2.2.4 Exercises

1) a) expand $A_{ij} y_j$

$$A_{ij} y_j \rightsquigarrow \left(\sum_{j=1}^{3} A_{ij} y_j \right)_{i=1,2,3} = Ay \quad \text{with } A = (A_{ij}), \; y = (y_j)$$

b) same as a)

c) expand

$$A_{\ell m} y_\ell \rightsquigarrow \left(\sum_{\ell=1}^{3} A_{\ell m} y_\ell \right)_{m=1,2,3} = y^t A$$

d) expand

$$A_{ij} y_i y_j \rightsquigarrow \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} y_i y_j = y^t A y$$

e) same as d)

2) a) expand

$$B_{ij} B_{ij} \rightsquigarrow \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \underbrace{B_{ij} B_{ij}}_{B_{ij}^2} \right) = (B_{11}^2 + B_{12}^2 + B_{13}^2) + (B_{21}^2 + B_{22}^2 + B_{23}^2) + (B_{31}^2 + B_{32}^2 + B_{33}^2)$$

b) expand

$B_{ij} B_{ij}$ where $B_{ij} = B_{ji}$ (i.e B is symmetric !)

$B_{ij} y_i y_j \rightsquigarrow$ as in a) then apply $B_{ij} = B_{ji}$

$$= (B_{11} y_1^2 + B_{22} y_2^2 + B_{33} y_3^2) + 2(B_{12} y_1 y_2 + B_{13} y_1 y_3 + B_{23} y_2 y_3)$$

c] expand

$B_{ij} B_{ij}$ where $B_{ij} = - B_{ji}$ for $i \neq j$ (i.e. B is skew symmetric!)

$B_{ij} y_i y_j \rightsquigarrow$ as in b] then apply $B_{ij} = - B_{ji}$, $i \neq j$

$$= B_{11} y_1^2 + B_{22} y_2^2 + B_{33} y_3^2$$

3] a] simplify

$$\delta_{ij} y_j \rightsquigarrow \left(\sum_{j=1}^{3} \delta_{ij} y_j \right)_{i=1,2,3} = (y_i) \rightsquigarrow y_i$$

b] simplify

$$\delta_{ij} C_{ij} \rightsquigarrow \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \delta_{ij} C_{ij} \right) = (\delta_{11} C_{11} + \delta_{12} C_{12} + \delta_{13} C_{13}) +$$

$$+ (\delta_{21} C_{21} + \delta_{22} C_{22} + \delta_{23} C_{23}) + (\delta_{31} C_{31} + \delta_{32} C_{32} + \delta_{33} C_{33})$$

$$= C_{11} + C_{22} + C_{33}$$

c] simplify

$\delta_{ij} y_i y_j \rightsquigarrow$ use b] $y_1^2 + y_2^2 + y_3^2$

d] simplify

$\delta_{ij} \delta_{ij} \rightsquigarrow$ use b] 3

e] simplify

$\delta_{ij} \delta_{ik} \delta_{jk} = \delta_{ii} \delta_{ii} \delta_{ii} \rightsquigarrow 3$

4) Verify $B_{ij} = A_{ki} A_{\ell j} B'_{k\ell}$ of Theorem 2.4

In the lecture we saw that

$$
\begin{tikzcd}
\mathbb{R}^3_{\mathcal{E}} \arrow[r, "B"] \arrow[d, shift right=1ex, "A"'] & \mathbb{R}^3_{\mathcal{E}} \arrow[d, shift left=1ex, "A"] \\
\mathbb{R}^3_{\mathcal{E}'} \arrow[r, "B'"'] \arrow[u, shift right=1ex, "A^t"'] & \mathbb{R}^3_{\mathcal{E}'} \arrow[u, shift left=1ex, "A^t"]
\end{tikzcd}
$$

$B = A^t B' A$

and

$B' = A B A^t$

we translate $B = A^t B' A$ into index notation:

$$B_{ij} = A_{ki} B'_{k\ell} A_{\ell j} = A_{ki} A_{\ell j} B'_{k\ell}$$