5.2.8 Exercises

1] Find the fundamental period:

a] $f(t) = 1 + \cos(t) + \cos(2t)$

Lemma: If $\left.\begin{matrix} f \text{ has fp } c_f \\ g \text{ — " — } c_g \end{matrix}\right\} \Rightarrow f+g$ has fp. $\mathrm{lcm}(c_f, c_g)$

easy to verify!

Hence, sine $f_1(t) = 1$ has any period, $f_2(t) = \cos(t)$ has period 2π and $f_3(t) = \cos(2t)$ has fp. $\frac{2\pi}{2} = \pi$, f has fp. $\boxed{2\pi}$

b] $f(t) = \sin(5t + \pi)$

The horizontal shift by π to the left does not affect the fp.

Hence, f has fp. $\boxed{\frac{2\pi}{5}}$

c] $\cos\left[\frac{3}{2}(\pi - t)\right] = \cos\left(\frac{3}{2}\pi - \frac{3}{2}t\right) = \cos\left(\frac{3}{2}t - \frac{3}{2}\pi\right)$

As in b], fp: $\frac{2\pi}{\frac{3}{2}} = \boxed{\frac{4\pi}{3}}$

e] $f(t) = \cos(\pi t)\sin(9\pi t) = \frac{1}{2}\sin(9\pi t - \pi t) + \frac{1}{2}\sin(9\pi t + \pi t)$

$= \frac{1}{2}\sin(8\pi t) + \frac{1}{2}\sin(10\pi t)$

Hence, we need to compute the $\mathrm{lcm}\left(\frac{2\pi}{8\pi}, \frac{2\pi}{10\pi}\right) =$

$= \mathrm{lcm}\left(\frac{1}{4}, \frac{1}{5}\right) = \boxed{1}$

2] If $f: \mathbb{R} \to \mathbb{R}$ has f.p. 2π, integr. and $d \in \mathbb{R}$, then

$$\int_d^{d+2\pi} f(t)\,dt = \int_0^{2\pi} f(t)\,dt$$

proof: $\int_0^{2p} f(t)\,dt = \int_0^{d} f(t)\,dt + \int_d^{2p} f(t)\,dt$

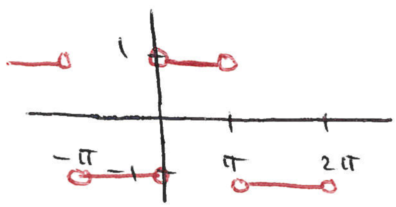
sub $s = t + 2p \Rightarrow ds = dt$, $t = s - 2p$

$s(0) = 2p$, $s(d) = d + 2p$

$$= \int_{2p}^{d+2p} \underbrace{f(s-2p)}_{\overset{\shortparallel}{f(s)}}\,ds + \int_{d}^{2p} f(t)\,dt = \int_{d}^{d+2p} f(t)\,dt$$

3] Find the Fourier expansion

a] $f(x) = \begin{cases} 1, & 0<t<\pi \\ -1, & \pi<t<2\pi \end{cases}$



Note: f has f.p. 2π, f is odd

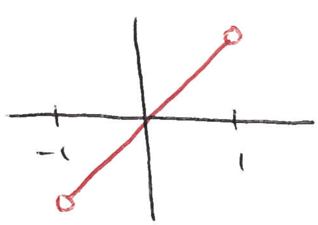
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{odd}}\,\underbrace{\cos(kx)}_{\text{even}}}_{\text{odd}}\,dx = 0 \quad \text{for } k=0,1,2,$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{\underbrace{f(t)}_{\text{odd}}\,\underbrace{\sin(kx)}_{\text{odd}}}^{\text{even}}\,dx = \frac{2}{\pi}\int_{0}^{\pi}\sin(kx)\,dx = -\frac{2\cos(kx)}{\pi k}\Big|_{0}^{\pi} =$$

$$= -\frac{2(-1)^k}{\pi k}, \quad k=1,2,\dots$$

$$\boxed{Sf(x) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(kx)}$$

b] $f(t) = t$, for $t \in (-1,1)$



Note: f has f.p. $p=1$, f is odd

hence, as above $a_k = 0$ for $k=0,1,2,\dots$

$$b_k = \frac{1}{1}\int_{-1}^{1} \overbrace{\underbrace{f(t)}_{\text{odd}}\,\underbrace{\sin\left(k\tfrac{\pi}{1}t\right)}_{\text{odd}}}^{\text{even}}\,dx = 2\int_{0}^{1}\underbrace{t}_{u}\,\underbrace{\sin(k\pi t)}_{v'}\,dt$$

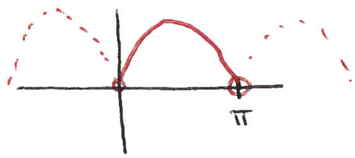
$$= 2\left[\frac{-t\cos(k\pi t)}{k\pi}\Big|_{0}^{1} + \frac{1}{k\pi}\int_{0}^{1}\cos(k\pi t)\,dt\right] = \frac{2}{k\pi}\left[(-1)^{k+1} + \frac{\sin(k\pi t)}{k\pi}\Big|_{0}^{1}\right]$$

$$= \frac{2}{k\pi}(-1)^{k+1}$$

Hence

$$\boxed{Sf(t) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(k\pi t)}$$

c| $f(t) = \sin t,\quad t \in [0, \pi)$



Note: f has f.p. π, also note that its periodic extension is even!

$$a_0 = \frac{1}{p}\int_0^{2p} f(t)\,dt = \frac{1}{\frac{\pi}{2}}\int_0^{\pi}\sin t\,dt = \frac{2}{\pi}\left[-\cos t\right]\Big|_0^{\pi} =$$

$$= \frac{2}{\pi}\left[1+1\right] = \frac{4}{\pi}$$

$$a_k = \frac{1}{p}\int_0^{2p} f(t)\cos\left(k\frac{\pi}{p}t\right)dt = \frac{2}{\pi}\int_0^{\pi}\sin t\cos(2kt)\,dt = \frac{1}{\pi}\int_0^{\pi}\left(\sin(\underbrace{t-2kt}_{(1-2k)t}) + \sin(\underbrace{t+2kt}_{(1+2k)t})\right)dt$$

$k>0$

$$= -\frac{1}{\pi}\left[\frac{\cos((1-2k)t)}{1-2k} + \frac{\cos((1+2k)t)}{1+2k}\right]_0^{\pi} = -\frac{1}{\pi}\left[\frac{\cos(\pi-2k\pi)}{1-2k} + \frac{\cos(\pi+2k\pi)}{1+2k}\right]$$

$$+\frac{1}{\pi}\left[\frac{1}{1-2k} + \frac{1}{1+2k}\right] = \frac{2}{\pi}\left[\frac{1}{1-2k} + \frac{1}{1+2k}\right] = \frac{4}{\pi}\cdot\frac{1}{1-4k^2}$$

$$b_k = \frac{1}{p}\int_0^{2p} f(t)\sin\left(k\frac{\pi}{p}t\right)dt = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\underbrace{f_e(t)}_{\text{even}}\,\underbrace{\sin(2kt)}_{\text{odd}}}_{\text{odd}}\,dt = 0$$

Hence

$$\boxed{Sf(t) = \frac{4}{\pi} + \sum_{k=1}^{\infty}\frac{4}{\pi}\cdot\frac{1}{1-4k^2}\cos(2kt)}$$

$$\boxed{= \frac{4}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kt)}{1-4k^2}}$$

4| show that

a| $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

Solution: we have shown in Example 5.9 that

$$4-t^2 = \frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\left(\frac{n\pi}{2}t\right) \quad \text{for } t=[-2,2]$$

Thus, by choosing $t=2$, we obtain

$$0 = \frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\underbrace{\frac{(-1)^{n+1}}{n^2}(-1)^n}_{\frac{-1}{n^2}}$$

hence $$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{8}{3}\cdot\frac{\pi^2}{16} = \frac{\pi^2}{6}$$

b) Show that $$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Solution: Recall from Example 5.10 that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = \frac{\pi^2}{12}$$ and from above that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$, thus adding these equations, we obtain

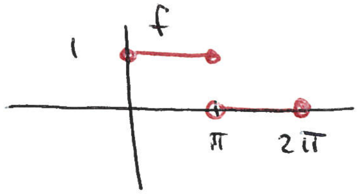
$$\underbrace{\frac{\pi^2}{6}+\frac{\pi^2}{12}}_{\frac{3\pi^2}{12}=\frac{\pi^2}{4}} = \sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = \sum_{n=1}^{\infty}\underbrace{\left[1+(-1)^{n+1}\right]}_{=\begin{cases}2 & n \text{ odd}\\ 0 & n \text{ even}\end{cases}}\frac{1}{n^2}$$

$$= 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

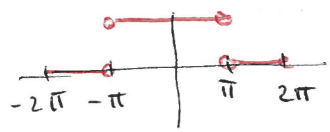
and thus

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

5] a] $f(t) = \begin{cases} 1, & 0 \le t \le \pi \\ 0, & \pi < t \le 2\pi \end{cases}$



i] half-range cosine expansion



f_e, Note $p = 2\pi$

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} dt = 1$$

$$a_k = \frac{2}{2\pi}\int_0^{2\pi} f(t)\cos\left(k\frac{\pi}{2\pi}t\right)dt = \frac{1}{\pi}\int_0^{\pi}\cos\left(\frac{k}{2}t\right)dt = \frac{2}{k\pi}\sin\left(\frac{k}{2}t\right)\Big|_0^{\pi}$$

$k > 0$

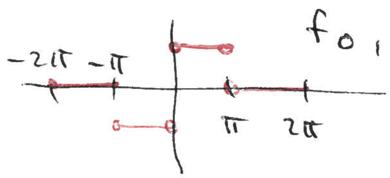
$$= \frac{2}{k\pi}\sin\left(k\frac{\pi}{2}\right) = \frac{2}{k\pi}\begin{cases} 0 & \text{if } k \equiv 0 \bmod 4 \\ 1 & k \equiv 1 \bmod 4 \\ 0 & k \equiv 2 \bmod 4 \\ -1 & k \equiv 3 \bmod 4 \end{cases}$$

or

$$a_{2\ell-1} = \frac{2}{(2\ell-1)\pi}(-1)^{\ell+1}, \quad \ell = 1, 2, 3, \dots$$

hence $$\boxed{Sf_e(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell+1}}{2\ell-1}\cos\left(\frac{(2\ell-1)}{2}x\right)}$$

ii] half-range sine expansion



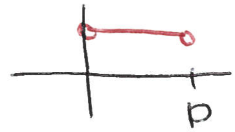
f_o, Note $p = 2\pi$

$$b_k = \frac{2}{2\pi}\int_0^{2\pi} f(t)\sin\left(k\frac{\pi}{2\pi}t\right)dt = \frac{1}{\pi}\int_0^{\pi}\sin\left(\frac{k}{2}t\right)dt =$$

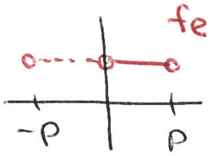
$$= -\frac{2}{k\pi}\cos\left(\frac{k}{2}t\right)\Big|_0^{\pi} = \frac{2}{k\pi}\left[1 - \cos\left(k\frac{\pi}{2}\right)\right]$$

hence $$\boxed{Sf_o(t) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1-\cos\left(\frac{k\pi}{2}\right)}{k}\sin\left(\frac{k}{2}t\right)}$$

b) $f(t)=1$ for $t\in(0,p)$, $p>0$



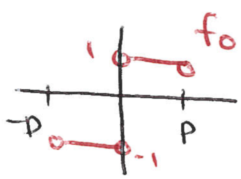
i) half-range cosine expansion



Note that fe periodically extended is $(f_e)_{ex}(x)=1 \ \forall x\in\mathbb{R}$, thus $\boxed{Sf_e(x)=1}$ which can be easily confirmed:

$$a_0=\frac{2}{p}\int_0^p dt = 2,\quad \underset{k>0}{a_k} = \frac{2}{p}\int_0^p \cos\left(\frac{k\pi}{p}t\right)dt = \frac{2}{k\pi}\sin\left(\frac{k\pi}{p}t\right)\Big|_0^p = 0$$

ii) half-range sine expansion



$$b_k = \frac{2}{p}\int_0^p \sin\left(\frac{k\pi}{p}t\right)dt = -\frac{2}{p}\frac{p}{k\pi}\cos\left(\frac{k\pi}{p}t\right)\Big|_0^p$$

$$= \frac{2}{k\pi}\left[\cos(k\pi)-1\right] = \frac{2}{k\pi}\left[(-1)^k-1\right]$$

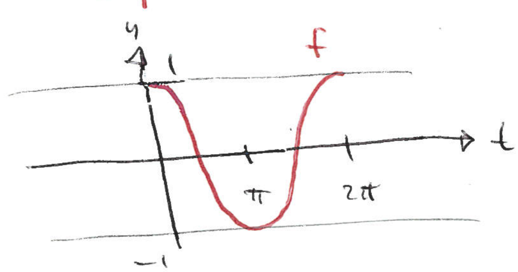
$$= \begin{cases} \frac{-4}{k\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

hence

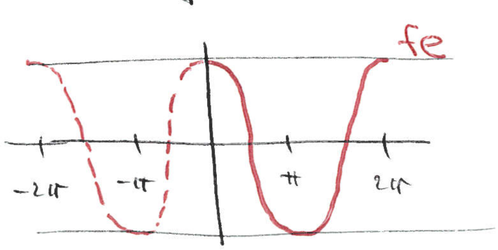
$$\boxed{Sf_o(t) = \sum_{\ell=1}^{\infty} \frac{-4}{(2\ell-1)\pi}\sin\left(\frac{(2\ell-1)\pi}{p}t\right)}$$

$$\boxed{= -\frac{4}{\pi}\sum_{\ell=1}^{\infty}\frac{\sin\left(\frac{(2\ell-1)\pi t}{p}\right)}{2\ell-1}.}$$

c) $f(t)=\cos(t)$, $t\in(0,2\pi)$



i) half-range cosine expansion



Note that the periodic extension of fe is cos, hence

$$Sf_e(x) = \cos x,$$

which can be easily confirmed:

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} \cos(t)\,dt = \frac{1}{\pi}\sin(t)\Big|_0^{2\pi} = 0$$

$$a_k = \frac{2}{2\pi}\int_0^{2\pi} \cos(t)\cos\left(k\frac{\pi}{2\pi}t\right)dt = \frac{1}{\pi}\int_0^{2\pi}\cos(t)\cos\left(\frac{kt}{2}\right)dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\Big(\cos(\underbrace{t-\tfrac{kt}{2}}_{\frac{t(2-k)}{2}}) + \cos(\underbrace{t+\tfrac{kt}{2}}_{\frac{t(2+k)}{2}})\Big)dt$$

$$= \begin{cases} \frac{1}{2\pi}\left[\frac{\sin\left(\frac{t(2-k)}{2}\right)}{\frac{2}{2-k}} + \frac{\sin\left(\frac{t(2+k)}{2}\right)}{\frac{2}{2+k}}\right]\Bigg|_0^{2\pi}, & k \neq 2 \\ \frac{1}{2\pi}\left[t + \frac{\sin(2t)}{2}\right]\Bigg|_0^{2\pi}, & k = 2 \end{cases}$$

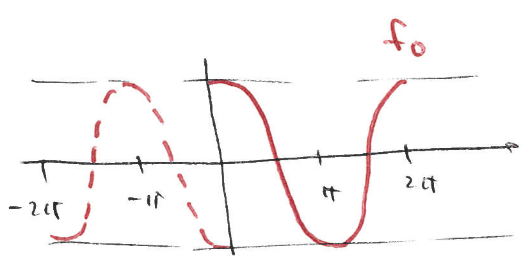
(note $k > 0$!)

$$= \begin{cases} 0, & k \neq 2 \\ 1, & k = 2 \end{cases}$$

Hence

$$Sf_e(x) = \sum_{k=1}^{\infty} a_k \cos\left(k\frac{\pi}{2\pi}x\right) = \cos(x)$$

ii] halfrange sine expansion



$$b_k = \frac{2}{2\pi}\int_0^{2\pi}\cos(t)\sin\left(k\frac{\pi}{2\pi}t\right)dt$$

$$= \frac{1}{\pi}\int_0^{2\pi}\sin\left(\frac{kt}{2}\right)\cos(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\Big(\sin(\underbrace{\tfrac{kt}{2}-t}_{\frac{t(k-2)}{2}}) + \sin(\underbrace{\tfrac{kt}{2}+t}_{\frac{t(k+2)}{2}})\Big)dt$$

$$= \begin{cases} \left. \dfrac{-\cos\left(\frac{kt}{2} - t\right)}{\frac{2}{k-2}} - \dfrac{\cos\left(\frac{kt}{2} + t\right)}{\frac{2}{k+2}} \right|_0^{2\pi}, & k \neq 2 \\ \left. -\dfrac{\cos(2t)}{2} \right|_0^{2\pi}, & k = 2 \end{cases}$$

$$= \begin{cases} \dfrac{-(-1)^k + 1}{\frac{2}{k-2}} + \dfrac{-(-1)^k + 1}{\frac{2}{k+2}}, & k \neq 2 \\ 0, & k = 2 \end{cases}$$

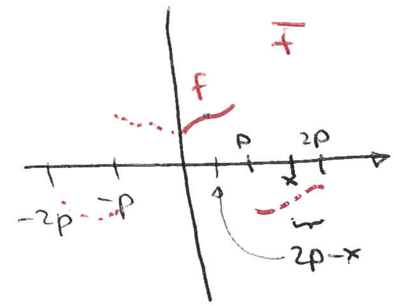
$$= \begin{cases} \dfrac{(1-(-1)^k)(k^2-4)}{2k}, & k \neq 2 \\ 0, & k = 2 \end{cases}$$

$$= \begin{cases} \dfrac{4k^2 - 4k - 3}{2k-1}, & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

hence $$\boxed{Sf_o(x) = \sum_{k=1}^{\infty} \frac{4k^2-4k-3}{2k-1} \sin\left(\frac{kt}{2}\right)}$$

6] If $a_k = 0$ for all $k = 0, 1, 2, \ldots$, then the expanded function is odd. However, since $a_0 \neq 0$, the function is obtained from an odd function that is shifted vertically by a_0.

7]



$$\boxed{\bar f(x) := \begin{cases} f(x), & x \in [0,p] \\ -f(2p-x), & x \in (p,2p] \end{cases}}$$

we compute the Fourier coefficients of $\bar f_e$: Note: $P = 2p$

$$a_0 = \frac{2}{2p}\int_0^{2p} \bar f(x)\,dx = \frac{1}{p}\int_0^{p} f(x)\,dx + \frac{1}{p}\int_p^{2p} -f(\underbrace{2p-x}_{})\,dx$$

sub $z = 2p - x \Rightarrow x = 2p - z$

$z(p) = p \quad z(2p) = 0$

$dz = -dx$

$$= \frac{1}{p}\int_0^{p} f(x)\,dx + \frac{1}{p}\int_p^{0} f(z)\,dz = 0$$

$$a_k = \frac{2}{2p}\int_0^{2p} \bar f(x)\cos\left(k\frac{\pi}{2p}x\right)dx = \frac{1}{p}\int_0^{p} f(x)\cos\left(\frac{k\pi x}{2p}\right)dx + \frac{1}{p}\int_p^{2p} -f(2p-x)\cos\left(\frac{k\pi x}{2p}\right)dx$$

sub $z = 2p - x$, $x = 2p - z$

$z(p) = p$, $z(2p) = 0$

$dz = -dx$

$$= \frac{1}{p}\int_0^{p} f(x)\cos\left(\frac{k\pi x}{2p}\right)dx - \frac{1}{p}\int_0^{p} f(z)\,\underbrace{\cos\left(\frac{k\pi}{2p}(2p-z)\right)}_{\underbrace{\cos(k\pi)}_{(-1)^k}\cos\left(\frac{k\pi z}{2p}\right) + \underbrace{\sin(k\pi)\sin\left(\frac{k\pi z}{2p}\right)}_{=0}}dz$$

$$= \frac{1}{p}\int_0^{p} f(x)\cos\left(\frac{k\pi x}{2p}\right)dx - (-1)^k\frac{1}{p}\int_0^{p} f(z)\cos\left(\frac{k\pi z}{2p}\right)dz$$

$$= \underbrace{\left[1-(-1)^k\right]}_{\begin{cases} 0, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}}\frac{1}{p}\int_0^{p} f(x)\cos\left(\frac{k\pi x}{2p}\right)dx$$

hence

$$\boxed{\begin{aligned} &a_{2\ell} = 0 \\ &a_{2\ell-1} = \frac{2}{p}\int_0^{p} f(x)\cos\left(\frac{(2\ell-1)\pi x}{2p}\right)dx, \quad \ell = 1,2,\dots \end{aligned}}$$

and $\boxed{S\bar f_e(x) = \sum_{\ell=1}^{\infty} a_{2\ell-1}\cos\left(\frac{(2\ell-1)\pi x}{2p}\right)dx}$