4.4.4 Exercises

1) $u_t(x,y,t) = k\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right)$

bcs $\quad u(0,y,t) = 0 \qquad\qquad (x,y) \in (0,1)\times(0,1)$

$u(1,y,t) = 0$

$u(x,0,t) = 0$

$u(x,1,t) = 0$

ic. $\quad u(x,y,0) = f(x,y) = x(1-x)(1-y)$

Solution

a) sep. variables: $u(x,y,t) = X(x)\,Y(y)\,T(t)$

$$X(x)Y(y)T'(t) = k\left(X''(x)Y(y)T(t) + X(x)Y''(y)T(t)\right) \quad \Big|\cdot \frac{1}{kX(x)Y(y)T(t)}$$

$$\Rightarrow \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\mu^2$$

$$\Rightarrow \boxed{T'(t) + \mu^2 k T(t) = 0} \qquad \frac{X''(x)}{X(x)} = \underbrace{-\mu^2 - \frac{Y''(y)}{Y(y)} = -\alpha^2}$$

$$\Downarrow$$

$$\Rightarrow \boxed{X''(x) + \alpha^2 X(x) = 0}$$

$$\boxed{Y''(y) + \beta^2 Y(y) = 0} \quad \Leftarrow \quad \frac{Y''(y)}{Y(y)} = -\mu^2 + \alpha^2 = -\beta^2$$

$$\boxed{\alpha^2 + \beta^2 = \mu^2}$$

b) transl. bcs:

$X(0) = 0,\ X(1) = 0,\ Y(0) = 0,\ Y(1) = 0$

c) solve SL problems

$$\underbrace{\begin{aligned} &X''(x) + \alpha^2 X(x) = 0 \\ &X(0) = 0 \\ &X(1) = 0 \end{aligned}} \qquad \Bigg| \qquad \underbrace{\begin{aligned} &Y''(y) + \beta^2 Y(y) = 0 \\ &Y(0) = 0 \\ &Y(1) = 0 \end{aligned}}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\boxed{\alpha_m = (m+1)\pi} \qquad\qquad \boxed{\beta_n = (n+1)\pi}$$

$$\boxed{X_m(x) = \sin\left((m+1)\pi x\right)} \qquad \boxed{Y_n(y) = \sin\left((n+1)\pi y\right)}$$

$m = 0,1,2,\ldots \qquad\qquad n = 0,1,2,\ldots$

d) solve time equation

$$T'(t) + \mu_{mn}^2 k T(t) = 0, \quad \mu_{mn}^2 = \alpha_m^2 + \beta_n^2 = [(m+1)^2 + (n+1)^2]\pi$$

$$\boxed{T_{mn}(t) = e^{-\mu_{mn}^2 kt} = e^{-[(m+1)^2 + (n+1)^2]\pi k t}}$$

e) general solution

$$u_{mn}(x,y,t) = X_m(x) Y_n(y) T_{mn}(t)$$

$$= \sin((m+1)\pi x)\sin((n+1)\pi x)\, e^{-[(m+1)^2+(n+1)^2]\pi k t}$$

$$\boxed{u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\, u_{mn}(x,y,t)}$$

f) apply initial condition

$$f(x,y) = u(x,y,0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn} X_m(x) Y_n(y), \quad T_{mn}(0) = 0\,!$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn} \sin((m+1)\pi x)\sin((n+1)\pi y)$$

where

$$\boxed{C_{mn}} = \frac{\int_0^1\int_0^1 f(x,y)\sin((m+1)\pi x)\sin((n+1)\pi x)\,dx\,dy}{\underbrace{\int_0^1 \sin^2((m+1)\pi x)\,dx}_{\frac{1}{2}}\ \underbrace{\int_0^1 \sin^2((n+1)\pi y)\,dy}_{\frac{1}{2}}}$$

$$\boxed{= 4\int_0^1\int_0^1 f(x,y)\sin((m+1)\pi x)\sin((n+1)\pi y)\,dx\,dy}$$

for $m,n = 0,1,2,\cdots$

2] $u_t(x,y,t) = k\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right)$

bcs: $u_x(0,y,t) = 0$ $\quad (x,y) \in (0,1) \times (0,2)$

$u(1,y,t) = 0$

$u(x,0,t) = 0$

$u(x,2,t) = 0$

ic: $u(x,y,0) = f(x,y) = (1-x^2)y(2-y)$

Solution:

a] sep. var: $u(x,y,t) = X(x)Y(y)T(t)$, as in 1a] we obtain

$\boxed{T'(t) + \mu^2 k T(t) = 0}$

$\boxed{X''(x) + \alpha^2 X(x) = 0}$

$\boxed{Y''(y) + \beta^2 Y(y) = 0}$

$\boxed{\alpha^2 + \beta^2 = \mu^2}$

b] transl. bcs

$X'(0) = 0, \ X(1) = 0, \ Y(0) = 0, \ Y(2) = 0$

c] solve SL Problems:

$X''(x) + \alpha^2 X(x) = 0$

$X'(0) = 0$

$X(1) = 0$

$\boxed{\alpha_m = \frac{(2m+1)}{2}\pi}$

$\boxed{X_m(x) = \cos\left(\frac{(2m+1)\pi}{2}x\right)}$

$Y''(y) + \beta^2 Y(y) = 0$

$Y(0) = 0$

$Y(2) = 0$

$\boxed{\beta_n = \frac{(n+1)\pi}{2}}$

$\boxed{Y_n(y) = \sin\left(\frac{(n+1)\pi}{2}y\right)}$

for $m,n = 0,1,2,\ldots$

d) solve time eqn:

$$T'(t) + \mu_{mn}^2 k T(t) = 0, \qquad \mu_{mn}^2 = \alpha_m^2 + \beta_n^2 = \frac{(2m+1)^2\pi^2}{4} + \frac{(n+1)^2\pi^2}{4}$$

$$\boxed{T_{mn}(t)} = e^{-\mu_{mn}^2 k t} = \boxed{e^{-\left[(2m+1)^2+(n+1)^2\right]\frac{\pi^2 k}{4}t}}$$

e) gen. sol.:

$$u_{mn}(x,y,t) = X_m(x)Y_n(y)T_{mn}(t)$$

$$= \cos\left(\frac{(2m+1)\pi}{2}x\right)\sin\left(\frac{(n+1)\pi}{2}y\right)e^{-\left[(2m+1)^2+(n+1)^2\right]\frac{\pi^2 k}{4}t}$$

$$\boxed{u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\, u_{mn}(x,y,t)}$$

f) apply ic:

$$f(x,y) = u(x,y,0) = \sum_{m=0}^{\infty} C_{mn} X_m(x) Y_n(y), \quad T_{mn}(0) = 1 \ !$$

$$= \sum_{m=0}^{\infty} C_{mn}\cos\left(\frac{(2m+1)\pi}{2}x\right)\sin\left(\frac{(n+1)\pi}{2}y\right)$$

where

$$\boxed{C_{mn}} = \frac{\int_0^2\int_0^1 f(x,y)\cos\left(\frac{(2m+1)\pi}{2}x\right)\sin\left(\frac{(n+1)\pi}{2}y\right)dx\,dy}{\underbrace{\int_0^1 \cos^2\left(\frac{(2m+1)\pi}{2}x\right)dx}_{\frac{1}{2}}\ \underbrace{\int_0^2 \sin\left(\frac{(n+1)\pi}{2}y\right)dy}_{\frac{2}{2}=1}}$$

$$\boxed{= 2\int_0^2\int_0^1 f(x,y)\cos\left(\frac{(2m+1)\pi}{2}x\right)\sin\left(\frac{(n+1)\pi}{2}y\right)dx\,dy}$$

for $m,n = 0,1,2,\dots$

3] $u_t(x,y,t) = k\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right)$

$(x,y) \in (0,1) \times (0,2)$

bcs: $u_x(0,y,t) = 0$

$u_x(1,y,t) = 0$

$u(x,0,t) = 0$

$u(x,2,t) = 0$

ic: $u(x,y,0) = f(x,y) = x^2(1-\frac{2}{3}x)y(2-y)$

Solution:

a] sep. var: $u(x,y,t) = X(x)Y(y)T(t)$, as in 1a] we obtain

$$\boxed{T'(t) + \mu^2 k T(t) = 0}$$

$$\boxed{X''(x) + \alpha^2 X(x) = 0}$$

$$\boxed{Y''(y) + \beta^2 Y(y) = 0}$$

$$\boxed{\alpha^2 + \beta^2 = \mu^2}$$

b] transl. bcs:

$X'(0) = 0,\ X'(1) = 0,\ Y(0) = 0,\ Y(2) = 0$

c] solve SL problems

$$\underbrace{\begin{array}{l} X''(x) + \alpha^2 X(x) = 0 \\ X'(0) = 0 \\ X'(1) = 0 \end{array}} \qquad \Bigg| \qquad \underbrace{\begin{array}{l} Y''(y) + \beta^2 Y(y) = 0 \\ Y(0) = 0 \\ Y(2) = 0 \end{array}}$$

$$\boxed{\alpha_m = m\pi} \qquad \boxed{\beta_n = \frac{(n+1)\pi}{2}}$$

$$\boxed{X_m(x) = \cos(m\pi x)} \qquad \boxed{Y_n(y) = \sin\left(\frac{(n+1)\pi}{2}y\right)}$$

for $m,n = 0,1,2,\dots$

d) solve time eqn:

$$T'(t) + \mu_{mn}^2 k T(t) = 0, \qquad \mu_{mn}^2 = \alpha_m^2 + \beta_n^2 = m^2\pi^2 + \frac{(n+1)^2\pi^2}{4}$$

$$\boxed{T_{mn}(t) = e^{-\mu_{mn}^2 kt} = e^{-\left[m^2 + \frac{(n+1)^2}{4}\right]\pi^2 kt}}$$

e) gen. sol:

$$u_{mn}(x,y,t) = X_m(x) Y_n(y) T_{mn}(t)$$

$$= \cos(m\pi x)\sin\left(\frac{(n+1)\pi}{2}y\right) e^{-\left[m^2 + \frac{(n+1)^2}{4}\right]\pi^2 kt}$$

$$\boxed{u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn} u_{mn}(x,y,t)}$$

f) apply bcs

$$f(x,y) = u(x,y,0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn} X_m(x) Y_n(y), \quad T_{mn}(0) = 1!$$

$$= \sum_{n=0}^{\infty} C_{0n}\sin\left(\frac{(n+1)\pi}{2}y\right) + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} C_{mn}\cos(m\pi x)\sin\left(\frac{(n+1)\pi}{2}y\right)$$

where

$$\boxed{C_{0n}} = \frac{\int_0^2\int_0^1 f(x,y)\sin\left(\frac{(n+1)\pi}{2}y\right)dx\,dy}{\underbrace{\int_0^2 \sin^2\left(\frac{(n+1)\pi}{2}y\right)dy}_{1}}$$

$$= \boxed{\int_0^2\int_0^1 f(x,y)\sin\left(\frac{(n+1)\pi}{2}y\right)dx\,dy}$$

for $n = 0, 1, 2, \dots$

$$\boxed{C_{mn}} = \frac{\int_0^2 \int_0^1 f(x,y) \cos(m\pi x) \sin\left(\frac{(n+1)\pi}{2} y\right) dx\,dy}{\underbrace{\int_0^1 \cos^2(m\pi x)\,dx}_{\frac{1}{2}} \; \underbrace{\int_0^2 \sin^2\left(\frac{(n+1)\pi}{2} y\right) dy}_{\frac{2}{2}=1}}$$

$$= \boxed{2 \int_0^2 \int_0^1 f(x,y) \cos(m\pi x) \sin\left(\frac{(n+1)\pi}{2} y\right) dx\,dy}$$

for $m = 1, 2, 3, \ldots$, $n = 0, 1, 2, \ldots$

4) $$u_t(r,\theta,t) = k\left(\frac{u_r(r,\theta,t) + r\,u_{rr}(r,\theta,t)}{r} + \frac{u_{\theta\theta}(r,\theta,t)}{r^2}\right)$$

$(r,\theta) \in (0,1) \times (0,\pi)$

bcs: $|u(0,\theta,t)| < \infty$

$u(1,\theta,t) = 0$

$u(r,0,t) = 0$

$u(r,\pi,t) = 0$

ic: $u(r,\theta,0) = f(r,\theta) = (r - r^3)\sin\theta$

Solution

a) sep. var: $u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$

$$R(r)\Theta(\theta)T'(t) = k\left(\frac{R'(r)\Theta(\theta)T(t) + r R''(r)\Theta(\theta)T(t)}{r} + \frac{R(r)\Theta''(\theta)T(t)}{r^2}\right)$$

Dividing by $k R(r)\Theta(\theta)T(t)$ yields

$$\frac{T'(t)}{kT(t)} = \frac{1}{r}\left(\frac{R'(r)}{R(r)} + \frac{r R''(r)}{R(r)}\right) + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = -\mu^2$$

$\Rightarrow \boxed{T'(t) + \mu^2 k T(t) = 0}$ and multiply the right eqn

by r^2:
$$r\frac{R'(r)}{R(r)} + r^2\frac{R''(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = -\mu^2 r$$

which is equivalent with

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \mu^2 r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \rho^2$$

Hence

$$\boxed{\Theta''(\theta) + \rho^2\,\Theta(\theta) = 0} \quad \text{and} \quad \boxed{r^2 R''(r) + rR'(r) + (\mu^2 r^2 - \rho^2)R(r) = 0}$$

b) transl. bcs:

$|R(0)| < \infty$, $R(1) = 0$, $\Theta(0) = 0$, $\Theta(\pi) = 0$

c) solve SL problems:

Note: the SL problem in θ has to be solved before the one in r!

$$\left.\begin{aligned} &\Theta''(\theta) + \rho^2\Theta(\theta) = 0 \\ &\Theta(0) = 0 \\ &\Theta(\pi) = 0 \end{aligned}\right\} \Rightarrow \boxed{\rho_m = m+1}, \quad \boxed{\Theta_m(\theta) = \sin((m+1)\theta)}$$

$m = 0, 1, 2, \ldots$

$$\left.\begin{aligned} &r^2 R''(r) + rR'(r) + (\mu^2 r^2 - \underbrace{\rho_m^2}_{=(m+1)^2})R(r) = 0 \\ &|R(0)| < \infty \\ &R(1) = 0 \end{aligned}\right\} \Rightarrow$$

$\boxed{\mu_{mn} > 0}$ solutions of $\boxed{J_{m+1}(\mu) = 0}$

$$\boxed{R_{mn}(r) = J_{m+1}(\mu_{mn} r)}$$

for $m, n = 0, 1, 2, \ldots$

d) solve time eqn

$T'(t) + \mu_{mn}^2 k\,T(t) = 0$,

$$\boxed{T_{mn}(t) = e^{-\mu_{mn}^2 kt}}$$

e) gen. sol.

$$u_{mn}(r,\theta,t) = R_{mn}(r)\,\Theta_m(\theta)\,T_{mn}(t) = J_{m+1}(\mu_{mn} r)\sin((m+1)\theta)\,e^{-\mu_{mn}^2 kt}$$

$$\boxed{u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\, u_{mn}(r,\theta,t)}$$

f] apply ic:

$$f(r,\theta) = u(r,\theta,0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\, J_{m+1}(\mu_{mn} r)\sin((m+1)\theta)$$

where

$$\boxed{C_{mn}} = \frac{\int_0^{\pi}\int_0^{1} f(r,\theta)\, J_{m+1}(\mu_{mn} r)\sin((m+1)\theta)\, r\,dr\,d\theta}{\underbrace{\int_0^{\pi}\sin^2((m+1)\theta)\,d\theta}_{\frac{\pi}{2}}\int_0^{1} J_{m+1}^2(\mu_{mn} r)\, r\,dr}$$

$$= \boxed{\frac{2\int_0^{\pi}\int_0^{1} f(r,\theta)\, J_{m+1}(\mu_{mn} r)\sin((m+1)\theta)\, r\,dr\,d\theta}{\pi\int_0^{1} J_{m+1}^2(\mu_{mn} r)\, r\,dr}}$$

for $m,n = 0,1,2,\dots$