

## 简 介

这是本人收集的大学数学习题集, 都是我自己或别人做过的题目, 当然很多方法都是别人的, 在这里非常感谢高等数学贴吧的吧友提供的题目和方法, 我相当于当了一回搬运工, 把他们的方法整理了一遍. 都是些难度比较高的题目, 题目会比较杂, 涉及到数学分析, 高等代数等, 主要题型是积分、极限、级数, 以及比较难的高等代数题, 题目偏向竞赛性质甚至高于竞赛. 纯粹  $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$  手打, 错误在所难免, 请读者见谅.

# 数学难题

向 禹

1. 设  $f(x)$  在  $(0, \pi)$  连续,  $\int_0^\pi f(x) \cos kx dx = \int_0^\pi f(x) \sin kx dx = 0$  对  $1 \leq k \leq n$  都成立. 证明:  $f(x)$  在  $(0, \pi)$  至少有  $2n$  个零点.

**证明** 由于  $f(x)$  连续, 我们只需要证明  $f(x)$  在  $(0, \pi)$  至少改变  $2n$  次符号. 为此, 只要说明对任何给定  $0 < x_1 < x_2 < \cdots < x_m < \pi, m \leq 2n-1$ , 存在形如

$$f(x) = \sum_{k=1}^n (a_k \cos kx + b_k \sin kx)$$

的函数恰好在上述点的附近改变符号: 比如在  $(0, x_1)$  为正, 在  $(x_1, x_2)$  为负, 在  $(x_2, x_3)$  为正, 在  $(x_3, x_4)$  为负,  $\cdots$ , 若  $m < 2n-1$ , 则记  $x_{m+1} = \cdots = x_{2n-1} = 0$ . 令

$$g(x) = \prod_{k=1}^{2n-1} \sin \frac{x-x_k}{2}$$

我们有

$$g(x) = \prod_{k=1}^{2n-1} \left( \sin \frac{x}{2} \cos \frac{x_k}{2} - \cos \frac{x}{2} \sin \frac{x_k}{2} \right) = C \prod_{k=1}^{2n-1} \left( \sin \frac{x}{2} - c_k \cos \frac{x}{2} \right)$$

其中  $C > 0, c_1, c_2, \cdots, c_{2n-1} \leq 0$ . 我们要证明有  $\alpha \in [0, \pi]$  使得  $f(x) = g(x) \cos \frac{x-\alpha}{2}$  具有形式

$$\sum_{k=1}^n (a_k \cos kx + b_k \sin kx)$$

此时  $f(x)$  满足要求.

用和差化积公式可知  $g(x) \cos \frac{x-\alpha}{2}$  是一个  $n$  阶的三角多项式. 所以只要证明存在  $\alpha \in [0, \pi]$  使得

$$\int_0^{2\pi} g(x) \cos \frac{x-\alpha}{2} dx = 0$$

这只要证明

$$\int_0^{2\pi} g(x) \cos \frac{x}{2} dx = 2 \int_0^\pi g(2x) \cos x dx$$

与

$$\int_0^{2\pi} g(x) \sin \frac{x}{2} dx = 2 \int_0^\pi g(2x) \sin x dx$$

异号 (只要其中有一个为零也认为是异号).

$$g(2x) = C \prod_{k=1}^{2n-1} (\sin x - c_k \cos x) = \sum_{k=1}^{2n-1} (-1)^k \alpha_k \sin^{2n-1-k} \cos^k x$$

其中  $\alpha_k$  非负. 由此立即得到  $\int_0^\pi g(2x) \sin x dx$  非负,  $\int_0^\pi g(2x) \cos x dx$  非正. 综上所述, 结论成立.  $\square$

2. 求极限

$$\lim_{n \rightarrow \infty} n^3 \left( \tan \int_0^\pi \sqrt[n]{\sin x} dx + \sin \int_0^\pi \sqrt[n]{\sin x} dx \right).$$

**解** 当  $x \rightarrow 0$  时,  $\tan x - \sin x \sim \frac{x^3}{2}$ ,  
于是

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^3 \left( \tan \int_0^\pi \sqrt[n]{\sin x} dx + \sin \int_0^\pi \sqrt[n]{\sin x} dx \right) \\ &= \lim_{n \rightarrow \infty} n^3 \left( \tan \int_0^\pi (\sqrt[n]{\sin x} - 1) dx - \sin \int_0^\pi (\sqrt[n]{\sin x} - 1) dx \right) \\ &= \lim_{n \rightarrow \infty} \frac{(n \int_0^\pi (\sqrt[n]{\sin x} - 1) dx)^3}{2} \\ &= \frac{(\int_0^\pi \ln \sin x dx)^3}{2} \\ &= -\frac{(\pi \ln 2)^3}{2} \end{aligned}$$

其中

$$\lim_{n \rightarrow \infty} n \int_0^\pi (\sqrt[n]{\sin x} - 1) dx = \lim_{n \rightarrow \infty} \int_0^\pi \frac{\sqrt[n]{\sin x} - 1}{1/n} dx = \int_0^\pi \ln(\sin x) dx = -\pi \ln 2$$

□

**3.** 把  $f(x) = \cos ax (a \notin \mathbb{Z})$  在  $[-\pi, \pi]$  上展开为 Fourier 级数.

**解** 把  $f$  延拓为整个数轴上的以  $2\pi$  为周期的函数. 记延拓以后的函数为  $\tilde{f}$ , 那么  $\tilde{f}$  是  $(-\infty, +\infty)$  上的周期为  $2\pi$  的连续偶函数. 因此

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi \cos ax \cos nx dx \\ &= \frac{1}{\pi} \int_0^\pi [\cos(a-n)x + \cos(a+n)x] dx \\ &= \frac{1}{\pi} \left[ \frac{\sin(a-n)\pi}{a-n} + \frac{\sin(a+n)\pi}{a+n} \right] \\ &= \frac{(-1)^n 2a \sin a\pi}{\pi(a^2 - n^2)} \quad (n = 0, 1, 2, \dots), \\ b_n &= 0 \quad (n = 1, 2, \dots). \end{aligned}$$

由 Dini 判别法, 即得  $\tilde{f}$  的 Fourier 展开式为

$$\tilde{f}(x) = \frac{\sin a\pi}{\pi} \left[ \frac{1}{a} + \sum_{n=1}^{\infty} (-1)^n \frac{2a}{a^2 - n^2} \cos nx \right] \quad (1)$$

限制在  $[-\pi, \pi]$  上, 就得到

$$\cos ax = \frac{\sin a\pi}{\pi} \left[ \frac{1}{a} + \sum_{n=1}^{\infty} (-1)^n \frac{2a}{a^2 - n^2} \cos nx \right] \quad (2)$$

如果在 (2) 式中取  $x = 0$  可得

$$\frac{\pi}{\sin a\pi} = \frac{1}{a} + \sum_{n=1}^{\infty} (-1)^n \frac{2a}{a^2 - n^2} \quad (a \notin \mathbb{Z}) \quad (3)$$

在 (2) 式中取  $x = \pi$  得

$$\cos a\pi = \frac{\sin a\pi}{\pi} \left[ \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2} \right] \quad (4)$$

再令  $a\pi = t$ , 得到

$$\cot t = \frac{1}{t} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2\pi^2}, t \neq 0, \pm\pi, \pm2\pi, \dots \quad (5)$$

在 (4) 式中令  $a\pi = t$  还可以得到

$$\frac{1}{\sin t} = \frac{1}{t} + \sum_{n=1}^{\infty} (-1)^n \frac{2t}{t^2 - n^2\pi^2}, t \neq 0, \pm\pi, \pm2\pi, \dots \quad (6)$$

□

#### 4. 计算积分

$$\int_0^{\infty} \left( \frac{a}{\sinh ax} - \frac{b}{\sinh bx} \right) \frac{dx}{x}.$$

**解** 首先由例 4 的 (6) 式, 令  $t = iax$ , 可得

$$\frac{a}{\sinh ax} = \frac{1}{x} + 2 \sum_{n=1}^{\infty} (-1)^n \frac{ax}{a^2x^2 + n^2\pi^2}$$

这里运用了公式  $\sinh x = \frac{\sin(ix)}{i}$ . 于是

$$\begin{aligned} I_1 &= \int_0^{\infty} \left( \frac{a}{\sinh ax} - \frac{1}{x} \right) \frac{dx}{x} = 2 \sum_{n=1}^{\infty} \int_0^{\infty} (-1)^n \frac{a^2x}{a^2x^2 + n^2\pi^2} dx \\ &= 2 \sum_{n=1}^{\infty} \frac{(-1)^n a}{n\pi} \arctan \left( \frac{ax}{n\pi} \right) \Big|_0^{\infty} \\ &= 2 \cdot \frac{\pi a}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \\ &= -a \ln 2. \end{aligned}$$

同理有  $I_2 = \int_0^{\infty} \left( \frac{b}{\sinh bx} - \frac{1}{x} \right) dx = -b \ln 2$ , 因此

$$\int_0^{\infty} \left( \frac{a}{\sinh ax} - \frac{b}{\sinh bx} \right) dx = (b - a) \ln 2.$$

□

#### 5. 计算积分

$$I = \int_0^{\infty} \frac{\ln x \ln(1+x)}{1+x^2} dx.$$

**解** 令  $x = \frac{1}{t}$  得

$$I = \int_0^{\infty} \frac{\ln \left( \frac{1}{t} \right) \ln \left( 1 + \frac{1}{t} \right)}{1+t^2} dt = \int_0^{\infty} \frac{\ln^2 t}{1+t^2} - \int_0^{\infty} \frac{\ln t \ln(1+t)}{1+t^2} dt.$$

于是

$$\begin{aligned}
 I &= \frac{1}{2} \int_0^{\infty} \frac{\ln^2 x}{1+x^2} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln^2 \tan x dx \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\ln \sin x - \ln \cos x)^2 dx \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\ln^2 \sin x + \ln^2 \cos x - 2 \ln \sin x \ln \cos x) dx \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (2 \ln^2 \sin x - 2 \ln \sin x \ln \cos x) dx \\
 &= \int_0^{\frac{\pi}{2}} \left( \left( -\ln 2 - \sum_{n=1}^{\infty} \frac{\cos 2nx}{n} \right)^2 - \left( -\ln 2 - \sum_{n=1}^{\infty} \frac{\cos 2nx}{n} \right) \left( -\ln 2 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos 2nx}{n} \right) \right) dx \\
 &= \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} \left( \frac{\cos^2 2nx}{n^2} + (-1)^{n-1} \frac{\cos^2 2nx}{n^2} \right) dx \\
 &= \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} \left( \frac{1 + \cos 4nx}{2n^2} + (-1)^{n-1} \frac{1 + \cos 4nx}{2n^2} \right) dx \\
 &= \frac{\pi}{4} \left( \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \right) \\
 &= \frac{\pi}{4} \left( \frac{\pi^2}{6} + \frac{\pi^2}{12} \right) = \frac{\pi^3}{16}.
 \end{aligned}$$

□

## 6. 计算积分

$$I = \int_0^{\infty} \frac{1}{x^4 - x^2 + 1} \ln^2 \frac{x^2}{x^4 - x^2 + 1} dx.$$

**解** 作变换  $x \rightarrow \frac{1}{x}$  可得

$$I = \int_0^{\infty} \frac{x^2}{x^4 - x^2 + 1} \ln^2 \frac{x^2}{x^4 - x^2 + 1} dx.$$

于是

$$\begin{aligned}
 I &= \frac{1}{2} \int_0^{\infty} \frac{x^2 + 1}{x^4 - x^2 + 1} \ln^2 \frac{x^2}{x^4 - x^2 + 1} dx \\
 &= \frac{1}{2} \int_0^{\infty} \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2} - 1} \ln^2 \left( x^2 + \frac{1}{x^2} - 1 \right) dx \stackrel{t=x-\frac{1}{x}}{=} \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\ln^2(t^2 + 1)}{t^2 + 1} dt \\
 &= \int_0^{\frac{\pi}{2}} \ln^2 \cos^2 u du = 4 \int_0^{\frac{\pi}{2}} \ln^2 \sin u du \\
 &= \frac{\pi^3}{6} + 2\pi \ln^2 2.
 \end{aligned}$$

□

## 7. 计算主值积分

$$\int_0^{\infty} \frac{\sin(\tan x)}{x} dx.$$

**解** 方法一

$$\begin{aligned}
 \int_0^{\infty} \frac{\sin(\tan x)}{x} dx &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sin(\tan x)}{x} dx \\
 &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\tan x) \sum_{k=-\infty}^{+\infty} \frac{1}{x+k\pi} dx \\
 &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\tan x) \left( \frac{1}{x} + 2x \sum_{k=1}^{+\infty} \frac{1}{x^2 - k^2\pi^2} \right) dx \\
 &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(\tan x)}{\tan x} dx \\
 &= \frac{1}{2} \Im \int_{-\infty}^{+\infty} \frac{e^{ix}}{x(x^2+1)} dx \\
 &= \frac{1}{2} \Im \left[ 2\pi i \left( -\frac{1}{2} \text{Res} f(0) + \text{Res} f(i) \right) \right] \\
 &= -\frac{\pi}{2} \frac{e+1}{e}.
 \end{aligned}$$

方法二 考虑围道积分  $\int_C \frac{e^{i \tan z}}{z} dz$ , 其中  $C$  是上半平面内高为  $R$  的矩形围道, 则

$$\text{V.P.} \int_{-\infty}^{\infty} \frac{e^{i \tan z}}{z} dz - \text{V.P.} \int_{-\infty}^{\infty} \frac{e^{i \tan(z+iR)}}{z+iR} dz = \pi i \text{Res}(f(z), z=0)$$

令  $R \rightarrow \infty, \tan(z+iR) \rightarrow i$ , 于是

$$\text{V.P.} \int_{-\infty}^{\infty} \frac{e^{i \tan z}}{z} dz = \pi i (1 - e^{-1}).$$

最后我们得到

$$\int_0^{\infty} \frac{\sin(\tan x)}{x} dx = \frac{\pi}{2} (1 - e^{-1})$$

□

## 8. 计算积分

$$\int_0^1 \frac{\sinh(a \ln x) \ln(1+x) \ln x}{x} dx$$

**解**

$$\begin{aligned}
 I(a) &= \int_0^1 \frac{\cosh(a \ln x) \ln(1+x)}{x} dx \\
 &= \int_0^{\infty} \cosh(ax) \ln(1+e^{-x}) dx \\
 &= \frac{1}{2} \int_0^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left( e^{-(k+a)x} + e^{-(k-a)x} \right) dx \\
 &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2 - a^2} = \frac{1}{2a} \left( \frac{\pi}{\sin \pi a} - \frac{1}{a} \right)
 \end{aligned}$$

这里我们利用了公式  $\frac{1}{\sin x} = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2\pi^2}$ .

因此

$$I'(a) = \int_0^1 \sinh(a \ln x) \frac{\ln(1+x) \ln x}{x} dx = \frac{1}{a^3} - \frac{\pi(1+a\pi \cot a\pi)}{2a^3 \sin \pi a}$$

□

## 9. 计算积分

$$\int_0^{\infty} \frac{\cos x - e^{-x^n}}{x} dx$$

**解** 首先有

$$F(s) = \int_0^{\infty} x^{s-1} (\cos x - e^{-x^n}) dx = \Gamma(s) \cos\left(\frac{\pi s}{2}\right) - \frac{1}{n} \Gamma\left(\frac{s}{n}\right).$$

其中

$$\int_0^{\infty} x^{s-1} \cos x dx = \Re \int_0^{\infty} x^{s-1} e^{ix} dx = \Re [i^s \cdot \Gamma(s)] = \Gamma(s) \cos\left(\frac{\pi s}{2}\right)$$

$$\int_0^{\infty} x^{s-1} e^{-x^n} dx = \frac{1}{n} \int_0^{\infty} x^{\frac{s}{n}-1} e^{-x} dx = \frac{1}{n} \Gamma\left(\frac{s}{n}\right)$$

于是

$$\begin{aligned} \int_0^{\infty} \frac{\cos x - e^{-x^n}}{x} dx &= \lim_{s \rightarrow 0} \left( \Gamma(s) \cos\left(\frac{\pi s}{2}\right) - \frac{1}{n} \Gamma\left(\frac{s}{n}\right) \right) \\ &= \lim_{s \rightarrow 0} \frac{\Gamma(s+1) \cos\left(\frac{\pi s}{2}\right) - \Gamma\left(\frac{s}{n}+1\right)}{s} \\ &= \lim_{s \rightarrow 0} \Gamma'(s+1) \cos\left(\frac{\pi s}{2}\right) - \frac{\pi}{2} \Gamma(s+1) \sin\left(\frac{\pi s}{2}\right) - \frac{1}{n} \Gamma'\left(\frac{s}{n}+1\right) \\ &= \left(1 - \frac{1}{n}\right) \Gamma'(1) \\ &= -\left(1 - \frac{1}{n}\right) \gamma \end{aligned}$$

这里  $\Gamma'(1) = -\gamma$ ,  $\Gamma(z) = \frac{1}{z} - \gamma + \frac{1}{6} \left(3\gamma^2 + \frac{\pi^2}{2}\right) + O(z^2)$ .

□

## 10. 计算积分

$$\int_0^{\infty} e^{-ax^2 - \frac{b}{x^2}} dx$$

**解**

$$\begin{aligned} \int_0^{\infty} e^{-ax^2 - \frac{b}{x^2}} dx &= \int_0^{\infty} e^{-(ax^2 + \frac{b}{x^2})} dx \\ &= \int_0^{\infty} e^{-\sqrt{ab} \left( \sqrt{\frac{a}{b}} x^2 + \sqrt{\frac{b}{a}} \frac{1}{x^2} \right)} dx \\ &= \sqrt[4]{\frac{b}{a}} \int_0^{\infty} e^{-\sqrt{ab} \left[ \left( \sqrt[4]{\frac{a}{b}} x \right)^2 + \left( \frac{1}{x} \sqrt[4]{\frac{b}{a}} \right)^2 \right]} d \left( \sqrt[4]{\frac{a}{b}} x \right) \\ &= \sqrt[4]{\frac{b}{a}} \left( \int_0^1 + \int_1^{\infty} \right) e^{-\sqrt{ab} \left( x^2 + \frac{1}{x^2} \right)} dx \end{aligned}$$



$$\begin{aligned}
 &= \sqrt[4]{\frac{b}{a}} \left( \int_1^\infty e^{-\sqrt{ab}(x^2 + \frac{1}{x^2})} \frac{1}{x^2} dx + \int_1^\infty e^{-\sqrt{ab}(x^2 + \frac{1}{x^2})} dx \right) \\
 &= \sqrt[4]{\frac{b}{a}} \int_1^\infty e^{-\sqrt{ab}(x^2 + \frac{1}{x^2})} \left( 1 + \frac{1}{x^2} \right) dx \\
 &= \sqrt[4]{\frac{b}{a}} \int_1^\infty e^{-\sqrt{ab}[(x - \frac{1}{x})^2 + 2]} d\left(x - \frac{1}{x}\right) \\
 &= \frac{1}{2} \sqrt[4]{\frac{b}{a}} e^{-2\sqrt{ab}} \int_0^\infty e^{-\sqrt{ab}x^2} dx \\
 &= \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}
 \end{aligned}$$

□

## 11. 计算积分

$$\int_0^\infty e^{-ax} \sin^{2n} x dx$$

**解** 由两次分部积分可得

$$\int_0^\infty e^{-ax} \sin^{2n} x dx = \frac{2n(2n-1)}{a^2 + 4n^2} \int_0^\infty e^{-ax} \sin^{2n-2} x dx$$

而  $\int_0^\infty e^{-ax} dx = \frac{1}{a}$ , 于是

$$\int_0^\infty e^{-ax} \sin^{2n} x dx = \frac{(2n)!}{a(a^2 + 2^2) \cdots (a^2 + 4n^2)}$$

□

## 12. 计算积分

$$\int_0^\infty \left( \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n)!!} \right) \left( \sum_{n=0}^\infty \frac{x^{2n}}{((2n)!!)^2} \right) dx$$

**解** 因为

$$\begin{aligned}
 \left( \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n)!!} \right) dx &= \frac{1}{2} \sum_{n=0}^\infty \frac{1}{n!} \left( -\frac{x^2}{2} \right)^n dx^2 \\
 &= \frac{1}{2} e^{-\frac{x^2}{2}} dx^2
 \end{aligned}$$

所以原积分

$$\begin{aligned}
 I &= \frac{1}{2} \int_0^\infty e^{-\frac{x^2}{2}} \sum_{n=0}^\infty \frac{(x^2)^n}{(2^2)^n (n!)^2} dx^2 = \int_0^\infty e^{-t} \sum_{n=0}^\infty \frac{t^n}{2^n (n!)^2} dt \\
 &= \sum_{n=0}^\infty \frac{\Gamma(n+1)}{2^n (n!)^2} = \sum_{n=0}^\infty \frac{1}{2^n n!} = e^{\frac{1}{2}}
 \end{aligned}$$

□

### 13. 计算积分

$$\int_0^{2\pi} e^{\cos x} \cos(\sin x - x) dx$$

解

$$\begin{aligned} \int_0^{2\pi} e^{\cos x} \cos(\sin x - x) dx &= \int_0^{2\pi} e^{\cos x} \cos(x - \sin x) dx \\ &= \Re \int_0^{2\pi} e^{\cos x} e^{i(x - \sin x)} dx \\ &= \Re \int_0^{2\pi} e^{e^{-ix}} e^{ix} dx \\ &= \Re \left( \frac{1}{i} \int_{|z|=1} e^{\frac{1}{z}} dz \right) \\ &= \Re \left( 2\pi i \cdot \frac{1}{i} \operatorname{Res} \left( e^{\frac{1}{z}}, z = 0 \right) \right) \\ &= 2\pi \end{aligned}$$

□

### 14. 计算积分

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos(\tan x - x)}{\cos x} dx$$

解

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{\cos(\tan x - x)}{\cos x} dx \\ &= \int_0^{\frac{\pi}{2}} [\cos(\tan x) + \tan x \sin(\tan x)] dx \\ &= \int_0^{\infty} \frac{\cos t + t \sin t}{1 + t^2} dt \quad (t = \tan x) \\ &= 2\pi i \cdot \operatorname{Res} \left( \frac{\cos t + t \sin t}{1 + t^2}, t = i \right) \\ &= \frac{\pi}{e} \end{aligned}$$

□

### 15. 计算积分

$$I(a) = \int_0^{\infty} \frac{\sin x}{\cosh ax + \cos x} \frac{x}{x^2 - \pi^2} dx$$

解 根据

$$\sum_{k=1}^{\infty} (-1)^{k+1} e^{-akx} \sin kx = \frac{1}{2} \frac{\sin x}{\cosh ax + \cos x}$$

得

$$I(a) = \int_0^{\infty} \frac{\sin x}{\cosh ax + \cos x} \frac{x}{x^2 - \pi^2} dx$$

$$\begin{aligned}
 &= 2 \sum_{k=1}^{\infty} (-1)^{k+1} \left( \int_0^{\infty} \frac{x}{x^2 - \pi^2} e^{-akx} \sin kx dx \right) \\
 &= 2 \sum_{k=1}^{\infty} (-1)^{k+1} \left( \int_0^{\infty} \frac{t}{t^2 - k^2 \pi^2} e^{-at} \sin t dt \right) \\
 &= \int_0^{\infty} \left( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2t}{t^2 - k^2 \pi^2} \right) e^{-at} \sin t dt \\
 &= \int_0^{\infty} \left( \frac{1}{t} - \csc t \right) e^{-at} \sin t dt \\
 &= \arctan \left( \frac{1}{a} \right) - \frac{1}{a}
 \end{aligned}$$

□

## 16. 计算积分

$$\int_0^{\infty} e^{-x^2} \ln x dx$$

解

$$\begin{aligned}
 \int_0^{\infty} e^{-x^2} \ln x dx &= \frac{1}{4} \int_0^{\infty} e^{-t} t^{-\frac{1}{2}} \ln t dt \\
 &= \frac{1}{4} \lim_{s \rightarrow 1} \partial_s \int_0^{\infty} e^{-t} t^{s-1-\frac{1}{2}} dt \\
 &= \frac{1}{4} \lim_{s \rightarrow 1} \partial_s \Gamma \left( s - \frac{1}{2} \right) \\
 &= \frac{1}{4} \lim_{s \rightarrow 1} \Gamma \left( s - \frac{1}{2} \right) \psi \left( s - \frac{1}{2} \right) \\
 &= \frac{1}{4} \Gamma \left( \frac{1}{2} \right) \psi \left( \frac{1}{2} \right) \\
 &= \frac{\sqrt{\pi}}{4} (\gamma + 2 \ln 2)
 \end{aligned}$$

□

## 17. 计算积分

$$I(a) = \int_0^{\infty} \frac{\cos x}{\tanh ax} dx$$

解 首先有

$$I'(a) = \int_0^{\infty} \frac{\cos x}{\cosh^2 ax} dx$$

考查围道积分

$$\int_C \frac{e^{iz}}{\cosh^2 az} dz$$

其中  $C$  为上半平面内高为  $\frac{2\pi i}{a}$  的围道, 那么有

$$\left( 1 - e^{-\frac{2\pi i}{a}} \right) \int_{-\infty}^{\infty} \frac{e^{iz}}{\cosh^2 az} dz = 2\pi i \left( \operatorname{Res} \left( f(z), z = \frac{\pi i}{2a} \right) + \operatorname{Res} \left( f(z), z = \frac{3\pi i}{2a} \right) \right)$$

于是

$$I'(a) = \frac{1}{2} \Re \int_{-\infty}^{\infty} \frac{e^{iz}}{\cosh^2 az} dz = \frac{\pi}{2n^2 \sinh\left(\frac{\pi}{2n}\right)}$$

最后得到

$$I(a) = \ln \coth\left(\frac{\pi}{2n}\right)$$

□

18. 设  $a > 0, b > 0$ , 计算积分

$$\int_0^{\infty} \frac{\cos ax}{x^2 + b^2} \left(\frac{\sin x}{x}\right)^n dx$$

解 根据留数定理

$$\begin{aligned} \int_0^{\infty} \frac{\cos ax}{x^2 + b^2} \left(\frac{\sin x}{x}\right)^n dx &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + b^2} \left(\frac{\sin x}{x}\right)^n dx \\ &= \frac{1}{2} \Re \left( 2\pi i \operatorname{Res} \left( \frac{e^{iaz}}{z^2 + b^2} \left(\frac{\sin z}{z}\right)^n, bi \right) \right) \\ &= \Re \left( \pi i \lim_{z \rightarrow bi} (z - bi) \frac{e^{iaz}}{z^2 + b^2} \left(\frac{\sin z}{z}\right)^n \right) \\ &= \frac{\pi e^{-ab}}{2b^{n+1}} \sinh^n b \left( \sin bi = \frac{e^{ibi} - e^{-ibi}}{2i} = -\frac{\sinh b}{i} \right) \end{aligned}$$

□

19. 计算积分

$$\int_0^1 \frac{\ln(x + \sqrt{1-x^2})}{x} dx$$

解 考虑积分

$$I(t) = \int_0^1 \frac{\ln(tx + \sqrt{1-x^2})}{x} dx$$

那么

$$\begin{aligned} I(0) &= \int_0^1 \frac{\ln(\sqrt{1-x^2})}{x} dx \\ &= \frac{1}{2} \left( \int_0^1 \frac{\ln(1+x)}{x} dx + \int_0^1 \frac{\ln(1-x)}{x} dx \right) \\ &= \frac{1}{2} \left( \frac{\pi^2}{12} - \frac{\pi^2}{6} \right) = -\frac{\pi^2}{24} \end{aligned}$$

而

$$I'(t) = \int_0^1 \frac{1}{tx + \sqrt{1-x^2}} d\theta = \int_0^{\infty} \frac{\cos \theta}{t \sin \theta + \cos \theta} d\theta = \frac{\pi}{2} \frac{1}{1+t^2} + \frac{t \ln t}{1+t^2}$$

上式对  $t$  积分得

$$I(t) = \frac{\pi}{2} \arctan t + \frac{1}{2} \ln(1+t^2) \ln t - \frac{1}{2} \int_0^t \frac{\ln(1+x^2)}{x} dx + C$$

其中

$$C = I(0) = -\frac{\pi^2}{24}, I = I(1) = \frac{\pi^2}{8} + 0 - \frac{1}{2} \cdot \frac{\pi^2}{24} - \frac{\pi^2}{24} = \frac{\pi^2}{16}$$

□

## 20. 计算不定积分

$$\int \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

解

$$\begin{aligned} \int \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx &= \int \frac{\sqrt{\tan x}}{\sqrt{\tan x} + 1} dx \\ &= \int \frac{2u^2}{(1+u)(1+u^4)} du \quad (u = \sqrt{\tan x}) \\ &= \int \left( \frac{1}{1+u} + \frac{-u^3 + u^2 + u - 1}{1+u^4} \right) dx \\ &= \ln(1+u) - \frac{1}{4} \ln(1+u^4) + \int \frac{d(u + \frac{1}{u})}{(u + \frac{1}{u})^2 - 1} + \frac{1}{2} \int \frac{d(u^2)}{1+(u^2)^2} \\ &= \ln(1+u) - \frac{1}{4} \ln(1+u^4) + \frac{1}{2} \ln \left( \frac{u^2 - u + 1}{u^2 + u + 1} \right) + \frac{1}{2} \arctan u^2 + C \\ &= \ln(1 + \sqrt{\tan x}) - \frac{1}{4} \ln(1 + \tan^2 x) + \frac{1}{2} \ln \left( \frac{\tan x - \sqrt{\tan x} + 1}{\tan^2 x + \sqrt{\tan x} + 1} \right) + \frac{1}{2} + C \end{aligned}$$

□

## 21. 计算积分

$$\iint_{x,y \geq 0} ye^{-(x+y)} \sin \left( \ln \frac{x}{y} \right) dx dy$$

解

$$\begin{aligned} \iint_{x,y \geq 0} ye^{-(x+y)} \sin \left( \ln \frac{x}{y} \right) dx dy &= \Im \iint_{x,y \geq 0} ye^{-(x+y)} e^{i \ln \left( \frac{x}{y} \right)} dx dy \\ &= \Im \iint_{x,y \geq 0} ye^{-(x+y)} \left( \frac{x}{y} \right)^i dx dy \\ &= \Im \int_0^\infty x^i e^{-x} dx \int_0^\infty y^{1-i} e^{-y} dy \\ &= \Im (\Gamma(1+i) \Gamma(2-i)) \\ &= \Im ((1-i)i \Gamma(i) \Gamma(1-i)) \\ &= \Im \left( (1-i)i \frac{\pi}{\sin(\pi i)} \right) \\ &= -\frac{\pi}{\sinh \pi} \end{aligned}$$

□

## 22. 计算积分

$$I(a) = \int_0^\infty \frac{\arctan x}{e^{ax} - 1} dx$$

解

$$\begin{aligned}
 I(a) &= \int_0^{\infty} \frac{\arctan x}{e^{ax} - 1} dx \\
 &= \int_0^{\infty} \arctan x \sum_{n=0}^{\infty} e^{-(n+1)ax} dx \\
 &= \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nax} \arctan x dx \\
 &= \frac{1}{a} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-nax}}{1+x^2} dx \\
 &= \frac{1}{a} \int_0^{\infty} e^{-au} \sum_{n=1}^{\infty} \frac{1}{n^2 + u^2} du \\
 &= \frac{1}{a} \int_0^{\infty} e^{-au} \frac{\coth(\pi u) - \frac{1}{\pi u}}{2u/\pi} du \\
 &= \frac{\pi}{2a} \int_0^{\infty} \left( \frac{1}{u} e^{-\frac{au}{\pi}} \right) \left( \coth u - \frac{1}{u} \right) du \\
 &= \frac{\pi}{2a} \int_0^{\infty} \left( \int_{\frac{a}{\pi}}^{\infty} e^{-us} ds \right) \left( \coth u - \frac{1}{u} \right) du \\
 &= \frac{\pi}{2a} \int_{\frac{a}{\pi}}^{\infty} \left[ \int_0^{\infty} e^{-us} \left( \coth u - \frac{1}{u} \right) du \right] ds
 \end{aligned}$$

而

$$\begin{aligned}
 &\int_0^{\infty} e^{-us} \left( \coth u - \frac{1}{u} \right) du \\
 &= \int_0^{\infty} e^{-us} \left( \frac{1}{u} - \frac{1+e^{-2u}}{1-e^{-2u}} \right) du \\
 &= \int_0^{\infty} e^{-\frac{sw}{2}} \left( \frac{1}{w} - \frac{1+e^{-w}}{1-e^{-w}} \right) dw \\
 &= \int_0^{\infty} \left[ \frac{e^{-w} - (e^{-w} - e^{-sw/2})}{w} - \frac{1}{2} \frac{e^{-sw/2} + e^{-sw/2} - e^{-sw/2} + e^{-(1+s/2)w}}{1-e^{-w}} \right] dw \\
 &= \int_0^{\infty} \left( \frac{e^{-w}}{w} - \frac{e^{-ws/2}}{1-e^{-w}} \right) dw + \int_0^{\infty} \frac{e^{-ws/2}}{2} dw + \int_0^{\infty} \frac{e^{-ws/2} - e^{-w}}{w} dw \\
 &= \psi\left(\frac{s}{2}\right) + \frac{1}{s} - \ln\left(\frac{s}{2}\right)
 \end{aligned}$$

因此

$$\begin{aligned}
 I(a) &= -\frac{\pi}{2a} \int_{\frac{a}{\pi}}^{\infty} \left[ \psi\left(\frac{s}{2}\right) + \frac{1}{s} - \ln\left(\frac{s}{2}\right) \right] ds \\
 &= \frac{\pi}{2a} \left[ 2 \ln \Gamma\left(\frac{s}{2}\right) + \ln(s) - s \ln\left(\frac{s}{2}\right) + s \right]_{\frac{a}{\pi}}^{\infty} \\
 &= \frac{\pi}{a} \ln \Gamma\left(\frac{a}{2\pi}\right) + \frac{\pi}{2a} \ln\left(\frac{a}{\pi}\right) - \frac{1}{2} \ln\left(\frac{a}{2\pi}\right) + \frac{1}{2} - \ln(4\pi)
 \end{aligned}$$

□

### 23. 计算不定积分

$$\int \left( \frac{\arctan x}{x - \arctan x} \right)^2 dx$$

解

$$\begin{aligned} \int \left( \frac{\arctan x}{x - \arctan x} \right)^2 dx &= \int \frac{t^2}{(\tan t - t)^2} \sec^2 t dt \\ &= \int \frac{t^2}{(\sin t - t \cos t)^2} dt \\ &= \int \left( -\frac{t}{\sin t} \right) \left( -\frac{t \sin t}{(\sin t - t \cos t)^2} \right) dt \\ &= -\frac{t}{\sin t} \frac{1}{\sin t - t \cos t} + \int \frac{dt}{\sin^2 t} \\ &= -\frac{(1 + \tan^2 t)t}{\tan t (\tan t - t)^2} - \frac{1}{\tan t} + C \\ &= -\frac{(1 + x^2) \arctan x}{x(x - \arctan x)} - \frac{1}{x} + C \\ &= -\frac{1 + x \arctan x}{x - \arctan x} + C \end{aligned}$$

□

### 24. 计算积分

$$I = \int_0^\infty \frac{x^n \sin x}{\cosh x - \cos x} dx$$

解 根据

$$\frac{\sin x}{\cosh x - \cos x} = 2 \sum_{k=1}^{\infty} e^{-kx} \sin kx$$

可得

$$\begin{aligned} I &= \int_0^\infty \sum_{k=1}^{\infty} x^n e^{-kx} \sin kx dx \\ &= \frac{\sin \left( \frac{(n+1)\pi}{4} \right) \Gamma(n+1)}{2^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} \frac{1}{k^{n+1}} \\ &= \frac{1}{2^{\frac{n+1}{2}}} \sin \left( \frac{(n+1)\pi}{4} \right) \Gamma(n+1) \zeta(n+1) \end{aligned}$$

□

### 25. 首先有恒等式

$$\begin{aligned} \sum_{k=1}^{\infty} a^k \cos(kx) &= \Re \sum_{k=1}^{\infty} a^k e^{ikx} \\ &= \Re \frac{e^{ix + \ln a}}{1 - e^{ix + \ln a}} \\ &= \Re \frac{ae^{ix}(1 - ae^{-ix})}{(1 - ae^{ix})(1 - ae^{-ix})} \end{aligned}$$

$$= \Re \frac{ae^{ix-a^2}}{1-2a\cos x+a^2}$$

$$= \frac{a\cos x - a^2}{1-2a\cos x+a^2}$$

于是

$$\int_0^\infty \frac{a - \cos x}{(1 - 2a \cos x + a^2)(1 + x^2)} dx = -\frac{1}{a} \int_0^\infty \sum_{k=1}^\infty a^k \frac{\cos(kx)}{1 + x^2} dx = -\frac{\pi}{2a} \sum_{k=1}^\infty \frac{a^k}{e^k} = \frac{e\pi}{2(a-e)}$$

26. 设  $0 < a < 1$ , 计算积分

$$I = \int_{-\infty}^\infty \frac{\sinh^2 ax}{\sinh^2 x} dx$$

解 首先有

$$\int_{-\infty}^\infty \frac{\sinh^2 ax}{\sinh^2 x} dx = 2 \int_{-\infty}^0 \frac{(e^{ax} - e^{-ax})^2 e^{2x}}{(e^{2x} - 1)^2} dt = \int_0^1 \frac{t^a - t^{-a} - 2}{(1-t)^2} dt$$

考虑积分

$$\begin{aligned} I(s) &= \int_0^1 (t^a - t^{-a} - 2)(1-t)^{s-2} dt \\ &= B(a+1, s-1) + B(-a+1, s-1) - 2B(1, s-1) \\ &= \frac{\Gamma(a+1)\Gamma(s-1)}{\Gamma(a+s)} + \frac{\Gamma(-a+1)\Gamma(s-1)}{\Gamma(-a+s)} - \frac{2\Gamma(s-1)}{\Gamma(s)} \\ &= \frac{\Gamma(s+1)}{s-1} \left( \frac{\Gamma(a+1) - \Gamma(a+s)}{s\Gamma(a+s)} + \frac{\Gamma(-a+1) - \Gamma(-a+s)}{s\Gamma(-a+s)} \right) \end{aligned}$$

其中  $\Gamma(s+1) = s\Gamma(s) = s(s-1)\Gamma(s-1)$ .

令  $s \rightarrow 0$  得

$$\begin{aligned} I &= I(0) = -\left( \frac{\Gamma'(a+1)}{\Gamma(a)} + \frac{\Gamma'(-a+1)}{\Gamma(-a)} \right) \\ &= a(\psi(-a+1) - \psi(a+1)) \\ &= a(\psi(a) - \psi(a+1) - \pi \cot a\pi) \\ &= 1 - a\pi \cot a\pi \end{aligned}$$

□

27. 计算积分

$$I = \int_0^\infty \frac{e^{-t} \cosh(a\sqrt{t})}{\sqrt{t}} dt$$

解

$$\begin{aligned} I &= \int_0^\infty \frac{e^{-t} \cosh(a\sqrt{t})}{\sqrt{t}} dt \\ &= 2 \int_0^\infty e^{-t^2} \cosh(at) dt = \int_0^\infty e^{-t^2} (e^{at} + e^{-at}) dt \end{aligned}$$



$$\begin{aligned}
 &= \int_0^\infty (e^{-t^2+at} + e^{-t^2-at}) dt \\
 &= \int_0^\infty e^{\frac{a^2}{4} - (t-\frac{a}{2})^2} dt + \int_0^\infty e^{\frac{a^2}{4} - (t+\frac{a}{2})^2} dt \\
 &= e^{\frac{a^2}{4}} \left( \int_0^\infty e^{-(t-\frac{a}{2})^2} dt + \int_0^\infty e^{-(t+\frac{a}{2})^2} dt \right) \\
 &= e^{\frac{a^2}{4}} \left( \int_{-\frac{a}{2}}^\infty e^{-x^2} dx + \int_{\frac{a}{2}}^\infty e^{-x^2} dx \right) \\
 &= e^{\frac{a^2}{4}} \left( \int_{-\infty}^{\frac{a}{2}} e^{-x^2} dx + \int_{\frac{a}{2}}^\infty e^{-x^2} dx \right) \\
 &= e^{\frac{a^2}{4}} \int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi} e^{\frac{a^2}{4}}
 \end{aligned}$$

□

## 28. 计算积分

$$I = \int_0^\infty e^{-ax} \left( \frac{1}{x} - \coth x \right) dx$$

解

$$\begin{aligned}
 I &= \int_0^\infty e^{-ax} \left( \frac{1}{x} - \coth x \right) dx \\
 &= a \int_0^\infty e^{-ax} \ln x dx - a \int_0^\infty e^{-ax} \ln(\sinh x) dx \\
 &= -\gamma - \ln a - a \int_0^\infty e^{-ax} \ln \left( \frac{e^x - e^{-x}}{2} \right) dx \\
 &= -\gamma - \ln a - a \int_0^\infty x e^{-ax} dx - a \ln 2 \int_0^\infty e^{-ax} dx - a \int_0^\infty e^{-ax} \ln(1 - e^{-2x}) dx \\
 &= -\gamma - \frac{1}{a} + \ln \left( \frac{2}{a} \right) - \sum_{k=1}^\infty \frac{a}{k} \int_0^\infty e^{-2kx - ax} dx \\
 &= -\gamma - \frac{1}{a} + \ln \left( \frac{2}{a} \right) - \sum_{k=0}^\infty \frac{a}{k(a+2k)} \\
 &= \psi \left( 1 + \frac{a}{2} \right) - \frac{1}{a} + \ln \left( \frac{2}{a} \right) \\
 &= \psi \left( \frac{a}{2} \right) + \ln \left( \frac{2}{a} \right) + \frac{1}{a}
 \end{aligned}$$

□

## 29. 计算积分

$$I = \int_0^1 \frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)(-\ln x)} dx$$

解

$$\begin{aligned}
 I &= \int_0^1 \frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)(-\ln x)} dx \\
 &= \int_0^1 \frac{(1-x^a)(1-x^b)}{1-x} dx \int_0^c x^y dy
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^c dy \int_0^1 \frac{x^y(1-x^a-x^b+x^{a+b})}{1-x} dx \\
 &= \int_0^c dy \left( \int_0^1 \frac{x^y}{1-x} dx - \int_0^1 \frac{a+y}{1-x} dx - \int_0^1 \frac{b+y}{1-x} dx + \int_0^1 \frac{a+b+y}{1-x} dx \right) \\
 &= \int_0^c dy \int_0^1 \sum_{n=0}^{\infty} (x^{y+n} - x^{a+y+n} - x^{b+y+n} + x^{a+b+y+n}) dx \\
 &= \int_0^c \sum_{n=0}^{\infty} \left( \frac{1}{y+n+1} - \frac{1}{a+y+n+1} - \frac{1}{b+y+n+1} + \frac{1}{a+b+y+n+1} \right) dy \\
 &= \int_0^c \sum_{n=0}^{\infty} [\psi(a+y+1) + \psi(b+y+1) - \psi(y+1) - \psi(a+b+y+1)] dy \\
 &= \ln \frac{\Gamma(b+c+1)\Gamma(a+c+1)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)\Gamma(a+b+c+1)}
 \end{aligned}$$

□

### 30. 计算积分

$$I = \int_{\frac{1}{2}}^1 \frac{\ln(2x-1)}{\sqrt[6]{x(1-x)(1-2x)^4}} dx$$

解

$$\begin{aligned}
 I &= \int_{\frac{1}{2}}^1 \frac{\ln(2x-1)}{\sqrt[6]{x(1-x)(1-2x)^4}} dx \stackrel{t=(2x-1)^2}{=} 4^{-\frac{4}{3}} \int_0^1 (1-t)^{-\frac{1}{6}} t^{-\frac{5}{6}} \ln t dt \\
 &= 4^{-\frac{4}{3}} \lim_{s \rightarrow 0} \partial_s \int_0^1 (1-t)^{-\frac{1}{6}} t^{s-\frac{5}{6}} dt \\
 &= 4^{-\frac{4}{3}} \lim_{s \rightarrow 0} \partial_s B\left(\frac{5}{6}, s + \frac{1}{6}\right) \\
 &= 4^{-\frac{4}{3}} \lim_{s \rightarrow 0} \partial_s \frac{\Gamma\left(\frac{5}{6}\right) \Gamma\left(s + \frac{1}{6}\right)}{\Gamma(1+s)} \\
 &= 4^{-\frac{4}{3}} \Gamma\left(\frac{5}{6}\right) \lim_{s \rightarrow 0} \frac{\Gamma\left(s + \frac{1}{6}\right)}{\Gamma(1+s)} \left[ \psi\left(s + \frac{1}{6}\right) - \psi(1+s) \right] \\
 &= 4^{-\frac{4}{3}} \Gamma\left(\frac{5}{6}\right) \Gamma\left(\frac{1}{6}\right) \left[ \psi\left(\frac{1}{6}\right) - \psi(1) \right] \\
 &= 4^{-\frac{4}{3}} \pi \csc\left(\frac{\pi}{6}\right) \left[ \psi\left(\frac{1}{6}\right) + \gamma \right] \\
 &= \frac{\pi}{\sqrt[3]{256}} \left[ 2\psi\left(\frac{1}{6}\right) + 2\gamma \right]
 \end{aligned}$$

□

### 31. 求极限

$$\lim_{n \rightarrow \infty} \sqrt{n} \int_0^{\infty} \cos^{2n-1} x e^{-\pi x} dx$$

解 首先有

$$\int_0^{\infty} \cos(nx) e^{-\pi x} dx = \frac{\pi}{n^2 + \pi^2}$$

接下来求出  $\cos^{2n-1} x$  的 Fourier 余弦级数即可.

$$\cos^{2n-1} x = \frac{2}{4^n} (e^{ix} + e^{-ix})^{2n-1} = \frac{1}{4^{n-1}} \sum_{k=0}^n \binom{2n-1}{k} \cos((2n-2k-1)x)$$

于是

$$\int_0^\infty \cos^{2n-1} x e^{-\pi x} dx = \frac{1}{4^{n-1}} \sum_{k=1}^n \binom{2n-1}{k} \frac{\pi}{(2k-1)^2 + \pi^2}$$

对于每个固定的  $0 \leq k \leq n$ , 根据 Stirling 公式我们有

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{4^{n-1}} \binom{2n-1}{k} = \frac{2}{\sqrt{\pi}}$$

因此原来的极限为

$$L = 2\sqrt{\pi} \sum_{k=1}^{\infty} \frac{1}{\pi^2 + (2k-1)^2} = \frac{\sqrt{\pi}}{2} \tanh\left(\frac{\pi^2}{2}\right)$$

□

**32.** 对固定的正整数  $n$ , 计算积分

$$I_n = \int_1^\infty \frac{dx}{x(x+1)(x+2)\cdots(x+n)}$$

**解** 用有理函数的部分分式法, 令

$$\frac{1}{x(x+1)(x+2)\cdots(x+n)} = \sum_{k=0}^n \frac{A_k}{x+k}$$

于是得到

$$1 = \sum_{k=0}^n A_k x(x+1)\cdots(x+k-1)(x+k+1)\cdots(x+n)$$

令  $x = -k$  得到

$$1 = A_k [(-k)(-k+1)\cdots(-k+k+1)][(-k+k+1)\cdots(-k+k+n)] = A_k [(-1)^k k!][(n-k)!]$$

于是  $A_k = \frac{(-1)^k}{n!} \binom{n}{k}$ , 原积分

$$\begin{aligned} I_n &= \int_1^\infty \frac{dx}{x(x+1)(x+2)\cdots(x+n)} \\ &= \int_1^\infty \sum_{k=0}^n \frac{A_k}{x+k} dx \\ &= \sum_{k=0}^n A_k \ln(x+k) \Big|_1^\infty \\ &= \sum_{k=0}^n \frac{(-1)^k}{n!} \ln(x+k) \Big|_1^\infty \end{aligned}$$

而  $\lim_{x \rightarrow \infty} \sum_{k=0}^n \frac{(-1)^k}{n!} \ln|x+k| = 0$ , 因此  $I_n = \sum_{k=1}^n \frac{(-1)^{k-1}}{n!} \ln|k+1|$ .

□

33. 设  $a > b > 0$ , 计算积分

$$\int_0^\pi \ln(a + b \cos x) dx$$

解 记  $I(b) = \int_0^\pi \ln(a + b \cos x) dx$ , 那么

$$\begin{aligned} I'(b) &= \int_0^\pi \frac{\cos x}{a + b \cos x} dx \\ &= \frac{1}{b} - \frac{a}{b} \int_0^\pi \frac{dx}{a + b \cos x} \\ &= \frac{\pi}{b} - \frac{2a}{b} \int_0^\infty \frac{dt}{(a+b) + (a-b)t^2} \quad (t = \tan(x/2)) \\ &= \frac{\pi}{b} - \frac{2a}{b} \cdot \frac{1}{\sqrt{a^2 - b^2}} \arctan \left( \sqrt{\frac{a-b}{a+b}} t \right) \Big|_0^\infty \\ &= \frac{\pi}{b} - \frac{2a}{b} \cdot \frac{1}{\sqrt{a^2 - b^2}} \cdot \frac{\pi}{2} = \frac{\pi}{2} \\ &= \frac{\pi}{b} - \frac{\pi a}{b \sqrt{a^2 - b^2}} \end{aligned}$$

□

34. 计算积分

$$I = \int_0^1 \frac{\ln 2 - \ln(1 + \sqrt{1-x^2})}{x} dx$$

解

$$\begin{aligned} I &= \int_0^1 \frac{\ln 2 - \ln(1 + \sqrt{1-x^2})}{x} dx \\ &= \int_0^1 \frac{1}{x} \ln \left( \frac{2}{1 + \sqrt{1-x^2}} \right) dx \quad \left( u = \frac{2}{1 + \sqrt{1-x^2}} \right) \\ &= \int_1^2 \frac{u}{2\sqrt{u-1}} \ln u \cdot \frac{2-u}{u^2 \sqrt{u-1}} du \\ &= \frac{1}{2} \int_1^2 \frac{2-u}{u(u-1)} \ln u du \\ &= \frac{1}{2} \int_1^2 \frac{\ln u}{u-1} du - \int_1^2 \frac{\ln u}{u} du \\ &= \frac{1}{2} \int_0^1 \frac{\ln(1+t)}{t} dt - \frac{\ln^2 2}{2} \\ &= \frac{\pi^2}{24} - \frac{\ln^2 2}{2} \end{aligned}$$

□

35. 计算积分

$$\int_0^1 \ln \Gamma(x) \cos(2n\pi x) dx$$

解

$$\int_0^1 \ln \Gamma(x) \cos(2n\pi x) dx = \int_0^1 \ln \Gamma(1-x) \cos(2n\pi x) dx$$

$$\begin{aligned}
 &= \frac{1}{2} \int_0^1 [\ln \Gamma(1-x) + \ln \Gamma(x)] \cos(2n\pi x) dx \\
 &= \frac{1}{2} \int_0^1 \ln \left( \frac{\pi}{\sin(\pi x)} \right) \cos(2n\pi x) dx \\
 &= \frac{1}{4n\pi} \int_0^\pi (\ln \pi - \ln \sin x) d(\sin(2nx)) \\
 &= -\frac{\ln \sin x}{4n\pi} \sin(2n\pi) \Big|_0^\pi + \frac{1}{4n\pi} \int_0^\pi \frac{\sin(2nx) \cos x}{\sin x} dx \\
 &= \frac{1}{8n\pi} \int_0^\pi \frac{\sin(2n+1)x + \sin(2n-1)x}{\sin x} dx = \frac{1}{4n}
 \end{aligned}$$

其中  $\int \frac{\sin(2n+1)x}{\sin x} dx = x + 2 \sum_{k=1}^n \frac{\sin 2kx}{2k} + C, \int_0^\pi \frac{\sin(2n+1)x}{\sin x} dx = \pi.$

□

### 36. 计算积分

$$I = \int_0^\infty \frac{\sin x - x - x^3}{x^3(x^2+1)} dx$$

**解** 首先我们有

$$I = \int_0^\infty \frac{\sin x - x - x^3}{x^3(x^2+1)} dx = \int_0^\infty \left( \frac{\sin x}{x^3} - \frac{\sin x}{x} + \frac{x \sin x}{1+x^2} - \frac{1}{x^2} \right) dx$$

其中

$$\begin{aligned}
 \int_0^\infty \frac{\sin x}{x} dx &= \frac{\pi}{2}, \int_0^\infty \frac{x \sin x}{1+x^2} dx = \frac{\pi}{2e}, \\
 \int_0^\infty \frac{\sin x - x}{x^3} dx &= -\frac{1}{2} \int_0^\infty (\sin x - x) d\left(\frac{1}{x^2}\right) \\
 &= \frac{1}{2} \int_0^\infty \frac{\cos x - 1}{x^2} dx \\
 &= -\frac{1}{2} \int_0^\infty \frac{\sin x}{x} dx \\
 &= -\frac{\pi}{4}
 \end{aligned}$$

故  $I = \frac{\pi}{2e} - \frac{3\pi}{4}.$

□

### 37. 计算积分

$$I = \int_0^\infty \frac{\arctan^2 x}{x(1+x^2)^2} dx$$

**解**

$$\begin{aligned}
 I &= \int_0^\infty \frac{\arctan^2 x}{x(1+x^2)^2} dx \\
 &= \int_0^{\frac{\pi}{2}} \frac{x^2 \cos^3 x}{\sin x} dx \\
 &= \int_0^{\frac{\pi}{2}} x^2 d\left(\frac{1}{4} \cos(2x) + \ln(\sin x)\right)
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{\pi^2}{16} - \int_0^{\frac{\pi}{2}} x \left( \frac{1}{2} \cos(2x) + 2 \ln(\sin x) \right) dx \\
 &= -\frac{\pi^2}{16} + \frac{1}{4} - 2 \int_0^{\frac{\pi}{2}} x \ln(\sin x) dx \\
 &= -\frac{\pi^2}{16} + \frac{1}{4} - 2 \int_0^{\frac{\pi}{2}} x \left( -\ln 2 - \sum_{k=1}^{\infty} \frac{\cos 2kx}{k} \right) dx \\
 &= -\frac{\pi^2}{16} + \frac{1}{4} + \frac{\pi^2 \ln 2}{4} \\
 &= -\frac{\pi^2}{16} + \frac{1}{4} + \frac{\pi^2 \ln 2}{4} - \sum_{k=0}^{\infty} \frac{1}{(2k-1)^3} \\
 &= -\frac{\pi^2}{16} + \frac{1}{4} + \frac{\pi^2 \ln 2}{4} - \frac{7}{8} \zeta(3)
 \end{aligned}$$

□

### 38. 计算积分

$$I = \int_{-\infty}^{\infty} \frac{e^x dx}{(e^{2x} + e^{2t})(x^2 + \pi^2)}$$

**解** 矩形围道  $C: -\infty - i\pi \rightarrow \infty - i\pi \rightarrow \infty + i\pi \rightarrow -\infty + i\pi$

于是由留数定理可得

$$\begin{aligned}
 &\oint_C \frac{e^z}{z(e^{2t} + e^{2z})} dz \\
 &= \int_{-\infty - i\pi}^{\infty - i\pi} \frac{e^x}{x(e^{2t} + e^{2x})} dx - \int_{-\infty + i\pi}^{\infty + i\pi} \frac{e^x}{x(e^{2t} + e^{2x})} dx \\
 &= - \int_{-\infty}^{\infty} \frac{2\pi i e^x}{(x^2 + \pi^2)(e^{2t} + e^{2x})} dx \\
 &= 2\pi i \left[ \sum \text{Res} \left( \frac{e^x}{x(e^{2t} + e^{2x})}, \left\{ x = t + \frac{\pi i}{2}, t - \frac{\pi i}{2}, 0 \right\} \right) \right] \\
 &= 2\pi i \left( -\frac{2\pi e^{-t}}{4t^2 + \pi^2} + \frac{1}{e^{2t} + 1} \right),
 \end{aligned}$$

所以原积分  $I = \int_{-\infty}^{\infty} \frac{e^x dx}{(e^{2x} + e^{2t})(x^2 + \pi^2)} = \left( \frac{2\pi e^{-t}}{4t^2 + \pi^2} - \frac{1}{e^{2t} + 1} \right).$

用同样的方法还可以求出另一个积分  $\int_{-\infty}^{\infty} \frac{dx}{(e^x + e^{-x})(x^2 + \pi^2)} = \frac{2}{\pi} - \frac{1}{2}.$

□

### 39. 计算积分

$$I = \int_1^{\infty} \frac{1}{x} \ln \left( \frac{x}{x-1} \right) \ln \left[ \ln \left( \frac{x}{x-1} \right) \right] dx$$

**解** 令  $t = \ln \left( \frac{x}{x-1} \right)$ , 得  $I = \int_0^{\infty} \frac{t \ln t}{e^t - 1} dt.$

$$\begin{aligned}
 f(p) &= \int_0^{\infty} \frac{t^p}{e^t - 1} dt \\
 &= \int_0^{\infty} t^p \sum_{n=1}^{\infty} e^{-nt} dt = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nt} t^p dt
 \end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{\Gamma(p+1)}{n^{p+1}} = \zeta(p+1)\Gamma(p+1),$$

$$\text{于是 } I = f'(1) = \zeta(2)\psi(2) - \zeta'(2) = \frac{1}{6}(1-\gamma)\pi^2 - \zeta'(2).$$

□

#### 40. 计算积分

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin^2(\tan x)}$$

解

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin^2(\tan x)} = \int_0^{\infty} \frac{dx}{(1+x^2)(1+\sin^2 x)} = \int_0^{\infty} \frac{dx}{(1+x^2)(2-\cos^2 x)}$$

考虑 Poisson 积分

$$\begin{aligned} I(\eta) &= \int_0^{\infty} \frac{dx}{(1+x^2)(\eta^2 - 2\eta \cos 2x + 1)} \\ &= \frac{1}{1-\eta^2} \int_0^{\infty} \left( 2 \sum_{k=0}^{\infty} \eta^k \cos 2kx - 1 \right) \frac{dx}{1+x^2} \\ &= \frac{\pi}{2} \frac{1}{1-\eta^2} \frac{e^2 + \eta}{e^2 - \eta}. \end{aligned}$$

同时有  $I(\eta) = \frac{1}{1+\eta^2} \int_0^{\infty} \frac{dx}{(1+x^2) \left( 1 - \frac{2\eta}{1+\eta^2} \cos 2x \right)}$ , 令  $\frac{2\eta}{1+\eta^2} = \frac{1}{3}$ , 解得  $\eta = 3 - 2\sqrt{2}$ , 于是最后得到

$$I = \frac{2}{3} I(3 - 2\sqrt{2}) = \frac{\pi}{2\sqrt{2}} \frac{e^2 + 3 - 2\sqrt{2}}{e^2 - 3 + 2\sqrt{2}}.$$

□

#### 41. 计算积分

$$\int_0^1 \frac{\ln x \ln(1+x^2)}{x} dx$$

解

$$\begin{aligned} I &= \int_0^1 \frac{\ln x \ln(1+x^2)}{x} dx \\ &= - \int_0^{\infty} t \ln(e^{-2t} + 1) dt \\ &= - \int_0^{\infty} t \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-2nt}}{n} dt \\ &= - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_0^{\infty} t e^{-2nt} dt \\ &= - \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \\ &= - \frac{3}{16} \zeta(3). \end{aligned}$$

□

#### 42. 计算积分

$$I = \int_0^{\frac{\pi}{2}} \ln(x^2 + \ln^2(\cos x)) dx$$

**解** 考虑参数积分

$$\begin{aligned} J(a) &= \int_{-\pi}^{\pi} \ln(ix + \ln(a \cos x)) dx \\ &= \oint_{|z|=1} \ln\left(\ln z + \ln\left(a \frac{z+z^{-1}}{2}\right)\right) \frac{-idz}{z} \quad z = e^{ix} \\ &= \oint_{|z|=1} \ln\left(\ln\left(a \frac{1+z^2}{2}\right)\right) \frac{-idz}{z} \\ &= 2\pi \operatorname{Res}\left[\ln\left(\ln\left(a \frac{1+z^2}{2}\right)\right), z=0\right] \\ &= 2\pi \ln\left(\ln\left(\frac{a}{2}\right)\right). \end{aligned}$$

于是得到

$$\begin{aligned} I(a) &= \int_0^{\frac{\pi}{2}} \ln(x^2 + \ln^2(a \cos x)) dx \\ &= 2\Re \int_0^{\frac{\pi}{2}} \ln(ix + \ln(a \cos x)) dx \\ &= \frac{1}{2} \Re \int_{-\pi}^{\pi} \ln(ix + \ln(a \cos x)) dx \\ &= \frac{1}{2} \Re J(a). \end{aligned}$$

$$\text{故原积分 } I = \int_0^{\frac{\pi}{2}} \ln(x^2 + \ln^2(\cos x)) dx = \frac{1}{2} \Re J(1) = \pi \ln(\ln 2).$$

□

#### 43. 计算积分

$$I = \int_0^{\pi} \sin(xt)(2 \sin t)^y dt$$

**解** 考查围道积分  $\oint_C z^{p-q-1}(z-z^{-1})^{p+q-2} dz$ , 这里  $C$  是单位圆在上半平面的部分  $C_R$  与从  $-1$  到  $1$  的实轴  $L$  构成的围道, 那么有

$$\begin{aligned} I_1 &= \int_{C_R} z^{p-q-1}(z-z^{-1})^{p+q-2} dz \xrightarrow{z=e^{it}} e^{i\pi \frac{p+q-1}{2}} \int_0^{\pi} e^{i(p-q)t} (2 \sin t)^{p+q-2} dt \\ I_2 &= \int_L z^{p-q-1}(z-z^{-1})^{p+q-2} dz = \int_{-1}^1 x^{p-q-1}(x-x^{-1})^{p+q-2} dx \\ &= \int_0^1 x^{p-q-1}(x-x^{-1})^{p+q-2} dx + \int_{-1}^0 x^{p-q-1}(x-x^{-1})^{p+q-2} dx \\ &= \frac{1}{2} e^{i\pi(p+q)} B(1-q, p+q-1) - \frac{1}{2} e^{i\pi(p-q)} B(1-q, p+q-1) \\ &= ie^{ip\pi} \sin(q\pi) B(1-q, p+q-1). \end{aligned}$$

由留数定理有  $I_1 + I_2 = 0$ , 再令  $x = p-q, y = p+q-2$ , 比较虚部可得

$$I = \int_0^{\pi} \sin(xt)(2 \sin t)^y dt = \frac{\pi \sin\left(\frac{x\pi}{2}\right) \Gamma(y+1)}{\Gamma\left(\frac{y+x}{2}+1\right) \Gamma\left(\frac{y-x}{2}+1\right)}.$$



同理还可以得到

$$I = \int_0^\pi \cos(xt)(2 \sin t)^y dt = \frac{\pi \cos\left(\frac{x\pi}{2}\right) \Gamma(y+1)}{\Gamma\left(\frac{y+x}{2}+1\right) \Gamma\left(\frac{y-x}{2}+1\right)}.$$

□

#### 44. 计算积分

$$I = \int_0^\pi \frac{\ln(\sin x)}{x^2 + \ln^2(\sin x)} dx$$

**解** 根据  $\frac{a}{a^2 + b^2} = \int_0^\infty e^{-ab} \cos by dy$  可得

$$\begin{aligned} -I &= \int_0^\pi \left( \int_0^\infty e^{y \ln(\sin x)} \cos xy dy \right) dx \\ &= \int_0^\infty dy \int_0^\pi \sin^y x \cos xy dx \\ &= \int_0^\infty 2^{-y} dy \int_0^\pi (2 \sin x)^y \cos xy dx \end{aligned}$$

由上一题可得  $\int_0^\pi (2 \sin x)^y \cos xy dx = \pi \cos\left(\frac{y\pi}{2}\right)$ , 再根据上一题有

$$I = -\pi \int_0^\infty 2^{-y} \cos\left(\frac{y\pi}{2}\right) dy = -\frac{4\pi \ln 2}{\pi^2 + 4 \ln^2 2}.$$

用类似的方法可以得到

$$\int_0^\pi \frac{x}{x^2 + \ln^2(\sin x)} dx = \frac{2\pi^2}{\pi^2 + 4 \ln^2 2}.$$

□

#### 45. 设 $b > 0$ , 计算积分

$$A = \int_0^\infty \frac{\cos bx}{x^s} dx \quad (0 < s < 1), \quad B = \int_0^\infty \frac{\sin bx}{x^s} dx \quad (0 < s < 2).$$

**解** 首先容易得到  $\frac{1}{x^s} = \frac{1}{\Gamma(s)} \int_0^\infty z^{s-1} e^{-xz} dz$ , 所以

$$\begin{aligned} A &= \frac{1}{\Gamma(s)} \int_0^\infty \cos bx dx \int_0^\infty z^{s-1} e^{-xz} dz \\ &= \frac{1}{\Gamma(s)} \int_0^\infty z^{s-1} dz \int_0^\infty e^{-xz} \cos bx dx \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{z^s dz}{z^2 + b^2} \\ &= \frac{b^{s-1}}{\Gamma(s)} \int_0^{\frac{\pi}{2}} \tan^s t dt \quad (z = b \tan t) \\ &= \frac{b^{s-1}}{2\Gamma(s)} B\left(\frac{s+1}{2}, \frac{1-s}{2}\right) \\ &= \frac{b^{s-1}}{2\Gamma(s)} \cdot \frac{\pi}{\sin\left(\frac{s+1}{2}\pi\right)} \end{aligned}$$

$$= \frac{\pi b^{s-1}}{2\Gamma(s)} \cos\left(\frac{s\pi}{2}\right)$$

类似地, 可以得到  $B = \frac{\pi b^{s-1}}{2\Gamma(s) \sin\left(\frac{s\pi}{2}\right)}$  (本题的另解见例 10).

□

#### 46. 计算积分

$$I = \int_0^\infty (1+x^n) \ln(1+e^{-x}) dx$$

解

$$\begin{aligned} I &= \int_0^\infty (1+x^n) \ln(1+e^{-x}) dx \\ &= \int_0^\infty \sum_{k=1}^\infty \frac{(-1)^{k-1} e^{-kx}}{k} dx + \int_0^\infty x^n \sum_{k=1}^\infty \frac{(-1)^{k-1} e^{-kx}}{k} dx \\ &= \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^2} + \Gamma(n+1) \sum_{k=0}^\infty \frac{(-1)^{k-1}}{k^{n+1}} \\ &= \frac{\pi^2}{12} + (1-2^{-n})\zeta(n+1)\Gamma(n+1) \end{aligned}$$

□

#### 47. 计算积分

$$I = \int_0^\infty \frac{\ln(x^3+1)}{x^3+1} dx$$

解 令  $t = x^3$  得  $I = \frac{1}{3} \int_0^\infty \frac{\ln(1+t)}{(1+t)t^{\frac{2}{3}}} dt$ , 考虑

$$\begin{aligned} I(a) &= \int_0^\infty \frac{(1+t)^a}{(1+t)t^{\frac{2}{3}}} dt \\ &= \int_0^1 y^{-\frac{1}{3}-a} (1-y)^{-\frac{2}{3}} dy \quad \left(1+t = \frac{1}{y}\right) \\ &= B\left(\frac{2}{3}-a, \frac{1}{3}\right) = \frac{\Gamma\left(\frac{2}{3}-a\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma(1-a)} \end{aligned}$$

于是  $\ln I(a) = \ln \Gamma\left(\frac{2}{3}-a\right) + \ln \Gamma\left(\frac{1}{3}\right) - \ln \Gamma(1-a)$ , 那么原积分

$$I = I'(0) = \frac{1}{3} I(0) \left[ -\psi\left(\frac{2}{3}\right) + \psi(1) \right] = \frac{\pi \ln 3}{\sqrt{3}} - \frac{\pi^2}{9}$$

其中  $I(0) = \frac{2\pi}{\sqrt{3}}$ ,  $\psi(1) = -\gamma$ ,  $\psi\left(\frac{2}{3}\right) = -\gamma - \frac{3}{2} \ln 3 + \frac{\pi}{2\sqrt{3}}$ ,  $\psi\left(\frac{2}{3}\right)$  的值可以借助  $\psi(1-x) = \psi(x) +$

$\pi \cot \pi x$  以及对公式  $\Gamma(3x) = \frac{3^{3x-\frac{1}{2}}}{2\pi} \Gamma(x) \Gamma\left(x + \frac{1}{3}\right) \Gamma\left(x + \frac{2}{3}\right)$  求导然后取  $x = \frac{1}{3}$  得到.

□

#### 48. 计算积分

$$I = \int_0^{\frac{\pi}{4}} \ln(1 - \tan^2 x) dx$$

**解**  $I = \int_0^{\frac{\pi}{4}} \ln(1 - \tan^2 x) dx = \int_0^{\frac{\pi}{4}} \ln(1 + \tan x) dx + \int_0^{\frac{\pi}{4}} \ln(1 - \tan x) dx$ , 其中

$$\begin{aligned} I_1 &= \int_0^{\frac{\pi}{4}} \ln(1 + \tan x) dx = \int_0^{\frac{\pi}{4}} \ln\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) dx \\ &= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan x}\right) dx \\ &= \frac{\pi}{4} \ln 2 - I_1 = \frac{\pi}{8} \ln 2 \\ I_2 &= \int_0^{\frac{\pi}{4}} \ln(1 - \tan x) dx = \int_0^{\frac{\pi}{4}} \ln\left(1 - \tan\left(\frac{\pi}{4} - x\right)\right) dx \\ &= \int_0^{\frac{\pi}{4}} \ln\left(1 - \frac{1 - \tan x}{1 + \tan x}\right) dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2 \tan x}{1 + \tan x}\right) dx \\ &= \frac{\pi}{4} \ln 2 + \int_0^{\frac{\pi}{4}} \ln(\tan x) dx - I_2 = \frac{\pi}{8} \ln 2 - G \end{aligned}$$

因此  $I = I_1 + I_2 = \frac{\pi}{4} \ln 2 - G$ .

□

#### 49. 计算积分

$$I = \int_0^{\infty} \frac{(1 - x^2) \arctan(x^2)}{x^4 + 4x^2 + 1} dx$$

**解** 设  $y \geq 0$ , 考虑含参积分

$$\begin{aligned} F(y) &= \int_0^{\infty} \frac{\arctan(yx^2)}{1 + x^2} dx \\ &= \int_0^{\infty} \int_0^y \frac{x^2}{(1 + t^2 x^4)(1 + x^2)} dt dx \\ &= \int_0^y dt \int_0^{\infty} \left[ \frac{1 + t^2 x^2}{(1 + t^2 x^4)(1 + t^2)} - \frac{1}{(1 + t^2)(1 + x^2)} \right] dx \\ &= \int_0^y dt \int_0^{\infty} \frac{1 + tx^2}{(1 + x^4)(1 + t^2)\sqrt{t}} dx - \int_0^y dt \int_0^{\infty} \frac{1}{(1 + t^2)(1 + x^2)} dx \\ &= \frac{\sqrt{2}\pi}{4} \int_0^y \frac{1 + t}{(1 + t^2)\sqrt{t}} dt - \frac{\pi}{2} \int_0^y \frac{1}{1 + t^2} dt \\ &= \frac{\pi}{2} \left[ \arctan(\sqrt{2y} + 1) + \arctan(\sqrt{2y} - 1) - \arctan y \right]. \end{aligned}$$

所以

$$\begin{aligned} I &= \int_0^{\infty} \frac{(1 - x^2) \arctan(x^2)}{x^4 + 4x^2 + 1} dx \\ &= \int_0^{\infty} \frac{(\sqrt{3} - 1) \arctan(x^2)}{2(x^2 + 2 - \sqrt{3})} dx - \int_0^{\infty} \frac{(\sqrt{3} + 1) \arctan(x^2)}{2(x^2 + 2 + \sqrt{3})} dx \\ &= \frac{\sqrt{2}}{2} \left[ F(2 - \sqrt{3}) - F(2 + \sqrt{3}) \right] \end{aligned}$$

$$\begin{aligned}
 &= -\frac{\sqrt{2}\pi}{4} \left[ \arctan(\sqrt{2y+1}) + \arctan(\sqrt{2y-1}) - \arctan y \right] \Big|_{y=2-\sqrt{3}}^{y=2+\sqrt{3}} \\
 &= -\frac{\sqrt{2}\pi}{4} \left[ -\arctan(\sqrt{3}-2) + \arctan(2-\sqrt{3}) \right] \\
 &= -\frac{\sqrt{2}\pi^2}{24}
 \end{aligned}$$

□

## 50. 计算积分

$$I = \int_0^\infty \frac{(1-e^{-6x})e^{-x}}{x(1+e^{-2x}+e^{-4x}+e^{-6x}+e^{-8x})} dx$$

解

$$\begin{aligned}
 I &= \int_0^\infty \frac{(1-e^{-6x})e^{-x}}{x(1+e^{-2x}+e^{-4x}+e^{-6x}+e^{-8x})} dx \\
 &= \int_0^\infty \frac{(1-e^{-6x})e^{-x}(1-e^{-2x})}{x(1-e^{-10x})} dx \\
 &= \int_0^\infty \sum_{n=0}^\infty \frac{(1-e^{-6x})e^{-(10n+1)x}(1-e^{-2x})}{x} \\
 &= \int_0^\infty \sum_{n=0}^\infty \frac{(e^{-(10n+1)x} - e^{-(10n+7)x}) - (e^{-(10n+3)x} - e^{-(10n+9)x})}{x} dx \\
 &= \sum_{n=0}^\infty \ln \left[ \frac{(10n+7)(10n+3)}{(10n+1)(10n+9)} \right] \\
 &= \sum_{n=0}^\infty \ln \left[ \frac{(\frac{7}{10}+n)(\frac{3}{10}+n)}{(\frac{1}{10}+n)(\frac{9}{10}+n)} \right] \\
 &= \lim_{n \rightarrow \infty} \ln \left[ \frac{\Gamma(\frac{1}{10})\Gamma(\frac{9}{10})\Gamma(\frac{3}{10}+n)\Gamma(\frac{7}{10}+n)}{\Gamma(\frac{3}{10})\Gamma(\frac{7}{10})\Gamma(\frac{1}{10}+n)\Gamma(\frac{9}{10}+n)} \right] \\
 &= \ln \left( \frac{\sin \frac{3\pi}{10}}{\sin \frac{\pi}{10}} \right) = \ln \left( \frac{\sqrt{5}+1}{\sqrt{5}-1} \right)
 \end{aligned}$$

□

51. 设  $a_0 = 1, a_1 = \frac{2}{3}$ ,

$$(n+1)a_{n+1} - (n-1)a_{n-1} = \frac{2}{3}a_n.$$

求极限  $\lim_{n \rightarrow \infty} n^{\frac{2}{3}}a_n$ .

解 设  $S(x) = \sum_{n=1}^\infty na_nx^{n-1}$ , 由递推关系可得

$$[(1-x^2)S(x)]' = \frac{2}{3}S(x)$$

解得

$$S(x) = \frac{2}{3}(1+x)^{-\frac{2}{3}}(1-x)^{-\frac{4}{3}}.$$

利用 Cauchy 乘积并比较系数得到当  $n \geq 1$  时有

$$\begin{aligned}
 a_n &= \frac{2}{3n\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})} \sum_{\substack{k+j=n-1 \\ k,j \geq 0}} (-1)^k \frac{\Gamma(k+\frac{2}{3})\Gamma(j+\frac{4}{3})}{k!j!} \\
 &= \frac{2}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})} \sum_{\substack{k+j=n-1 \\ k,j \geq 0}} (-1)^k \binom{n-1}{k} \int_0^1 t^{k-\frac{1}{3}}(1-t)^{j+\frac{1}{3}} dt \\
 &= \frac{2}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})} \int_0^1 (1-2t)^{n-1} \frac{\sqrt[3]{1-t}}{\sqrt[3]{t}} dt \\
 &= \frac{1}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})} \int_{-1}^1 \frac{\sqrt[3]{1+s}}{\sqrt[3]{1-s}} s^{n-1} ds \\
 &= \frac{1}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})} \left( \int_0^1 \frac{\sqrt[3]{2}}{\sqrt[3]{1-s}} s^{n-1} ds + \int_0^1 \frac{\sqrt[3]{1+s}-\sqrt[3]{2}}{\sqrt[3]{1-s}} s^{n-1} ds + \int_{-1}^0 \frac{\sqrt[3]{1+s}}{\sqrt[3]{1-s}} s^{n-1} ds \right) \\
 &\triangleq F_n + G_n + H_n
 \end{aligned}$$

容易说明  $nG_n, nH_n$  有界,  $\lim_{n \rightarrow \infty} n^{\frac{2}{3}} a_n = \lim_{n \rightarrow \infty} n^{\frac{2}{3}} F_n = \frac{\sqrt[3]{2}}{\Gamma(\frac{1}{3})}$ .

□

## 52. 计算三重积分

$$I = \int_{-\infty}^{\infty} dt \int_t^{\infty} dw \int_w^{\infty} e^{-(u^2+t^2+\frac{1}{2}w^2)} du$$

**解** 记  $\Omega = (u, w, t) | -\infty < t \leq w \leq u < \infty$ , 做广义球坐标变换

$$\begin{cases} u = r \cos \theta \sin \varphi \\ w = \sqrt{2}r \sin \theta \sin \varphi \\ t = r \cos \varphi \\ r \geq 0, \theta \in [-\pi, \pi], \varphi \in [0, \pi] \end{cases}$$

设  $\alpha \in (0, \frac{\pi}{2})$  满足  $\sin \alpha = \frac{1}{\sqrt{3}}$ ,  $\varphi(\theta) \in (0, \pi)$  满足  $\cot \varphi(\theta) = \sqrt{2} \sin \theta$ , 则原积分

$$\begin{aligned}
 I &= \iiint_{\Omega} e^{-(u^2+t^2+\frac{1}{2}w^2)} du dw dt \\
 &= \int_0^{\infty} dr \int_{-\pi+\alpha}^{\alpha} d\theta \int_{\varphi(\theta)}^{\pi} \sqrt{2} e^{-r^2} r^2 \sin \varphi d\varphi \\
 &= \frac{\sqrt{2}\pi}{4} \int_{-\pi+\alpha}^{\alpha} d\theta \int_{\varphi(\theta)}^{\pi} \sin \varphi d\varphi \\
 &= \frac{\sqrt{2}\pi}{4} \int_{-\pi+\alpha}^{\alpha} [1 + \cos \varphi(\theta)] d\theta \\
 &= \frac{\sqrt{2}\pi}{4} \int_{-\pi+\alpha}^{\alpha} \left( 1 + \frac{\sqrt{2} \sin \theta}{\sqrt{1+2\sin^2 \theta}} \right) d\theta \\
 &= \frac{\sqrt{2}\pi}{4} \left( \pi - 2 \int_0^{\frac{2}{3}} \frac{ds}{\sqrt{1-s^2}} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\sqrt{2\pi}}{4} \left[ \pi - 2 \arcsin \left( \frac{2}{3} \right) \right] \\
 &= \sqrt{2\pi} \arcsin \left( \frac{1}{\sqrt{6}} \right)
 \end{aligned}$$

□

### 53. 计算积分

$$I = \int_0^1 \frac{\sqrt[n]{x^m(1-x)^{n-m}}}{(1+x)^3} dx$$

解

$$\begin{aligned}
 I &= \int_0^1 \frac{\sqrt[n]{x^m(1-x)^{n-m}}}{(1+x)^3} dx \\
 &= \int_0^1 \left( \frac{x}{1+x} \right)^{\frac{m}{n}} \left( \frac{1-x}{1+x} \right)^{\frac{n-m}{n}} \frac{dx}{(1+x)^2} \\
 &= 2^{-\frac{n+m}{n}} \int_0^1 t^{\frac{m}{n}} (1-t)^{\frac{n-m}{n}} dt \quad \left( t = \frac{x}{1+x} \right) \\
 &= \frac{2^{-\frac{n+m}{n}}}{\Gamma(3)} \Gamma\left(\frac{m+n}{n}\right) \Gamma\left(\frac{2n-m}{n}\right) \\
 &= 2^{-\frac{n+m}{n}} \cdot \frac{m}{n} \frac{n-m}{n} \cdot \Gamma\left(\frac{m}{n}\right) \cdot \Gamma\left(1 - \frac{m}{n}\right) \\
 &= 2^{-\frac{n+m}{n}} \cdot \frac{m(n-m)}{n^2} \cdot \frac{\pi}{\sin\left(\frac{m\pi}{n}\right)}
 \end{aligned}$$

□

### 54. 计算积分

$$I = \int_0^1 \sin(\pi x) \ln \Gamma(x) dx$$

解

$$\begin{aligned}
 I &= \int_0^1 \sin(\pi x) \ln \Gamma(x) dx \\
 &= \int_0^1 \sin \pi(1-x) \ln \Gamma(1-x) dx \\
 &= \int_0^1 \sin(\pi x) \ln \Gamma(1-x) dx \\
 &= \frac{1}{2} \int_0^1 \sin(\pi x) \ln[\Gamma(x)\Gamma(1-x)] dx \\
 &= \frac{1}{2} \int_0^1 \sin(\pi x) \ln \left( \frac{\pi}{\sin(\pi x)} \right) dx \\
 &= \ln \pi \int_0^1 \sin(\pi x) dx - \int_0^1 \sin(\pi x) \ln(\sin \pi x) dx \\
 &= \frac{\ln \pi}{\pi} - \frac{1}{2\pi} \int_0^\pi \sin t \ln(\sin t) dt \\
 &= \frac{\ln \pi}{\pi} - \frac{1}{2\pi} \left( \cos t - \ln \left( \cos \frac{t}{2} \right) + \ln \left( \sin \frac{t}{2} \right) - \cos t \ln(\sin t) \right) \Big|_0^\pi
 \end{aligned}$$

$$= \frac{\ln \pi}{\pi} - \frac{2 \ln 2 - 2}{2\pi} = \frac{\ln\left(\frac{\pi}{2}\right) + 1}{\pi}$$

□

### 55. 计算积分

$$I = \int_0^{\frac{\pi}{4}} \frac{\ln(\sin x) \ln(\cos x)}{\sin 2x} dx$$

解

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \frac{\ln(\sin x) \ln(\cos x)}{\sin 2x} dx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\ln(\sin x) \ln(\cos x)}{\sin 2x} dx \\ &= \frac{1}{16} \int_0^{\frac{\pi}{2}} \frac{\cos x \ln(\sin^2 x) \ln(\cos^2 x)}{\sin x \cos^2 x} dx \\ &= \frac{1}{32} \int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)} dx \\ &= \frac{1}{32} \int_0^1 \ln x \ln(1-x) \left( \frac{1}{x} + \frac{1}{1-x} \right) dx \\ &= \frac{1}{16} \int_0^1 \frac{\ln x \ln(1-x)}{x} dx \\ &= -\frac{1}{16} \int_0^1 \left( \ln x \sum_{k=1}^{\infty} \frac{x^{k-1}}{k} \right) dx \\ &= \frac{1}{16} \zeta(3) \end{aligned}$$

□

### 56. 计算积分

$$\int_0^{\frac{\pi}{2}} \frac{\arctan(\sqrt{2} \tan x)}{\tan x} dx$$

解 令  $I(a) = \int_0^{\frac{\pi}{2}} \frac{\arctan(a \tan x)}{\tan x} dx$ , 则  $I(0) = 0$ ,

$$\begin{aligned} I'(a) &= \int_0^{\frac{\pi}{2}} \frac{\tan x}{(1+a^2 \tan^2 x) \tan x} dx \\ &= \frac{1}{1-a^2} \int_0^{\infty} \frac{(1+a^2 u^2) - a^2(1+u^2)}{(1+a^2 u^2)(1+u^2)} du \\ &= \frac{1}{1-a^2} \int_0^{\infty} \frac{1}{1+u^2} du - \frac{a^2}{1-a^2} \int_0^{\infty} \frac{1}{1+a^2 u^2} du \\ &= \frac{\pi}{2} \left( \frac{1}{1-a^2} - \frac{a}{1-a^2} \right) \\ &= \frac{\pi}{2} \frac{1}{1+a} \end{aligned}$$

于是原积分  $I = I(\sqrt{2}) = \int_0^{\sqrt{2}} I'(a) da = \frac{\pi}{2} \ln(1+\sqrt{2})$ .

□

57. 计算积分

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+e^x)(1+e^{-x})} dx$$

解

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{x^2}{(1+e^x)(1+e^{-x})} dx &= 2 \int_0^{\infty} \frac{x^2 e^x}{(1+e^x)^2} dx \\ &= \frac{-2x^2}{1+e^x} \Big|_0^{\infty} + 4 \int_0^{\infty} \frac{x}{1+e^x} dx \\ &= 4 \int_0^{\infty} \frac{x e^{-x}}{1+e^x} dx \\ &= 4 \int_0^{\infty} x e^{-x} \sum_{k=0}^{\infty} (-e)^{-kx} dx \\ &= 4 \sum_{k=1}^{\infty} \int_0^{\infty} (-1)^{k-1} x e^{-kx} dx \\ &= 4 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{3} \end{aligned}$$

□

58. 计算积分

$$\int_0^{\frac{\pi}{2}} \frac{x e^{-\tan^2 x} \sin(4x)}{\cos^8 x} dx$$

解

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \frac{x e^{-\tan^2 x} \sin(4x)}{\cos^8 x} dx \\ &= \int_0^{\infty} \arctan t \cdot e^{-t^2} 2 \frac{2t}{1+t^2} \frac{1-t^2}{1+t^2} d(\arctan t) \\ &= 2 \int_0^{\infty} e^{-t^2} (1-t^4) \arctan t d(t^2) \\ &= -2 \int_0^{\infty} (1-t^4) \arctan t d(e^{-t^2}) \\ &= -2 (1-t^4) \arctan t \Big|_0^{\infty} + 2 \int_0^{\infty} e^{-t^2} [(1-t^2) - 4t^3 \arctan t] dt \\ &= 2 \int_0^{\infty} e^{-t^2} (1-t^2) dt - 4 \int_0^{\infty} e^{-t^2} \arctan t d(t^2) \\ &= 2 \int_0^{\infty} e^{-t^2} (1-t^2) dt + 4 \int_0^{\infty} t^2 \arctan t d(e^{-t^2}) \\ &= 2 \int_0^{\infty} e^{-t^2} (1-t^2) dt - 4 \int_0^{\infty} e^{-t^2} \left[ \frac{t^2}{1+t^2} + 2t \arctan t \right] dt \\ &= 2 \int_0^{\infty} e^{-t^2} (1-t^2) dt - 4 \int_0^{\infty} e^{-t^2} dt \\ &= -2 \int_0^{\infty} e^{-t^2} (1+t^2) dt \\ &= -\frac{3}{2} \sqrt{\pi} \end{aligned}$$



□

### 59. 计算积分

$$\int_0^{\pi} e^{p \cos x} \cos(p \sin x) \cos qx dx$$

解

$$\begin{aligned} \int_0^{\pi} e^{p \cos x} \cos(p \sin x) \cos qx dx &= \Re \int_0^{\pi} e^{p \cos x} e^{i p \sin x} \cos qx dx \\ &= \Re \int_0^{\pi} e^{i p \sin x + p \cos x} \cos qx dx \\ &= \Re \int_0^{\pi} e^{p e^{ix}} \cos qx dx \\ &= \Re \int_0^{\pi} \sum_{k=0}^{\infty} \left( \frac{p^k e^{ikx}}{k!} \right) \cos qx dx \\ &= \sum_{k=0}^{\infty} \frac{p^k}{k!} \int_0^{\pi} \cos qx \cos kx dx \\ &= \sum_{k=0}^{\infty} \frac{p^k}{2k!} \left[ \frac{\sin(k-q)x}{k-q} + \frac{\sin(k+q)x}{k+q} \right] \Big|_0^{\pi} \\ &= \frac{\pi p^q}{2 q!} \end{aligned}$$

□

### 60. 求极限

$$\lim_{n \rightarrow \infty} \left( \sum_{k=0}^n \frac{1}{3k+1} - \frac{1}{3} \ln n \right)$$

解 首先有

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{3k+1} &= 1 + \frac{1}{3} \left( \sum_{k=1}^n \left( \frac{1}{k+1/3} - \frac{1}{k} \right) \right) \\ &= 1 + \frac{1}{3} \sum_{k=1}^n \left( \frac{1}{k+1/3} - \frac{1}{k} \right) + \frac{1}{p} \left( \sum_{k=1}^n \frac{1}{k} - \ln n \right) + \frac{1}{3} \ln n \end{aligned}$$

于是

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{3k+1} - \frac{1}{3} \ln n &= 1 + \frac{1}{3} \sum_{k=1}^n \left( \frac{1}{k+1/3} - \frac{1}{k} \right) + \frac{1}{p} \left( \sum_{k=1}^n \frac{1}{k} \right) \\ &= 1 + \frac{1}{3} \sum_{k=1}^n \left( \int_0^1 x^{k+1/3-1} dx - \int_0^1 x^{k-1} dx \right) + \frac{1}{3} \left( \sum_{k=1}^n \frac{1}{k} - \ln n \right) \\ &= 1 + \frac{1}{3} \left( \int_0^1 \frac{x^{1/3} - 1}{1-x} dx \right) + \frac{1}{3} \left( \sum_{k=1}^n \frac{1}{k} - \ln n \right) \\ &= 1 + \int_0^1 \frac{x^{1/3} - 1}{1-x} dx + \frac{1}{3} \left( \sum_{k=1}^n \frac{1}{k} - \ln n \right) \end{aligned}$$

$$\begin{aligned}
 &= 1 - \int_0^1 \frac{x^2}{x^2 + x + 1} dx + \frac{1}{3} \left( \sum_{k=1}^n \frac{1}{k} - \ln n \right) \\
 &= \frac{\pi\sqrt{3}}{18} + \frac{1}{2} \ln 3 + \frac{1}{3} \left( \sum_{k=1}^n \frac{1}{k} - \ln n \right)
 \end{aligned}$$

因此  $\lim_{n \rightarrow \infty} \left( \sum_{k=0}^n \frac{1}{3k+1} - \frac{1}{3} \ln n \right) = \frac{1}{3} \gamma + \frac{\pi\sqrt{3}}{18} + \frac{1}{2} \ln 3.$

□

61. 设  $a_0, a_1 > 0, a_{n+2} = \frac{1}{a_n} + \frac{1}{a_{n+1}}$ , 证明:  $\lim_{n \rightarrow \infty} a_n = \sqrt{2}.$

解 令

$$M = \max \left\{ a_0, a_1, \frac{2}{a_0}, \frac{2}{a_1} \right\}$$

则利用递推公式可以归纳证明  $\frac{2}{M} \leq a_n \leq M$ . 设  $a_n$  的上下极限为  $L, \ell$ . 则

$$\frac{2}{M} \leq \ell \leq M.$$

且

$$\ell = \lim_{n \rightarrow \infty} a_{n+2} \geq \lim_{n \rightarrow \infty} \frac{1}{a_{n+1}} + \lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{2}{L} \quad (1)$$

$$L = \lim_{n \rightarrow \infty} a_{n+2} \leq \lim_{n \rightarrow \infty} \frac{1}{a_{n+1}} + \lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{2}{\ell} \quad (2)$$

因此  $L\ell = 2$ .

由上极限的性质, 存在子列  $a_{m_k+3}$  使得  $\lim_{k \rightarrow \infty} a_{m_k+3} = L$ . 进一步抽取子列 (为方便起见, 子列记号不变), 可以使  $a_{m_k+2}, a_{m_k+1}, a_{m_k}$  都收敛, 设极限依次为  $x, y, z$ , 则  $\ell \leq x, y, z \leq L$ . 于是由递推公式可得

$$L = \frac{1}{x} + \frac{1}{y}, \quad x = \frac{1}{y} + \frac{1}{z}$$

由 (1) 式结合  $\ell \leq x, y \leq L$  以及  $L = \frac{2}{\ell}$  得到  $x = y = \ell$ . 类似地, 可由第二式得到  $y = z = L$ . 所以  $L = \ell$ , 从而  $L = \ell = \lim_{n \rightarrow \infty} a_n = \sqrt{2}.$

□

62. 计算积分

$$\int_0^1 \frac{(\frac{1}{2} - x) \ln(1-x)}{x^2 - x + 1} dx$$

解 首先有  $I = \int_0^1 \frac{(\frac{1}{2} - x) \ln(1-x)}{x^2 - x + 1} dx = \int_0^1 \frac{(x - \frac{1}{2}) \ln x}{x^2 - x + 1} dx$ , 利用

$$\sum_{n=1}^{\infty} x^{n-1} \sin(na\pi) = \frac{\sin(a\pi)}{x^2 - 2\cos(a\pi) + 1}$$

令  $a = \frac{1}{3}$  可得

$$\begin{aligned} I &= \frac{2}{\sqrt{3}} \int_0^1 \sum_{n=1}^{\infty} x^{n-1} \sin\left(\frac{n\pi}{3}\right) \left(x - \frac{1}{2}\right) \ln x dx \\ &= \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{3}\right) \int_0^1 x^{n-1} \left(x - \frac{1}{2}\right) \ln x dx \\ &= \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{3}\right) \left[ \frac{1}{2n^2} - \frac{1}{(n+1)^2} \right] \\ &= \frac{\pi^2}{36} \end{aligned}$$

接下来计算  $J = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{3}\right) \left[ \frac{1}{2n^2} - \frac{1}{(n+1)^2} \right]$ .

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{(n+1)^2} = \sum_{n=2}^{\infty} \frac{\sin\left(\frac{n\pi}{3} - \frac{\pi}{3}\right)}{n^2} = \sum_{n=2}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right) \cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{n\pi}{3}\right) \sin\left(\frac{\pi}{3}\right)}{n^2}$$

于是

$$J = \frac{\sin\left(\frac{\pi}{3}\right)}{2} + \sum_{n=2}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right) \cos\left(\frac{\pi}{3}\right)}{n^2} = \frac{\sqrt{3}}{2} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n^2}$$

利用  $x^2$  的 Fourier 展开式  $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \frac{3x^2 - 6\pi x + 2\pi^2}{12}$ , 令  $x = \frac{\pi}{3}$  得  $I = \sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n^2} = \frac{\pi^2}{36}$ .  $\square$

### 63. 计算积分

$$\int_0^{\pi} \sin(a \sin x) \csc x dx$$

解

$$\begin{aligned} \int_0^{\pi} \sin(a \sin x) \csc x dx &= \Im \int_0^{\pi} e^{a \cos x} e^{ia \sin x} \csc x dx \\ &= \Im \int_0^{\pi} \frac{e^{ia \sin x + a \cos x}}{\sin x} dx \\ &= \Im \int_0^{\pi} \frac{e^{ae^{ix}}}{\sin x} dx \\ &= \Im \int_0^{\pi} \frac{\sum_{k=0}^{\infty} \frac{a^k e^{ikx}}{k!}}{\sin x} dx \\ &= \sum_{k=0}^{\infty} \frac{a^k}{k!} \int_0^{\pi} \frac{\sin kx}{\sin x} dx \\ &= \pi \sum_{k=0}^{\infty} \frac{a^{2k+1}}{(2k+1)!} = \pi \sinh(a) \end{aligned}$$

其中

$$\int_0^{\pi} \frac{\sin nx}{\sin x} dx = \begin{cases} \pi, & n = 2k + 1 \\ 0, & n = 2k \end{cases}$$

$\square$

## 64. 求极限

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \int_0^{\frac{\pi}{2}} x \left( \frac{\sin nx}{\sin x} \right)^4 dx$$

解 由

$$\lim_{x \rightarrow 0} x^2 \left( \frac{1}{\sin^4 x} - \frac{1}{x^4} \right) = \frac{2}{3}$$

知存在常数  $C > 0$  使得

$$\left| \frac{x \sin^4 nx}{\sin^4 x} - \frac{\sin^4 nx}{x^3} \right| \leq \frac{C \sin^4 nx}{x} \leq Cn, \quad \forall x \in \left(0, \frac{\pi}{2}\right).$$

由此得到

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^2} \int_0^{\frac{\pi}{2}} x \left( \frac{\sin nx}{\sin x} \right)^4 dx &= \lim_{n \rightarrow \infty} \frac{1}{n^2} \int_0^{\frac{\pi}{2}} \frac{\sin^4 nx}{x^3} dx \\ &= \lim_{n \rightarrow \infty} \int_0^{\frac{n\pi}{2}} \frac{\sin^4 x}{x^3} dx = \int_0^\infty \frac{\sin^4 x}{x^3} dx \\ &= \int_0^\infty \frac{2 \sin^3 x \cos x}{x^2} dx \\ &= \int_0^\infty \frac{6 \sin^2 x \cos^2 x - 2 \sin^4 x}{x} dx \\ &= \int_0^\infty \frac{\cos 2x - \cos 4x}{x} dx = \ln 2. \end{aligned}$$

□

## 65. 求极限

$$\lim_{n \rightarrow \infty} \left( \int_0^\pi \frac{\sin^2 nx}{\sin x} dx - \sum_{k=1}^n \frac{1}{k} \right)$$

解 先考虑积分  $I(n) = \int_0^\pi \frac{\sin^2 nx}{\sin x} dx$ , 有

$$I(n) - I(n-1) = \int_0^\pi \frac{\sin^2 nx}{\sin x} dx = \frac{2}{2n-1}$$

于是

$$I(n) = 2 \sum_{k=1}^n \frac{1}{2k-1} \sim \ln 2 + \ln(2n) + \gamma = \ln(4n) + \gamma, \quad n \rightarrow \infty$$

$$\text{故 } \lim_{n \rightarrow \infty} \left( \int_0^\pi \frac{\sin^2 nx}{\sin x} dx - \sum_{k=1}^n \frac{1}{k} \right) = \ln 4.$$

□

66. 把方程  $\tan x = x$  的正根按从小到大顺序排成数列  $x_n$ , 求极限

$$\lim_{n \rightarrow \infty} x_n^2 \sin(x_{n+1} - x_n)$$

解 首先容易得到  $x_n \in \left((n-1)\pi, (n-1)\pi + \frac{\pi}{2}\right)$ , 于是  $x_n - (n-1)\pi \in \left(0, \frac{\pi}{2}\right)$ , 故

$$x_n = \tan x_n = \tan(x_n - (n-1)\pi)$$

所以  $\arctan x_n = x_n - (n-1)\pi$ , 且  $x_n - (n-1)\pi \rightarrow \frac{\pi}{2}, n \rightarrow \infty$ .

$$\begin{aligned}\lim_{n \rightarrow \infty} x_n^2 \sin(x_{n+1} - x_n) &= \lim_{n \rightarrow \infty} x_n^2 \sin(\arctan x_{n+1} - \arctan x_n + \pi) \\ &= - \lim_{n \rightarrow \infty} n^2 \pi^2 \sin \left[ \arctan \left( \frac{x_{n+1} - x_n}{1 + x_n x_{n+1}} \right) \right] \\ &= - \lim_{n \rightarrow \infty} n^2 \pi^2 \frac{x_{n+1} - x_n}{1 + x_n x_{n+1}} = - \lim_{n \rightarrow \infty} (x_{n+1} - x_n) \\ &= - \lim_{n \rightarrow \infty} [x_{n+1} - n\pi - (x_n - (n-1)\pi)] - \pi \\ &= -\pi.\end{aligned}$$

□

### 67. 求极限

$$\lim_{x \rightarrow 0} \frac{\Gamma(x+1) - \sin x \Gamma(\sin x)}{x^4 \Gamma(\sin x)}$$

解

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\Gamma(x+1) - \sin x \Gamma(\sin x)}{x^4 \Gamma(\sin x)} &= \lim_{x \rightarrow 0} \frac{\Gamma(x - \sin x + \sin x + 1) - \Gamma(\sin x + 1)}{x^3 \Gamma(\sin x + 1)} \\ &= \lim_{x \rightarrow 0} \frac{\Gamma(x - \sin x + \sin x + 1) - \Gamma(\sin x + 1)}{6(x - \sin x) \Gamma(\sin x + 1)} \\ &= \frac{\psi(1)}{6} = -\frac{\gamma}{6}.\end{aligned}$$

□

### 68. 计算主值积分

$$\int_0^\infty \sin(x) \sin(\sqrt{x}) dx$$

解

$$\begin{aligned}&\int_0^\infty \sin(x) \sin(\sqrt{x}) dx \\ &= \frac{1}{2} \int_0^\infty [\cos(x - \sqrt{x}) - \cos(x + \sqrt{x})] dx \\ &= \frac{1}{2} \int_0^\infty \left[ \cos \left( \left( \sqrt{x} - \frac{1}{2} \right)^2 - \frac{1}{4} \right) - \cos \left( \left( \sqrt{x} + \frac{1}{2} \right)^2 - \frac{1}{4} \right) \right] dx \\ &= \frac{1}{2} \int_{-\frac{1}{2}}^\infty (2t+1) \cos \left( t^2 - \frac{1}{4} \right) dt - \frac{1}{2} \int_{\frac{1}{2}}^\infty (2t-1) \cos \left( t^2 - \frac{1}{4} \right) dt \\ &= \frac{1}{2} \int_{-\frac{1}{2}}^\infty (2t+1) \cos \left( t^2 - \frac{1}{4} \right) dt + \frac{1}{2} \int_{-\infty}^{-\frac{1}{2}} (2t+1) \cos \left( t^2 - \frac{1}{4} \right) dt \\ &= \frac{1}{2} \int_{-\infty}^\infty \cos \left( t^2 - \frac{1}{4} \right) dt \\ &= \frac{1}{2} \int_{-\infty}^\infty \left[ \cos \left( \frac{1}{4} \right) \cos t^2 + \sin \left( \frac{1}{4} \right) \sin t^2 \right] dt \\ &= \frac{1}{2} \sqrt{\frac{\pi}{2}} \left[ \cos \left( \frac{1}{4} \right) + \sin \left( \frac{1}{4} \right) \right]\end{aligned}$$

$$= \frac{\sqrt{\pi}}{2} \cos\left(\frac{\pi-1}{4}\right)$$

□

## 69. 计算积分

$$\int_0^{\infty} \frac{1 - \cos x}{x(e^x - 1)} dx$$

**解** 考虑积分  $I(a) = \int_0^{\infty} \frac{1 - \cos(ax)}{x(e^x - 1)} dx$ , 则

$$\begin{aligned} I'(a) &= \int_0^{\infty} \frac{\sin(ax)}{e^x - 1} dx \\ &= \int_0^{\infty} \sin(ax) \sum_{n=1}^{\infty} e^{-nx} dx \\ &= \sum_{n=1}^{\infty} \sin(ax) e^{-nx} dx \\ &= \sum_{n=1}^{\infty} \frac{a}{n^2 + a^2} \end{aligned}$$

因此

$$I = I(1) = \int_0^1 \sum_{n=1}^{\infty} \frac{a}{n^2 + a^2} da = \frac{1}{2} \sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right) = \frac{1}{2} \ln\left(\frac{\sinh(\pi)}{\pi}\right)$$

□

## 70. 求和

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{1+n^2}\right)$$

**解**

$$\begin{aligned} \sum_{n=1}^{\infty} \arctan\left(\frac{1}{1+n^2}\right) &= \sum_{n=1}^{\infty} \arg\left(1 + \frac{i}{1+n^2}\right) \\ &= \arg \prod_{n=1}^{\infty} \left(1 + \frac{1+i}{n^2}\right) \\ &= \arg \frac{\prod_{n=1}^{\infty} \left(1 + \frac{1+i}{n^2}\right)}{\prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right)} \\ &= \arg \frac{\prod_{n=1}^{\infty} \left(1 + \frac{1+i}{n^2}\right)}{\frac{\sinh(\pi)}{\pi}}. \end{aligned}$$

令  $(1+i)\pi^2 = (u+iv)^2$ ,  $u = \sqrt{\frac{1+\sqrt{2}}{2}}\pi$ ,  $v = \sqrt{\frac{\sqrt{2}-1}{2}}\pi$ , 则

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{1+n^2}\right) = \arg\left[\frac{\pi}{\sinh(\pi)} \frac{\sinh[\pi(u+iv)]}{u+iv}\right]$$

$$\begin{aligned}
 &= \arg \left( \frac{\pi \sinh(u + iv)}{(u + iv) \sinh(\pi)} \right) \\
 &= \arg(\tanh u + i \tan v) - \frac{1}{2} \arg(1 + i) \\
 &= \arctan \left( \frac{\tan v}{\tanh u} \right) - \frac{\pi}{8} \\
 &= \arctan \left( \frac{\tan \sqrt{\frac{1+\sqrt{2}}{2}} \pi}{\tan \sqrt{\frac{\sqrt{2}-1}{2}} \pi} \right) - \frac{\pi}{8}.
 \end{aligned}$$

利用公式  $\arctan a - \arctan b = \arctan \left( \frac{a-b}{1+ab} \right)$ , ( $a, b > 0$ ) 以及下面几个等式

$$\begin{aligned}
 \frac{2}{n^2} &= \frac{2}{1 + (n-1)(n+1)} \\
 \frac{1}{2n^2} &= \frac{2}{1 + (2n-1)(2n+1)} \\
 \frac{2n}{n^4 + n^2 + 2} &= \frac{2n}{1 + (n^2 - n + 1)(n^2 + n + 1)}
 \end{aligned}$$

可求出下面的反正切和

$$\begin{aligned}
 \sum_{n=1}^{\infty} \arctan \left( \frac{2}{n^2} \right) &= \sum_{n=1}^{\infty} [\arctan(n+1) - \arctan(n-1)] = \frac{\pi}{2} \\
 \sum_{n=1}^{\infty} \arctan \left( \frac{1}{2n^2} \right) &= \sum_{n=1}^{\infty} [\arctan(2n+1) - \arctan(2n-1)] = \frac{\pi}{4} \\
 \sum_{n=1}^{\infty} \arctan \left( \frac{2n}{n^4 + n^2 + 2} \right) &= \sum_{n=1}^{\infty} [\arctan(n^2 + n + 1) - \arctan(n^2 - n + 1)] = \frac{\pi}{4}
 \end{aligned}$$

□

71. 数列  $\{a_n\}$  定义为  $a_1 = 2, a_2 = 8, a_n = 4a_{n-1} - a_{n-2} (n = 2, 3, \dots)$ , 求和  $\sum_{n=1}^{\infty} \operatorname{arccot}(a_n^2)$ .

**解** 利用递推式可得

$$\begin{aligned}
 a_n(4a_{n-1}) &= a_{n-1}a_n \\
 \Rightarrow a_n(a_n + a_{n-2}) &= a_{n-1}(a_{n+1} + a_{n-1}) \\
 \Rightarrow a_n^2 - a_{n-1}a_{n+1} &= a_{n-1}^2 - a_na_{n-2}
 \end{aligned}$$

根据上述递推关系可得, 对  $\forall n \geq 2$ ,

$$a_n^2 - a_{n-1}a_{n+1} = a_{n-1}^2 - a_na_{n-2} = \dots = a_2^2 - a_1a_3 = 4.$$

根据反余切公式  $\operatorname{arccot} a - \operatorname{arccot} b = \operatorname{arccot} \left( \frac{1+ab}{b-a} \right)$  可得

$$\operatorname{arccot} \left( \frac{a_{n+1}}{a_n} \right) - \operatorname{arccot} \left( \frac{a_n}{a_{n-1}} \right) = \operatorname{arccot} \left( \frac{1 + \frac{a_{n+1}}{a_n} \cdot \frac{a_n}{a_{n-1}}}{\frac{a_n}{a_{n-1}} - \frac{a_{n+1}}{a_n}} \right)$$

$$\begin{aligned}
 &= \operatorname{arccot} \left[ \frac{a_n(a_{n-1} + a_{n+1})}{a_n^2 - a_{n-1}a_{n+1}} \right] \\
 &= \operatorname{arccot} \left[ \frac{a_n(4a_n)}{4} \right] \\
 &= \operatorname{arccot} a_n^2.
 \end{aligned}$$

由特征根方法可得  $\{a_n\}$  的通项公式为  $a_n = \frac{1}{\sqrt{3}} \left[ (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \right]$ , 于是

$$\begin{aligned}
 \sum_{n=1}^{\infty} \operatorname{arccot}(a_n^2) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \operatorname{arccot}(a_k^2) \\
 &= \operatorname{arccot} a_1^2 + \lim_{n \rightarrow \infty} \sum_{k=2}^n \left[ \operatorname{arccot} \left( \frac{a_{k+1}}{a_k} \right) - \operatorname{arccot} \left( \frac{a_k}{a_{k-1}} \right) \right] \\
 &= \operatorname{arccot} a_1^2 + \lim_{n \rightarrow \infty} \left[ \operatorname{arccot} \left( \frac{a_{n+1}}{a_n} \right) - \operatorname{arccot} \left( \frac{a_2}{a_1} \right) \right] \\
 &= \lim_{n \rightarrow \infty} \operatorname{arccot} \left( \frac{a_{n+1}}{a_n} \right) = \operatorname{arccot}(2 + \sqrt{3}) = \frac{\pi}{12}.
 \end{aligned}$$

□

## 72. 计算积分

$$\int_0^{\infty} \frac{(\arctan x)^3}{x^{\frac{3}{2}}} dx.$$

**解** 考虑积分  $I(a) = \int_0^{\pi} \frac{\cos(ax)}{\sin^b x} dx = 2 \int_0^{\frac{\pi}{2}} \frac{\cos(2ax)}{\sin^b(2x)} dx$ .

那么

$$\begin{aligned}
 I''(a) &= -8 \int_0^{\frac{\pi}{2}} x^2 \frac{\cos(2ax)}{\sin^b(2x)} dx, \\
 I''\left(\frac{1}{2}\right) \Big|_{b=\frac{1}{2}} &= -\frac{8}{\sqrt{2}} \int_0^{\frac{\pi}{2}} x^2 \sqrt{\cot x} dx = -4\sqrt{2}A.
 \end{aligned}$$

回到原积分

$$\begin{aligned}
 \int_0^{\infty} \frac{(\arctan x)^3}{x^{\frac{3}{2}}} dx &= 6 \int_0^{\infty} \frac{(\arctan x)^2}{\sqrt{x}(1+x^2)} dx \\
 &= 6 \int_0^{\frac{\pi}{2}} x^2 \sqrt{\cot x} dx \\
 &= 6A.
 \end{aligned}$$

由例 42 得  $I(a) = \frac{\pi \cdot 2^b \cdot \cos\left(\frac{\pi a}{2}\right) \Gamma(1-b)}{\Gamma\left(\frac{a}{2} - \frac{b}{2} + 1\right) \Gamma\left(-\frac{a}{2} - \frac{b}{2} + 1\right)}$ , 且

$$\begin{aligned}
 I''\left(\frac{1}{2}\right) \Big|_{b=\frac{1}{2}} &= -\frac{5\pi^3}{2} + \pi \ln^2 2 + \pi^2 \ln 2, \\
 A &= \frac{\sqrt{2}}{8} \left( \frac{5\pi^3}{12} - \pi \ln^2 2 - \pi^2 \ln 2 \right).
 \end{aligned}$$



最后得到

$$\int_0^{\infty} \frac{(\arctan x)^3}{x^{\frac{3}{2}}} dx = 6A = \frac{5\pi^3}{8\sqrt{2}} - \frac{3\pi \ln^2 2}{2\sqrt{2}} - \frac{3\pi^2 \ln 2}{2\sqrt{2}}.$$

□

### 73. 计算积分

$$\int_0^{\infty} \frac{1}{\sinh^{\frac{1}{4}} x \cosh x} dx.$$

解

$$\begin{aligned} \int_0^{\infty} \frac{1}{\sinh^{\frac{1}{4}} x \cosh x} dx &= \int_0^{\infty} \frac{1}{\sinh^x \frac{1}{4} (1 + \sinh^2 x)} d(\sinh x) \\ &= \int_0^{\infty} \frac{1}{x^{\frac{1}{4}} (1 + x^2)} dx \xrightarrow{t = \frac{1}{1+x^2}} \frac{1}{2} \int_0^1 t^{\frac{3}{8}} (1-t)^{\frac{1}{8}} dt \\ &= \frac{1}{2} B\left(\frac{5}{8}, \frac{3}{8}\right) = \frac{\pi}{2 \sin\left(\frac{3}{8}\pi\right)} \\ &= \frac{\pi}{\sqrt{2 + \sqrt{2}}}. \end{aligned}$$

□

### 74. 计算积分

$$\int_0^{\infty} \frac{\ln x}{\sqrt{x} \sqrt{x+1} \sqrt{2x+1}} dx.$$

解

$$\begin{aligned} I &= - \int_0^{\infty} \frac{\ln(2u)}{\sqrt{u} \sqrt{u+1} \sqrt{2u+1}} du \\ &= - \int_0^{\infty} \frac{\ln(2)}{\sqrt{u} \sqrt{u+1} \sqrt{2u+1}} du - \int_0^{\infty} \frac{\ln(u)}{\sqrt{u} \sqrt{u+1} \sqrt{2u+1}} du \\ &= - \int_0^{\infty} \frac{\ln(2)}{\sqrt{u} \sqrt{u+1} \sqrt{2u+1}} du - I \\ \Rightarrow I &= - \frac{\ln 2}{2} \int_0^{\infty} \frac{dx}{\sqrt{x} \sqrt{x+1} \sqrt{2x+1}} \\ &= - \frac{\ln 2}{2} \int_1^{\infty} \frac{du}{\sqrt{u-1} \sqrt{u+1} \sqrt{u}} \\ &= - \frac{\ln 2}{2} \int_1^{\infty} \frac{du}{\sqrt{u^2-1} \sqrt{u}} \\ &= - \frac{\ln 2}{2} \int_1^0 \frac{t^{3/2} (-1)}{1-t^2} \frac{(-1)}{t^2} dt \\ &= - \frac{\ln 2}{2} \int_0^1 \frac{dt}{\sqrt{t} \sqrt{1-t^2}} \\ &= - \frac{\ln 2}{4} \int_0^1 \frac{dw}{w^{3/4} \sqrt{1-w}} \\ &= - \frac{\ln 2}{4} B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{\ln 2}{4} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \end{aligned}$$

$$= -\frac{\ln 2}{2\sqrt{2}} \frac{\pi^{3/2}}{\Gamma^2\left(\frac{3}{4}\right)}.$$

□

### 75. 计算积分

$$\int_0^\infty \frac{\sin\left(\frac{\pi}{4} - x\right)}{\sqrt{x}(x^2 + 1)} dx$$

解

$$\begin{aligned} & \int_0^\infty \frac{\sin\left(\frac{\pi}{4} - x\right)}{\sqrt{x}(x^2 + 1)} dx \\ &= \frac{\sqrt{2}}{2} \int_0^\infty \frac{\cos x - \sin x}{\sqrt{x}(x^2 + 1)} dx = \frac{\sqrt{2}}{2} \int_{-\infty}^\infty \frac{\cos(x^2) - \sin(x^2)}{x^4 + 1} dx \\ &= 2\pi i \cdot \frac{\sqrt{2}}{2} \left[ \operatorname{Res}\left(\frac{\cos(x^2) - \sin(x^2)}{x^4 + 1}, \frac{1+i}{\sqrt{2}}\right) + \operatorname{Res}\left(\frac{\cos(x^2) - \sin(x^2)}{x^4 + 1}, \frac{-1+i}{\sqrt{2}}\right) \right] \\ &= \sqrt{2}\pi i \cdot \left[ -\frac{1+i}{8} (\sqrt{2} \cosh(1) - \sqrt{2} \sinh(1)) + \frac{1+i}{8} (\sqrt{2} \sinh(1) - \sqrt{2} \cosh(1)) \right] \\ &= \frac{\pi}{2e}. \end{aligned}$$

□

### 76. 计算积分

$$\int_0^1 \frac{\ln(1+x)}{x(x^2+1)} dx$$

解 令  $f(a) = \int_0^1 \frac{\ln(1+ax)}{x(x^2+1)} dx$ , 则

$$f'(a) = \int_0^1 \frac{dx}{(1+ax)(1+x^2)} = \frac{\pi}{4(1+a^2)} - \frac{a \ln 2}{2(1+a^2)} + \frac{a \ln(1+a)}{1+a^2}.$$

于是

$$\begin{aligned} I &= \int_0^1 \frac{\ln(1+x)}{x(x^2+1)} dx = f(1) = \int_0^1 f'(a) da \\ &= \frac{\pi^2}{16} - \frac{1}{4} \ln^2 2 + \int_0^1 \frac{a \ln(1+a)}{1+a^2} da \end{aligned}$$

令  $g(b) = \int_0^1 \frac{a \ln(1+ba)}{1+a^2} da$ , 则

$$g'(b) = \int_0^1 \frac{a^2}{(1+ba)(1+a^2)} da = \frac{\ln(1+b)}{b} - f'(b)$$

因此

$$g(1) = \int_0^1 \frac{\ln(1+b)}{b} db - I = \frac{\pi^2}{12} - I$$

所以

$$I = \frac{\pi^2}{16} - \frac{1}{4} \ln^2 2 + \frac{\pi^2}{12} - I \Rightarrow I = \frac{7\pi^2}{96} - \frac{\ln^2 2}{8}.$$

□

## 77. 计算积分

$$\int_0^{\infty} \frac{x^2}{2e^x - 1} dx.$$

解

$$\begin{aligned} \int_0^{\infty} \frac{x^2}{2e^x - 1} dx &= \frac{1}{2} \int_0^{\infty} \frac{x^2 e^{-x}}{1 - \frac{1}{2}e^{-x}} dx \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \int_0^{\infty} x^2 e^{-x} \left( \frac{1}{2} e^{-x} \right)^n dx \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^{\infty} x^2 e^{-(n+1)x} dx \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n (n+1)^3} \\ &= \lim_{x \rightarrow \frac{1}{2}} \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^3} = 2\text{Li}_3\left(\frac{1}{2}\right) \\ &= 2 \cdot \frac{21\zeta(3) + 4\ln^3 2 - 2\pi^2 \ln 2}{24} \\ &= \frac{7}{4}\zeta(3) + \frac{1}{3}\ln^3 2 - \frac{\pi^2}{6} \ln 2. \end{aligned}$$

□

## 78. 计算积分

$$\int_0^{\frac{\pi}{2}} \ln(1 + \sin x) \ln(1 + \cos x) \tan x dx.$$

解

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \ln(1 + \sin x) \ln(1 + \cos x) \tan x dx \\ &= \int_0^1 \ln\left(1 + \sqrt{1-x^2}\right) \frac{\ln(1+x)}{x} dx \\ &= -\text{Li}_2(-x) \ln\left(1 + \sqrt{1-x^2}\right) \Big|_0^1 + \int_0^1 \text{Li}(-x) \left( \frac{1}{x} - \frac{1}{x\sqrt{1-x^2}} \right) dx \\ &= \text{Li}_3(-1) - \int_0^1 \frac{\text{Li}_2(-x)}{x\sqrt{1-x^2}} dx \\ &= -\frac{3}{4}\zeta(3) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} B\left(\frac{k}{2}, \frac{1}{2}\right) \end{aligned}$$

利用  $\frac{(-1)^{k-1}}{k^2} B\left(\frac{k}{2}, \frac{1}{2}\right) x^{k-1} = \frac{\pi - 2 \arcsin x}{\sqrt{1-x^2}}$  可得

$$\begin{aligned} \frac{(-1)^{k-1}}{k^2} B\left(\frac{k}{2}, \frac{1}{2}\right) &= \int_0^{\frac{\pi}{2}} \frac{\pi x \cos x - x^2 \cos x}{\sin x} dx \\ &= \frac{\pi^2}{2} \ln 2 - \frac{\pi^2}{4} \ln 2 + \frac{7}{8}\zeta(3) \\ &= \frac{\pi^2}{4} \ln 2 + \frac{7}{8}\zeta(3) \end{aligned}$$

因此

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \ln(1 + \sin x) \ln(1 + \cos x) \tan x dx \\ &= -\frac{3}{4}\zeta(3) + \frac{1}{2} \left( \frac{\pi^2}{4} \ln 2 + \frac{7}{8}\zeta(3) \right) \\ &= \frac{\pi^2}{8} \ln 2 - \frac{5}{16}\zeta(3). \end{aligned}$$

□

#### 79. 计算积分

$$\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4} \arccos\left(\frac{2x}{1+x^2}\right) dx.$$

解

$$\begin{aligned} & \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4} \arccos\left(\frac{2x}{1+x^2}\right) dx \\ &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\tan^4 t}{1-\tan^2 t} \left(\frac{\pi}{2} - t\right) dt \\ &= \pi \int_0^{\frac{\pi}{6}} \frac{\tan^4 t}{1-\tan^2 t} dt \\ &= -\pi \int_0^{\frac{\pi}{6}} (1 + \tan^2 t) dt + \pi \int_0^{\frac{\pi}{6}} \frac{1}{1-\tan^2 t} dt \\ &= -\frac{\pi}{\sqrt{3}} + \pi \int_0^{\frac{\pi}{6}} \frac{1 + \cos 2t}{2 \cos 2t} dt \\ &= -\frac{\pi}{\sqrt{3}} + \frac{\pi^2}{12} + \frac{\pi}{4} \ln \left( \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right). \end{aligned}$$

□

#### 80. 计算积分

$$\int_0^\infty e^{-at} \sin(bt) \frac{\ln t}{t} dt.$$

解 考虑积分

$$\begin{aligned} I(s) &= \int_0^\infty t^{s-1} e^{-at} \sin(bt) dt \\ &= \Im \int_0^\infty e^{-at+ibt} t^{s-1} dt \\ &= \Im \left[ \frac{1}{(a-ib)^s} \int_0^\infty x^{s-1} e^{-x} dx \right] \quad (x = (a-ib)t) \\ &= \Im \left[ \frac{\Gamma(s)(a+ib)^s}{(a^2+b^2)^s} \right] \\ &= \frac{\Gamma(s)}{(a^2+b^2)^s} \Im \left[ e^{\frac{s}{2} \ln(a^2+b^2) + is \arctan(\frac{b}{a})} \right] \\ &= \frac{\Gamma(s)}{(a^2+b^2)^{\frac{s}{2}}} \sin \left[ s \arctan \left( \frac{b}{a} \right) \right] \end{aligned}$$

那么

$$\int_0^{\infty} e^{-at} \sin(bt) \frac{\ln t}{t} dt = I'(0) = -\left(\frac{\ln(a^2 + b^2)}{2} + \gamma\right) \arctan\left(\frac{b}{a}\right).$$

□

81. 设  $a, b > 0$ , 计算积分

$$\int_0^{\infty} e^{-x^2} \left[ \cos\left(\frac{a^2}{x^2}\right) + \sin\left(\frac{a^2}{x^2}\right) \right] dx$$

**解** 由例 11 的结论得

$$\begin{aligned} \int_0^{\infty} e^{-x^2 + i \frac{a^2}{x^2}} dx &= \frac{\sqrt{\pi}}{2} e^{-2a\sqrt{i}} \\ &= \frac{\sqrt{\pi}}{2} e^{-\sqrt{2}a(1-i)} \\ &= \frac{\sqrt{\pi}}{2} e^{-\sqrt{2}a} e^{i\sqrt{2}a} \\ &= \frac{\sqrt{\pi}}{2} e^{-\sqrt{2}a} \left[ \cos(\sqrt{2}a) + i \sin(\sqrt{2}a) \right]. \end{aligned}$$

因此

$$\int_0^{\infty} e^{-x^2} \left[ \cos\left(\frac{a^2}{x^2}\right) + \sin\left(\frac{a^2}{x^2}\right) \right] dx = \frac{\sqrt{\pi}}{2} e^{-\sqrt{2}a} \left[ \cos(\sqrt{2}a) + \sin(\sqrt{2}a) \right].$$

□

82. 计算积分

$$\int_0^{\infty} \sin(x^2) \operatorname{erf}(x) dx.$$

**解** 考虑积分  $f(a) = \int_0^{\infty} \sin(x^2) \operatorname{erf}(ax) dx$ , 则

$$\begin{aligned} f'(a) &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} x e^{-a^2 x^2} \sin(x^2) dx \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} t e^{-a^2 t} \sin t dt \\ &= \frac{1}{\sqrt{\pi}(1+a^4)}. \end{aligned}$$

$$\begin{aligned} \int_0^{\infty} \sin(x^2) \operatorname{erf}(x) dx &= I(1) = \int_0^1 \frac{1}{\sqrt{\pi}(1+a^4)} da + I(0) \\ &= \frac{\pi + 2 \coth^{-1}(\sqrt{2})}{4\sqrt{2}\pi} + \frac{1}{2} \int_0^1 \sin(x^2) dx \\ &= \frac{\pi + \coth^{-1}(\sqrt{2})}{2\sqrt{2}\pi}. \end{aligned}$$

□

83. 计算积分

$$\int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{2x}{3}\right)}{\tan x} dx.$$

解

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{2x}{3}\right)}{\tan x} dx &= \frac{3}{2} \int_0^{\frac{\pi}{3}} \frac{\sin x}{\tan\left(\frac{3x}{2}\right)} dx \\ &= \frac{3}{2} \int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{2u}{1+u^2}}{\frac{3u-u^3}{1-3u^2}} \frac{2du}{1+u^2} \\ &= 6 \int_0^{\frac{1}{\sqrt{3}}} \frac{1-3u^2}{(3-u^2)(1+u^2)^2} du \\ &= \left[ \left( \frac{6u}{u^2+1} + \sqrt{3} \ln(\sqrt{3}-u) - \sqrt{3} \ln(u+\sqrt{3}) \right) \right]_0^{\frac{1}{\sqrt{3}}} \\ &= \frac{\sqrt{3}}{4} (3 - 2 \ln 2). \end{aligned}$$

□

84. 已知  $K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$ , 计算积分

$$\int_0^1 \frac{K(k)}{1+k} dk$$

解

$$\begin{aligned} I &= \int_0^1 \frac{K(k)}{1+k} dk \\ &= \int_0^1 \frac{1}{\sqrt{1-t^2}} dt \int_0^1 \frac{1}{(1+k)\sqrt{1-k^2t^2}} dk \\ &= \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{\ln(1+\sqrt{1-t^2})}{\sqrt{1-t^2}} dt \\ &= \sum_{n=0}^{\infty} \int_0^1 t^{2n} \ln(1+\sqrt{1-t^2}) dt \\ &= \sum_{n=0}^{\infty} \frac{\sqrt{\pi} (n+\frac{1}{2})! - n!}{(2n+1)^2 n!} \\ &= \frac{\pi^2}{8}. \end{aligned}$$

□

85. 求和

$$\sum_{n=1}^{\infty} \frac{1}{2^n (1 + \sqrt[n]{2})}.$$

解 首先注意到

$$\frac{1}{2^n (\sqrt[n]{2} - 1)} - \frac{1}{2^n (\sqrt[n]{2} + 1)} = \frac{1}{2^{n-1} (\sqrt[n]{2} - 1)}.$$

于是得到

$$\frac{1}{2^n \left( \sqrt[n]{2} + 1 \right)} = \left[ \frac{1}{2^n \left( \sqrt[n]{2} - 1 \right)} - 1 \right] - \left[ \frac{1}{2^{n-1} \left( \sqrt[n-1]{2} - 1 \right)} - 1 \right]$$

且当  $n = 1$  时,

$$\frac{1}{2^{n-1} \left( \sqrt[n-1]{2} - 1 \right)} - 1 = 0.$$

因此可求得部分和

$$\sum_{n=1}^m \frac{1}{2^n \left( 1 + \sqrt[n]{2} \right)} = \frac{1}{2^m \left( \sqrt[m]{2} - 1 \right)} - 1.$$

令  $m \rightarrow \infty$  可得

$$\sum_{n=1}^{\infty} \frac{1}{2^n \left( 1 + \sqrt[n]{2} \right)} = \frac{1}{\ln 2} - 1.$$

□

## 86. 计算积分

$$\int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x} dx.$$

**解** 首先考虑积分

$$\begin{aligned} I(s) &= \int_0^1 x^{p-1} (1-x)^{1-s} dx - \int_0^1 x^{q-1} (1-x)^{1-s} dx \\ &= B(p, s) - B(q, s) \\ &= \Gamma(s) \left[ \frac{\Gamma(p)}{\Gamma(p+s)} - \frac{\Gamma(q)}{\Gamma(q+s)} \right] \\ &= \Gamma(1+s) \left[ \frac{\Gamma(p) - \Gamma(p+s)}{s} - \frac{\Gamma(q) - \Gamma(q+s)}{s} \right]. \end{aligned}$$

于是

$$\int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x} dx = \lim_{s \rightarrow 0} I(s) = \psi(q) - \psi(p).$$

特别地,

$$\begin{aligned} \int_0^1 \frac{1 - x^{q-1}}{1-x} dx &= \gamma + \psi(q), \\ \int_0^1 \frac{x^{p-1} - x^{-p}}{1-x} dx &= \psi(1-p) - \psi(p) = \pi \cot \pi p. \end{aligned}$$

□

## 87. 求和

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(k+1)(2k+1)}.$$

**解** 根据  $\cot x$  的幂级数展开得到

$$\sum_{k=1}^{\infty} \zeta(2k) x^{2k} = \frac{1 - \pi x \cot \pi x}{2}.$$

于是逐项积分得

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)x^{2k+2}}{(2k+1)(2k+2)} = \int_0^x dv \int_0^v \frac{1 - \pi u \cot \pi u}{2} du.$$

注意到

$$\begin{aligned} \int_0^1 dv \int_0^v \frac{1 - \pi u \cot \pi u}{2} du &= - \int_0^1 dv \int_0^v \frac{\pi u \cot \pi u}{2} du + \int_0^1 dv \int_0^v \frac{du}{2} \\ &= -\frac{1}{2} \int_0^1 \pi u \cot \pi u du \int_u^1 dv + \frac{1}{4} \\ &= \frac{1}{4} - \frac{\pi}{2} \int_0^1 u(1-u) \cot \pi u du \\ &= \frac{1}{4} + \frac{\pi}{2} \int_0^1 u(1-u) \cot \pi u du \\ &= \frac{1}{4}. \end{aligned}$$

因此  $\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(k+1)(2k+1)} = \frac{1}{2}.$

□

88. 求和

$$\sum_{k=1}^{\infty} \frac{1}{k^4 + k^2 + 1}.$$

**解** 令  $\omega = e^{\frac{2\pi i}{3}}$ , 则

$$n^4 + n^2 + 1 = (n^2 - \omega)(n^2 - \omega^2),$$

于是

$$\frac{1}{1 + n^2 + n^4} = \frac{1}{i\sqrt{3}} \left( \frac{1}{n^2 - \omega} - \frac{1}{n^2 - \omega^2} \right).$$

利用

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a} = \frac{-1 + \pi\sqrt{a} \coth(\pi\sqrt{a})}{2a}$$

可得

$$\sum_{k=1}^{\infty} \frac{1}{k^4 + k^2 + 1} = \frac{1}{\sqrt{3}} \Im \left( \sum_{n=1}^{\infty} \frac{1}{n^2 - \omega} \right) = \frac{1}{6} \left[ -3 + \pi\sqrt{3} \tanh \left( \frac{\pi\sqrt{3}}{2} \right) \right].$$

□

89. 求和

$$\sum_{n=1}^{\infty} \frac{\sinh \pi}{\cosh(2n\pi) - \cosh \pi}.$$

**解** 在等式

$$\sum_{n=1}^{\infty} e^{-nt} \sin nx = \frac{1}{2} \frac{\sin x}{\cosh t - \cos x}$$



中令  $x = \pi i, t = 2k\pi$  得

$$\frac{\sinh \pi}{\cosh(2k\pi) - \cosh \pi} = 2 \sum_{n=1}^{\infty} e^{-2nk\pi} \sinh(n\pi).$$

于是

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sinh \pi}{\cosh(2n\pi) - \cosh \pi} &= 2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} e^{-2nk\pi} \sinh(n\pi) \\ &= 2 \sum_{n=1}^{\infty} \sinh(n\pi) \sum_{k=1}^{\infty} e^{-2nk\pi} \\ &= 2 \sum_{n=1}^{\infty} \frac{\sinh(n\pi)}{e^{2n\pi} - 1} \\ &= \sum_{n=1}^{\infty} e^{-n\pi} = \frac{1}{e^{\pi} - 1}. \end{aligned}$$

□

## 90. 求和

$$\sum_{n=1}^{\infty} \frac{\zeta(2n) - \zeta(3n)}{n}.$$

解

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\zeta(2n) - \zeta(3n)}{n} &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{\infty} \left( \frac{1}{k^{2n}} - \frac{1}{k^{3n}} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=2}^{\infty} \left( \frac{1}{k^{2n}} - \frac{1}{k^{3n}} \right) \\ &= \sum_{k=2}^{\infty} \left[ \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{1}{k^2} \right)^n - \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{1}{k^3} \right)^n \right] \\ &= \sum_{k=2}^{\infty} \left[ -\ln \left( 1 - \frac{1}{k^2} \right) + \ln \left( 1 - \frac{1}{k^3} \right) \right] \\ &= -\ln \left[ \prod_{k=2}^{\infty} \ln \left( 1 - \frac{1}{k^2} \right) \right] + \ln \left[ \prod_{k=2}^{\infty} \ln \left( 1 - \frac{1}{k^3} \right) \right] \\ &= -\ln \left( \frac{1}{2} \right) + \ln \left[ \frac{\cosh \left( \frac{\sqrt{3}\pi}{2} \right)}{3\pi} \right] \\ &= \ln \left[ \frac{2 \cosh \left( \frac{\sqrt{3}\pi}{2} \right)}{3\pi} \right]. \end{aligned}$$

□

## 91. 求和

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3 + 1}.$$

解

$$\begin{aligned}
 S &= \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3 + 1} \\
 &= \sum_{n=1}^{\infty} (-1)^n \frac{1}{3} \left( \frac{2n-1}{n^2-n+1} + \frac{1}{n+1} \right) \\
 &= \frac{1}{3} \sum_{n=1}^{\infty} (-1)^n \frac{2n-1}{n^2-n+1} + \frac{1}{3} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n+1} \\
 &= \frac{1}{3} N + \frac{1}{3} (\ln 2 - 1).
 \end{aligned}$$

其中  $N = \sum_{n=1}^{\infty} (-1)^n \frac{2n-1}{n^2-n+1}$ , 考虑函数  $f(z) = \pi \csc(\pi z) \frac{2z-1}{z^2-z+1}$ ,

$$\begin{aligned}
 \sum_{n=1}^{\infty} \operatorname{Res}(f(z), z=n) &= \sum_{n=1}^{\infty} \frac{2n-1}{n^2-n+1} \lim_{z \rightarrow n} \frac{\pi}{\sin(\pi z)} (z-n) \\
 &= \sum_{n=1}^{\infty} \frac{2n-1}{n^2-n+1} \lim_{z \rightarrow n} \frac{\pi}{\pi \cos(\pi z)} \\
 &= \sum_{n=1}^{\infty} (-1)^n \frac{2n-1}{n^2-n+1} = N.
 \end{aligned}$$

$$\begin{aligned}
 \operatorname{Res}\left(f(z), z = \frac{1+\sqrt{3}i}{2}\right) &= \frac{\pi \left(2^{\frac{1+\sqrt{3}i}{2}} - 1\right)}{\sin\left(\pi \frac{1+\sqrt{3}i}{2}\right)} \lim_{z \rightarrow \frac{1+\sqrt{3}i}{2}} \frac{z - \frac{1+\sqrt{3}i}{2}}{z^2 - z + 1} \\
 &= \frac{\sqrt{3}\pi i}{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)} \cdot \frac{-i}{\sqrt{3}} = \frac{\pi}{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)} \\
 &\Rightarrow N + \frac{\pi}{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)} = 0.
 \end{aligned}$$

于是

$$S = \frac{1}{3} \left( -\frac{\pi}{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)} \right) + \frac{1}{3} (\ln 2 - 1) = \frac{1}{3} \left( \ln 2 - 1 - \frac{\pi}{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)} \right).$$

□

## 92. 计算积分

$$\int_0^{\infty} \frac{x e^{-x}}{e^x + e^{-x} - 1} dx.$$

解

$$\begin{aligned}
 \int_0^{\infty} \frac{x e^{-x}}{e^x + e^{-x} - 1} dx &= \int_0^{\infty} \frac{x e^{-2x}}{e^{-2x} - e^{-x} + 1} dx \\
 &= \int_0^{\infty} \frac{x e^{-2x} (1 + e^{-x})}{1 + e^{-3x}} dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^\infty \sum_{n=0}^\infty (-1)^n e^{-(2n+1)x} (1 + e^{-x}) x dx \\
 &= \sum_{n=0}^\infty (-1)^n \int_0^\infty [x e^{-(3n+2)x} + x e^{-(3n+3)x}] dx \\
 &= \sum_{n=0}^\infty (-1)^n \left[ \frac{1}{(3n+2)^2} + \frac{1}{(3n+3)^2} \right] \\
 &= \frac{1}{36} \left[ \psi' \left( \frac{1}{3} \right) - \psi' \left( \frac{5}{6} \right) \right] + \frac{\pi^2}{108}.
 \end{aligned}$$

□

93. 设  $a > 0$ , 计算积分

$$\int_0^\infty [e^{-x} - (1+x)^{-a}] \frac{dx}{x}.$$

解 首先根据 Frullani 积分可得

$$\int_0^\infty \int_1^s e^{-tz} dt dz = \int_0^\infty \frac{e^{-z} - e^{-sz}}{z} dz = \ln s.$$

于是可得

$$\begin{aligned}
 \Gamma'(a) &= \int_0^\infty e^{-s} s^{a-1} \ln s ds \\
 &= \int_0^\infty e^{-s} s^{a-1} \int_0^\infty \frac{e^{-z} - e^{-zs}}{z} dz ds \\
 &= \int_0^\infty \left( e^{-z} \int_0^\infty s^{a-1} e^{-s} ds - \int_0^\infty s^{a-1} e^{-s(1+z)} ds \right) \frac{dz}{z} \\
 &= \Gamma(a) \int_0^\infty [e^{-x} - (1+x)^{-a}] \frac{dx}{x}
 \end{aligned}$$

因此

$$\int_0^\infty [e^{-x} - (1+x)^{-a}] \frac{dx}{x} = \frac{\Gamma'(a)}{\Gamma(a)} = \psi(a).$$

□

94. 计算积分

$$\int_0^1 \left( \frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx.$$

解 首先考虑积分

$$\begin{aligned}
 I(s) &= \int_0^1 \left( \frac{x^{q-1}}{(1-ax)^{1-s}} - \frac{x^{-q}}{(a-x)^{1-s}} \right) dx \\
 &= \int_0^1 \frac{x^{q-1}}{(1-ax)^{1-s}} dx - \int_0^1 \frac{x^{-q}}{(a-x)^{1-s}} dx \\
 &= a^{-q} \int_0^a \frac{t^{q-1}}{(1-t)^{1-s}} dt - a^{-q+s} \int_0^{\frac{1}{a}} \frac{t^{-q}}{(1-t)^{1-s}} dt.
 \end{aligned}$$

令  $s \rightarrow 0$  得到

$$\int_0^1 \left( \frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx = a^{-q} \left( \int_0^a \frac{t^{q-1}}{1-t} dt - \int_0^{\frac{1}{a}} \frac{t^{-q}}{1-t} dt \right).$$

上式右端括号的部分对参数  $a$  求导为零, 因此是一个独立于  $a$  的常数, 于是

$$\begin{aligned}\int_0^1 \left( \frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx &= a^{-q} \left( \int_0^1 \frac{t^{q-1}}{1-t} dt - \int_0^1 \frac{t^{-q}}{1-t} dt \right) \\ &= a^{-q} \int_0^1 \frac{t^{q-1} - t^{-q}}{1-t} dt \\ &= a^{-q} [\psi(1-q) - \psi(q)] \\ &= a^{-q} \pi \cot \pi q.\end{aligned}$$

□

## 95. 计算积分

$$\int_0^\infty \frac{\ln^2 x}{a^2 + x^2} dx.$$

**解** 方法一

$$\begin{aligned}\int_0^\infty \frac{\ln^2 x}{a^2 + x^2} dx &= \lim_{s \rightarrow 0} \partial_s^2 \int_0^\infty \frac{x^s}{a^2 + x^2} dx \\ &= \frac{1}{a} \lim_{s \rightarrow 0} \partial_s^2 \left( a^s \int_0^\infty \frac{x^s}{1+x^2} dx \right) \\ &= \frac{1}{2a} \lim_{s \rightarrow 0} \partial_s^2 \left[ a^s \int_0^1 t^{-\frac{s+1}{2}} (1-t)^{\frac{s+1}{2}-1} dt \right] \quad \left( t = \frac{1}{1+x^2} \right) \\ &= \frac{1}{2a} \lim_{s \rightarrow 0} \partial_s^2 \left[ a^s B \left( \frac{s+1}{2}, 1 - \frac{s+1}{2} \right) \right] \\ &= \frac{1}{2a} \lim_{s \rightarrow 0} \partial_s^2 \left[ a^s \Gamma \left( \frac{s+1}{2} \right) \Gamma \left( 1 - \frac{s+1}{2} \right) \right] \\ &= \frac{\pi}{2a} \lim_{s \rightarrow 0} \partial_s^2 \left[ a^s \sec \left( \frac{\pi s}{2} \right) \right] \\ &= \frac{\pi}{8a} (\pi^2 + 4 \ln^2 a).\end{aligned}$$

方法二

$$\begin{aligned}\int_0^\infty \frac{\ln^2 x}{a^2 + x^2} dx &= \frac{1}{a} \int_0^{\frac{\pi}{2}} \ln^2(a \tan t) dt \quad (x = a \tan t) \\ &= \frac{1}{a} \int_0^{\frac{\pi}{2}} [\ln^2 a + \ln^2(\tan t) + 2 \ln a \ln(\tan t)] dt \\ &= \frac{1}{a} \left( \frac{\pi^3}{8} + \frac{\pi}{2} \ln^2 a \right).\end{aligned}$$

其中  $\int_0^{\frac{\pi}{2}} \ln^2(\tan t) dt = \frac{\pi^3}{8}$  参见例 6.

□

## 96. 计算积分

$$\int_0^\infty \frac{\sinh x}{x \cosh^3 x} dx.$$

**解**

$$\int_0^\infty \frac{\sinh x}{x \cosh^3 x} dx$$

$$\begin{aligned}
 &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sinh x}{x \cosh^3 x} dx \\
 &= \pi i \sum_{k=0}^{\infty} \operatorname{Res} \left( \frac{\sinh x}{x \cosh^3 x}, x = \frac{\pi i}{2} + 2k\pi i \right) + \pi i \sum_{k=1}^{\infty} \operatorname{Res} \left( \frac{\sinh x}{x \cosh^3 x}, x = -\frac{\pi i}{2} + 2k\pi i \right) \\
 &= \pi i \left( -\frac{8i}{\pi^3} \right) \pi^3 \left( \sum_{k=0}^{\infty} \frac{1}{(4k+1)^3} + \sum_{k=0}^{\infty} \frac{1}{(4k+3)^3} \right) \\
 &= \frac{7}{\pi^2} \zeta(3).
 \end{aligned}$$

□

### 97. 计算积分

$$\int_0^{\infty} e^{-x^2} \ln x dx.$$

解

$$\begin{aligned}
 \int_0^{\infty} e^{-x^2} \ln x dx &= \frac{1}{4} \int_0^{\infty} e^{-t} t^{-\frac{1}{2}} \ln t dt \\
 &= \frac{1}{4} \lim_{s \rightarrow 1} \partial_s \int_0^{\infty} e^{-t} t^{s-1-\frac{1}{2}} dt \\
 &= \frac{1}{4} \lim_{s \rightarrow 1} \partial_s \left[ \Gamma \left( s - \frac{1}{2} \right) \right] \\
 &= \frac{1}{4} \lim_{s \rightarrow 1} \Gamma \left( s - \frac{1}{2} \right) \psi \left( s - \frac{1}{2} \right) \\
 &= \frac{1}{4} \Gamma \left( \frac{1}{2} \right) \psi \left( \frac{1}{2} \right) \\
 &= \frac{\sqrt{\pi}}{4} (\gamma + 2 \ln 2).
 \end{aligned}$$

□

### 98. 计算积分

$$\int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x} dx.$$

解 方法一首先有

$$\frac{1}{\sin x} = \frac{2ie^{-ix}}{1 - e^{-2ix}} = 2ie^{-ix} \sum_{k=0}^{\infty} e^{-2kix} = 2i \sum_{k=0}^{\infty} e^{-(2k+1)ix}.$$

于是

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x} dx &= 2i \sum_{k=0}^{\infty} \int_0^{\frac{\pi}{2}} x^2 e^{-(2k+1)ix} dx \\
 &= 2i \sum_{k=0}^{\infty} (-1)^k \left[ \frac{(2\pi k + \pi)^2 + 8i(-1)^k - 4i\pi(2k+1) - 8}{4(2k+1)^2} \right] \\
 &= 2\pi G - \frac{7}{2} \zeta(3).
 \end{aligned}$$

方法二

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x} dx &= \int_0^{\frac{\pi}{2}} x^2 d \left[ \ln \left( \tan \frac{x}{2} \right) \right] \\
 &= -2 \int_0^{\frac{\pi}{2}} x \ln \left( \tan \frac{x}{2} \right) dx \\
 &= 4 \int_0^{\frac{\pi}{2}} x \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{2k-1} dx \\
 &= 4 \left[ \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2} - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \right] \\
 &= 2\pi G - \frac{7}{2} \zeta(3).
 \end{aligned}$$

□

### 99. 计算积分

$$\int_0^1 \frac{1}{(1+x\sqrt{2})^{\sqrt{2}}} dx.$$

解

$$\begin{aligned}
 \int_0^1 \frac{1}{(1+x\sqrt{2})^{\sqrt{2}}} dx &= \frac{1}{\sqrt{2}} \int_0^1 \frac{t^{\frac{\sqrt{2}}{2}-1}}{(1+t)^{\sqrt{2}}} dt \quad (t = x\sqrt{2}) \\
 &= \frac{1}{\sqrt{2}} \int_0^{\frac{1}{2}} \frac{u^{\frac{\sqrt{2}}{2}-1} (1-u)^{\sqrt{2}}}{(1-u)^{\frac{\sqrt{2}}{2}-1} (1-u)^2} du \quad \left(u = \frac{t}{1+t}\right) \\
 &= \frac{1}{\sqrt{2}} \int_{\frac{1}{2}}^1 \frac{v^{\frac{\sqrt{2}}{2}-1} (1-v)^{\sqrt{2}}}{(1-v)^{\frac{\sqrt{2}}{2}-1} (1-v)^2} dv \quad (u = 1-v) \\
 &= \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \int_0^1 u^{\frac{\sqrt{2}}{2}-1} (1-u)^{\frac{\sqrt{2}}{2}-1} du \\
 &= \frac{1}{2\sqrt{2}} B\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\
 &= \frac{1}{2\sqrt{2}} \frac{\Gamma\left(\frac{1}{\sqrt{2}}\right) \Gamma\left(\frac{1}{\sqrt{2}}\right)}{\Gamma(\sqrt{2})} \\
 &= \frac{1}{2\sqrt{2}} \frac{\Gamma^2\left(\frac{1}{\sqrt{2}}\right)}{\Gamma(\sqrt{2})}.
 \end{aligned}$$

□

### 100. 计算积分

$$\int_0^{\infty} \frac{\ln^2 x}{1+x^4} dx.$$

解

$$\int_0^{\infty} \frac{\ln^2 x}{1+x^4} dx = \lim_{s \rightarrow 0} \partial_s^2 \int_0^{\infty} \frac{x^s}{1+x^4} dx$$

$$\begin{aligned}
 & \frac{1}{4} \lim_{s \rightarrow 0} \partial_s^2 \int_0^\infty t^{\frac{s+1}{4}-1} (1-t)^{-\frac{s+1}{4}} dt \\
 &= \frac{1}{4} \lim_{s \rightarrow 0} \partial_s^2 \left[ B\left(\frac{s+1}{4}, 1 - \frac{s+1}{4}\right) \right] \\
 &= \frac{1}{4} \lim_{s \rightarrow 0} \partial_s^2 \left[ \Gamma\left(\frac{s+1}{4}\right) \Gamma\left(1 - \frac{s+1}{4}\right) \right] \\
 &= \frac{\pi}{4} \lim_{s \rightarrow 0} \partial_s^2 \left[ \csc\left(\frac{s+1}{4}\pi\right) \right] \\
 &= \frac{\pi^3}{64} \lim_{s \rightarrow 0} \left[ 1 + 2 \cot^2\left(\frac{s+1}{4}\pi\right) \csc\left(\frac{s+1}{4}\pi\right) \right] \\
 &= \frac{\pi^3}{64} \left[ 1 + 2 \cot^2\left(\frac{\pi}{4}\right) \csc\left(\frac{\pi}{4}\right) \right] \\
 &= \frac{3\sqrt{2}\pi^3}{64}.
 \end{aligned}$$

□

#### 101. 计算积分

$$\int_0^\infty \frac{e^{a \cos x} \sin(a \sin x)}{x} dx.$$

解

$$\begin{aligned}
 \int_0^\infty \frac{e^{a \cos x} \sin(a \sin x)}{x} dx &= \Im \int_0^\infty \frac{e^{ae^{ix}}}{x} dx \\
 &= \Im \int_0^\infty \frac{1}{x} \sum_{k=0}^\infty \frac{a^k e^{ikx}}{k!} dx \\
 &= \Im \int_0^\infty \frac{1}{x} \sum_{k=0}^\infty \frac{a^k \sin(kx)}{k!} dx \\
 &= \frac{\pi}{2} \sum_{k=1}^\infty \frac{a^k}{k!} \\
 &= \frac{\pi}{2} (e^a - 1).
 \end{aligned}$$

□

#### 102. 计算积分

$$\int_0^\infty \frac{\sin x}{1 + \cosh x - \sinh x} dx.$$

解

$$\begin{aligned}
 \int_0^\infty \frac{\sin x}{1 + \cosh x - \sinh x} dx &= \int_0^\infty \frac{\sin x}{1 + e^{-x}} dx \\
 &= \sum_{n=0}^\infty (-1)^n \int_0^\infty e^{-nx} \sin x dx = \sum_{n=0}^\infty \frac{(-1)^n}{n^2 + 1} \\
 &= \sum_{n=0}^\infty \frac{1}{1 + (2n)^2} - \sum_{n=0}^\infty \frac{1}{1 + (2n+1)^2} \\
 &= \frac{1}{2} + \frac{\pi}{4} \coth\left(\frac{\pi}{2}\right) - \frac{\pi}{4} \tanh\left(\frac{\pi}{2}\right)
 \end{aligned}$$

$$= \frac{1}{2} + \frac{\pi}{2} \operatorname{csch} \pi.$$

□

### 103. 计算积分

$$\int_0^\infty \frac{\sin(x^n) \ln x}{x} dx.$$

**解** 令  $u = x^n$ , 则原积分  $I = \int_0^\infty \frac{\sin u \ln u}{u} du$ , 考虑积分

$$\begin{aligned} f(t) &= \int_0^\infty \frac{\sin x}{x^t} dx \\ &= \frac{1}{\Gamma(t)} \int_0^\infty \left( \int_0^\infty e^{-xu} \sin x dx \right) u^{t-1} du \\ &= \frac{1}{\Gamma(t)} \int_0^\infty \frac{u^{t-1}}{1+u^2} du \\ &= \frac{1}{\Gamma(t)} \int_0^{\frac{\pi}{2}} \tan^{t-1} \theta d\theta \\ &= \frac{1}{2\Gamma(t)} \frac{\pi}{\sin\left(\frac{\pi t}{2}\right)}. \end{aligned}$$

于是

$$\begin{aligned} I = -f'(1) &= \frac{\pi}{2} \frac{\psi(t)\Gamma(t) \sin\left(\frac{\pi t}{2}\right) - \frac{\pi}{2} \cos\left(\frac{\pi t}{2}\right) \Gamma(t)}{\left[\Gamma(t) \sin\left(\frac{\pi t}{2}\right)\right]^2} \Big|_{t=1} \\ &= \frac{\pi}{2} \psi(1) = -\frac{\gamma\pi}{2}. \end{aligned}$$

□

**104.** 设  $f : [0, 1] \rightarrow \mathbb{R}$  是连续函数, 且  $\int_0^1 f^3(x) dx = 0$ . 求证:

$$\int_0^1 f^4(x) dx \geq \frac{27}{4} \left( \int_0^1 f(x) dx \right)^4.$$

**解** 令

$$I_n = \int_0^1 f^n(x) dx$$

由 Cauchy 不等式得

$$I_2 \geq I_1^2$$

再由 Cauchy 不等式得

$$\left( \int_0^1 (r + f^2(x)) f(x) dx \right)^2 \leq \int_0^1 (r + f^2(x))^2 dx \int_0^1 f^2(x) dx$$

展开得到

$$r^2 I_1^2 \leq r^2 I_2 + 2r I_2^2 + I_2 I_4$$



也即

$$(I_1^2 - I_2)r^2 - 2I_2^2r - I_2I_4 \leq 0$$

于是上式左边的最大值也小于等于 0, 最大值在  $r = \frac{I_2^2}{I_1^2 - I_2}$  取到, 即满足

$$\frac{I_4^4}{I_1^2 - I_2} - \frac{2I_2^4}{I_1^2 - I_2} - I_2I_4 \leq 0$$

即

$$I_4 \geq \frac{I_2^3}{I_2 - I_1^2}$$

所以只要证明

$$\frac{I_2^3}{I_2 - I_1^2} \geq \frac{27}{4}I_1^4$$

注意到

$$(I_2 - I_1^2)I_1^4 = \frac{1}{2}(2I_2 - 2I_1^2)I_1^2 \cdot I_1^2 \leq \frac{4}{27}I_2^3$$

即

$$\frac{I_2^3}{I_2 - I_1^2} \geq \frac{27}{4}I_1^4$$

故有

$$\int_0^1 f^4(x)dx \geq \frac{27}{4} \left( \int_0^1 f(x)dx \right)^4.$$

□

**105.** 设  $f$  是在  $[0, 1]$  上非负的连续的凹函数, 且  $f(0) = 1$ , 求证:

$$2 \int_0^1 x^2 f(x)dx + \frac{1}{12} \leq \left( \int_0^1 f(x)dx \right)^2$$

**证明** 设

$$F(x) = \int_0^x f(t)dt, I = \int_0^1 x^2 f(x)dx, U = \int_0^1 f(x)dx$$

由于

$$f(ax) = f(ax + (1-a) \cdot 0) \geq af(x) + (1-a)f(0) = af(x) + 1-a$$

上式对  $a$  从 0 到 1 积分得

$$\int_0^1 f(tx)dt \geq \frac{1}{2}f(x) + \frac{1}{2}$$

换元即得

$$2F(x) \geq xf(x) + x$$

另外我们有

$$I = \int_0^1 x^2 f(x)dx = x^2 F(x)|_0^1 - 2 \int_0^1 x F(x)dx \leq F(1) - 2 \int_0^1 x(xf(x) + x)dx$$

即

$$2I \leq F(1) - \frac{1}{3} = U - \frac{1}{3}$$

所以

$$U^2 - 2I - \frac{1}{12} \geq \left(2I + \frac{1}{3}\right)^2 - 2I - \frac{1}{12} = \left(2I - \frac{1}{6}\right)^2 \geq 0$$

□

106. 设  $f : [0, 1] \rightarrow \mathbb{R}$  是可积函数, 且  $|f(x)| \leq 1$ ,  $\int_0^1 xf(x)dx = 0$ . 令  $F(x) = \int_0^x f(y)dy \geq 0$ , 求证:

$$\int_0^1 f^2(x)dx + 5 \int_0^1 F^2(x)dx \geq 10 \int_0^1 f(x)F(x)dx.$$

**证明** 由于

$$\begin{aligned} \int_0^1 \left( \int_0^x f(y)dy \right) dx &= \int_0^1 \left( \int_y^1 f(y)dx \right) dy \\ \int_0^1 \left( \int_0^x f(x)f(y)dy \right) dx &= \int_0^1 \left( \int_y^1 f(x)f(y)dx \right) dy \end{aligned}$$

故有

$$\begin{aligned} \int_0^1 F(x)dx &= \int_0^1 (1-y)f(y)dy \\ \int_0^1 f(x)F(x)dx &= F^2(1) - \int_0^1 f(y)F(y)dy \end{aligned}$$

利用  $\int_0^1 xf(x)dx = 0$  得

$$\int_0^1 F(x)dx = \int_0^1 f(x)dx = F(1), \int_0^1 f(x)F(x)dx = \frac{1}{2}F^2(1) \geq 0$$

利用  $A^2 + B^2 \geq 0$  则有

$$\int_0^1 f^2(x)dx + \int_0^1 F^2(x)dx \geq 2 \int_0^1 f(x)F(x)dx$$

另外由 Cauchy 不等式得

$$4 \int_0^1 F^2(x)dx \geq \left( \int_0^1 2F(x)dx \right)^2 = 4F^2(1) = 8 \int_0^1 f(x)F(x)dx$$

相加即证得原式.

注意到有更一般的式子

$$A \int_0^1 f^2(x)dx + B \int_0^1 F^2(x)dx \geq 2(A+B) \int_0^1 f(x)F(x)dx$$

等号成立当且仅当  $A = B = 0$  或  $f(x) = F(x) = 0$ .

注意到

$$\int_0^1 f(x)dx = F(1) = \int_0^1 \frac{d(xF(x))}{dx} dx = \int_0^1 xf(x)dx + \int_0^1 F(x)dx = \int_0^1 F(x)dx$$

$$F^2(1) = \int_0^1 \frac{dF^2(x)}{dx} dx = 2 \int_0^1 f(x)F(x)dx$$

由 Cauchy 不等式得

$$\int_0^1 f^2(x)dx \geq \left( \int_0^1 f(x)dx \right)^2 = F^2(1) = 2 \int_0^1 f(x)F(x)dx$$

$$\int_0^1 F^2(x)dx \geq \left( \int_0^1 f(x)dx \right)^2 = F^2(1) = 2 \int_0^1 f(x)F(x)dx$$

相加即可.

□

**107.** 设函数  $f \in C(a, b)$  不恒为零, 满足  $0 \leq f(x) \leq M$ , 试证明:

$$\left( \int_a^b f(x)dx \right)^2 \leq \left( \int_a^b f(x) \sin x dx \right)^2 + \left( \int_a^b f(x) \cos x dx \right)^2 + \frac{M^2(b-a)^4}{12}$$

**证明** 令

$$A = \left( \int_a^b f(x)dx \right)^2 = \iint_D f(x)f(y)dxdy$$

$$B = \left( \int_a^b f(x) \sin x dx \right)^2 = \iint_D f(x)f(y) \sin x \sin y dxdy$$

$$C = \left( \int_a^b f(x) \cos x dx \right)^2 = \iint_D f(x)f(y) \cos x \cos y dxdy$$

这里区域  $D = \{(x, y) | a \leq x \leq b, a \leq y \leq b\}$ .

则有

$$B + C = \iint_D f(x)f(y)(\sin x \sin y + \cos x \cos y)dxdy = \iint_D f(x)f(y) \cos(x-y)dxdy$$

$$\begin{aligned} A - (B + C) &= \iint_D f(x)f(y)[1 - \cos(x-y)]dxdy \\ &= 2 \iint_D f(x)f(y) \sin^2\left(\frac{x-y}{2}\right)dxdy \\ &\leq \frac{M^2}{2} \iint_D (x-y)^2 dxdy \\ &= \frac{M^2}{2} \int_a^b dy \int_a^b (x-y)^2 dx \\ &= \frac{M^2(b-a)^4}{12} \end{aligned}$$

□

### 108. 求极限

$$\lim_{x \rightarrow +\infty} \sqrt{x} \int_0^{\frac{\pi}{4}} e^{x(\cos t - 1)} \cos t \, dt$$

**解** 一方面有

$$\cos t - 1 = -\frac{1}{2}t^2 + o(t^2) \quad t \rightarrow 0$$

于是

$$e^{x(\cos t - 1)} = e^{-\frac{1}{2}xt^2 + o(xt^2)}$$

且注意到

$$\lim_{x \rightarrow 0} \cos x = 1$$

于是对任意  $\varepsilon > 0$ , 存在  $\delta > 0$ , 当  $0 < x < \delta$  时

$$\cos t > 1 - \varepsilon$$

$$\sqrt{x} \int_0^\delta e^{x(\cos t - 1)} \cos t \, dt \leq \sqrt{x} \int_0^\delta e^{-\frac{1}{2}xt^2 + o(xt^2)} \, dt = \sqrt{2} \int_0^{\frac{\delta}{\sqrt{2}}\sqrt{x}} e^{-y^2 + o(y^2)} \, dy \rightarrow \sqrt{\frac{\pi}{2}}$$

另外

$$\sqrt{x} \int_0^\delta e^{x(\cos t - 1)} \cos t \, dt \geq (1 - \varepsilon) \sqrt{x} \int_0^\delta e^{-\frac{1}{2}xt^2 + o(xt^2)} \, dt \rightarrow \sqrt{\frac{\pi}{2}}(1 - \varepsilon)$$

不难得到

$$\sqrt{x} \int_\delta^{\frac{\pi}{4}} e^{x(\cos t - 1)} \cos t \, dt = 0$$

这里只需要用

$$\cos t - 1 = -2 \sin^2\left(\frac{t}{2}\right) \leq -2\left(\frac{x}{\pi}\right)^2$$

最后, 由  $\varepsilon$  的任意性知

$$\lim_{x \rightarrow +\infty} \sqrt{x} \int_0^{\frac{\pi}{4}} e^{x(\cos t - 1)} \cos t \, dt = \sqrt{\frac{\pi}{2}}$$

□

### 109. 求极限

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} \left( \sum_{k=0}^n \frac{n^k}{k!} - \sum_{k=n+1}^{\infty} \frac{n^k}{k!} \right)$$

**解** 我们有

$$e^n = \sum_{k=0}^n \frac{n^k}{k!} + \sum_{k=n+1}^{\infty} \frac{n^k}{k!} = \sum_{k=0}^n \frac{n^k}{k!} + \frac{1}{n!} \int_0^n e^t (n-t)^n \, dt$$

所以

$$\begin{aligned} \sum_{k=n+1}^{\infty} \frac{n^k}{k!} &= \frac{1}{n!} \int_0^n e^t (n-t)^n \, dt \\ \sum_{k=0}^n \frac{n^k}{k!} &= e^n - \frac{1}{n!} \int_0^n e^t (n-t)^n \, dt \end{aligned}$$

因此只要计算

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} \left( e^n - \frac{2}{n!} \int_0^n e^t (n-t)^n \, dt \right)$$

我们有

$$\begin{aligned}\int_0^n e^t (n-t)^n dt &= n^{n+1} \int_0^1 e^{nz} (1-z)^n dz \\ &= n^{n+1} \int_0^1 e^{n(z+\ln(1-z))} dz \\ &= n^{n+1} \int_0^1 e^{-\frac{1}{2}nz^2 - \frac{1}{3}nz^3 + o(nz^3)} dz \\ &= n^{n+1} \int_0^1 e^{-\frac{1}{2}nz^2} \left(1 - \frac{1}{3}nz^3 + o(nz^3)\right) dz\end{aligned}$$

$$\begin{aligned}&\lim_{n \rightarrow \infty} \frac{n!}{n^n} \left( e^n - \frac{2}{n!} \int_0^n e^t (n-t)^n dt \right) \\ &= \frac{n!e^n}{n^n} - 2n \left[ \int_0^1 e^{-\frac{1}{2}nz^2} \left(1 - \frac{1}{3}nz^2 + o(nz^2)\right) dz \right] \\ &= \left( \sqrt{2\pi n} e^{\frac{\theta_n}{12n}} - 2n \int_0^1 e^{-\frac{1}{2}nz^2} dz \right) + 2n \int_0^1 e^{-\frac{1}{2}nz^2} \left( \frac{1}{3}nz^2 + o(nz^3) \right) dz \quad \theta_n \in (0, 1)\end{aligned}$$

显然有

$$\begin{aligned}\lim_{n \rightarrow \infty} \left( \sqrt{2\pi n} e^{\frac{\theta_n}{12n}} - 2n \int_0^1 e^{-\frac{1}{2}nz^2} dz \right) &= 0 \\ \lim_{n \rightarrow \infty} 2n \int_0^1 e^{-\frac{1}{2}nz^2} \left( \frac{1}{3}nz^2 + o(nz^3) \right) dz &= \lim_{n \rightarrow \infty} \frac{4}{3} \left( \int_0^{\frac{n}{2}} e^{-z} z dz + o\left(\frac{1}{n}\right) \right) = \frac{4}{3}\end{aligned}$$

所以

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} \left( \sum_{k=0}^n \frac{n^k}{k!} - \sum_{k=n+1}^{\infty} \frac{n^k}{k!} \right) = \frac{4}{3}$$

□

## 110. 计算积分

$$\int_0^1 \frac{\arctan \sqrt{x^2+2}}{(x^2+1)\sqrt{x^2+2}} dx$$

解

$$\begin{aligned}\frac{\pi^2}{16} &= \int_0^1 \int_0^1 \frac{dx dy}{(1+x^2)(1+y^2)} \\ &= \int_0^1 \int_0^1 \left[ \frac{1}{(1+x^2)(2+x^2+y^2)} + \frac{1}{(1+y^2)(2+x^2+y^2)} \right] dx dy \\ &= 2 \int_0^1 \int_0^1 \frac{1}{(1+x^2)(2+x^2+y^2)} dy dx \\ &= 2 \int_0^1 \frac{1}{(1+x^2)\sqrt{2+x^2}} \arctan \frac{1}{\sqrt{2+x^2}} dx \\ &= 2 \int_0^1 \left[ \frac{\pi}{2(1+x^2)\sqrt{2+x^2}} - \frac{\arctan \sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}} \right] dx \\ &= \frac{\pi^2}{6} - 2 \int_0^1 \frac{\arctan \sqrt{x^2+2}}{(x^2+1)\sqrt{x^2+2}} dx\end{aligned}$$

$$\Rightarrow \int_0^1 \frac{\arctan \sqrt{x^2+2}}{(x^2+1)\sqrt{x^2+2}} dx = \frac{5}{96}\pi^2$$

□

111. 设  $f(x) \in C^2(0, 1)$  且  $\lim_{x \rightarrow 1^-} f(x) = 0$ . 若存在  $M > 0$ , 使得  $(1-x)^2|f''(x)| \leq M (0 < x < 1)$ , 证明

$$\lim_{x \rightarrow 1^-} (1-x)f'(x) = 0$$

**证明** 对  $t, x \in (0, 1), t > x$ , 由 Taylor 公式得

$$f(t) = f(x) + f'(x)(t-x) + f''(\xi)\frac{(t-x)^2}{2}, x < \xi < t$$

并取  $t = x + (1-x)\delta, 0 < \delta < \frac{1}{2}$ , 我们有

$$\begin{aligned} f(t) - f(x) &= \delta(1-x)f'(x) + \frac{\delta^2}{2}f''(\xi)(1-x)^2 \\ \Leftrightarrow (1-x)f'(x) &= \frac{f(t) - f(x)}{\delta} - \frac{\delta}{2}f''(\xi)(1-x)^2 \\ |f'(x)(1-x)| &\leq \frac{|f(t) - f(x)|}{\delta} + \frac{\delta}{2}|f''(\xi)(1-x)^2| \end{aligned}$$

注意到

$$\begin{aligned} \xi &= x + (t-x)\theta, 0 < \theta < 1 \\ \Rightarrow (1-\xi)^2 &= (1-x)^2(1-\delta\theta)^2 > \frac{1}{4}(1-x)^2 \end{aligned}$$

这里是由于  $0 < \delta\theta < \frac{1}{2}$ . 结合条件  $(1-x)^2|f''(x)| \leq M (0 < x < 1)$  得

$$\begin{aligned} \frac{\delta}{2}|f''(\xi)|(1-\xi)^2 &\cdot \frac{(1-x)^2}{(1-\xi)^2} \cdot \frac{\delta}{2} < 2M\delta \\ \Rightarrow |f'(x)(1-x)| &\leq \frac{|f(t) - f(x)|}{\delta} + 2M\delta \end{aligned}$$

现在, 对  $\forall \varepsilon$ , 取  $\delta = \frac{\varepsilon}{4M}$ , 对上述  $\delta\varepsilon$ , 存在  $\eta > 0$ , 对  $\forall 0 < 1-x < \eta$  有

$$|f(t) - f(x)| < \frac{\delta\varepsilon}{2}$$

这样, 对  $\forall 0 < 1-x < \eta$ , 就有

$$|f'(x)(1-x)| < \varepsilon$$

□

112. 计算积分

$$\int_0^{\frac{\pi}{2}} x \ln(\sin x) \ln(\cos x) dx$$

**解** 首先设

$$I = \int_0^{\frac{\pi}{2}} x \ln(\sin x) \ln(\cos x) dx$$

显然有

$$I = \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \ln(\sin x) \ln(\cos x) dx$$

根据 Fourier 级数

$$\ln(2 \cos x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos 2nx}{n}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

而另一方面

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos 2nx \ln(\sin x) dx &= \int_0^{\frac{\pi}{2}} \ln(\sin x) d\left(\frac{\sin 2nx}{2n}\right) \\ &= \frac{1}{2n} \sin 2nx \cdot \ln(\sin x) \Big|_0^{\frac{\pi}{2}} - \frac{1}{2n} \int_0^{\frac{\pi}{2}} \frac{\cos x \cdot \sin 2nx}{\sin x} dx \\ &= -\frac{1}{4n} \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x + \sin(2n-1)x}{\sin x} dx \\ &= -\frac{\pi}{4n} \end{aligned}$$

所以

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \ln(\sin x) \ln(\cos x) dx &= \int_0^{\frac{\pi}{2}} \ln(2 \cos x) \ln(\sin x) dx - \ln 2 \cdot \int_0^{\frac{\pi}{2}} \ln(\sin x) dx \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{\pi}{4n^2} + \frac{\pi}{2} \ln^2 2 \\ &= \frac{\pi}{2} \ln^2 2 - \frac{1}{48} \pi^2 \end{aligned}$$

于是

$$I = \frac{(\pi \ln 2)^2}{8} - \frac{\pi^4}{192}$$

□

### 113. 计算积分

$$\int_0^{\infty} \frac{1}{(x^4 + (1 + 2\sqrt{2})x^2 + 1)(x^{100} - x^{98} + \dots + 1)} dx$$

**解** 记

$$I = \int_0^{\infty} \frac{1}{(x^4 + (1 + 2\sqrt{2})x^2 + 1)(x^{100} - x^{98} + \dots + 1)} dx$$

把  $x$  换成  $\frac{1}{x}$  得

$$I = \int_0^{\infty} \frac{x^{102}}{(x^4 + (1 + 2\sqrt{2})x^2 + 1)(x^{100} - x^{98} + \dots + 1)} dx$$

注意到

$$x^{100} - x^{98} + \dots + 1 = \frac{1 + x^{102}}{1 + x^2}$$

于是

$$I = \frac{1}{2} \int_0^{\infty} \frac{1 + x^2}{x^4 + (1 + 2\sqrt{2})x^2 + 1} dx$$

$$\begin{aligned}
 &= \frac{1}{2} \int_0^{\infty} \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2} + 1 + 2\sqrt{2}} dx \\
 &= \frac{\pi}{2(1 + \sqrt{2})}
 \end{aligned}$$

□

#### 114. 求极限

$$\lim_{n \rightarrow \infty} \frac{1}{(2n-1)^{2017}} \sum_{k=0}^{n-1} \int_{2k\pi}^{(2k+1)\pi} x^{2016} \sin^3 x \cos^2 x dx$$

**解** 由推广的积分第一中值定理, 对每个正整数  $n$ ,  $\exists \theta_n \in (0, 1)$  使得

$$\int_{2n\pi}^{(2n+1)\pi} x^{2016} \sin^3 x \cos^2 x dx = ((2n + \theta_n)\pi)^{2016} \int_{2n\pi}^{(2n+1)\pi} \sin^3 x \cos^2 x dx$$

由此得

$$\begin{aligned}
 &\int_{2n\pi}^{(2n+1)\pi} x^{2016} \sin^3 x \cos^2 x dx \\
 &= ((2n\pi)^{2016} + o(n^{2016})) \int_{2n\pi}^{(2n+1)\pi} \sin^3 x \cos^2 x dx \\
 &= ((2n\pi)^{2016} + o(n^{2016})) \left( \frac{\cos 5x}{80} - \frac{\cos 3x}{48} - \frac{\cos x}{8} \right) \Big|_{2n\pi}^{(2n+1)\pi} \\
 &= \frac{4}{15} ((2n\pi)^{2016} + o(n^{2016})) \quad n \rightarrow \infty
 \end{aligned}$$

另外

$$(2n+1)^{2017} - (2n-1)^{2017} = 4034(2n)^{2016} + o(n^{2016}) \quad n \rightarrow \infty$$

于是由 Stolz 定理得

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} \frac{1}{(2n-1)^{2017}} \sum_{k=0}^{n-1} \int_{2k\pi}^{(2k+1)\pi} x^{2016} \sin^3 x \cos^2 x dx \\
 &= \lim_{n \rightarrow \infty} \frac{\int_{2n\pi}^{(2n+1)\pi} x^{2016} \sin^3 x \cos^2 x dx}{(2n+1)^{2017} - (2n-1)^{2017}} \\
 &= \frac{2}{30510} \lim_{n \rightarrow \infty} \frac{(2n\pi)^{2016} + o(n^{2016})}{(2n)^{2016} + o(n^{2016})} \\
 &= \frac{2\pi^{2016}}{30510}
 \end{aligned}$$

更一般的结果是

$$\lim_{n \rightarrow \infty} \frac{1}{(2n-1)^{p+1}} \sum_{k=0}^{n-1} \int_{2k\pi}^{(2k+1)\pi} x^p \sin^3 x \cos^2 x dx = \frac{2\pi^p}{15(p+1)}.$$

□

#### 115. 求极限

$$\lim_{y \rightarrow +\infty} \left( \ln^2 y - 2 \int_0^y \frac{\ln x}{\sqrt{x^2 + 1}} dx \right)$$



**解** 首先有

$$\ln^2 y - 2 \int_0^y \frac{\ln x}{\sqrt{x^2 + 1}} dx = \ln^2 y - \frac{y}{\sqrt{y^2 + 1}} \cdot \ln^2 y + \int_0^y \frac{\ln^2 x}{(x^2 + 1)^{\frac{3}{2}}} dx$$

注意到

$$\lim_{y \rightarrow +\infty} \left( \ln^2 y - \frac{y}{\sqrt{y^2 + 1}} \cdot \ln^2 y \right) = 0$$

于是我们只需要计算积分

$$\int_0^\infty \frac{\ln^2 x}{(x^2 + 1)^{\frac{3}{2}}} dx$$

记

$$\begin{aligned} I(p, q) &= \int_0^\infty \frac{x^p}{(1 + x^2)^q} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\tan^p \theta}{\sec^{2q} \theta} \sec^2 \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^{2q-p-2} \theta d\theta \\ &= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{2q-p-1}{2}\right) \\ &= \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{2q-p-1}{2}\right)}{\Gamma(q)} \end{aligned}$$

于是

$$\int_0^\infty \frac{\ln^2 x}{(x^2 + 1)^{\frac{3}{2}}} dx = \frac{\partial^2 I}{\partial p^2} \left(0, \frac{3}{2}\right)$$

注意到

$$I\left(p, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{1+p}{2}\right) \Gamma\left(1 - \frac{p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

求导可得

$$\frac{\partial I}{\partial p} \left(p, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{1+p}{2}\right) \Gamma\left(1 - \frac{p}{2}\right)}{2\Gamma\left(\frac{1}{2}\right)} \left[ \psi\left(\frac{1+p}{2}\right) - \psi\left(1 - \frac{p}{2}\right) \right]$$

再求导

$$\frac{\partial^2 I}{\partial p^2} \left(p, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{1+p}{2}\right) \Gamma\left(1 - \frac{p}{2}\right)}{4\Gamma\left(\frac{1}{2}\right)} \left[ \left\{ \psi\left(\frac{1+p}{2}\right) - \psi\left(1 - \frac{p}{2}\right) \right\}^2 + \left\{ \psi'\left(\frac{1+p}{2}\right) + \psi'\left(1 - \frac{p}{2}\right) \right\} \right]$$

代入  $p = 0$  可得

$$\frac{\partial^2 I}{\partial p^2} \left(0, \frac{3}{2}\right) = \frac{1}{4} \left[ \left\{ \psi\left(\frac{1}{2}\right) - \psi(1) \right\}^2 + \left\{ \psi'\left(\frac{1}{2}\right) + \psi'(1) \right\} \right]$$

根据公式

$$\psi(s) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+s} \right)$$

可得

$$\begin{aligned}\psi\left(\frac{1}{2}\right) - \psi(1) &= \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+\frac{1}{2}} \right) \\ &= -2 \sum_{n=0}^{\infty} \left( \frac{1}{2n+1} - \frac{1}{2n+2} \right) \\ &= -2 \ln 2\end{aligned}$$

类似的有

$$\begin{aligned}\psi'\left(\frac{1}{2}\right) + \psi'(1) &= \sum_{n=0}^{\infty} \left[ \frac{1}{(n+\frac{1}{2})^2} + \frac{1}{(n+1)^2} \right] \\ &= 4 \sum_{n=0}^{\infty} \left[ \frac{1}{(2n+1)^2} + \frac{1}{(2n+2)^2} \right] \\ &= 4\zeta(2)\end{aligned}$$

因此最后得到

$$\lim_{y \rightarrow +\infty} \left( \ln^2 y - 2 \int_0^y \frac{\ln x}{\sqrt{x^2+1}} dx \right) = \zeta(2) + \ln^2 2 = \frac{\pi^2}{6} + \ln^2 2$$

□

**116.** 设  $-1 < a < 1$ , 计算积分

$$\int_0^1 \frac{\cosh(a \ln x) \ln(1+x)}{x} dx$$

解

$$\begin{aligned}\int_0^1 \frac{\cosh(a \ln x) \ln(1+x)}{x} dx &= \int_{-\infty}^0 \cosh(at) \ln(1+e^t) dt \\ &= \int_{-\infty}^0 \frac{e^{at} + e^{-at}}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} e^{kt} dt \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_{-\infty}^0 [e^{(k+a)t} + e^{(k-a)t}] dt \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left( \frac{1}{k+a} + \frac{1}{k-a} \right) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2 - a^2} \\ &= \frac{1}{2a} \left( \frac{\pi}{\sin(\pi a)} - \frac{1}{a} \right)\end{aligned}$$

□

**117.** 计算积分

$$\int_0^{\infty} \operatorname{si}(x) \tan^{-1} \left( \frac{\sin x}{2 - \cos x} \right) dx$$

其中  $\text{si}(x) = -\int_x^\infty \frac{\sin t}{t} dt$ .

**解** 考虑积分

$$I(a) = \int_0^\infty \text{si}(x) \tan^{-1} \left( \frac{\sin x}{a - \cos x} \right) dx$$

则

$$\begin{aligned} I'(a) &= -\int_0^\infty \text{si}(x) \frac{\sin x}{a^2 - 2a \cos x + 1} dx \\ &= -\sum_{n=1}^\infty \frac{1}{a^{n+1}} \int_0^\infty \text{si}(x) \sin(nx) dx \\ &= -\sum_{n=1}^\infty \frac{1}{a^{n+1}} \left[ \text{si}(x) \frac{-\cos(nx)}{n} \Big|_0^\infty + \frac{1}{n} \int_0^\infty \frac{\sin x \cos(nx)}{x} dx \right] \\ &= -\sum_{n=1}^\infty \frac{1}{a^{n+1}} \left[ -\frac{\pi}{2n} + \frac{1}{2n} \int_0^\infty \frac{\sin(n+1)x - \sin(n-1)x}{x} dx \right] \\ &= -\sum_{n=1}^\infty \frac{1}{a^{n+1}} \left[ -\frac{\pi}{2n} + \frac{\pi}{4n} (\text{sgn}(1-n) + \text{sgn}(1+n)) \right] \\ &= \frac{\pi}{2} \sum_{n=2}^\infty \frac{1}{n a^{n+1}} + \frac{\pi}{4a^2} \\ &\Rightarrow I(a) = -\frac{\pi}{2} \sum_{n=2}^\infty \frac{1}{n^2 a^n} - \frac{\pi}{4} + C \end{aligned}$$

注意到  $a \rightarrow \infty$  时,  $I(a) \rightarrow 0$ , 因此  $C = 0$ .

$$\begin{aligned} I(2) &= \int_0^\infty \text{si}(x) \tan^{-1} \left( \frac{\sin x}{2 - \cos x} \right) dx \\ &= \frac{\pi}{2} \left( -\frac{1}{2} + \sum_{n=1}^\infty \frac{1}{n^2 2^n} \right) - \frac{\pi}{8} \\ &= \frac{\pi}{8} - \frac{\pi}{2} \sum_{n=1}^\infty \frac{1}{n^2 \cdot 2^n} \\ &= -\frac{\pi}{2} \cdot \frac{1}{12} (\pi^2 - 6 \ln^2 2) + \frac{\pi}{8} \\ &= -\frac{\pi^3}{24} + \frac{\pi}{4} \ln^2 2 + \frac{\pi}{8} \end{aligned}$$

□

**118.** 求和

$$\sum_{n=0}^\infty \left( \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots \right)^2$$

**解** 首先注意到

$$\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots = \int_0^1 (x^n - x^{n+1} + x^{n+2} - \cdots) dx = \int_0^1 \frac{x^n}{1+x} dx$$

于是可得

$$\begin{aligned}
 \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots \right)^2 &= \sum_{n=0}^{\infty} \left( \int_0^1 \frac{x^n}{x+1} dx \right) \left( \int_0^1 \frac{y^n}{y+1} dy \right) \\
 &= \int_0^1 \int_0^1 \frac{1}{(1+x)(1+y)} \left( \sum_{n=0}^{\infty} (xy)^n \right) dx dy \\
 &= \int_0^1 \int_0^1 \frac{1}{(1+x)(1+y)(1-xy)} dx dy \\
 &= \int_0^1 \frac{1}{1+x} \left( \int_0^1 \frac{1}{(1+y)(1-xy)} dy \right) dx \\
 &= \int_0^1 \frac{1}{1+x} \left( \frac{\ln 2 - \ln(1-x)}{1+x} \right) dx \\
 &= \left( \frac{(1-x)\ln(1-x)}{2(1+x)} + \frac{\ln(1+x)}{2} - \frac{\ln 2}{1+x} \right) \Big|_0^1 \\
 &= \ln 2
 \end{aligned}$$

□

#### 119. 计算积分

$$\int_{-\infty}^{\infty} \frac{e^{\frac{ax}{1+x^2}} \sin\left(\frac{a}{1+x^2}\right)}{1+x^2} dx$$

解

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{e^{\frac{ax}{1+x^2}} \sin\left(\frac{a}{1+x^2}\right)}{1+x^2} dx &= \int_{-\infty}^{\infty} \frac{e^{\frac{-ax}{1+x^2}} \sin\left(\frac{a}{1+x^2}\right)}{1+x^2} dx \\
 &= \Im \int_{-\infty}^{\infty} \frac{e^{\frac{a(i-x)}{1+x^2}}}{1+x^2} dx \\
 &= \Im \left[ 2\pi i \operatorname{Res} \left( \frac{e^{\frac{a(i-x)}{1+x^2}}}{1+x^2}, x=i \right) \right] \\
 &= \Im \left( \pi e^{\frac{ia}{2}} \right) \\
 &= \pi \sin\left(\frac{a}{2}\right)
 \end{aligned}$$

□

#### 120. 计算积分

$$\int_0^1 \frac{t^2-1}{(t^2+1)\ln t} dt$$

解

$$\begin{aligned}
 \int_0^1 \frac{t^2-1}{(t^2+1)\ln t} dt &= \int_0^{\infty} \frac{1-e^{-2x}}{x(1+e^{-2x})} e^{-x} dx \\
 &= \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} \frac{1-e^{-2x}}{x} e^{-(2n+1)x} dx
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} (-1)^n \ln \left( \frac{2n+3}{2n+1} \right) \quad \text{Frullani积分} \\
 &= \sum_{n=0}^{\infty} \ln \frac{(4n+3)(4n+3)}{(4n+1)(4n+5)} \\
 &= \sum_{n=0}^{\infty} \ln \frac{(n+\frac{3}{4})^2}{(n+\frac{1}{4})(n+\frac{5}{4})} \\
 &= \lim_{n \rightarrow \infty} \ln \left[ \frac{\Gamma(\frac{1}{4}) \Gamma(\frac{5}{4}) \Gamma^2(n+\frac{3}{4})}{\Gamma^2(\frac{3}{4}) \Gamma(n+\frac{1}{4}) \Gamma(n+\frac{5}{4})} \right] \\
 &= 2 \ln \left[ \frac{\Gamma(\frac{1}{4})}{2\Gamma(\frac{3}{4})} \right]
 \end{aligned}$$

□

## 121. 计算积分

$$\int_0^{\infty} \frac{\ln x}{\cosh^2 x} dx$$

**解** 设  $a > 0$ , 考虑积分

$$\begin{aligned}
 I(a) &= \int_0^{\infty} \frac{x^a}{\cosh^2 x} dx \\
 &= 4 \int_0^{\infty} \frac{x^a}{(e^x + e^{-x})^2} dx \\
 &= 4 \int_0^{\infty} x^a \frac{e^{-2x}}{(1 + e^{-2x})^2} dx \\
 &= 4 \int_0^{\infty} x^a \sum_{n=1}^{\infty} (-1)^{n-1} n e^{-2nx} dx \\
 &= 4 \sum_{n=1}^{\infty} (-1)^{n-1} n \int_0^{\infty} x^a e^{-2nx} dx \\
 &= 4 \sum_{n=1}^{\infty} (-1)^{n-1} n \frac{\Gamma(a+1)}{(2n)^{a+1}} \\
 &= \frac{2\Gamma(a+1)}{2^a} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^a} \\
 &= \frac{2\Gamma(a+1)\eta(a)}{2^a}
 \end{aligned}$$

积分求导得

$$\begin{aligned}
 I'(a) &= \int_0^{\infty} \frac{x^a \ln x}{\cosh^2 x} dx \\
 &= 2 \frac{[\Gamma'(a+1)\eta(a) + \Gamma(a+1)\eta'(a)] \cdot 2^a - \Gamma(a+1)\eta(a) \ln 2 \cdot 2^a}{2^{2a}}
 \end{aligned}$$

因此

$$\int_0^{\infty} \frac{\ln x}{\cosh^2 x} dx = I'(0) = 2(\Gamma'(0)\eta(0) + \eta'(0) - \eta(0) \ln 2)$$

利用关系式

$$\eta(s) = (1 - 2^{1-s}) \zeta(s)$$

以及

$$\zeta(0) = -\frac{1}{2} \text{ 和 } \zeta'(0) = -\frac{1}{2} \ln(2\pi)$$

可得

$$\int_0^\infty \frac{\ln x}{\cosh^2 x} dx = 2 \left[ -\gamma \left( \frac{1}{2} \right) + \frac{1}{2} \ln \left( \frac{\pi}{2} \right) - \frac{1}{2} \ln 2 \right] = \ln \left( \frac{\pi}{4} \right) - \gamma$$

□

**122.** 设  $f(x)$  是连续实值函数, 且满足

$$\int_0^1 f(x) dx = \int_0^1 x f(x) dx = \cdots = \int_0^1 x^{n-1} f(x) dx = 1$$

证明:

$$\int_0^1 f^2(x) dx \geq n^2$$

**证明** 考虑多项式

$$P(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$$

如果多项式  $P(x)$  也满足上面的条件, 那么

$$\int_0^1 P^2(x) dx = a_0 + a_1 + \cdots + a_{n-1}$$

为了求出系数  $a_i$ , 再次利用条件

$$\begin{aligned} \int_0^1 x^k P(x) dx &= 1 \quad k = 0, 1, \dots, n-1 \\ \Rightarrow \frac{a_0}{k+1} + \frac{a_1}{k+2} + \cdots + \frac{a_{n-1}}{k+n} &= 1 \quad k = 0, 1, \dots, n-1 \end{aligned}$$

设

$$H(x) = \frac{a_0}{x+1} + \frac{a_1}{x+2} + \cdots + \frac{a_{n-1}}{x+n}$$

则显然有

$$H(0) = H(1) = \cdots = H(n-1) = 0$$

于是

$$H(x) = \frac{Ax(x-1)(x-2)\cdots(x-n+1)}{(x+1)(x+2)\cdots(x+n)}$$

对比系数可得  $A = -1$  以及

$$a_k = (-1)^{n-k+1} \frac{(n+k)!}{(k!)^2(n-k+1)!} \quad k = 0, 1, \dots, n-1$$

用数学归纳法可以证明

$$\sum_{k=0}^{n-1} a_k = n^2$$

所以, 多项式  $P(x)$  满足上面的性质, 则

$$\int_0^1 P^2(x) dx = a_0 + a_1 + \cdots + a_{n-1} = n^2$$

取满足以上条件的多项式  $P(x)$ , 由 Cauchy 不等式得

$$\begin{aligned} \int_0^1 P^2(x) dx \int_0^1 f^2(x) dx &\geq \left( \int_0^1 P(x) f(x) dx \right)^2 = n^4 \\ \Rightarrow \int_0^1 f^2(x) dx &\geq n^2 \end{aligned}$$

□

### 123. 计算积分

$$\int_0^\pi \left( \frac{e + \cos x}{1 + 2e \cdot \cos x + e^2} \right)^2 dx$$

解

$$\begin{aligned} &\int_0^\pi \left( \frac{e + \cos x}{1 + 2e \cdot \cos x + e^2} \right)^2 dx \\ &= \frac{1}{2} \oint_{|z|=1} \left( \frac{e + \frac{z^2+1}{2z}}{1 + 2e \frac{z^2+1}{2z} + e^2} \right)^2 \frac{dz}{iz} \\ &= \frac{1}{2} \cdot 2\pi i \cdot \text{Res} \left[ \frac{1}{iz} \left( \frac{e + \frac{z^2+1}{2z}}{1 + 2e \frac{z^2+1}{2z} + e^2} \right)^2, |z| < 1 \right] \quad \left( z = -\frac{1}{e}, 0 \right) \\ &= \frac{\pi}{4} \left( \frac{3e^2 - 1}{e^2(e^2 - 1)} + \frac{1}{e^2} \right) \\ &= \frac{\pi}{2} \left( \frac{2e^2 - 1}{e^2 - 1} \right) \end{aligned}$$

□

### 124. 求和

$$\sum_{n=1}^{\infty} \frac{1}{\sinh(2^n)}$$

解

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{\sinh(2^n)} &= \sum_{n=1}^{\infty} \frac{2}{e^{2^n} - e^{-2^n}} \\ &= \sum_{n=1}^{\infty} \frac{2}{e^{2^n} (1 - e^{-2 \cdot 2^n})} \\ &= 2 \sum_{n=1}^{\infty} e^{-2^n} \sum_{k=0}^{\infty} e^{-2 \cdot 2^n \cdot k} \\ &= 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} e^{-(2k+1) \cdot 2^n} \end{aligned}$$

$$= 2 \sum_{m=1}^{\infty} e^{-2m} = \frac{2}{e^2 - 1}$$

□

125. 证明: 对  $n \geq 1$ ,

$$\frac{n(n+1)(n+2)}{3} < \sum_{k=1}^n \frac{1}{\ln^2(1+\frac{1}{k})} < \frac{n}{4} + \frac{n(n+1)(n+2)}{3}$$

**证明** 求导容易证明对  $x \in (0, 1]$  有

$$\frac{2x}{2+x} < \ln(1+x) < \frac{x}{\sqrt{1+x}}$$

因此对  $1 \leq k \leq n$

$$k(k+1) = \frac{1+\frac{1}{k}}{(\frac{1}{k})^2} < \frac{1}{\ln^2(1+\frac{1}{k})} < \left(\frac{2+\frac{1}{k}}{\frac{2}{k}}\right)^2 = k(k+1) + \frac{1}{4}$$

求和即得

$$\frac{n(n+1)(n+2)}{3} < \sum_{k=1}^n \frac{1}{\ln^2(1+\frac{1}{k})} < \frac{n}{4} + \frac{n(n+1)(n+2)}{3}$$

□

126. 求极限

$$\lim_{n \rightarrow \infty} \sum_{k=2}^{n-1} \frac{\zeta(k)}{\Gamma(n-k)}, \lim_{n \rightarrow \infty} \sum_{k=1}^{n-2} (-1)^{k-1} \frac{\zeta(n-k)}{\Gamma(n-k)}$$

**解** 首先注意到

$$\begin{aligned} \sum_{k=2}^{\infty} (\zeta(k) - 1) &= \sum_{k=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n^k} = \sum_{n=2}^{\infty} \sum_{k=2}^{\infty} \frac{1}{n^k} \\ &= \sum_{n=2}^{\infty} \frac{\frac{1}{n^2}}{1 - \frac{1}{n}} = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \\ &= \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n} \right) = 1 \end{aligned}$$

而且对任意  $z \in \mathbb{C}$ ,

$$\sum_{k=1}^{\infty} \frac{z^{k-1}}{\Gamma(k)} = \sum_{k=0}^{\infty} \frac{z^{k-1}}{(k-1)!} = e^z$$

这两个绝对收敛级数的 Cauchy 乘积为

$$\sum_{n=3}^{\infty} \left( \sum_{k=2}^{n-1} \frac{(\zeta(k) - 1) z^{n-k+1}}{\Gamma(n-k)} \right) = e^z$$

根据 Mertens 定理知该级数是收敛的, 因此通项趋于 0, 即

$$\lim_{n \rightarrow \infty} \sum_{k=2}^{n-1} \frac{(\zeta(k) - 1) z^{n-k+1}}{\Gamma(n-k)} = 0$$



因此

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{k=2}^{n-1} \frac{\zeta(k) z^{n-k+1}}{\Gamma(n-k)} &= \lim_{n \rightarrow \infty} \sum_{k=2}^{n-1} \frac{z^{n-k+1}}{\Gamma(n-k)} + \lim_{n \rightarrow \infty} \sum_{k=2}^{n-1} \frac{(\zeta(k) - 1) z^{n-k+1}}{\Gamma(n-k)} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^{n-2} \frac{z^{k-1}}{\Gamma(k)} = e^z\end{aligned}$$

在这个结果中取  $z = 1$  和  $z = -1$  可得

$$\lim_{n \rightarrow \infty} \sum_{k=2}^{n-1} \frac{\zeta(k)}{\Gamma(n-k)} = e, \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{n-2} \frac{(-1)^{k-1} \zeta(n-k)}{\Gamma(k)} = \lim_{n \rightarrow \infty} \sum_{k=1}^{n-2} \frac{(-1)^{n-k-1}}{\Gamma(n-k)} = e^{-1}$$

□

**127.** 设  $k$  是一个整数,  $k \geq 2$ , 计算积分

$$I_k = \int_0^{+\infty} \frac{\ln|1-x|}{x^{1+\frac{1}{k}}} dx$$

**解** 首先有

$$\begin{aligned}\int_1^{+\infty} \frac{\ln|1-x|}{x^{1+\frac{1}{k}}} dx &= \int_1^{+\infty} \frac{\ln(x-1)}{x^{1+\frac{1}{k}}} dx \\ &= \int_0^1 \frac{\ln(\frac{1}{t}-1)}{(\frac{1}{t})^{1+\frac{1}{k}}} d\left(\frac{1}{t}\right) \\ &= \int_0^1 \frac{\ln(1-t)}{t^{1-\frac{1}{k}}} dt - \int_0^1 \frac{\ln t}{t^{1-\frac{1}{k}}} dt\end{aligned}$$

而

$$\int_0^1 \frac{\ln t}{t^{1-\frac{1}{k}}} dt = \left[ -k^2 t^{\frac{1}{k}} + k t^{\frac{1}{k}} \ln t \right] \Big|_{t=0^+}^1 = -k^2$$

进一步, 如果  $\alpha \in (0, 2)$ , 则

$$-\int_0^1 \frac{\ln(1-t)}{t^\alpha} dt = \int_0^1 \frac{1}{t^\alpha} \sum_{j=1}^{\infty} \frac{t^j}{j} dt = \sum_{j=1}^{\infty} \frac{1}{j} \int_0^1 t^{j-\alpha} dt = \sum_{j=1}^{\infty} \frac{1}{j(j+1-\alpha)}$$

因此

$$\begin{aligned}I_k &= k^2 + \int_0^1 \frac{\ln(1-t)}{t^{1-\frac{1}{k}}} dt + \int_0^1 \frac{\ln(1-t)}{t^{1+\frac{1}{k}}} dt \\ &= k^2 - \sum_{j=1}^{\infty} \frac{1}{j(j+\frac{1}{k})} - \sum_{j=1}^{\infty} \frac{1}{j(j-\frac{1}{k})} \\ &= k^2 + 2 \sum_{j=1}^{\infty} \frac{1}{\frac{1}{k^2} - j^2} = k\pi \cot\left(\frac{\pi}{k}\right)\end{aligned}$$

最后一步我们利用了例 4 的等式 (5).

□

128. 设  $f(x)$  是  $[0, 1]$  上的  $n$  阶连续可微函数, 满足  $f\left(\frac{1}{2}\right) = f^{(i)}\left(\frac{1}{2}\right) = 0$ , 其中  $i$  是不超过  $n$  的偶数, 证明

$$\left(\int_0^1 f(x) dx\right)^2 \leq \frac{1}{(2n+1)4^n(n!)^2} \int_0^1 \left(f^{(n)}(x)\right)^2 dx$$

**证明** 如果  $g \in C^n([0, 1])$ , 则对任意  $a \in (0, 1)$ , 由分部积分可得

$$\int_0^a g(x) dx = \sum_{i=0}^{n-1} \frac{(-1)^i a^{i+1} g^{(i)}(a)}{(i+1)!} + \frac{(-1)^n}{n!} \int_0^a x^n g^{(n)}(x) dx$$

因此

$$\int_0^{\frac{1}{2}} f(x) dx = \sum_{i=0}^{n-1} \frac{(-1)^i \left(\frac{1}{2}\right)^{i+1} f^{(i)}\left(\frac{1}{2}\right)}{(i+1)!} + \frac{(-1)^n}{n!} \int_0^{\frac{1}{2}} x^n f^{(n)}(x) dx$$

以及

$$\int_{\frac{1}{2}}^1 f(x) dx = \int_0^{\frac{1}{2}} f(1-x) dx = \sum_{i=0}^{n-1} \frac{(-1)^i \left(\frac{1}{2}\right)^{i+1} f^{(i)}\left(\frac{1}{2}\right)}{(i+1)!} + \frac{1}{n!} \int_0^{\frac{1}{2}} x^n f^{(n)}(1-x) dx$$

由于  $f^{(i)}\left(\frac{1}{2}\right) = 0$ , 其中  $i$  是小于  $n$  的偶数, 于是

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^{\frac{1}{2}} f(x) dx + \int_{\frac{1}{2}}^1 f(x) dx \\ &= \frac{1}{n!} \left( \int_0^{\frac{1}{2}} x^n f^{(n)}(x) dx + \int_0^{\frac{1}{2}} x^n f^{(n)}(1-x) dx \right) \end{aligned}$$

最后由 Cauchy 不等式得

$$\begin{aligned} \left(\int_0^1 f(x) dx\right)^2 &\leq \frac{2}{(n!)^2} \left[ \left(\int_0^{\frac{1}{2}} x^n f^{(n)}(x) dx\right)^2 + \left(\int_0^{\frac{1}{2}} x^n f^{(n)}(1-x) dx\right)^2 \right] \\ &\leq \left[ \int_0^{\frac{1}{2}} x^{2n} dx \int_0^{\frac{1}{2}} \left(f^{(n)}(x)\right)^2 dx + \int_0^{\frac{1}{2}} x^{2n} dx \int_0^{\frac{1}{2}} \left(f^{(n)}(1-x)\right)^2 dx \right] \\ &\leq \frac{1}{(2n+1)4^n(n!)^2} \int_0^1 \left(f^{(n)}(x)\right)^2 dx \end{aligned}$$

□

129. 求和

$$\sum_{p=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+n)^4 + ((m+n)(m+p))^2}$$

**解** 令上述和为  $S$ ,  $a = m + n$ ,  $b = m + p$ , 由对称性

$$\begin{aligned} S &= \sum_{m=1}^{\infty} \sum_{a>m} \sum_{b>m} \frac{1}{a^2(a^2+b^2)} \\ &= \sum_{m=1}^{\infty} \sum_{a>m} \sum_{b>m} \left( \frac{1}{a^2 b^2} - \frac{1}{b^2(a^2+b^2)} \right) \end{aligned}$$

$$= \sum_{m=1}^{\infty} \sum_{a>m} \sum_{b>m} \frac{1}{a^2 b^2} - S$$

因此

$$\begin{aligned} S &= \frac{1}{2} \sum_{m=1}^{\infty} \sum_{a>m} \sum_{b>m} \frac{1}{a^2 b^2} \\ &= \sum_{m=1}^{\infty} \sum_{a>b>m} \frac{1}{a^2 b^2} + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{a>m} \frac{1}{a^4} \\ &= \sum_{a>b \geq 1} \frac{1}{a^2 b^2} \sum_{m=1}^{b-1} 1 + \frac{1}{2} \sum_{a \geq 1} \frac{1}{a^4} \sum_{m=1}^{a-1} 1 \\ &= \sum_{a>b \geq 1} \frac{1}{a^2 b^2} - \sum_{a>b \geq 1} \frac{1}{a^2 b^2} + \frac{1}{2} \sum_{a \geq 1} \frac{1}{a^3} - \frac{1}{2} \sum_{a \geq 1} \frac{1}{a^4} \\ &= \zeta(2, 1) - \zeta(2, 2) + \frac{1}{2} \zeta(3) - \frac{1}{2} \zeta(4) \\ &= \frac{3}{2} \zeta(3) - \frac{5}{4} \zeta(4) \end{aligned}$$

这里我们运用了基本结论  $\zeta(2, 1) = \zeta(3)$ ,  $\zeta(2, 2) = \frac{3}{4} \zeta(4)$ .

□

**130.** 设  $f$  是  $[0, 1]$  上二阶连续可导的实值函数, 满足  $f\left(\frac{1}{2}\right) = 0$ , 证明

$$\int_0^1 (f''(x))^2 dx \geq 320 \left( \int_0^1 f(x) dx \right)^2$$

**证明** 利用 Taylor 公式可得

$$f(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \int_{\frac{1}{2}}^x f''(t)(x-t) dt$$

由于  $f\left(\frac{1}{2}\right) = 0$ , 于是有

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \left( \int_{\frac{1}{2}}^x f''(t)(x-t) dt \right) dx \\ &= \int_{x=0}^{\frac{1}{2}} \int_{t=x}^{\frac{1}{2}} f''(t)(t-x) dt dx + \int_{x=\frac{1}{2}}^1 \int_{t=\frac{1}{2}}^x f''(t)(x-t) dt dx \\ &= \int_{t=0}^{\frac{1}{2}} \int_{x=0}^t f''(t)(t-x) dx dt + \int_{t=\frac{1}{2}}^1 \int_{x=t}^1 f''(t)(x-t) dx dt \\ &= \int_{t=0}^{\frac{1}{2}} f''(t) \left[ -\frac{(t-x)^2}{2} \right]_{x=0}^t dt + \int_{t=\frac{1}{2}}^1 f''(t) \left[ \frac{(x-t)^2}{2} \right]_{x=t}^1 dt \\ &= \frac{1}{2} \int_{t=0}^{\frac{1}{2}} f''(t) t^2 dt + \frac{1}{2} \int_{t=\frac{1}{2}}^1 f''(t) (1-t)^2 dt \\ &= \frac{1}{2} \int_{t=0}^1 f''(t) h(t) dt \end{aligned}$$

其中

$$h(t) = \begin{cases} t^2, & t \in [0, \frac{1}{2}] \\ (1-t)^2, & t \in [\frac{1}{2}, 1] \end{cases}$$

因此由 Cauchy 不等式得

$$\left( \int_0^1 f(x) dx \right)^2 \leq \frac{1}{4} \int_0^1 (h(t))^2 dt \cdot \int_0^1 (f''(t))^2 dt = \frac{1}{320} \int_0^1 (f''(t))^2 dt$$

□

**131.**  $\{x\}$  表示  $x$  的小数部分, 计算积分

$$\int_0^{\frac{\pi}{2}} \frac{\{\tan x\}}{\tan x} dx$$

**解** 令  $t_n = \arctan(n)$ , 则

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{\{\tan x\}}{\tan x} dx = \int_0^{\frac{\pi}{2}} \left( 1 - \frac{[\tan x]}{\tan x} \right) dx \\ &= \frac{\pi}{2} - \sum_{n=1}^{\infty} n \int_{t_n}^{t_{n+1}} \frac{dx}{\tan x} \\ &= \frac{\pi}{2} - \sum_{n=1}^{\infty} n (\ln(\sin t_{n+1}) - \ln(\sin t_n)) \end{aligned}$$

由于  $t > 0$  时,  $\sin(\arctan t) = \frac{1}{\sqrt{1+t^2}}$ , 那么对  $N \geq 2$  有

$$\begin{aligned} &\sum_{n=1}^{N-1} n (\ln(\sin t_{n+1}) - \ln(\sin t_n)) \\ &= \sum_{n=1}^{N-1} [(n+1) \ln(\sin t_{n+1}) - n \ln(\sin t_n)] + \sum_{n=1}^{N-1} \ln(\sin t_{n+1}) \\ &= N \ln(\sin t_N) - \sum_{n=0}^{N-1} \ln(\sin t_{n+1}) \\ &= N \ln \left( \frac{1}{\sqrt{1+\frac{1}{N^2}}} \right) - \ln \left( \prod_{n=1}^N \frac{1}{\sqrt{1+\frac{1}{n^2}}} \right) \end{aligned}$$

令  $N \rightarrow \infty$ , 可得

$$\begin{aligned} I &= \frac{\pi}{2} - \frac{1}{2} \ln \left( \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n^2} \right) \right) \\ &= \frac{\pi}{2} - \frac{1}{2} \ln \left( \frac{\sin(\pi i)}{\pi i} \right) \\ &= \frac{\pi}{2} - \frac{1}{2} \ln \left( \frac{\sinh \pi}{\pi} \right) \\ &= \frac{1}{2} \ln \left( \frac{2\pi}{1 - e^{-2\pi}} \right) \end{aligned}$$

□

**132.** 对实数  $a$  和  $b$  以及整数  $n \geq 1$ , 令  $\gamma_n(a, b) = -\ln(n+a) + \sum_{k=1}^n \frac{1}{k+b}$ , 证明极限  $\gamma(a, b) = \lim_{n \rightarrow \infty} \gamma_n(a, b)$  存在且有限, 并求极限

$$\lim_{n \rightarrow \infty} \left( \ln \left( \frac{e}{n+a} \right) + \sum_{k=1}^n \frac{1}{k+b} - \gamma(a, b) \right)^n$$

**解** 如果  $x > 0$ , 我们有

$$\psi(x) = -\gamma + \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{x-1+k} \right) = \ln x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right)$$

这就意味着 (其中  $b$  不能是负整数)

$$\begin{aligned} \gamma_n(a, b) + \psi(b+1) &= -\ln(n+a) + \sum_{k=1}^n \frac{1}{k+b} + \psi(b+1) \\ &= -\ln(n+a) + \psi(b+n+1) \\ &= -\ln(n+a) + \ln(b+n+1) - \frac{1}{2(b+n+1)} + O\left(\frac{1}{n^2}\right) \\ &= \ln\left(\frac{b+n+1}{n+a}\right) - \frac{1}{2(b+n+1)} + O\left(\frac{1}{n^2}\right) \\ &= \frac{b-a+\frac{1}{2}}{n} + O\left(\frac{1}{n^2}\right) \end{aligned}$$

因此  $\gamma(a, b) = \lim_{n \rightarrow \infty} \gamma_n(a, b) = -\psi(b+1)$ , 进一步有

$$\begin{aligned} \ln\left(\frac{e}{n+a}\right) + \sum_{k=1}^n \frac{1}{k+b} - \gamma(a, b) &= 1 - \ln(n+a) + \sum_{k=1}^n \frac{1}{k+b} + \psi(b+1) \\ &= 1 + \frac{b-a+\frac{1}{2}}{n} + O\left(\frac{1}{n^2}\right) \end{aligned}$$

于是最后得到

$$\lim_{n \rightarrow \infty} \left( \ln\left(\frac{e}{n+a}\right) + \sum_{k=1}^n \frac{1}{k+b} - \gamma(a, b) \right)^n = e^{b-a+\frac{1}{2}}$$

□

**133.** 设  $f$  是  $[0, 1]$  上的连续非负函数, 证明

$$\int_0^1 f^3(x) dx \geq 4 \left( \int_0^1 x^2 f(x) dx \right) \left( \int_0^1 x f^2(x) dx \right)$$

**证明** 这里我们证明一个更一般的结论: 设  $f, g$  是  $[0, 1]$  上的连续非负函数,  $a$  和  $b$  是非负实数, 则

$$\int_0^1 f^{a+b}(x) dx \int_0^1 g^{a+b}(x) dx \geq \left( \int_0^1 f^a(x) g^b(x) dx \right) \left( \int_0^1 f^b(x) g^a(x) dx \right)$$

设  $A, B$  是非负实数, 则

$$(A^a - B^a)(A^b - B^b) \geq 0$$

这就意味着

$$A^{a+b} + B^{a+b} \geq A^a B^b + A^b B^a$$

令  $A = f(x)g(y)$ ,  $B = f(y)g(x)$ , 并在  $[0, 1] \times [0, 1]$  上积分, 我们有

$$\begin{aligned} & \int_0^1 \left( \int_0^1 [f(x)g(y)]^{a+b} dx \right) dy + \int_0^1 \left( \int_0^1 [f(y)g(x)]^{a+b} dy \right) dx \\ & \geq \int_0^1 \left( \int_0^1 (f(x)g(y))^a (f(y)g(x))^b dx \right) dy + \int_0^1 \left( \int_0^1 (f(x)g(y))^b (f(y)g(x))^a dy \right) dx \end{aligned}$$

也就是

$$\begin{aligned} & \left( \int_0^1 f^{a+b}(x) dx \right) \left( \int_0^1 g^{a+b}(y) dy \right) + \left( \int_0^1 f^{a+b}(y) dy \right) \left( \int_0^1 g^{a+b}(x) dx \right) \\ & \geq \left( \int_0^1 f^a(x) g^b(x) dx \right) \left( \int_0^1 f^a(y) g^b(y) dy \right) + \left( \int_0^1 f^a(y) g^b(y) dy \right) \left( \int_0^1 f^a(x) g^b(x) dx \right) \end{aligned}$$

得证, 那么在待证式中取  $g(x) = x, a = 2, b = 1$  即可.

□

134. 设  $H_n = \sum_{k=1}^n \frac{1}{k}$ , 求和

$$\sum_{n=1}^{\infty} H_n \left( \zeta(3) - \sum_{k=1}^n \frac{1}{k^3} \right)$$

解

$$\begin{aligned} \sum_{n=1}^{\infty} H_n \left( \zeta(3) - \sum_{k=1}^n \frac{1}{k^3} \right) &= \sum_{n=1}^{\infty} H_n \sum_{k=n+1}^{\infty} \frac{1}{k^3} = \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{n=1}^{k-1} H_n \\ &= \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{n=1}^{k-1} \sum_{j=1}^n \frac{1}{j} = \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{j=1}^{k-1} \frac{1}{j} \sum_{n=j}^{k-1} 1 \\ &= \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{j=1}^{k-1} \frac{k-j}{j} = \sum_{k=2}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k-1} \frac{1}{j} - \sum_{k=2}^{\infty} \frac{k-1}{k^3} \\ &= \zeta(2, 1) - \zeta(2) + \zeta(3) = 2\zeta(3) - \zeta(2) \end{aligned}$$

□

135. 设  $f$  是  $[0, 1]$  上的非负函数, 证明

$$\frac{3}{4} \left( \int_0^1 f(x) dx \right)^2 \leq \frac{1}{16} + \int_0^1 f^3(x) dx$$

证明 首先注意到对  $t \geq 0$  有

$$t^3 - \frac{3}{4}t^2 + \frac{1}{6} = \frac{(4t+1)(2t-1)^2}{16} \geq 0$$

由于  $f$  是非负函数, 则

$$\int_0^1 \left( f^3(x) - \frac{3}{4}f^2(x) + \frac{1}{16} \right) dx \geq 0$$

那么由 Cauchy 不等式得

$$\int_0^1 f^3(x) dx + \frac{1}{6} \geq \frac{3}{4} \int_0^1 f^2(x) dx \geq \frac{3}{4} \left( \int_0^1 f(x) dx \right)^2$$

□

### 136. 求极限

$$\lim_{n \rightarrow \infty} n^2 \int_{((n+1)!)^{-1/(n+1)}}^{((n)!)^{-1/n}} \Gamma(nx) dx$$

**解** 我们将证明如果  $f$  是  $(a, b)$  上的实值连续函数且  $e \in (a, b)$ , 则

$$\lim_{n \rightarrow \infty} n^2 \int_{((n+1)!)^{-1/(n+1)}}^{((n)!)^{-1/n}} f(nx) dx = ef(e)$$

令  $b_n = n(n!)^{-1/n}$ ,  $a_n = n((n+1)!)^{-1/(n+1)}$ , 那么由积分平均值定理可得

$$\lim_{n \rightarrow \infty} n^2 \int_{((n+1)!)^{-1/(n+1)}}^{((n)!)^{-1/n}} f(nx) dx = n \int_{a_n}^{b_n} f(t) dt = n(b_n - a_n) f(t_n)$$

对某个  $t_n \in (a_n, b_n)$  成立. 再由 Stirling 公式得

$$\ln(n!) = n \ln n - n + \frac{1}{2} \ln n + \ln \sqrt{2\pi} + O\left(\frac{1}{n}\right)$$

因此

$$b_n = ne^{-\frac{\ln(n!)}{n}} = e - \frac{e \ln n}{2n} - \frac{e \ln \sqrt{2\pi}}{n} + O\left(\frac{\ln^2 n}{n^2}\right)$$

$$b_n - a_n = b_n - \frac{nb_{n+1}}{n+1} = \frac{e}{n} + O\left(\frac{\ln n}{n^2}\right) = e$$

也就意味着

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} t_n = e$$

再由  $f$  在  $e$  处的连续性

$$\lim_{n \rightarrow \infty} n(b_n - a_n) f(t_n) = ef(e)$$

而这里的话,  $\Gamma$  函数是  $(0, +\infty)$  上的实值连续函数, 因而极限是  $e\Gamma(e)$ .

□

### 137. 求和

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1}$$

**解** 首先有熟知的等式

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} = \frac{\pi}{4} \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{3}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{8}$$

因此

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \left( \frac{\pi}{4} + \sum_{k=1}^n \frac{(-1)^k}{2k-1} \right)$$

$$\begin{aligned}
 &= -\frac{\pi^2}{16} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sum_{k=1}^n \frac{(-1)^k}{2k-1} \\
 &= -\frac{\pi^2}{16} + \sum_{1 \leq k \leq n} \frac{(-1)^{n+k}}{(2k-1)(2n-1)} \\
 &= -\frac{\pi^2}{16} + \frac{1}{2} \left( \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \right) \\
 &= -\frac{\pi^2}{16} + \frac{1}{2} \left( \frac{\pi^2}{16} + \frac{\pi^2}{8} \right) = \frac{\pi^2}{32}
 \end{aligned}$$

□

138. 设  $f$  是  $[0, +\infty)$  上的有界非负连续函数, 求极限

$$\lim_{n \rightarrow \infty} n \left( \sqrt[n]{\int_0^{\infty} f^{n+1}(x) e^{-x} dx} - \sqrt[n]{\int_0^{\infty} f^n(x) e^{-x} dx} \right)$$

**解** 令  $h(t) = f(-\ln t)$ , 则  $h$  是  $(0, 1]$  上有界非负连续函数, 令  $a_n = \int_0^1 h^n(t) dt$ , 则

$$L = \lim_{n \rightarrow \infty} n \left( \sqrt[n]{\int_0^{\infty} f^{n+1}(x) e^{-x} dx} - \sqrt[n]{\int_0^{\infty} f^n(x) e^{-x} dx} \right) = \lim_{n \rightarrow \infty} n (\sqrt[n]{a_{n+1}} - \sqrt[n]{a_n})$$

我们将证明  $L = M \ln M$ , 这里  $M = \sup_{x \in [0, +\infty)} f(x) = \sup_{t \in (0, 1]} h(t) \geq 0$ , 如果  $M = 0$ , 则显然  $f = h = 0$ , 则  $L = 0$ , 下设  $M > 0$ . 对  $0 < \varepsilon < M$ , 存在  $(0, 1]$  内的一个非空区间  $I$  使得对任意  $t \in I, h(t) \geq M - \varepsilon > 0$ , 因此

$$(M - \varepsilon) |I|^{\frac{1}{n}} = ((M - \varepsilon)^n |I|)^{\frac{1}{n}} \leq \sqrt[n]{a_n} \leq (M^n |(0, 1]|)^{\frac{1}{n}} = M$$

且  $|I|^{\frac{1}{n}} \rightarrow 1$  ( $|I| > 0$ ) 意味着  $\sqrt[n]{a_n} \rightarrow M$ . 现在我们考虑数列  $\left\{ \frac{a_{n+1}}{a_n} \right\}$ , 它是有界的, 因为

$$\frac{a_{n+1}}{a_n} = \frac{1}{a_n} \int_0^1 h^{n+1}(t) dt \leq \frac{M}{a_n} \int_0^1 h^n(t) dt + M$$

它还是单调递增的, 因为利用 Cauchy 不等式得

$$a_{n+1}^2 = \left( \int_0^1 h^{\frac{n+2}{2}}(t) h^{\frac{n}{2}}(t) dt \right) \leq \int_0^1 h^{n+2}(t) dt \int_0^1 h^n(t) dt = a_{n+2} a_n$$

所以  $\left\{ \frac{a_{n+1}}{a_n} \right\}$  存在极限  $M'$ , 由 Stolz-Cesàro 定理,

$$\ln M = \lim_{n \rightarrow \infty} \ln(\sqrt[n]{a_n}) = \lim_{n \rightarrow \infty} \frac{\ln a_n}{n} = \lim_{n \rightarrow \infty} (\ln a_{n+1} - \ln a_n) = \lim_{n \rightarrow \infty} \ln \left( \frac{a_{n+1}}{a_n} \right) = \ln M'$$

最后得到

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} n (\sqrt[n]{a_{n+1}} - \sqrt[n]{a_n}) \\
 &= \lim_{n \rightarrow \infty} n \sqrt[n]{a_n} \left( \exp \left( \frac{1}{n} \ln \left( \frac{a_{n+1}}{a_n} \right) \right) - 1 \right) \\
 &= \lim_{n \rightarrow \infty} \sqrt[n]{a_n} \ln \left( \frac{a_{n+1}}{a_n} \right) = M \ln M
 \end{aligned}$$

□



139. 计算二重积分

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(x+y)}{2 - \cos x - \cos y} dx dy$$

解 首先有

$$\frac{1 - \cos(x+y)}{2 - \cos x - \cos y} = \frac{1 - \cos(x+y)}{2 - 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)}$$

作二重积分换元  $x = u + v, y = u - v$ , 则  $\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = 2$ , 于是积分域变为正方形  $(u, v) : -\pi \leq u \pm v \leq \pi$ , 由对称性

$$\begin{aligned} I &= 4 \iint_{0 \leq u+v \leq \pi} \frac{1 - \cos 2u}{1 - \cos u \cos v} du dv \\ &= 4 \int_0^{\pi} \left( \frac{1 - \cos 2u}{\cos u} \int_0^{\pi-u} \frac{dv}{\sec u - \cos v} \right) du \\ &= 4 \int_0^{\pi} \left( \frac{1 - \cos 2u}{\cos u} \frac{2}{\sqrt{\sec^2 u - 1}} \arctan \left( \sqrt{\frac{\sec u + 1}{\sec u - 1}} \tan \frac{v}{2} \right) \Big|_{v=0}^{\pi-u} \right) du \\ &= 16 \int_0^{\pi} \sin u \arctan \left( \cot^2 \left( \frac{u}{2} \right) \right) du \\ &= 64 \int_0^{\infty} \frac{w}{(1+w^2)^2} \arctan(w^2) dw \quad \left( w = \cot \left( \frac{u}{2} \right) \right) \\ &= 32 \int_0^{\infty} \frac{\arctan t}{(1+t)^2} dt \\ &= 8\pi \end{aligned}$$

□

140. 记  $f(n) = \sum_{k=1}^n k^k$ , 令  $g(n) = \sum_{k=1}^n f(k)$ , 求极限

$$\lim_{n \rightarrow \infty} \left( \frac{g(n+2)}{g(n+1)} - \frac{g(n+1)}{g(n)} \right)$$

解 容易得到

$$g(n) = \sum_{k=1}^n (n+1-k) k^k = \sum_{k=1}^n (n+1-k) k^k = \sum_{k=1}^n k(n+1-k)^{n+1-k}$$

于是我们有

$$\frac{g(n)}{n} = \sum_{k=1}^n \frac{k}{n^{k-1}} \left( 1 - \frac{k-1}{n} \right)^{n+1-k} = 1 + \frac{2}{n} \left( 1 - \frac{1}{n} \right)^{n-1} + h(n) + o\left(\frac{1}{n}\right)$$

因为

$$0 \leq h(n) = \sum_{k=3}^n \frac{k}{n^{k-1}} \left( 1 - \frac{k-1}{n} \right)^{n+1-k} \leq \frac{3}{n^2} + \frac{4}{n^3} + \frac{1}{n^4} \sum_{k=5}^n k \leq \frac{3}{n^2} + \frac{4}{n^3} + \frac{n^2+n}{2n^4}$$

因此

$$\begin{aligned}\frac{g(n+1)}{n^{n+1}} &= \left(1 + \frac{1}{n}\right)^{n+1} \cdot \frac{g(n+1)}{(n+1)^{n+1}} \\ &= e \left(1 + \frac{2}{n} + o\left(\frac{1}{n}\right)\right) \left(1 + \frac{2e^{-1}}{n} + o\left(\frac{1}{n}\right)\right) \\ &= e + \frac{e+4}{2n} + o\left(\frac{1}{n}\right)\end{aligned}$$

且

$$\begin{aligned}\frac{g(n+2)}{n^{n+2}} &= \left(1 + \frac{2}{n}\right)^{n+2} \cdot \frac{g(n+2)}{(n+2)^{n+2}} \\ &= e \left(1 + \frac{2}{n} + o\left(\frac{1}{n}\right)\right) \left(1 + \frac{2e^{-1}}{n} + o\left(\frac{1}{n}\right)\right) \\ &= e^2 + \frac{2e^2 + 2e}{n} + o\left(\frac{1}{n}\right)\end{aligned}$$

最后得到

$$\begin{aligned}&\frac{g(n+2)}{g(n+1)} - \frac{g(n+1)}{g(n)} \\ &= \frac{g(n+2)g(n) - g^2(n+1)}{g(n+1)g(n)} \\ &= \left(\frac{n}{e} + o(1)\right) \left[ \left(e^2 + \frac{2e^2 + 2e}{n} + o\left(\frac{1}{n}\right)\right) \left(1 - \frac{2e^{-1}}{n} + o\left(\frac{1}{n}\right)\right) - \left(e + \frac{e+4}{2n} + o\left(\frac{1}{n}\right)\right)^2 \right] \\ &= \left(\frac{n}{e} + o(1)\right) \left(\frac{e^2}{n} + o\left(\frac{1}{n}\right)\right) = e + o\left(\frac{1}{n}\right) \rightarrow e\end{aligned}$$

□

#### 141. 计算二重积分

$$I = \int_0^\infty \left( \int_x^\infty e^{-(x-y)^2} \sin^2(x^2 + y^2) \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \right) dx$$

**解** 转化为极坐标可得

$$\begin{aligned}I &= \int_0^\infty \left( \int_{\pi/4}^{\pi/2} e^{-\rho^2(1-\sin 2\theta)} \sin^2 \rho^2 \frac{\rho^2 \cos 2\theta}{\rho^4} d\theta \right) d\rho \\ &= \int_0^\infty \frac{e^{-\rho^2} \sin^2 \rho^2}{2\rho^3} \left( \int_{\pi/4}^{\pi/2} e^{\rho^2 \sin 2\theta} d(\rho^2 \sin 2\theta) \right) d\rho \\ &= \int_0^\infty \frac{e^{-\rho^2} \sin^2 \rho^2}{2\rho^3} \left( e^{\rho^2 \sin 2\theta} \right) \Big|_{\theta=\pi/4}^{\pi/2} d\rho \\ &= \int_0^\infty \frac{(e^{-\rho^2} - 1) \sin^2 \rho^2}{2\rho^3} d\rho \\ &= \frac{1}{4} \int_0^\infty \frac{(e^{-t} - 1) \sin^2 t}{t^2} dt\end{aligned}$$

注意到

$$\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = - \int_0^{\infty} \sin^2 t d\left(\frac{1}{t}\right) = \int_0^{\infty} \frac{\sin 2t}{t} dt = \frac{\pi}{2}$$

对积分  $\int_0^{\infty} \frac{e^{-t} \sin^2 t}{t^2} dt$ , 考虑参数积分

$$I(a) = \int_0^{\infty} \frac{e^{-t} \sin^2 at}{t^2} dt = \int_0^{\infty} \frac{e^{-t} (1 - \cos 2at)}{2t^2} dt$$

$$I'(a) = \int_0^{\infty} \frac{e^{-t} \sin 2at}{t} dt, I''(a) = 2 \int_0^{\infty} e^{-t} \cos 2at dt = \frac{2}{1 + 4a^2}$$

注意到  $I(0) = I'(0) = 0$ , 于是

$$I'(a) = \arctan 2a, I''(a) = a \arctan 2a - \frac{1}{4} \ln(1 + 4a^2)$$

原积分为

$$I = \frac{1}{4} \left( I(1) - \frac{\pi}{2} \right) = \frac{1}{4} \arctan 2 - \frac{1}{16} \ln 5 - \frac{\pi}{8}$$

□

#### 142. 求和

$$S = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{2n+1} \left( \frac{(n-1)!}{(2n-1)!!} \right)^2$$

**解** 首先我们有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^{2n-1}}{2n+1} \left( \frac{(n-1)!}{(2n-1)!!} \right)^2 &= \sum_{n=1}^{\infty} \frac{2}{2n+1} \left( \frac{(2n-2)!!}{(2n-1)!!} \right)^2 \\ &= \sum_{n=1}^{\infty} \frac{2}{2n+1} \int_0^{\frac{\pi}{2}} \sin^{2n-1} x dx \int_0^{\frac{\pi}{2}} \sin^{2n-1} y dy \\ &= \sum_{n=1}^{\infty} \frac{2}{2n+1} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin^{2n-1} x \sin^{2n-1} y dx dy \end{aligned}$$

利用对数函数的幂级数公式不难得到

$$\sum_{n=1}^{\infty} \frac{2 \sin^{2n-1} x \sin^{2n-1} y}{2n+1} = \frac{1}{\sin^2 x \sin^2 y} \left( \ln \frac{1 + \sin x \sin y}{1 - \sin x \sin y} - 2 \sin x \sin y \right)$$

考虑参变量积分

$$I(a) = \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 y} \left( \ln \frac{1 + a \sin y}{1 - a \sin y} - 2a \sin y \right) dy \quad |a| < 1$$

则可得

$$\begin{aligned} I(0) &= 0 \\ I'(a) &= \int_0^{\frac{\pi}{2}} \frac{1}{\sin y} \left( \frac{1}{1 + a \sin y} + \frac{1}{1 - a \sin y} - 2 \right) dy \\ &= 2a^2 \int_0^{\frac{\pi}{2}} \frac{\sin y}{1 - a^2 \sin^2 y} dy \end{aligned}$$

$$\begin{aligned}
 &= 2a^2 \int_0^1 \frac{dt}{1-a^2(1-t^2)} \quad (t = \cos y) \\
 &= 2 \int_0^{\frac{\pi}{2}} \frac{dt}{t^2 + (1-a^2)/a^2} \\
 &= \frac{2a}{\sqrt{1-a^2}} \arctan \frac{a}{\sqrt{1-a^2}}
 \end{aligned}$$

那么

$$\begin{aligned}
 I(\sin x) &= \int_0^{\sin x} \frac{2a}{\sqrt{1-a^2}} \arctan \frac{a}{\sqrt{1-a^2}} da \\
 &= 2 \int_0^x u \sin u du \quad (a = \sin u) \\
 &= 2(\sin x - x \cos x)
 \end{aligned}$$

于是

$$\begin{aligned}
 S &= \int_0^{\frac{\pi}{2}} \left[ \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x \sin^2 y} \left( \ln \frac{1 + \sin x \sin y}{1 - \sin x \sin y} - 2 \sin x \sin y \right) dy \right] dx \\
 &= \int_0^{\frac{\pi}{2}} \frac{I(\sin x)}{\sin^2 x} dx = 2 \int_0^{\frac{\pi}{2}} \frac{\sin x - x \cos x}{\sin^2 x} dx \\
 &= -2(\sin x - x \cos) \cot x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} x \cos x dx \\
 &= 2 \int_0^{\frac{\pi}{2}} x d(\sin x) = \pi - 2
 \end{aligned}$$

□

### 143. 计算二重积分

$$\int_0^\infty \left( \int_y^\infty \frac{(x-y)^2 \ln \left( \frac{x+y}{x-y} \right)}{xy \sinh(x+y)} dx \right) dy$$

**解** 作积分换元  $u = x + y, v = \frac{x-y}{x+y}$ , 积分化为

$$\begin{aligned}
 I &= \int_0^1 \left( \int_0^\infty \frac{-(uv)^2 \ln v}{(u^2(1-v^2)/4) \sinh u} \left| \det \frac{\partial(x, y)}{\partial(u, v)} \right| du \right) dv \\
 &= \int_0^1 \frac{-2v^2 \ln v}{1-v^2} dv \int_0^\infty \frac{u}{\sinh u} du \\
 &= \frac{\pi^4}{16} - \frac{\pi^2}{2}
 \end{aligned}$$

其中

$$\begin{aligned}
 \int_0^1 \frac{-2v^2 \ln v}{1-v^2} dv &= 2 \int_0^1 \sum_{k=1}^\infty v^{2k} \ln v dv \\
 &= 2 \sum_{k=1}^\infty \int_0^\infty e^{-(2k+1)t} t dt \quad (t = -\ln v)
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = 2 \left( \frac{\pi^2}{8} - 1 \right) \\
 \int_0^{\infty} \frac{u}{\sinh u} du &= \int_0^{\infty} \frac{2ue^{-u}}{1-e^{-2u}} du \\
 &= 2 \sum_{k=0}^{\infty} \int_0^{\infty} ue^{-(2k+1)u} du = \frac{\pi^2}{4}
 \end{aligned}$$

□

144. 求极限

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n \left( \frac{x \ln(1+x/n)}{1+x} \right) dx$$

**解** 令  $t = \frac{x}{n}$ , 则

$$\begin{aligned}
 \frac{1}{n} \int_0^n \left( \frac{x \ln(1+x/n)}{1+x} \right) dx &= \int_0^1 \frac{nt}{1+nt} \ln(1+t) dt \\
 &= \int_0^1 \ln(1+t) dt - \int_0^1 \frac{t}{1+nt} \frac{\ln(1+t)}{t} dt \\
 &= 2 \ln 2 - 1 - \int_0^1 \frac{t}{1+nt} \frac{\ln(1+t)}{t} dt
 \end{aligned}$$

由于

$$0 \leq \int_0^1 \frac{t}{1+nt} \frac{\ln(1+t)}{t} dt \leq \frac{1}{n} \int_0^1 \frac{\ln(1+t)}{t} dt < \frac{1}{n}$$

因此

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n \left( \frac{x \ln(1+x/n)}{1+x} \right) dx = 2 \ln 2 - 1$$

□

145. 计算二重积分

$$I = \int_0^{\infty} \frac{1}{x} \int_0^x \frac{\cos(x-y) - \cos x}{y} dy dx$$

**解** 考虑参变量积分

$$I(t) = \int_0^{\infty} \frac{1}{x} \int_0^x \frac{\cos(x-ty) - \cos x}{y} dy dx$$

则

$$\begin{aligned}
 I(0) &= 0 \\
 I'(t) &= \int_0^{\infty} \frac{1}{x} \int_0^x \sin(x-ty) dy dx \\
 &= \int_0^{\infty} \frac{1}{x} \left( \frac{1}{t} \cos(x-ty) \Big|_{y=0}^{y=x} \right) dx \\
 &= \int_0^{\infty} \frac{\cos(1-t)x - \cos x}{tx} dx \\
 &= -\frac{\ln(1-t)}{t}
 \end{aligned}$$

上面最后一步我们利用了 Frullani 积分, 于是

$$\begin{aligned} I &= \int_0^\infty \frac{1}{x} \int_0^x \frac{\cos(x-y) - \cos x}{y} dy dx \\ &= - \int_0^1 \frac{\ln(1-t)}{t} dt = \int_0^1 \sum_{k=1}^\infty \frac{t^{k-1}}{k} dt = \sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6} \end{aligned}$$

□

146. 设函数  $f: [0, 1] \rightarrow \mathbb{R}$  是连续可微函数, 证明不等式

$$\int_0^1 [f'(x)]^2 dx \geq 12 \left( \int_0^1 f(x) dx - 2 \int_0^{1/2} f(x) dx \right)^2$$

**证明** 利用 Cauchy 不等式得

$$\begin{aligned} \int_0^{1/2} [f'(x)]^2 dx \int_0^{1/2} x^2 dx &\geq \left( \int_0^{1/2} x f'(x) dx \right)^2 = \left[ \frac{1}{2} f\left(\frac{1}{2}\right) - \int_0^{1/2} f(x) dx \right]^2 \\ \Rightarrow \int_0^{1/2} [f'(x)]^2 dx &\geq 24 \left[ \frac{1}{2} f\left(\frac{1}{2}\right) - \int_0^{1/2} f(x) dx \right]^2 \end{aligned}$$

再利用 Cauchy 不等式得

$$\begin{aligned} \int_{1/2}^1 [f'(x)]^2 dx \int_{1/2}^1 (1-x)^2 dx &\geq \left[ -\frac{1}{2} f\left(\frac{1}{2}\right) + \int_{1/2}^1 f(x) dx \right]^2 \\ \Rightarrow \int_{1/2}^1 [f'(x)]^2 dx &\geq 24 \left[ -\frac{1}{2} f\left(\frac{1}{2}\right) + \int_{1/2}^1 f(x) dx \right]^2 \end{aligned}$$

两式相加, 利用不等式  $2(a^2 + b^2) \geq (a + b)^2$  得

$$\begin{aligned} \int_0^1 [f'(x)]^2 dx &\geq 24 \left[ \left( \frac{1}{2} f\left(\frac{1}{2}\right) - \int_0^{1/2} f(x) dx \right)^2 + \left( -\frac{1}{2} f\left(\frac{1}{2}\right) + \int_{1/2}^1 f(x) dx \right)^2 \right] \\ &\geq 12 \left( \int_0^1 f(x) dx - 2 \int_0^{1/2} f(x) dx \right)^2 \end{aligned}$$

特别地, 当  $\int_0^{1/2} f(x) dx = 0$  时, 我们有

$$\int_0^1 [f'(x)]^2 dx \geq 12 \left( \int_0^1 f(x) dx \right)^2$$

□

147. 求和

$$\sum_{n=1}^\infty \frac{H_{n+2}}{n(n+2)}$$

**解** 首先注意到

$$H_{n+2} = \sum_{k=1}^{n+2} \frac{1}{k} = \int_0^1 \sum_{k=0}^n x^k dx = \int_0^1 \frac{1-x^{n+2}}{1-x} dx$$

于是

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_{n+2}}{n(n+2)} &= \int_0^1 \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{1-x^{n+2}}{n(n+2)} dx \\ &= \int_0^1 \frac{1}{1-x} \left( \frac{3}{4} - \frac{x}{2} - \frac{x^2}{4} - 2(1-x^2) \ln(1-x) \right) dx \\ &= \int_0^1 \left( \frac{x+3}{4} + \frac{1}{2}(1+x) \ln(1-x) \right) dx \\ &= \frac{7}{4} \end{aligned}$$

□

**148.** 求和

$$\sum_{n=1}^{\infty} \arctan(\sinh n) \cdot \arctan\left(\frac{\sinh 1}{\cosh n}\right)$$

**解** 注意到

$$\begin{aligned} \arctan(\sinh n) &= \arctan\left(\frac{e^n - e^{-n}}{2}\right) = \arctan\left(\frac{e^n - e^{-n}}{1 + e^n \cdot e^{-n}}\right) \\ &= \arctan(e^n) - \arctan(e^{-n}) = 2 \arctan(e^n) - \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} \arctan\left(\frac{\sinh 1}{\cosh n}\right) &= \arctan\left(\frac{e - e^{-1}}{e^n + e^{-n}}\right) = \arctan\left(\frac{e^{n+1} - e^{n-1}}{1 + e^{n+1} \cdot e^{n-1}}\right) \\ &= \arctan(e^{n+1}) - \arctan(e^{n-1}) \end{aligned}$$

因此

$$\begin{aligned} &\sum_{n=1}^{\infty} \arctan(\sinh n) \cdot \arctan\left(\frac{\sinh 1}{\cosh n}\right) \\ &= \sum_{n=1}^{\infty} \left[ 2 \arctan(e^n) - \frac{\pi}{2} \right] [\arctan(e^{n+1}) - \arctan(e^{n-1})] \\ &= 2 \left[ \lim_{n \rightarrow \infty} \arctan(e^n) \arctan(e^{n+1}) - \frac{\pi}{4} \arctan(e) \right] \\ &\quad - \frac{\pi}{2} \left[ \lim_{n \rightarrow \infty} (\arctan(e^n) + \arctan(e^{n+1})) - \frac{\pi}{4} - \arctan(e) \right] \\ &= 2 \left( \frac{\pi^2}{4} - \frac{\pi}{4} \arctan(e) \right) - \frac{\pi}{2} \left( \frac{3}{4}\pi - \arctan(e) \right) = \frac{\pi^2}{8} \end{aligned}$$

□

**149.** 求和

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2}\right)$$

解

$$\begin{aligned}
 \tan \left[ \sum_{n=1}^{\infty} \arctan \left( \frac{1}{n^2} \right) \right] &= \tan \left[ \sum_{n=1}^{\infty} \arg \left( 1 + \frac{i}{n^2} \right) \right] \\
 &= \tan \left[ \arg \left( \prod_{n=1}^{\infty} \left( 1 + \frac{(\pi(1+i)/\sqrt{2})^2}{n^2 \pi^2} \right) \right) \right] \\
 &= \tan \left[ \arg \left( \frac{\sinh(\pi(1+i)/\sqrt{2})}{\pi(1+i)/\sqrt{2}} \right) \right] \\
 &= \tan \left[ \arctan \left( \frac{\tan(\pi/\sqrt{2}) - \tanh(\pi/\sqrt{2})}{\tan(\pi/\sqrt{2}) + \tanh(\pi/\sqrt{2})} \right) \right] \\
 &= \frac{\tan(\pi/\sqrt{2}) - \tanh(\pi/\sqrt{2})}{\tan(\pi/\sqrt{2}) + \tanh(\pi/\sqrt{2})}
 \end{aligned}$$

因此

$$\sum_{n=1}^{\infty} \arctan \left( \frac{1}{n^2} \right) = \arctan \left( \frac{\tan(\pi/\sqrt{2}) - \tanh(\pi/\sqrt{2})}{\tan(\pi/\sqrt{2}) + \tanh(\pi/\sqrt{2})} \right)$$

□

## 150. 计算积分

$$\int_0^1 \frac{\arctan x}{\sqrt{1-x^2}} dx$$

解 考虑参变量积分  $I(a) = \int_0^1 \frac{\arctan(ax)}{\sqrt{1-x^2}} dx$ , 则

$$I(0) = 0, \quad I(\infty) = \frac{\pi}{2} \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \arcsin 1 = \frac{\pi^2}{4}$$

$$\begin{aligned}
 I'(a) &= \int_0^1 \frac{x dx}{(1+a^2 x^2) \sqrt{1-x^2}} \\
 &= \int_0^1 \frac{dt}{1+a^2 - a^2 t^2} \quad (t = \sqrt{1-x^2}) \\
 &= \frac{1}{a\sqrt{1+a^2}} \operatorname{arctanh} \left( \frac{at}{\sqrt{1+a^2}} \right) \Big|_0^1 \\
 &= \frac{1}{a\sqrt{1+a^2}} \operatorname{arctanh} \left( \frac{a}{\sqrt{1+a^2}} \right) = \frac{\operatorname{arcsinh} a}{a\sqrt{1+a^2}}
 \end{aligned}$$

于是

$$\begin{aligned}
 \int_0^1 \frac{\arctan x}{\sqrt{1-x^2}} dx &= \int_0^1 I'(a) da = - \int_0^1 \frac{\operatorname{arcsinh} a}{\sqrt{1+1/a^2}} d\left(\frac{1}{a}\right) \\
 &= - \int_0^1 \operatorname{arcsinh} a d\left(\operatorname{arcsinh} \left(\frac{1}{a}\right)\right)
 \end{aligned}$$



$$\begin{aligned}
 &= -\operatorname{arcsinh}^2 1 + \int_0^1 \frac{1}{\sqrt{1+a^2}} \operatorname{arcsinh} \left( \frac{1}{a} \right) da \\
 &= \int_1^\infty \frac{\operatorname{arcsinh} u}{u \sqrt{1+u^2}} du - \operatorname{arcsinh}^2 1 = \int_1^\infty I'(u) du - \operatorname{arcsinh}^2 1 \\
 &= I(\infty) - I(1) - \operatorname{arcsinh}^2 1 = \frac{1}{2} [I(\infty) - \operatorname{arcsinh}^2 1] \\
 &= \frac{1}{2} \left[ \frac{\pi^2}{4} - \ln^2(1 + \sqrt{2}) \right] = \frac{\pi^2}{8} - \frac{1}{2} \ln^2(1 + \sqrt{2})
 \end{aligned}$$

□

### 151. 求和

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2}$$

**解** 首先注意到

$$\begin{aligned}
 \frac{1}{\binom{2n}{n}(2n+1)^2} &= \frac{(n!)^2}{(2n+1)!(2n+1)} \\
 &= \frac{1}{2n+1} \cdot \frac{n!}{(2n+1)!! \cdot 2^n} \\
 &= \frac{1}{2^{2n}(2n+1)} \cdot \frac{(2n)!!}{(2n+1)!!} \\
 &= \frac{1}{2^{2n}(2n+1)} \int_0^{\frac{\pi}{2}} \cos^{2n+1} x dx
 \end{aligned}$$

于是

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2} &= \sum_{n=0}^{\infty} \frac{1}{2n+1} \int_0^{\frac{\pi}{2}} \left( \frac{\cos x}{2} \right)^{2n+1} dx \\
 &= \int_0^{\frac{\pi}{2}} \ln \left( \frac{2 + \cos x}{2 - \cos x} \right) dx
 \end{aligned}$$

考虑参数积分

$$\begin{aligned}
 I(t) &= \int_0^{\frac{\pi}{2}} \ln \left( \frac{2 + t \cos x}{2 - t \cos x} \right) dx \\
 I'(t) &= \int_0^{\frac{\pi}{2}} \frac{\cos x}{2 + t \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - t \cos x} dx \\
 &= \int_0^{\frac{\pi}{2}} \frac{4 \cos x}{4 - t^2 \cos^2 x} dx = 4 \int_0^{\frac{\pi}{2}} \frac{d(\sin x)}{(4 - t^2) + t^2 \sin^2 x} \\
 &= \frac{4}{t \sqrt{4 - t^2}} \arctan \frac{t}{\sqrt{4 - t^2}}
 \end{aligned}$$

因此

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2} &= I(1) = \int_0^1 \frac{4}{t \sqrt{4 - t^2}} \arctan \frac{t}{\sqrt{4 - t^2}} dt + I(0) \\
 &= 2 \int_0^{\frac{\pi}{6}} \frac{u}{\sin u} du \quad (t = 2 \sin u)
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \int_0^{\frac{\pi}{6}} u d\left(\ln\left(\tan \frac{u}{2}\right)\right) = -\frac{\pi}{3} \ln(2 + \sqrt{3}) - 2 \int_0^{\frac{\pi}{6}} \ln\left(\tan \frac{u}{2}\right) du \\
 &= 4 \int_0^{\frac{\pi}{6}} \sum_{k=1}^{\infty} \frac{\cos(2k-1)u}{2k-1} du - \frac{\pi}{3} \ln(2 + \sqrt{3}) \\
 &= \frac{8}{3} G - \frac{\pi}{3} \ln(2 + \sqrt{3})
 \end{aligned}$$

□

**152.** 设  $H_n$  表示第  $n$  个调和数, 即  $H_n = \sum_{k=1}^n \frac{1}{k}$ ,  $E_n = H_n^2 - \sum_{k=1}^n \frac{H_{\max(k, n-k)}}{k}$ , 求  $\lim_{n \rightarrow \infty} E_n$ .

**解** 令  $H_n^{(2)} = \sum_{k=1}^n \frac{1}{k^2}$ , 我们将要证明的是

$$E_n = \frac{1}{2} H_{\left[\frac{n}{2}\right]}^{(2)}$$

那么就有

$$\lim_{n \rightarrow \infty} E_n = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{12}$$

由于  $E_1 = \frac{1}{2} H_0^{(2)} = 0$ , 且对  $n > 0$  有

$$\frac{1}{2} H_{\left[\frac{n+1}{2}\right]}^{(2)} - \frac{1}{2} H_{\left[\frac{n}{2}\right]}^{(2)} = \begin{cases} 0, & n \text{ 是偶数} \\ \frac{2}{(n+1)^2}, & n \text{ 是奇数} \end{cases}$$

只需要证明差分  $E_{n+1} - E_n$  满足相同的性质即可, 注意到

$$E_n = H_n^2 - \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{H_{n-k}}{k} - \sum_{k=\left[\frac{n}{2}\right]+1}^n \frac{H_k}{k}$$

先假定  $n$  是正整数, 则  $\left[\frac{n}{2}\right] = \left[\frac{n+1}{2}\right] = \frac{n}{2}$ , 且

$$\begin{aligned}
 E_{n+1} - E_n &= H_{n+1}^2 - H_n^2 - \sum_{k=1}^{\frac{n}{2}} \frac{H_{n+1-k}}{k} - \frac{H_{n+1}}{n+1} \\
 &= \frac{H_{n+1} + H_n}{n+1} - \sum_{k=1}^{\frac{n}{2}} \frac{1}{k(n+1-k)} - \frac{H_{n+1}}{n+1} \\
 &= \frac{H_n}{n+1} - \frac{1}{n+1} \sum_{k=1}^{\frac{n}{2}} \left( \frac{1}{n+1-k} + \frac{1}{k} \right) = 0
 \end{aligned}$$

再假定  $n$  是奇数, 则  $\left[\frac{n}{2}\right] = \frac{n-1}{2}$ ,  $\left[\frac{n+1}{2}\right] = \frac{n+1}{2}$ , 且

$$E_{n+1} - E_n = H_{n+1}^2 - H_n^2 - \sum_{k=1}^{\frac{n-1}{2}} \frac{H_{n+1-k} - H_{n-k}}{k} - \frac{H_{n+1}}{n+1} + \frac{2H_{\frac{n+1}{2}}}{n+1}$$

$$\begin{aligned}
 &= \frac{H_{n+1} + H_n}{n+1} - \frac{1}{n+1} \sum_{k=1}^{\frac{n-1}{2}} \left( \frac{1}{n+1-k} + \frac{1}{k} \right) - \frac{H_{n+1}}{n+1} \\
 &= \frac{H_n}{n+1} - \frac{1}{n+1} \left( H_n - \frac{2}{n+1} \right) = \frac{2}{(n+1)^2}
 \end{aligned}$$

□

### 153. 求和

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)}$$

**解** 注意到对  $0 \leq x < 1$  有

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n} x^n &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{\infty} \left( \frac{\sqrt{x}}{k} \right)^{2n} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\sqrt{x}}{k} \right)^{2n} \\
 &= -\ln \prod_{k=1}^{\infty} \left( 1 - \left( \frac{\sqrt{x}}{k} \right)^2 \right) = -\ln \left( \frac{\sin(\pi\sqrt{x})}{\pi\sqrt{x}} \right)
 \end{aligned}$$

于是我们有

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)} &= \int_0^1 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n} x^n dx = -\int_0^1 \ln \left( \frac{\sin(\pi\sqrt{x})}{\pi\sqrt{x}} \right) dx \\
 &= -\int_0^1 \ln(\sin \pi\sqrt{x}) dx + \int_0^1 \ln \pi dx + \frac{1}{2} \int_0^1 \ln x dx \\
 &= -\frac{2}{\pi^2} \int_0^{\pi} t \ln(\sin t) dt + \ln \pi - \frac{1}{2} = \ln(2\pi) - \frac{1}{2}
 \end{aligned}$$

□

### 154. 设 $r$ 是一个整数, 求和

$$\sum_{n=-\infty}^{\infty} \arctan \left( \frac{\sinh r}{\cosh n} \right)$$

**解** 首先有

$$\begin{aligned}
 \arctan \left( \frac{\sinh r}{\cosh n} \right) &= \arctan \left( \frac{e^r - e^{-r}}{e^n + e^{-n}} \right) = \arctan \left( \frac{e^{-(n-r)} - e^{-(n+r)}}{1 + e^{-2n}} \right) \\
 &= \arctan(e^{-(n-r)}) - \arctan(e^{-(n+r)})
 \end{aligned}$$

不失一般性, 不妨设  $r \geq 0$ , 我们有

$$\begin{aligned}
 &\sum_{n=-\infty}^{\infty} \arctan \left( \frac{\sinh r}{\cosh n} \right) \\
 &= 2 \sum_{n=1}^{\infty} \arctan \left( \frac{\sinh r}{\cosh n} \right) + \arctan(\sinh r) \\
 &= 2 \sum_{n=1}^{\infty} \left( \arctan(e^{-(n-r)}) - \arctan(e^{-(n+r)}) \right) + \arctan(e^r) - \arctan(e^{-r})
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \sum_{m \geq 1-r} \arctan(e^{-m}) - 2 \sum_{m \geq 1+r} \arctan(e^{-m}) + \arctan(e^r) - \arctan(e^{-r}) \\
 &= 2 \sum_{1-r \leq m \leq r} \arctan(e^{-m}) + \arctan(e^r) - \arctan(e^{-r}) \\
 &= 2 \sum_{-r \leq m \leq r} \arctan(e^{-m}) - \arctan(e^r) - \arctan(e^{-r}) \\
 &= 2 \sum_{1 \leq m \leq r} [\arctan(e^m) + \arctan(e^{-m})] + 2 \arctan(1) - \arctan(e^r) - \arctan(e^{-r}) \\
 &= 2 \sum_{1 \leq m \leq r} \frac{\pi}{2} + 2 \cdot \frac{\pi}{4} - \frac{\pi}{2} = \pi r
 \end{aligned}$$

□

155. 求和

$$\sum_{n=1}^{\infty} \operatorname{arcsinh} \left( \frac{1}{\sqrt{2^{n+2}} + 2 + \sqrt{2^{n+1}} + 2} \right)$$

解 记

$$a_n = \frac{1}{\sqrt{2^{n+2}} + 2 + \sqrt{2^{n+1}} + 2}, \quad b_n = \frac{\sqrt{2^{n+1}} + 1 - \sqrt{3}}{2^{\frac{n+1}{2}}}$$

不难得到

$$b_{n+1} \sqrt{1 + b_n^2} - b_n \sqrt{1 + b_{n+1}^2} = a_n$$

根据基本性质

$$\operatorname{arcsinh} \left( x \sqrt{1 + y^2} - y \sqrt{1 + x^2} \right) = \operatorname{arcsinh}(x) - \operatorname{arcsinh}(y)$$

我们得到

$$\sum_{n=1}^N \operatorname{arcsinh}(a_n) = \sum_{n=1}^N (\operatorname{arcsinh}(b_{n+1}) - \operatorname{arcsinh}(b_n)) = \operatorname{arcsinh}(b_{N+1}) - \operatorname{arcsinh}(b_1)$$

现在  $b_1 = 0, b_{N+1} \rightarrow \frac{1}{\sqrt{2}}$ , 因此

$$\begin{aligned}
 \sum_{n=1}^{\infty} \operatorname{arcsinh} \left( \frac{1}{\sqrt{2^{n+2}} + 2 + \sqrt{2^{n+1}} + 2} \right) &= \lim_{N \rightarrow \infty} \operatorname{arcsinh}(b_{N+1}) \\
 &= \operatorname{arcsinh} \left( \frac{1}{\sqrt{2}} \right) = \frac{\ln(2 + \sqrt{3})}{2}
 \end{aligned}$$

□

156. 设  $F_k$  表示第  $k$  个 Fibonacci 数, 求和

$$\sum_{n=1}^{\infty} \left( \arctan \frac{1}{F_{4n-3}} + \arctan \frac{1}{F_{4n-2}} + \arctan \frac{1}{F_{4n-1}} - \arctan \frac{1}{F_{4n}} \right)$$

解 令  $\varphi = \frac{\sqrt{5} + 1}{2}$ , 对  $n > 0$  有

$$\arctan(\varphi^{-(2n+1)}) + \arctan(\varphi^{-(2n-1)}) = \arctan \left( \frac{\varphi^{-(2n+1)} + \varphi^{-(2n-1)}}{1 - \varphi^{-4n}} \right)$$

$$\begin{aligned}
 &= \arctan \left( \frac{\varphi^{-1} + \varphi}{\varphi^{2n} - \varphi^{-2n}} \right) = \arctan \left( \frac{\sqrt{5}}{\varphi^{2n} - \varphi^{-2n}} \right) \\
 &= \arctan \left( \frac{1}{F_{2n}} \right)
 \end{aligned}$$

进一步, 对  $n > 1$ ,

$$\begin{aligned}
 \arctan \frac{1}{F_{2n}} + \arctan \frac{1}{F_{2n-1}} &= \arctan \left( \frac{F_{2n}^{-1} + F_{2n-1}^{-1}}{1 - F_{2n}^{-1} F_{2n-1}^{-1}} \right) \\
 &= \arctan \left( \frac{F_{2n} + F_{2n-1}}{F_{2n} F_{2n-1} - 1} \right) = \arctan \left( \frac{F_{2n+1}}{F_{2n} F_{2n-1} - 1} \right) \\
 &= \arctan \frac{1}{F_{2n-2}}
 \end{aligned}$$

因此

$$\begin{aligned}
 &\sum_{n=1}^N \left( \arctan \frac{1}{F_{4n-2}} - \arctan \frac{1}{F_{4n}} \right) \\
 &= \sum_{n=1}^{2N} (-1)^{n+1} \arctan \frac{1}{F_{2n}} \\
 &= \sum_{n=1}^{2N} (-1)^{n+1} \left( \arctan \left( \varphi^{-(2n+1)} \right) + \arctan \left( \varphi^{-(2n-1)} \right) \right) \\
 &= \sum_{n=2}^{2N+1} (-1)^{n+1} \arctan \left( \varphi^{-(2n+1)} \right) + \sum_{n=1}^{2N} (-1)^{n+1} \arctan \left( \varphi^{-(2n-1)} \right) \\
 &= \arctan \left( \varphi^{-1} \right) - \arctan \left( \varphi^{-(4N+1)} \right) \\
 &= \frac{\pi}{2} - \arctan \left( \varphi \right) - \arctan \left( \varphi^{-(4N+1)} \right)
 \end{aligned}$$

并且

$$\begin{aligned}
 \sum_{n=1}^N \left( \arctan \frac{1}{F_{4n-3}} + \arctan \frac{1}{F_{4n-1}} \right) &= \sum_{n=1}^{2N} \arctan \frac{1}{F_{2n-1}} \\
 &= \arctan \frac{1}{F_1} + \sum_{n=2}^{2N} \left( \arctan \frac{1}{F_{2n-2}} - \arctan \frac{1}{F_{2n}} \right) \\
 &= \arctan \frac{1}{F_1} + \arctan \frac{1}{F_2} - \arctan \frac{1}{F_{4N}} \\
 &= \frac{\pi}{2} - \arctan \frac{1}{F_{4N}}
 \end{aligned}$$

最后得到

$$\sum_{n=1}^{\infty} \left( \arctan \frac{1}{F_{4n-3}} + \arctan \frac{1}{F_{4n-2}} + \arctan \frac{1}{F_{4n-1}} - \arctan \frac{1}{F_{4n}} \right) = \pi - \arctan \left( \varphi \right)$$

□

**157.** 记  $C_n$  是第  $n$  个 Catalan 数, 即  $C_n = \frac{1}{n+1} \binom{2n}{n}$ , 证明:

$$(a) \sum_{n=0}^{\infty} \frac{2^n}{C_n} = 5 + \frac{3\pi}{2}; \quad (b) \sum_{n=0}^{\infty} \frac{3^n}{C_n} = 22 + 8\sqrt{3}\pi.$$

**解** 首先有展开式

$$f(z) = \left( \arcsin\left(\frac{z}{2}\right) \right)^2 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{z^{2n}}{n^2 \binom{2n}{n}}$$

因此

$$\begin{aligned} D_z(z^3 D_z(z D_z f)) &= D_z\left(z^3 D_z\left(\frac{1}{2} \sum_{n=1}^{\infty} \frac{z^{2n}}{n^2 \binom{2n}{n}}\right)\right) \\ &= D_z(z^3 D_z(z D_z f)) = D_z\left(z^3 D_z\left(\sum_{n=1}^{\infty} \frac{z^{2n}}{n \binom{2n}{n}}\right)\right) \\ &= 2 D_z\left(\sum_{n=1}^{\infty} \frac{z^{2n+2}}{\binom{2n}{n}}\right) = 4z \sum_{n=1}^{\infty} \frac{z^{2n}}{C_n} \end{aligned}$$

因此

$$\sum_{n=1}^{\infty} \frac{z^{2n}}{C_n} = 1 + \frac{1}{4z} D_z(z^3 D_z(z D_z f)) = g(z) = \frac{2(8+z^2)}{(4-z^2)^2} + \frac{24z \arcsin(\frac{z}{2})}{(4-z^2)^{\frac{5}{2}}}$$

意味着 (a)  $\sum_{n=0}^{\infty} \frac{2^n}{C_n} = 5 + \frac{3\pi}{2}$ ; (b)  $\sum_{n=0}^{\infty} \frac{3^n}{C_n} = 22 + 8\sqrt{3}\pi$ .

□

**158.** 求极限

$$\lim_{n \rightarrow \infty} n \prod_{m=1}^n \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right)$$

**解** 令  $z = -\frac{1}{2} + i$ , 则

$$\begin{aligned} e^{\gamma z} \prod_{m=1}^n \left(1 + \frac{z}{m}\right) e^{-\frac{z}{m}} \cdot e^{\gamma \bar{z}} \prod_{m=1}^n \left(1 + \frac{\bar{z}}{m}\right) e^{-\frac{\bar{z}}{m}} &= e^{H_n - \gamma} \prod_{m=1}^n \left|1 + \frac{z}{m}\right|^2 \\ &= \frac{e^{H_n - \gamma}}{n} \prod_{m=1}^n \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right) \end{aligned}$$

其中  $H_n = \sum_{k=1}^n \frac{1}{k}$ , 由于  $H_n = \ln n + \gamma + o(1)$ , 则  $\frac{e^{H_n - \gamma}}{n} \rightarrow 1$ .

进一步, 由  $\Gamma$  函数的定义

$$\frac{1}{z\Gamma(z)} = e^{\gamma z} \prod_{m=1}^{\infty} \left(1 + \frac{z}{m}\right) e^{-\frac{z}{m}}$$

因此

$$\lim_{n \rightarrow \infty} \prod_{m=1}^n \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right) = \frac{1}{z\Gamma(z)} \cdot \frac{1}{\bar{z}\Gamma(\bar{z})} = \frac{1}{\Gamma(z+1)} \cdot \frac{1}{\Gamma(\bar{z}+1)}$$

由于  $z+1 = \frac{1}{2} + i, \bar{z}+1 = \frac{1}{2} - i = 1 - (z+1)$ , 由余元公式

$$\lim_{n \rightarrow \infty} \prod_{m=1}^n \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right) = \frac{\sin(\pi(z+1))}{\pi} = \frac{\cos(\pi i)}{\pi} = \frac{\cosh(\pi)}{\pi}$$

□

**159.**  $\psi$  表示 Digamma 函数,  $\varphi = \frac{1+\sqrt{5}}{2}$ , 证明

$$\sum_{n=1}^{\infty} \frac{\psi(n+\varphi) - \psi\left(n - \frac{1}{\varphi}\right)}{n^2 + n - 1} = \frac{\pi^2}{2\sqrt{5}} + \frac{\pi^2 \tan^2\left(\frac{\sqrt{5}\pi}{2}\right)}{\sqrt{5}} + \frac{4\pi \tan\left(\frac{\sqrt{5}\pi}{2}\right)}{5}$$

**证明** 由于  $\psi(n+z) = \psi(z) + \sum_{k=0}^{n-1} \frac{1}{k+z}$ , 那么如果  $a$  和  $b$  不是非正整数, 以及任意  $N \geq 0$  有

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\psi(n+a) - \psi(n+b)}{(n+a)(n+b)} \\ &= (\psi(a) - \psi(b)) \sum_{n=0}^N \frac{1}{(n+a)(n+b)} \\ & \quad + \frac{b-a}{2} \left( \left( \sum_{n=0}^N \frac{1}{(n+a)(n+b)} \right)^2 - \sum_{n=0}^N \frac{1}{((n+a)(n+b))^2} \right) \end{aligned}$$

现在  $a = \varphi, b = -\frac{1}{\varphi}$ , 令  $N \rightarrow \infty$ , 我们得到

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\psi(n+\varphi) - \psi\left(n - \frac{1}{\varphi}\right)}{n^2 + n - 1} \\ &= \left( \psi(\varphi) - \psi\left(-\frac{1}{\varphi}\right) \right) \left( 1 + \sum_{n=0}^{\infty} \frac{1}{n^2 + n - 1} \right) \\ & \quad - \frac{\sqrt{5}}{2} \left( \left( \sum_{n=0}^N \frac{1}{n^2 + n - 1} \right)^2 - \sum_{n=0}^N \frac{1}{(n^2 + n - 1)^2} \right) \end{aligned}$$

(1) 利用余元公式可以得出

$$\psi(\varphi) - \psi\left(-\frac{1}{\varphi}\right) = \psi(\varphi) - \psi(1-\varphi) = \frac{\pi}{\tan(\pi\varphi)} = \pi \tan\left(\frac{\sqrt{5}\pi}{2}\right)$$

(2) 利用留数定理

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n^2 + n - 1} &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + n - 1} \\ &= \frac{\pi}{2} \sum \operatorname{Res} \left( \frac{\cot(\pi z)}{z^2 + z - 1}, z \right) \\ &= -\frac{\pi}{2} \left( \operatorname{Res} \left( \frac{\cot(\pi z)}{z^2 + z - 1}, -\varphi \right) + \operatorname{Res} \left( \frac{\cot(\pi z)}{z^2 + z - 1}, \frac{1}{\varphi} \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= -\frac{\pi \cot(\pi\varphi)}{\sqrt{5}} = \frac{\pi \tan\left(\frac{\sqrt{5}\pi}{2}\right)}{\sqrt{5}} \\
 \sum_{n=0}^{\infty} \frac{1}{(n^2 + n - 1)^2} &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + n - 1)^2} \\
 &= \frac{\pi}{2} \sum \operatorname{Res} \left( \frac{\cot(\pi z)}{(z^2 + z - 1)^2}, z \right) \\
 &= -\frac{\pi}{2} \left( \operatorname{Res} \left( \frac{\cot(\pi z)}{(z^2 + z - 1)^2}, -\varphi \right) + \operatorname{Res} \left( \frac{\cot(\pi z)}{(z^2 + z - 1)^2}, \frac{1}{\varphi} \right) \right) \\
 &= -\frac{\pi}{5} \left( -\frac{\pi}{\sin^2(\pi\varphi)} - \frac{2 \cot(\pi\varphi)}{\sqrt{5}} \right) \\
 &= \frac{\pi^2}{5} + \frac{\pi^2 \tan^2\left(\frac{\sqrt{5}\pi}{2}\right)}{5} - \frac{2\pi \tan\left(\frac{\sqrt{5}\pi}{2}\right)}{5\sqrt{5}}
 \end{aligned}$$

上述三个式子相加证得原式.

□

#### 160. 计算积分

$$\int_0^1 \left[ \frac{1 + \sqrt{1-x}}{x} + \frac{2}{\ln(1-x)} \right] dx$$

**解** 方法一 令  $\ln(1-x) = -t$ , 有

$$\begin{aligned}
 \int_0^1 \left[ \frac{1 + \sqrt{1-x}}{x} + \frac{2}{\ln(1-x)} \right] dx &= \int_0^{\infty} e^{-t} \left( \frac{1}{1 - e^{-\frac{t}{2}}} - \frac{2}{t} \right) dt \\
 &= \int_0^{\infty} \left[ -2 \left( \frac{e^{-t}}{t} - \frac{1}{e^t - 1} \right) - \frac{e^{-\frac{t}{2}}}{1 + e^{\frac{t}{2}}} \right] dt \\
 &= -2 \int_0^{\infty} \left( \frac{e^{-t}}{t} - \frac{1}{e^t - 1} \right) dt - \int_0^{\infty} \frac{e^{-\frac{t}{2}}}{1 + e^{\frac{t}{2}}} dt \\
 &= 2\gamma - \int_0^{\infty} \left( e^{-\frac{t}{2}} - \frac{e^{-\frac{t}{2}}}{1 + e^{-\frac{t}{2}}} \right) dt \\
 &= 2\gamma - 2 + 2 \ln 2
 \end{aligned}$$

方法二 令  $1-x \rightarrow x$ , 有

$$\begin{aligned}
 \int_0^1 \left[ \frac{1 + \sqrt{1-x}}{x} + \frac{2}{\ln(1-x)} \right] dx &= \int_0^1 \left( \frac{1 + \sqrt{x}}{1-x} - \frac{2}{\ln x} \right) dx \\
 &= \int_0^1 \int_0^1 \left( \frac{x^t \ln x}{1-x} + \frac{x^{t+\frac{1}{2}} \ln x}{1-x} + 2x^t \right) dx dt \\
 &= - \int_0^{\infty} \left[ \psi'(t+1) + \psi' \left( t + \frac{2}{3} \right) - \frac{2}{t+1} \right] dt \\
 &= -\psi(1) - \psi \left( \frac{3}{2} \right) = 2\gamma - 2 + 2 \ln 2
 \end{aligned}$$

□



### 161. 计算积分

$$\int_0^{\pi} \arctan\left(\frac{\ln(\sin x)}{x}\right) dx$$

**解** 考虑参数积分

$$I(s) = \int_0^{\pi} \arctan\left(\frac{\ln(s \sin x)}{x}\right) dx$$

那么在积分下求导有

$$\begin{aligned} I'(s) &= \frac{1}{s} \int_0^{\pi} \frac{x}{x^2 + \ln^2(s \sin x)} dx = -\frac{1}{s} \int_0^{\pi} \frac{1}{\ln\left(\frac{se^{ix}-s}{2i}\right)} dx \\ &= -\frac{1}{s} \Im \int_{|z|=1} \frac{1}{\ln\left(\frac{sz-s}{2i}\right)} \frac{dz}{2iz} = -\frac{1}{s} \Im \frac{\pi}{\ln\left(-\frac{s}{2i}\right)} \\ &= -\frac{1}{s} \Im \left( \frac{\pi}{\ln\left(\frac{s}{2}\right) + \frac{\pi i}{2}} \cdot \frac{\ln\left(\frac{s}{2}\right) - \frac{\pi i}{2}}{\ln\left(\frac{s}{2}\right) - \frac{\pi i}{2}} \right) \\ &= \frac{1}{2s} \frac{\pi^2}{\ln^2\left(\frac{s}{2}\right) + \frac{\pi^2}{4}} \end{aligned}$$

因此

$$\begin{aligned} I(1) &= I(\infty) + \frac{\pi^2}{2} \int_{s=\infty}^{s=1} \frac{1}{\ln^2\left(\frac{s}{2}\right) + \frac{\pi^2}{4}} d\left(\ln\left(\frac{s}{2}\right)\right) \\ &= \frac{\pi^2}{2} + \frac{\pi^2}{2} \cdot \frac{2}{\pi} \arctan\left(\frac{2 \ln\left(\frac{s}{2}\right)}{\pi}\right) \Big|_{\infty}^1 \\ &= -\pi \arctan\left(\frac{2 \ln 2}{\pi}\right) \end{aligned}$$

□

### 162. 证明

$$G = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1) \cdot 4^n} \left(1 - \frac{2}{4^n}\right)$$

其中  $G$  是 Catalan 常数, 对  $s > 1$ ,  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ ,  $\zeta(0) = -\frac{1}{2}$ .

**证明** 根据  $\cot x$  的幂级数展开得

$$\sum_{n=1}^{\infty} \zeta(2n) x^{2n} = \frac{1 - \pi x \cot \pi x}{2}$$

于是

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\zeta(2n) x^{2n}}{2n+1} &= \frac{1}{x} \int_0^x \frac{1 - \pi t \cot \pi t}{2} dt \\ &= \frac{1}{2} - \frac{1}{2\pi x} \int_0^{\pi x} u \cot u du \\ &= \frac{1}{2} - \frac{1}{2\pi x} \int_0^{\pi x} u d(\ln(\sin u)) \end{aligned}$$

$$= \frac{1}{2} - \frac{1}{2} \ln(\sin \pi x) + \frac{1}{2\pi x} \int_0^{\pi x} \ln(\sin u) du$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1) \cdot 4^n} &= \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n+1} \left(\frac{1}{2}\right)^{2n} \\ &= \frac{1}{2} - \frac{1}{2} \ln\left(\sin \frac{\pi}{2}\right) + \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \ln(\sin u) du \\ &= \frac{1}{2} - \frac{1}{2} \ln 2 \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1) \cdot 4^n} \frac{2}{4^n} &= 2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n+1} \left(\frac{1}{4}\right)^{2n} \\ &= 2 \left[ \frac{1}{2} - \frac{1}{2} \ln\left(\sin \frac{\pi}{4}\right) + \frac{2}{\pi} \int_0^{\frac{\pi}{4}} \ln(\sin u) du \right] \\ &= 1 + \frac{1}{2} \ln 2 + \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \ln(\sin u) du \\ &= 1 + \frac{1}{2} \ln 2 + \frac{4}{\pi} \left( -\frac{\pi}{4} \ln 2 - \frac{G}{2} \right) \\ &= 1 - \frac{1}{2} \ln 2 - \frac{2}{\pi} G \end{aligned}$$

最后得到

$$\begin{aligned} \frac{\pi}{2} \sum_{n=1}^{\infty} \zeta(2n) x^{2n} &= \frac{\pi}{2} \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1) 4^n} - \frac{2\zeta(2n)}{(2n+1) 4^{2n}} \right] \\ &= \frac{\pi}{2} \left( \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \ln 2 - 1 + \frac{1}{2} \ln 2 + \frac{2}{\pi} G \right) \\ &= G \end{aligned}$$

其中利用  $\ln(\sin u)$  的 Fourier 展开式有

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \ln(\sin u) du &= \int_0^{\frac{\pi}{4}} \left( -\ln 2 - \sum_{n=1}^{\infty} \frac{\cos 2nu}{n} \right) du \\ &= -\frac{\pi}{4} \ln 2 - \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{2n^2} \\ &= -\frac{\pi}{4} \ln 2 - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \\ &= -\frac{\pi}{4} \ln 2 - \frac{G}{2}. \end{aligned}$$

□

### 163. 计算积分

$$\int_0^{\infty} \left( \frac{\sin x}{x} \right)^3 \frac{1}{3 + \cos 2x} dx$$

解

$$\begin{aligned}
 & \int_0^{\infty} \left( \frac{\sin x}{x} \right)^3 \frac{1}{3 + \cos 2x} dx \\
 &= \sum_{n=0}^{\infty} \int_{\frac{n\pi}{2}}^{\frac{n+1}{2}\pi} \left( \frac{\sin x}{x} \right)^3 \frac{1}{3 + \cos 2x} dx \\
 &= \sum_{n=0}^{\infty} \int_{n\pi}^{\frac{2n+1}{2}\pi} \left( \frac{\sin x}{x} \right)^3 \frac{1}{3 + \cos 2x} dx + \sum_{n=1}^{\infty} \int_{\frac{2n-1}{2}\pi}^{n\pi} \left( \frac{\sin x}{x} \right)^3 \frac{1}{3 + \cos 2x} dx \\
 &= \sum_{n=0}^{\infty} \int_0^{\frac{\pi}{2}} \left( \frac{\sin(x+n\pi)}{x+n\pi} \right)^3 \frac{1}{3 + \cos 2(x+n\pi)} dx \\
 &\quad + \sum_{n=1}^{\infty} \int_{-\frac{\pi}{2}}^0 \left( \frac{\sin(x+n\pi)}{x+n\pi} \right)^3 \frac{1}{3 + \cos 2(x+n\pi)} dx \\
 &= \sum_{n=0}^{\infty} \int_0^{\frac{\pi}{2}} \left( \frac{\sin x}{x+n\pi} \right)^3 \frac{(-1)^n}{3 + \cos 2x} dx + \sum_{n=1}^{\infty} \int_0^{\frac{\pi}{2}} \left( \frac{\sin x}{x-n\pi} \right)^3 \frac{(-1)^n}{3 + \cos 2x} dx \\
 &= \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{3 + \cos 2x} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(x+n\pi)^3} dx
 \end{aligned}$$

由已知等式  $\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x+n\pi}$  得

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(x+n\pi)^3} = \frac{(-1)^2}{2!} \frac{d^2}{dx^2} \left( \frac{1}{\sin x} \right) = \frac{1 + \cos^2 x}{2 \sin^3 x}$$

因此

$$\int_0^{\infty} \left( \frac{\sin x}{x} \right)^3 \frac{1}{3 + \cos 2x} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1 + \cos^2 x}{3 + \cos 2x} dx = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{3 + \cos 2x}{3 + \cos 2x} dx = \frac{\pi}{8}.$$

□

#### 164. 计算积分

$$\int_0^{\pi} \frac{x^2 \sin^2 x}{(1 + |\cos x|)^2} dx$$

解

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \frac{x^2 \sin^2 x}{(1 + |\cos x|)^2} dx &= \int_0^{\frac{\pi}{2}} \frac{4x^2 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}}{4 \cos^4 \frac{x}{2}} dx \\
 &= \int_0^{\frac{\pi}{2}} x^2 \left( \sec^2 \frac{x}{2} - 1 \right) dx = 2 \int_0^{\frac{\pi}{2}} x^2 d \left( \tan \frac{x}{2} \right) - \frac{\pi^3}{24} \\
 &= \frac{\pi^2}{2} - \frac{\pi^3}{24} - 4 \int_0^{\frac{\pi}{2}} x \tan \frac{x}{2} dx \\
 &= \frac{\pi^2}{2} - \frac{\pi^3}{24} + 8 \int_0^{\frac{\pi}{2}} x d \left( \ln \left( \cos \frac{x}{2} \right) \right) \\
 &= \frac{\pi^2}{2} - \frac{\pi^3}{24} - 2\pi \ln 2 - 8 \int_0^{\frac{\pi}{2}} \ln \left( \cos \frac{x}{2} \right) dx
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\pi^2}{2} - \frac{\pi^3}{24} - 2\pi \ln 2 - 8 \int_0^{\frac{\pi}{2}} \left( -\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n} \right) dx \\
 &= \frac{\pi^2}{2} - \frac{\pi^3}{24} + 2\pi \ln 2 + 8 \sum_{n=1}^{\infty} \frac{(-1)^n \sin \frac{n\pi}{2}}{n^2} \\
 &= \frac{\pi^2}{2} - \frac{\pi^3}{24} + 2\pi \ln 2 - 8G
 \end{aligned}$$

$$\begin{aligned}
 \int_{\frac{\pi}{2}}^{\pi} \frac{x^2 \sin^2 x}{(1 + |\cos x|)^2} dx &= \int_{\frac{\pi}{2}}^{\pi} \frac{4x^2 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}}{4 \sin^4 \frac{x}{2}} dx \\
 &= \int_{\frac{\pi}{2}}^{\pi} x^2 \left( \csc^2 \frac{x}{2} - 1 \right) dx = -2 \int_{\frac{\pi}{2}}^{\pi} x^2 d \left( \cot \frac{x}{2} \right) - \frac{\pi^3}{24} \\
 &= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 4 \int_{\frac{\pi}{2}}^{\pi} x \cot \frac{x}{2} dx \\
 &= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 8 \int_{\frac{\pi}{2}}^{\pi} x d \left( \ln \left( \sin \frac{x}{2} \right) \right) \\
 &= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 2\pi \ln 2 - 8 \int_{\frac{\pi}{2}}^{\pi} \ln \left( \sin \frac{x}{2} \right) dx \\
 &= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 2\pi \ln 2 - 8 \int_{\frac{\pi}{2}}^{\pi} \left( -\ln 2 - \sum_{n=1}^{\infty} \frac{\cos nx}{n} \right) dx \\
 &= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 6\pi \ln 2 - 8 \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2} \\
 &= \frac{\pi^2}{2} - \frac{7\pi^3}{24} + 6\pi \ln 2 - 8G
 \end{aligned}$$

因此

$$\int_0^{\pi} \frac{x^2 \sin^2 x}{(1 + |\cos x|)^2} dx = \pi^2 - \frac{\pi^3}{3} + 8\pi \ln 2 - 16G$$

□

165. 设  $a > 0$ , 计算积分

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+1)!}{(n+2)! n! 4^{2n+3}}$$

解

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+1)!}{(n+2)! n! 4^{2n+3}} &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^{2n+3}} \frac{(2n+2)!}{[(n+1)!]^2 (n+2)} \\
 &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{4^{2n+1}} \frac{(2n)!}{[(n)!]^2 (n+1)} \\
 &= -2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^{2n+2}} \frac{(2n)!}{[(n)!]^2 (n+1)} \\
 &= -2 \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\left(-\frac{1}{16}\right)^{n+1}}{(n+1)} - \frac{1}{8}
 \end{aligned}$$

令  $y^2 - y + x = 0$ , 应用 Lagrange 反演公式得

$$y = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{n+1}}{n+1}$$

再令  $x = -\frac{1}{16}$  得  $y = \frac{2-\sqrt{5}}{4}$ , 故

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+1)!}{(n+2)! n! 4^{2n+3}} = -2 \cdot \frac{2-\sqrt{5}}{4} - \frac{1}{8} = \frac{4\sqrt{5}-9}{8}$$

□

#### 166. 求和

$$\int_0^{\frac{\pi}{2}} \frac{\sin ax}{\sin x} dx$$

解

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin ax}{\sin x} dx &= \int_0^{\frac{\pi}{2}} \frac{e^{i(a-1)x} - e^{-i(a-1)x}}{1 - e^{-2ix}} dx = \frac{i}{2} \int_{-1}^1 \frac{(-t)^{-\frac{a-1}{2}-1} - (-t)^{\frac{a+1}{2}-1}}{1+t} dt \\ &= \frac{i}{2} \left[ \int_{-1}^0 \frac{(-t)^{-\frac{a-1}{2}-1} - (-t)^{\frac{a+1}{2}-1}}{1+t} dt + \int_0^1 \frac{(-t)^{-\frac{a-1}{2}-1} - (-t)^{\frac{a+1}{2}-1}}{1+t} dt \right] \\ &= \frac{i}{2} \int_0^1 \frac{s^{-\frac{a-1}{2}-1} - s^{\frac{a+1}{2}-1}}{1-s} ds + \frac{1}{2} \int_0^1 \frac{e^{i\frac{a\pi}{2}} t^{-\frac{a-1}{2}-1} - e^{-i\frac{a\pi}{2}} t^{\frac{a+1}{2}-1}}{1+t} dt \\ &= \frac{i}{2} \left[ \psi\left(\frac{1+a}{2}\right) - \psi\left(\frac{1-a}{2}\right) \right] - \frac{1}{2} \left[ e^{i\frac{a\pi}{2}} \beta\left(\frac{1+a}{2}\right) - e^{-i\frac{a\pi}{2}} \beta\left(\frac{1-a}{2}\right) \right] \\ &= \frac{i\pi}{2} \tan\left(\frac{a\pi}{2}\right) - \frac{1}{2} \left[ \left(e^{i\frac{a\pi}{2}} + e^{-i\frac{a\pi}{2}}\right) \beta\left(\frac{1+a}{2}\right) - e^{-i\frac{a\pi}{2}} \pi \sec\left(\frac{a\pi}{2}\right) \right] \\ &= \frac{\pi}{2} - \cos\left(\frac{a\pi}{2}\right) \beta\left(\frac{1+a}{2}\right) \end{aligned}$$

□

#### 167. 计算积分

$$\int_0^1 \left( \frac{\arctan x}{x} \right)^3 dx$$

解

$$\begin{aligned} \int_0^1 \left( \frac{\arctan x}{x} \right)^3 dx &= \int_0^{\frac{\pi}{4}} \frac{t^3 d(\sin t)}{\sin^3 t} = -\frac{1}{2} \int_0^{\frac{\pi}{4}} t^3 d\left(\frac{1}{\sin^2 t}\right) \\ &= -\frac{1}{2} \left( \frac{2\pi^3}{4^3} - 3 \int_0^{\frac{\pi}{4}} \frac{t^2}{\sin^2 t} dt \right) = -\frac{\pi^3}{64} - \frac{3}{2} \int_0^{\frac{\pi}{4}} t^2 d(\cot t) \\ &= -\frac{\pi^3}{64} - \frac{3}{2} \left( \frac{\pi^2}{16} - 2 \int_0^{\frac{\pi}{4}} t \cot t dt \right) \\ &= -\frac{\pi^3}{64} - \frac{3\pi^2}{32} + 3 \left( -\frac{\pi}{8} \ln 2 - \int_0^{\frac{\pi}{4}} \ln(\sin t) dt \right) \end{aligned}$$

$$\begin{aligned}
 &= -\frac{\pi^3}{64} - \frac{3\pi^2}{32} - \frac{3\pi}{8} \ln 2 - 3 \int_0^{\frac{\pi}{4}} \ln(\sin t) dt \\
 &= -\frac{\pi^3}{64} - \frac{3\pi^2}{32} + \frac{3\pi}{8} \ln 2 + \frac{3}{2} G
 \end{aligned}$$

□

#### 168. 求和

**解** 首先有

$$\begin{aligned}
 \frac{16^n}{(2n+1)^2 n^2 \binom{2n}{n}^2} &= \frac{16^n}{(2n+1)^2 n^2} \left[ \frac{(n!)^2}{(2n)!} \right]^2 \\
 &= \frac{16^n}{(2n+1)^2 n^2} \left[ \frac{n!}{(2n-1)!! \cdot 2^n} \right]^2 \\
 &= \frac{2}{n(2n+1)} \cdot \frac{(2n)!!}{(2n+1)!!} \cdot \frac{(2n-2)!!}{(2n-1)!!} \\
 &= \frac{2}{n(2n+1)} \int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx \int_0^{\frac{\pi}{2}} \sin^{2n-1} y dy
 \end{aligned}$$

记

$$I(t) = \sum_{n=1}^{\infty} \frac{t^{2n+1}}{n(2n+1)} \int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx \int_0^{\frac{\pi}{2}} \sin^{2n-1} y dy$$

则

$$\begin{aligned}
 I'(t) &= \sum_{n=1}^{\infty} \frac{t^{2n}}{n} \int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx \int_0^{\frac{\pi}{2}} \sin^{2n-1} y dy \\
 &= - \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin y} \ln(1 - t^2 \sin^2 x \sin^2 y) dx dy
 \end{aligned}$$

于是

$$S = -2 \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin y} \ln(1 - t^2 \sin^2 x \sin^2 y) dy dx dt$$

考虑

$$f(u) = \int_0^{\frac{\pi}{2}} \frac{1}{\sin y} \ln(1 - u \sin^2 y) dy$$

则

$$f'(u) = - \int_0^{\frac{\pi}{2}} \frac{\sin y}{1 - u \sin^2 y} dy = - \frac{1}{\sqrt{u - u^2}} \arctan \sqrt{\frac{u}{1-u}}$$

于是

$$\begin{aligned}
 S &= 2 \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{t^2 \sin^2 x} \frac{\sin x}{\sqrt{u - u^2}} \arctan \sqrt{\frac{u}{1-u}} du dx dt \\
 &= 2 \int_0^1 \int_0^{\frac{\pi}{2}} \sin x \arctan^2 \left( \sqrt{\frac{u}{1-u}} \right) \Big|_{u=0}^{t^2 \sin^2 x} dx dt
 \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^1 \int_0^{\frac{\pi}{2}} \sin x \arctan^2 \left( \frac{t \sin x}{\sqrt{1-t^2 \sin^2 x}} \right) dx dt \\
&= 2 \int_0^{\frac{\pi}{2}} \int_0^x z^2 \cos z dz dx \quad \left( t = \frac{\sin z}{\sin x} \right) \\
&= 2 \int_0^{\frac{\pi}{2}} (2x \cos x + x^2 \sin x - 2 \sin x) dx \\
&= 4\pi - 12
\end{aligned}$$

□

## 169. 计算积分

$$I = \int_0^1 \exp \left( 4 \sqrt{\frac{t-t^2}{1+8t}} \right) \cdot \sqrt{\frac{1-8t+16t^2}{t+7t^2-8t^3}} dt$$

解 设  $t = x^2$ , 那么有

$$\begin{aligned}
I &= 2 \int_0^1 \exp \left( 4 \sqrt{\frac{t-t^2}{1+8t}} \right) \cdot \sqrt{\frac{1-8t+16t^2}{t+7t^2-8t^3}} dt \\
&= \int_0^1 e^{4xy} \frac{|1-4x^2|}{\sqrt{(1-x^2)(1+8x^2)}} dx \\
&= \int_0^{\frac{1}{2}} e^{4xy} \frac{1-4x^2}{\sqrt{(1-x^2)(1+8x^2)}} dx + \int_{\frac{1}{2}}^1 e^{4xy} \frac{4x^2-1}{\sqrt{(1-x^2)(1+8x^2)}} dx \\
&= \int_0^{\frac{1}{2}} e^{4xy} \frac{1-4x^2}{\sqrt{(1-x^2)(1+8x^2)}} dx + \int_0^{\frac{1}{2}} e^{4xy} \frac{3(1-4y^2)}{(1+8y^2)\sqrt{(1-y^2)(1+8y^2)}} dy \\
&= \int_0^{\frac{1}{2}} e^{4xy} \frac{1-4x^2}{\sqrt{(1-x^2)(1+8x^2)}} dx + \int_0^{\frac{1}{2}} e^{4xy} \frac{3(1-4x^2)}{(1+8x^2)\sqrt{(1-x^2)(1+8x^2)}} dx \\
&= \int_0^{\frac{1}{2}} e^{4xy} \frac{4(1-4x^2)(1+2x^2)}{(1+8x^2)\sqrt{(1-x^2)(1+8x^2)}} dx \\
&= \int_0^{\frac{1}{2}} e^{4xy} (4xy)'_x dx = \left[ \exp \left( 4x \sqrt{\frac{1-x^2}{1+8x^2}} \right) \right] \Big|_0^{\frac{1}{2}} = 1
\end{aligned}$$

□

计算积分

$$\int_0^\infty \frac{x^2}{(1+x^2)^4} \ln^2 \left( \frac{2x}{1+x^2} \right) dx$$

解

$$\begin{aligned}
\int_0^\infty \frac{x^2}{(1+x^2)^4} \ln^2 \left( \frac{2x}{1+x^2} \right) dx &= \int_0^{\frac{\pi}{2}} \sin^2 t \cos^4 t \ln^2 (\sin 2t) dt \quad (x = \tan t) \\
&= \int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t \ln^2 (\sin 2t) dt
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin^2 t + \cos^2 t) \sin^2 t \cos^2 t \ln^2 (\sin 2t) dt \\
 &= \frac{1}{8} \int_0^{\frac{\pi}{2}} \sin^2 2t \ln^2 (\sin 2t) dt = \frac{1}{8} \int_0^{\frac{\pi}{2}} \sin^2 t \ln^2 (\sin t) dt \\
 &= \frac{1}{16} \int_0^{\frac{\pi}{2}} \ln^2 (\sin t) dt - \frac{1}{6} \int_0^{\frac{\pi}{2}} \cos 2t \ln^2 (\sin t) dt \\
 &= \frac{1}{16} \left( \frac{\pi}{2} \ln^2 2 + \frac{\pi^3}{24} \right) - \frac{1}{32} \int_0^{\frac{\pi}{2}} \ln^2 (\sin t) d(\sin 2t) \\
 &= \frac{\pi}{32} \ln^2 2 + \frac{\pi^3}{384} + \frac{1}{32} \int_0^{\frac{\pi}{2}} \sin 2t \frac{2 \ln (\sin t)}{\sin t} \cos t dt \\
 &= \frac{\pi}{32} \ln^2 2 + \frac{\pi^3}{384} + \frac{1}{8} \int_0^{\frac{\pi}{2}} \cos^2 t \ln (\sin t) dt \\
 &= \frac{\pi}{32} \ln^2 2 + \frac{\pi^3}{384} - \frac{\pi}{32} \ln 2 - \frac{\pi}{64}
 \end{aligned}$$

□

### 170. 计算积分

$$I = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{du dv dw}{1 - \cos u \cos v \cos w}$$

**解** 首先作换元

$$\tan \frac{u}{2} = x, \quad \tan \frac{v}{2} = y, \quad \tan \frac{w}{2} = z$$

再转化成极坐标

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

然后令  $\varphi = 2\phi$ , 于是按照上面的换元我们得到

$$\begin{aligned}
 I &= \frac{8}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty \frac{dx dy dz}{(1+x^2)(1+y^2)(1+z^2) - (1-x^2)(1-y^2)(1-z^2)} \\
 &= \frac{4}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty \frac{dx dy dz}{x^2 + y^2 + z^2 + x^2 y^2 z^2} \\
 &= \frac{4}{\pi^3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^\infty \frac{\sin \theta dr d\theta d\phi}{1 + r^2 \sin^4 \theta \cos^2 \theta \sin^2 \phi \cos^2 \phi} \\
 &= \frac{4}{\pi^3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^\infty \frac{\sin \theta dr d\theta d\varphi}{1 + \frac{1}{4} r^2 \sin^4 \theta \cos^2 \theta \sin^2 \varphi}
 \end{aligned}$$

令  $t = r \sin \theta \frac{1}{2} \cos \theta \sin \varphi$ , 则

$$\begin{aligned}
 I &= \frac{4}{\pi^3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^\infty \frac{dr d\theta d\varphi}{(1+t^4) \sqrt{\frac{1}{2} \cos \theta \sin \varphi}} \\
 &= \frac{4\sqrt{2}}{\pi^3} \int_0^\infty \frac{dt}{1+t^4} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos \theta}} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{\sin \varphi}} \\
 &= \frac{4\sqrt{2}}{\pi^3} \cdot \frac{\pi}{2\sqrt{2}} \cdot \frac{\Gamma(\frac{1}{4}) \Gamma(\frac{1}{2})}{2\Gamma(\frac{3}{4})} \cdot \frac{\Gamma(\frac{1}{4}) \Gamma(\frac{1}{2})}{2\Gamma(\frac{3}{4})} \\
 &= \frac{\Gamma^4(\frac{1}{4})}{4\pi^3}
 \end{aligned}$$

□



171. 计算积分

$$\int_0^1 \frac{\ln^2(1-x) \ln x}{x} dx$$

解

$$\begin{aligned} \int_0^1 \frac{\ln^2(1-x) \ln x}{x} dx &= - \int_0^1 \ln(1-x) \ln(x) d(\text{Li}_2(x)) \\ &= \int_0^1 \text{Li}_2(x) \frac{\ln(1-x)}{x} dx - \int_0^1 \text{Li}_2(x) \frac{\ln x}{1-x} dx \\ &= -\frac{1}{2} \text{Li}_2^2(1) - \int_0^1 \frac{\ln x}{1-x} \sum_{n=1}^{\infty} \frac{x^n}{n^2} dx \\ &= -\frac{1}{2} \text{Li}_2^2(1) + \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \\ &= -\frac{\pi^4}{72} + \frac{\pi^4}{120} = -\frac{\pi^4}{180} \end{aligned}$$

□

172. 计算积分

$$\int_0^{\infty} \int_0^{\infty} \frac{\sin x \sin y \sin(x+y)}{xy(x+y)} dx dy$$

解 考虑参数积分

$$I(t) = \int_0^{\infty} \int_0^{\infty} \frac{\sin x \sin y \sin t(x+y)}{xy(x+y)} dx dy, \quad 0 < t < 1$$

则

$$\begin{aligned} I'(t) &= \int_0^{\infty} \int_0^{\infty} \frac{\sin x \sin y \cos t(x+y)}{xy} dx dy \\ &= \int_0^{\infty} \int_0^{\infty} \frac{\sin x \sin y [\cos(tx) \cos(ty) - \sin(tx) \sin(ty)]}{xy} dx dy \end{aligned}$$

其中

$$\begin{aligned} \int_0^{\infty} \int_0^{\infty} \frac{\sin x \sin y \cos(tx) \cos(ty)}{xy} dx dy &= \int_0^{\infty} \frac{\sin x \cos(tx)}{x} dx \int_0^{\infty} \frac{\sin y \cos(ty)}{y} dy \\ &= \left( \int_0^{\infty} \frac{\sin x \cos(tx)}{x} dx \right)^2 \\ &= \left( \frac{1}{2} \int_0^{\infty} \frac{\sin(1+t)x + \sin(1-t)x}{x} dx \right)^2 \\ &= \left( \frac{1}{2} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) \right)^2 = \frac{\pi^2}{4} \quad (\text{Dirichlet Integral}) \end{aligned}$$

$$\begin{aligned} \int_0^{\infty} \int_0^{\infty} \frac{\sin x \sin y \sin(tx) \sin(ty)}{xy} dx dy &= \int_0^{\infty} \frac{\sin x \sin(tx)}{x} dx \int_0^{\infty} \frac{\sin y \sin(ty)}{y} dy \\ &= \left( \int_0^{\infty} \frac{\sin x \sin(tx)}{x} dx \right)^2 \end{aligned}$$

$$\begin{aligned}
 &= \left( \frac{1}{2} \int_0^\infty \frac{\cos(1-t)x - \cos(1+t)x}{x} dx \right)^2 \\
 &= \frac{1}{4} \ln^2 \left( \frac{1-t}{1+t} \right) \quad (\text{Frullani Integral})
 \end{aligned}$$

于是

$$\begin{aligned}
 I &= I(0) + \int_0^1 I'(t) dt = \frac{\pi^2}{4} - \frac{1}{4} \int_0^1 \ln^2 \left( \frac{1-t}{1+t} \right) dt \\
 &= \frac{\pi^2}{4} - 2 \int_0^1 \frac{\ln^2 u}{(1+u)^2} du = \frac{\pi^2}{6}
 \end{aligned}$$

□

### 173. 计算积分

$$\int_0^1 \frac{\arccos(\sqrt{x}) \operatorname{Li}_2(x)}{x} dx$$

解

$$\begin{aligned}
 \int_0^1 \frac{\arccos(\sqrt{x}) \operatorname{Li}_2(x)}{x} dx &= 2 \int_0^1 \frac{\arccos(x) \operatorname{Li}_2(x^2)}{x} dx \\
 &= 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 x^{2n-1} \arccos(x) dx \\
 &= \sum_{n=1}^{\infty} \frac{1}{n^3} \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}} dx = \sum_{n=1}^{\infty} \frac{1}{n^3} \int_0^{\frac{\pi}{2}} \sin^{2n} x dx \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\frac{\pi}{2}} \sin^{2n} x dx \int_0^\infty e^{-ny} dy \int_0^\infty e^{-nz} dz \\
 &= - \int_0^\infty \int_0^\infty \int_0^{\frac{\pi}{2}} \ln(1 - e^{-(y+z)} \sin^2 x) dx dy dz \\
 &= \pi \int_0^\infty \int_0^\infty \ln \left( \frac{2}{1 + \sqrt{1 - e^{-(y+z)}}} \right) dy dz \\
 &= -\pi \int_0^1 \int_0^1 \frac{1}{uv} \ln \left( \frac{1 + \sqrt{1 - uv}}{2} \right) du dv \\
 &= -\frac{\pi}{2} \int_0^1 \frac{dv}{v} \int_0^1 \frac{\ln u}{\sqrt{1 - uv} + 1 - uv} du \\
 &= -\pi \int_0^1 \frac{\ln v}{v} \left( \ln 2 - \ln(1 + \sqrt{1 - v}) \right) dv \\
 &= -\frac{\pi}{2} \int_0^1 \left( \ln 2 - \ln(1 + \sqrt{1 - v}) \right) d(\ln^2 v) \\
 &= \frac{\pi}{4} \int_0^1 \frac{\ln^2 v}{\sqrt{1 - v} + 1 - v} dv \\
 &= \frac{\pi}{2} \int_0^1 \frac{\ln^2(1 - t^2)}{1 + t} dt = \frac{\pi}{2} (I_1 + 2I_2 + I_3)
 \end{aligned}$$

其中

$$I_1 = \int_0^1 \frac{\ln^2(1+t)}{1+t} dt = \frac{1}{3} \ln^3 2$$

$$I_2 = \int_0^1 \frac{\ln(1+t)\ln(1-t)}{1+t} dt = \frac{1}{3} \ln^3 2 - \frac{\pi^2}{12} \ln 2 + \frac{1}{8} \zeta(3)$$

$$I_3 = \int_0^1 \frac{\ln^2(1-t)}{1+t} dt = \frac{1}{3} \ln^3 2 - \frac{\pi^2}{6} \ln 2 + \frac{7}{4} \zeta(3)$$

因此原积分

$$I = \frac{\pi}{2} (I_1 + 2I_2 + I_3) = \frac{2\pi}{3} \ln^3 2 - \frac{\pi^3}{6} \ln 2 + \pi \zeta(3)$$

□

#### 174. 计算积分

$$\int_0^{\frac{1}{2}} \frac{x \ln \left( \frac{\ln 2 - \ln(1+2x)}{\ln 2 - \ln(1-x)} \right)}{3+4x^2} dx$$

**解** 首先有

$$\begin{aligned} & \int_0^{\frac{1}{2}} \frac{x \ln \left( \frac{\ln 2 - \ln(1+2x)}{\ln 2 - \ln(1-x)} \right)}{3+4x^2} dx = \frac{1}{4} \int_0^1 \frac{x \ln \left( \frac{\ln 2 - \ln(1+x)}{\ln 2 - \ln(1-x)} \right)}{3+x^2} dx \\ &= \frac{1}{4} \int_0^1 \frac{x}{3+x^2} \ln \left( \frac{\ln \frac{1+x}{2}}{\ln \frac{1-x}{2}} \right) dx = \frac{1}{4} \int_{-1}^0 \frac{x}{3+x^2} \ln \left( \frac{\ln \frac{1+x}{2}}{\ln \frac{1-x}{2}} \right) dx \\ &= \frac{1}{8} \int_{-1}^1 \frac{x}{3+x^2} \ln \left( \frac{\ln \frac{1+x}{2}}{\ln \frac{1-x}{2}} \right) dx = \frac{1}{4} \left[ \int_{-1}^1 \frac{x}{3+x^2} \ln \left( \left| \ln \frac{1+x}{2} \right| \right) dx \right] \\ &= \frac{1}{2} \Re \left[ \int_0^1 \frac{2t-1}{3+(2t-1)^2} \ln(\ln t) dx \right] = \frac{1}{8} \int_0^1 \frac{(2t-1) \ln(-\ln t)}{t^2-t+1} dt \\ &= \frac{1}{8} \int_0^1 \ln(-\ln t) d(\ln(t^2-t+1)) \quad (x=2t-1) \\ &= -\frac{1}{8} \int_0^1 \frac{\ln(t^2-t+1)}{t \ln t} dt = \frac{1}{8} \int_0^\infty \frac{\ln(e^{-2s}-e^{-s}+1)}{s} ds \quad (t=e^{-s}) \\ &= \frac{1}{8} \int_0^\infty \frac{\ln(1+e^{-3s}) - \ln(1+e^{-s})}{s} ds \end{aligned}$$

考虑参数积分  $I(a, b) = \int_0^\infty \frac{\ln(1+e^{-as}) - \ln(1+e^{-bs})}{s} ds$ , 则  $I(b, b) = 0$ ,

$$I'_a(a, b) = -\int_0^\infty \frac{e^{-as}}{1+e^{-as}} ds = -\frac{1}{a} \ln 2$$

于是

$$I(a, b) = -\ln 2 \int_b^a \frac{1}{u} du = -\ln 2 \ln \frac{a}{b}$$

原积分  $I = \frac{1}{8} I(3, 1) = -\frac{1}{8} \ln 2 \ln 3$ .

□

#### 175. 计算积分

$$\int_0^1 (1 + \ln x) \arctan(x) \ln \left( \ln \frac{1}{x} \right) dx$$

**解** 首先有

$$\begin{aligned}\int_0^1 (1 + \ln x) \arctan(x) \ln\left(\frac{1}{x}\right) dx &= \int_0^\infty (1 - t) \arctan(e^{-t}) \ln(t) e^{-t} dt \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \int_0^\infty (1 - t) e^{-2nt} \ln t dt\end{aligned}$$

考虑积分

$$\begin{aligned}I(a) &= \int_0^\infty t^a e^{-2nt} dt = \frac{\Gamma(a+1)}{(2n)^{a+1}}, \\ I'(a) &= \frac{\Gamma'(a+1)}{(2n)^{a+1}} - \frac{\Gamma(a+1)}{(2n)^{a+1}} \ln(2n).\end{aligned}$$

于是

$$\begin{aligned}& \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \int_0^\infty (1 - t) e^{-2nt} \ln t dt \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} (I'(0) - I'(1)) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \left( -\gamma - \frac{\ln(2n)}{2n} - \frac{1-\gamma}{(2n)^2} + \frac{\ln(2n)}{(2n)^2} \right) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \left( -\gamma \frac{2n-1}{(2n)^2} - \frac{(2n-1)\ln(2n)}{(2n)^2} - \frac{1}{(2n)^2} \right) \\ &= \gamma \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n \ln(2n)}{(2n)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n)^2} \\ &= -\frac{\pi^2}{48} \gamma + \frac{1}{8} \zeta'(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \\ &= -\frac{\pi^2}{48} \gamma + \frac{1}{8} \zeta'(2) + \frac{\pi^2}{48} + \frac{1}{2} \ln 2 - \frac{\pi}{4}\end{aligned}$$

□

## 176. 计算积分

$$\int_0^\infty x \sin x \ln(1 - e^{-x}) dx$$

**解**

$$\begin{aligned}\int_0^\infty x \sin x \ln(1 - e^{-x}) dx &= -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^\infty x e^{-nx} \sin x dx \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} \Im \int_0^\infty x e^{-(n-i)x} dx \\ &= -\Im \sum_{n=1}^{\infty} \frac{1}{n(n-i)^2} = -\Im \sum_{n=1}^{\infty} \frac{(n+i)^2}{n(n^2+1)^2} \\ &= -\sum_{n=1}^{\infty} \frac{2}{(n^2+1)^2} = 1 - \frac{\pi}{2 \tanh(\pi)} - \frac{\pi^2}{2 \sinh^2(\pi)}\end{aligned}$$

□

177. 设  $f(x) : (1, +\infty) \rightarrow \mathbb{R}$ , 且是连续可导的函数, 满足

$$f(x) \leq x^2 \ln x, \quad f'(x) > 0, x \in (1, +\infty).$$

证明: 积分  $\int_1^{+\infty} \frac{1}{f'(x)} dx$  发散.

**证明** 如果  $f'(x)$  有界, 结论显然成立, 不妨设  $f'(x)$  无界, 这时  $f(x)$  单调趋于  $+\infty$ . 对  $\forall A > 0$ , 由 Cauchy 不等式得

$$\left( \int_{e^{A/2}}^{e^A} \frac{dx}{f'(x)} \right) \left( \int_{e^{A/2}}^{e^A} \frac{f'(x)}{x^2 \ln^2 x} dx \right) \geq \left( \int_{e^{A/2}}^{e^A} \frac{dx}{x \ln x} \right)^2 = \ln^2 2$$

由  $f(x) \leq x^2 \ln x$  得  $f(e^x) \leq xe^{2x}$ , 因此

$$\begin{aligned} \int_{e^{A/2}}^{e^A} \frac{f'(x)}{x^2 \ln^2 x} dx &= \int_{\frac{A}{2}}^A \frac{f'(e^t) e^t}{t^2 e^{2t}} dt = \int_{\frac{A}{2}}^A \frac{d[f(e^t)]}{t^2 e^{2t}} \\ &= \frac{f(e^t)}{t^2 e^{2t}} \Big|_{\frac{A}{2}}^A + \int_{\frac{A}{2}}^A \frac{2t^2 e^{-2t} + 2t e^{-2t}}{t^4} f(e^t) dt \\ &\leq \frac{f(e^A)}{A^2 e^{2A}} + \int_{\frac{A}{2}}^A \frac{2t^2 e^{-2t} + 2t e^{-2t}}{t^4} t e^{2t} dt \\ &\leq \frac{1}{A} + 2 \left( \ln 2 + \frac{1}{A} \right) = 2 \ln 2 + \frac{3}{A}. \end{aligned}$$

取  $A$  充分大, 则  $\int_{e^{A/2}}^{e^A} \frac{f'(x)}{x^2 \ln^2 x} dx \leq 2$ , 因此

$$\int_{e^{A/2}}^{e^A} \frac{dx}{f'(x)} \geq \frac{\ln^2 2}{2}$$

对任意充分大的  $A$  都成立, 于是积分  $\int_1^{+\infty} \frac{1}{f'(x)} dx$  发散.

□

178. 计算积分

$$\int_0^1 \sinh(\ln \sqrt{x}) \frac{\ln x \ln(1+x)}{x} dx$$

**解**

$$\begin{aligned} &\int_0^1 \sinh(\ln \sqrt{x}) \frac{\ln x \ln(1+x)}{x} dx \\ &= 4 \int_0^\infty x \sinh(x) \ln(1+e^{-2x}) dx \quad (x \rightarrow e^{-2x}) \\ &= 2 \int_0^\infty x (e^x - e^{-x}) \ln(1+e^{-2x}) dx \\ &= -2 \int_0^\infty \left( \frac{1}{x} - x \right) \ln(x) \ln(1+x^2) \frac{dx}{x} \quad (x \rightarrow -\ln x) \\ &= -2 \int_0^\infty (1-x^2) \ln(x) \ln(1+x^2) \frac{dx}{x^2} \end{aligned}$$

$$\begin{aligned}
 &= 2 \int_0^1 \ln(x) \ln(1+x^2) dx - 2 \int_0^1 \ln(x) \ln(1+x^2) \frac{dx}{x^2} \\
 &= 2 \int_0^1 \ln(x) \ln(1+x^2) d\left(\frac{1}{x}\right) - 2 \int_0^1 \ln(1+x^2) dx - 4 \int_0^1 \ln(x) \frac{x^2}{1+x^2} dx \\
 &= -2 \int_0^1 \ln(1+x^2) \frac{dx}{x^2} - 4 \int_0^1 \frac{\ln(x)}{1+x^2} dx - 2 \ln(2) + 4 \int_0^1 \frac{x^2}{1+x^2} dx - 4 \int_0^1 \ln(x) \frac{x^2}{1+x^2} dx \\
 &= 2 \int_0^1 \ln(1+x^2) d\left(\frac{1}{x}\right) - 4 \int_0^1 \ln(x) dx - 2 \ln(2) + 4 - \pi \\
 &= 2 \ln(2) - 4 \int_0^1 \frac{dx}{1+x^2} + 4 - 2 \ln(2) + 4 - \pi = 8 - 2\pi
 \end{aligned}$$

□

### 179. 计算积分

$$I = \int_0^1 \frac{\ln(1+x)}{(1+x^2)(1+x^3)} dx$$

解

$$\begin{aligned}
 I &= \int_0^1 \frac{\ln(1+x)}{(1+x^2)(1+x^3)} dx \\
 &= \int_0^1 \frac{x^2 [\ln(1+x) - \ln x]}{(1+x^2)(1+x^3)} dx \\
 &= \frac{1}{2} \left( \int_0^\infty \frac{\ln(1+x)}{1+x^2} dx - \int_0^\infty \frac{(x^2+1-1) \ln x}{(1+x^2)(1+x^3)} dx \right) \\
 &= \int_0^1 \frac{\ln(1+x)}{1+x^2} dx - \int_0^\infty \frac{\ln x}{1+x^2} dx - \int_0^\infty \frac{\ln x}{(1+x^2)(1+x^3)} dx
 \end{aligned}$$

注意到  $\frac{1}{(1+x^2)(1+x^3)} = \frac{1+x}{1+x^2} - \frac{x^2+x-1}{1+x^3}$ , 考虑

$$\begin{aligned}
 J(a) &= \int_0^\infty \frac{x^a}{(1+x^2)(1+x^3)} dx \\
 &= \frac{1}{2} \left( \int_0^\infty \frac{x^{a+1} + x^a}{1+x^2} dx - \int_0^\infty \frac{x^{a+2} + x^{a+1} - x^a}{1+x^3} dx \right) \\
 &= \frac{1}{4} \int_0^\infty \frac{x^{\frac{a}{2}} + x^{\frac{a-1}{2}}}{1+x} dx - \frac{1}{6} \int_0^\infty \frac{x^{\frac{a}{3}} + x^{\frac{a-1}{3}} - x^{\frac{a-2}{3}}}{1+x} dx \\
 &= \frac{B(1+\frac{a}{2}, -\frac{a}{2}) + B(\frac{1+a}{2}, \frac{1-a}{2})}{4} - \frac{B(1+\frac{a}{3}, -\frac{a}{3}) + B(\frac{2+a}{3}, \frac{1-a}{3}) - B(\frac{1+a}{3}, \frac{2-a}{3})}{6}
 \end{aligned}$$

那么

$$J'(0) = \int_0^\infty \frac{\ln x}{(1+x^2)(1+x^3)} dx = -\frac{37\pi^2}{432}$$

又

$$\begin{aligned}
 \int_0^\infty \frac{\ln(1+x)}{1+x^2} dx &= 2 \int_0^1 \frac{\ln(1+x)}{1+x^2} dx - \int_0^1 \frac{\ln x}{1+x^2} dx \\
 &= \frac{\pi \ln 2}{4} + \int_0^1 \frac{\arctan x}{x} dx = \frac{\pi \ln 2}{4} + \gamma
 \end{aligned}$$

$$I = \int_0^\infty \frac{\ln(1+x)}{(1+x^2)(1+x^3)} dx = \frac{1}{2} \left( \int_0^\infty \frac{\ln(1+x)}{1+x^2} dx - \frac{37\pi^2}{432} \right) = \frac{\pi \ln 2}{8} + \frac{\gamma}{2} - \frac{37\pi^2}{864}$$

□

#### 180. 计算积分

$$\int_0^1 \frac{\ln^2 x}{1-x+x^2} dx$$

解

$$\begin{aligned} \int_0^1 \frac{\ln^2 x}{1-x+x^2} dx &= \int_0^1 \frac{1+x}{1+x^3} \ln^2 x dx \\ &= \int_0^1 \frac{\ln^2 x}{1+x^3} dx + \int_0^1 \frac{x \ln^2 x}{1+x^3} dx \\ &= \frac{1}{27} \left( \beta''\left(\frac{1}{3}\right) + \beta''\left(\frac{2}{3}\right) \right) \\ &= \frac{1}{216} \left( \psi''\left(\frac{2}{3}\right) - \psi''\left(\frac{1}{3}\right) + \psi''\left(\frac{5}{6}\right) - \psi''\left(\frac{1}{6}\right) \right) \\ &= \frac{\pi}{216} \left( \frac{d^2}{dz^2} (\cos \pi z) \Big|_{z=\frac{1}{3}} + \frac{d^2}{dz^2} (\cos \pi z) \Big|_{z=\frac{1}{6}} \right) \\ &= \frac{\pi}{216} \left( \frac{8\pi^2}{3\sqrt{3}} + 8\sqrt{3}\pi^2 \right) = \frac{10\pi^3}{81\sqrt{3}} \end{aligned}$$

□

#### 181. 计算积分

$$\int_0^\infty \int_0^\infty \frac{\ln(x) \ln(y)}{\sqrt{xy}} \cos(x+y) dx dy$$

解

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \frac{\ln(x) \ln(y)}{\sqrt{xy}} \cos(x+y) dx dy \\ &= \Re \int_0^\infty \int_0^\infty \frac{\ln(x) \ln(y)}{\sqrt{xy}} e^{-ix} e^{-iy} dx dy \\ &= \Re \left( \int_0^\infty \frac{\ln x}{\sqrt{x}} e^{-ix} dx \right)^2 \end{aligned}$$

$$\begin{aligned} \int_0^\infty \frac{\ln x}{\sqrt{x}} e^{-ix} dx &= \lim_{s \rightarrow 1} \partial_s \int_0^\infty e^{-ix} x^{s-1-\frac{1}{2}} dx \\ &= \lim_{s \rightarrow 1} \partial_s \left( e^{-i\frac{\pi}{2}(s-\frac{1}{2})} \Gamma\left(s-\frac{1}{2}\right) \right) \\ &= \lim_{s \rightarrow 1} e^{-i\frac{\pi}{2}(s-\frac{1}{2})} \Gamma\left(s-\frac{1}{2}\right) \left[ -i\frac{\pi}{2} + \psi\left(s-\frac{1}{2}\right) \right] \\ &= e^{-i\frac{\pi}{4}} \sqrt{\pi} \left[ -i\frac{\pi}{2} - \gamma - 2\ln 2 \right] \end{aligned}$$

于是

$$I = \Re \left[ e^{-i\frac{\pi}{2}} \pi \left( -i\frac{\pi}{2} - \gamma - 2\ln 2 \right)^2 \right] = \pi^2 (\gamma + 2\ln 2)$$

□

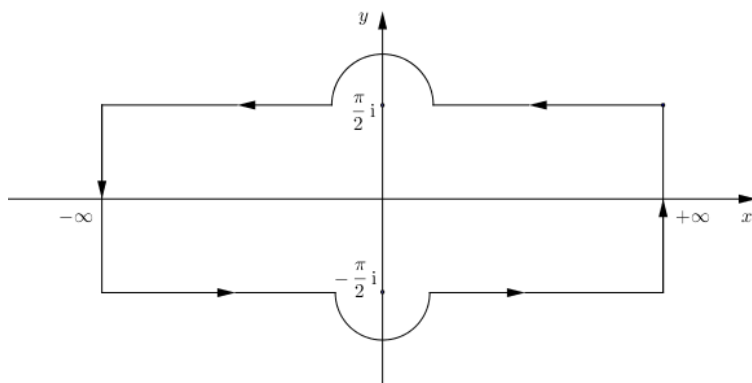
## 182. 计算积分

$$\int_0^1 \frac{1}{1+a^2x^2} \left[ \left(1-\frac{x}{2}\right) \ln \frac{1+x}{1-x} + \frac{\pi^2x^2}{4} \right]^{-1} dx$$

**解** 先作换元  $x \rightarrow \tanh x$ , 可得

$$\begin{aligned} & \int_0^1 \frac{1}{1+a^2x^2} \left[ \left(1-\frac{x}{2}\right) \ln \frac{1+x}{1-x} + \frac{\pi^2x^2}{4} \right]^{-1} dx \\ &= \int_0^\infty \frac{1}{1+a^2 \tanh^2 x} \left( \frac{\coth^2 x - 1}{(\coth x - x)^2 + \frac{\pi^2x^2}{4}} \right) dx \\ &= \frac{1}{2} \int_{-\infty}^\infty \frac{1}{1+a^2 \tanh^2 x} \left( \frac{\coth^2 x - 1}{(\coth x - x)^2 + \frac{\pi^2x^2}{4}} \right) dx \end{aligned}$$

于是我们考虑函数  $f(z) = \frac{1}{1+a^2 \coth^2 z} \cdot \frac{\tanh^2 z - 1}{\tanh z - z}$  的如下围道积分



注意到  $f(z) = \frac{1}{1+a^2 \coth^2 z} \cdot \frac{\tanh^2 z - 1}{\tanh z - z}$  的极点为  $z=0, z=\pm \frac{\pi}{2}i$  (这两个极点在围道边界上), 以及  $1+a^2 \coth^2 z=0$  的根  $z=\pm i \cdot \operatorname{arccoth}\left(\frac{i}{a}\right) = \pm i \cdot \arctan(a)$ , 因此根据留数定理有

$$\begin{aligned} & \int_{-\infty-\frac{\pi}{2}i}^{\infty-\frac{\pi}{2}i} f(z) dz - \int_{-\infty+\frac{\pi}{2}i}^{\infty+\frac{\pi}{2}i} f(z) dz \\ &= 2\pi i (\operatorname{Res}[f(z), z=0] + (\operatorname{Res}[f(z), i \cdot \arctan(a)] + \operatorname{Res}[f(z), z=-i \cdot \arctan(a)])) \\ & \quad + \pi i \left( \operatorname{Res}\left[f(z), z=\frac{\pi}{2}i\right] + \operatorname{Res}\left[f(z), z=-\frac{\pi}{2}i\right] \right) \\ &= 2\pi i \left( \frac{3}{a^2} - \frac{a}{2(a-\arctan(a))} - \frac{a}{2(a-\arctan(a))} \right) + \pi i(1+1) \\ &= 2\pi i \left( \frac{3}{a^2} - \frac{\arctan(a)}{a-\arctan(a)} \right) \end{aligned}$$

注意到  $\tanh\left(z \pm \frac{\pi}{2}i\right) = \coth z, \coth\left(z \pm \frac{\pi}{2}i\right) = \tanh z$ , 于是

$$\begin{aligned} & \int_{-\infty-\frac{\pi}{2}i}^{\infty-\frac{\pi}{2}i} f(z) dz - \int_{-\infty+\frac{\pi}{2}i}^{\infty+\frac{\pi}{2}i} f(z) dz = \int_{-\infty}^{\infty} f\left(x - \frac{\pi}{2}i\right) dx - \int_{-\infty}^{\infty} f\left(x + \frac{\pi}{2}i\right) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{1+a^2 \tanh^2 x} \cdot \frac{\coth^2 x - 1}{\coth x - (x - \frac{\pi}{2}i)^2} dx - \int_{-\infty}^{\infty} \frac{1}{1+a^2 \tanh^2 x} \cdot \frac{\coth^2 x - 1}{\coth x - (x + \frac{\pi}{2}i)^2} dx \end{aligned}$$



$$= -\pi i \int_{-\infty}^{\infty} \frac{1}{1+a^2 \tanh^2 x} \cdot \frac{\coth^2 x - 1}{(\coth x - x)^2 + \frac{\pi^2}{4}} dx = 2\pi i \left( \frac{3}{a^2} - \frac{\arctan(a)}{a - \arctan(a)} \right)$$

因此

$$\begin{aligned} \int_0^{\infty} \frac{1}{1+a^2 \tanh^2 x} \cdot \frac{\coth^2 x - 1}{(\coth x - x)^2 + \frac{\pi^2}{4}} dx &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+a^2 \tanh^2 x} \cdot \frac{\coth^2 x - 1}{(\coth x - x)^2 + \frac{\pi^2}{4}} dx \\ &= \frac{\arctan(a)}{a - \arctan(a)} - \frac{3}{a^2} \end{aligned}$$

□

### 183. 计算积分

$$\int_0^{\infty} \frac{\ln(x)}{1+e^x} dx$$

解

$$\begin{aligned} \int_0^{\infty} \frac{\ln(x)}{1+e^x} dx &= \int_0^1 \frac{\ln(x)}{1+e^x} dx + \int_1^{\infty} \frac{\ln(x)}{1+e^x} dx \\ &= -\ln(x) \ln\left(\frac{1+e^{-x}}{2}\right) \Big|_0^1 + \int_0^1 \ln\left(\frac{1+e^{-x}}{2}\right) \frac{dx}{x} - \ln(x) \ln(1+e^{-x}) \Big|_1^{\infty} + \int_1^{\infty} \ln(1+e^{-x}) \frac{dx}{x} \\ &= \int_0^1 \ln\left(\frac{1-e^{-xy}}{y}\right) \Big|_{y=1}^{y=2} \frac{dx}{x} + \int_1^{\infty} \ln(1-e^{-xy}) \Big|_{y=1}^{y=2} \frac{dx}{x} \\ &= \int_0^1 \int_1^2 \left( \frac{1}{e^{xy}-1} - \frac{1}{xy} \right) dy dx + \int_1^{\infty} \int_1^2 \frac{dx dy}{e^{xy}-1} \\ &= \int_1^2 \frac{dy}{y} \left[ \ln\left(\frac{1-e^{-xy}}{x}\right) \Big|_{x=0}^{x=1} + \ln(1-e^{-xy}) \Big|_{x=1}^{x=\infty} \right] \\ &= -\int_1^2 \frac{\ln(y)}{y} dy = -\frac{\ln^2 2}{2} \end{aligned}$$

□

### 184. 计算积分

$$\int_0^{\frac{\pi}{2}} \frac{x(1+\sin^2 x) \cos x}{(1+3\sin^2 x)(\sin^2 x+3)} dx$$

解

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{x(1+\sin^2 x) \cos x}{(1+3\sin^2 x)(\sin^2 x+3)} dx \\ &= \frac{1}{4} \int_0^{\frac{\pi}{2}} \left( \frac{x}{1+3\sin^2 x} + \frac{x}{\sin^2 x+3} \right) d(\sin x) \\ &= \frac{1}{4\sqrt{3}} \left[ \int_0^{\frac{\pi}{2}} x d \left[ \arctan(\sqrt{3} \sin x) \right] + \int_0^{\frac{\pi}{2}} x d \left[ \arctan\left(\frac{\sin x}{\sqrt{3}}\right) \right] \right] \end{aligned}$$

记

$$J = \int_0^{\frac{\pi}{2}} x d \left[ \arctan(\sqrt{3} \sin x) \right] = x \arctan(a \sin x) \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \arctan(a \sin x) dx$$

$$K(a) = \int_0^{\frac{\pi}{2}} \arctan(a \sin x) dx = \int_0^1 \frac{\arctan(ax)}{\sqrt{1-x^2}} dx,$$

$$K'(a) = \int_0^1 \frac{x}{(1+a^2x^2)\sqrt{1-x^2}} dx = \frac{\operatorname{arcsinh}(a)}{a\sqrt{1+a^2}}$$

$$K(a) = \int_0^a \frac{\operatorname{arcsinh}(t)}{t\sqrt{1+t^2}} dt = - \int_0^a \operatorname{arcsinh}(t) d \left[ \operatorname{arcsinh} \left( \frac{1}{t} \right) \right] + \int_0^a \frac{\operatorname{arcsinh} \left( \frac{1}{t} \right)}{\sqrt{1+t^2}} dt$$

$$= -\operatorname{arcsinh}(a) \operatorname{arcsinh} \left( \frac{1}{a} \right) + \int_{\frac{1}{a}}^{\infty} \frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}} du$$

$$= -\operatorname{arcsinh}(a) \operatorname{arcsinh} \left( \frac{1}{a} \right) + \int_0^{\infty} \frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}} du - \int_0^{\frac{1}{a}} \frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}} du$$

$$\int_0^a \frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}} du + \int_0^{\frac{1}{a}} \frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}} du = -\operatorname{arcsinh}(a) \operatorname{arcsinh} \left( \frac{1}{a} \right) + \int_0^{\infty} \frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}} du$$

故

$$I = \frac{1}{4\sqrt{3}} \left[ x \arctan(a \sin x) \Big|_0^{\frac{\pi}{2}} + x \arctan \left( \frac{\sin x}{a} \right) \Big|_0^{\frac{\pi}{2}} \right] - \left( \int_0^a \frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}} du + \int_0^{\frac{1}{a}} \frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}} du \right)$$

$$= \frac{1}{4\sqrt{3}} \left[ \frac{\pi^2}{4} + \operatorname{arcsinh}(a) \operatorname{arcsinh} \left( \frac{1}{a} \right) \Big|_{a=\sqrt{3}} - \int_0^{\infty} \frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}} du \right]$$

$$= \frac{1}{8\sqrt{3}} \ln(3) \ln(2 + \sqrt{3})$$

其中

$$\int_0^{\infty} \frac{\operatorname{arcsinh}(u)}{u\sqrt{1+u^2}} du = \int_0^{\infty} \frac{t}{\sinh(t)} dt = \int_0^{\infty} \sum_{k=0}^{\infty} 2te^{-(2k+1)t} dt = \sum_{k=0}^{\infty} \frac{2}{(2k+1)^2} = \frac{\pi^2}{4}$$

□

**185.** 设  $a, b > 0$ , 计算积分

$$\int_0^{\frac{\pi}{4}} \frac{x \ln \left( \frac{\cos x + \sin x}{\cos x - \sin x} \right)}{\cos x (\cos x + \sin x)} dx$$

**解** 对  $x \in [0, \frac{\pi}{2}]$ ,

$$\frac{x \ln \left( \frac{\cos x + \sin x}{\cos x - \sin x} \right)}{\cos x (\cos x + \sin x)} = \frac{x \ln \left( \frac{1 + \tan x}{1 - \tan x} \right)}{\cos^2 x (1 + \tan x)} = \frac{x (1 + \tan^2 x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right)}{1 + \tan x}$$

于是

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \frac{x (1 + \tan^2 x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right)}{1 + \tan x} dx \\ &= \int_0^1 \frac{\arctan x \ln \left( \frac{1+x}{1-x} \right)}{1+x} dx \\ &= \int_0^1 \frac{\arctan \left( \frac{x-1}{1+x} \right) \ln x}{1+x} dx \end{aligned}$$

$$= \int_0^1 \frac{\arctan(x) \ln x}{1+x} dx - \frac{\pi}{4} \int_0^1 \frac{\ln x}{1+x} dx$$

其中  $\int_0^1 \frac{\ln x}{1+x} dx = -\frac{\pi^2}{12}$ , 记  $F(x) = \int_0^x \frac{\ln t}{1+t} dt = \int_0^1 \frac{x \ln(xy)}{1+xy} dy$ , 则  $F(1) = -\frac{\pi^2}{12}$ ,

$$\begin{aligned} \int_0^1 \frac{\arctan(x) \ln x}{1+x} dx &= [F(x) \arctan(x)]_0^1 - \int_0^1 \frac{F(x)}{1+x} dx \\ &= -\frac{\pi^3}{48} - \int_0^1 \int_0^1 \frac{x \ln(xy)}{(1+xy)(1+x^2)} dx dy \\ &= -\frac{\pi^3}{48} - \int_0^1 \int_0^1 \frac{x \ln(x)}{(1+xy)(1+x^2)} dx dy - \int_0^1 \int_0^1 \frac{x \ln(y)}{(1+xy)(1+x^2)} dx dy \\ &= -\frac{\pi^3}{48} - \int_0^1 \left[ \frac{\ln(x) \ln(1+xy)}{1+x^2} \right]_{y=0}^{y=1} dx - \\ &\quad \int_0^1 \left[ -\frac{\ln(y) \ln(1+xy)}{1+y^2} + \frac{\ln(y) \ln(1+x^2)}{2(1+y^2)} + \frac{y \ln(y) \arctan x}{1+y^2} \right]_{x=0}^{x=1} dy \\ &= -\frac{\pi^3}{48} - \int_0^1 \frac{\ln(x) \ln(1+x)}{1+x^2} dx + \int_0^1 \frac{\ln(y) \ln(1+y)}{1+y^2} dy \\ &\quad - \frac{\ln(2)}{2} \int_0^1 \frac{\ln y}{1+y^2} dy - \frac{\pi}{4} \int_0^1 \frac{y \ln y}{1+y^2} dy \\ &= \frac{G}{2} - \frac{\pi^3}{64} \end{aligned}$$

于是  $I = \frac{G \ln 2}{2} + \frac{\pi^3}{192}$ .

□

**186.** 设  $0 < a < \frac{\pi}{2}$  计算积分

$$\int_0^\infty \frac{x}{1+x^2} \frac{\cos(ax)}{\sinh(\pi x)} dx$$

**解**

$$\begin{aligned} \int_0^\infty \frac{x}{1+x^2} \frac{\cos(ax)}{\sinh(\pi x)} dx &= \int_0^\infty \frac{\cos(ax)}{\sinh(\pi x)} \int_0^\infty e^{-t} \sin(xt) dt dx \\ &= \int_0^\infty e^{-t} dt \int_0^\infty \frac{\sin(tx) \cos(ax)}{\sinh(\pi x)} dx \\ &= \frac{1}{2} \int_0^\infty e^{-t} dt \int_0^\infty \frac{\sin(t+a)x + \sin(t-a)x}{\sinh(\pi x)} dx \\ &= \frac{1}{2} \int_0^\infty e^{-t} \left( \tanh \frac{t+a}{2} + \tanh \frac{t-a}{2} \right) dt \\ &= \frac{1}{4} \left( \int_0^1 \frac{e^a - s}{e^a + s} ds + \int_0^1 \frac{e^{-a} - s}{e^{-a} + s} ds \right) \\ &= -\frac{1}{2} \left( 1 - e^a \ln \left( \frac{1+e^a}{e^a} \right) - e^{-a} \ln \left( \frac{1+e^{-a}}{e^{-a}} \right) \right) \\ &= -\frac{1}{2} - \frac{ae^a}{2} + \cosh(a) \ln(1+e^a) \end{aligned}$$

其中

$$\int_0^\infty \frac{\sin(vx)}{\sinh(\mu x)} dx = 2 \sum_{k=0}^\infty \int_0^\infty e^{-(2k+1)\mu x} \sin(vx) dx = 2 \sum_{k=0}^\infty \frac{v}{v^2 + (2k+1)^2 \mu^2} = \frac{\pi}{2\mu} \tanh\left(\frac{v\pi}{2\mu}\right)$$

□

### 187. 计算积分

$$\int_0^1 \int_0^1 \frac{(1-x^2) dx dy}{(1+x^2 y^2) \ln^2(xy)}$$

解

$$\begin{aligned} I &= \int_0^1 \int_0^1 \int_0^\infty \int_0^\infty \frac{1-x^2}{1+x^2 y^2} (xy)^{w+z} dw dz dx dy \\ &= \int_0^1 \int_0^1 \int_0^\infty \int_0^\infty \sum_{k=0}^\infty (-x^2 y^2)^k (1-x^2) (xy)^{w+z} dw dz dx dy \\ &= \sum_{k=0}^\infty (-1)^k \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 (x^{2k+w+z} y^{2k+w+z} - x^{2k+w+z+2} y^{2k+w+z+2}) dx dy dw dz \\ &= \sum_{k=0}^\infty (-1)^k \int_0^\infty \int_0^\infty \frac{2}{(2k+w+z+1)^2 (2k+2+z+3)} dw dz \\ &= \sum_{k=0}^\infty (-1)^k \frac{1}{2} \ln \frac{k+\frac{3}{2}}{k+\frac{1}{2}} + \sum_{k=1}^\infty (-1)^k k \left( \ln \frac{2k-1}{2k+1} - 1 \right) \\ &= \frac{1}{2} \sum_{k=0}^\infty (-1)^k \ln \frac{k+\frac{3}{2}}{k+\frac{1}{2}} = \frac{1}{2} \prod_{k=0}^\infty \frac{2k+\frac{3}{2}}{2k+\frac{1}{2}} \cdot \frac{2k+\frac{3}{2}}{2k+\frac{5}{2}} = \frac{1}{2} \ln \frac{\Gamma(\frac{1}{4}) \Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4}) \Gamma(\frac{3}{4})} = \ln \frac{\Gamma(\frac{1}{4})}{2\Gamma(\frac{3}{4})} \end{aligned}$$

因为

$$\lim_{n \rightarrow \infty} \prod_{k=1}^{2n} \left(1 - \frac{2}{2k+1}\right)^{k \cdot (-1)^k} = \exp\left(\frac{2G}{\pi} - \frac{1}{2}\right) \lim_{n \rightarrow \infty} \prod_{k=1}^{2n+1} \left(1 - \frac{2}{2k+1}\right)^{k \cdot (-1)^k} = \exp\left(\frac{2G}{\pi} + \frac{1}{2}\right)$$

所以

$$\begin{aligned} \sum_{k=1}^\infty (-1)^k k \left( \ln \frac{2k-1}{2k+1} - 1 \right) &= \frac{2G}{\pi} - \frac{1}{2} \\ I &= \ln \frac{\Gamma(\frac{1}{4})}{2\Gamma(\frac{3}{4})} + \frac{2G}{\pi} - \frac{1}{2} \end{aligned}$$

□

### 188. 计算积分

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos x) - \ln(1+\cos y)}{\cos x - \cos y} dx dy$$

解 考虑参数积分

$$I(a) = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\ln(1+a \cos x) - \ln(1+a \cos y)}{\cos x - \cos y} dx dy, I(0) = 0$$

则

$$\begin{aligned} I'(a) &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{(1+a \cos x)(1+a \cos y)} dx dy \\ &= \left( \int_0^{\frac{\pi}{2}} \frac{1}{1+a \cos x} dx \right)^2 \\ &= \frac{4}{1-a^2} \arctan^2 \left( \sqrt{\frac{1-a}{1+a}} \right) \end{aligned}$$

于是

$$\begin{aligned} I &= I(1) = \int_0^1 \frac{4}{1-a^2} \arctan^2 \left( \sqrt{\frac{1-a}{1+a}} \right) da \\ &= \int_0^{\frac{\pi}{2}} \frac{t^2}{\sin t} dt = -2 \int_0^{\frac{\pi}{2}} t \ln \left( \tan \frac{t}{2} \right) dt \quad a = \cos t \\ &= 4 \sum_{k=1}^{\infty} \int_0^{\frac{\pi}{2}} \frac{t \cos(2k-1)t}{2k-1} dt \\ &= 4 \sum_{k=1}^{\infty} \frac{\pi}{2} \frac{(-1)^{k-1}}{(2k-1)^2} - 4 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \\ &= 2\pi G - \frac{7}{2} \zeta(3) \end{aligned}$$

□

**189.** 设  $a > 0, b > 0$ , 计算积分

$$\int_0^{\infty} \frac{\cos(x^a) - \cos(x^b)}{x} dx$$

**解** 对任意  $0 < s < 1$ ,

$$\int_0^{\infty} \frac{\cos x}{x^{1-s}} dx = \Re \left( \int_0^{\infty} x^{s-1} e^{ix} dx \right) = \Re [i^s \cdot \Gamma(s)] = \Gamma(s) \cos \left( \frac{\pi s}{2} \right)$$

于是

$$\begin{aligned} \int_0^{\infty} \frac{\cos(x^a)}{x^{1-s}} dx &= \frac{1}{a} \int_0^{\infty} t^{\frac{s}{a}-1} \cos t dt = \frac{1}{a} \Gamma \left( \frac{s}{a} \right) \cos \left( \frac{\pi s}{2a} \right) \\ \int_0^{\infty} \frac{\cos(x^b)}{x^{1-s}} dx &= \frac{1}{b} \Gamma \left( \frac{s}{b} \right) \cos \left( \frac{\pi s}{2b} \right) \end{aligned}$$

于是

$$\begin{aligned} \int_0^{\infty} \frac{\cos(x^a) - \cos(x^b)}{x} dx &= \lim_{s \rightarrow 0} \left[ \frac{1}{a} \Gamma \left( \frac{s}{a} \right) \cos \left( \frac{\pi s}{2a} \right) - \frac{1}{b} \Gamma \left( \frac{s}{b} \right) \cos \left( \frac{\pi s}{2b} \right) \right] \\ &= \lim_{s \rightarrow 0} \left[ \frac{1}{a} \left( \frac{a}{s} - \gamma + o(1) \right) - \frac{1}{b} \left( \frac{b}{s} - \gamma + o(1) \right) \right] \\ &= \left( \frac{1}{b} - \frac{1}{a} \right) \gamma \end{aligned}$$

□

### 190. 求极限

$$\lim_{n \rightarrow \infty} n \left[ \left( \int_0^1 \frac{1}{1+x^n} dx \right)^n - \frac{1}{2} \right]$$

**解** 首先有

$$\begin{aligned} I_n &= \int_0^1 \frac{1}{1+x^n} dx = \frac{1}{n} \int_0^1 \frac{t^{\frac{1}{n}-1}}{1+t} dt \\ &= \frac{1}{n} \int_0^1 t^{\frac{1}{n}} \left( \frac{1}{t} - \frac{1}{1+t} \right) dt = 1 - \frac{1}{n} \int_0^1 \frac{t^{\frac{1}{n}}}{1+t} dt \\ &= 1 - \sum_{k=0}^{\infty} \frac{1}{n^{k+1} k!} \int_0^1 \frac{\ln^k x}{1+x} dx \end{aligned}$$

因此不难得到

$$I(n) = 1 - \frac{\ln 2}{n} + \frac{\pi^2}{12n^2} + o\left(\frac{1}{n^2}\right)$$

故

$$\begin{aligned} I^n(n) &= e^{n \ln \left[ 1 - \frac{\ln 2}{n} + \frac{\pi^2}{12n^2} + o\left(\frac{1}{n^2}\right) \right]} = e^{n \left[ -\frac{\ln 2}{n} + \frac{\pi^2}{12n^2} - \frac{\ln^2 2}{2n^2} + o\left(\frac{1}{n^2}\right) \right]} \\ &= \frac{1}{2} \left[ 1 + \left( \frac{\pi^2}{12} - \frac{1}{2} \ln^2 2 \right) \frac{1}{n} + o\left(\frac{1}{n}\right) \right] \end{aligned}$$

于是最后得到

$$\lim_{n \rightarrow \infty} n \left[ I^n(n) - \frac{1}{2} \right] = \frac{\pi^2}{24} - \frac{1}{4} \ln^2 2$$

□

### 191. 计算积分

$$\int_0^1 \frac{1-x}{1+x} \cdot \frac{2k+3+x^2}{1+x^2} \cdot \frac{dx}{\ln x}$$

**解** 记

$$I(k) = \int_0^1 \frac{1-x}{1+x} \cdot \frac{2k+3+x^2}{1+x^2} \cdot \frac{dx}{\ln x}$$

则

$$\begin{aligned} I(-1) &= \int_0^1 \frac{1-x}{1+x} \frac{dx}{\ln x} = - \int_0^{\infty} \frac{1-e^{-t}}{1+e^{-t}} \frac{e^{-t}}{t} dt \\ &= \int_0^{\infty} \frac{1-e^{-t}}{t} \sum_{k=1}^{\infty} (-e^{-t})^k dt = \sum_{k=1}^{\infty} (-1)^k \int_0^{\infty} \frac{e^{-kt} - e^{-(k+1)t}}{t} dt \\ &= \sum_{k=1}^{\infty} (-1)^k \ln \frac{k+1}{k} = \sum_{k=1}^{\infty} \ln \left( \frac{2k+1}{2k} \cdot \frac{2k-1}{2k} \right) \\ &= \ln \left( \prod_{k=1}^{\infty} \frac{2k+1}{2k} \cdot \frac{2k-1}{2k} \right) = -\ln \left( \frac{\pi}{2} \right) \end{aligned}$$

$$I'(k) = \int_0^1 \frac{1-x}{1+x} \cdot \frac{2}{1+x^2} \cdot \frac{dx}{\ln x} = \int_1^{\infty} \frac{1-x}{1+x} \cdot \frac{2}{1+x^2} \cdot \frac{dx}{\ln x}$$

$$\begin{aligned}
 &= \int_0^\infty \frac{1-x}{1+x} \cdot \frac{1}{1+x^2} \cdot \frac{dx}{\ln x} = - \int_0^\infty \frac{dx}{(1+x)(1+x^2)} \int_0^1 x^y dy \\
 &= \frac{1}{2} \int_0^1 \int_0^\infty \left( \frac{x^{y+1}}{1+x^2} - \frac{x^y}{1+x^2} - \frac{x^y}{1+x} \right) dx dy \\
 &= -\frac{\pi}{4} \int_0^1 \frac{dy}{\cos \frac{\pi y}{4} (\cos \frac{\pi y}{4} + \sin \frac{\pi y}{4})} = -\ln 2
 \end{aligned}$$

于是

$$I(k) = I(-1) + \int_{-1}^k (-\ln 2) dt = -\ln\left(\frac{\pi}{2}\right) - (k+1)\ln 2 = -\ln(2^k \pi)$$

□

192. 求和

$$\sum_{n=1}^{\infty} \arctan \frac{10n}{(3n^2+2)(9n^2-1)}$$

解

$$\begin{aligned}
 S &= \sum_{n=1}^{\infty} \arctan \frac{10n}{(3n^2+2)(9n^2-1)} \\
 &= \sum_{n=1}^{\infty} \arg \left( 1 + \frac{10in}{(3n^2+2)(9n^2-1)} \right) \\
 &= \arg \prod_{n=1}^{\infty} \left( 1 + \frac{10in}{(3n^2+2)(9n^2-1)} \right) \\
 &= \arg \prod_{n=1}^{\infty} \left( \frac{(1-\frac{i}{n})(1+\frac{i}{3n-1})(1+\frac{i}{3n+1})(1+\frac{i}{3n})}{1+\frac{2}{3n^2}} \right) \\
 &= \lim_{m \rightarrow \infty} \sum_{n=1}^m \left[ \arctan \left( \frac{1}{3n-1} \right) + \arctan \left( \frac{1}{3n} \right) + \arctan \left( \frac{1}{3n+1} \right) - \arctan \left( \frac{1}{n} \right) \right] \\
 &= -\arctan(1) + \lim_{m \rightarrow \infty} \sum_{n=m+1}^{3m+1} \left[ \frac{1}{n} + O\left(\frac{1}{n^3}\right) \right] = \ln 3 - \frac{\pi}{4}
 \end{aligned}$$

□

193. 设  $f(x)$  是  $[0, +\infty)$  上正的连续函数, 且  $\int_0^\infty \frac{dx}{f(x)}$  收敛. 记  $F(x) = \int_0^x f(t) dt$ , 求证

$$\int_0^\infty \frac{x}{F(x)} dx < 2 \int_0^\infty \frac{dx}{f(x)}$$

证明 由 Cauchy 不等式得

$$\left( \int_0^x f(t) dt \right) \left( \int_0^x \frac{t^2}{f(t)} dt \right) \geq \left( \int_0^x t dt \right)^2 = \frac{1}{4} x^4$$

所以

$$\int_0^\infty \frac{x}{F(x)} dx \leq \int_0^\infty \frac{4}{x^3} \int_0^x \frac{t^2}{f(t)} dt dx$$

注意到

$$\lim_{x \rightarrow 0^+} \int_0^x \frac{t^2}{f(t)} dt = \int_0^\xi \frac{1}{f(t)} dt < \int_0^\infty \frac{dt}{f(t)}$$

故

$$\begin{aligned} \int_0^A \frac{4}{x^3} \int_0^x \frac{t^2}{f(t)} dt &= \int_0^A \left( \int_0^x \frac{t^2}{f(t)} dt \right) d\left(-\frac{2}{x^2}\right) \\ &= \int_0^A \frac{2}{f(x)} dx - \frac{2}{A^2} \int_0^A \frac{t^2}{f(t)} dt < \int_0^A \frac{2}{f(x)} dx - 2 \int_\xi^A \frac{t^2}{f(t)} dt \end{aligned}$$

令  $A \rightarrow +\infty$ , 得到

$$\int_0^\infty \frac{4}{x^3} \int_0^x \frac{t^2}{f(t)} dt dx \leq \int_0^\infty \frac{2}{f(x)} dx - 2 \int_\xi^\infty \frac{t^2}{f(t)} dt < \int_0^\infty \frac{2}{f(x)} dx$$

另外, 当我们取  $f(x) = x^a + 1 (a > 1)$  时, 有  $\int_0^{+\infty} \frac{dx}{x^a + 1} = \frac{\pi}{a \sin \frac{\pi}{a}}$  收敛. 此时有

$$\lim_{a \rightarrow 1} \frac{\int_0^\infty \frac{x}{f(x)} dx}{\int_0^\infty \frac{dx}{f(x)}} = \lim_{a \rightarrow 1} \frac{\int_0^\infty \frac{x}{x^{a+1}/(a+1)+x} dx}{\int_0^\infty \frac{dx}{x^a+1}} = \lim_{a \rightarrow 1} \frac{\int_0^\infty \frac{1}{x^{a/(a+1)+1}} dx}{\int_0^\infty \frac{dx}{x^a+1}} = \lim_{a \rightarrow 1} (a+1)^{\frac{1}{a}} = 2$$

□

**194.** 设  $f(x)$  是  $[0, +\infty)$  上周期为  $T$  的局部可积函数, 且  $\int_0^a \frac{f(x)}{x} dx$  收敛, 其中  $0 < a < \pi$ , 证明

$$\lim_{n \rightarrow \infty} \frac{1}{\ln n} \int_0^a \frac{f(nx)}{\sin x} dx = \frac{1}{T} \int_0^T f(x) dx$$

**证明** 由于  $f(x)$  局部可积故有界,  $\exists M > 0$ , 使得  $|f(x)| < M$ , 而  $\int_0^a \frac{f(nx)}{x} dx = \int_0^{na} \frac{f(t)}{t} dt$  ( $n \in \mathbb{N}_+$ ). 由于  $\int_0^a \frac{f(x)}{x} dx$  收敛, 故  $\int_0^{na} \frac{f(t)}{t} dt = \int_0^a \frac{f(nx)}{x} dx$  存在, 而

$$\left| \int_0^a \frac{f(nx)}{\sin x} dx - \int_0^a \frac{f(nx)}{x} dx \right| = \left| \int_0^a f(nx) \left( \frac{1}{\sin x} - \frac{1}{x} \right) dx \right| \leq M \int_0^a \frac{x - \sin x}{x \sin x} dx$$

由于  $\lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} = 0$ , 故  $\int_0^a \frac{x - \sin x}{x \sin x} dx$  存在且为有限数, 从而

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\ln n} \int_0^a \frac{f(nx)}{\sin x} dx &= \lim_{n \rightarrow \infty} \frac{1}{\ln n} \int_0^a \frac{f(nx)}{x} dx = \lim_{n \rightarrow \infty} \frac{1}{\ln n} \int_0^{na} \frac{f(t)}{t} dt \\ &= \lim_{n \rightarrow \infty} \frac{1}{\ln(na) - \ln a} \int_0^{na} \frac{f(t)}{t} dt = \lim_{x \rightarrow +\infty} \frac{1}{\ln x} \int_0^x \frac{f(t)}{t} dt \\ &= \frac{1}{T} \int_0^T f(x) dx \end{aligned}$$

□

**195.** 计算积分

$$\int_0^\infty [\operatorname{si}(x) \cos x - \operatorname{ci}(x) \sin x]^2 dx$$



其中  $\text{si}(x) = -\int_x^\infty \frac{\sin t}{t} dt$ ,  $\text{ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt$ .

解

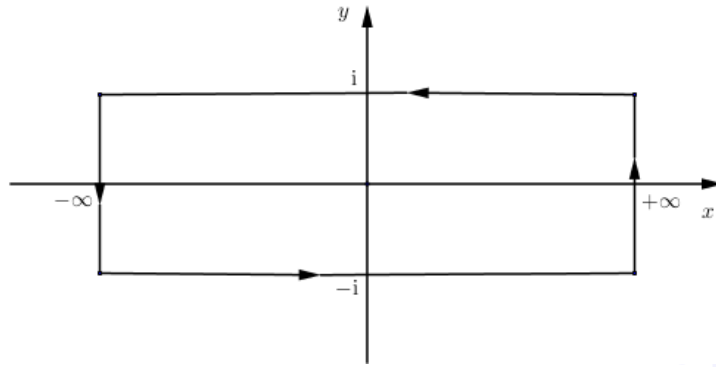
$$\begin{aligned}
 \int_0^\infty [\text{si}(x) \cos x - \text{ci}(x) \sin x]^2 dx &= \int_0^\infty \left[ \sin x \int_x^\infty \frac{\cos t}{t} dt - \cos x \int_x^\infty \frac{\sin t}{t} dt \right]^2 dx \\
 &= \int_0^\infty \left[ \int_x^\infty \frac{\sin(x-t)}{t} dt \right]^2 dx = \int_0^\infty \left( \int_0^\infty \frac{\sin \theta}{x+\theta} d\theta \right)^2 dx \\
 &= \int_0^\infty \left( \int_0^\infty \sin \theta d\theta \int_0^\infty e^{-(x+\theta)y} dy \right)^2 dx \\
 &= \int_0^\infty \left( \int_0^\infty e^{-xy} dy \int_0^\infty e^{-y\theta} \sin \theta d\theta \right)^2 dx \\
 &= \int_0^\infty \left( \int_0^\infty \frac{e^{-xy}}{1+y^2} dy \right)^2 dx \\
 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-xy}}{1+y^2} \cdot \frac{e^{-xz}}{1+z^2} dx dy dz \\
 &= \int_0^\infty \int_0^\infty \frac{dy dz}{(1+y^2)(1+z^2)} \int_0^\infty e^{-(y+z)x} dx \\
 &= \int_0^\infty \int_0^\infty \frac{dy dz}{(1+y^2)(1+z^2)(y+z)} \\
 &= \int_0^\infty \frac{dz}{(1+z^2)} \left( z \int_0^\infty \frac{dy}{1+y^2} + \int_0^\infty \left( \frac{1}{y+z} - \frac{y}{1+y^2} \right) dy \right) \\
 &= \int_0^\infty \frac{dz}{(1+z^2)} \left( \frac{\pi}{2} z - \ln z \right) \\
 &= \frac{\pi}{2} \int_0^\infty \frac{z dz}{(1+z^2)^2} - \int_0^\infty \frac{\ln z}{(1+x^2)^2} dz \\
 &= \frac{\pi}{4} - \left( \frac{d}{ds} \int_0^\infty \frac{z^{s-1}}{(1+z^2)^2} dz \right)_{s=1} \\
 &= \frac{\pi}{4} - \frac{1}{2} \left[ \frac{d}{ds} B \left( 2 - \frac{s}{2}, \frac{s}{2} \right) \right]_{s=1} \\
 &= \frac{\pi}{4} - \frac{1}{2} \left[ \frac{d}{ds} \left( \left( 1 - \frac{s}{2} \right) \frac{\pi}{\sin \frac{\pi s}{2}} \right) \right]_{s=1} = \frac{\pi}{2}
 \end{aligned}$$

□

#### 196. 计算积分

$$\int_{-\infty}^\infty \frac{e^{7\pi x}}{(e^{3\pi x} + e^{-3\pi x})^3 (1+x^2)} dx$$

解 考虑函数  $f(z) = \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z}$  的如下围道积分, 以  $y = \pm i$  和  $x = \pm \infty$  为边的矩形围道 (实质是令长方形的长趋于  $\infty$ ).



易知在  $x = \pm\infty$  两条围道的积分是 0, 在矩形内部的所有极点为  $z = 0, \pm\frac{i}{6}, \pm\frac{i}{2}, \frac{5}{6}i$ , 那么由留数定理有

$$\begin{aligned}
 & \int_{-\infty-i}^{+\infty-i} \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z} dz - \int_{-\infty+i}^{+\infty+i} \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z} dz \\
 &= \int_{-\infty}^{+\infty} \frac{e^{7\pi(x-i)}}{(e^{3\pi(x-i)} + e^{-3\pi(x-i)})^3 (x-i)} dx - \int_{-\infty}^{+\infty} \frac{e^{7\pi(x+i)}}{(e^{3\pi(x+i)} + e^{-3\pi(x+i)})^3 (x+i)} dx \\
 &= 2i \int_{-\infty}^{\infty} \frac{e^{7\pi x}}{(e^{3\pi x} + e^{-3\pi x})^3 (1+x^2)} dx \\
 &= 2\pi i \sum \text{res} \left( \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z}, z = 0, \pm\frac{i}{6}, \pm\frac{i}{2}, \pm\frac{5}{6}i \right)
 \end{aligned}$$

其中

$$\begin{aligned}
 & \text{res} \left( \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z}, z = 0 \right) = \frac{1}{8} \\
 & \text{res} \left( \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z}, z = \frac{i}{6} \right) = -\frac{e^{\frac{\pi i}{6}} (-18 + 21\pi i + 10\pi^2)}{18\pi^3} \\
 & \text{res} \left( \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z}, z = -\frac{i}{6} \right) = \frac{e^{\frac{5\pi i}{6}} (-18 + 21\pi i + 10\pi^2)}{18\pi^3} \\
 & \text{res} \left( \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z}, z = \frac{i}{2} \right) = \frac{i(-2 + 7\pi i + 10\pi^2)}{54\pi^3} \\
 & \text{res} \left( \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z}, z = -\frac{i}{2} \right) = -\frac{i(-2 - 7\pi i + 10\pi^2)}{54\pi^3} \\
 & \text{res} \left( \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z}, z = \frac{5i}{6} \right) = -\frac{e^{\frac{5\pi i}{6}} (-18 + 105\pi i + 250\pi^2)}{2250\pi^3} \\
 & \text{res} \left( \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z}, z = -\frac{5i}{6} \right) = \frac{e^{\frac{1}{6}\pi i} (-18 - 205\pi i + 250\pi^2)}{2250\pi^3}
 \end{aligned}$$

于是最后得到

$$\int_{-\infty}^{\infty} \frac{e^{7\pi x}}{(e^{3\pi x} + e^{-3\pi x})^3 (1+x^2)} dx = \pi \sum \text{res} \left( \frac{e^{7\pi z}}{(e^{3\pi z} + e^{-3\pi z})^3 z}, z = 0, \pm\frac{i}{6}, \pm\frac{i}{2}, \pm\frac{5}{6}i \right)$$

$$= \frac{\pi}{8} + \frac{4(837 + 5\pi(161 - 75\sqrt{3}\pi))}{3375\pi^2}$$

□

197. 设  $s > 0, n$  是正整数, 计算积分

$$\int_0^1 \int_0^1 \frac{(xy)^{s-1} - y^n}{(1-xy) \ln(xy)} dx dy$$

解 令  $u = xy, x = \frac{u}{y}, dx = \frac{du}{y}$ ,

$$\begin{aligned} \int_0^1 \int_0^1 \frac{(xy)^{s-1} - y^n}{(1-xy) \ln(xy)} dx dy &= \int_0^1 \int_u^1 \frac{u^{s-1} - y^n}{(1-u) \ln u} \frac{du dy}{y} \\ &= \int_0^1 \frac{1}{(1-u) \ln u} \left( \int_u^1 \left( \frac{u^{s-1}}{y} - y^{n-1} \right) dy \right) du \\ &= \int_0^1 \frac{1}{(1-u) \ln u} \left( u^{s-1} \ln y - \frac{y^n}{n} \right) \Big|_{y=u}^1 du \\ &\quad - \int_0^1 \left( \frac{u^{s-1}}{1-u} + \frac{1}{\ln u} \right) du + \int_0^1 \left( \frac{1-u^n}{n(1-u)} - 1 \right) \frac{du}{\ln u} \\ &= - \int_0^1 \left( \frac{u^{s-1}}{1-u} + \frac{1}{\ln u} \right) du - \frac{1}{n} \sum_{k=1}^{n-1} \int_0^1 \frac{u^k - 1}{\ln u} du \\ &= \frac{\Gamma'(s)}{\Gamma(s)} - \frac{\ln(n!)}{n} \end{aligned}$$

□

198. 求和

$$\sum_{k=1}^{\infty} \left( H_k - \ln k - \gamma - \frac{1}{2k} + \frac{1}{12k^2} \right)$$

解 首先有

$$\sum_{k=1}^n H_k = \sum_{k=1}^n \sum_{j=1}^k \frac{1}{j} = \sum_{j=1}^n \frac{1}{j} \sum_{k=j}^n 1 = \sum_{j=1}^n \frac{n+1-j}{j} = (n+1)H_n - n$$

因此

$$\begin{aligned} \sum_{k=1}^{\infty} \left( H_k - \ln k - \gamma - \frac{1}{2k} + \frac{1}{12k^2} \right) &= (n+1)H_n - n - \ln(n!) - n\gamma - \frac{H_n}{2} \\ &= \left( n + \frac{1}{2} \right) \left( \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \right) - n - n\gamma - \left( n \ln n - n + \frac{\ln(2\pi)}{2} + \frac{\ln n}{2} + O\left(\frac{1}{n}\right) \right) \\ &= \frac{1+\gamma}{2} - \frac{\ln(2\pi)}{2} + O\left(\frac{1}{n}\right) \end{aligned}$$

其中

$$H_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

$$\ln(n!) = n \ln n - n + \frac{\ln(2\pi)}{2} + \frac{\ln n}{2} + O\left(\frac{1}{n}\right)$$

且

$$\sum_{k=1}^{\infty} \frac{1}{12k^2} = \frac{1}{12} \zeta(2) = \frac{\pi^2}{72}$$

因此

$$\sum_{k=1}^{\infty} \left( H_k - \ln k - \gamma - \frac{1}{2k} + \frac{1}{12k^2} \right) = \frac{1+\gamma}{2} - \frac{\ln(2\pi)}{2} + \frac{\pi^2}{72}$$

□

199. 求和

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n+m} \frac{H_{n+m}}{n+m}$$

解 首先有

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx \quad \frac{1}{n} = \int_0^{\infty} e^{-ny} dy$$

于是

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n+m} \frac{H_{n+m}}{n+m} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^1 \frac{1-x^{n+m}}{1-x} dx \int_0^{\infty} (-e^y)^{m+n} dy \\ &= \int_0^1 \int_0^{\infty} \left[ \frac{1}{(1-x)(1+e^y)^2} - \frac{x^2}{(1-x)(x+e^y)^2} \right] dy dx \\ &= \int_0^1 \left( \frac{\ln 2 - \frac{1}{2}}{1-x} - \frac{1}{1-x} \left( \ln(1+x) - 1 + \frac{1}{1+x} \right) \right) dx \\ &= \int_0^1 \frac{\ln 2 - \ln(1+x)}{1-x} dx - \int_0^1 \frac{1}{2(1+x)} dx \\ &= \frac{\pi^2}{12} - \frac{\ln^2 2}{2} - \frac{\ln 2}{2} \end{aligned}$$

□

200. 求和

$$\sum_{n=0}^{\infty} (-1)^n \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{n+k} \right)^2$$

解 首先有

$$\begin{aligned} \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{n+k} \right)^2 &= \left( (-1)^n \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k} \right)^2 \\ &= \left( \sum_{k=n+1}^{\infty} \int_0^1 (-x)^{k-1} dx \right)^2 = \int_0^1 \frac{x^n}{1+x} dx \int_0^1 \frac{y^n}{1+y} dy \end{aligned}$$

于是

$$\sum_{n=0}^{\infty} (-1)^n \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{n+k} \right)^2 = \sum_{n=0}^{\infty} (-1)^n \int_0^1 \frac{x^n}{1+x} dx \int_0^1 \frac{y^n}{1+y} dy$$

$$\begin{aligned}
 &= \int_0^1 \int_0^1 \frac{1}{(1+x)(1+y)(1+xy)} dx dy \\
 &= \int_0^1 \frac{1}{1+x} \left( \int_0^1 \frac{1}{(1+y)(1+xy)} dy \right) dx \\
 &= \int_0^1 \frac{1}{1+x} \frac{\ln 2 - \ln(1+x)}{1-x} dx \\
 &= \frac{1}{2} \int_0^1 \frac{\ln(1+x) - \ln(1-x)}{1+x} dx \\
 &= \frac{\pi^2}{24}
 \end{aligned}$$

□

**201.** 设  $n$  和  $q$  都是正整数, 满足  $2n > q \geq 1$ , 令

$$f(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e^{-t(x_1^{2n} + \cdots + x_q^{2n})}}{1 + x_1^{2n} + \cdots + x_q^{2n}} dx_1 \cdots dx_q$$

证明

$$\lim_{t \rightarrow +\infty} t^{\frac{q}{2n}} f(t) = n^{-q} \Gamma^q \left( \frac{1}{2n} \right)$$

**证明** 令  $x_i^{2n} = u_i, i = 1, \cdots, q$ , 则

$$\begin{aligned}
 f(t) &= 2^q \int_{[0, \infty)^q} \frac{e^{-t(x_1^{2n} + \cdots + x_q^{2n})}}{1 + x_1^{2n} + \cdots + x_q^{2n}} dx_1 \cdots dx_q \\
 &= n^{-q} \int_{[0, \infty)^q} \frac{\prod_{i=1}^q u_i^{\frac{1}{2n}-1} e^{-tu_i}}{1 + u_1 + \cdots + u_q} du_1 \cdots du_q \\
 &= n^{-q} \int_{[0, \infty)^q} \prod_{i=1}^q u_i^{\frac{1}{2n}-1} e^{-tu_i} \left( \int_0^\infty e^{-s(1+u_1+\cdots+u_q)} ds \right) du_1 \cdots du_q \\
 &= n^{-q} \int_0^\infty e^{-s} \left( \int_0^\infty u^{\frac{1}{2n}-1} e^{-(s+t)u} du \right)^q ds \\
 &= n^{-q} \int_0^\infty e^{-s} \left( \frac{1}{(s+t)^{\frac{1}{2n}}} \int_0^\infty r^{\frac{1}{2n}-1} e^{-r} dr \right)^q ds \\
 &= n^{-q} \int_0^\infty e^{-s} \left( \frac{\Gamma(\frac{1}{2n})}{(s+t)^{\frac{1}{2n}}} \right)^q ds \\
 &= n^{-q} \Gamma^q \left( \frac{1}{2n} \right) \int_0^\infty \frac{e^{-s}}{(s+t)^{\frac{q}{2n}}} ds
 \end{aligned}$$

因此

$$\lim_{t \rightarrow +\infty} t^{\frac{q}{2n}} f(t) = n^{-q} \Gamma^q \left( \frac{1}{2n} \right) \lim_{t \rightarrow +\infty} \int_0^\infty e^{-s} \left( \frac{t}{s+t} \right)^{\frac{q}{2n}} ds = n^{-q} \Gamma^q \left( \frac{1}{2n} \right)$$

□

**202.** 求和

$$\sum_{n=1}^{\infty} 4^n \sin^4(2^{-n}\theta)$$

**解** 我们归纳证明对非负整数  $N$ , 有

$$\sum_{n=1}^N 4^n \sin^4(2^{-n}\theta) = 4^N \sin^2(2^{-N}\theta) - \sin^2\theta$$

等式对  $N = 0$  显然, 假定对  $N \geq 0$  成立, 则

$$\begin{aligned} \sum_{n=1}^{N+1} 4^n \sin^4(2^{-n}\theta) &= 4^N \sin^2(2^{-N}\theta) - \sin^2\theta + 4^{N+1} \sin^4(2^{-(N+1)}\theta) \\ &= 4^N \sin^2(2^{-N}\theta) - \sin^2\theta + 4^N (1 - \cos(2^{-N}\theta))^2 \\ &= 4^N (2 - 2\cos(2^{-N}\theta) - \sin^2\theta) \\ &= 4^{N+1} \sin^2(2^{-(N+1)}\theta) - \sin^2\theta \end{aligned}$$

因此令  $N \rightarrow \infty$  得

$$\sum_{n=1}^{\infty} 4^n \sin^4(2^{-n}\theta) = \theta^2 - \sin^2\theta$$

□

**203. 证明**

$$\prod_{n=2}^{\infty} \left( \left( \frac{n^2-1}{n^2} \right)^{2(n^2-1)} \left( \frac{n+1}{n-1} \right)^n \right) = \pi$$

**解** 先考虑有限乘积

$$\begin{aligned} \prod_{n=2}^N \left( \frac{n+1}{n-1} \right)^n &= \frac{3^2}{1^2} \cdot \frac{4^3}{2^3} \cdot \frac{5^4}{3^4} \cdots \frac{N^{N-1}}{(N-2)^{N-1}} \cdot \frac{(N+1)^N}{(N-1)^N} \\ &= \frac{N^{N-1} (N+1)^N}{2((N-1)!)^2} = \frac{N^{2N+1}}{2(N!)^2} \left( 1 + \frac{1}{N} \right)^N \\ &= \frac{e^{2N+1}}{4\pi} (1 + o(1)) \end{aligned}$$

再考虑另一个有限乘积

$$\begin{aligned} \prod_{n=2}^N \left( \frac{n^2-1}{n^2} \right)^{n^2-1} &= \prod_{n=2}^N \frac{(n-1)^{n^2-1} (n+1)^{n^2-1}}{n^{2(n^2-1)}} \\ &= \frac{1^3 \cdot 3^3}{2^6} \cdot \frac{2^8 \cdot 4^8}{3^{16}} \cdot \frac{3^{15} \cdot 5^{15}}{4^{30}} \cdots \frac{(N-1)^{N^2-1} (N+1)^{N^2-1}}{N^{2(N^2-1)}} \\ &= \frac{((N-1)!)^2 (N+1)^{N^2-1}}{N^{N^2+2N+2}} = \frac{(N!)^2}{N^{2N+1}} \left( 1 + \frac{1}{N} \right)^{N^2-1} \\ &= \frac{2\pi}{e^{2N}} e^{N-\frac{1}{2}} (1 + o(1)) = \frac{2\pi}{e^{N+\frac{1}{2}}} (1 + o(1)) \end{aligned}$$

因此最后得到

$$\prod_{n=2}^{\infty} \left( \left( \frac{n^2-1}{n^2} \right)^{2(n^2-1)} \left( \frac{n+1}{n-1} \right)^n \right) = \left( \prod_{n=2}^{\infty} \left( \frac{n^2-1}{n^2} \right)^{n^2-1} \right)^2 \prod_{n=2}^{\infty} \left( \frac{n+1}{n-1} \right)^n$$

$$= \frac{4\pi^2}{e^{2N+1}} (1 + o(1)) \frac{e^{2N+1}}{4\pi} (1 + o(1)) \rightarrow \pi$$

□

204. 证明

$$\prod_{n=2}^{\infty} \left( \frac{1}{e} \left( \frac{n^2}{n^2-1} \right)^{n^2-1} \right) = \frac{e\sqrt{e}}{2\pi}$$

**证明** 取对数后得

$$\begin{aligned} \sum_{n=2}^{\infty} \left( -1 + (n^2-1) \ln \left( \frac{1}{1-\frac{1}{n^2}} \right) \right) &= \sum_{n=2}^{\infty} \left( -1 + (n^2-1) \sum_{k=1}^{\infty} \frac{1}{kn^{2k}} \right) \\ &= \sum_{n=2}^{\infty} \left( -1 + \sum_{k=1}^{\infty} \frac{1}{kn^{2k-2}} - \sum_{k=1}^{\infty} \frac{1}{kn^{2k}} \right) \\ &= \sum_{k=2}^{\infty} \left( \sum_{n=2}^{\infty} \frac{1}{(k+1)n^{2k}} - \sum_{k=1}^5 \frac{1}{kn^{2k}} \right) \\ &= \sum_{k=2}^{\infty} \left( \sum_{n=2}^{\infty} \frac{1}{(k+1)n^{2k}} - \sum_{k=1}^5 \frac{1}{kn^{2k}} \right) \end{aligned}$$

其中

$$\sum_{k=1}^{\infty} \frac{\zeta(2k) - 1}{k} = \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \frac{1}{kn^{2k}} = \sum_{n=2}^{\infty} \ln \left( \frac{n^2}{n^2-1} \right) = \ln 2$$

□

205. 设

$$L = \lim_{n \rightarrow \infty} \int_0^1 \sqrt[n]{x^n + (1-x)^n} dx.$$

(a) 求  $L$ .

(b) 求极限

$$\lim_{n \rightarrow \infty} n^2 \left( \int_0^1 \sqrt[n]{x^n + (1-x)^n} dx - L \right)$$

**解**

(a) 令  $t = \frac{x}{1-x}$ ,  $x = \frac{t}{1+t}$ ,  $dx = \frac{dt}{(1+t)^2}$ , 且

$$\int_0^1 \sqrt[n]{x^n + (1-x)^n} dx = 2 \int_0^{\frac{1}{2}} (1-x) \sqrt[n]{1 + \left( \frac{x}{1-x} \right)^n} dx = 2 \int_0^1 \frac{\sqrt[n]{1+t^n}}{(1+t)^3} dt$$

对任意  $t \in [0, 1]$ , 且

$$\frac{\sqrt[n]{1+t^n}}{(1+t)^3} \rightarrow \frac{1}{(1+t)^3} \quad 0 \leq \frac{\sqrt[n]{1+t^n}}{(1+t)^3} \leq \frac{2}{(1+t)^3}$$

因此, 由 Lebesgue 控制收敛定理,

$$L = \lim_{n \rightarrow \infty} \int_0^1 \sqrt[n]{x^n + (1-x)^n} dx = 2 \int_0^1 \frac{dt}{(1+t)^3} = \frac{3}{4}$$

(b) 令  $s = t^n$ , 则  $t = s^{\frac{1}{n}}$ ,  $dt = \frac{1}{n}s^{\frac{1}{n}-1}$ , 且

$$\begin{aligned} n^2 \left( \int_0^1 \sqrt[n]{x^n + (1-x)^n} dx - L \right) &= 2n^2 \int_0^1 \frac{\sqrt[n]{1+t^n} - 1}{(1+t)^3} dt \\ &= 2n \int_0^1 \frac{(1+s)^{\frac{1}{n}} - 1}{\left(1+s^{\frac{1}{n}}\right)^3} s^{\frac{1}{n}-1} ds \\ &= 2n \int_0^1 \frac{\exp\left(\frac{1}{n} \ln(1+s)\right) - 1}{\left(1+s^{\frac{1}{n}}\right)^3} s^{\frac{1}{n}-1} ds \\ &= 2 \int_0^1 \frac{\exp(h_n(s)) \ln(1+s)}{\left(1+s^{\frac{1}{n}}\right)^3} s^{\frac{1}{n}-1} ds \end{aligned}$$

其中根据中值定理,  $0 < h_n(s) < \frac{1}{n} \ln(1+s)$ , 对  $s \in (0, 1]$ ,

$$\frac{\exp(h_n(s)) \ln(1+s)}{\left(1+s^{\frac{1}{n}}\right)^3} s^{\frac{1}{n}-1} \rightarrow \frac{\ln(1+s)}{8s} \quad 0 \leq \frac{\exp(h_n(s)) \ln(1+s)}{\left(1+s^{\frac{1}{n}}\right)^3} \leq \frac{2 \ln(1+s)}{s}$$

再由 Lebesgue 控制收敛定理得

$$\lim_{n \rightarrow \infty} n^2 \left( \int_0^1 \sqrt[n]{x^n + (1-x)^n} dx - L \right) = \frac{1}{4} \int_0^1 \frac{\ln(1+s)}{s} ds = \frac{\pi^2}{48}$$

□

## 206. 计算积分

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\ln(2 - \sin \theta \cos \varphi) \sin \theta}{2 - 2 \sin \theta \cos \varphi + \sin^2 \theta \cos^2 \varphi} d\theta d\varphi$$

解

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\ln(2 - \sin \theta \cos \varphi) \sin \theta}{1 + (1 - \sin \theta \cos \varphi)^2} d\theta d\varphi \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\ln(2-x)}{1 + (1-x)^2} \frac{1}{\sqrt{1-x^2-y^2}} dy dx \\ &= \int_0^1 \frac{\ln(2-x)}{1 + (1-x)^2} \left[ \arctan \left( \frac{y}{\sqrt{1-x^2-y^2}} \right) \right] \Big|_{y=0}^{\sqrt{1-x^2}} dx \\ &= \frac{\pi}{2} \int_0^1 \frac{\ln(2-x)}{1 + (1-x)^2} dx = \frac{\pi}{2} \int_0^1 \frac{\ln(1+x)}{1+x^2} dx \\ &= \frac{\pi^2}{16} \ln 2 \end{aligned}$$

□

## 207. 计算主值积分

$$\int_0^\infty \frac{\ln(1+\cos x)}{e^x + 1} dx$$



**解** 首先有

$$\begin{aligned}\int_0^\infty \frac{\ln(1+\cos x)}{e^x+1} dx &= -\ln^2 2 - 2 \sum_{n=1}^\infty \frac{(-1)^n}{n} \int_0^\infty \frac{\cos(nx)}{e^x+1} dx \\ &= -\ln^2 2 + 2 \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(-1)^{m+n} m}{n(m^2+n^2)}\end{aligned}$$

于是

$$\begin{aligned}2I &= -2\ln^2 2 + 2 \left[ \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(-1)^{m+n} m}{n(m^2+n^2)} + \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(-1)^{m+n} n}{m(m^2+n^2)} \right] \\ &= -2\ln^2 2 + 2 \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(-1)^{m+n}}{m^2+n^2} \left( \frac{n}{m} + \frac{m}{n} \right) \\ &= -2\ln^2 2 + 2 \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(-1)^{m+n}}{mn} \\ &= -2\ln^2 2 + 2 \left( \sum_{n=1}^\infty \frac{(-1)^n}{n} \right)^2 = 0\end{aligned}$$

□

## 208. 计算积分

$$\int_0^1 \cos(\pi x) \ln \left( \frac{\Gamma(x)}{\Gamma^2(\frac{x}{2})} \right) dx$$

**解** 利用傅里叶展开公式

$$\begin{aligned}\ln \left( \frac{\Gamma(x)}{\Gamma^2(\frac{x}{2})} \right) &= \sum_{n=1}^\infty \frac{\ln(2n\pi) + \gamma}{n\pi} \sin(2n\pi x) + \sum_{n=1}^\infty \frac{\cos(2n\pi x)}{2n} + \frac{\ln(2\pi)}{2} \\ &\quad - 2 \left[ \sum_{n=1}^\infty \frac{\ln(2n\pi) + \gamma}{n\pi} \sin(n\pi x) + \sum_{n=1}^\infty \frac{\cos(n\pi x)}{2n} + \frac{\ln(2\pi)}{2} \right]\end{aligned}$$

再利用三角积分的正交性有

$$\int_0^1 \cos(\pi x) \sin(2n-1)\pi x dx = 0, \int_0^1 \cos(\pi x) \cos(n\pi x) dx = 0 \quad (n \neq 1)$$

于是

$$\begin{aligned}I &= \int_0^1 \cos(\pi x) \left( -\frac{\ln 2}{\pi} \sum_{n=1}^\infty \frac{\sin(2n\pi x)}{n} - \cos(\pi x) \right) dx \\ &= -\frac{1}{2} - \frac{\ln 2}{\pi} \int_0^1 \frac{\pi - 2\pi x}{2} \cos(\pi x) dx = -\frac{1}{2} - \frac{2\ln 2}{\pi^2}\end{aligned}$$

□

## 209. 计算积分

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sqrt{1 - \tan x} dx$$

解

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sqrt{1 - \tan x} dx &= \int_0^1 \frac{\sqrt{y(1-y)}}{1+y^2} dy \quad y = \tan x \\
 &= \int_0^\infty \frac{\sqrt{t}}{(1+t)(1+2t+2t^2)} dt \quad t = \frac{y}{1-y} \\
 &= \int_0^\infty \frac{2z^2}{(1+z^2)(1+2z^2+2z^4)} dz \\
 &= 2 \int_0^\infty \left( \frac{2z^2}{1+2z^2+2z^4} + \frac{1}{1+2z^2+2z^4} - \frac{1}{1+z^2} \right) dz \\
 &= \int_{-\infty}^\infty \left( \frac{2z^2}{1+2z^2+2z^4} + \frac{1}{1+2z^2+2z^4} - \frac{1}{1+z^2} \right) dz \\
 &= I_1 + I_2 - \pi
 \end{aligned}$$

其中

$$\begin{aligned}
 I_1 &= \int_{-\infty}^\infty \frac{2z^2}{1+2z^2+2z^4} dz = \int_{-\infty}^\infty \frac{2}{1+\frac{1}{2z^2}+z^2} dz \\
 &= \int_{-\infty}^\infty \frac{1}{\left(z - \frac{1}{\sqrt{2}z}\right)^2 + 1 + \sqrt{2}} dz = \int_{-\infty}^\infty \frac{1}{z^2 + 1 + \sqrt{2}} dz \\
 &= \frac{\pi}{\sqrt{1+\sqrt{2}}} \\
 I_2 &= \frac{1}{2} \int_{-\infty}^\infty \frac{1}{z^4 + z^2 + \frac{1}{2}} dz = \pi \sqrt{\frac{\sqrt{2}-1}{2}}
 \end{aligned}$$

于是

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sqrt{1 - \tan x} dx = \frac{\pi}{\sqrt{2}} \sqrt{\frac{2+\sqrt{2}}{2}} - \pi$$

□

## 210. 计算积分

$$\int_0^{\frac{\pi}{2}} x \csc^2(x) \arctan(a \tan x) dx$$

解 首先作换元  $u = \tan x$  得

$$\int_0^{\frac{\pi}{2}} x \csc^2(x) \arctan(a \tan x) dx = \int_0^\infty \frac{\tan^{-1} u \tan^{-1}(au)}{u^2} du$$

记

$$I(\alpha, \beta) = \int_0^\infty \frac{\tan^{-1}(\alpha u) \tan^{-1}(\beta u)}{u^2} du$$

则

$$\begin{aligned}
 \frac{\partial^2 I}{\partial \alpha \partial \beta} &= \int_0^\infty \frac{du}{(1+\alpha^2 u^2)(1+\beta^2 u^2)} \\
 &= \frac{1}{\alpha^2 - \beta^2} \int_0^\infty \left( \frac{\alpha^2}{1+\alpha^2 u^2} - \frac{\beta^2}{1+\beta^2 u^2} \right) du
 \end{aligned}$$

$$= \frac{\pi}{2(\alpha + \beta)}$$

于是原式求两次积分即可.

□

## 211. 计算积分

$$\int_0^1 \frac{\arctan x}{x} \frac{1-x^4}{1+x^4} dx$$

**解** 记

$$I(a) = \int_0^1 \frac{\arctan(ax)}{x} \frac{1-x^4}{1+x^4} dx$$

则

$$\begin{aligned} I'(a) &= \int_0^1 \frac{1}{x} \frac{x}{1+(ax)^2} \frac{1-x^4}{1+x^4} dx \\ &= \int_0^1 \left( \frac{1}{1+a^2x^2} \frac{a^4-1}{a^4+1} + \frac{1}{1+x^4} \frac{2-2a^2x^2}{1+a^4} \right) dx \end{aligned}$$

于是

$$\begin{aligned} I(a) &= \int_0^1 \left( \int_0^1 \left( \frac{1}{1+a^2x^2} \frac{1-a^4}{a^4+1} + \frac{1}{1+x^4} \frac{2-2a^2x^2}{1+a^4} \right) dx \right) da \\ &= I(a) = \int_0^1 \left( - \int_0^1 \frac{1}{1+a^2x^2} \frac{1-a^4}{a^4+1} da + \int_0^1 \frac{1}{1+x^4} \frac{2-2a^2x^2}{1+a^4} da \right) dx \\ &= -I(a) + \int_0^1 \left( \int_0^1 \frac{1}{1+x^4} \frac{2-2a^2x^2}{1+a^4} da \right) dx \end{aligned}$$

因此

$$\begin{aligned} I(a) &= \int_0^1 \left( \int_0^1 \frac{1}{1+x^4} \frac{2-2a^2x^2}{1+a^4} da \right) dx \\ &= \left( \int_0^1 \frac{1}{1+x^4} dx \right)^2 - \left( \int_0^1 \frac{x^2}{1+x^4} dx \right)^2 \\ &= \left( \int_0^1 \frac{1+x^2}{1+x^4} dx \right) \left( \int_0^1 \frac{1-x^2}{1+x^4} dx \right) = \frac{\pi}{4} \ln(\sqrt{2}+1) \end{aligned}$$

一般形式为

$$\begin{aligned} I(n) &= \int_0^1 \frac{\arctan x}{x} \frac{1-x^n}{1+x^n} dx = G - \frac{\pi}{2n} + \frac{2}{n} \int_0^{\frac{\pi}{4}} \ln(1+\tan^n x) dx \\ &= G - \frac{\pi}{2n} \ln 2 + \frac{2}{n} \int_0^{\frac{\pi}{4}} \ln(\cos^n x + \sin^n x) dx - \frac{2}{n} \int_0^{\frac{\pi}{4}} \ln(\cos^n x) dx \\ &= G - \frac{\pi}{2n} \ln 2 + \frac{2}{n} \int_0^{\frac{\pi}{4}} \ln(\cos^n x + \sin^n x) dx - 2 \left( \frac{G}{2} - \frac{\pi}{4} \ln 2 \right) \\ &= \frac{n-1}{2n} \pi \ln 2 + \frac{2}{n} \int_0^{\frac{\pi}{4}} \ln(\cos^n x + \sin^n x) dx \end{aligned}$$

□

## 212. 计算积分

$$\int_0^\pi \sqrt{\tan \frac{\theta}{2}} \ln(\sin \theta) d\theta$$

解

$$\begin{aligned} \int_0^\pi \sqrt{\tan \frac{\theta}{2}} \ln(\sin \theta) d\theta &= \int_0^\infty \frac{2\sqrt{t}}{1+t^2} \ln\left(\frac{2t}{1+t^2}\right) dt \quad t = \tan \frac{\theta}{2} \\ &= \int_0^\infty \frac{2\sqrt{1/t}}{1+t^2} \ln\left(\frac{2}{t+1/t}\right) dt \\ &= \int_0^\infty \frac{\sqrt{1/t} + \sqrt{1/t^3}}{t+1/t} \ln\left(\frac{2}{t+1/t}\right) dt \\ &= \int_{-\infty}^\infty \frac{2}{x^2+2} \ln\left(\frac{2}{x^2+2}\right) dx \\ &= 2\sqrt{2} \int_0^{\frac{\pi}{2}} \ln(\cos^2 u) du = -2\sqrt{2}\pi \ln 2 \quad (x = \sqrt{2} \tan u) \end{aligned}$$

类似的有

$$\int_0^\pi \sqrt{\tan \frac{\theta}{2}} \ln^2(\sin \theta) d\theta = \int_{-\infty}^\infty \frac{2}{x^2+2} \ln^2\left(\frac{x^2+2}{2}\right) dx = \frac{\sqrt{2}}{3}\pi^3 + 4\sqrt{2}\pi \ln^2 2$$

□

## 213. 证明下列两个积分等式:

$$\begin{aligned} (1) \quad & \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{1}{2}x^2} dx = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{z^2}{2\sin^2 x}} dz; \\ (2) \quad & \left( \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{1}{2}x^2} dx \right)^2 = \frac{1}{\pi} \int_0^{\frac{\pi}{4}} e^{-\frac{z^2}{2\sin^2 x}} dz. \end{aligned}$$

**证明** 我们只证明 (2) 式, (1) 式同理. (2) 式等价于

$$\frac{1}{2} \left( \int_z^\infty e^{-\frac{1}{2}x^2} dx \right)^2 - \int_0^{\frac{\pi}{4}} e^{-\frac{z^2}{2\sin^2 x}} dx = 0$$

令  $f(z) = \frac{1}{2} \left( \int_z^\infty e^{-\frac{1}{2}x^2} dx \right)^2 - \int_0^{\frac{\pi}{4}} e^{-\frac{z^2}{2\sin^2 x}} dx$ , 则

$$\begin{aligned} f'(z) &= -e^{-\frac{1}{2}z^2} \int_z^\infty e^{-\frac{1}{2}x^2} dx - \int_0^{\frac{\pi}{4}} e^{-\frac{z^2}{2\sin^2 x}} \csc^2 x (-z \csc^2 x) dx \\ &= -e^{-\frac{1}{2}z^2} \int_z^\infty e^{-\frac{1}{2}x^2} dx - \int_0^{\frac{\pi}{4}} e^{-\frac{z^2}{2}(\cot^2 x + 1)} z d(\cot x) \\ &= -e^{-\frac{1}{2}z^2} \int_z^\infty e^{-\frac{1}{2}x^2} dx + e^{-\frac{1}{2}z^2} \int_1^\infty e^{-\frac{z^2}{2}u^2} z du = 0 \end{aligned}$$

因此  $f(z) = f(0) = 0$ .

□

## 214. 设 $p, q > 0$ , 计算积分

$$\int_0^\infty \frac{e^{-qx} \sin(px)}{1+e^{-qx}} dx$$

解

$$\begin{aligned} \int_0^\infty \frac{e^{-qx} \sin(px)}{1+e^{-qx}} dx &= \int_0^\infty e^{-qx} \sin(px) \sum_{k=0}^\infty (-1)^k e^{-kqx} dx \\ &= \Im \sum_{k=0}^\infty (-1)^k \int_0^\infty e^{-[(k+1)q-ip]x} dx = \Im \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)q-ip} \\ &= p \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^2 q^2 + p^2} = \frac{1}{2p} \left[ 1 - \frac{p^2}{q^2} \sum_{k=-\infty}^\infty \frac{(-1)^k}{k^2 + \frac{p^2}{q^2}} \right] \\ &= \frac{1}{2p} - \frac{\pi}{2q} \operatorname{csc} h \left( \frac{\pi p}{q} \right) \end{aligned}$$

其中最后一步运用了  $\sum_{k=-\infty}^\infty \frac{(-1)^k}{k^2 + a^2} = \frac{\pi}{a} \operatorname{csc} h(\pi a) (a > 0)$ .

□

## 215. 计算积分

$$\int_0^1 \frac{\ln(1-x) \ln^2(1+x)}{x} dx$$

解 令  $a = \ln(1-x)$ ,  $b = \ln(1+x)$ , 利用公式

$$6ab = (a+b)^3 + (a-b)^3 - 2a^3$$

可得

$$I := \int_0^1 \frac{\ln(1-x) \ln^2(1+x)}{x} dx = \frac{I_1 + I_2 - 2I_3}{6}$$

其中

$$\begin{aligned} I_1 &= \int_0^1 \frac{\ln^3(1-x^2)}{x} dx = \frac{1}{2} \int_0^1 \frac{\ln^3 t}{1-t} dt \quad (t = 1-x^2) \\ I_2 &= \int_0^1 \frac{\ln^3\left(\frac{1-x}{1+x}\right)}{x} dx = 2 \int_0^1 \frac{\ln^3 t}{(1-t)(1+t)} dt \\ &= \int_0^1 \frac{\ln^3 t}{1-t} dt + \int_0^1 \frac{\ln^3 t}{1+t} dt \quad \left(t = \frac{1-x}{1+x}\right) \\ I_3 &= \int_0^1 \frac{\ln^3(1-x)}{x} dx = \int_0^1 \frac{\ln^3 t}{1-t} dt \quad (t = 1-x) \end{aligned}$$

因此可得原积分

$$\begin{aligned} I &= \frac{1}{6} \left( \left( \frac{1}{2} + 1 - 2 \right) \int_0^1 \frac{\ln^3 t}{1-t} dt + \int_0^1 \frac{\ln^3 t}{1+t} dt \right) \\ &= \frac{1}{6} \left( -\frac{1}{2} \sum_{n=0}^\infty \int_0^1 t^n \ln^3 t dt + \sum_{n=0}^\infty (-1)^n \int_0^1 t^n \ln^3 t dt \right) \\ &= \frac{1}{6} \left( \frac{6}{2} \sum_{n=0}^\infty \frac{1}{(n+1)^4} - 6 \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^4} \right) \\ &= \frac{\zeta(4)}{2} - \frac{7\zeta(4)}{8} = -\frac{\pi^4}{240} \end{aligned}$$

其中由分部积分不难得到  $\int_0^1 t^n \ln^m t dt = \frac{(-1)^m m!}{(n+1)^{m+1}}$ .

□

## 216. 求和

$$S = \sum_{n=0}^{\infty} \left[ \left( 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} \right) \frac{1}{9^n (2n+1)} \right]$$

**解** 首先有

$$\begin{aligned} S &= 3 \sum_{n=0}^{\infty} \frac{1}{3^{2n+1} (2n+1)} \int_0^1 \frac{1-x^{2n+2}}{1-x^2} dx \\ &= \frac{3}{2} \int_0^1 \frac{\ln 2 - x \ln \frac{3+x}{3-x}}{1-x^2} dx = \frac{3}{2} \int_0^1 \frac{\ln 2 - x \ln 2 + x (\ln 2 - \ln \frac{3+x}{3-x})}{1-x^2} dx \\ &= \frac{3}{2} \ln^2 2 - \frac{3}{4} \int_0^1 \left( \ln 2 - \ln \frac{3+x}{3-x} \right) d(\ln(1-x^2)) \\ &= \frac{3}{2} \ln^2 2 - \frac{3}{4} \int_0^1 [\ln(1-x) + \ln(1+x)] \left( \frac{1}{3+x} + \frac{1}{3-x} \right) dx \\ &= \frac{3}{2} \ln^2 2 - \frac{3}{4} I \end{aligned}$$

其中

$$\begin{aligned} \int_0^1 \frac{\ln(1-x)}{3-x} dx &= \int_2^3 \frac{\ln(x-2)}{x} dx = \int_2^3 \frac{\ln(1-\frac{2}{x}) + \ln x}{x} dx \\ &= \int_2^3 \frac{\ln(1-\frac{2}{x})}{x} dx + \frac{1}{2} \ln^2 3 - \frac{1}{2} \ln^2 2 \\ &= \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln(1-2x)}{x} dx + \frac{1}{2} \ln^2 3 - \frac{1}{2} \ln^2 2 \\ &= \text{Li}_2\left(\frac{2}{3}\right) - \text{Li}_2(1) + \frac{1}{2} \ln^2 3 - \frac{1}{2} \ln^2 2 \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{\ln(1+x)}{3-x} dx &= \int_2^3 \frac{\ln(4-x)}{x} dx = \int_2^3 \frac{\ln(1-\frac{x}{4}) + \ln 4}{x} dx \\ &= \text{Li}_2\left(\frac{1}{2}\right) - \text{Li}_2\left(\frac{3}{4}\right) + \ln 4 \ln 3 - \ln 4 \ln 2 \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{\ln(1-x)}{3+x} dx &= \int_3^4 \frac{\ln(4-x)}{x} dx = \int_3^4 \frac{\ln(1-\frac{x}{4}) + \ln 4}{x} dx \\ &= \text{Li}_2\left(\frac{3}{4}\right) - \text{Li}_2(1) + \ln 4 \ln 4 - \ln 4 \ln 3 \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{\ln(1+x)}{3+x} dx &= \int_3^4 \frac{\ln(x-2)}{x} dx = \int_3^4 \frac{\ln(1-\frac{2}{x}) + \ln x}{x} dx \\ &= \int_{\frac{1}{4}}^{\frac{1}{3}} \frac{\ln(1-2x)}{x} dx + \frac{1}{2} \ln^2 4 - \frac{1}{2} \ln^2 3 \end{aligned}$$

$$= \text{Li}_2\left(\frac{1}{2}\right) - \text{Li}_2\left(\frac{2}{3}\right) + \frac{1}{2} \ln^2 4 - \frac{1}{2} \ln^2 3$$

因此

$$\begin{aligned} I &= \text{Li}_2\left(\frac{2}{3}\right) - \text{Li}_2(1) + \frac{1}{2} \ln^2 3 - \frac{1}{2} \ln^2 2 + \text{Li}_2\left(\frac{1}{2}\right) - \text{Li}_2\left(\frac{3}{4}\right) + \ln 4 \ln 3 - \ln 4 \ln 2 \\ &\quad + \text{Li}_2\left(\frac{3}{4}\right) - \text{Li}_2(1) + \ln 4 \ln 4 - \ln 4 \ln 3 + \text{Li}_2\left(\frac{1}{2}\right) - \text{Li}_2\left(\frac{2}{3}\right) + \frac{1}{2} \ln^2 4 - \frac{1}{2} \ln^2 3 \\ &= -2\text{Li}_2(1) - \frac{1}{2} \ln^2 2 + 2\text{Li}_2\left(\frac{1}{2}\right) - 2\ln^2 2 + 4\ln^2 2 + 2\ln^2 2 \\ &= -\frac{\pi^2}{3} + \frac{7}{2} \ln^2 2 + 2\frac{\pi^2 - 6\ln^2 2}{12} = -\frac{\pi^2}{6} + \frac{5}{2} \ln^2 2 \end{aligned}$$

因此

$$S = \frac{3}{2} \ln^2 2 - \frac{3}{4} I = \frac{\pi^2}{8} - \frac{3}{8} \ln^2 2$$

□

## 217. 计算积分

$$\int_0^1 \frac{\arctan x}{1+x^2} \ln\left(\frac{1+x^2}{1+x}\right) dx$$

**解** 令  $t = \arctan x$ ,

$$\begin{aligned} \int_0^1 \frac{\arctan x}{1+x^2} \ln\left(\frac{1+x^2}{1+x}\right) dx &= -\int_0^{\frac{\pi}{4}} t \ln[\cos t (\cos t + \sin t)] dt \\ &= -\frac{1}{2} \int_0^{\frac{\pi}{4}} t \ln[\cos^2 t (\cos t + \sin t)^2] dt \\ &= -\frac{1}{2} \int_0^{\frac{\pi}{4}} t \ln\left[\frac{1+\cos 2t}{2} (1+\sin 2t)\right] dt \\ &= -\frac{1}{2} \int_0^{\frac{\pi}{4}} t \ln[(1+\cos 2t)(1+\sin 2t)] dt + \frac{1}{64} \pi^2 \ln 2 \\ &= -\frac{1}{8} \int_0^{\frac{\pi}{2}} t \ln[(1+\cos t)(1+\sin t)] dt + \frac{1}{64} \pi^2 \ln 2 \\ &= -\frac{\pi}{32} \int_0^{\frac{\pi}{2}} \ln[(1+\cos t)(1+\sin t)] dt + \frac{1}{64} \pi^2 \ln 2 \\ &= -\frac{\pi}{16} \int_0^{\frac{\pi}{2}} \ln(1+\cos t) dt + \frac{1}{64} \pi^2 \ln 2 \end{aligned}$$

而

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \ln(1+\cos t) dt &= \int_0^{\frac{\pi}{2}} \ln\left(2\cos^2 \frac{t}{2}\right) dt = \frac{\pi}{2} \ln 2 + 2 \int_0^{\frac{\pi}{2}} \ln\left(\cos \frac{t}{2}\right) dt \\ &= \frac{\pi}{2} \ln 2 + 4 \int_0^{\frac{\pi}{4}} \ln(\cos t) dt = 2G - \frac{\pi}{2} \ln 2 \end{aligned}$$

因此

$$\int_0^1 \frac{\arctan x}{1+x^2} \ln\left(\frac{1+x^2}{1+x}\right) dx = \frac{1}{64} \pi (3\pi \ln 2 - 8G).$$

□

## 218. 计算积分

$$\int_0^{\pi/3} ((\sqrt{3} \cos \theta - \sin \theta) \sin \theta)^{\frac{1}{2}} \cos \theta d\theta$$

**解** 首先有

$$\begin{aligned} (\sqrt{3} \cos \theta - \sin \theta) \sin \theta &= 2 \sin(\frac{\pi}{3} - \theta) \sin \theta \\ &= \cos(\frac{\pi}{3} - 2\theta) - \cos(\frac{\pi}{3}) = \frac{1}{2} - 2 \sin^2(\frac{\pi}{6} - \theta) \\ &= \frac{1}{2} - 2 \sin^2(\theta - \frac{\pi}{6}) \end{aligned}$$

于是

$$\begin{aligned} &\int_0^{\pi/3} ((\sqrt{3} \cos \theta - \sin \theta) \sin \theta)^{\frac{1}{2}} \cos \theta d\theta \\ &= \int_0^{\pi/3} \left( \frac{1}{2} - 2 \sin^2(\theta - \frac{\pi}{6}) \right)^{\frac{1}{2}} \cos \theta d\theta \\ &= \frac{1}{\sqrt{2}} \int_{-\pi/6}^{\pi/6} (1 - 4 \sin^2 \theta)^{\frac{1}{2}} \cos(\theta + \frac{\pi}{6}) d\theta \\ &= \frac{1}{\sqrt{2}} \int_{-\pi/6}^{\pi/6} (1 - 4 \sin^2 \theta)^{\frac{1}{2}} \left( \frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta \right) d\theta \\ &= \frac{\sqrt{3}}{2\sqrt{2}} \int_{-\pi/6}^{\pi/6} (1 - 4 \sin^2 \theta)^{\frac{1}{2}} \cos \theta d\theta \\ &= \frac{\sqrt{3}}{2\sqrt{2}} \int_{-\pi/6}^{\pi/6} (1 - 4 \sin^2 \theta)^{\frac{1}{2}} d \sin \theta \\ &= \frac{\sqrt{3}}{4\sqrt{2}} \int_{-\pi/2}^{\pi/2} (1 - \sin^2 \theta)^{\frac{1}{2}} d \sin \theta \\ &= \frac{\sqrt{3}}{4\sqrt{2}} \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = \frac{\pi\sqrt{3}}{8\sqrt{2}} \end{aligned}$$

□

## 219. 求和

$$\sum_{n=1}^{\infty} \frac{H_n - H_{2n}}{n(2n+1)}$$

**解** 首先不难得到

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n - H_{2n}}{n(2n+1)} &= 2 \sum_{n=1}^{\infty} (H_n - H_{2n}) \left( \frac{1}{2n} - \frac{1}{2n+1} \right) \\ &= 2 \sum_{n=1}^{\infty} \left( \frac{1}{2n} - \frac{1}{2n+1} \right) \int_0^1 \frac{x^{2n} - x^n}{1-x} dx \\ &= \int_0^1 \frac{\sqrt{x} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} - \ln \frac{1+x}{1-x} - \ln(1+x)}{1-x} dx \\ &\quad + \int_0^1 \left( \frac{1}{\sqrt{x}} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} - \frac{1}{x} \ln \frac{1+x}{1-x} \right) dx \end{aligned}$$



其中

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} dx &= 2 \int_0^1 \ln \frac{1+t}{1-t} dt = 4 \ln 2 \\ \int_0^1 \frac{1}{x} \ln \frac{1+x}{1-x} dx &= \text{Li}_2(1) - \text{Li}_2(-1) = \frac{\pi^2}{4}\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{(\sqrt{x}-1)}{1-x} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} dx &= -2 \int_0^1 \frac{t}{1+t} \ln \frac{1+t}{1-t} dt \\ &= 2 \int_0^1 \left[ \frac{\ln(1+t)}{1+t} - \frac{\ln(1-t)}{1+t} \right] dt - 2 \int_0^1 \ln \frac{1+t}{1-t} dt \\ &= \ln^2 2 + 2\text{Li}_2\left(\frac{1}{2}\right) - 4 \ln 2 = \frac{\pi^2}{6} - 4 \ln 2\end{aligned}$$

又

$$\begin{aligned}\int_0^1 \frac{\ln \frac{1+\sqrt{x}}{1-\sqrt{x}} - \ln \frac{1+x}{1-x} - \ln(1+x)}{1-x} dx &= 2 \int_0^1 \frac{1}{1-x} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} dx \\ &= 2 \int_0^1 \frac{\ln(1+\sqrt{x}) - \ln 2}{1-x} dx - 2 \int_0^1 \frac{\ln(1+x) - \ln 2}{1-x} dx\end{aligned}$$

其中

$$\begin{aligned}\int_0^1 \frac{\ln(1+\sqrt{x}) - \ln 2}{1-x} dx &= 2 \int_0^1 \frac{t}{1-t^2} \ln \frac{1+t}{2} dt \\ &= \int_0^1 \frac{1}{1-t} \ln \frac{1+t}{2} dt - \int_0^1 \frac{1}{1+t} \ln \frac{1+t}{2} dt \\ &= -\text{Li}_2\left(\frac{1}{2}\right) + \frac{1}{2} \ln^2 2 = \ln^2 2 - \frac{\pi^2}{12} \\ \int_0^1 \frac{\ln(1+x) - \ln 2}{1-x} dx &= -\text{Li}_2\left(\frac{1}{2}\right) = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}\end{aligned}$$

最后得到

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n - H_{2n}}{n(2n+1)} &= 4 \ln 2 - \frac{\pi^2}{4} + \frac{\pi^2}{6} - 4 \ln 2 + 2 \left( \ln^2 2 - \frac{\pi^2}{12} \right) - 2 \left( \frac{\ln^2 2}{2} - \frac{\pi^2}{12} \right) \\ &= \ln^2 2 - \frac{\pi^2}{12}\end{aligned}$$

□

## 220. 计算积分

**解** 利用基本积分

$$\int_0^{\infty} \frac{x^{\alpha} dx}{1+2x \cos \beta + x^2}$$

$$\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$$

可得

$$\int_0^{\infty} \frac{x^a}{x^2 + 2(\cos b)x + 1} dx$$

$$\begin{aligned}
 &= \int_0^\infty \frac{x^a}{(x + e^{ib})(x + e^{-ib})} dx \\
 &= \frac{1}{-e^{ib} + e^{-ib}} \int_0^\infty \frac{x^a}{e^{ib} + x} dx + \frac{1}{-e^{-ib} + e^{ib}} \int_0^\infty \frac{x^a}{e^{-ib} + x} dx \\
 &= \frac{1}{-2i \sin b} \int_0^\infty \frac{(e^{ib}u)^a}{1+u} du + \frac{1}{2i \sin b} \int_0^\infty \frac{(e^{-ib}u)^a}{1+u} du \\
 &= \frac{e^{iab}}{-2i \sin b} \frac{\pi}{\sin(\pi a + \pi)} + \frac{e^{-iab}}{2i \sin b} \frac{\pi}{\sin(\pi a + \pi)} \\
 &= \frac{\pi}{\sin \pi a \sin b} \left( \frac{e^{iab} - e^{-iab}}{2i} \right) = \frac{\pi \sin ab}{\sin \pi a \sin b}
 \end{aligned}$$

□

## 221. 计算积分

$$\int_0^1 \frac{\ln 2 - \ln(1+x^2)}{1-x} dx$$

**解** 首先由分部积分得  $I = - \int_0^1 \frac{2x \ln(1-x)}{1+x^2} dx$ , 令

$$I(a) = \int_0^1 \frac{-2x \ln(1-ax)}{1+x^2} dx$$

$$\begin{aligned}
 I'(a) &= \int_0^1 \frac{2x^2}{(1+x^2)(1-ax)} dx \\
 &= -\frac{2}{1+a^2} \int_0^1 \left( \frac{ax}{1+x^2} + \frac{1}{1+x^2} - \frac{1}{1-ax} \right) dx \\
 &= -\frac{a \ln 2}{1+a^2} - \frac{\pi}{2} \frac{1}{1+a^2} - \frac{2 \ln(1-a)}{a(1+a^2)}
 \end{aligned}$$

$$\begin{aligned}
 I(1) &= \int_0^1 \left( -\frac{a \ln 2}{1+a^2} - \frac{\pi}{2} \frac{1}{1+a^2} - \frac{2 \ln(1-a)}{a(1+a^2)} \right) da \\
 &= -\frac{1}{2} \ln^2 2 - \frac{\pi}{2} \arctan 1 + 2\text{Li}_2(1) - \int_0^1 \frac{-2a \ln(1-a)}{1+a^2} da \\
 &= \frac{5\pi^2}{24} - \frac{1}{2} \ln^2 2 - I(1)
 \end{aligned}$$

于是

$$I(1) = \frac{5\pi^2}{48} - \frac{1}{4} \ln^2 2$$

□

## 222. 计算积分

$$\int_0^\infty e^{-\sqrt{x}} \ln \left( 1 + \frac{1}{\sqrt{x}} \right) dx$$

**解** 首先有

$$\int_0^\infty e^{-\sqrt{x}} \ln \left( 1 + \frac{1}{\sqrt{x}} \right) dx = 2 \int_0^\infty u e^{-u} \ln \left( 1 + \frac{1}{u} \right) du$$

$$= 2 \int_0^{\infty} u e^{-u} \ln(1+u) du - 2 \int_0^{\infty} u e^{-u} \ln u du$$

利用分部积分可得第一个积分

$$\begin{aligned} & \int_0^{\infty} u e^{-u} \ln(1+u) du \\ &= [-u e^{-u} \ln(1+u)] \Big|_0^{\infty} + \int_0^{\infty} e^{-u} \left( \ln(1+u) + \frac{u}{1+u} \right) du \\ &= \int_0^{\infty} e^{-u} \left( \ln(1+u) + 1 - \frac{1}{1+u} \right) du \\ &= \int_0^{\infty} e^{-u} \left( \ln(1+u) \frac{1}{1+u} \right) du + \int_0^{\infty} e^{-u} du = 1 \end{aligned}$$

$$\begin{aligned} \int_0^{\infty} u e^{-u} \ln u du &= [-u e^{-u} \ln u] \Big|_0^{\infty} + \int_0^{\infty} e^{-u} (\ln u + 1) du \\ &= \int_0^{\infty} e^{-u} \ln u du + 1 = -\gamma + 1 \end{aligned}$$

因此原积分

$$\int_0^{\infty} e^{-\sqrt{x}} \ln \left( 1 + \frac{1}{\sqrt{x}} \right) dx = 2\gamma$$

□

### 223. 计算积分

$$\int_0^{\frac{\pi}{2}} x \cos(8x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) dx$$

解

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} x \cos(8x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) dx \\ &= \int_0^{\frac{\pi}{4}} x \cos(8x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \cos(8x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) dx \\ &= \int_0^{\frac{\pi}{4}} x \cos(8x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) dx - \int_{\frac{\pi}{4}}^0 \left( \frac{\pi}{2} - x \right) \cos(8x) \ln \left( \frac{1 + \cot x}{1 - \cot x} \right) dx \\ &= \int_0^{\frac{\pi}{4}} x \cos(8x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) dx + \int_0^{\frac{\pi}{4}} \left( \frac{\pi}{2} - x \right) \cos(8x) \ln \left( \frac{1 + \tan x}{\tan x - 1} \right) dx \\ &= \int_0^{\frac{\pi}{4}} x \cos(8x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) dx + \int_0^{\frac{\pi}{4}} \left( \frac{\pi}{2} - x \right) \cos(8x) \left[ \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) + i \right] dx \\ &= \int_0^{\frac{\pi}{4}} x \cos(8x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) dx + \int_0^{\frac{\pi}{4}} \left( \frac{\pi}{2} - x \right) \cos(8x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) dx \\ &\quad + \pi i \int_0^{\frac{\pi}{4}} \left( \frac{\pi}{2} - x \right) \cos(8x) dx \\ &= \frac{\pi}{2} \int_0^{\frac{\pi}{4}} \cos(8x) \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) dx = \frac{\pi}{16} \int_0^{\frac{\pi}{4}} \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) d \sin(8x) \\ &= \frac{\pi}{16} \left[ \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) \sin(8x) \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin(8x) d \ln \left( \frac{1 + \tan x}{1 - \tan x} \right) \right] \end{aligned}$$

$$\begin{aligned}
 &= -\frac{\pi}{16} \int_0^{\frac{\pi}{4}} \sin(8x) \left( \frac{1}{1+\tan x} + \frac{1}{1-\tan x} \right) \sec^2 x dx \\
 &= -\frac{\pi}{8} \int_0^{\frac{\pi}{4}} \frac{\sin(8x)}{\cos(2x)} dx = \frac{\pi}{12}.
 \end{aligned}$$

□

## 224. 计算积分

$$\int_0^{\frac{\pi}{4}} \frac{\tan^{2e} x - 2 \sin^2 x}{\sin(2x) \ln \tan x} dx$$

**解** 根据

$$\sin^2 x = \frac{\tan^2 x}{1 + \tan^2 x}, \sin(2x) = \frac{2 \tan x}{1 + \tan^2 x}$$

通过两次换元  $x \rightarrow \tan x, \ln x \rightarrow -x$  可得

$$\begin{aligned}
 &\int_0^{\frac{\pi}{4}} \frac{\tan^{2e} x - 2 \sin^2 x}{\sin(2x) \ln \tan x} dx \\
 &= \int_0^{\pi/4} \frac{\tan^{2e} x (1 + \tan^2 x) - 2 \tan^2 x}{2 \tan x \ln \tan x} dx \\
 &= \frac{1}{2} \int_0^1 \frac{x^{2e-1} (1 + x^2) - 2x}{\ln x} \frac{1}{x^2 + 1} dx \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \int_0^1 \frac{x^{2e-1} (1 + x^2) - 2x}{\ln x} x^{2n} dx \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \int_0^1 \frac{(x^{2(n+e)-1} - x^{2n+1}) + (x^{2(n+e)+1} - x^{2n+1})}{\ln x} dx \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left[ \ln \frac{n+e}{n+1} + \ln \frac{n+e+1}{n+1} \right] \\
 &= \frac{1}{2} \left( 1 + \ln \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n)^2 - 1} \right) \\
 &= \frac{1}{2} \left( 1 + \ln \frac{\pi}{2} \right)
 \end{aligned}$$

□

## 225. 计算积分

$$\int_0^{\frac{\pi}{2}} \frac{\ln \cos x}{\tan x} \cdot \ln \left( \frac{\ln \sin x}{\ln \cos x} \right) dx$$

**解** 考虑积分

$$I = \int_0^{\frac{\pi}{2}} \frac{\ln(\cos x)}{\tan x} \ln \frac{|\ln \sin x|}{|\ln \cos x|} dx$$

拆开后令  $x \rightarrow \frac{\pi}{2}$  不难得到

$$I = \int_0^{\frac{\pi}{2}} (\cot x \ln(\cos x) - \tan x \ln(\sin x) \ln |\ln(\sin x)|) dx$$

而

$$\int \cot(x) \ln(\cos x) - \tan(x) \ln(\sin x) dx = \ln(|\sin x|) \ln(|\cos x|) + C$$

分部积分可得

$$\begin{aligned}
 I &= - \int_0^{\frac{\pi}{2}} \cot x \ln(\cos x) dx \\
 &= - \int_0^1 \frac{t \log(t)}{1-t^2} dt = - \frac{1}{2} \int_0^1 \frac{\log(1-t^2)}{t} dt \\
 &= - \frac{1}{2} \int_0^1 \frac{\log(1+t)}{t} dt - \frac{1}{2} \int_0^1 \frac{\log(1+t)}{t} dt \\
 &= \frac{\text{Li}_2(1) + \text{Li}_2(-1)}{2} \\
 &= \frac{1}{2} \left( \frac{\pi^2}{6} - \frac{\pi^2}{12} \right) \\
 &= \frac{\pi^2}{24}
 \end{aligned}$$

□

226. 求和

$$\sum_{n=1}^{\infty} \sum_{m=1}^{2n} \frac{(-1)^n}{nm}$$

解 首先有

$$\begin{aligned}
 \sum_{n=1}^{\infty} \sum_{m=1}^{2n} \frac{(-1)^n}{nm} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} H_{2n} = -2 \sum_{n=1}^{\infty} (-1)^n \int_0^1 (1-t)^{2n-1} \ln t dt \\
 &= -2 \sum_{n=1}^{\infty} (-1)^n \int_0^1 (1-t)^{2n-1} \ln t dt = 2 \int_0^1 \frac{(1-t) \ln t}{1+(1-t)^2} dt \\
 &= 2 \int_0^1 \frac{t \ln(1-t)}{1+t^2} dt = I
 \end{aligned}$$

令  $I(a) = 2 \int_0^1 \frac{t \ln(1-at)}{1+t^2} dt$ , 则

$$I(0) = 0$$

$$\begin{aligned}
 I'(a) &= - \int_0^1 \frac{2t^2}{(1-at)(1+t^2)} dt = \frac{2}{1+a^2} \left( \int_0^1 \frac{1+at}{1+t^2} dt - \int_0^1 \frac{1}{1-at} dt \right) \\
 &= - \frac{\pi + 2a \ln 2}{2(1+a^2)} - \frac{2 \ln(1-a)}{a} + \frac{2a \ln(1-a)}{1+a^2}
 \end{aligned}$$

$$\begin{aligned}
 I &= I(1) = \int_0^1 I'(a) da \\
 &= \int_0^1 \frac{\pi + 2a \ln 2}{2(1+a^2)} da + \int_0^1 \frac{2 \ln(1-a)}{a} da - \int_0^1 \frac{2a \ln(1-a)}{1+a^2} da \\
 &= \frac{\pi^2}{8} + \frac{1}{2} \ln^2 2 - \frac{\pi^2}{3} - I = \frac{1}{2} \ln^2 2 - \frac{5}{24} \pi^2 - I
 \end{aligned}$$

因此最后得到

$$\sum_{n=1}^{\infty} \sum_{m=1}^{2n} \frac{(-1)^n}{nm} = I = \frac{1}{4} \ln^2 2 - \frac{5}{48} \pi^2$$

□

227. 设  $n$  是一个正整数, 证明

$$\lim_{x \rightarrow 0} \frac{\int_0^x \sin^n \frac{1}{t} dt}{x} = \lim_{x \rightarrow 0} \frac{\int_0^x \cos^n \frac{1}{t} dt}{x} = \begin{cases} \frac{(n-1)!!}{n!!}, & n \text{ 为偶数} \\ 0, & n \text{ 为奇数} \end{cases}$$

**证明** 先考虑复杂的  $n$  为偶数的情形, 这个时候只需要考虑  $x \rightarrow 0^+$  即可, 以正弦为例 (余弦同理)

$$\lim_{x \rightarrow 0^+} \frac{\int_0^x \sin^n \frac{1}{t} dt}{x} = \lim_{x \rightarrow 0^+} \frac{\int_{\frac{1}{x}}^{+\infty} \frac{\sin^n t}{t^2} dt}{x} = \lim_{x \rightarrow +\infty} x \int_x^{+\infty} \frac{\sin^n t}{t^2} dt$$

对  $\forall x > 0, \exists k \in \mathbb{N}, \text{s.t. } (k-1)\pi \leq x < k\pi$ , 则  $x \rightarrow +\infty$  时  $k \rightarrow +\infty$ , 于是

$$x \int_x^{+\infty} \frac{\sin^n t}{t^2} dt = x \int_x^{k\pi} \frac{\sin^n t}{t^2} dt + x \int_{k\pi}^{+\infty} \frac{\sin^n t}{t^2} dt$$

其中

$$\left| x \int_x^{k\pi} \frac{\sin^n t}{t^2} dt \right| \leq \left| x \int_x^{k\pi} \frac{1}{t^2} dt \right| = \left| \frac{k\pi - x}{x} \right| \leq \left| \frac{\pi}{x} \right| \rightarrow 0, x \rightarrow +\infty$$

$$\begin{aligned} \int_{k\pi}^{+\infty} \frac{\sin^n t}{t^2} dt &= \sum_{i=k}^{+\infty} \int_{i\pi}^{(i+1)\pi} \frac{\sin^n t}{t^2} dt = \int_0^\pi \sin^n t \sum_{i=k}^{\infty} \frac{1}{(t+i\pi)^2} dt \\ &= \frac{1}{\pi^2} \int_0^\pi \sin^2 t \sum_{i=k}^{\infty} \frac{1}{(i + \frac{t}{\pi})^2} dt \end{aligned}$$

不难得到当  $k \rightarrow +\infty$  时,

$$\sum_{i=k}^{\infty} \frac{1}{(i+1)^2} \sim \sum_{i=k}^{\infty} \frac{1}{(i + \frac{t}{\pi})^2} \sim \sum_{i=k}^{\infty} \frac{1}{i^2} \sim \frac{1}{k}$$

于是当  $x \rightarrow +\infty$  时,

$$x \int_{k\pi}^{+\infty} \frac{\sin^n t}{t^2} dt = \frac{x}{\pi^2} \int_0^\pi \sin^n t \sum_{i=k}^{\infty} \frac{1}{(i + \frac{t}{\pi})^2} dt \sim \frac{k\pi}{\pi^2} \cdot \frac{1}{k} \int_0^\pi \sin^n t dt = \frac{(n-1)!!}{n!!}$$

这就是  $n$  是偶数的极限, 而当  $n$  是奇数的时候, 正项级数  $\sum_{i=k}^{\infty} \frac{1}{(i + \frac{t}{\pi})^2}$  会变成交错级数  $\sum_{i=k}^{\infty} \frac{(-1)^i}{(i + \frac{t}{\pi})^2}$ ,

这个交错级数的绝对值不会超过  $\frac{1}{(k + \frac{t}{\pi})^2} < \frac{1}{k^2}$ , 因此最后的极限是 0.

□

228. 求和

$$\sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \frac{(-1)^{m-n}}{(m^2 - n^2)^2}$$

**解** 首先有当  $m^2 \neq n^2$  时,

$$\frac{1}{(m^2 - n^2)^2} = \int_0^1 \int_0^1 x^{m+n-1} y^{m-n-1} \log x \log y dx dy$$

因此可得

$$\sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \frac{(-1)^{m-n}}{(m^2 - n^2)^2} = \int_0^1 \int_0^1 \frac{-\log x \log y}{(1-x^2)(1+xy)} dx dy$$

进一步利用因式分解可得

$$\begin{aligned} & \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \frac{(-1)^{m-n}}{(m^2 - n^2)^2} \\ &= \int_0^1 \int_0^1 \frac{-\log x \log y}{(1-x^2)(1+xy)} dx dy \\ &= \int_0^1 \int_0^1 \left( \frac{-\log x \log y}{2(1+x)(1-y)} + \frac{-\log x \log y}{2(1-x)(1+y)} + \frac{y^2 \log x \log y}{(1-y^2)(1+xy)} \right) dx dy \\ &= -2 \cdot \frac{\pi^4}{144} + \frac{\pi^4}{480} = -\frac{17\pi^4}{1440} \end{aligned}$$

□

**229.** 给定  $0 \leq a \leq 2$ , 设  $\{a_n\}_{n \geq 1}$  是由  $a_1 = a, a_{n+1} = 2^n - \sqrt{2^n(2^n - a_n)}$  所定义的数列, 求  $\sum_{n=1}^{\infty} a_n^2$ .

**解** 令

$$\alpha = 4 \arcsin \sqrt{\frac{a}{2}} = \begin{cases} \arccos(2a^2 - 4a + 1), & a \in [0, 1] \\ 2\pi - \arccos(2a^2 - 4a + 1), & a \in [1, 2] \end{cases}$$

然后利用二倍角公式  $2 \cos^2 \left( \frac{\theta}{2} \right) = 1 + \cos \theta$ , 不难得到

$$a_n = 2^{n-1} \left( 1 - \cos \frac{\alpha}{2^n} \right)$$

对  $N \in \mathbb{N}$  有

$$\begin{aligned} \sum_{n=1}^N a_n^2 &= \sum_{n=1}^N 4^{n-1} \left( 1 + \cos^2 \frac{\alpha}{2^n} - 2 \cos \frac{\alpha}{2^n} \right) \\ &= \sum_{n=1}^N 4^{n-1} \left( 1 + \frac{1 + \cos(\alpha/2^{n-1})}{2} - 2 \cos \frac{\alpha}{2^n} \right) \\ &= \frac{1}{2} \sum_{n=1}^N 4^n \left( 1 - \cos \frac{\alpha}{2^n} \right) - \frac{1}{2} \sum_{n=1}^N 4^{n-1} \left( 1 - \cos \frac{\alpha}{2^n} \right) \\ &= \frac{1}{2} \sum_{n=1}^N 4^{n-1} \left( 1 - \cos \frac{\alpha}{2^n} \right) - \frac{1}{2} \sum_{n=0}^{N-1} 4^n \left( 1 - \cos \frac{\alpha}{2^n} \right) \\ &= \frac{1}{2} \left( 4^N \left( 1 - \cos \frac{\alpha}{2^N} \right) - (1 - \cos \alpha) \right) \end{aligned}$$

因此

$$\begin{aligned} \sum_{n=1}^{\infty} a_n^2 &= \frac{1}{2} \left( \lim_{N \rightarrow \infty} 4^N \left( 1 - \cos \frac{\alpha}{2^N} \right) - (1 - \cos \alpha) \right) \\ &= \frac{\alpha^2}{4} + a^2 - 2a = 4 \arcsin^2 \sqrt{\frac{a}{2}} + a^2 - 2a. \end{aligned}$$

□

**230.** 设函数  $f, g$  是  $[0, 1]$  上的连续实值函数, 证明存在  $c \in (0, 1)$  使得

$$\int_0^1 f(x) dx \int_0^c xg(x) dx = \int_0^1 g(x) dx \int_0^c xf(x) dx$$

**解** 令

$$h(t) = g(t) \int_0^1 f(x) dx - f(t) \int_0^1 g(x) dx$$

只需要证明存在  $c \in (0, 1)$  使得

$$\int_0^c th(t) dt = \int_0^c tg(t) dt \int_0^1 f(x) dx - \int_0^c tf(t) dt \int_0^1 g(x) dx = 0$$

令

$$u(s) = \int_0^s H(x) dx = \int_0^s \int_0^x h(t) dt dx$$

则  $u$  是  $[0, 1]$  上连续可微的函数且  $u'(0) = H(0) = 0$ ,

$$\begin{aligned} u'(1) &= H(1) = \int_0^1 h(t) dt \\ &= \int_0^1 g(t) dt \int_0^1 f(x) dx - \int_0^1 f(t) dt \int_0^1 g(x) dx = 0 \end{aligned}$$

由 Flett 中值定理, 存在  $c \in (0, 1)$  使得

$$u'(c) = H(c) = \frac{u(c) - u(0)}{c - 0} = \frac{u(c)}{c}$$

因此

$$\int_0^c th(t) dt = \int_0^c t d(H(t)) = [tH(t)]_0^c - \int_0^c H(t) dt = cH(c) - u(c) = 0$$

□

**231.** 设  $\mu$  为一正实数, 证明

$$\lim_{x \rightarrow 1^+} (\ln x)^{\frac{1}{\mu}} \sum_{k=1}^{\infty} x^{-(2k-1)\mu} = \frac{\Gamma\left(\frac{1}{\mu}\right)}{2\mu}$$

**解** 首先我们证明下面一个定理

**定理 1.** 如果  $f: \mathbb{R}^+ \rightarrow \mathbb{R}$  非负且单调递减, 且积分  $\int_0^{\infty} f(x) dx < +\infty$ . 则对任意满足  $a > \max(b, 0)$  的实数  $a, b$ ,

$$\lim_{h \rightarrow 0^+} h \sum_{k=1}^{\infty} f((ka-b)h) = \frac{1}{a} \int_0^{\infty} f(t) dt$$

Proof: 不失一般性不妨假定  $a = 1$ . 对  $h > 0$ , 令  $x_k = (k-b)h, k \in \mathbb{N}$ , 由于  $f$  单调减, 则

$$hf(x_{k+1}) \leq \int_{x_k}^{x_{k+1}} f(t) dt \leq hf(x_k)$$

求和即得

$$h \sum_{k=2}^n f(x_k) \leq \int_{x_1}^{x_n} f(t) dt \leq h \sum_{k=1}^{n-1} f(x_k)$$



令  $n \rightarrow \infty$ ,

$$h \sum_{k=1}^{\infty} f(x_k) - hf(x_1) \leq \int_0^{\infty} f(t) dt \leq h \sum_{k=1}^{\infty} f(x_k)$$

那么定理自然就得证了. 现在令  $f(t) = e^{-t^\mu}, t \geq 0$ , 则

$$\lim_{h \rightarrow 0^+} h \sum_{k=1}^{\infty} e^{-(ak-b)^\mu h^\mu} = \frac{1}{a} \int_0^{\infty} e^{-t^\mu} dt = \frac{\Gamma\left(\frac{1}{\mu}\right)}{a\mu}$$

□

**232.** 设  $\mu > 0, 0 < b < a$ , 计算积分

$$\int_0^{\pi} \frac{\sin^{\mu-1} x}{(a + b \cos x)^\mu} dx$$

**解**

$$\begin{aligned} \int_0^{\pi} \frac{\sin^{\mu-1} x}{(a + b \cos x)^\mu} dx &= \int_0^{\frac{\pi}{2}} \frac{\sin^{\mu-1} x}{(a + b \cos x)^\mu} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin^{\mu-1} x}{(a + b \cos x)^\mu} dx \\ &= \int_0^{\frac{\pi}{2}} \sin^{\mu-1} x \left( \frac{1}{(a + b \cos x)^\mu} + \frac{1}{(a - b \cos x)^\mu} \right) dx \\ &= a^{-\mu} \int_0^{\frac{\pi}{2}} \sin^{\mu-1} x \left( \sum_{n=0}^{\infty} \binom{-\mu}{n} \left(-\frac{b}{a} \cos x\right)^n + \sum_{n=0}^{\infty} \binom{-\mu}{n} \left(\frac{b}{a} \cos x\right)^n \right) dx \\ &= 2a^{-\mu} \sum_{n=0}^{\infty} \binom{-\mu}{2n} \left(\frac{b}{a}\right)^{2n} \int_0^{\frac{\pi}{2}} \sin^{\mu-1} x \cos^{2n} x dx \\ &= a^{-\mu} \sum_{n=0}^{\infty} \frac{(-\mu)(-\mu-1)\cdots(-\mu-2n+1)}{(2n)!} \left(\frac{b}{a}\right)^{2n} B\left(\frac{\mu}{2}, \frac{2n+1}{2}\right) \\ &= a^{-\mu} \sum_{n=0}^{\infty} \frac{\mu(\mu+1)\cdots(\mu+2n-1)}{(2n)!} \left(\frac{b}{a}\right)^{2n} \frac{\Gamma\left(\frac{\mu}{2}\right) \Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{2n+1+\mu}{2}\right)} \\ &= a^{-\mu} \Gamma\left(\frac{\mu}{2}\right) \sum_{n=0}^{\infty} \frac{\mu(\mu+1)\cdots(\mu+2n-1)}{(2n)!!} \left(\frac{b}{a}\right)^{2n} \frac{\sqrt{\pi}}{(2n+\mu-1)(2n+\mu-3)\cdots(\mu+1) \Gamma\left(\frac{\mu+1}{2}\right)} \\ &= a^{-\mu} \Gamma\left(\frac{\mu}{2}\right) \frac{\sqrt{\pi}}{\Gamma\left(\frac{\mu+1}{2}\right)} \sum_{n=0}^{\infty} \frac{\mu(\mu+2)\cdots(\mu+2n-2)}{2^n n!} \left(\frac{b}{a}\right)^{2n} \\ &= a^{-\mu} \Gamma\left(\frac{\mu}{2}\right) \frac{\sqrt{\pi}}{\Gamma\left(\frac{\mu+1}{2}\right)} \sum_{n=0}^{\infty} \frac{\frac{\mu}{2} \left(\frac{\mu}{2}+2\right)\cdots\left(\frac{\mu}{2}+2n-2\right)}{n!} \left(\frac{b}{a}\right)^{2n} \\ &= a^{-\mu} \Gamma\left(\frac{\mu}{2}\right) \frac{\sqrt{\pi}}{\Gamma\left(\frac{\mu+1}{2}\right)} \sum_{n=0}^{\infty} \binom{-\frac{\mu}{2}}{n} \left(-\frac{b^2}{a^2}\right)^n = a^{-\mu} \Gamma\left(\frac{\mu}{2}\right) \frac{\sqrt{\pi}}{\Gamma\left(\frac{\mu+1}{2}\right)} \left(1 - \frac{b^2}{a^2}\right)^{-\frac{\mu}{2}} \\ &= (a^2 - b^2)^{-\frac{\mu}{2}} B\left(\frac{\mu}{2}, \frac{1}{2}\right) \end{aligned}$$

□

**233.** 设  $n$  是正整数, 计算积分

$$\int_0^{\infty} \frac{\ln^n x}{\sqrt{x}(1-x)^2} dx.$$

解

$$\begin{aligned}\int_0^\infty \frac{\ln^n x}{\sqrt{x}(1-x)^2} dx &= 2^{n+1} \int_0^\infty \frac{\ln^n t}{(1-t^2)^2} dt \\ &= 2^{n+1} \left( \int_0^1 \frac{\ln^n t}{(1-t^2)^2} dt + \int_1^\infty \frac{\ln^n t}{(1-t^2)^2} dt \right) \\ &= 2^{n+1} \int_0^1 \frac{(-1)^n t^2 + 1}{(1-t^2)^2} \ln^n t dt.\end{aligned}$$

如果  $n$  是奇数, 令  $\ln t = -z$ , 则

$$\begin{aligned}I &= -2^{n+1} \int_0^\infty \frac{z^n e^{-z}}{1-e^{-2z}} dz \\ &= -2^{n+1} n! (1-2^{-n-1}) \zeta(n+1) = -n! (2^{n+1}-1) \zeta(n+1).\end{aligned}$$

如果  $n$  是偶数, 则利用分部积分可得

$$\begin{aligned}I &= 2^{n+1} \int_0^1 \frac{t^2+1}{(1-t^2)^2} \ln^n t dt = 2^{n+1} \int_0^1 \ln^n t d\left(\frac{1}{1-t^2}\right) \\ &= -n2^{n+1} \int_0^1 \frac{\ln^{n-1} t}{1-t^2} dt = n2^{n+1} \int_0^\infty \frac{z^{n-1} e^{-z}}{1-e^{-2z}} dz \\ &= n2^{n+1} (n-1)! (1-2^{-n}) \zeta(n) = n! (2^{n+1}-2) \zeta(n).\end{aligned}$$

□

### 234. 计算极限

$$\lim_{a \rightarrow +\infty} \frac{1}{a} \int_0^a \frac{t}{1+a^2 \cos^2 t} dt.$$

解 首先注意到

$$\int_0^{\frac{\pi}{2}} \frac{dt}{1+a^2 \cos^2 t} = \left[ \frac{1}{\sqrt{a^2+1}} \arctan\left(\frac{\tan t}{\sqrt{a^2+1}}\right) \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2\sqrt{a^2+1}}$$

记  $n = \left\lceil \frac{a}{\pi} \right\rceil$  表示不小于  $\frac{a}{\pi}$  的最小整数, 我们有

$$\begin{aligned}\frac{1}{a} \int_0^a \frac{t}{1+a^2 \cos^2 t} dt &= \sum_{k=0}^{n-1} \frac{1}{a} \int_{k\pi}^{(k+1)\pi} \frac{t}{1+a^2 \cos^2 t} dt + \frac{1}{a} \int_{n\pi}^a \frac{t}{1+a^2 \cos^2 t} dt \\ &= \sum_{k=0}^{n-1} \frac{1}{a} \int_0^\pi \frac{k\pi+t}{1+a^2 \cos^2 t} dt + \frac{1}{a} \int_0^{a-n\pi} \frac{n\pi+t}{1+a^2 \cos^2 t} dt.\end{aligned}$$

对最后一项,

$$0 \leq \frac{1}{a} \int_0^{a-n\pi} \frac{n\pi+t}{1+a^2 \cos^2 t} dt \leq \frac{1}{a} \int_0^\pi \frac{dt}{1+a^2 \cos^2 t} = \frac{\pi}{a\sqrt{a^2+1}}.$$

对前面的求和项,

$$\frac{k\pi}{a} \int_0^\pi \frac{dt}{1+a^2 \cos^2 t} \leq \frac{1}{a} \int_0^\pi \frac{k\pi+t}{1+a^2 \cos^2 t} dt \leq \frac{(k+1)\pi}{a} \int_0^\pi \frac{dt}{1+a^2 \cos^2 t}.$$

对  $k$  求和可得

$$\frac{n(n-1)\pi}{2a\sqrt{a^2+1}} \leq \sum_{k=0}^{n-1} \frac{1}{a} \int_0^\pi \frac{k\pi+t}{1+a^2\cos^2 t} dt \leq \frac{n(n+1)\pi}{2a\sqrt{a^2+1}}.$$

由于  $n\pi \leq a < (n+1)\pi$ , 我们有

$$\lim_{n \rightarrow \infty} \frac{n(n-1)\pi}{2a\sqrt{a^2+1}} = \lim_{n \rightarrow \infty} \frac{n(n+1)\pi}{2a\sqrt{a^2+1}} = \frac{1}{2}.$$

因此由夹逼准则可得

$$\lim_{a \rightarrow +\infty} \sum_{k=0}^{n-1} \frac{1}{a} \int_0^\pi \frac{k\pi+t}{1+a^2\cos^2 t} dt = \frac{1}{2}.$$

也就是原极限为  $\frac{1}{2}$ .

□

### 235. 计算极限

$$\int_1^2 \cdots \int_1^2 \frac{1}{1+(x_1 \cdots x_n)^{1/n}} dx_1 \cdots dx_n.$$

解

$$\begin{aligned} & \int_1^2 \cdots \int_1^2 \frac{1}{1+(x_1 \cdots x_n)^{1/n}} dx_1 \cdots dx_n \\ &= \frac{1}{2} \int_1^2 \cdots \int_1^2 \frac{1}{1+\frac{(x_1 \cdots x_n)^{1/n}-1}{2}} dx_1 \cdots dx_n \\ &= \frac{1}{2} \int_1^2 \cdots \int_1^2 \sum_{k=0}^{\infty} (-1)^k \left( \frac{(x_1 \cdots x_n)^{1/n}-1}{2} \right)^k dx_1 \cdots dx_n \\ &= \frac{1}{2} \int_1^2 \cdots \int_1^2 \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^k} \binom{k}{j} (-1)^{k-j} (x_1 \cdots x_n)^{\frac{j}{n}} dx_1 \cdots dx_n \\ &= \frac{1}{2} \int_0^1 \cdots \int_0^1 \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (-1)^j (x_1 \cdots x_n)^{\frac{j}{n}} dx_1 \cdots dx_n \\ &= \frac{1}{2} \int_1^2 \cdots \int_1^2 \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (-1)^j \left( \frac{n}{n+j} \right)^n \left( 2^{\frac{j+n}{n}} - 1 \right)^n dx_1 \cdots dx_n \end{aligned}$$

对任意  $\varepsilon > 0$ , 注意到级数  $\sum_{k=0}^{\infty} (-1)^k \left( \frac{(x_1 \cdots x_n)^{1/n}-1}{2} \right)^k$  是收敛的, 因此可取充分大的  $K$ , 使得

$$\left| \sum_{k=K+1}^{\infty} (-1)^k \left( \frac{(x_1 \cdots x_n)^{1/n}-1}{2} \right)^k \right| < \varepsilon$$

即  $\int_1^2 \cdots \int_1^2 \left| \sum_{k=K+1}^{\infty} (-1)^k \left( \frac{(x_1 \cdots x_n)^{1/n}-1}{2} \right)^k \right| dx < \varepsilon$ , 因此

$$\frac{1}{2} \int_0^1 \cdots \int_0^1 \sum_{k=0}^K \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (-1)^j (x_1 \cdots x_n)^{\frac{j}{n}} dx_1 \cdots dx_n$$

$$\begin{aligned}
 &= \frac{1}{2} \sum_{k=0}^K \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (-1)^j \int_1^2 \cdots \int_1^2 (x_1 \cdots x_n)^{\frac{j}{n}} dx_1 \cdots dx_n \\
 &= \frac{1}{2} \sum_{k=0}^K \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (-1)^j \left( \int_1^2 x^{\frac{j}{n}} dx \right)^n \\
 &= \frac{1}{2} \sum_{k=0}^K \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (-1)^j \left( \frac{n}{n+j} \right)^n \left( 2^{\frac{j+n}{n}} - 1 \right)^n
 \end{aligned}$$

注意到对任意  $1 \leq j \leq k$  均有

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left( \frac{n}{n+j} \right)^n \left( 2^{\frac{j+n}{n}} - 1 \right)^n &= e^{-j} \lim_{n \rightarrow \infty} \left( 2^{\frac{j+n}{n}} - 2 + 1 \right)^n \\
 &= e^{-j} \lim_{n \rightarrow \infty} e^{n \ln \left( 2^{\frac{j+n}{n}} - 2 + 1 \right)} = e^{-j} \lim_{n \rightarrow \infty} e^{n \left( 2^{1+\frac{j}{n}} - 2 \right)} \\
 &= e^{-j} \lim_{n \rightarrow \infty} e^{2n \frac{j}{n} \ln 2} = \left( \frac{4}{e} \right)^j
 \end{aligned}$$

因此

$$\begin{aligned}
 \frac{1}{2} \sum_{k=0}^K \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (-1)^j \left( \frac{n}{n+j} \right)^n \left( 2^{\frac{j+n}{n}} - 1 \right)^n &\rightarrow \frac{1}{2} \sum_{k=0}^K \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} \left( -\frac{4}{e} \right)^j \\
 &= \frac{1}{2} \sum_{k=0}^K \frac{1}{2^k} \left( 1 - \frac{4}{e} \right)^k = \frac{1}{2} \frac{1 - \frac{1}{2} \left( 1 - \frac{4}{e} \right)^K}{1 - \frac{1}{2} \left( 1 - \frac{4}{e} \right)} \\
 &= \frac{e}{4+e} \left[ 1 - \frac{1}{2} \left( 1 - \frac{4}{e} \right)^K \right]
 \end{aligned}$$

令  $K \rightarrow \infty$ , 最后得到原极限等于  $\frac{e}{4+e}$ .

□

### 236. 计算积分

$$\int_0^\infty \left( \sqrt{x} - \sqrt{\sqrt{1+x^2} - 1} \right) \sin x dx.$$

**解** 首先分部积分得

$$I = \int_0^\infty \frac{\cos x}{2\sqrt{x}} dx - \frac{1}{2} \int_0^\infty \frac{x \cos x}{\sqrt{(x^2+1)(\sqrt{x^2+1}-1)}} dx.$$

而

$$\int_0^\infty \frac{\cos x}{2\sqrt{x}} dx = \int_0^\infty \cos(x^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}.$$

注意到

$$\frac{x}{\sqrt{(x^2+1)(\sqrt{x^2+1}-1)}} = \sqrt{\frac{\sqrt{x^2+1}+1}{x^2+1}} = \frac{2\sqrt{2}}{\pi} \int_0^\infty \frac{t^2+1}{x^2+(t^2+1)^2} dt.$$

当  $a > 0$  时, 我们有  $\int_0^\infty \frac{\cos x}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-a}$ , 于是

$$\begin{aligned} \int_0^\infty \frac{x \cos x}{\sqrt{(x^2 + 1)(\sqrt{x^2 + 1} - 1)}} dx &= \frac{2\sqrt{2}}{\pi} \int_0^\infty \int_0^\infty \frac{(t^2 + 1) \cos x}{x^2 + (t^2 + 1)^2} dx dt \\ &= \sqrt{2} \int_0^\infty e^{-(t^2 + 1)} dt = \frac{1}{e} \sqrt{\frac{\pi}{2}}. \end{aligned}$$

于是得到

$$\int_0^\infty \left( \sqrt{x} - \sqrt{\sqrt{1 + x^2} - 1} \right) \sin x dx = \frac{e - 1}{2e} \sqrt{\frac{\pi}{2}}.$$

□

### 237. 计算积分

$$\int_0^1 \frac{x \arccos x}{1 + x^4} dx.$$

**解** 首先令  $y = \arccos x$ , 则

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{y \cos y \sin y}{1 + \cos^4 y} dy = -\frac{1}{2} \int_0^{\frac{\pi}{2}} y d(\arctan(\cos^2 y)) = \frac{1}{2} \int_0^{\frac{\pi}{2}} \arctan(\cos^2 y) dy \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} dy \int_0^1 \frac{\cos^2 y}{1 + t^2 \cos^4 y} dt = \frac{1}{2} \int_0^1 dt \int_0^{\frac{\pi}{2}} \frac{\cos^2 y}{1 + t^2 \cos^4 y} dy. \end{aligned}$$

其中

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 y}{1 + t^2 \cos^4 y} dy = \int_0^\infty \frac{du}{u^4 + 2u^2 + t^2 + 1} = \frac{\pi \sqrt{\sqrt{t^2 + 1} - 1}}{2\sqrt{2}t \sqrt{t^2 + 1}}.$$

于是原积分

$$\begin{aligned} I &= \frac{\pi}{4\sqrt{2}} \int_0^1 \frac{\sqrt{\sqrt{t^2 + 1} - 1}}{t \sqrt{t^2 + 1}} dt = \int_0^1 \frac{t \sqrt{\sqrt{t^2 + 1} - 1}}{t^2 \sqrt{t^2 + 1}} dt \\ &= \frac{\pi}{4\sqrt{2}} \int_0^1 \frac{\sqrt{\sqrt{t^2 + 1} - 1}}{t^2} d(\sqrt{t^2 + 1}) = \frac{\pi}{4\sqrt{2}} \int_1^{\sqrt{2}} \frac{\sqrt{s - 1}}{s^2 - 1} ds \\ &= \frac{\pi}{4\sqrt{2}} \int_1^{\sqrt{2}} \frac{ds}{(s + 1) \sqrt{s - 1}} = \frac{\pi}{4} \arctan \left( \sqrt{\frac{\sqrt{2} - 1}{2}} \right). \end{aligned}$$

□

### 238. 计算积分

$$\int_0^1 \frac{\arctan \sqrt{x}}{(3 - x) \sqrt{2 - x}} dx.$$

**解** 考虑参数积分  $J(a) = \int_0^1 \frac{\arctan \sqrt{ax}}{(3 - x) \sqrt{2 - x}} dx$ , 则  $J(0) = 0, I = J(1)$ .

$$J'(a) = \frac{1}{2\sqrt{a}} \int_0^1 \frac{1}{(3 - x)(1 + ax)} \sqrt{\frac{x}{2 - x}} dx \quad \left( t = \sqrt{\frac{x}{2 - x}} \right)$$

$$= \frac{1}{\sqrt{a}} \frac{1}{3a} \int_0^1 \left( \frac{3}{3+t^2} - \frac{1}{1+(1+2a)t^2} \right) dt = \frac{1}{(1+3a)\sqrt{a}} \left( \frac{\pi}{2\sqrt{3}} - \frac{\arctan \sqrt{1+2a}}{\sqrt{1+2a}} \right).$$

于是

$$\begin{aligned} J(1) &= \frac{\pi^2}{9} - 2 \int_0^1 \frac{\arctan \sqrt{1+2a^2}}{(1+3a^2)\sqrt{1+2a^2}} da \\ &= \frac{\pi^2}{9} - 2 \int_0^1 \arctan \sqrt{1+2a^2} d \left( \arctan \frac{a}{\sqrt{1+2a^2}} \right) = 2 \int_0^1 \frac{\arctan \frac{a}{\sqrt{1+2a^2}}}{(1+a^2)\sqrt{1+2a^2}} a da \\ &= \pi \int_0^1 \frac{a da}{(1+a^2)\sqrt{1+2a^2}} - 2 \int_0^1 \frac{\arctan \frac{\sqrt{1+2a^2}}{a}}{(1+a^2)\sqrt{1+2a^2}} da \\ &= \frac{\pi^2}{12} - 2 \int_1^\infty \frac{\arctan \sqrt{2+s^2}}{(1+s^2)\sqrt{2+s^2}} ds = \frac{\pi^2}{12} - \frac{\pi^2}{16} = \frac{\pi^2}{48}. \end{aligned}$$

其中

$$\begin{aligned} \int_1^\infty \frac{\arctan \sqrt{2+s^2}}{(1+s^2)\sqrt{2+s^2}} ds &= \int_1^\infty \int_1^\infty \frac{1}{1+s^2} \frac{1}{2+s^2+p^2} dp ds \\ &= \frac{1}{2} \int_1^\infty \int_1^\infty \frac{1}{1+s^2} \frac{1}{1+p^2} dp ds = \frac{\pi^2}{32}. \end{aligned}$$

□

### 239. 计算积分

$$\int_0^1 \int_0^1 \frac{dx dy}{\left( \left[ \frac{x}{y} \right] + 1 \right)^2}.$$

**解** 换元  $x = u, y = uv$ , 则

$$\begin{aligned} I &= \int_0^1 \int_0^1 \frac{u}{\left( \left[ \frac{1}{v} \right] + 1 \right)^2} du dv + \int_1^\infty \int_0^{\frac{1}{v}} \frac{u}{\left( \left[ \frac{1}{v} \right] + 1 \right)^2} du dv \\ &= \frac{1}{2} \int_0^1 \frac{dv}{\left( \left[ \frac{1}{v} \right] + 1 \right)^2} + \frac{1}{2} \int_1^\infty \frac{dv}{v^2 \left( \left[ \frac{1}{v} \right] + 1 \right)^2} \\ &= \frac{1}{2} \int_0^1 \frac{dv}{\left( \left[ \frac{1}{v} \right] + 1 \right)^2} + \frac{1}{2} \int_0^1 \frac{dv}{v^2} = \frac{1}{2} \int_1^\infty \frac{dv}{v^2 \left( [v] + 1 \right)^2} + \frac{1}{2} \\ &= \frac{1}{2} \sum_{n=1}^\infty \frac{1}{(n+1)^2} \int_n^{n+1} \frac{dv}{v^2} + \frac{1}{2} = \frac{1}{2} \sum_{n=1}^\infty \left( \frac{1}{n(n+1)} - \frac{1}{(n+1)^2} - \frac{1}{(n+1)^3} \right) + \frac{1}{2} \\ &= 2 - \frac{\zeta(2) + \zeta(3)}{2}. \end{aligned}$$

□

### 240. 计算积分

$$\int_0^\pi \ln(1 - \cos x) \ln(1 + \cos x) dx.$$

**解**

$$\int_0^\pi \log(1 - \cos x) \log(1 + \cos x) dx = \int_0^\pi \log(1 + \cos x) \left( -\log 2 - 2 \sum_{n=1}^\infty \frac{\cos nx}{n} \right) dx$$

$$\begin{aligned}
 &= \pi \log^2 2 - 2 \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\pi} \cos nx \log(1 + \cos x) dx \\
 &= \pi \log^2 2 - 2 \sum_{n=1}^{\infty} \frac{\pi(-1)^{n-1}}{n^2} \\
 &= \pi \log^2 2 - 2\pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \\
 &= \pi \log^2 2 - \frac{\pi^3}{6}.
 \end{aligned}$$

□

#### 241. 计算积分

$$\int_{-\pi}^{\pi} \frac{x \sin x}{1 + a^2 - 2a \cos x} dx.$$

解

$$\begin{aligned}
 I &= 2 \int_0^{\pi} \frac{x \sin x}{1 + a^2 - 2a \cos x} dx = \frac{2}{2a} \int_0^{\pi} \frac{x \sin x}{\frac{1+a^2}{2a} - \cos x} dx \\
 &= \frac{1}{a} \int_0^{\pi} \frac{x \sin x}{A - \cos x} dx = \frac{x \ln(A - \cos x)}{a} \Big|_0^{\pi} - \frac{1}{a} \int_0^{\pi} \ln(A - \cos x) dx \\
 &= \frac{\pi \ln(A + 1)}{a} - \frac{1}{a} \int_0^{\pi} \ln(A - \cos x) dx \\
 &= \frac{\pi \ln(A + 1)}{a} - \frac{\pi}{a} \ln \left( \frac{A + \sqrt{A^2 - 1}}{2} \right) = \frac{2\pi}{a} \ln \left( 1 + \frac{1}{a} \right).
 \end{aligned}$$

□

#### 242.

$$\int_0^{2\pi} \frac{\cos^2 3x}{1 + a^2 - 2a \cos 2x} dx.$$

解 利用公式

$$\sum_{k=0}^{\infty} a^k \sin kx = \frac{a \sin x}{1 + a^2 - 2a \cos x}$$

可得

$$I = \sum_{k=0}^{\infty} a^{k-1} \int_0^{2\pi} \frac{\cos^2(3x) \sin 2kx}{\sin 2x} dx.$$

考虑积分

$$J_k = \int_0^{2\pi} \frac{\cos^2(3x) \sin 2kx}{\sin 2x} dx,$$

则

$$J_{k+2} - J_k = 2 \int_0^{2\pi} \cos^2(3x) \cos(2k+2)x dx = \begin{cases} 0, & k \neq 2 \\ \pi, & k = 2 \end{cases}.$$

于是

$$J_1 = \pi, J_0 = 0 \implies J_{2p+1} = \pi, J_0 = J_2 = 0, J_4 = J_6 = J_8 = J_{10} = \cdots = \pi.$$

因此最后得到

$$I = \pi(1 + a^3 + a^4 + \cdots) = \pi \left( \frac{1}{1-a} - a \right) = \pi \frac{1-a+a^2}{1-a}.$$

□

243. 设  $n$  是一个自然数, 计算

$$\int_0^{2\pi} \frac{(1+2\cos x)^n \cos nx}{3+2\cos x} dx.$$

解 对任意自然数  $k < n$ ,

$$\begin{aligned} \alpha_k(n) &= \int_0^{2\pi} (1+2\cos x)^k \cos nx dx \\ &= \int_0^{2\pi} (1+2\cos x)^{k-1} \cos nx dx + \int_0^{2\pi} (1+2\cos x)^{k-1} (2\cos x \cos nx) dx \\ &= \alpha_{k-1}(n) + \int_0^{2\pi} (1+2\cos x)^{k-1} \cos(n-1)x dx + \int_0^{2\pi} (1+2\cos x)^{k-1} \cos(n+1)x dx \\ &= \alpha_{k-1}(n-1) + \alpha_{k-1}(n) + \alpha_{k-1}(n+1). \end{aligned}$$

且

$$\alpha_0(n) = \int_0^{2\pi} \cos nx dx = 2\pi\delta_{n,0}.$$

由递推公式以及初始条件可知对任意  $k < n$  有  $\alpha_k(n) = 0$ . 再令

$$f_k(n) := \int_0^{2\pi} \frac{(1+2\cos x)^k \cos nx}{3+2\cos x} dx.$$

那么当  $0 \leq k \leq n$  时,

$$\begin{aligned} f_k(n) + f_k(n+2) &= 2 \int_0^{2\pi} \frac{(1+2\cos x)^k \cos x \cos(n+1)x}{3+2\cos x} dx \\ &= \int_0^{2\pi} (1+2\cos x)^k \cos(n+1)x dx - 3 \int_0^{2\pi} \frac{(1+2\cos x)^k \cos(n+1)x}{3+2\cos x} dx \\ &= -3f_k(n+1). \end{aligned}$$

因此我们得到递推关系

$$f_k(n+2) + 3f_k(n+1) + f_k(n) = 0.$$

由特征根方法可知解形式为

$$f_k(n) = c_1(k)\lambda_1^n + c_2(k)\lambda_2^n,$$

其中  $\lambda_1 = \frac{-3+\sqrt{5}}{2}$ ,  $\lambda_2 = \frac{-3-\sqrt{5}}{2}$ , 由 Riemann-Lebesgue 引理可知  $\lim_{n \rightarrow \infty} f_k(n) = 0$ , 因此  $c_2(k) = 0$ ,  $f_k(n) = c_1(k)\lambda_1^n$ ,  $0 \leq k \leq n$ .

$$\begin{aligned} f_k(n) &= \int_0^{2\pi} \frac{(1+2\cos x)^{k-1} (1+2\cos x) \cos nx}{3+2\cos x} dx \\ &= \int_0^{2\pi} (1+2\cos x)^{k-1} \cos nx dx - 2f_{k-1}(n) = -2f_{k-1}(n). \end{aligned}$$



这是一个关于指标  $k$  的等比数列, 于是  $f_k(n) = c(-2)^k \lambda_1^n$ . 当  $k = n = 0$  时,

$$c = \int_0^{2\pi} \frac{1}{3 + 2 \cos x} dx = \frac{2\pi}{\sqrt{5}}.$$

因此最后得到原积分

$$f_n(n) = \frac{2\pi}{\sqrt{5}} (-2)^n \lambda_1^n = \frac{2\pi}{\sqrt{5}} (3 - \sqrt{5})^n.$$

□

**244.** 设  $\operatorname{Re}(\alpha) \geq 1$ , 计算积分

$$\int_{-\infty}^{\infty} |\sin x|^{\alpha-1} \frac{\sin x}{x} dx.$$

**解**

$$\begin{aligned} \int_{-\infty}^{+\infty} |\sin x|^{\alpha-1} \frac{\sin x}{x} dx &= 2 \int_0^{+\infty} |\sin x|^{\alpha-1} \frac{\sin x}{x} dx \\ &= 2 \sum_{n=0}^{+\infty} \left( \int_{2n\pi}^{(2n+1)\pi} (\sin x)^{\alpha-1} \frac{\sin x}{x} dx + \int_{(2n+1)\pi}^{(2n+2)\pi} (-\sin x)^{\alpha-1} \frac{\sin x}{x} dx \right) \\ &= 2 \sum_{n=0}^{+\infty} \left( \int_0^{\pi} \frac{(\sin x)^{\alpha}}{x + 2n\pi} dx + \int_{-(2n+2)\pi}^{-(2n+1)\pi} \frac{(\sin x)^{\alpha}}{x} dx \right) \\ &= 2 \int_0^{\pi} (\sin x)^{\alpha} \sum_{n=0}^{+\infty} \left( \frac{1}{x + 2n\pi} + \frac{1}{x - (2n+2)\pi} \right) dx \\ &= 2 \int_0^{\pi} (\sin x)^{\alpha} \left( \frac{1}{x} + \sum_{n=1}^{+\infty} \frac{2x}{x^2 - 4n^2\pi^2} \right) dx \\ &= \int_0^{\pi} (\sin x)^{\alpha} \cot\left(\frac{x}{2}\right) dx \\ &= 2 \int_0^{\frac{\pi}{2}} (\sin 2x)^{\alpha} \cot(x) dx \\ &= 2^{\alpha} \cdot 2 \int_0^{\frac{\pi}{2}} (\sin x)^{\alpha-1} (\cos x)^{\alpha+1} dx \\ &= 2^{\alpha} B\left(\frac{\alpha}{2}, \frac{\alpha}{2} + 1\right) \\ &= 2^{\alpha} \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2} + 1\right)}{\Gamma(\alpha + 1)} \\ &= 2^{\alpha-1} \frac{\Gamma^2\left(\frac{\alpha}{2}\right)}{\Gamma(\alpha)}. \end{aligned}$$

□

**245.** 设  $\alpha$  是一个正实数, 证明

$$\int_0^{\pi} x^{\alpha-2} \sin x dx \geq \frac{\pi^{\alpha} (\alpha + 6)}{\alpha (\alpha + 2) (\alpha + 3)}.$$

由于函数  $f(x) = \frac{1 - \cos x}{x^2}$  在  $(0, \frac{\pi}{2}]$  上是凹函数, 因此它在  $x = \frac{\pi}{2}$  处的切线在曲线  $y = f(x)$  的上方,

$$\frac{1 - \cos x}{x^2} = f(x) \leq f'\left(\frac{\pi}{2}\right) \left(x - \frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = \frac{2(6 - \pi)}{\pi^2} - \frac{4(4 - \pi)x}{\pi^3}.$$

因此对  $x \in \left[0, \frac{1}{2}\right]$ ,

$$\cos \pi s \geq 1 - 2(6 - \pi)s^2 + 4(4 - \pi)s^3 \geq 1 - 6s^2 + 4s^3,$$

且

$$\sin \pi t = \pi \int_0^t \cos \pi s ds \geq \pi \int_0^t (1 - 6s^2 + 4s^3) ds = \pi (1 - 2t^3 + t^4) := p(t).$$

进一步, 由对称性  $\sin \pi t = \sin \pi(1-t)$  以及  $p(t) = p(1-t)$  可知上述不等式对  $s \in [0, 1]$  都成立. 则

$$\begin{aligned} \int_0^\pi x^{\alpha-2} \sin x dx &= \pi^{\alpha-1} \int_0^1 t^{\alpha-2} \sin \pi t dt \geq \pi^{\alpha-1} \int_0^1 t^{\alpha-2} p(t) dt \\ &= \pi^\alpha \int_0^1 (t^{\alpha-1} - 2t^{\alpha+1} + t^{\alpha+2}) dt = \pi^\alpha \left( \frac{1}{\alpha} - \frac{2}{\alpha+2} + \frac{1}{\alpha+3} \right) \\ &= \frac{\pi^\alpha (\alpha+6)}{\alpha(\alpha+2)(\alpha+3)}. \end{aligned}$$

**246.** 对  $n \in \mathbb{N}^*$ , 设  $H_n = \sum_{k=1}^n \frac{1}{k}$ , 且

$$S_n = \sum_{k=1}^n \frac{(-1)^{n-k}}{k} \sum_{j=1}^k H_j.$$

求极限  $\lim_{n \rightarrow \infty} \frac{S_n}{\ln n}$  与  $\lim_{n \rightarrow \infty} (S_{2n} - S_{2n-1})$ .

**解** 易得  $\sum_{j=1}^k H_j = k(H_k - 1) + H_k$ , 因此

$$S_{2n} = \sum_{k=1}^{2n} (-1)^k H_k - \sum_{k=1}^{2n} (-1)^k + \sum_{k=1}^{2n} (-1)^k \frac{H_k}{k} = \frac{H_n}{2} + \sum_{k=1}^{2n} (-1)^k \frac{H_k}{k}$$

且

$$S_{2n-1} = - \sum_{k=1}^{2n-1} (-1)^k H_k - \sum_{k=1}^{2n-1} (-1)^k - \sum_{k=1}^{2n-1} (-1)^k \frac{H_k}{k} = H_{2n} - \frac{H_n}{2} - 1 - \sum_{k=1}^{2n-1} (-1)^k \frac{H_k}{k}.$$

因此

$$S_{2n} - S_{2n-1} = H_n + 1 - H_{2n} - \frac{H_{2n}}{2n} + 2 \sum_{k=1}^{2n} (-1)^k \frac{H_k}{k}.$$

当  $N \rightarrow \infty$  时,

$$\begin{aligned} \sum_{k=1}^N (-1)^k \frac{H_k}{k} &= \sum_{k=1}^N \frac{(-1)^k}{k^2} + \sum_{k=2}^N (-1)^k \frac{H_{k-1}}{k} \\ &= \sum_{k=1}^N \frac{(-1)^k}{k^2} + \frac{1}{2} \sum_{k=2}^N \sum_{j=1}^{k-1} \frac{(-1)^k}{k} \left( \frac{1}{j} + \frac{1}{k-j} \right) \\ &= \sum_{k=1}^N \frac{(-1)^k}{k^2} + \frac{1}{2} \sum_{j=1}^{N-1} \sum_{k=j+1}^N \frac{(-1)^k}{j(k-j)} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=1}^N \frac{(-1)^k}{k^2} + \frac{1}{2} \sum_{j=1}^{N-1} \frac{(-1)^j}{j} \sum_{i=1}^{N-j} \frac{(-1)^i}{i} \\
 &= \sum_{k=1}^N \frac{(-1)^k}{k^2} + \frac{1}{2} \left( \sum_{j=1}^{N-1} \frac{(-1)^j}{j} \right)^2 - \frac{1}{2} \sum_{j=1}^{N-1} \frac{(-1)^j}{j} \sum_{i=N-j+1}^{N-1} \frac{(-1)^i}{i} \\
 &\rightarrow -\frac{\pi^2}{12} + \frac{\ln^2 2}{2}.
 \end{aligned}$$

其中

$$\left| \sum_{j=1}^{N-1} \frac{(-1)^j}{j} \sum_{i=N-j+1}^{N-1} \frac{(-1)^i}{i} \right| \leq \frac{1}{\sqrt{N}} \rightarrow 0.$$

由于  $H_n = \ln n + \gamma + o(1)$ , 我们有

$$\lim_{n \rightarrow \infty} (S_{2n} - S_{2n-1}) = \ln n + 1 - \ln(2n) - \frac{\pi^2}{6} + o(1) \rightarrow \ln^2 2 - \ln 2 + 1 - \frac{\pi^2}{6}.$$

且

$$\lim_{n \rightarrow \infty} \frac{S_n}{\ln n} = \lim_{n \rightarrow \infty} \frac{H_n}{2 \ln n} = \frac{1}{2}.$$

□

**247.** 设常数  $a > 0$ , 证明

$$\lim_{x \rightarrow 0^+} \frac{1}{x} \int_0^x \left| \sin \frac{1}{t^2} \right|^a dt = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin^a x dx.$$

**证明**

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} \frac{1}{x} \int_0^x \left| \sin \frac{1}{t^2} \right|^a dt &= \lim_{x \rightarrow 0^+} \frac{1}{x} \int_{x^{-2}}^{\infty} \frac{|\sin u|^a}{2u^{3/2}} du \\
 &= \lim_{x \rightarrow +\infty} \sqrt{x} \int_x^{\infty} \frac{|\sin u|^a}{2u^{3/2}} du \\
 &= \lim_{k \rightarrow \infty} \sqrt{k} \pi \int_{k\pi}^{\infty} \frac{|\sin u|^a}{2u^{3/2}} du \\
 &= \lim_{k \rightarrow \infty} \sqrt{k} \pi \sum_{i=k}^{\infty} \int_{i\pi}^{(i+1)\pi} \frac{|\sin u|^a}{2u^{3/2}} du \\
 &= \lim_{k \rightarrow \infty} \frac{\sqrt{k} \pi}{2} \int_0^{\pi} |\sin u|^a \sum_{i=k}^{\infty} \frac{1}{(u + i\pi)^{3/2}} du \\
 &= \lim_{k \rightarrow \infty} \frac{\sqrt{k}}{2\pi} \int_0^{\pi} \sin^a u \sum_{i=k}^{\infty} \frac{1}{(i + \frac{u}{\pi})^{3/2}} du \\
 &= \lim_{k \rightarrow \infty} \frac{\sqrt{k}}{2\pi} \frac{2}{\sqrt{k}} \int_0^{\pi} \sin^a u du \\
 &= \frac{1}{\pi} \int_0^{\pi} \sin^a u du = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin^a u du.
 \end{aligned}$$

其中当  $k \rightarrow \infty$  时,

$$\sum_{i=k}^{\infty} \frac{1}{(i + \frac{u}{\pi})^{3/2}} \sim \int_k^{\infty} \frac{dx}{x^{3/2}} = \frac{2}{\sqrt{k}}.$$

□

248. 证明

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \frac{\arctan x}{(1+x^2)^n} dx = \frac{\pi^2}{6}.$$

**证明** 首先利用 Wallis 公式可知

$$I_n = \int_0^{\infty} \frac{\arctan x}{(1+x^2)^n} dx = \int_0^{\frac{\pi}{2}} t \cos^{2n-2} t dt \rightarrow 0.$$

再由分部积分可得

$$\begin{aligned} I_n &= \left. \frac{x \arctan x}{(1+x^2)^n} \right|_0^{\infty} - \int_0^{\infty} \frac{x}{(1+x^2)^{n+1}} dx + 2n \int_0^{\infty} \frac{x^2 \arctan x}{(1+x^2)^{n+1}} dx \\ &= \left. \frac{1}{2n(1+x^2)^n} \right|_0^{\infty} + 2n \int_0^{\infty} \frac{\arctan x}{(1+x^2)^n} dx - 2n \int_0^{\infty} \frac{\arctan x}{(1+x^2)^{n+1}} dx \\ &= -\frac{1}{2n} + 2nI_n - 2nI_{n+1}. \end{aligned}$$

由此可得递推关系  $\frac{I_n}{n} = 2(I_n - I_{n+1}) - \frac{1}{2n^2}$ , 于是

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{I_n}{n} &= 2I_1 - 2 \lim_{n \rightarrow \infty} I_n - \sum_{n=1}^{\infty} \frac{1}{2n^2} \\ &= 2 \int_0^{\infty} \frac{\arctan x}{1+x^2} dx - \frac{\pi^2}{12} = \frac{\pi^2}{6}. \end{aligned}$$

□

249. 设  $f: [0, 1] \rightarrow \mathbb{R}$  有连续非负的二阶导数, 且  $\int_0^1 f(x) dx = 0$ , 证明:

$$10 \int_0^1 x^3 f(x) dx + 6 \int_0^1 x f(x) dx \geq 15 \int_0^1 x^2 f(x) dx.$$

**证明** 令  $P(x) = (x(1-x))^3$ , 则  $P^{(k)}(0) = P^{(k)}(1) = 0, k = 0, 1, 2$ . 因此由分部积分可得

$$\begin{aligned} \int_0^1 P^{(3)}(x) f(x) dx &= - \int_0^1 P^{(2)}(x) f'(x) dx \\ &= \int_0^1 P^{(1)}(x) f''(x) dx = - \int_0^1 P(x) f'''(x) dx \leq 0. \end{aligned}$$

易得  $P^{(3)}(x) = -12(10x^3 - 15x^2 + 6x - 1/2)$ , 因此

$$\int_0^1 \left( 10x^3 - 15x^2 + 6x - \frac{1}{2} \right) f(x) dx \geq 0.$$

注意到  $\int_0^1 f(x) dx = 0$ , 我们得到

$$10 \int_0^1 x^3 f(x) dx + 6 \int_0^1 x f(x) dx \geq 15 \int_0^1 x^2 f(x) dx.$$

□

250. 求和  $\sum_{n=1}^{\infty} \frac{4^n ((n-1)!)^2}{(2n)!}$ .

解

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{4^n ((n-1)!)^2}{(2n)!} &= \sum_{n=1}^{\infty} \frac{4^n}{2n} \frac{\Gamma(n) \Gamma(n)}{\Gamma(2n)} = \sum_{n=1}^{\infty} \frac{4^n}{2n} B(n, n) \\ &= \sum_{n=1}^{\infty} \frac{4^n}{2n} \int_0^1 x^{n-1} (1-x)^{n-1} dx \\ &= -\frac{1}{2} \int_0^1 \frac{\ln(1-4x(1-x))}{x(1-x)} dx = -2 \int_0^{\frac{1}{2}} \frac{\ln(1-2x)}{x(1-x)} dx \\ &= -2 \int_0^{\frac{1}{2}} \frac{\ln(1-2x)}{x} dx - 2 \int_0^{\frac{1}{2}} \frac{\ln(1-2x)}{1-x} dx. \end{aligned}$$

其中

$$\int_0^{\frac{1}{2}} \frac{\ln(1-2x)}{x} dx = \int_0^1 \frac{\ln(1-t)}{t} dt = -\text{Li}_2(1) = -\frac{\pi^2}{6}$$

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{\ln(1-2x)}{1-x} dx &= \int_0^1 \frac{\ln(1-t)}{2-t} dt = \int_0^1 \frac{\ln t}{1+t} dt \\ &= \sum_{n=0}^{\infty} (-1)^n \int_0^1 t^n \ln t dt = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2} \\ &= -\frac{\pi^2}{12} \end{aligned}$$

□

251. 计算积分

$$\int \frac{x}{\sqrt{e^x + (x+2)^2}} dx$$

解

$$\begin{aligned} \int \frac{x}{\sqrt{e^x + (x+2)^2}} dx &= \int \frac{x e^{-\frac{x}{2}}}{\sqrt{1 + e^{-x} (x+2)^2}} dx \\ &= -2 \int \frac{d((x+2)e^{-\frac{x}{2}})}{\sqrt{1 + e^{-x} (x+2)^2}} \\ &= -2 \ln \left( (x+2)e^{-\frac{x}{2}} + \sqrt{1 + e^{-x} (x+2)^2} \right) + C. \end{aligned}$$

□

252. 计算积分

$$\int_0^1 (-x^2 + x)^{2017} [2017x] dx.$$

解

$$\begin{aligned}
 \int_0^1 (-x^2 + x)^{2017} [2017x] dx &= \sum_{k=0}^{2016} k \int_{\frac{k}{2017}}^{\frac{k+1}{2017}} x^{2017} (1-x)^{2017} dx \\
 &= \sum_{k=1}^{2016} \sum_{i=1}^k \int_{\frac{k}{2017}}^{\frac{k+1}{2017}} x^{2017} (1-x)^{2017} dx \\
 &= \sum_{i=1}^{2016} \sum_{k=i}^{2016} \int_{\frac{k}{2017}}^{\frac{k+1}{2017}} x^{2017} (1-x)^{2017} dx \\
 &= \sum_{i=1}^{2016} \int_{\frac{i}{2017}}^1 x^{2017} (1-x)^{2017} dx \\
 &= \sum_{i=1}^{2016} \int_0^{\frac{i}{2017}} x^{2017} (1-x)^{2017} dx \\
 &= \frac{1}{2} \sum_{i=1}^{2016} \int_0^1 x^{2017} (1-x)^{2017} dx \\
 &= 1008 B(2018, 2018) = 1008 \frac{(2017!)^2}{4035!}.
 \end{aligned}$$

□

### 253. 计算积分

$$\int_{-1}^1 \frac{1}{x} \sqrt{\frac{1+x}{1-x}} \ln \frac{2x^2 + 2x + 1}{2x^2 - 2x + 1} dx.$$

解 首先有

$$\begin{aligned}
 I &= \int_{-1}^1 \frac{1}{x} \sqrt{\frac{1+x}{1-x}} \ln \frac{2x^2 + 2x + 1}{2x^2 - 2x + 1} dx \\
 &= \int_0^1 \frac{1}{x} \sqrt{\frac{1+x}{1-x}} \ln \frac{2x^2 + 2x + 1}{2x^2 - 2x + 1} dx + \int_{-1}^0 \frac{1}{x} \sqrt{\frac{1+x}{1-x}} \ln \frac{2x^2 + 2x + 1}{2x^2 - 2x + 1} dx \\
 &= \int_0^1 \frac{1}{x} \left( \sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}} \right) \ln \frac{2x^2 + 2x + 1}{2x^2 - 2x + 1} dx \\
 &= 2 \int_0^1 \frac{1}{x \sqrt{1-x^2}} \ln \frac{2x^2 + 2x + 1}{2x^2 - 2x + 1} dx \\
 &= 2 \int_0^{\frac{\pi}{2}} \frac{1}{\cos t} \ln \frac{2 \cos^2 t + 2 \cos t + 1}{2 \cos^2 t - 2 \cos t + 1} dt
 \end{aligned}$$

考虑含参变量积分

$$I(a) = \int_0^{\frac{\pi}{2}} \frac{2}{\cos t} \ln \frac{2 \cos^2 t + a \cos t + 1}{2 \cos^2 t - a \cos t + 1} dt.$$

则  $I(0) = 0$ , 且

$$\begin{aligned}
 I'(a) &= 2 \int_0^{\frac{\pi}{2}} \left( \frac{1}{2 \cos^2 t + a \cos t + 1} + \frac{1}{2 \cos^2 t - a \cos t + 1} \right) dt \\
 &= 4 \int_0^{\frac{\pi}{2}} \frac{1 + 2 \cos^2 t}{(1 + 2 \cos^2 t)^2 - a^2 \cos^2 t} dt
 \end{aligned}$$

$$\begin{aligned}
 &= 4 \int_0^{\frac{\pi}{2}} \frac{1 + 2 \cos^2 t}{4 \cos^4 t + (4 - a^2) \cos^2 t + 1} dt \\
 &\quad \vdots \\
 &= \frac{2\pi(3 + \sqrt{9 - a^2})}{\sqrt{9 - a^2} \sqrt{6 - a^2} + 2\sqrt{9 - a^2}}.
 \end{aligned}$$

这里这个积分的计算非常复杂, 我算了好几遍. 分母是四次的积分, 分子分母除以  $\cos^4 t$ , 然后  $u = \tan t$ , 于是

$$\begin{aligned}
 I &= I(2) = \int_0^2 \frac{2\pi(3 + \sqrt{9 - a^2})}{\sqrt{9 - a^2} \sqrt{6 - a^2} + 2\sqrt{9 - a^2}} da \\
 &= \int_0^{\arcsin \frac{2}{3}} \frac{2\pi(3 + 3 \cos y)}{\sqrt{6 - 9 \sin^2 y} + 6 \cos y} dy \\
 &= 12\pi \int_0^{\arcsin \frac{2}{3}} \frac{\cos^2 \frac{y}{2}}{\sqrt{12 \cos^2 \frac{y}{2} - 36 \sin^2 \frac{y}{2} \cos^2 \frac{y}{2}}} dy \\
 &= 24\pi \int_0^{\arcsin \frac{2}{3}} \frac{d(\sin(y/2))}{\sqrt{12 - 36 \sin^2 \frac{y}{2}}} \\
 &= 24\pi \int_0^{\sqrt{\frac{3-\sqrt{5}}{6}}} \frac{du}{\sqrt{12 - 36u^2}} = 4\pi \arcsin \frac{\sqrt{5} - 1}{2}.
 \end{aligned}$$

□

## 254. 求极限

$$\lim_{n \rightarrow \infty} n \left( \sum_{k=1}^n \cos^n \sqrt{\frac{k}{n}} - \frac{\sqrt{e} + 1}{e - 1} \right).$$

**解** 对固定的  $k$ , 我们有

$$\begin{aligned}
 \cos^n \sqrt{\frac{k}{n}} &= \left( 1 - \frac{k}{2n} + \frac{k^2}{24n^2} + o\left(\frac{1}{n^2}\right) \right)^n \\
 &= \exp \left( n \ln \left( 1 - \frac{k}{2n} + \frac{k^2}{24n^2} + o\left(\frac{1}{n^2}\right) \right) \right) \\
 &= \exp \left( n \left( -\frac{k}{2n} - \frac{k^2}{12n^2} + o\left(\frac{1}{n^2}\right) \right) \right) \\
 &= e^{-\frac{k}{2}} e^{-\frac{k^2}{12n} + o(\frac{1}{n})} = e^{-\frac{k}{2}} \left( 1 - \frac{k^2}{12n} + o\left(\frac{1}{n}\right) \right).
 \end{aligned}$$

注意到

$$\sum_{k=1}^{\infty} e^{-\frac{k}{2}} = \frac{e^{-\frac{1}{2}}}{1 - e^{-\frac{1}{2}}} = \frac{1}{\sqrt{e} - 1} = \frac{\sqrt{e} + 1}{e - 1}, \quad \sum_{k=1}^{\infty} k^2 e^{-\frac{k}{2}} = \frac{e + \sqrt{e}}{(\sqrt{e} - 1)^3}$$

因此

$$\sum_{k=1}^n \cos^n \sqrt{\frac{k}{n}} = \frac{\sqrt{e} + 1}{e - 1} - \frac{1}{12n} \frac{e + \sqrt{e}}{(\sqrt{e} - 1)^3} + o\left(\frac{1}{n}\right).$$

因此

$$\lim_{n \rightarrow \infty} n \left( \sum_{k=1}^n \cos^n \sqrt{\frac{k}{n}} - \frac{\sqrt{e} + 1}{e - 1} \right) = -\frac{e + \sqrt{e}}{12(\sqrt{e} - 1)^3}$$

事实上用这样的方法, 我们还可以作更精细的渐进展开, 严格计算其实我们已经利用了 Tannery 定理.

□

## 255. 计算积分

$$I = \int_0^1 \frac{x \ln \left( \frac{1+x}{1-x} \right)}{(\pi^2 + \ln^2 \left( \frac{1+x}{1-x} \right))^2} dx.$$

先换元  $x \rightarrow \tanh x$  可得

$$\begin{aligned} I &= 2 \int_0^\infty \frac{x \tanh x}{(\pi^2 + 4x^2)^2 \operatorname{sech}^2 x} dx = -\frac{1}{4} \int_0^\infty \frac{\sinh x}{\cosh^3 x} d \left( \frac{1}{\pi^2 + 4x^2} \right) \\ &= \frac{1}{4} \int_0^\infty \frac{\cosh^2 x - 3 \sinh^2 x}{(\pi^2 + 4x^2) \cosh^4 x} dx = \frac{1}{8} \int_{-\infty}^\infty \frac{\cosh^2 x - 3 \sinh^2 x}{(\pi^2 + 4x^2) \cosh^4 x} dx. \end{aligned}$$

关键就是化到这一步, 下面就套路来了, 令  $f(z) = \frac{\sinh^2 z - 3 \cosh^2 z}{z \sinh^4 z}$ , 考虑以  $x = \pm\infty, y = \pm\frac{\pi}{2}i$  为四边的逆时针矩形围道, 利用留数定理可得

$$\int_{-\infty - \frac{\pi}{2}i}^{\infty - \frac{\pi}{2}i} f(x) dx - \int_{-\infty + \frac{\pi}{2}i}^{\infty + \frac{\pi}{2}i} f(x) dx = 2\pi i \cdot \operatorname{res}(f(z), z=0)$$

注意到

$$\begin{aligned} f\left(x - \frac{\pi}{2}i\right) &= \frac{\sinh^2\left(x - \frac{\pi}{2}i\right) - 3 \cosh^2\left(x - \frac{\pi}{2}i\right)}{\left(x - \frac{\pi}{2}i\right) \sinh^4\left(x - \frac{\pi}{2}i\right)} = \frac{3 \sinh^3 x - \cosh^2 x}{\left(x - \frac{\pi}{2}i\right) \cosh^4 x} \\ f\left(x + \frac{\pi}{2}i\right) &= \frac{3 \sinh^3 x - \cosh^2 x}{\left(x + \frac{\pi}{2}i\right) \cosh^4 x}, \operatorname{res}(f(z), z=0) = -\frac{1}{15} \end{aligned}$$

于是

$$\int_{-\infty}^{\infty} \left( f\left(x - \frac{\pi}{2}i\right) - f\left(x + \frac{\pi}{2}i\right) \right) dx = \pi i \int_{-\infty}^{\infty} \frac{3 \sinh^3 x - \cosh^2 x}{\left(x^2 + \frac{\pi^2}{4}\right) \cosh^4 x} dx = -\frac{2\pi i}{15}$$

原积分等于  $\frac{1}{240}$ .

## 256. 求和

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left( n \sum_{k=n+1}^{\infty} \frac{1}{k^2} - 1 \right).$$

**解** 考虑  $a_n = (-1)^{(n-1)}$ ,  $b_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2} - \frac{1}{n}$ ,  $T_n = a_1 + \cdots + a_n = \frac{1 - (-1)^n}{4} - \frac{(-1)^n}{2}n$ . 由 Abel 求和公式可得

$$\begin{aligned} S_N &= \sum_{n=1}^N (-1)^{n-1} \left( n \sum_{k=n+1}^{\infty} \frac{1}{k^2} - 1 \right) \\ &= \sum_{n=1}^N (-1)^{n-1} n \left( \sum_{k=n+1}^{\infty} \frac{1}{k^2} - \frac{1}{n} \right) = T_N b_N + \sum_{n=1}^{N-1} T_n (b_n - b_{n+1}) \end{aligned}$$



$$= \left( \frac{1 - (-1)^N}{4} - \frac{(-1)^N}{2} N \right) \left( \sum_{k=N+1}^{\infty} \frac{1}{k^2} - \frac{1}{N} \right) - \sum_{n=1}^{N-1} \left( \frac{1 - (-1)^n}{4} - \frac{(-1)^n}{2} n \right) \left( \frac{1}{n+1} - \frac{1}{n} + \frac{1}{(n+1)^2} \right).$$

令  $N \rightarrow \infty$ , 由 Stolz 定理不难得到

$$\sum_{k=N+1}^{\infty} \frac{1}{k^2} - \frac{1}{N} \sim \frac{1}{2N^2}.$$

因此

$$\left( \frac{1 - (-1)^N}{4} - \frac{(-1)^N}{2} N \right) \left( \sum_{k=N+1}^{\infty} \frac{1}{k^2} - \frac{1}{N} \right) \rightarrow 0.$$

以及

$$\begin{aligned} & - \left( \frac{1 - (-1)^n}{4} - \frac{(-1)^n}{2} n \right) \left( \frac{1}{n+1} - \frac{1}{n} + \frac{1}{(n+1)^2} \right) \\ &= \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{4n(n+1)^2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(n+1)} \\ &= \sum_{n=1}^{\infty} \frac{1}{8n^2(2n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^n}{2} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \frac{\pi^2}{16} - \frac{\ln 2}{2} - \frac{1}{2}. \end{aligned}$$

最后一步利用  $\zeta(2) = \frac{\pi^2}{6}$  和幂级数即可. □

## 257. 证明

$$\int_0^1 \frac{\arctan(x)}{x} \ln \left( \frac{1+x^2}{(1-x)^2} \right) dx = \frac{\pi^3}{16}$$

**证明** 记  $H_k = \sum_{j=1}^k \frac{1}{j}$ ,  $k \geq 1$ . 对任意  $x \in (0, 1)$  有

$$\begin{aligned} \arctan(x) \ln(1+x^2) &= \frac{i}{2} (\ln(1-ix) - \ln(1+ix)) (\ln(1-ix) + \ln(1+ix)) \\ &= \frac{i}{2} (\ln^2(1-ix) - \ln^2(1+ix)) \\ &= -\operatorname{Im}(\ln^2(1-ix)) = -2\operatorname{Im} \left( \sum_{k=1}^{\infty} \frac{H_k (ix)^{k+1}}{k+1} \right). \end{aligned}$$

这里我们用到了

$$\begin{aligned} -\ln(1-t) &= \sum_{k=1}^{\infty} \frac{t^k}{k} \Rightarrow -\frac{\ln(1-t)}{1-t} = \sum_{k=1}^{\infty} H_k t^k \\ &\Rightarrow \ln^2(1-t) = 2 \sum_{k=1}^{\infty} \frac{H_k t^{k+1}}{k+1}. \end{aligned}$$

因此,

$$\int_0^1 \frac{\arctan(x) \ln(1+x^2)}{x} dx = -2\operatorname{Im} \left( \int_0^1 \sum_{k=1}^{\infty} \frac{H_k i^{k+1} x^k}{k+1} dx \right)$$

$$\begin{aligned}
 &= -2\text{Im} \left( \sum_{k=1}^{\infty} \frac{H_k i^{k+1}}{k+1} \int_0^1 x^k dx \right) \\
 &= -2\text{Re} \left( \sum_{k=1}^{\infty} \frac{H_k i^k}{(k+1)^2} \right).
 \end{aligned}$$

另一方面,

$$\begin{aligned}
 \int_0^1 \frac{\arctan(x) \ln(1-x)}{x} dx &= \int_0^1 \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k} \ln(1-x)}{2k+1} dx \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \int_0^1 x^{2k} \ln(1-x) dx \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \int_0^1 \ln(1-x) d(x^{2k+1} - 1) \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \int_0^1 \frac{x^{2k+1} - 1}{x-1} dx \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k H_{2k+1}}{(2k+1)^2} = -\text{Re} \left( \sum_{k=0}^{\infty} \frac{H_{k+1} i^k}{(k+1)^2} \right).
 \end{aligned}$$

因此,

$$\begin{aligned}
 \int_0^1 \frac{\arctan(x)}{x} \ln \left( \frac{1+x^2}{(1-x)^2} \right) dx &= -2\text{Re} \left( \sum_{k=1}^{\infty} \frac{H_k i^k}{(k+1)^2} \right) + 2\text{Re} \left( \sum_{k=0}^{\infty} \frac{H_{k+1} i^k}{(k+1)^2} \right) \\
 &= 2\text{Re} \left( \sum_{k=0}^{\infty} \frac{i^k}{(k+1)^3} \right) = 2\text{Im} \left( \sum_{k=1}^{\infty} \frac{i^{k-1}}{k^3} \right) \\
 &= 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} = \frac{\pi^3}{16}.
 \end{aligned}$$

□