2.6.6 Exercises

1) Let $r_1(t) = e^{-t}\cos(t)$,
$r_2(t) = e^{-t}\sin(t)$
$r_3(t) = e^{-t}$.

Find the equation of the tangent line to the curve defined by $\vec{r}$ at $(1,0,1)$.

Solution: a) compute $\vec{r}\,'$

$$\vec{r}\,'(t) = \left(r_i'(t) \right) = \begin{pmatrix} -e^{-t}\cos(t) - e^{-t}\sin(t) \\ -e^{-t}\sin(t) + e^{-t}\cos(t) \\ -e^{-t} \end{pmatrix}$$

b) ~~For~~ which $t \in \mathbb{R}$ is $\vec{r}(t) = (1,0,1)$

$\vec{r}(t) = (1,0,1) \Leftrightarrow e^{-t}\cos(t) = 1$ and $e^{-t}\sin(t) = 0$ and $e^{-t} = 1$
$\Rightarrow t = 0$

c) tangent line

$$\vec{x}(\lambda) = \vec{r}(0) + \lambda \vec{r}\,'(0)$$

i.e. $\boxed{\vec{x}(\lambda) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}}$

2) Evaluate the integral $\int \begin{pmatrix} e^t \\ 2t \\ \ln t \end{pmatrix} dt$

Solution

$$\boxed{\int \begin{pmatrix} e^t \\ 2t \\ \ln t \end{pmatrix} dt} = \begin{pmatrix} \int e^t\,dt \\ \int 2t\,dt \\ \int \ln t\,dt \end{pmatrix} = \boxed{\begin{pmatrix} e^t + C_1 \\ t^2 + C_2 \\ t\ln t - t + C_3 \end{pmatrix}}$$

Recall $\int \ln x\,dx = \int \underbrace{1}_{u'} \cdot \underbrace{\ln x}_{v}\,dx = \underbrace{x}_{u}\,\underbrace{\ln x}_{v} - \int \underbrace{x}_{u} \cdot \underbrace{\tfrac{1}{x}}_{v'}\,dx$

$$= x\ln x - x + C$$

3] Prove: If u, v are diff vector functions, then

$$(u \cdot v)' = u' \cdot v + u v'$$

<u>proof</u>: Recall: $(u \cdot v)(t) := u(t) \cdot v(t) = u_i(t) v_i(t)$

Hence, $(u \cdot v)'(t) = [u_i(t) v_i(t)]' = u_i'(t) v_i(t) +$
Product Rule for 1-d fu!

$+ u_i(t) v_i'(t) = [u_i' v_i + u_i v_i'](t)$.

Hence $[u \cdot v]' = u' \cdot v + u \cdot v'$.

4] Prove: If u, v are diff. vector functions, then

$$(u \times v)' = u' \times v + u \times v'$$

<u>proof</u>: Recall: $(u \times v)(t) := u(t) \times v(t) = \varepsilon_{ijk} u_j(t) v_k(t)$

Hence, $(u \times v)'(t) = \left(\varepsilon_{ijk} u_j(t) v_j(t)\right)' = \varepsilon_{ijk} u_j'(t) v_j(t) +$
product rule for 1-d fu.!

$+ \varepsilon_{ijk} u_j(t) v_j'(t) = u'(t) \times v(t) + u(t) \times v'(t)$.

Hence, $(u \times v)' = u' \times v + u \times v'$.

5] Show that any vector field of the form

$$\vec{F}(x_1, x_2, x_3) = \begin{pmatrix} F_1(x_1) \\ F_2(x_2) \\ F_3(x_3) \end{pmatrix}$$

is irrotational.

<u>Solution</u>: Show that $\operatorname{curl}(\vec{F}) = 0$:

$$\operatorname{curl}(\vec{F}) = \varepsilon_{ijk} F_{k,j} = \begin{pmatrix} \varepsilon_{123} F_{3,2} + \varepsilon_{132} F_{2,3} \\ \varepsilon_{213} F_{3,1} + \varepsilon_{231} F_{1,3} \\ \varepsilon_{312} F_{2,1} + \varepsilon_{321} F_{1,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

since $F_{i,j} = 0$ for $i \neq j$ by definition of F_i.

6] Determine whether the vector field

$$\vec{F}(x_1, x_2, x_3) = \begin{pmatrix} 2x_1 x_2 \\ x_1^2 + 2x_2 x_3 \\ x_2^2 \end{pmatrix}$$

is conservative. If it is conservative, find a potential function f.

<u>Solution</u>: a] $\vec{F}$ is conservative:

$$\operatorname{curl}(\vec{F}) = \varepsilon_{ijk} F_{k,j} = \begin{pmatrix} F_{3,2} - F_{2,3} \\ F_{1,3} - F_{3,1} \\ F_{2,1} - F_{1,2} \end{pmatrix} = \begin{pmatrix} 2x_2 - 2x_2 \\ 0 - 0 \\ 2x_1 - 2x_1 \end{pmatrix} = \theta$$

b] compute a potential function of $\vec{F}$:

By Thm 2.22, $\vec{F}$ possesses a potential function f, with

$$f_{,1}(x_1, x_2, x_3) = 2x_1 x_2 \qquad (*)$$
$$f_{,2}(x_1, x_2, x_3) = x_1^2 + 2x_2 x_3 \qquad (**)$$
$$f_{,3}(x_1, x_2, x_3) = x_2^2 \qquad (***)$$

Integrating eqn. $(*)$ with respect to x_1, we obtain

$$f(x_1, x_2, x_3) = x_1^2 x_2 + C(x_2, x_3) \qquad (+)$$

Differentiating $(+)$ with respect to x_2, we obtain

$$f_{,2}(x_1, x_2, x_3) = x_1^2 + C_{,2}(x_2, x_3)$$

which by $(**)$ equals $x_1^2 + 2x_2 x_3$. Hence

$$C_{,2}(x_2, x_3) = 2x_2 x_3 \qquad (++)$$

Integrating (++) with respect to x_2, we obtain

$$c(x_2, x_3) = x_2^2 x_3 + d(x_3)$$

and therefore

$$f(x_1, x_2, x_3) = x_1^2 x_2 + x_2^2 x_3 + d(x_3) \quad (+++)$$

Differentiating (+++) with respect to x_3, yields

$$f_{,3}(x_1, x_2, x_3) = x_2^2 + d_{,3}(x_3)$$

which, by (***), equals x_2^2, hence $d_{,3}(x_3) = 0$.

Hence, $d(x_3) = C$, and therefore

$$\boxed{f(x_1, x_2, x_3) = x_1^2 x_2 + x_2^2 x_3 + C}, \quad C \in \mathbb{R}.$$

7| Prove: $\operatorname{curl}(f\vec{F}) = (\nabla f) \times \vec{F} + f \operatorname{curl}(\vec{F})$

<u>proof</u>: $\operatorname{curl}(f\vec{F}) = \varepsilon_{ijk}(f\vec{F})_{k,j} = \varepsilon_{ijk}(f\vec{F}_k)_{,j} =$

$$\underset{\text{product rule for l-d}}{=} \varepsilon_{ijk}(f_{,j}F_k + f F_{k,j}) = \varepsilon_{ijk} f_{,j} F_k + \varepsilon_{ijk} f F_{k,j}$$

$$= (\nabla f) \times \vec{F} + f \operatorname{curl}(\vec{F})$$

8| Prove: $\operatorname{curl}(\nabla f) = 0$

<u>proof</u>: $\operatorname{curl}(\nabla f) = \varepsilon_{ijk}(\nabla f)_{k,j} = \varepsilon_{ijk}(f_{,k})_{,j} =$

$$= \varepsilon_{ijk} f_{,kj} = \begin{pmatrix} \varepsilon_{123} f_{,32} + \varepsilon_{132} f_{,23} \\ \varepsilon_{213} f_{,31} + \varepsilon_{231} f_{,13} \\ \varepsilon_{312} f_{,21} + \varepsilon_{321} f_{,12} \end{pmatrix} =$$

$$= \begin{pmatrix} f_{,32} - f_{,23} \\ -f_{,31} + f_{,13} \\ f_{,21} - f_{,12} \end{pmatrix} = \Theta$$ provided the 2nd-order partials are continuous (Thm. 2.18)

9| Prove $\nabla\cdot(\nabla\times\vec{F}) = 0$ (see Thm. 2.23)

proof: $\underbrace{\nabla\cdot(\underbrace{\nabla\times\vec{F}}_{\mathrm{curl}(F)})}_{\mathrm{div}} = \nabla\cdot(\varepsilon_{ijk} F_{k,j}) = (\varepsilon_{ijk} F_{k,j})_{i,i}$

$= \varepsilon_{ijk} F_{k,ji} = (\varepsilon_{123} F_{3,21} + \varepsilon_{132} F_{2,31}) + (\varepsilon_{213} F_{3,12} + \varepsilon_{231} F_{1,32})$

$+ (\varepsilon_{312} F_{2,13} + \varepsilon_{321} F_{1,23}) =$

$= \cancel{F_{3,21}} - \cancel{F_{2,31}} - \cancel{F_{3,12}} + \cancel{F_{1,32}} + \cancel{F_{2,13}} - \cancel{F_{1,23}} = 0$

10| Prove $\nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$

proof: $\underbrace{\nabla\times(\underbrace{\nabla\times\vec{F}}_{})}_{\text{curl}(\text{ curl}(\vec{F}))} = \varepsilon_{ijk}(\nabla\times\vec{F})_{k,j} = \varepsilon_{ijk}(\varepsilon_{krs} F_{s,r})_{,j}$

$= \underbrace{\varepsilon_{kij}}_{=\varepsilon_{ijk}} \varepsilon_{krs} F_{s,rj} \xlongequal[\varepsilon-\delta \text{ identity}]{} (\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}) F_{s,rj} =$

$= \underbrace{\delta_{ir}\underbrace{\delta_{js} F_{s,rj}}_{F_{j,rj}}}_{F_{j,ij}} - \underbrace{\delta_{is}\underbrace{\delta_{jr} F_{s,rj}}_{F_{s,jj}}}_{F_{i,jj}} =$ (use Eqn 2.123, p.77)

$$= F_{j,ij} - F_{i,jj} = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$$