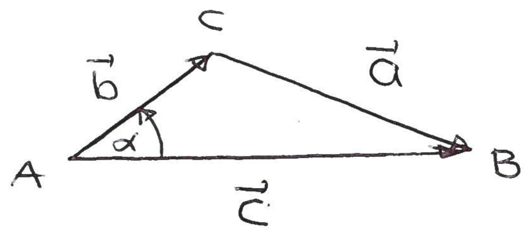
2.3.3 Exercises

1] Prove $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$

Solution:



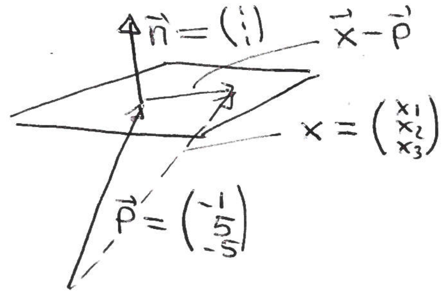
Let $\vec{a} := \vec{c} - \vec{b}$

$a := \|\vec{a}\|$, $b := \|\vec{b}\|$, $c := \|\vec{c}\|$

Then,
$$a^2 = \|\vec{a}\|^2 = (\vec{c}-\vec{b})\cdot(\vec{c}-\vec{b}) = \vec{c}\cdot\vec{c} - 2\vec{c}\cdot\vec{b} + \vec{b}\cdot\vec{b}$$
$$= \|\vec{c}\|^2 + \|\vec{b}\|^2 - 2\|\vec{c}\|\,\|\vec{b}\|\cos(\alpha)$$
$$= c^2 + b^2 - 2cb\cos(\alpha)$$

2] Find eqn of plane through $(-1, 5, -5)$, parallel to $x_1 + x_2 + x_3 = 2$

Solution: pick $\vec{n} := \begin{pmatrix}1\\1\\1\end{pmatrix}$, the normal vector of the given plane.



Then, the new plane is given by

$$\vec{n}(\vec{x} - \vec{p}) = 0 \Leftrightarrow \vec{n}\vec{x} = \vec{n}\vec{p}$$
$$\Leftrightarrow x_1 + x_2 + x_3 = -1 + 5 - 5 = -1$$

i.e. the plane is given by $\boxed{x_1 + x_2 + x_3 = -1}$

3] Find a unit vector that is normal to both $e_1 + e_2$ and $e_1 + e_3$.

Solution: a] with vector product

$$n = \frac{(e_1+e_2)\times(e_1\times e_3)}{\|(e_1+e_2)\times(e_1\times e_3)\|} = \frac{e_1\times e_1 + e_1\times e_3 + e_2\times e_1 + e_2\times e_3}{\| \quad \cdots \quad \|}$$

$$= \frac{0 - e_2 - e_3 + e_1}{\| \ \cdots \ \|} \qquad \text{by Thm 2.12 (1,3,4), Thm 2.13}$$

$$= \frac{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}}{\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}}$$

b] using row reduction of matrices, reduced row-echelon form

Note $x \perp e_1+e_2,\ e_1+e_3 \Leftrightarrow (e_1+e_2)x = 0$ and $(e_1+e_3)x = 0$

$\Leftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$. We therefore use row-reduction to transform the matrix into reduced row-echelon form!

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$ is in red. row-echelon form!

hence $\left.\begin{aligned} x_1 \quad + x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}\right\} \Rightarrow x_1 = -x_3,\ x_2 = x_3$

hence $S = \left\{ \begin{pmatrix} -x_3 \\ x_3 \\ x_3 \end{pmatrix} \middle| \ x_3 \in \mathbb{R} \right\}$, and any vector of S is orthogonal to both e_1+e_2 and e_1+e_3. We therefore pick one with length 1. Note that there are <u>two</u> choices!

$$\boxed{\vec{n} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$$

or

$$\boxed{\vec{n}\,' = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \qquad (= -\vec{n})}$$

4] Find the direction cosines and direction angles of

$$\vec{a} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

<u>Solution:</u> We first compute $\|\vec{a}\| = \sqrt{4+1+4} = 3$. Then

$\cos(\theta_1) = \frac{a_1}{\|a\|} = \frac{2}{3} \Rightarrow \theta_1 = 0.841$	$= 48.2^\circ$
$\cos(\theta_2) = \frac{a_2}{\|a\|} = \frac{-1}{3} \Rightarrow \theta_2 = 1.91$	$= 109.47^\circ$
$\cos(\theta_3) = \frac{a_3}{\|a\|} = \frac{2}{3} \Rightarrow \theta_3 = 0.841$	$= 48.2^\circ$

5] Find the scalar and vector projection of $\vec{b}$ unto $\vec{a}$, where $\vec{a} = \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix}$.

Solution: we first compute $\|\vec{a}\| = \sqrt{4+9+36} = 7$

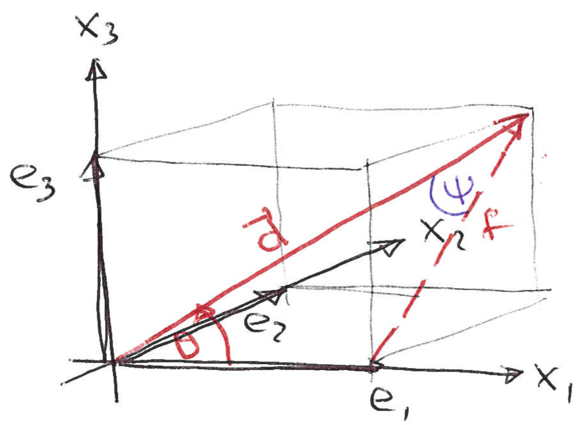
Then

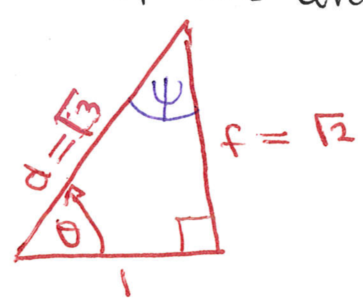
$$\boxed{\mathrm{comp}_{\vec{a}}(\vec{b})} = \frac{\vec{a}\cdot\vec{b}}{\|a\|} = \frac{-10-3-24}{7} = \frac{-37}{7} = \boxed{-\frac{37}{7}}$$

$$\boxed{\mathrm{proj}_{\vec{a}}(\vec{b})} = \mathrm{comp}_{\vec{a}}(\vec{b}) \cdot \frac{\vec{a}}{\|\vec{a}\|} = -\frac{37}{7} \cdot \frac{\begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}}{7} = \boxed{-\frac{37}{49}\begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}}$$

6] Find the angle between a spacial diagonal of a cube and one of its edges.

Solution: Note that length and angles are invariant under congruence operations, moreover, angles are invariant under dilations. Therefore, we can confine ourselves to the special case of the unit cube:





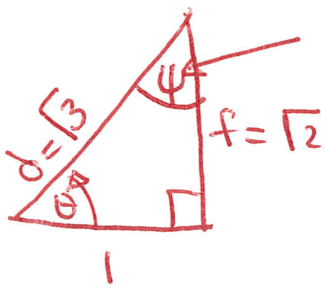
$$\theta = \arccos\left(\frac{\vec{d}\cdot\vec{e}_1}{\|\vec{d}\|\|\vec{e}_1\|}\right) = \arccos\left(\frac{1}{\sqrt{3}}\right)$$

Hence, $\boxed{\theta = 0.9553}$

$\boxed{\theta = 54.74°}$

7] Find the angle between a spacial diagonal of a cube and a diagonal of one of its faces.

Solution: we continue our computation of 6]:
The angle to be computed is the angle ψ in our diagram



$$\psi = \frac{\pi}{2} - \theta$$

$$=$$

$$=$$

in degree $\qquad \psi = 90° - 54.74° = 35.264°$