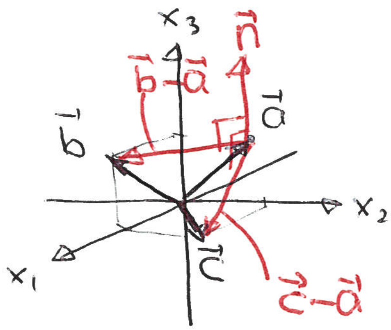
2.4.4 Exercises

1] Find an equation of the plane through the points (0,1,1), (1,0,1) and (1,1,0)

Solution: Let $\vec{a} = \begin{pmatrix}0\\1\\1\end{pmatrix}$, $\vec{b} = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $\vec{c} = \begin{pmatrix}1\\1\\0\end{pmatrix}$



To compute a normal vector to the plane determined by the three points, consider

$$\vec{b}-\vec{a} = \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\vec{c}-\vec{a} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

which are direction vector of the plane and

$$\vec{n} = (\vec{b}-\vec{a}) \times (\vec{c}-\vec{a}) = \begin{pmatrix}1\\-1\\0\end{pmatrix} \times \begin{pmatrix}1\\0\\-1\end{pmatrix} =$$

$$= \begin{vmatrix} e_1 & 1 & 1 \\ e_2 & -1 & 0 \\ e_3 & 0 & -1 \end{vmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

Hence, the plane is given by $\vec{n}(\vec{x}-\vec{a}) = 0$

$$\boxed{x_1 + x_2 + x_3 = 2}$$

2] Find a parametric equation for the line of intersection of the two planes $x_1 + x_2 + x_3 = 1$ and $x_1 + 2x_2 + 2x_3 = 1$ and find the angle between these planes.

Solution a) line of intersection:
In order to find the line of intersection, we solve the (under-determined) system of linear equations using elementary row-reduction:

$$\left.\begin{array}{l} x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + 2x_3 = 1 \end{array}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

This matrix is in reduced row-echelon form.

Hence, if $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is an element of the solution set of the system of equation, it follows that

$$\left.\begin{array}{l} x_1 \qquad\quad = 1 \\ \quad x_2 + x_3 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x_1 = 1 \\ x_2 = -x_3 \end{array}\right.$$

Hence

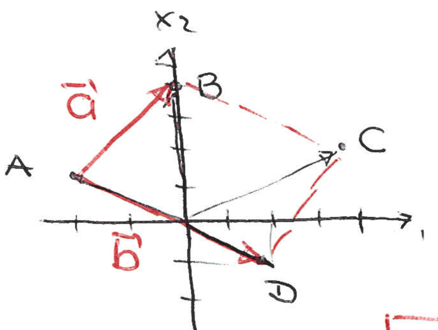
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

i.e. the intersection of the two planes is

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$

which is a line in parametric form (see Def 2.10, p 55)

3] Find the area A of the parallelogram with vertices $A(-2,1)$, $B(0,4)$, $C(4,2)$, $D(2,-1)$.



$$\vec{a} = \vec{OB} - \vec{OA} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

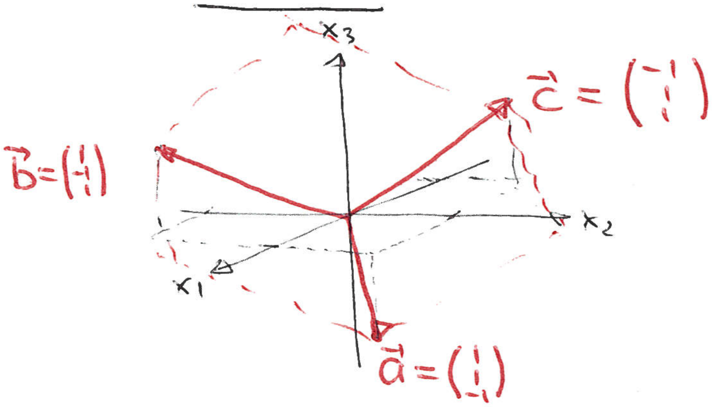
$$\vec{b} = \vec{OD} - \vec{OA} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \equiv \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$$

Then, by Example 2.8 on p 57,

$$\boxed{A = \|\vec{a} \times \vec{b}\| = \left\| \begin{vmatrix} e_1 & 2 & 4 \\ e_2 & 3 & -2 \\ e_3 & 0 & 0 \end{vmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \\ -16 \end{pmatrix} \right\| = 16}$$

4] Find the volume of the parallelepiped spanned by the vectors $\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}$.

Solution



By Example 2.10 (p. 59) the volume of the parallelepiped spanned by these vectors $\vec{a}, \vec{b}, \vec{c}$ is

$$\boxed{V = |(\vec{a} \times \vec{b}) \cdot \vec{c}|} \underset{p.\,59}{\overset{\text{Thm 2.15}}{=\!=}}$$

$$= \left| \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} \right| \underset{R_3 \to R_3 + R_1}{\overset{R_2 \to R_2 - R_1}{=\!=}}$$

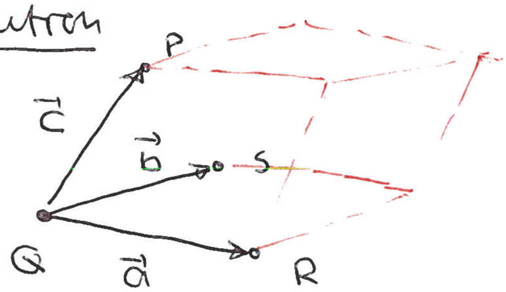
$$= \begin{vmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{vmatrix} = \left| -2 \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} \right| = |-2 \cdot 2| = \boxed{4}$$

5] a] Suppose that P is a point not on the plane that passes through the points Q, R, and S. Show that the distance d from P to the plane is given by

$$d = \frac{|(a \times b) \cdot c|}{\|a \times b\|}$$

when $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{QS}$, $\vec{c} = \overrightarrow{QP}$

Solution



Note that $\vec{a}, \vec{b}, \vec{c}$ span a parallelepiped of volume

$V = |(\vec{a} \times b) \cdot \vec{c}|$ and

and base $\|\vec{a}\times\vec{b}\|$, and height d. Thus

$$V = \text{base} \times d$$

$$\Leftrightarrow \quad |(\vec{a}\times b)\cdot\vec{c}| = \|\vec{a}\times\vec{b}\|\, d$$

and thus

$$d = \frac{|(\vec{a}\times\vec{b})\cdot\vec{c}|}{\|\vec{a}\times\vec{b}\|}$$

5b) Use this formula to find the distance from the point $P(2,1,4)$ to the plane through $Q(1,0,0)$, $R(0,2,0)$ and $S(0,0,3)$.

Solution: $\vec{a} = \overrightarrow{QR} = \begin{pmatrix}0\\2\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$, $\vec{b} = \overrightarrow{QS} = \begin{pmatrix}0\\0\\3\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\3\end{pmatrix}$, $\vec{c} = \overrightarrow{QP} = \begin{pmatrix}2\\1\\4\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\4\end{pmatrix}$. Then

$$\boxed{d} = \frac{\left|\begin{vmatrix}-1 & -1 & 1\\ 2 & 0 & 1\\ 0 & 3 & 4\end{vmatrix}\right|}{\left\|\begin{vmatrix}e_1 & -1 & -1\\ e_2 & 2 & 0\\ e_3 & 0 & 3\end{vmatrix}\right\|} \overset{C_1 \to C_1 - 2C_3}{=\!=\!=} \frac{\left|\begin{vmatrix}-3 & -1 & 1\\ 0 & 0 & 1\\ -8 & 3 & 4\end{vmatrix}\right|}{\left\|\begin{pmatrix}6\\3\\2\end{pmatrix}\right\|} = \frac{\left|-\begin{vmatrix}-3 & -1\\ -8 & 3\end{vmatrix}\right|}{\sqrt{36+9+4}}$$

$$= \frac{|-9-8|}{7} = \boxed{\frac{17}{7}}$$

6) Show that

a) $\varepsilon_{ijk}\varepsilon_{kji} = -6$

Solution:

$$\varepsilon_{ijk}\varepsilon_{kji} = \sum_i\sum_j\sum_k \varepsilon_{ijk}\varepsilon_{kji} = \overbrace{\varepsilon_{123}}^{1}\overbrace{\varepsilon_{321}}^{-1} + \overbrace{\varepsilon_{132}}^{-1}\overbrace{\varepsilon_{231}}^{1} +$$

$$+ \underbrace{\varepsilon_{231}}_{1}\underbrace{\varepsilon_{132}}_{-1} + \underbrace{\varepsilon_{213}}_{-1}\underbrace{\varepsilon_{312}}_{1} + \underbrace{\varepsilon_{312}}_{1}\underbrace{\varepsilon_{213}}_{-1} + \underbrace{\varepsilon_{321}}_{-1}\underbrace{\varepsilon_{123}}_{1}$$

$= -6$

b) $\varepsilon_{kks} = 0$

Solution: clear, since kks is not a permutation of 1, 2, 3 for any value of $k, s \in \{1,2,3\}$.

c) $\varepsilon_{ijk} a_j a_k = 0$

Solution: By Cor 2.1 (p 60)

$\varepsilon_{ijk} a_j a_k = \vec{a} \times \vec{a} = \vec{0}$ by Thm 2.11 (p. 57)

7) Write in indicial notation:

a) the trace of the matrix A:

Solution: $\boxed{tr(A) = A_{ii}}$

b) the determinant of the matrix A:

Solution:

$$\boxed{|A| = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}}$$

8) Write the following matrix equation in expanded and indicial notation:

a) $AB = C$

Solution :

i] expanded notation

$$\left(\sum_{j=1}^{3} A_{ij} B_{jk} \right)_{i,k=1,2,3} = \left(C_{ik} \right)_{i,k=1,2,3}$$

ii] indicial notation

$$A_{ij} B_{jk} = C_{ik}$$

b] $b^t A b = C$

Solution :

i] expanded notation

$$\sum_{j=1}^{3} \sum_{i=1}^{3} b_i A_{ij} b_j = c$$

ii] indicial notation

$$b_i A_{ij} b_j = c$$

9] Let A and B be two matrices. Show that the ordinary matrix product $(AB)_{il} = A_{ij} B_{l}$ is an inner product.

Solution. Let $T := A \otimes B$ denote the outer product of A and B: $T_{ijkl} = A_{ij} B_{kl}$. We identify j and k: $T_{ijjl} = A_{ij} B_{jl}$, which by Def 2.16 (p. 61) is an inner product and is the ordinary matrix product.