\ &= -\frac{1}{2}-\frac{\ln 2}{\pi}\int_0^1\frac{\pi-2\pi x}{2}\cos(\pi x)\mathrm{d}x=-\frac{1}{2}-\frac{2\ln 2}{\pi^2}\end{aligned}$$

□

209. 计算积分

$$\int_0^{\frac{\pi}{4}}\sqrt{\tan x}\sqrt{1-\tan x}\mathrm{d}x$$

解

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}}\sqrt{\tan x}\sqrt{1-\tan x}\mathrm{d}x &= \int_0^1\frac{\sqrt{y(1-y)}}{1+y^2}\mathrm{d}y \quad y=\tan x\\
&= \int_0^{\infty}\frac{\sqrt{t}}{(1+t)\left(1+2t+2t^2\right)}\mathrm{d}t \quad t=\frac{y}{1-y}\\
&= \int_0^{\infty}\frac{2z^2}{\left(1+z^2\right)\left(1+2z^2+2z^4\right)}\mathrm{d}z\\
&= 2\int_0^{\infty}\left(\frac{2z^2}{1+2z^2+2z^4}+\frac{1}{1+2z^2+2z^4}-\frac{1}{1+z^2}\right)\mathrm{d}z\\
&= \int_{-\infty}^{\infty}\left(\frac{2z^2}{1+2z^2+2z^4}+\frac{1}{1+2z^2+2z^4}-\frac{1}{1+z^2}\right)\mathrm{d}z\\
&= I_1+I_2-\pi
\end{aligned}
$$

其中

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{\infty}\frac{2z^2}{1+2z^2+2z^4}\mathrm{d}z = \int_{-\infty}^{\infty}\frac{2}{1+\frac{1}{2z^2}+z^2}\mathrm{d}z\\
&= \int_{-\infty}^{\infty}\frac{1}{\left(z-\frac{1}{\sqrt{2}z}\right)^2+1+\sqrt{2}}\mathrm{d}z = \int_{-\infty}^{\infty}\frac{1}{z^2+1+\sqrt{2}}\mathrm{d}z\\
&= \frac{\pi}{\sqrt{1+\sqrt{2}}}\\
I_2 &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{z^4+z^2+\frac{1}{2}}\mathrm{d}z = \pi\sqrt{\frac{\sqrt{2}-1}{2}}
\end{aligned}
$$

于是

$$
\int_0^{\frac{\pi}{4}}\sqrt{\tan x}\sqrt{1-\tan x}\mathrm{d}x = \frac{\pi}{\sqrt[4]{2}}\sqrt{\frac{2+\sqrt{2}}{2}}-\pi
$$

□

210. 计算积分

$$
\int_0^{\frac{\pi}{2}} x\csc^2(x)\arctan(a\tan x)\,\mathrm{d}x
$$

解 首先作换元 $u=\tan x$ 得

$$
\int_0^{\frac{\pi}{2}} x\csc^2(x)\arctan(a\tan x)\,\mathrm{d}x = \int_0^{\infty}\frac{\tan^{-1}u\tan^{-1}(au)}{u^2}\mathrm{d}u
$$

记

$$
I(\alpha,\beta) = \int_0^{\infty}\frac{\tan^{-1}(\alpha u)\tan^{-1}(\beta u)}{u^2}\mathrm{d}u
$$

则

$$
\begin{aligned}
\frac{\partial^2 I}{\partial\alpha\partial\beta} &= \int_0^{\infty}\frac{\mathrm{d}u}{\left(1+\alpha^2u^2\right)\left(1+\beta^2u^2\right)}\\
&= \frac{1}{\alpha^2-\beta^2}\int_0^{\infty}\left(\frac{\alpha^2}{1+\alpha^2u^2}-\frac{\beta^2}{1+\beta^2u^2}\right)\mathrm{d}u
\end{aligned}
$$

$$= \frac{\pi}{2(\alpha+\beta)}$$

于是原式求两次积分即可.

□

211. 计算积分

$$\int_0^1 \frac{\arctan x}{x}\frac{1-x^4}{1+x^4}\mathrm{d}x$$

解 记

$$I(a) = \int_0^1 \frac{\arctan(ax)}{x}\frac{1-x^4}{1+x^4}\mathrm{d}x$$

则

$$\begin{aligned}
I'(a) &= \int_0^1 \frac{1}{x}\frac{x}{1+(ax)^2}\frac{1-x^4}{1+x^4}\mathrm{d}x \\
&= \int_0^1 \left(\frac{1}{1+a^2x^2}\frac{a^4-1}{a^4+1} + \frac{1}{1+x^4}\frac{2-2a^2x^2}{1+a^4}\right)\mathrm{d}x
\end{aligned}$$

于是

$$\begin{aligned}
I(a) &= \int_0^1 \left(\int_0^1 \left(\frac{1}{1+a^2x^2}\frac{1-a^4}{a^4+1} + \frac{1}{1+x^4}\frac{2-2a^2x^2}{1+a^4}\right)\mathrm{d}x\right)\mathrm{d}a \\
&= I(a) = \int_0^1 \left(-\int_0^1 \frac{1}{1+a^2x^2}\frac{1-a^4}{a^4+1}\mathrm{d}a + \int_0^1 \frac{1}{1+x^4}\frac{2-2a^2x^2}{1+a^4}\mathrm{d}a\right)\mathrm{d}x \\
&= -I(a) + \int_0^1 \left(\int_0^1 \frac{1}{1+x^4}\frac{2-2a^2x^2}{1+a^4}\mathrm{d}a\right)\mathrm{d}x
\end{aligned}$$

因此

$$\begin{aligned}
I(a) &= \int_0^1 \left(\int_0^1 \frac{1}{1+x^4}\frac{2-2a^2x^2}{1+a^4}\mathrm{d}a\right)\mathrm{d}x \\
&= \left(\int_0^1 \frac{1}{1+x^4}\mathrm{d}x\right)^2 - \left(\int_0^1 \frac{x^2}{1+x^4}\mathrm{d}x\right)^2 \\
&= \left(\int_0^1 \frac{1+x^2}{1+x^4}\mathrm{d}x\right)\left(\int_0^1 \frac{1-x^2}{1+x^4}\mathrm{d}x\right) = \frac{\pi}{4}\ln\left(\sqrt{2}+1\right)
\end{aligned}$$

一般形式为

$$\begin{aligned}
I(n) &= \int_0^1 \frac{\arctan x}{x}\frac{1-x^n}{1+x^n}\mathrm{d}x = G - \frac{\pi}{2n} + \frac{2}{n}\int_0^{\frac{\pi}{4}} \ln\left(1+\tan^n x\right)\mathrm{d}x \\
&= G - \frac{\pi}{2n}\ln 2 + \frac{2}{n}\int_0^{\frac{\pi}{4}} \ln\left(\cos^n x + \sin^n x\right)\mathrm{d}x - \frac{2}{n}\int_0^{\frac{\pi}{4}} \ln\left(\cos^n x\right)\mathrm{d}x \\
&= G - \frac{\pi}{2n}\ln 2 + \frac{2}{n}\int_0^{\frac{\pi}{4}} \ln\left(\cos^n x + \sin^n x\right)\mathrm{d}x - 2\left(\frac{G}{2} - \frac{\pi}{4}\ln 2\right) \\
&= \frac{n-1}{2n}\pi\ln 2 + \frac{2}{n}\int_0^{\frac{\pi}{4}} \ln\left(\cos^n x + \sin^n x\right)\mathrm{d}x
\end{aligned}$$

□

212. 计算积分

$$\int_0^{\pi}\sqrt{\tan\frac{\theta}{2}}\ln(\sin\theta)\,\mathrm{d}\theta$$

解

$$\begin{aligned}\int_0^{\pi}\sqrt{\tan\frac{\theta}{2}}\ln(\sin\theta)\,\mathrm{d}\theta &= \int_0^{\infty}\frac{2\sqrt{t}}{1+t^2}\ln\left(\frac{2t}{1+t^2}\right)\mathrm{d}t \quad t=\tan\frac{\theta}{2}\\ &= \int_0^{\infty}\frac{2\sqrt{1/t}}{1+t^2}\ln\left(\frac{2}{t+1/t}\right)\mathrm{d}t\\ &= \int_0^{\infty}\frac{\sqrt{1/t}+\sqrt{1/t^3}}{t+1/t}\ln\left(\frac{2}{t+1/t}\right)\mathrm{d}t\\ &= \int_{-\infty}^{\infty}\frac{2}{x^2+2}\ln\left(\frac{2}{x^2+2}\right)\mathrm{d}x\\ &= 2\sqrt{2}\int_0^{\frac{\pi}{2}}\ln\left(\cos^2 u\right)\mathrm{d}u = -2\sqrt{2}\pi\ln 2 \quad \left(x=\sqrt{2}\tan u\right)\end{aligned}$$

类似的有

$$\int_0^{\pi}\sqrt{\tan\frac{\theta}{2}}\ln^2(\sin\theta)\,\mathrm{d}\theta = \int_{-\infty}^{\infty}\frac{2}{x^2+2}\ln^2\left(\frac{x^2+2}{2}\right)\mathrm{d}x = \frac{\sqrt{2}}{3}\pi^3+4\sqrt{2}\pi\ln^2 2$$

□

213. 证明下列两个积分等式:

(1) $\displaystyle\frac{1}{\sqrt{2\pi}}\int_z^{\infty}\mathrm{e}^{-\frac12 x^2}\mathrm{d}x = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\mathrm{e}^{-\frac{z^2}{2\sin^2 x}}\mathrm{d}z;$

(2) $\displaystyle\left(\frac{1}{\sqrt{2\pi}}\int_z^{\infty}\mathrm{e}^{-\frac12 x^2}\mathrm{d}x\right)^2 = \frac{1}{\pi}\int_0^{\frac{\pi}{4}}\mathrm{e}^{-\frac{z^2}{2\sin^2 x}}\mathrm{d}z.$

证明 我们只证明 (2) 式,(1) 式同理.(2) 式等价于

$$\frac12\left(\int_z^{\infty}\mathrm{e}^{-\frac12 x^2}\mathrm{d}x\right)^2-\int_0^{\frac{\pi}{4}}\mathrm{e}^{-\frac{z^2}{2\sin^2 x}}\mathrm{d}x=0$$

令 $f(z)=\displaystyle\frac12\left(\int_z^{\infty}\mathrm{e}^{-\frac12 x^2}\mathrm{d}x\right)^2-\int_0^{\frac{\pi}{4}}\mathrm{e}^{-\frac{z^2}{2\sin^2 x}}\mathrm{d}x$, 则

$$\begin{aligned}f'(z) &= -\mathrm{e}^{-\frac12 z^2}\int_z^{\infty}\mathrm{e}^{-\frac12 x^2}\mathrm{d}x-\int_0^{\frac{\pi}{4}}\mathrm{e}^{-\frac{z^2}{2}\csc^2 x}\left(-z\csc^2 x\right)\mathrm{d}x\\ &= -\mathrm{e}^{-\frac12 z^2}\int_z^{\infty}\mathrm{e}^{-\frac12 x^2}\mathrm{d}x-\int_0^{\frac{\pi}{4}}\mathrm{e}^{-\frac{z^2}{2}\left(\cot^2 x+1\right)}z\,\mathrm{d}(\cot x)\\ &= -\mathrm{e}^{-\frac12 z^2}\int_z^{\infty}\mathrm{e}^{-\frac12 x^2}\mathrm{d}x+\mathrm{e}^{-\frac12 z^2}\int_1^{\infty}\mathrm{e}^{-\frac{z^2}{2}u^2}z\,\mathrm{d}u=0\end{aligned}$$

因此 $f(z)=f(0)=0$.

□

214. 设 $p,q>0$, 计算积分

$$\int_0^{\infty}\frac{\mathrm{e}^{-qx}\sin(px)}{1+\mathrm{e}^{-qx}}\mathrm{d}x$$

解

$$
\begin{aligned}
\int_0^\infty \frac{\mathrm{e}^{-qx}\sin(px)}{1+\mathrm{e}^{-qx}}\mathrm{d}x &= \int_0^\infty \mathrm{e}^{-qx}\sin(px)\sum_{k=0}^\infty (-1)^k \mathrm{e}^{-kqx}\mathrm{d}x \\
&= \Im\sum_{k=0}^\infty (-1)^k \int_0^\infty \mathrm{e}^{-[(k+1)q-\mathrm{i}p]x}\mathrm{d}x = \Im\sum_{k=0}^\infty \frac{(-1)^k}{(k+1)q-\mathrm{i}p} \\
&= p\sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^2q^2+p^2} = \frac{1}{2p}\left[1-\frac{p^2}{q^2}\sum_{k=-\infty}^\infty \frac{(-1)^k}{k^2+\frac{p^2}{q^2}}\right] \\
&= \frac{1}{2p}-\frac{\pi}{2q}\operatorname{csc h}\left(\frac{\pi p}{q}\right)
\end{aligned}
$$

其中最后一步运用了 $\sum\limits_{k=-\infty}^\infty \frac{(-1)^k}{k^2+a^2} = \frac{\pi}{a}\operatorname{csc h}(\pi a)(a>0)$.

□

215. 计算积分

$$
\int_0^1 \frac{\ln(1-x)\ln^2(1+x)}{x}\mathrm{d}x
$$

解 令 $a=\ln(1-x), b=\ln(1+x)$, 利用公式

$$
6ab = (a+b)^3+(a-b)63-2a^3
$$

可得

$$
I := \int_0^1 \frac{\ln(1-x)\ln^2(1+x)}{x}\mathrm{d}x = \frac{I_1+I_2-2I_3}{6}
$$

其中

$$
\begin{aligned}
I_1 &= \int_0^1 \frac{\ln^3(1-x^2)}{x}\mathrm{d}x = \frac{1}{2}\int_0^1 \frac{\ln^3 t}{1-t}\mathrm{d}t \quad (t=1-x^2) \\
I_2 &= \int_0^1 \frac{\ln^3\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x = 2\int_0^1 \frac{\ln^3 t}{(1-t)(1+t)}\mathrm{d}t \\
&= \int_0^1 \frac{\ln^3 t}{1-t}\mathrm{d}t + \int_0^1 \frac{\ln^3 t}{1+t}\mathrm{d}t \quad \left(t=\frac{1-x}{1+x}\right) \\
I_3 &= \int_0^1 \frac{\ln^3(1-x)}{x}\mathrm{d}x = \int_0^1 \frac{\ln^3 t}{1-t}\mathrm{d}t \ (t=1-x)
\end{aligned}
$$

因此可得原积分

$$
\begin{aligned}
I &= \frac{1}{6}\left(\left(\frac{1}{2}+1-2\right)\int_0^1 \frac{\ln^3 t}{1-t}\mathrm{d}t + \int_0^1 \frac{\ln^3 t}{1+t}\mathrm{d}t\right) \\
&= \frac{1}{6}\left(-\frac{1}{2}\sum_{n=0}^\infty \int_0^1 t^n\ln^3 t\,\mathrm{d}t + \sum_{n=0}^\infty (-1)^n\int_0^1 t^n\ln^3 t\,\mathrm{d}t\right) \\
&= \frac{1}{6}\left(\frac{6}{2}\sum_{n=0}^\infty \frac{1}{(n+1)^4} - 6\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^4}\right) \\
&= \frac{\zeta(4)}{2}-\frac{7\zeta(4)}{8} = -\frac{\pi^4}{240}
\end{aligned}
$$

其中由分部积分不难得到 $\int_0^1 t^n \ln^m t\,\mathrm{d}t = \frac{(-1)^m\, m!}{(n+1)^{m+1}}$. □

216. 求和

$$S=\sum_{n=0}^{\infty}\left[\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}\right)\frac{1}{9^n\,(2n+1)}\right]$$

解 首先有

$$\begin{aligned}
S&=3\sum_{n=0}^{\infty}\frac{1}{3^{2n+1}\,(2n+1)}\int_0^1\frac{1-x^{2n+2}}{1-x^2}\mathrm{d}x\\
&=\frac{3}{2}\int_0^1\frac{\ln 2-x\ln\frac{3+x}{3-x}}{1-x^2}\mathrm{d}x=\frac{3}{2}\int_0^1\frac{\ln 2-x\ln 2+x\left(\ln 2-\ln\frac{3+x}{3-x}\right)}{1-x^2}\mathrm{d}x\\
&=\frac{3}{2}\ln^2 2-\frac{3}{4}\int_0^1\left(\ln 2-\ln\frac{3+x}{3-x}\right)\mathrm{d}\left(\ln\left(1-x^2\right)\right)\\
&=\frac{3}{2}\ln^2 2-\frac{3}{4}\int_0^1\left[\ln(1-x)+\ln(1+x)\right]\left(\frac{1}{3+x}+\frac{1}{3-x}\right)\mathrm{d}x\\
&=\frac{3}{2}\ln^2 2-\frac{3}{4}I
\end{aligned}$$

其中

$$\begin{aligned}
\int_0^1\frac{\ln(1-x)}{3-x}\mathrm{d}x&=\int_2^3\frac{\ln(x-2)}{x}\mathrm{d}x=\int_2^3\frac{\ln\left(1-\frac{2}{x}\right)+\ln x}{x}\mathrm{d}x\\
&=\int_2^3\frac{\ln\left(1-\frac{2}{x}\right)}{x}\mathrm{d}x+\frac{1}{2}\ln^2 3-\frac{1}{2}\ln^2 2\\
&=\int_{\frac{1}{3}}^{\frac{1}{2}}\frac{\ln(1-2x)}{x}\mathrm{d}x+\frac{1}{2}\ln^2 3-\frac{1}{2}\ln^2 2\\
&=\mathrm{Li}_2\left(\frac{2}{3}\right)-\mathrm{Li}_2(1)+\frac{1}{2}\ln^2 3-\frac{1}{2}\ln^2 2
\end{aligned}$$

$$\begin{aligned}
\int_0^1\frac{\ln(1+x)}{3-x}\mathrm{d}x&=\int_2^3\frac{\ln(4-x)}{x}\mathrm{d}x=\int_2^3\frac{\ln\left(1-\frac{x}{4}\right)+\ln 4}{x}\mathrm{d}x\\
&=\mathrm{Li}_2\left(\frac{1}{2}\right)-\mathrm{Li}_2\left(\frac{3}{4}\right)+\ln 4\ln 3-\ln 4\ln 2
\end{aligned}$$

$$\begin{aligned}
\int_0^1\frac{\ln(1-x)}{3+x}\mathrm{d}x&=\int_3^4\frac{\ln(4-x)}{x}\mathrm{d}x=\int_3^4\frac{\ln\left(1-\frac{x}{4}\right)+\ln 4}{x}\mathrm{d}x\\
&=\mathrm{Li}_2\left(\frac{3}{4}\right)-\mathrm{Li}_2(1)+\ln 4\ln 4-\ln 4\ln 3
\end{aligned}$$

$$\begin{aligned}
\int_0^1\frac{\ln(1+x)}{3+x}\mathrm{d}x&=\int_3^4\frac{\ln(x-2)}{x}\mathrm{d}x=\int_3^4\frac{\ln\left(1-\frac{2}{x}\right)+\ln x}{x}\mathrm{d}x\\
&=\int_{\frac{1}{4}}^{\frac{1}{3}}\frac{\ln(1-2x)}{x}\mathrm{d}x+\frac{1}{2}\ln^2 4-\frac{1}{2}\ln^2 3
\end{aligned}$$

$$= \mathrm{Li}_2\left(\frac{1}{2}\right) - \mathrm{Li}_2\left(\frac{2}{3}\right) + \frac{1}{2}\ln^2 4 - \frac{1}{2}\ln^2 3$$

因此

$$\begin{aligned}
I =& \mathrm{Li}_2\left(\frac{2}{3}\right) - \mathrm{Li}_2(1) + \frac{1}{2}\ln^2 3 - \frac{1}{2}\ln^2 2 + \mathrm{Li}_2\left(\frac{1}{2}\right) - \mathrm{Li}_2\left(\frac{3}{4}\right) + \ln 4\ln 3 - \ln 4\ln 2 \\
&+ \mathrm{Li}_2\left(\frac{3}{4}\right) - \mathrm{Li}_2(1) + \ln 4\ln 4 - \ln 4\ln 3 + \mathrm{Li}_2\left(\frac{1}{2}\right) - \mathrm{Li}_2\left(\frac{2}{3}\right) + \frac{1}{2}\ln^2 4 - \frac{1}{2}\ln^2 3 \\
=& -2\mathrm{Li}_2(1) - \frac{1}{2}\ln^2 2 + 2\mathrm{Li}_2\left(\frac{1}{2}\right) - 2\ln^2 2 + 4\ln^2 2 + 2\ln^2 2 \\
=& -\frac{\pi^2}{3} + \frac{7}{2}\ln^2 2 + 2\frac{\pi^2 - 6\ln^2 2}{12} = -\frac{\pi^2}{6} + \frac{5}{2}\ln^2 2
\end{aligned}$$

因此

$$S = \frac{3}{2}\ln^2 2 - \frac{3}{4}I = \frac{\pi^2}{8} - \frac{3}{8}\ln^2 2$$

□

217. 计算积分

$$\int_0^1 \frac{\arctan x}{1+x^2}\ln\left(\frac{1+x^2}{1+x}\right)\mathrm{d}x$$

解 令 $t = \arctan x$,

$$\begin{aligned}
\int_0^1 \frac{\arctan x}{1+x^2}\ln\left(\frac{1+x^2}{1+x}\right)\mathrm{d}x &= -\int_0^{\frac{\pi}{4}} t\ln[\cos t(\cos t + \sin t))]\mathrm{d}t \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{4}} t\ln[\cos^2 t(\cos t + \sin t)^2]\mathrm{d}t \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{4}} t\ln\left[\frac{1+\cos 2t}{2}(1+\sin 2t)\right]\mathrm{d}t \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{4}} t\ln[(1+\cos 2t)(1+\sin 2t)]\mathrm{d}t + \frac{1}{64}\pi^2\ln 2 \\
&= -\frac{1}{8}\int_0^{\frac{\pi}{2}} t\ln[(1+\cos t)(1+\sin t)]\mathrm{d}t + \frac{1}{64}\pi^2\ln 2 \\
&= -\frac{\pi}{32}\int_0^{\frac{\pi}{2}} \ln[(1+\cos t)(1+\sin t)]\mathrm{d}t + \frac{1}{64}\pi^2\ln 2 \\
&= -\frac{\pi}{16}\int_0^{\frac{\pi}{2}} \ln(1+\cos t)\mathrm{d}t + \frac{1}{64}\pi^2\ln 2
\end{aligned}$$

而

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \ln(1+\cos t)\,\mathrm{d}t &= \int_0^{\frac{\pi}{2}} \ln\left(2\cos^2\frac{t}{2}\right)\mathrm{d}t = \frac{\pi}{2}\ln 2 + 2\int_0^{\frac{\pi}{2}} \ln\left(\cos\frac{t}{2}\right)\mathrm{d}t \\
&= \frac{\pi}{2}\ln 2 + 4\int_0^{\frac{\pi}{4}} \ln(\cos t)\,\mathrm{d}t = 2G - \frac{\pi}{2}\ln 2
\end{aligned}$$

因此

$$\int_0^1 \frac{\arctan x}{1+x^2}\ln\left(\frac{1+x^2}{1+x}\right)\mathrm{d}x = \frac{1}{64}\pi(3\pi\ln 2 - 8G).$$

□

218. 计算积分

$$\int_0^{\pi/3}\left((\sqrt{3}\cos\theta-\sin\theta)\sin\theta\right)^{\frac{1}{2}}\cos\theta\mathrm{d}\theta$$

解 首先有

$$\begin{aligned}\left(\sqrt{3}\cos\theta-\sin\theta\right)\sin\theta&=2\sin(\tfrac{\pi}{3}-\theta)\sin\theta\\&=\cos(\tfrac{\pi}{3}-2\theta)-\cos(\tfrac{\pi}{3})=\tfrac{1}{2}-2\sin^2(\tfrac{\pi}{6}-\theta)\\&=\tfrac{1}{2}-2\sin^2(\theta-\tfrac{\pi}{6})\end{aligned}$$

于是

$$\begin{aligned}&\int_0^{\pi/3}\left((\sqrt{3}\cos\theta-\sin\theta)\sin\theta\right)^{\frac{1}{2}}\cos\theta\,\mathrm{d}\theta\\=&\int_0^{\pi/3}\left(\frac{1}{2}-2\sin^2(\theta-\frac{\pi}{6})\right)^{\frac{1}{2}}\cos\theta\,\mathrm{d}\theta\\=&\frac{1}{\sqrt{2}}\int_{-\pi/6}^{\pi/6}\left(1-4\sin^2\theta\right)^{\frac{1}{2}}\cos(\theta+\frac{\pi}{6})\,\mathrm{d}\theta\\=&\frac{1}{\sqrt{2}}\int_{-\pi/6}^{\pi/6}\left(1-4\sin^2\theta\right)^{\frac{1}{2}}\left(\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta\right)\mathrm{d}\theta\\=&\frac{\sqrt{3}}{2\sqrt{2}}\int_{-\pi/6}^{\pi/6}\left(1-4\sin^2\theta\right)^{\frac{1}{2}}\cos\theta\,\mathrm{d}\theta\\=&\frac{\sqrt{3}}{2\sqrt{2}}\int_{-\pi/6}^{\pi/6}\left(1-4\sin^2\theta\right)^{\frac{1}{2}}\,\mathrm{d}\sin\theta\\=&\frac{\sqrt{3}}{4\sqrt{2}}\int_{-\pi/2}^{\pi/2}\left(1-\sin^2\theta\right)^{\frac{1}{2}}\,\mathrm{d}\sin\theta\\=&\frac{\sqrt{3}}{4\sqrt{2}}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,\mathrm{d}\theta=\frac{\pi\sqrt{3}}{8\sqrt{2}}\end{aligned}$$

□

219. 求和

$$\sum_{n=1}^{\infty}\frac{H_n-H_{2n}}{n\left(2n+1\right)}$$

解 首先不难得到

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n-H_{2n}}{n\left(2n+1\right)}=&2\sum_{n=1}^{\infty}\left(H_n-H_{2n}\right)\left(\frac{1}{2n}-\frac{1}{2n+1}\right)\\=&2\sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{2n+1}\right)\int_0^1\frac{x^{2n}-x^n}{1-x}\mathrm{d}x\\=&\int_0^1\frac{\sqrt{x}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}-\ln\frac{1+x}{1-x}-\ln\left(1+x\right)}{1-x}\mathrm{d}x\\&+\int_0^1\left(\frac{1}{\sqrt{x}}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}-\frac{1}{x}\ln\frac{1+x}{1-x}\right)\mathrm{d}x\end{aligned}$$

其中

$$\int_0^1 \frac{1}{\sqrt{x}}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}\mathrm{d}x = 2\int_0^1 \ln\frac{1+t}{1-t}\mathrm{d}t = 4\ln 2$$

$$\int_0^1 \frac{1}{x}\ln\frac{1+x}{1-x}\mathrm{d}x = \mathrm{Li}_2(1) - \mathrm{Li}_2(-1) = \frac{\pi^2}{4}$$

$$\begin{aligned}\int_0^1 \frac{\left(\sqrt{x}-1\right)}{1-x}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}\mathrm{d}x &= -2\int_0^1 \frac{t}{1+t}\ln\frac{1+t}{1-t}\mathrm{d}t \\ &= 2\int_0^1 \left[\frac{\ln(1+t)}{1+t} - \frac{\ln(1-t)}{1+t}\right]\mathrm{d}t - 2\int_0^1 \ln\frac{1+t}{1-t}\mathrm{d}t \\ &= \ln^2 2 + 2\mathrm{Li}_2\left(\frac{1}{2}\right) - 4\ln 2 = \frac{\pi^2}{6} - 4\ln 2\end{aligned}$$

又

$$\int_0^1 \frac{\ln\frac{1+\sqrt{x}}{1-\sqrt{x}} - \ln\frac{1+x}{1-x} - \ln(1+x)}{1-x}\mathrm{d}x = 2\int_0^1 \frac{1}{1-x}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}\mathrm{d}x$$

$$=2\int_0^1 \frac{\ln\left(1+\sqrt{x}\right)-\ln 2}{1-x}\mathrm{d}x - 2\int_0^1 \frac{\ln(1+x)-\ln 2}{1-x}\mathrm{d}x$$

其中

$$\begin{aligned}\int_0^1 \frac{\ln\left(1+\sqrt{x}\right)-\ln 2}{1-x}\mathrm{d}x &= 2\int_0^1 \frac{t}{1-t^2}\ln\frac{1+t}{2}\mathrm{d}t \\ &= \int_0^1 \frac{1}{1-t}\ln\frac{1+t}{2}\mathrm{d}t - \int_0^1 \frac{1}{1+t}\ln\frac{1+t}{2}\mathrm{d}t \\ &= -\mathrm{Li}_2\left(\frac{1}{2}\right) + \frac{1}{2}\ln^2 2 = \ln^2 2 - \frac{\pi^2}{12}\end{aligned}$$

$$\int_0^1 \frac{\ln(1+x)-\ln 2}{1-x}\mathrm{d}x = -\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}$$

最后得到

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n - H_{2n}}{n(2n+1)} &= 4\ln 2 - \frac{\pi^2}{4} + \frac{\pi^2}{6} - 4\ln 2 + 2\left(\ln^2 2 - \frac{\pi^2}{12}\right) - 2\left(\frac{\ln^2 2}{2} - \frac{\pi^2}{12}\right) \\ &= \ln^2 2 - \frac{\pi^2}{12}\end{aligned}$$

□

220. 计算积分

$$\int_0^\infty \frac{x^\alpha \mathrm{d}x}{1+2x\cos\beta + x^2}$$

解 利用基本积分

$$\int_0^\infty \frac{x^{p-1}}{1+x}\mathrm{d}x = \frac{\pi}{\sin p\pi}$$

可得

$$\int_0^\infty \frac{x^a}{x^2+2(\cos b)x+1}\mathrm{d}x$$

$$
\begin{aligned}
&=\int_0^{\infty}\frac{x^a}{(x+\mathrm{e}^{\mathrm{i}b})(x+e^{-\mathrm{i}b})}\mathrm{d}x\\
&=\frac{1}{-\mathrm{e}^{\mathrm{i}b}+\mathrm{e}^{-\mathrm{i}b}}\int_0^{\infty}\frac{x^a}{\mathrm{e}^{\mathrm{i}b}+x}\mathrm{d}x+\frac{1}{-\mathrm{e}^{-\mathrm{i}b}+\mathrm{e}^{\mathrm{i}b}}\int_0^{\infty}\frac{x^a}{\mathrm{e}^{-\mathrm{i}b}+x}\mathrm{d}x\\
&=\frac{1}{-2\mathrm{i}\sin b}\int_0^{\infty}\frac{(\mathrm{e}^{\mathrm{i}b}u)^a}{1+u}\mathrm{d}u+\frac{1}{2\mathrm{i}\sin b}\int_0^{\infty}\frac{(\mathrm{e}^{-\mathrm{i}b}u)^a}{1+u}\mathrm{d}u\\
&=\frac{\mathrm{e}^{\mathrm{i}ab}}{-2\mathrm{i}\sin b}\frac{\pi}{\sin(\pi a+\pi)}+\frac{\mathrm{e}^{-\mathrm{i}ab}}{2\mathrm{i}\sin b}\frac{\pi}{\sin(\pi a+\pi)}\\
&=\frac{\pi}{\sin\pi a\sin b}\left(\frac{\mathrm{e}^{\mathrm{i}ab}-\mathrm{e}^{-\mathrm{i}ab}}{2\mathrm{i}}\right)=\frac{\pi\sin ab}{\sin\pi a\sin b}
\end{aligned}
$$

□

221. 计算积分

$$
\int_0^1\frac{\ln 2-\ln\left(1+x^2\right)}{1-x}\mathrm{d}x
$$

解 首先由分部积分得 $I=-\int_0^1\frac{2x\ln(1-x)}{1+x^2}\mathrm{d}x$, 令

$$
I(a)=\int_0^1\frac{-2x\ln(1-ax)}{1+x^2}\mathrm{d}x
$$

$$
\begin{aligned}
I'(a)&=\int_0^1\frac{2x^2}{(1+x^2)(1-ax)}\mathrm{d}x\\
&=-\frac{2}{1+a^2}\int_0^1\left(\frac{ax}{1+x^2}+\frac{1}{1+x^2}-\frac{1}{1-ax}\right)\mathrm{d}x\\
&=-\frac{a\ln 2}{1+a^2}-\frac{\pi}{2}\frac{1}{1+a^2}-\frac{2\ln(1-a)}{a(1+a^2)}
\end{aligned}
$$

$$
\begin{aligned}
I(1)&=\int_0^1\left(-\frac{a\ln 2}{1+a^2}-\frac{\pi}{2}\frac{1}{1+a^2}-\frac{2\ln(1-a)}{a(1+a^2)}\right)\mathrm{d}a\\
&=-\frac{1}{2}\ln^2 2-\frac{\pi}{2}\arctan 1+2\mathrm{Li}_2(1)-\int_0^1\frac{-2a\ln(1-a)}{1+a^2}\mathrm{d}a\\
&=\frac{5\pi^2}{24}-\frac{1}{2}\ln^2 2-I(1)
\end{aligned}
$$

于是

$$
I(1)=\frac{5\pi^2}{48}-\frac{1}{4}\ln^2 2
$$

□

222. 计算积分

$$
\int_0^{\infty}\mathrm{e}^{-\sqrt{x}}\ln\left(1+\frac{1}{\sqrt{x}}\right)\mathrm{d}x
$$

解 首先有

$$
\int_0^{\infty}\mathrm{e}^{-\sqrt{x}}\ln\left(1+\frac{1}{\sqrt{x}}\right)\mathrm{d}x=2\int_0^{\infty}u\mathrm{e}^{-u}\ln\left(1+\frac{1}{u}\right)\mathrm{d}u
$$

$$= 2\int_0^{\infty} ue^{-u}\ln(1+u)\mathrm{d}u - 2\int_0^{\infty} ue^{-u}\ln u\mathrm{d}u$$

利用分部积分可得第一个积分

$$\begin{aligned}
&\int_0^{\infty} ue^{-u}\ln(1+u)\,\mathrm{d}u \\
&= \left[-ue^{-u}\ln(1+u)\right]\Big|_0^{\infty} + \int_0^{\infty} e^{-u}\left(\ln(1+u) + \frac{u}{1+u}\right)\mathrm{d}u \\
&= \int_0^{\infty} e^{-u}\left(\ln(1+u) + 1 - \frac{1}{1+u}\right)\mathrm{d}u \\
&= \int_0^{\infty} e^{-u}\left(\ln(1+u)\frac{1}{1+u}\right)\mathrm{d}u + \int_0^{\infty} e^{-u}\mathrm{d}u = 1
\end{aligned}$$

$$\begin{aligned}
\int_0^{\infty} ue^{-u}\ln u\mathrm{d}u &= \left[-ue^{-u}\ln u\right]\Big|_0^{\infty} + \int_0^{\infty} e^{-u}(\ln u + 1)\,\mathrm{d}u \\
&= \int_0^{\infty} e^{-u}\ln u\mathrm{d}u + 1 = -\gamma + 1
\end{aligned}$$

因此原积分

$$\int_0^{\infty} e^{-\sqrt{x}}\ln\left(1+\frac{1}{\sqrt{x}}\right)\mathrm{d}x = 2\gamma$$

□

223. 计算积分

$$\int_0^{\frac{\pi}{2}} x\cos(8x)\ln\left(\frac{1+\tan x}{1-\tan x}\right)\mathrm{d}x$$

解

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} x\cos(8x)\ln\left(\frac{1+\tan x}{1-\tan x}\right)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} x\cos(8x)\ln\left(\frac{1+\tan x}{1-\tan x}\right)\mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos(8x)\ln\left(\frac{1+\tan x}{1-\tan x}\right)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} x\cos(8x)\ln\left(\frac{1+\tan x}{1-\tan x}\right)\mathrm{d}x - \int_{\frac{\pi}{4}}^{0} \left(\frac{\pi}{2}-x\right)\cos(8x)\ln\left(\frac{1+\cot x}{1-\cot x}\right)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} x\cos(8x)\ln\left(\frac{1+\tan x}{1-\tan x}\right)\mathrm{d}x + \int_0^{\frac{\pi}{4}} \left(\frac{\pi}{2}-x\right)\cos(8x)\ln\left(\frac{1+\tan x}{\tan x-1}\right)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} x\cos(8x)\ln\left(\frac{1+\tan x}{1-\tan x}\right)\mathrm{d}x + \int_0^{\frac{\pi}{4}} \left(\frac{\pi}{2}-x\right)\cos(8x)\left[\ln\left(\frac{1+\tan x}{1-\tan x}\right)+\mathrm{i}\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} x\cos(8x)\ln\left(\frac{1+\tan x}{1-\tan x}\right)\mathrm{d}x + \int_0^{\frac{\pi}{4}} \left(\frac{\pi}{2}-x\right)\cos(8x)\ln\left(\frac{1+\tan x}{1-\tan x}\right)\mathrm{d}x \\
&\quad + \pi\mathrm{i}\int_0^{\frac{\pi}{4}} \left(\frac{\pi}{2}-x\right)\cos(8x)\mathrm{d}x \\
&= \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \cos(8x)\ln\left(\frac{1+\tan x}{1-\tan x}\right)\mathrm{d}x = \frac{\pi}{16}\int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\tan x}{1-\tan x}\right)\mathrm{d}\sin(8x) \\
&= \frac{\pi}{16}\left[\ln\left(\frac{1+\tan x}{1-\tan x}\right)\sin(8x)\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin(8x)\mathrm{d}\ln\left(\frac{1+\tan x}{1-\tan x}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&=-\frac{\pi}{16}\int_0^{\frac{\pi}{4}}\sin(8x)\left(\frac{1}{1+\tan x}+\frac{1}{1-\tan x}\right)\sec^2 x\mathrm{d}x\\
&=-\frac{\pi}{8}\int_0^{\frac{\pi}{4}}\frac{\sin(8x)}{\cos(2x)}\mathrm{d}x=\frac{\pi}{12}.
\end{aligned}
$$

□

224. 计算积分

$$\int_0^{\frac{\pi}{4}}\frac{\tan^{2\mathrm{e}}x-2\sin^2 x}{\sin(2x)\ln\tan x}\mathrm{d}x$$

解 根据

$$\sin^2=\frac{\tan^2 x}{1+\tan^2 x},\sin(2x)=\frac{2\tan x}{1+\tan^2 x}$$

通过两次换元 $x\to\tan x,\ln x\to -x$ 可得

$$
\begin{aligned}
&\int_0^{\frac{\pi}{4}}\frac{\tan^{2\mathrm{e}}x-2\sin^2 x}{\sin(2x)\ln\tan x}\mathrm{d}x\\
=&\int_0^{\pi/4}\frac{\tan^{2\mathrm{e}}x(1+\tan^2 x)-2\tan^2 x}{2\tan x\ln\tan x}\mathrm{d}x\\
=&\frac{1}{2}\int_0^1\frac{x^{2\mathrm{e}-1}(1+x^2)-2x}{\ln x}\frac{1}{x^2+1}\mathrm{d}x\\
=&\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1\frac{x^{2\mathrm{e}-1}(1+x^2)-2x}{\ln x}x^{2n}\mathrm{d}x\\
=&\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1\frac{(x^{2(n+\mathrm{e})-1}-x^{2n+1})+(x^{2(n+\mathrm{e})+1}-x^{2n+1})}{\ln x}\mathrm{d}x\\
=&\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left[\ln\frac{n+\mathrm{e}}{n+1}+\ln\frac{n+\mathrm{e}+1}{n+1}\right]\\
=&\frac{1}{2}\left(1+\ln\prod_{n=1}^{\infty}\frac{(2n)^2}{(2n)^2-1}\right)\\
=&\frac{1}{2}\left(1+\ln\frac{\pi}{2}\right)
\end{aligned}
$$

□

225. 计算积分

$$\int_0^{\frac{\pi}{2}}\frac{\ln\cos x}{\tan x}\cdot\ln\left(\frac{\ln\sin x}{\ln\cos x}\right)\mathrm{d}x$$

解 考虑积分

$$I=\int_0^{\frac{\pi}{2}}\frac{\ln(\cos x)}{\tan x}\ln\frac{|\ln\sin x|}{|\ln\cos x|}\mathrm{d}x$$

拆开后令 $x\to\frac{\pi}{2}$ 不难得到

$$I=\int_0^{\frac{\pi}{2}}(\cot x\ln(\cos x)-\tan x\ln(\sin x)\ln|\ln(\sin x)|)\mathrm{d}x$$

而

$$\int \cot(x)\ln(\cos x)-\tan(x)\ln(\sin x)\,dx=\ln(|\sin x|)\ln(|\cos x|)+C$$

分部积分可得

$$\begin{aligned}
I&=-\int_0^{\frac{\pi}{2}}\cot x\ln(\cos x)\,\mathrm{d}x\\
&=-\int_0^1\frac{t\log(t)}{1-t^2}\mathrm{d}t=-\frac{1}{2}\int_0^1\frac{\log(1-t^2)}{t}\,dt\\
&=-\frac{1}{2}\int_0^1\frac{\log(1+t)}{t}\mathrm{d}t-\frac{1}{2}\int_0^1\frac{\log(1+t)}{t}\mathrm{d}t\\
&=\frac{\mathrm{Li}_2(1)+\mathrm{Li}_2(-1)}{2}\\
&=\frac{1}{2}\left(\frac{\pi^2}{6}-\frac{\pi^2}{12}\right)\\
&=\frac{\pi^2}{24}
\end{aligned}$$

□

226. 求和

$$\sum_{n=1}^{\infty}\sum_{m=1}^{2n}\frac{(-1)^n}{nm}$$

解 首先有

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{2n}\frac{(-1)^n}{nm}&=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}H_{2n}=-2\sum_{n=1}^{\infty}(-1)^n\int_0^1(1-t)^{2n-1}\ln t\,\mathrm{d}t\\
&=-2\sum_{n=1}^{\infty}(-1)^n\int_0^1(1-t)^{2n-1}\ln t\,\mathrm{d}t=2\int_0^1\frac{(1-t)\ln t}{1+(1-t)^2}\mathrm{d}t\\
&=2\int_0^1\frac{t\ln(1-t)}{1+t^2}\mathrm{d}t=I
\end{aligned}$$

令 $I(a)=2\int_0^1\frac{t\ln(1-at)}{1+t^2}\mathrm{d}t$, 则

$$\begin{aligned}
&I(0)=0\\
&I'(a)=-\int_0^1\frac{2t^2}{(1-at)(1+t^2)}\mathrm{d}t=\frac{2}{1+a^2}\left(\int_0^1\frac{1+at}{1+t^2}\mathrm{d}t-\int_0^1\frac{1}{1-at}\mathrm{d}t\right)\\
&\quad=-\frac{\pi+2a\ln 2}{2(1+a^2)}-\frac{2\ln(1-a)}{a}+\frac{2a\ln(1-a)}{1+a^2}
\end{aligned}$$

$$\begin{aligned}
I&=I(1)=\int_0^1 I'(a)\,\mathrm{d}a\\
&=\int_0^1\frac{\pi+2a\ln 2}{2(1+a^2)}\mathrm{d}a+\int_0^1\frac{2\ln(1-a)}{a}\mathrm{d}a-\int_0^1\frac{2a\ln(1-a)}{1+a^2}\mathrm{d}a\\
&=\frac{\pi^2}{8}+\frac{1}{2}\ln^2 2-\frac{\pi^2}{3}-I=\frac{1}{2}\ln^2 2-\frac{5}{24}\pi^2-I
\end{aligned}$$

因此最后得到

$$\sum_{n=1}^{\infty}\sum_{m=1}^{2n}\frac{(-1)^n}{nm}=I=\frac{1}{4}\ln^2 2-\frac{5}{48}\pi^2$$

□

227. 设 n 是一个正整数, 证明

$$\lim_{x\to 0}\frac{\int_0^x \sin^n \frac{1}{t}\mathrm{d}t}{x}=\lim_{x\to 0}\frac{\int_0^x \cos^n \frac{1}{t}\mathrm{d}t}{x}=\begin{cases}\frac{(n-1)!!}{n!!}, & n\text{为偶数}\\ 0, & n\text{为奇数}\end{cases}$$

证明 先考虑复杂的 n 为偶数的情形, 这个时候只需要考虑 $x\to 0^+$ 即可, 以正弦为例 (余弦同理)

$$\lim_{x\to 0^+}\frac{\int_0^x \sin^n \frac{1}{t}\mathrm{d}t}{x}=\lim_{x\to 0^+}\frac{\int_{\frac{1}{x}}^{+\infty}\frac{\sin^n t}{t^2}\mathrm{d}t}{x}=\lim_{x\to+\infty}x\int_x^{+\infty}\frac{\sin^n t}{t^2}\mathrm{d}t$$

对 $\forall x>0,\exists k\in\mathbb{N},\text{s.t.}(k-1)\pi\leqslant x<k\pi$, 则 $x\to+\infty$ 时 $k\to+\infty$, 于是

$$x\int_x^{+\infty}\frac{\sin^n t}{t^2}\mathrm{d}t=x\int_x^{k\pi}\frac{\sin^n t}{t^2}\mathrm{d}t+x\int_{k\pi}^{+\infty}\frac{\sin^n t}{t^2}\mathrm{d}t$$

其中

$$\left|x\int_x^{k\pi}\frac{\sin^n t}{t^2}\mathrm{d}t\right|\leqslant\left|x\int_x^{k\pi}\frac{1}{x^2}\mathrm{d}t\right|=\left|\frac{k\pi-x}{x}\right|\leqslant\left|\frac{\pi}{x}\right|\to 0,x\to+\infty$$

$$\begin{aligned}\int_{k\pi}^{+\infty}\frac{\sin^n t}{t^2}\mathrm{d}t&=\sum_{i=k}^{+\infty}\int_{i\pi}^{(i+1)\pi}\frac{\sin^n t}{t^2}\mathrm{d}t=\int_0^{\pi}\sin^n t\sum_{i=k}^{\infty}\frac{1}{(t+i\pi)^2}\mathrm{d}t\\&=\frac{1}{\pi^2}\int_0^{\pi}\sin^2 t\sum_{i=k}^{\infty}\frac{1}{\left(i+\frac{t}{\pi}\right)^2}\mathrm{d}t\end{aligned}$$

不难得到当 $k\to+\infty$ 时,

$$\sum_{i=k}^{\infty}\frac{1}{(i+1)^2}\sim\sum_{i=k}^{\infty}\frac{1}{\left(i+\frac{t}{\pi}\right)^2}\sim\sum_{i=k}^{\infty}\frac{1}{i^2}\sim\frac{1}{k}$$

于是当 $x\to+\infty$ 时,

$$x\int_{k\pi}^{+\infty}\frac{\sin^n t}{t^2}\mathrm{d}t=\frac{x}{\pi^2}\int_0^{\pi}\sin^n t\sum_{i=k}^{\infty}\frac{1}{\left(i+\frac{t}{\pi}\right)^2}\mathrm{d}t\sim\frac{k\pi}{\pi^2}\cdot\frac{1}{k}\int_0^{\pi}\sin^n t\mathrm{d}t=\frac{(n-1)!!}{n!!}$$

这就是 n 是偶数的极限, 而当 n 是奇数的时候, 正项级数 $\sum\limits_{i=k}^{\infty}\frac{1}{\left(i+\frac{t}{\pi}\right)^2}$ 会变成交错级数 $\sum\limits_{i=k}^{\infty}\frac{(-1)^i}{\left(i+\frac{t}{\pi}\right)^2}$, 这个交错级数的绝对值不会超过 $\frac{1}{\left(k+\frac{t}{\pi}\right)^2}<\frac{1}{k^2}$, 因此最后的极限是 0.

□

228. 求和

$$\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\frac{(-1)^{m-n}}{(m^2-n^2)^2}$$

解 首先有当 $m^2\neq n^2$ 时,

$$\frac{1}{(m^2-n^2)^2}=\int_0^1\int_0^1 x^{m+n-1}y^{m-n-1}\log x\log y\,\mathrm{d}x\mathrm{d}y$$

因此可得

$$\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\frac{(-1)^{m-n}}{(m^2-n^2)^2}=\int_0^1\int_0^1\frac{-\log x\log y}{(1-x^2)(1+xy)}\mathrm{d}x\mathrm{d}y$$

进一步利用因式分解可得

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\frac{(-1)^{m-n}}{(m^2-n^2)^2}\\&=\int_0^1\int_0^1\frac{-\log x\log y}{(1-x^2)(1+xy)}\mathrm{d}x\mathrm{d}y\\&=\int_0^1\int_0^1\left(\frac{-\log x\log y}{2(1+x)(1-y)}+\frac{-\log x\log y}{2(1-x)(1+y)}+\frac{y^2\log x\log y}{(1-y^2)(1+xy)}\right)\mathrm{d}x\mathrm{d}y\\&=-2\cdot\frac{\pi^4}{144}+\frac{\pi^4}{480}=-\frac{17\pi^4}{1440}\end{aligned}$$

□

229. 给定 $0\leqslant a\leqslant 2$, 设 $\{a_n\}_{n\geqslant 1}$ 是由 $a_1=a, a_{n+1}=2^n-\sqrt{2^n(2^n-a_n)}$ 所定义的数列, 求 $\sum\limits_{n=1}^{\infty}a_n^2$.

解 令

$$\alpha=4\arcsin\sqrt{\frac{a}{2}}=\begin{cases}\arccos\left(2a^2-4a+1\right), & a\in[0,1]\\ 2\pi-\arccos\left(2a^2-4a+1\right), & a\in[1,2]\end{cases}$$

然后利用二倍角公式 $2\cos^2\left(\frac{\theta}{2}\right)=1+\cos\theta$, 不难得到

$$a_n=2^{n-1}\left(1-\cos\frac{\alpha}{2^n}\right)$$

对 $N\in\mathbb{N}$ 有

$$\begin{aligned}\sum_{n=1}^{N}a_n^2&=\sum_{n=1}^{N}4^{n-1}\left(1+\cos^2\frac{\alpha}{2^n}-2\cos\frac{\alpha}{2^n}\right)\\&=\sum_{n=1}^{N}4^{n-1}\left(1+\frac{1+\cos\left(\alpha/2^{n-1}\right)}{2}-2\cos\frac{\alpha}{2^n}\right)\\&=\frac{1}{2}\sum_{n=1}^{N}4^{n}\left(1-\cos\frac{\alpha}{2^n}\right)-\frac{1}{2}\sum_{n=1}^{N}4^{n-1}\left(1-\cos\frac{\alpha}{2^n}\right)\\&=\frac{1}{2}\sum_{n=1}^{N}4^{n-1}\left(1-\cos\frac{\alpha}{2^n}\right)-\frac{1}{2}\sum_{n=0}^{N-1}4^{n}\left(1-\cos\frac{\alpha}{2^n}\right)\end{aligned}$$

$$= \frac{1}{2}\left(4^N\left(1-\cos\frac{\alpha}{2^N}\right)-(1-\cos\alpha)\right)$$

因此

$$\begin{aligned}\sum_{n=1}^{\infty} a_n^2 &= \frac{1}{2}\left(\lim_{N\to\infty} 4^N\left(1-\cos\frac{\alpha}{2^N}\right)-(1-\cos\alpha)\right)\\ &= \frac{\alpha^2}{4}+a^2-2a = 4\arcsin^2\sqrt{\frac{a}{2}}+a^2-2a.\end{aligned}$$

□

230. 设函数 f, g 是 $[0,1]$ 上的连续实值函数, 证明存在 $c \in (0,1)$ 使得

$$\int_0^1 f(x)\,\mathrm{d}x \int_0^c xg(x)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x \int_0^c xf(x)\,\mathrm{d}x$$

解 令

$$h(t) = g(t)\int_0^1 f(x)\,\mathrm{d}x - f(t)\int_0^1 g(x)\,\mathrm{d}x$$

只需要证明存在 $c \in (0,1)$ 使得

$$\int_0^c th(t)\,\mathrm{d}t = \int_0^c tg(t)\,\mathrm{d}t \int_0^1 f(x)\,\mathrm{d}x - \int_0^c tf(t)\,\mathrm{d}t \int_0^1 g(x)\,\mathrm{d}x = 0$$

令

$$u(s) = \int_0^s H(x)\,\mathrm{d}x = \int_0^s \int_0^x h(t)\,\mathrm{d}t\,\mathrm{d}x$$

则 u 是 $[0,1]$ 上连续可微的函数且 $u'(0) = H(0) = 0$,

$$\begin{aligned}u'(1) = H(1) &= \int_0^1 h(t)\,\mathrm{d}t\\ &= \int_0^1 g(t)\,\mathrm{d}t \int_0^1 f(x)\,\mathrm{d}x - \int_0^1 f(t)\,\mathrm{d}t \int_0^1 g(x)\,\mathrm{d}x = 0\end{aligned}$$

由 Flett 中值定理, 存在 $c \in (0,1)$ 使得

$$u'(c) = H(c) = \frac{u(c)-u(0)}{c-0} = \frac{u(c)}{c}$$

因此

$$\int_0^c th(t)\,\mathrm{d}t = \int_0^c t\,d(H(t)) = [tH(t)]_0^c - \int_0^c H(t)\,\mathrm{d}t = cH(c) - u(c) = 0$$

□

231. 设 μ 为一正实数, 证明

$$\lim_{x\to 1^+} (\ln x)^{\frac{1}{\mu}} \sum_{k=1}^{\infty} x^{-(2k-1)^\mu} = \frac{\Gamma\left(\frac{1}{\mu}\right)}{2\mu}$$

解 首先我们证明下面一个定理

定理 1. 如果 $f:\mathbb{R}^{+}\to\mathbb{R}$ 非负且单调递减, 且积分 $\int_0^{\infty} f(x)\mathrm{d}x<+\infty$. 则对任意满足 $a>\max(b,0)$ 的实数 a,b,

$$\lim_{h\to 0^+} h\sum_{k=1}^{\infty} f((ka-b)h)=\frac{1}{a}\int_0^{\infty} f(t)\,\mathrm{d}t$$

Proof: 不失一般性不妨假定 $a=1$. 对 $h>0$, 令 $x_k=(k-b)h, k\in\mathbb{N}$, 由于 f 单调减, 则

$$hf(x_{k+1})\leqslant\int_{x_k}^{x_{k+1}} f(t)\,\mathrm{d}t\leqslant hf(x_k)$$

求和即得

$$h\sum_{k=2}^{n} f(x_k)\leqslant\int_{x_1}^{x_n} f(t)\,\mathrm{d}t\leqslant h\sum_{k=1}^{n-1} f(x_k)$$

令 $n\to\infty$,

$$h\sum_{k=1}^{\infty} f(x_k)-hf(x_1)\leqslant\int_0^{\infty} f(t)\,\mathrm{d}t\leqslant h\sum_{k=1}^{\infty} f(x_k)$$

那么定理自然就得证了. 现在令 $f(t)=\mathrm{e}^{-t^{\mu}}, t\geqslant 0$, 则

$$\lim_{h\to 0^+} h\sum_{k=1}^{\infty}\mathrm{e}^{-(ak-b)^{\mu}h^{\mu}}=\frac{1}{a}\int_0^{\infty}\mathrm{e}^{-t^{\mu}}\mathrm{d}t=\frac{\Gamma\left(\frac{1}{\mu}\right)}{a\mu}$$

□

232. 设 $\mu>0, 0<b<a$, 计算积分

$$\int_0^{\pi}\frac{\sin^{\mu-1}x}{(a+b\cos x)^{\mu}}\mathrm{d}x$$

解

$$\int_0^{\pi}\frac{\sin^{\mu-1}x}{(a+b\cos x)^{\mu}}\mathrm{d}x=\int_0^{\frac{\pi}{2}}+\int_{\frac{\pi}{2}}^{\pi}\frac{\sin^{\mu-1}x}{(a+b\cos x)^{\mu}}\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{2}}\sin^{\mu-1}x\left(\frac{1}{(a+b\cos x)^{\mu}}+\frac{1}{(a-b\cos x)^{\mu}}\right)\mathrm{d}x$$

$$=a^{-\mu}\int_0^{\frac{\pi}{2}}\sin^{\mu-1}x\left(\sum_{n=0}^{\infty}\binom{-\mu}{n}\left(-\frac{b}{a}\cos x\right)^n+\sum_{n=0}^{\infty}\binom{-\mu}{n}\left(\frac{b}{a}\cos x\right)^n\right)\mathrm{d}x$$

$$=2a^{-\mu}\sum_{n=0}^{\infty}\binom{-\mu}{2n}\left(\frac{b}{a}\right)^{2n}\int_0^{\frac{\pi}{2}}\sin^{\mu-1}x\cos^{2n}x\mathrm{d}x$$

$$=a^{-\mu}\sum_{n=0}^{\infty}\frac{(-\mu)(-\mu-1)\cdots(-\mu-2n+1)}{(2n)!}\left(\frac{b}{a}\right)^{2n}\mathrm{B}\left(\frac{\mu}{2},\frac{2n+1}{2}\right)$$

$$=a^{-\mu}\sum_{n=0}^{\infty}\frac{\mu(\mu+1)\cdots(\mu+2n-1)}{(2n)!}\left(\frac{b}{a}\right)^{2n}\frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{2n+1+\mu}{2}\right)}$$

$$=a^{-\mu}\Gamma\left(\frac{\mu}{2}\right)\sum_{n=0}^{\infty}\frac{\mu(\mu+1)\cdots(\mu+2n-1)}{(2n)!!}\left(\frac{b}{a}\right)^{2n}\frac{\sqrt{\pi}}{(2n+\mu-1)(2n+\mu-3)\cdots(\mu+1)\Gamma\left(\frac{\mu+1}{2}\right)}$$

$$
\begin{aligned}
&=a^{-\mu}\Gamma\left(\frac{\mu}{2}\right)\frac{\sqrt{\pi}}{\Gamma\left(\frac{\mu+1}{2}\right)}\sum_{n=0}^{\infty}\frac{\mu\left(\mu+2\right)\cdots\left(\mu+2n-2\right)}{2^n n!}\left(\frac{b}{a}\right)^{2n}\\
&=a^{-\mu}\Gamma\left(\frac{\mu}{2}\right)\frac{\sqrt{\pi}}{\Gamma\left(\frac{\mu+1}{2}\right)}\sum_{n=0}^{\infty}\frac{\frac{\mu}{2}\left(\frac{\mu}{2}+2\right)\cdots\left(\frac{\mu}{2}+2n-2\right)}{n!}\left(\frac{b}{a}\right)^{2n}\\
&=a^{-\mu}\Gamma\left(\frac{\mu}{2}\right)\frac{\sqrt{\pi}}{\Gamma\left(\frac{\mu+1}{2}\right)}\sum_{n=0}^{\infty}\binom{-\frac{\mu}{2}}{n}\left(-\frac{b^2}{a^2}\right)^{n}=a^{-\mu}\Gamma\left(\frac{\mu}{2}\right)\frac{\sqrt{\pi}}{\Gamma\left(\frac{\mu+1}{2}\right)}\left(1-\frac{b^2}{a^2}\right)^{-\frac{\mu}{2}}\\
&=\left(a^2-b^2\right)^{-\frac{\mu}{2}}B\left(\frac{\mu}{2},\frac{1}{2}\right)
\end{aligned}
$$

□

233. 设 n 是正整数, 计算积分

$$\int_0^{\infty}\frac{\ln^n x}{\sqrt{x}\left(1-x\right)^2}\mathrm{d}x.$$

解

$$
\begin{aligned}
\int_0^{\infty}\frac{\ln^n x}{\sqrt{x}\left(1-x\right)^2}\mathrm{d}x&=2^{n+1}\int_0^{\infty}\frac{\ln^n t}{\left(1-t^2\right)^2}\mathrm{d}t\\
&=2^{n+1}\left(\int_0^1\frac{\ln^n t}{\left(1-t^2\right)^2}\mathrm{d}t+\int_1^{\infty}\frac{\ln^n t}{\left(1-t^2\right)^2}\mathrm{d}t\right)\\
&=2^{n+1}\int_0^1\frac{\left(-1\right)^n t^2+1}{\left(1-t^2\right)^2}\ln^n t\,\mathrm{d}t.
\end{aligned}
$$

如果 n 是奇数, 令 $\ln t=-z$, 则

$$
\begin{aligned}
I&=-2^{n+1}\int_0^{\infty}\frac{z^n\mathrm{e}^{-z}}{1-\mathrm{e}^{-2z}}\mathrm{d}z\\
&=-2^{n+1}n!\left(1-2^{-n-1}\right)\zeta\left(n+1\right)=-n!\left(2^{n+1}-1\right)\zeta\left(n+1\right).
\end{aligned}
$$

如果 n 是偶数, 则利用分部积分可得

$$
\begin{aligned}
I&=2^{n+1}\int_0^1\frac{t^2+1}{\left(1-t^2\right)^2}\ln^n t\,\mathrm{d}t=2^{n+1}\int_0^1\ln^n t\,\mathrm{d}\left(\frac{1}{1-t^2}\right)\\
&=-n2^{n+1}\int_0^1\frac{\ln^{n-1}}{1-t^2}\mathrm{d}t=n2^{n+1}\int_0^{\infty}\frac{z^{n-1}\mathrm{e}^{-z}}{1-\mathrm{e}^{-2z}}\mathrm{d}z\\
&=n2^{n+1}\left(n-1\right)!\left(1-2^{-n}\right)\zeta\left(n\right)=n!\left(2^{n+1}-2\right)\zeta\left(n\right).
\end{aligned}
$$

□

234. 计算极限

$$\lim_{a\to+\infty}\frac{1}{a}\int_0^a\frac{t}{1+a^2\cos^2 t}\mathrm{d}t.$$

解 首先注意到

$$\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}t}{1+a^2\cos^2 t}=\left[\frac{1}{\sqrt{a^2+1}}\arctan\left(\frac{\tan t}{\sqrt{a^2+1}}\right)\right]\Bigg|_0^{\frac{\pi}{2}}=\frac{\pi}{2\sqrt{a^2+1}}$$

记 $n=\left\lceil\frac{a}{\pi}\right\rceil$ 表示不小于 $\frac{a}{\pi}$ 的最小整数, 我们有

$$\begin{aligned}\frac{1}{a}\int_0^a\frac{t}{1+a^2\cos^2 t}\mathrm{d}t&=\sum_{k=0}^{n-1}\frac{1}{a}\int_{k\pi}^{(k+1)\pi}\frac{t}{1+a^2\cos^2 t}\mathrm{d}t+\frac{1}{a}\int_{n\pi}^{a}\frac{t}{1+a^2\cos^2 t}\mathrm{d}t\\&=\sum_{k=0}^{n-1}\frac{1}{a}\int_0^{\pi}\frac{k\pi+t}{1+a^2\cos^2 t}\mathrm{d}t+\frac{1}{a}\int_0^{a-n\pi}\frac{n\pi+t}{1+a^2\cos^2 t}\mathrm{d}t.\end{aligned}$$

对最后一项,

$$0\leqslant\frac{1}{a}\int_0^{a-n\pi}\frac{n\pi+t}{1+a^2\cos^2 t}\mathrm{d}t\leqslant\frac{1}{a}\int_0^{\pi}\frac{\mathrm{d}t}{1+a^2\cos^2 t}=\frac{\pi}{a\sqrt{a^2+1}}.$$

对前面的求和项,

$$\frac{k\pi}{a}\int_0^{\pi}\frac{\mathrm{d}t}{1+a^2\cos^2 t}\leqslant\frac{1}{a}\int_0^{\pi}\frac{k\pi+t}{1+a^2\cos^2 t}\mathrm{d}t\leqslant\frac{(k+1)\pi}{a}\int_0^{\pi}\frac{\mathrm{d}t}{1+a^2\cos^2 t}.$$

对 k 求和可得

$$\frac{n(n-1)\pi}{2a\sqrt{a^2+1}}\leqslant\sum_{k=0}^{n-1}\frac{1}{a}\int_0^{\pi}\frac{k\pi+t}{1+a^2\cos^2 t}\mathrm{d}t\leqslant\frac{n(n+1)\pi}{2a\sqrt{a^2+1}}.$$

由于 $n\pi\leqslant a<(n+1)\pi$, 我们有

$$\lim_{n\to\infty}\frac{n(n-1)\pi}{2a\sqrt{a^2+1}}=\lim_{n\to\infty}\frac{n(n+1)\pi}{2a\sqrt{a^2+1}}=\frac{1}{2}.$$

因此由夹逼准则可得

$$\lim_{a\to+\infty}\sum_{k=0}^{n-1}\frac{1}{a}\int_0^{\pi}\frac{k\pi+t}{1+a^2\cos^2 t}\mathrm{d}t=\frac{1}{2}.$$

也就是原极限为 $\frac{1}{2}$. □

235. 计算极限

$$\int_1^2\cdots\int_1^2\frac{1}{1+(x_1\cdots x_n)^{1/n}}\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

解

$$\begin{aligned}&\int_1^2\cdots\int_1^2\frac{1}{1+(x_1\cdots x_n)^{1/n}}\mathrm{d}x_1\cdots\mathrm{d}x_n\\&=\frac{1}{2}\int_1^2\cdots\int_1^2\frac{1}{1+\frac{(x_1\cdots x_n)^{1/n}-1}{2}}\mathrm{d}x_1\cdots\mathrm{d}x_n\\&=\frac{1}{2}\int_1^2\cdots\int_1^2\sum_{k=0}^{\infty}(-1)^k\left(\frac{(x_1\cdots x_n)^{1/n}-1}{2}\right)^k\mathrm{d}x_1\cdots\mathrm{d}x_n\\&=\frac{1}{2}\int_1^2\cdots\int_1^2\sum_{k=0}^{\infty}(-1)^k\frac{1}{2^k}\binom{k}{j}(-1)^{k-j}(x_1\cdots x_n)^{\frac{j}{n}}\mathrm{d}x_1\cdots\mathrm{d}x_n\\&=\frac{1}{2}\int_0^1\cdots\int_0^1\sum_{k=0}^{\infty}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j(x_1\cdots x_n)^{\frac{j}{n}}\mathrm{d}x_1\cdots\mathrm{d}x_n\end{aligned}$$

$$=\frac{1}{2}\int_1^2\cdots\int_1^2\sum_{k=0}^{\infty}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{n}{n+j}\right)^n\left(2^{\frac{j+n}{n}}-1\right)^n\mathrm{d}x_1\cdots\mathrm{d}x_n$$

对任意 $\varepsilon>0$, 注意到级数 $\displaystyle\sum_{k=0}^{\infty}(-1)^k\left(\frac{(x_1\cdots x_n)^{1/n}-1}{2}\right)^k$ 是收敛的, 因此可取充分大的 K, 使得

$$\left|\sum_{k=K+1}^{\infty}(-1)^k\left(\frac{(x_1\cdots x_n)^{1/n}-1}{2}\right)^k\right|<\varepsilon$$

即 $\displaystyle\int_1^2\cdots\int_1^2\left|\sum_{k=K+1}^{\infty}(-1)^k\left(\frac{(x_1\cdots x_n)^{1/n}-1}{2}\right)^k\right|\mathrm{d}x<\varepsilon$, 因此

$$\begin{aligned}
&\frac{1}{2}\int_0^1\cdots\int_0^1\sum_{k=0}^{K}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j(x_1\cdots x_n)^{\frac{j}{n}}\mathrm{d}x_1\cdots\mathrm{d}x_n\\
=&\frac{1}{2}\sum_{k=0}^{K}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\int_1^2\cdots\int_1^2(x_1\cdots x_n)^{\frac{j}{n}}\mathrm{d}x_1\cdots\mathrm{d}x_n\\
=&\frac{1}{2}\sum_{k=0}^{K}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\int_1^2 x^{\frac{j}{n}}\mathrm{d}x\right)^n\\
=&\frac{1}{2}\sum_{k=0}^{K}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{n}{n+j}\right)^n\left(2^{\frac{j+n}{n}}-1\right)^n
\end{aligned}$$

注意到对任意 $1\leqslant j\leqslant k$ 均有

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{n}{n+j}\right)^n\left(2^{\frac{j+n}{n}}-1\right)^n&=\mathrm{e}^{-j}\lim_{n\to\infty}\left(2^{\frac{j+n}{n}}-2+1\right)^n\\
&=\mathrm{e}^{-j}\lim_{n\to\infty}\mathrm{e}^{n\ln\left(2^{\frac{j+n}{n}}-2+1\right)}=\mathrm{e}^{-j}\lim_{n\to\infty}\mathrm{e}^{n\left(2^{1+\frac{j}{n}}-2\right)}\\
&=\mathrm{e}^{-j}\lim_{n\to\infty}\mathrm{e}^{2n\frac{j}{n}\ln 2}=\left(\frac{4}{\mathrm{e}}\right)^j
\end{aligned}$$

因此

$$\begin{aligned}
\frac{1}{2}\sum_{k=0}^{K}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{n}{n+j}\right)^n\left(2^{\frac{j+n}{n}}-1\right)^n&\to\frac{1}{2}\sum_{k=0}^{K}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(-\frac{4}{\mathrm{e}}\right)^j\\
&=\frac{1}{2}\sum_{k=0}^{K}\frac{1}{2^k}\left(1-\frac{4}{\mathrm{e}}\right)^k=\frac{1}{2}\frac{1-\frac{1}{2}\left(1-\frac{4}{\mathrm{e}}\right)^K}{1-\frac{1}{2}\left(1-\frac{4}{\mathrm{e}}\right)}\\
&=\frac{\mathrm{e}}{4+\mathrm{e}}\left[1-\frac{1}{2}\left(1-\frac{4}{\mathrm{e}}\right)^K\right]
\end{aligned}$$

令 $K\to\infty$, 最后得到原极限等于 $\dfrac{\mathrm{e}}{4+\mathrm{e}}$. □

236. 计算积分

$$\int_0^{\infty}\left(\sqrt{x}-\sqrt{\sqrt{1+x^2}-1}\right)\sin x\,\mathrm{d}x.$$

解 首先分部积分得

$$I=\int_0^{\infty}\frac{\cos x}{2\sqrt{x}}\mathrm{d}x-\frac{1}{2}\int_0^{\infty}\frac{x\cos x}{\sqrt{\left(x^2+1\right)\left(\sqrt{x^2+1}-1\right)}}\mathrm{d}x.$$

而

$$\int_0^{\infty}\frac{\cos x}{2\sqrt{x}}\mathrm{d}x=\int_0^{\infty}\cos\left(x^2\right)\mathrm{d}x=\frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

注意到

$$\frac{x}{\sqrt{\left(x^2+1\right)\left(\sqrt{x^2+1}-1\right)}}=\sqrt{\frac{\sqrt{x^2+1}+1}{x^2+1}}=\frac{2\sqrt{2}}{\pi}\int_0^{\infty}\frac{t^2+1}{x^2+\left(t^2+1\right)^2}\mathrm{d}t.$$

当 $a>0$ 时, 我们有 $\int_0^{\infty}\frac{\cos x}{x^2+a^2}\mathrm{d}x=\frac{\pi}{2a}\mathrm{e}^{-a}$, 于是

$$\begin{aligned}\int_0^{\infty}\frac{x\cos x}{\sqrt{\left(x^2+1\right)\left(\sqrt{x^2+1}-1\right)}}\mathrm{d}x&=\frac{2\sqrt{2}}{\pi}\int_0^{\infty}\int_0^{\infty}\frac{\left(t^2+1\right)\cos x}{x^2+\left(t^2+1\right)^2}\mathrm{d}x\mathrm{d}t\\&=\sqrt{2}\int_0^{\infty}\mathrm{e}^{-\left(t^2+1\right)}\mathrm{d}t=\frac{1}{\mathrm{e}}\sqrt{\frac{\pi}{2}}.\end{aligned}$$

于是得到

$$\int_0^{\infty}\left(\sqrt{x}-\sqrt{\sqrt{1+x^2}-1}\right)\sin x\mathrm{d}x=\frac{\mathrm{e}-1}{2\mathrm{e}}\sqrt{\frac{\pi}{2}}.$$

□

237. 计算积分

$$\int_0^1\frac{x\arccos x}{1+x^4}\mathrm{d}x.$$

解 首先令 $y=\arccos x$, 则

$$\begin{aligned}I&=\int_0^{\frac{\pi}{2}}\frac{y\cos y\sin y}{1+\cos^4 y}\mathrm{d}y=-\frac{1}{2}\int_0^{\frac{\pi}{2}}y\mathrm{d}\left(\arctan\left(\cos^2 x\right)\right)=\frac{1}{2}\int_0^{\frac{\pi}{2}}\arctan\left(\cos^2 y\right)\mathrm{d}y\\&=\frac{1}{2}\int_0^{\frac{\pi}{2}}\mathrm{d}y\int_0^1\frac{\cos^2 y}{1+t^2\cos^4 y}\mathrm{d}t=\frac{1}{2}\int_0^1\mathrm{d}t\int_0^{\frac{\pi}{2}}\frac{\cos^2 y}{1+t^2\cos^4 y}\mathrm{d}y.\end{aligned}$$

其中

$$\int_0^{\frac{\pi}{2}}\frac{\cos^2 y}{1+t^2\cos^4 y}\mathrm{d}y=\int_0^{\infty}\frac{\mathrm{d}u}{u^4+2u^2+t^2+1}=\frac{\pi\sqrt{\sqrt{t^2+1}-1}}{2\sqrt{2}t\sqrt{t^2+1}}.$$

于是原积分

$$\begin{aligned}I&=\frac{\pi}{4\sqrt{2}}\int_0^1\frac{\sqrt{\sqrt{t^2+1}-1}}{t\sqrt{t^2+1}}\mathrm{d}t=\int_0^1\frac{t\sqrt{\sqrt{t^2+1}-1}}{t^2\sqrt{t^2+1}}\mathrm{d}t\\&=\frac{\pi}{4\sqrt{2}}\int_0^1\frac{\sqrt{\sqrt{t^2+1}-1}}{t^2}\mathrm{d}\left(\sqrt{t^2+1}\right)=\frac{\pi}{4\sqrt{2}}\int_1^{\sqrt{2}}\frac{\sqrt{s-1}}{s^2-1}\mathrm{d}s\\&=\frac{\pi}{4\sqrt{2}}\int_1^{\sqrt{2}}\frac{\mathrm{d}s}{(s+1)\sqrt{s-1}}=\frac{\pi}{4}\arctan\left(\sqrt{\frac{\sqrt{2}-1}{2}}\right).\end{aligned}$$

□

238. 计算积分

$$\int_0^1 \frac{\arctan\sqrt{x}}{(3-x)\sqrt{2-x}}\mathrm{d}x.$$

解 考虑参数积分 $J(a)=\int_0^1 \frac{\arctan\sqrt{ax}}{(3-x)\sqrt{2-x}}\mathrm{d}x$, 则 $J(0)=0, I=J(1)$.

$$\begin{aligned}
J'(a)&=\frac{1}{2\sqrt{a}}\int_0^1 \frac{1}{(3-x)(1+ax)}\sqrt{\frac{x}{2-x}}\mathrm{d}x \quad \left(t=\sqrt{\frac{x}{2-x}}\right)\\
&=\frac{1}{\sqrt{a}}\frac{1}{3a}\int_0^1\left(\frac{3}{3+t^2}-\frac{1}{1+(1+2a)t^2}\right)\mathrm{d}t=\frac{1}{(1+3a)\sqrt{a}}\left(\frac{\pi}{2\sqrt{3}}-\frac{\arctan\sqrt{1+2a}}{\sqrt{1+2a}}\right).
\end{aligned}$$

于是

$$\begin{aligned}
J(1)&=\frac{\pi^2}{9}-2\int_0^1\frac{\arctan\sqrt{1+2a^2}}{(1+3a^2)\sqrt{1+2a^2}}\mathrm{d}a\\
&=\frac{\pi^2}{9}-2\int_0^1\arctan\sqrt{1+2a^2}\,\mathrm{d}\left(\arctan\frac{a}{\sqrt{1+2a^2}}\right)=2\int_0^1\frac{\arctan\frac{a}{\sqrt{1+2a^2}}}{(1+a^2)\sqrt{1+2a^2}}a\mathrm{d}a\\
&=\pi\int_0^1\frac{a\mathrm{d}a}{(1+a^2)\sqrt{1+2a^2}}-2\int_0^1\frac{\arctan\frac{\sqrt{1+2a^2}}{a}}{(1+a^2)\sqrt{1+2a^2}}\mathrm{d}a\\
&=\frac{\pi^2}{12}-2\int_1^\infty\frac{\arctan\sqrt{2+s^2}}{(1+s^2)\sqrt{2+s^2}}\mathrm{d}s=\frac{\pi^2}{12}-\frac{\pi^2}{16}=\frac{\pi^2}{48}.
\end{aligned}$$

其中

$$\begin{aligned}
\int_1^\infty\frac{\arctan\sqrt{2+s^2}}{(1+s^2)\sqrt{2+s^2}}\mathrm{d}s&=\int_1^\infty\int_1^\infty\frac{1}{1+s^2}\frac{1}{2+s^2+p^2}\mathrm{d}p\mathrm{d}s\\
&=\frac{1}{2}\int_1^\infty\int_1^\infty\frac{1}{1+s^2}\frac{1}{1+p^2}\mathrm{d}p\mathrm{d}s=\frac{\pi^2}{32}.
\end{aligned}$$

□

239. 计算积分

$$\int_0^1\int_0^1\frac{\mathrm{d}x\mathrm{d}y}{\left(\left[\frac{x}{y}\right]+1\right)^2}.$$

解 换元 $x=u, y=uv$, 则

$$\begin{aligned}
I&=\int_0^1\int_0^1\frac{u}{\left(\left[\frac{1}{v}\right]+1\right)^2}\mathrm{d}u\mathrm{d}v+\int_1^\infty\int_0^{\frac{1}{v}}\frac{u}{\left(\left[\frac{1}{v}\right]+1\right)^2}\mathrm{d}u\mathrm{d}v\\
&=\frac{1}{2}\int_0^1\frac{\mathrm{d}v}{\left(\left[\frac{1}{v}\right]+1\right)^2}+\frac{1}{2}\int_1^\infty\frac{\mathrm{d}v}{v^2\left(\left[\frac{1}{v}\right]+1\right)^2}\\
&=\frac{1}{2}\int_0^1\frac{\mathrm{d}v}{\left(\left[\frac{1}{v}\right]+1\right)^2}+\frac{1}{2}\int_0^1\frac{\mathrm{d}v}{v^2}=\frac{1}{2}\int_1^\infty\frac{\mathrm{d}v}{v^2([v]+1)^2}+\frac{1}{2}\\
&=\frac{1}{2}\sum_{n=1}^\infty\frac{1}{(n+1)^2}\int_n^{n+1}\frac{\mathrm{d}v}{v^2}+\frac{1}{2}=\frac{1}{2}\sum_{n=1}^\infty\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)^2}-\frac{1}{(n+1)^3}\right)+\frac{1}{2}\\
&=2-\frac{\zeta(2)+\zeta(3)}{2}.
\end{aligned}$$

□

240. 计算积分

$$\int_0^{\pi} \ln(1-\cos x)\ln(1+\cos x)\,\mathrm{d}x.$$

解

$$\begin{aligned}
\int_0^{\pi} \log(1-\cos x)\log(1+\cos x)\ \mathrm{d}x &= \int_0^{\pi} \log(1+\cos x)\left(-\log 2 - 2\sum_{n=1}^{\infty}\frac{\cos nx}{n}\right)\mathrm{d}x \\
&= \pi\log^2 2 - 2\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\pi}\cos nx\log(1+\cos x)\ \mathrm{d}x \\
&= \pi\log^2 2 - 2\sum_{n=1}^{\infty}\frac{\pi(-1)^{n-1}}{n^2} \\
&= \pi\log^2 2 - 2\pi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} \\
&= \pi\log^2 2 - \frac{\pi^3}{6}.
\end{aligned}$$

□

241. 计算积分

$$\int_{-\pi}^{\pi}\frac{x\sin x}{1+a^2-2a\cos x}\mathrm{d}x.$$

解

$$\begin{aligned}
I &= 2\int_0^{\pi}\frac{x\sin x}{1+a^2-2a\cos x}\mathrm{d}x = \frac{2}{2a}\int_0^{\pi}\frac{x\sin x}{\frac{1+a^2}{2a}-\cos x}\mathrm{d}x \\
&= \frac{1}{a}\int_0^{\pi}\frac{x\sin x}{A-\cos x}\mathrm{d}x = \frac{x\ln(A-\cos x)}{a}\Big|_0^{\pi} - \frac{1}{a}\int_0^{\pi}\ln(A-\cos x)\,\mathrm{d}x \\
&= \frac{\pi\ln(A+1)}{a} - \frac{1}{a}\int_0^{\pi}\ln(A-\cos x)\,\mathrm{d}x \\
&= \frac{\pi\ln(A+1)}{a} - \frac{\pi}{a}\ln\left(\frac{A+\sqrt{A^2-1}}{2}\right) = \frac{2\pi}{a}\ln\left(1+\frac{1}{a}\right).
\end{aligned}$$

□

242.

$$\int_0^{2\pi}\frac{\cos^2 3x}{1+a^2-2a\cos 2x}\mathrm{d}x.$$

解 利用公式

$$\sum_{k=0}^{\infty}a^k\sin kx = \frac{a\sin x}{1+a^2-2a\cos x}$$

可得

$$I = \sum_{k=0}^{\infty}a^{k-1}\int_0^{2\pi}\frac{\cos^2(3x)\sin 2kx}{\sin 2x}\mathrm{d}x.$$

考虑积分

$$J_k = \int_0^{2\pi}\frac{\cos^2(3x)\sin 2kx}{\sin 2x}\mathrm{d}x,$$

则

$$J_{k+2}-J_k=2\int_0^{2\pi}\cos^2(3x)\cos(2k+2)x\mathrm{d}x=\begin{cases}0, & k\neq 2\\ \pi, & k=2\end{cases}.$$

于是

$$J_1=\pi, J_0=0 \implies J_{2p+1}=\pi, J_0=J_2=0, J_4=J_6=J_8=J_{10}=\cdots=\pi.$$

因此最后得到

$$I=\pi(1+a^3+a^4+\cdots)=\pi\left(\frac{1}{1-a}-a\right)=\pi\frac{1-a+a^2}{1-a}.$$

□

243. 设 n 是一个自然数, 计算

$$\int_0^{2\pi}\frac{(1+2\cos x)^n\cos nx}{3+2\cos x}\mathrm{d}x.$$

解 对任意自然数 $k<n$,

$$\begin{aligned}\alpha_k(n)&=\int_0^{2\pi}(1+2\cos x)^k\cos nx\mathrm{d}x\\&=\int_0^{2\pi}(1+2\cos x)^{k-1}\cos nx\mathrm{d}x+\int_0^{2\pi}(1+2\cos x)^{k-1}(2\cos x\cos nx)\mathrm{d}x\\&=\alpha_{k-1}(n)+\int_0^{2\pi}(1+2\cos x)^{k-1}\cos(n-1)x\mathrm{d}x+\int_0^{2\pi}(1+2\cos x)^{k-1}\cos(n+1)x\mathrm{d}x\\&=\alpha_{k-1}(n-1)+\alpha_{k-1}(n)+\alpha_{k-1}(n+1).\end{aligned}$$

且

$$\alpha_0(n)=\int_0^{2\pi}\cos nx\mathrm{d}x=2\pi\delta_{n,0}.$$

由递推公式以及初始条件可知对任意 $k<n$ 有 $\alpha_k(n)=0$. 再令

$$f_k(n):=\int_0^{2\pi}\frac{(1+2\cos x)^k\cos nx}{3+2\cos x}\mathrm{d}x.$$

那么当 $0\leqslant k\leqslant n$ 时,

$$\begin{aligned}f_k(n)+f_k(n+2)&=2\int_0^{2\pi}\frac{(1+2\cos x)^k\cos x\cos(n+1)x}{3+2\cos x}\mathrm{d}x\\&=\int_0^{2\pi}(1+2\cos x)^k\cos(n+1)x\,dx-3\int_0^{2\pi}\frac{(1+2\cos x)^k\cos(n+1)x}{3+2\cos x}\mathrm{d}x\\&=-3f_k(n+1).\end{aligned}$$

因此我们得到递推关系

$$f_k(n+2)+3f_k(n+1)+f_k(n)=0.$$

由特征根方法可知解形式为

$$f_k(n)=c_1(k)\lambda_1^n+c_2(k)\lambda_2^n,$$

其中 $\lambda_1=\dfrac{-3+\sqrt{5}}{2},\lambda_2=\dfrac{-3-\sqrt{5}}{2}$, 由 Riemann-Lebesgue 引理可知 $\lim\limits_{n\to\infty}f_k(n)=0$, 因此 $c_2(k)=0$, $f_k(n)=c_1(k)\lambda_1^n, 0\leqslant k\leqslant n$.

$$
\begin{aligned}
f_k(n)&=\int_0^{2\pi}\frac{(1+2\cos x)^{k-1}(1+2\cos x)\cos nx}{3+2\cos x}\mathrm{d}x\\
&=\int_0^{2\pi}(1+2\cos x)^{k-1}\cos nx\mathrm{d}x-2f_{k-1}(n)=-2f_{k-1}(n).
\end{aligned}
$$

这是一个关于指标 k 的等比数列, 于是 $f_k(n)=c(-2)^k\lambda_1^n$. 当 $k=n=0$ 时,

$$
c=\int_0^{2\pi}\frac{1}{3+2\cos x}\mathrm{d}x=\frac{2\pi}{\sqrt{5}}.
$$

因此最后得到原积分

$$
f_n(n)=\frac{2\pi}{\sqrt{5}}(-2)\lambda_1^n=\frac{2\pi}{\sqrt{5}}\left(3-\sqrt{5}\right)^n.
$$

□

244. 设 $\mathrm{Re}(\alpha)\geqslant 1$, 计算积分

$$
\int_{-\infty}^{\infty}|\sin x|^{\alpha-1}\frac{\sin x}{x}\mathrm{d}x.
$$

解

$$
\begin{aligned}
\int_{-\infty}^{+\infty}|\sin x|^{\alpha-1}\frac{\sin x}{x}\mathrm{d}x&=2\int_0^{+\infty}|\sin x|^{\alpha-1}\frac{\sin x}{x}\mathrm{d}x\\
&=2\sum_{n=0}^{+\infty}\left(\int_{2n\pi}^{(2n+1)\pi}(\sin x)^{\alpha-1}\frac{\sin x}{x}\mathrm{d}x+\int_{(2n+1)\pi}^{(2n+2)\pi}(-\sin x)^{\alpha-1}\frac{\sin x}{x}\mathrm{d}x\right)\\
&=2\sum_{n=0}^{+\infty}\left(\int_0^{\pi}\frac{(\sin x)^{\alpha}}{x+2n\pi}\mathrm{d}x+\int_{-(2n+2)\pi}^{-(2n+1)\pi}\frac{(\sin x)^{\alpha}}{x}\mathrm{d}x\right)\\
&=2\int_0^{\pi}(\sin x)^{\alpha}\sum_{n=0}^{+\infty}\left(\frac{1}{x+2n\pi}+\frac{1}{x-(2n+2)\pi}\right)\mathrm{d}x\\
&=2\int_0^{\pi}(\sin x)^{\alpha}\left(\frac{1}{x}+\sum_{n=1}^{+\infty}\frac{2x}{x^2-4n^2\pi^2}\right)\mathrm{d}x\\
&=\int_0^{\pi}(\sin x)^{\alpha}\cot\left(\frac{x}{2}\right)\mathrm{d}x\\
&=2\int_0^{\frac{\pi}{2}}(\sin 2x)^{\alpha}\cot(x)\mathrm{d}x\\
&=2^{\alpha}\cdot 2\int_0^{\frac{\pi}{2}}(\sin x)^{\alpha-1}(\cos x)^{\alpha+1}\mathrm{d}x\\
&=2^{\alpha}\mathrm{B}\left(\frac{\alpha}{2},\frac{\alpha}{2}+1\right)\\
&=2^{\alpha}\frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}+1\right)}{\Gamma(\alpha+1)}\\
&=2^{\alpha-1}\frac{\Gamma^2\left(\frac{\alpha}{2}\right)}{\Gamma(\alpha)}.
\end{aligned}
$$

□

245. 设 α 是一个正实数, 证明

$$\int_0^\pi x^{\alpha-2}\sin x\mathrm{d}x \geqslant \frac{\pi^\alpha(\alpha+6)}{\alpha(\alpha+2)(\alpha+3)}.$$

由于函数 $f(x)=\dfrac{1-\cos x}{x^2}$ 在 $\left(0,\dfrac{\pi}{2}\right]$ 上是凹函数, 因此它在 $x=\dfrac{\pi}{2}$ 处的切线在曲线 $y=f(x)$ 的上方,

$$\frac{1-\cos x}{x^2}=f(x)\leqslant f'\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)+f\left(\frac{\pi}{2}\right)=\frac{2(6-\pi)}{\pi^2}-\frac{4(4-\pi)x}{\pi^3}.$$

因此对 $x\in\left[0,\dfrac{1}{2}\right]$,

$$\cos\pi s\geqslant 1-2(6-\pi)s^2+4(4-\pi)s^3\geqslant 1-6s^2+4s^3,$$

且

$$\sin\pi t=\pi\int_0^t\cos\pi s\mathrm{d}s\geqslant\pi\int_0^t\left(1-6s^2+4s^3\right)\mathrm{d}s=\pi\left(1-2t^3+t^4\right):=p(t).$$

进一步, 由对称性 $\sin\pi t=\sin\pi(1-t)$ 以及 $p(t)=p(1-t)$ 可知上述不等式对 $s\in[0,1]$ 都成立. 则

$$\begin{aligned}\int_0^\pi x^{\alpha-2}\sin x\mathrm{d}x&=\pi^{\alpha-1}\int_0^1 t^{\alpha-2}\sin\pi t\mathrm{d}t\geqslant\pi^{\alpha-1}\int_0^1 t^{\alpha-2}p(t)\mathrm{d}t\\&=\pi^\alpha\int_0^1\left(t^{\alpha-1}-2t^{\alpha+1}+t^{\alpha+2}\right)\mathrm{d}t=\pi^\alpha\left(\frac{1}{\alpha}-\frac{2}{\alpha+2}+\frac{1}{\alpha+3}\right)\\&=\frac{\pi^\alpha(\alpha+6)}{\alpha(\alpha+2)(\alpha+3)}.\end{aligned}$$

246. 对 $n\in\mathbb{N}^*$, 设 $H_n=\sum\limits_{k=1}^n\dfrac{1}{k}$, 且

$$S_n=\sum_{k=1}^n\frac{(-1)^{n-k}}{k}\sum_{j=1}^k H_j.$$

求极限 $\lim\limits_{n\to\infty}\dfrac{S_n}{\ln n}$ 与 $\lim\limits_{n\to\infty}(S_{2n}-S_{2n-1})$.

解 易得 $\sum\limits_{j=1}^k H_j=k(H_k-1)+H_k$, 因此

$$S_{2n}=\sum_{k=1}^{2n}(-1)^k H_k-\sum_{k=1}^{2n}(-1)^k+\sum_{k=1}^{2n}(-1)^k\frac{H_k}{k}=\frac{H_n}{2}+\sum_{k=1}^{2n}(-1)^k\frac{H_k}{k}$$

且

$$S_{2n-1}=-\sum_{k=1}^{2n-1}(-1)^k H_k-\sum_{k=1}^{2n-1}(-1)^k-\sum_{k=1}^{2n-1}(-1)^k\frac{H_k}{k}=H_{2n}-\frac{H_n}{2}-1-\sum_{k=1}^{2n-1}(-1)^k\frac{H_k}{k}.$$

因此

$$S_{2n}-S_{2n-1}=H_n+1-H_{2n}-\frac{H_{2n}}{2n}+2\sum_{k=1}^{2n}(-1)^k\frac{H_k}{k}.$$

当 $N\to\infty$ 时,

$$\sum_{k=1}^N(-1)^k\frac{H_k}{k}=\sum_{k=1}^N\frac{(-1)^k}{k^2}+\sum_{k=2}^N(-1)^k\frac{H_{k-1}}{k}$$

$$
\begin{aligned}
&=\sum_{k=1}^{N}\frac{(-1)^k}{k^2}+\frac{1}{2}\sum_{k=2}^{N}\sum_{j=1}^{k-1}\frac{(-1)^k}{k}\left(\frac{1}{j}+\frac{1}{k-j}\right)\\
&=\sum_{k=1}^{N}\frac{(-1)^k}{k^2}+\frac{1}{2}\sum_{j=1}^{N-1}\sum_{k=j+1}^{N}\frac{(-1)^k}{j\,(k-j)}\\
&=\sum_{k=1}^{N}\frac{(-1)^k}{k^2}+\frac{1}{2}\sum_{j=1}^{N-1}\frac{(-1)^j}{j}\sum_{i=1}^{N-j}\frac{(-1)^i}{i}\\
&=\sum_{k=1}^{N}\frac{(-1)^k}{k^2}+\frac{1}{2}\left(\sum_{j=1}^{N-1}\frac{(-1)^j}{j}\right)^2-\frac{1}{2}\sum_{j=1}^{N-1}\frac{(-1)^j}{j}\sum_{i=N-j+1}^{N-1}\frac{(-1)^i}{i}\\
&\to-\frac{\pi^2}{12}+\frac{\ln^2 2}{2}.
\end{aligned}
$$

其中

$$
\left|\sum_{j=1}^{N-1}\frac{(-1)^j}{j}\sum_{i=N-j+1}^{N-1}\frac{(-1)^i}{i}\right|\leqslant\frac{1}{\sqrt{N}}\to 0.
$$

由于 $H_n=\ln n+\gamma+o(1)$, 我们有

$$
\lim_{n\to\infty}\left(S_{2n}-S_{2n-1}\right)=\ln n+1-\ln(2n)-\frac{\pi^2}{6}+o(1)\to\ln^2 2-\ln 2+1-\frac{\pi^2}{6}.
$$

且

$$
\lim_{n\to\infty}\frac{S_n}{\ln n}=\lim_{n\to\infty}\frac{H_n}{2\ln n}=\frac{1}{2}.
$$

□

247. 设常数 $a>0$, 证明

$$
\lim_{x\to 0^+}\frac{1}{x}\int_0^x\left|\sin\frac{1}{t^2}\right|^a\mathrm{d}t=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin^a x\,\mathrm{d}x.
$$

证明

$$
\begin{aligned}
\lim_{x\to 0^+}\frac{1}{x}\int_0^x\left|\sin\frac{1}{t^2}\right|^a\mathrm{d}t&=\lim_{x\to 0^+}\frac{1}{x}\int_{x^{-2}}^{\infty}\frac{|\sin u|^a}{2u^{3/2}}\mathrm{d}u\\
&=\lim_{x\to+\infty}\sqrt{x}\int_x^{\infty}\frac{|\sin u|^a}{2u^{3/2}}\mathrm{d}u\\
&=\lim_{k\to\infty}\sqrt{k\pi}\int_{k\pi}^{\infty}\frac{|\sin u|^a}{2u^{3/2}}\mathrm{d}u\\
&=\lim_{k\to\infty}\sqrt{k\pi}\sum_{i=k}^{\infty}\int_{i\pi}^{(i+1)\pi}\frac{|\sin u|^a}{2u^{3/2}}\mathrm{d}u\\
&=\lim_{k\to\infty}\frac{\sqrt{k\pi}}{2}\int_0^{\pi}|\sin u|^a\sum_{i=k}^{\infty}\frac{1}{(u+i\pi)^{3/2}}\mathrm{d}u\\
&=\lim_{k\to\infty}\frac{\sqrt{k}}{2\pi}\int_0^{\pi}\sin^a u\sum_{i=k}^{\infty}\frac{1}{\left(i+\frac{u}{\pi}\right)^{3/2}}\mathrm{d}u\\
&=\lim_{k\to\infty}\frac{\sqrt{k}}{2\pi}\frac{2}{\sqrt{k}}\int_0^{\pi}\sin^a u\,\mathrm{d}u
\end{aligned}
$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin^a u \mathrm{d}u = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^a u \mathrm{d}u.$$

其中当 $k \to \infty$ 时,

$$\sum_{i=k}^{\infty} \frac{1}{\left(i + \frac{u}{\pi}\right)^{3/2}} \sim \int_k^{\infty} \frac{\mathrm{d}x}{x^{3/2}} = \frac{2}{\sqrt{k}}.$$

□

248. 证明

$$\sum_{n=1}^{\infty} \frac{1}{n}\int_0^{\infty} \frac{\arctan x}{\left(1+x^2\right)^n}\mathrm{d}x = \frac{\pi^2}{6}.$$

证明 首先利用 Wallis 公式可知

$$I_n = \int_0^{\infty} \frac{\arctan x}{\left(1+x^2\right)^n}\mathrm{d}x = \int_0^{\frac{\pi}{2}} t\cos^{2n-2} t \mathrm{d}t \to 0.$$

再由分部积分可得

$$\begin{aligned}
I_n &= \left.\frac{x\arctan x}{\left(1+x^2\right)^n}\right|_0^{\infty} - \int_0^{\infty} \frac{x}{\left(1+x^2\right)^{n+1}}\mathrm{d}x + 2n\int_0^{\infty} \frac{x^2\arctan x}{\left(1+x^2\right)^{n+1}}\mathrm{d}x \\
&= \left.\frac{1}{2n\left(1+x^2\right)^n}\right|_0^{\infty} + 2n\int_0^{\infty} \frac{\arctan x}{\left(1+x^2\right)^n}\mathrm{d}x - 2n\int_0^{\infty} \frac{\arctan x}{\left(1+x^2\right)^{n+1}}\mathrm{d}x \\
&= -\frac{1}{2n} + 2nI_n - 2nI_{n+1}.
\end{aligned}$$

由此可得递推关系 $\frac{I_n}{n} = 2(I_n - I_{n+1}) - \frac{1}{2n^2}$, 于是

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{I_n}{n} &= 2I_1 - 2\lim_{n\to\infty} I_n - \sum_{n=1}^{\infty} \frac{1}{2n^2} \\
&= 2\int_0^{\infty} \frac{\arctan x}{1+x^2}\mathrm{d}x - \frac{\pi^2}{12} = \frac{\pi^2}{6}.
\end{aligned}$$

□

249. 设 $f:[0,1]\to\mathbb{R}$ 有连续非负的三阶导数, 且 $\int_0^1 f(x)\mathrm{d}x = 0$, 证明:

$$10\int_0^1 x^3 f(x)\mathrm{d}x + 6\int_0^1 xf(x)\mathrm{d}x \geqslant 15\int_0^1 x^2 f(x)\mathrm{d}x.$$

证明 令 $P(x) = (x(1-x))^3$, 则 $P^{(k)}(0) = P^{(k)}(1) = 0, k = 0,1,2.$ 因此由分部积分可得

$$\begin{aligned}
\int_0^1 P^{(3)}(x) f(x)\mathrm{d}x &= -\int_0^1 P^{(2)}(x) f'(x)\mathrm{d}x \\
&= \int_0^1 P^{(1)}(x) f''(x)\mathrm{d}x = -\int_0^1 P(x) f'''(x)\mathrm{d}x \leqslant 0.
\end{aligned}$$

易得 $P^{(3)}(x) = -12\left(10x^3 - 15x^2 + 6x - 1/2\right)$, 因此

$$\int_0^1 \left(10x^3 - 15x^2 + 6x - \frac{1}{2}\right) f(x)\mathrm{d}x \geqslant 0.$$

注意到 $\int_0^1 f(x)\mathrm{d}x = 0$, 我们得到

$$10\int_0^1 x^3 f(x)\,\mathrm{d}x + 6\int_0^1 xf(x)\,\mathrm{d}x \geqslant 15\int_0^1 x^2 f(x)\,\mathrm{d}x.$$

□

250. 求和 $\sum_{n=1}^{\infty}\frac{4^n((n-1)!)^2}{(2n)!}$.

解

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{4^n((n-1)!)^2}{(2n)!} &= \sum_{n=1}^{\infty}\frac{4^n}{2n}\frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} = \sum_{n=1}^{\infty}\frac{4^n}{2n}B(n,n)\\
&= \sum_{n=1}^{\infty}\frac{4^n}{2n}\int_0^1 x^{n-1}(1-x)^{n-1}\,\mathrm{d}x\\
&= -\frac{1}{2}\int_0^1\frac{\ln(1-4x(1-x))}{x(1-x)}\mathrm{d}x = -2\int_0^{\frac{1}{2}}\frac{\ln(1-2x)}{x(1-x)}\mathrm{d}x\\
&= -2\int_0^{\frac{1}{2}}\frac{\ln(1-2x)}{x}\mathrm{d}x - 2\int_0^{\frac{1}{2}}\frac{\ln(1-2x)}{1-x}\mathrm{d}x.
\end{aligned}$$

其中

$$\int_0^{\frac{1}{2}}\frac{\ln(1-2x)}{x}\mathrm{d}x = \int_0^1\frac{\ln(1-t)}{t}\mathrm{d}t = -\mathrm{Li}_2(1) = -\frac{\pi^2}{6}$$

$$\begin{aligned}
\int_0^{\frac{1}{2}}\frac{\ln(1-2x)}{1-x}\mathrm{d}x &= \int_0^1\frac{\ln(1-t)}{2-t}\mathrm{d}t = \int_0^1\frac{\ln t}{1+t}\mathrm{d}t\\
&= \sum_{n=0}^{\infty}(-1)^n\int_0^1 t^n\ln t\,\mathrm{d}t = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)^2}\\
&= -\frac{\pi^2}{12}
\end{aligned}$$

□

251. 计算积分

$$\int\frac{x}{\sqrt{\mathrm{e}^x+(x+2)^2}}\mathrm{d}x$$

解

$$\begin{aligned}
\int\frac{x}{\sqrt{\mathrm{e}^x+(x+2)^2}}\mathrm{d}x &= \int\frac{x\mathrm{e}^{-\frac{x}{2}}}{\sqrt{1+\mathrm{e}^{-x}(x+2)^2}}\mathrm{d}x\\
&= -2\int\frac{\mathrm{d}\left((x+2)\mathrm{e}^{-\frac{x}{2}}\right)}{\sqrt{1+\mathrm{e}^{-x}(x+2)^2}}\\
&= -2\ln\left((x+2)\mathrm{e}^{-\frac{x}{2}}+\sqrt{1+\mathrm{e}^{-x}(x+2)^2}\right)+C.
\end{aligned}$$

□

252. 计算积分

$$\int_0^1 \left(-x^2+x\right)^{2017} \lfloor 2017x \rfloor \mathrm{d}x.$$

解

$$\begin{aligned}
\int_0^1 \left(-x^2+x\right)^{2017} \lfloor 2017x \rfloor \mathrm{d}x &= \sum_{k=0}^{2016} k \int_{\frac{k}{2017}}^{\frac{k+1}{2017}} x^{2017}(1-x)^{2017}\,\mathrm{d}x \\
&= \sum_{k=1}^{2016}\sum_{i=1}^{k} \int_{\frac{k}{2017}}^{\frac{k+1}{2017}} x^{2017}(1-x)^{2017}\,\mathrm{d}x \\
&= \sum_{i=1}^{2016}\sum_{k=i}^{2016} \int_{\frac{k}{2017}}^{\frac{k+1}{2017}} x^{2017}(1-x)^{2017}\,\mathrm{d}x \\
&= \sum_{i=1}^{2016} \int_{\frac{i}{2017}}^{1} x^{2017}(1-x)^{2017}\,\mathrm{d}x \\
&= \sum_{i=1}^{2016} \int_0^{\frac{i}{2017}} x^{2017}(1-x)^{2017}\,\mathrm{d}x \\
&= \frac{1}{2}\sum_{i=1}^{2016} \int_0^1 x^{2017}(1-x)^{2017}\,\mathrm{d}x \\
&= 1008B(2018,2018) = 1008\frac{(2017!)^2}{4035!}.
\end{aligned}$$

□

253. 计算积分

$$\int_{-1}^1 \frac{1}{x}\sqrt{\frac{1+x}{1-x}} \ln\frac{2x^2+2x+1}{2x^2-2x+1}\mathrm{d}x.$$

解 首先有

$$\begin{aligned}
I &= \int_{-1}^1 \frac{1}{x}\sqrt{\frac{1+x}{1-x}} \ln\frac{2x^2+2x+1}{2x^2-2x+1}\mathrm{d}x \\
&= \int_0^1 \frac{1}{x}\sqrt{\frac{1+x}{1-x}} \ln\frac{2x^2+2x+1}{2x^2-2x+1}\mathrm{d}x + \int_{-1}^0 \frac{1}{x}\sqrt{\frac{1+x}{1-x}} \ln\frac{2x^2+2x+1}{2x^2-2x+1}\mathrm{d}x \\
&= \int_0^1 \frac{1}{x}\left(\sqrt{\frac{1+x}{1-x}}+\sqrt{\frac{1-x}{1+x}}\right) \ln\frac{2x^2+2x+1}{2x^2-2x+1}\mathrm{d}x \\
&= 2\int_0^1 \frac{1}{x\sqrt{1-x^2}} \ln\frac{2x^2+2x+1}{2x^2-2x+1}\mathrm{d}x \\
&= 2\int_0^{\frac{\pi}{2}} \frac{1}{\cos t} \ln\frac{2\cos^2 t+2\cos t+1}{2\cos^2 t-2\cos t+1}\mathrm{d}t
\end{aligned}$$

考虑含参变量积分

$$I(a) = \int_0^{\frac{\pi}{2}} \frac{2}{\cos t} \ln\frac{2\cos^2 t+a\cos t+1}{2\cos^2 t-a\cos t+1}\mathrm{d}t.$$

则 $I(0)=0$, 且

$$I'(a) = 2\int_0^{\frac{\pi}{2}} \left(\frac{1}{2\cos^2 t+a\cos t+1}+\frac{1}{2\cos^2 t-a\cos t+1}\right)\mathrm{d}t$$

$$
\begin{aligned}
&= 4\int_0^{\frac{\pi}{2}} \frac{1+2\cos^2 t}{\left(1+2\cos^2 t\right)^2 - a^2\cos^2 t}\mathrm{d}t \\
&= 4\int_0^{\frac{\pi}{2}} \frac{1+2\cos^2 t}{4\cos^4 t + \left(4-a^2\right)\cos^2 t + 1}\mathrm{d}t \\
&\quad\vdots \\
&= \frac{2\pi\left(3+\sqrt{9-a^2}\right)}{\sqrt{9-a^2}\sqrt{6-a^2+2\sqrt{9-a^2}}}.
\end{aligned}
$$

这里这个积分的计算非常复杂, 我算了好几遍. 分母是四次的积分, 分子分母除以 $\cos^4 t$, 然后 $u = \tan t$, 于是

$$
\begin{aligned}
I = I(2) &= \int_0^2 \frac{2\pi\left(3+\sqrt{9-a^2}\right)}{\sqrt{9-a^2}\sqrt{6-a^2+2\sqrt{9-a^2}}}\mathrm{d}a \\
&= \int_0^{\arcsin\frac{2}{3}} \frac{2\pi(3+3\cos y)}{\sqrt{6-9\sin^2 y+6\cos y}}\mathrm{d}y \\
&= 12\pi\int_0^{\arcsin\frac{2}{3}} \frac{\cos^2\frac{y}{2}}{\sqrt{12\cos^2\frac{y}{2}-36\sin^2\frac{y}{2}\cos^2\frac{y}{2}}}\mathrm{d}y \\
&= 24\pi\int_0^{\arcsin\frac{2}{3}} \frac{\mathrm{d}(\sin(y/2))}{\sqrt{12-36\sin^2\frac{y}{2}}} \\
&= 24\pi\int_0^{\sqrt{\frac{3-\sqrt{5}}{6}}} \frac{\mathrm{d}u}{\sqrt{12-36u^2}} = 4\pi\arcsin\frac{\sqrt{5}-1}{2}.
\end{aligned}
$$

□

254. 求极限

$$
\lim_{n\to\infty} n\left(\sum_{k=1}^n \cos^n\sqrt{\frac{k}{n}} - \frac{\sqrt{\mathrm{e}}+1}{\mathrm{e}-1}\right).
$$

解 对固定的 k, 我们有

$$
\begin{aligned}
\cos^n\sqrt{\frac{k}{n}} &= \left(1-\frac{k}{2n}+\frac{k^2}{24n^2}+o\left(\frac{1}{n^2}\right)\right)^n \\
&= \exp\left(n\ln\left(1-\frac{k}{2n}+\frac{k^2}{24n^2}+o\left(\frac{1}{n^2}\right)\right)\right) \\
&= \exp\left(n\left(-\frac{k}{2n}-\frac{k^2}{12n^2}+o\left(\frac{1}{n^2}\right)\right)\right) \\
&= \mathrm{e}^{-\frac{k}{2}}\mathrm{e}^{-\frac{k^2}{12n}+o\left(\frac{1}{n}\right)} = \mathrm{e}^{-\frac{k}{2}}\left(1-\frac{k^2}{12n}+o\left(\frac{1}{n}\right)\right).
\end{aligned}
$$

注意到

$$
\sum_{k=1}^\infty \mathrm{e}^{-\frac{k}{2}} = \frac{\mathrm{e}^{-\frac{1}{2}}}{1-\mathrm{e}^{-\frac{1}{2}}} = \frac{1}{\sqrt{\mathrm{e}}-1} = \frac{\sqrt{\mathrm{e}}+1}{\mathrm{e}-1},\quad \sum_{k=1}^\infty k^2\mathrm{e}^{-\frac{k}{2}} = \frac{\mathrm{e}+\sqrt{\mathrm{e}}}{(\sqrt{\mathrm{e}}-1)^3}
$$

因此

$$
\sum_{k=1}^n \cos^n\sqrt{\frac{k}{n}} = \frac{\sqrt{\mathrm{e}}+1}{\mathrm{e}-1} - \frac{1}{12n}\frac{\mathrm{e}+\sqrt{\mathrm{e}}}{(\sqrt{\mathrm{e}}-1)^3} + o\left(\frac{1}{n}\right).
$$

因此

$$\lim_{n\to\infty} n\left(\sum_{k=1}^{n}\cos^n\sqrt{\frac{k}{n}}-\frac{\sqrt{e}+1}{e-1}\right)=-\frac{e+\sqrt{e}}{12\left(\sqrt{e}-1\right)^3}$$

事实上用这样的方法,我们还可以作更精细的渐进展开,严格计算其实我们已经利用了 Tannery 定理.

□

255. 计算积分

$$I=\int_0^1\frac{x\ln\left(\frac{1+x}{1-x}\right)}{\left(\pi^2+\ln^2\left(\frac{1+x}{1-x}\right)\right)^2}\mathrm{d}x.$$

先换元 $x\to\tanh x$ 可得

$$\begin{aligned}I&=2\int_0^{\infty}\frac{x\tanh x}{\left(\pi^2+4x^2\right)^2\operatorname{sech}^2 x}\mathrm{d}x=-\frac{1}{4}\int_0^{\infty}\frac{\sinh x}{\cosh^3 x}\mathrm{d}\left(\frac{1}{\pi^2+4x^2}\right)\\&=\frac{1}{4}\int_0^{\infty}\frac{\cosh^2 x-3\sinh^2 x}{\left(\pi^2+4x^2\right)\cosh^4 x}\mathrm{d}x=\frac{1}{8}\int_{-\infty}^{\infty}\frac{\cosh^2 x-3\sinh^2 x}{\left(\pi^2+4x^2\right)\cosh^4 x}\mathrm{d}x.\end{aligned}$$

关键就是化到这一步,下面就套路来了,令 $f(z)=\frac{\sinh^2 z-3\cosh^2 z}{z\sinh^4 z}$, 考虑以 $x=\pm\infty, y=\pm\frac{\pi}{2}\mathrm{i}$ 为四边的逆时针矩形围道,利用留数定理可得

$$\int_{-\infty-\frac{\pi}{2}\mathrm{i}}^{\infty-\frac{\pi}{2}\mathrm{i}} f(x)\,\mathrm{d}x-\int_{-\infty+\frac{\pi}{2}\mathrm{i}}^{\infty+\frac{\pi}{2}\mathrm{i}} f(x)\,\mathrm{d}x=2\pi\mathrm{i}\cdot\operatorname{res}\left(f(z),z=0\right)$$

注意到

$$f\left(x-\frac{\pi}{2}\mathrm{i}\right)=\frac{\sinh^2\left(x-\frac{\pi}{2}\mathrm{i}\right)-3\cosh^2\left(x-\frac{\pi}{2}\mathrm{i}\right)}{\left(x-\frac{\pi}{2}\mathrm{i}\right)\sinh^4\left(x-\frac{\pi}{2}\mathrm{i}\right)}=\frac{3\sinh^3 x-\cosh^2 x}{\left(x-\frac{\pi}{2}\mathrm{i}\right)\cosh^4 x}$$

$$f\left(x+\frac{\pi}{2}\mathrm{i}\right)=\frac{3\sinh^3 x-\cosh^2 x}{\left(x+\frac{\pi}{2}\mathrm{i}\right)\cosh^4 x},\operatorname{res}\left(f(z),z=0\right)=-\frac{1}{15}$$

于是

$$\int_{-\infty}^{\infty}\left(f\left(x-\frac{\pi}{2}\mathrm{i}\right)-f\left(x+\frac{\pi}{2}\mathrm{i}\right)\right)\mathrm{d}x=\pi\mathrm{i}\int_{-\infty}^{\infty}\frac{3\sinh^3 x-\cosh^2 x}{\left(x^2+\frac{\pi^2}{4}\right)\cosh^4 x}\mathrm{d}x=-\frac{2\pi\mathrm{i}}{15}$$

原积分等于 $\frac{1}{240}$.

256. 求和

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(n\sum_{k=n+1}^{\infty}\frac{1}{k^2}-1\right).$$

解 考虑 $a_n=(-1)^{(n-1)}, b_n=\sum_{k=n+1}^{\infty}\frac{1}{k^2}-\frac{1}{n}, T_n=a_1+\cdots+a_n=\frac{1-(-1)^n}{4}-\frac{(-1)^n}{2}n$. 由 Abel 求和公式可得

$$\begin{aligned}S_N&=\sum_{n=1}^{N}(-1)^{n-1}\left(n\sum_{k=n+1}^{\infty}\frac{1}{k^2}-1\right)\\&=\sum_{n=1}^{N}(-1)^{n-1}n\left(\sum_{k=n+1}^{\infty}\frac{1}{k^2}-\frac{1}{n}\right)=T_N b_N+\sum_{n=1}^{N-1}T_n\left(b_n-b_{n+1}\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1-(-1)^N}{4} - \frac{(-1)^N}{2}N\right)\left(\sum_{k=N+1}^{\infty}\frac{1}{k^2} - \frac{1}{N}\right)\\
&\quad - \sum_{n=1}^{N-1}\left(\frac{1-(-1)^n}{4} - \frac{(-1)^n}{2}n\right)\left(\frac{1}{n+1} - \frac{1}{n} + \frac{1}{(n+1)^2}\right).
\end{aligned}
$$

令 $N \to \infty$, 由 Stolz 定理不难得到

$$
\sum_{k=N+1}^{\infty}\frac{1}{k^2} - \frac{1}{N} \sim \frac{1}{2N^2}.
$$

因此

$$
\left(\frac{1-(-1)^N}{4} - \frac{(-1)^N}{2}N\right)\left(\sum_{k=N+1}^{\infty}\frac{1}{k^2} - \frac{1}{N}\right) \to 0.
$$

以及

$$
\begin{aligned}
&-\left(\frac{1-(-1)^n}{4} - \frac{(-1)^n}{2}n\right)\left(\frac{1}{n+1} - \frac{1}{n} + \frac{1}{(n+1)^2}\right)\\
&= \sum_{n=1}^{\infty}\frac{1-(-1)^n}{4n\,(n+1)^2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{2n\,(n+1)}\\
&= \sum_{n=1}^{\infty}\frac{1}{8n^2\,(2n-1)} - \sum_{n=1}^{\infty}\frac{(-1)^n}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{\pi^2}{16} - \frac{\ln 2}{2} - \frac{1}{2}.
\end{aligned}
$$

最后一步利用 $\zeta(2) = \frac{\pi^2}{6}$ 和幂级数即可. □

257. 证明

$$
\int_0^1 \frac{\arctan(x)}{x}\ln\left(\frac{1+x^2}{(1-x)^2}\right)\mathrm{d}x = \frac{\pi^3}{16}
$$

证明 记 $H_k = \sum_{j=1}^{k}\frac{1}{j}, k \geq 1$. 对任意 $x \in (0,1)$ 有

$$
\begin{aligned}
\arctan(x)\ln\left(1+x^2\right) &= \frac{\mathrm{i}}{2}\left(\ln(1-\mathrm{i}x) - \ln(1+\mathrm{i}x)\right)\left(\ln(1-\mathrm{i}x) + \ln(1+\mathrm{i}x)\right)\\
&= \frac{\mathrm{i}}{2}\left(\ln^2(1-\mathrm{i}x) - \ln^2(1+\mathrm{i}x)\right)\\
&= -\mathrm{Im}\left(\ln^2(1-\mathrm{i}x)\right) = -2\mathrm{Im}\left(\sum_{k=1}^{\infty}\frac{H_k\,(\mathrm{i}x)^{k+1}}{k+1}\right).
\end{aligned}
$$

这里我们用到了

$$
\begin{aligned}
-\ln(1-t) = \sum_{k=1}^{\infty}\frac{t^k}{k} &\Rightarrow -\frac{\ln(1-t)}{1-t} = \sum_{k=1}^{\infty}H_k t^k\\
&\Rightarrow \ln^2(1-t) = 2\sum_{k=1}^{\infty}\frac{H_k t^{k+1}}{k+1}.
\end{aligned}
$$

因此,

$$
\int_0^1 \frac{\arctan(x)\ln\left(1+x^2\right)}{x}\mathrm{d}x = -2\mathrm{Im}\left(\int_0^1\sum_{k=1}^{\infty}\frac{H_k \mathrm{i}^{k+1}x^k}{k+1}\mathrm{d}x\right)
$$

$$
\begin{aligned}
&= -2\mathrm{Im}\left(\sum_{k=1}^{\infty} \frac{H_k \mathrm{i}^{k+1}}{k+1} \int_0^1 x^k \mathrm{d}x\right) \\
&= -2\mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{H_k \mathrm{i}^k}{(k+1)^2}\right).
\end{aligned}
$$

另一方面,

$$
\begin{aligned}
\int_0^1 \frac{\arctan(x)\ln(1-x)}{x}\mathrm{d}x &= \int_0^1 \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}\ln(1-x)}{2k+1}\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \int_0^1 x^{2k}\ln(1-x)\,\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \int_0^1 \ln(1-x)\,\mathrm{d}\left(x^{2k+1}-1\right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \int_0^1 \frac{x^{2k+1}-1}{x-1}\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k H_{2k+1}}{(2k+1)^2} = -\mathrm{Re}\left(\sum_{k=0}^{\infty} \frac{H_{k+1}\mathrm{i}^k}{(k+1)^2}\right).
\end{aligned}
$$

因此,

$$
\begin{aligned}
\int_0^1 \frac{\arctan(x)}{x}\ln\left(\frac{1+x^2}{(1-x)^2}\right)\mathrm{d}x &= -2\mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{H_k \mathrm{i}^k}{(k+1)^2}\right) + 2\mathrm{Re}\left(\sum_{k=0}^{\infty} \frac{H_{k+1}\mathrm{i}^k}{(k+1)^2}\right) \\
&= 2\mathrm{Re}\left(\sum_{k=0}^{\infty} \frac{\mathrm{i}^k}{(k+1)^3}\right) = 2\mathrm{Im}\left(\sum_{k=1}^{\infty} \frac{\mathrm{i}^{k-1}}{k^3}\right) \\
&= 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} = \frac{\pi^3}{16}.
\end{aligned}
$$

□