4.3.3 Exercises

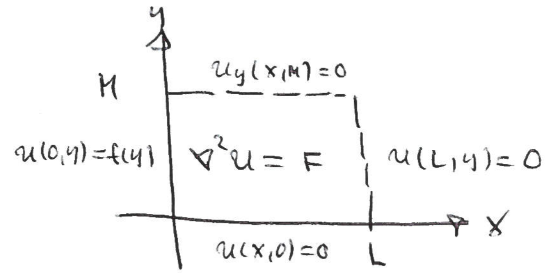
1) $u_{xx} + u_{yy} = F(x,y) \qquad \left(= \frac{1}{10} xy(L-x)(M-y)\right)$

$u(0,y) = f(y) \qquad (= y)$

$u(L,y) = 0$

$u(x,0) = 0$

$u_y(x,M) = 0$



Solution:

a) decompose BVP: $u = v + w$

A. Poisson Eqn with 4 hom. bcs

$v_{xx} + v_{yy} = F(x,y)$

$\left.\begin{array}{l} v(0,y)=0 \\ v(L,y)=0 \end{array}\right]$ hom. bcs in x

$\left.\begin{array}{l} v(x,0)=0 \\ v_y(x,M)=0 \end{array}\right]$ hom bcs in y

B. Laplace Eqn with given bcs

$w_{xx} + w_{yy} = 0$

$w(0,y) = f(y)$

$w(L,y) = 0$

$\left.\begin{array}{l} w(x,0)=0 \\ w_y(x,M)=0 \end{array}\right]$ hom. bcs in y

b) solve subproblem

A. Poisson eqn with hom. bcs

i) compute Eigenvalues, -functions

$$\left.\begin{array}{l} v_{xx} + v_{yy} = 0 \\ v(0,y)=0 \\ v(L,y)=0 \\ v(x,0)=0 \\ v_y(x,M)=0 \end{array}\right\} \xrightarrow[\text{var.}]{\text{sep.}} \left\{\begin{array}{l} X''(x) + \mu X = 0 \\ X(0)=0 \\ X(L)=0 \end{array}\right.$$

$$\boxed{\mu_m = \left[\frac{(m+1)\pi}{L}\right]^2}$$

$$\boxed{X_m(x) = \sin\left(\frac{(m+1)\pi}{L} x\right)}$$

$m = 0, 1, 2, \ldots$

$\Downarrow$ sep. var

$$\left.\begin{array}{l} Y''(y) + \rho Y(y) = 0 \\ Y(0) = 0 \\ Y'(M) = 0 \end{array}\right\}$$

$$\boxed{\rho_n = \left[\frac{(2n+1)\pi}{2M}\right]^2}$$

$$\boxed{Y_n(y) = \sin\left(\frac{(2n+1)\pi}{2M} y\right)}$$

$n = 0, 1, 2, \ldots$

ii] Eigenfunction expansion of solution

$$\boxed{v(x,y)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} X_m(x) Y_n(y) =$$

$$\boxed{= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \sin\left(\frac{(m+1)\pi}{L}x\right)\sin\left(\frac{(2n+1)\pi}{2M}y\right)}$$

with

$$\boxed{A_{mn}} = -\frac{\int_0^M\int_0^L F(x,y)\, X_m(x)\, Y_n(y)\, dx\, dy}{(\mu_m+\rho_n)\int_0^L X_m^2(x)\,dx \int_0^M Y_n^2(y)\,dy}$$

$$= -\frac{\int_0^M\int_0^L F(x,y)\sin\left(\frac{(m+1)\pi}{L}x\right)\sin\left(\frac{(2n+1)\pi}{2M}y\right)dx\,dy}{\left[\frac{(m+1)^2\pi^2}{L^2}+\frac{(2n+1)^2\pi^2}{4M^2}\right]\underbrace{\int_0^L \sin^2\left(\frac{(m+1)\pi}{L}x\right)dx}_{\frac{L}{2}}\ \underbrace{\int_0^M \sin^2\left(\frac{(2n+1)\pi}{2M}y\right)dy}_{\frac{M}{2}}}$$

$$\boxed{= -\frac{16LM}{\left(4M^2(m+1)^2+L^2(2n+1)^2\right)\pi^2}\int_0^M\int_0^L F(x,y)\sin\left(\frac{(m+1)\pi}{L}x\right)\sin\left(\frac{(2n+1)\pi}{2M}y\right)dx\,dy}$$

B. Laplace Eqn

i] sep. variables $w(x,y) = X(x)\,Y(y)$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \mu \Rightarrow \begin{cases} X''(x)-\mu X(x) = 0 \\ Y''(y)+\mu Y(y) = 0 \end{cases}$$

ii] transl. bc: $X(L)=0$, $Y(0)=0$, $Y'(M)=0$

iii] solve SL problem

$\boxed{\mu_n = \frac{(2n+1)^2\pi^2}{4M^2}}$, $\boxed{Y_n(y) = \sin\left(\frac{(2n+1)\pi}{2M}y\right)}$, $n=0,1,2,\dots$

iv] solve remaining eqn

$X''(x) - \mu_n X(x) = 0$, $\mu_n > 0$, char eqn $r^2 = \frac{(2n+1)^2\pi^2}{4M^2}$

$X(L) = 0$ Note bc on non-zero boundary!

$$X_n(x) = C_1 \cosh\left(\frac{(2n+1)\pi}{2H}(x-L)\right) + C_2 \sinh\left(\frac{(2n+1)\pi}{2H}(x-L)\right)$$

$$0 = X_n(L) = C_1 \Rightarrow C_1 = 0, \quad C_2 \neq 0$$

Hence

$$\boxed{X_n(x) = \sinh\left(\frac{(2n+1)\pi}{2H}(x-L)\right)}, \quad n = 0,1,2,\ldots$$

v) gen. solution

$$w_n(x,y) = X_n(x)\,Y_n(y) = \sinh\left(\frac{(2n+1)\pi}{2H}(x-L)\right)\sin\left(\frac{(2n+1)\pi}{2H}y\right)$$

$$\boxed{w(x,y) = \sum_{n=0}^{\infty} B_n\, w_n(x,y) = \sum_{n=0}^{\infty} B_n X_n(x)\,Y_n(y)}$$

vi) apply nonhom. bc. $\quad \sinh\left(\frac{(2n+1)\pi}{2H}(-L)\right) = -\sinh\left(\frac{(2n+1)\pi L}{2H}\right)$

$$f(y) = w(0,y) = \sum_{n=0}^{\infty} B_n X_n(0)\,Y_n(y)$$

$$= \sum_{n=0}^{\infty}\left[-B_n \sinh\left(\frac{(2n+1)\pi L}{2H}\right)\right]\sin\left(\frac{(2n+1)\pi}{2H}y\right)$$

hence

$$-B_n \sinh\left(\frac{(2n+1)\pi L}{2H}\right) = \frac{\int_0^H f(y)\sin\left(\frac{(2n+1)\pi}{2H}y\right)dy}{\underbrace{\int_0^H \sin^2\left(\frac{(2n+1)\pi}{2H}y\right)dy}_{=\frac{H}{2}}}$$

and therefore

$$\boxed{B_n = -\frac{2}{H\sinh\left(\frac{(2n+1)\pi L}{2H}\right)}\int_0^H f(y)\sin\left(\frac{(2n+1)\pi}{2H}y\right)dy}$$

for $n = 0,1,2,\ldots$

c] combine solutions of subproblems

$$\boxed{\begin{aligned} u(x,y) &= v(x,y) + w(x,y) \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \sin\left(\frac{(m+1)\pi}{L}x\right)\sin\left(\frac{(2n+1)\pi}{2M}y\right) \\ &\quad + \sum_{n=0}^{\infty} B_n \sinh\left(\frac{(2n+1)\pi}{2M}(x-L)\right)\sin\left(\frac{(2n+1)\pi}{2M}y\right) \end{aligned}}$$

where

$$\boxed{A_{mn} = -\frac{16LM}{(4M^2(m+1)^2 + L^2(2n+1)^2\pi^2}\int_0^M\int_0^L F(x,y)\sin\left(\frac{(m+1)\pi}{L}x\right)\cdot \sin\left(\frac{(2n+1)\pi}{2M}y\right)dx\,dy}$$

$$\boxed{B_n = -\frac{2}{M\sinh\left(\frac{(2n+1)\pi L}{2M}\right)}\int_0^M f(y)\sin\left(\frac{(2n+1)\pi}{2M}y\right)dy}$$

for $m, n = 0, 1, 2, \ldots$

2] $u_{xx} + u_{yy} = F(x,y)$

$u(0,y) = f(y)$

$u(L,y) = g(y)$

$u(x,0) = 0$

$u_y(x,M) = 0$

<u>Solution</u>

a] decompose BVP: $u = v + w$

A. Poisson Eqtn with 4 hom. bcs

$$v_{xx} + v_{yy} = F(x,y)$$

$$v(0,y) = 0$$
$$v(L,y) = 0$$
$$v(x,0) = 0$$
$$v_y(x,M) = 0$$

B. Laplace Eqtn with given bcs

$$w_{xx} + w_{yy} = 0$$

$$w(0,y) = f(y)$$
$$w(L,y) = g(y)$$
$$w(x,0) = 0$$
$$w_y(x,M) = 0$$

b) solve subproblems

A. Poisson eqtn with hom. bcs.

Note this problems was already solved with the identical boundary conditions on Exercise 1! Hence

$$\boxed{v(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \sin\left(\frac{(m+1)\pi}{L} x\right) \sin\left(\frac{(2n+1)\pi}{2M} y\right)}$$

with

$$\boxed{A_{mn} = -\frac{16LM}{(4M^2(m+1)^2 + L^2(2n+1)^2)\pi^2} \int_0^M \int_0^L F(x,y) \sin\left(\frac{(m+1)\pi}{L} x\right) \sin\left(\frac{(2n+1)\pi}{2M} y\right) dx\,dy}$$

B. Laplace equation

since this equation has 2 nonhom. bcs., we have to decompose it further

B1: $w^{(1)}_{xx} + w^{(1)}_{yy} = 0$

$$w^{(1)}(0,y) = f(y)$$
$$w^{(1)}(L,y) = 0$$
$$w^{(1)}(x,0) = 0$$
$$w^{(1)}_y(x,M) = 0$$

B2: $w^{(2)}_{xx} + w^{(2)}_{yy} = 0$

$$w^{(2)}(0,y) = 0$$
$$w^{(2)}(L,y) = g(y)$$
$$w^{(2)}(x,0) = 0$$
$$w^{(2)}_y(x,M) = 0$$

B1: Laplace eqn

Note this equation was already solved in Exercise 1 hence

$$\boxed{w^{(1)}(x,y) = \sum_{n=0}^{\infty} B_n \sinh\left(\frac{(2n+1)\pi}{2M}(x-L)\right)\sin\left(\frac{(2n+1)\pi}{2M}y\right)}$$

with

$$\boxed{B_n = -\frac{2}{M\sinh\left(\frac{(2n+1)\pi L}{2M}\right)} \int_0^M f(y)\sin\left(\frac{(2n+1)\pi}{2M}y\right)dy}$$

for $n = 0, 1, 2, \ldots$

B2: Laplace eqn

i] sep. variables: $w^{(2)}(x,y) = X(x)Y(y)$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \mu \Rightarrow \begin{cases} X''(x) - \mu X(x) = 0 \\ Y''(y) + \mu Y(y) = 0 \end{cases}$$

ii] transl. bcs: $X(0) = 0$, $Y(0) = 0$, $Y'(M) = 0$

iii] solve SL problem:

$$\left.\begin{array}{l} Y''(y) + \mu Y(y) = 0 \\ Y(0) = 0 \\ Y'(M) = 0 \end{array}\right\} \quad \boxed{\mu_n = \frac{(2n+1)^2\pi^2}{4M^2}} \quad \boxed{Y_n(x) = \sin\left(\frac{(2n+1)\pi}{2M}y\right)}$$

$n = 0, 1, 2, \ldots$

iv] solve remaining eqn

$X''(x) - \mu_n X(x) = 0$, $\mu_n > 0$, char. eqn $r^2 = \frac{(2n+1)^2\pi^2}{4M^2} > 0$

$X(0) = 0$

$$X_n(x) = C_1 \cosh\left(\frac{(2n+1)\pi}{2M}x\right) + C_2 \sinh\left(\frac{(2n+1)\pi}{2M}x\right)$$

$$0 = X_n(0) = C_1 \Rightarrow C_1 = 0, \quad C_2 \neq 0,$$

hence

$$\boxed{X_n(x) = \sinh\left(\frac{(2n+1)\pi}{2M}x\right)}, \quad n = 0, 1, 2, \ldots$$

v] gen. solution

$$w_n^{(2)}(x,y) = X_n(x)\,Y_n(y) = \sinh\left(\frac{(2n+1)\pi}{2M}x\right)\sin\left(\frac{(2n+1)\pi}{2M}y\right)$$

$$\boxed{w^{(2)}(x,y) = \sum_{n=0}^{\infty} D_n\, w_n^{(2)}(x,y) = \sum_{n=0}^{\infty} D_n X_n(x)\, Y_n(y)}$$

vi] apply nonhom. bc:

$$g(y) = w^2(L,y) = \sum_{n=0}^{\infty}\left[D_n \sinh\left(\frac{(2n+1)\pi L}{2M}\right)\right]\sin\left(\frac{(2n+1)\pi}{2M}y\right)$$

hence

$$D_n \sinh\left(\frac{(2n+1)\pi L}{2M}\right) = \frac{\int_0^M g(y)\sin\left(\frac{(2n+1)\pi}{2M}y\right)dy}{\underbrace{\int_0^M \sin^2\left(\frac{(2n+1)\pi}{2M}y\right)dy}_{=\frac{M}{2}}}$$

and therefore

$$\boxed{D_n = \frac{2}{M\sinh\left(\frac{(2n+1)\pi L}{2M}\right)}\int_0^M g(y)\sin\left(\frac{(2n+1)\pi}{2M}y\right)dy}$$

for $n = 0, 1, 2, \ldots$

c] combine solutions of subproblems

$$\boxed{\begin{aligned} u(x,y) &= v(x,y) + w^{(1)}(x,y) + w^{(2)}(x,y) = \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \sin\left(\frac{(m+1)\pi}{L}x\right)\sin\left(\frac{(2n+1)\pi}{2M}y\right) \\ &+ \sum_{n=0}^{\infty} B_n \sinh\left(\frac{(2n+1)\pi}{2M}(x-L)\right)\sin\left(\frac{(2n+1)\pi}{2M}y\right) \\ &+ \sum_{n=0}^{\infty} D_n \sinh\left(\frac{(2n+1)\pi}{2M}x\right)\sin\left(\frac{(2n+1)\pi}{2M}y\right) \end{aligned}}$$

where

$$\boxed{A_{mn} = -\frac{16LM}{(4M^2(m+1)^2 + L^2(2n+1)^2)\pi^2}\int_0^M\int_0^L F(x,y)\sin\left(\frac{(m+1)\pi}{L}x\right)\sin\left(\frac{(2n+1)\pi}{2M}y\right)dx\,dy}$$

$$\boxed{B_n = -\frac{2}{M\sinh\left(\frac{(2n+1)\pi L}{2M}\right)}\int_0^M f(y)\sin\left(\frac{(2n+1)\pi}{2M}y\right)dy}$$

$$\boxed{D_n = \frac{2}{M\sinh\left(\frac{(2n+1)\pi L}{2M}\right)}\int_0^M g(y)\sin\left(\frac{(2n+1)\pi}{2M}y\right)dy}$$

3] $u_{xx} + u_{yy} = F(x,y)$

$u(0,y) = f(y)$

$u(L,y) = g(y)$

$u_y(x,0) = 0$

$u_y(x,M) = 0$

Solution:

a] decompose BVP: $u = v + w$

A. Poisson eqtn with 4 hom. bcs

$$v_{xx} + v_{yy} = F(x,y)$$
$$v(0,y) = 0$$
$$v(L,y) = 0$$
$$v_y(x,0) = 0$$
$$v_y(x,M) = 0$$

B. Laplace eqn with given bcs

$$w_{xx} + w_{yy} = 0$$
$$w(0,y) = f(y)$$
$$w(L,y) = g(y)$$
$$w_y(x,0) = 0$$
$$w_y(x,M) = 0$$

b] solve subproblem

A. Poisson eqtn

i] compute Eigenvalues, -functions

$$\underbrace{\left.\begin{array}{l} v_{xx} + v_{yy} = 0 \\ v(0,y) = 0 \\ v(L,y) = 0 \\ v_y(x,0) = 0 \\ v_y(x,M) = 0 \end{array}\right\}}_{} \xrightarrow[\text{var.}]{\text{sep}} \underbrace{\left\{\begin{array}{l} X''(x) + \mu X(x) = 0 \\ X(0) = 0 \\ X(L) = 0 \end{array}\right.}_{}$$

$$\boxed{\mu_m = \left[\frac{(m+1)\pi}{L}\right]^2}, \quad \boxed{X_m(x) = \sin\left(\frac{(m+1)\pi}{L}x\right)}$$

$\Downarrow$ sep var.

$$\left.\begin{array}{l} y''(y) + \rho y(y) = 0 \\ y'(0) = 0 \\ y'(M) = 0 \end{array}\right\} \Rightarrow \boxed{\rho_n = \left[\frac{n\pi}{M}\right]^2}, \quad \boxed{y_n(y) = \cos\left(\frac{n\pi}{M}y\right)}$$

for $m, n = 0, 1, 2, \ldots$

ii] Eigenfunction expansion of solution

$$\boxed{v(x,y)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} X_m(x) y_n(y) =$$
$$\boxed{= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \sin\left(\frac{(m+1)\pi}{L}x\right)\cos\left(\frac{n\pi}{M}y\right)}$$

with

$$A_{mn} = -\frac{\int_0^M\int_0^L F(x,y)\, X_m(x)\, Y_n(y)\, dx\, dy}{(\mu_m + \rho_n)\int_0^L X_m^2(x)\,dx \int_0^M Y_n^2(y)\, dy}$$

Note $\rho_0 = 0$ is Eigenvalue, $Y_0(y) = 1$ is Eigenfunction

$$\boxed{A_{m0}} = -\frac{\int_0^M\int_0^L F(x,y)\, X_m(x)\, dx\, dy}{\mu_m \cdot \frac{L}{2}\cdot M}$$

$$= \boxed{-\frac{2L}{(m+1)^2\pi^2 M}\int_0^M\int_0^L F(x,y)\sin\left(\frac{(m+1)\pi}{L}x\right)dx\,dy}, \quad m = 0,1,2,\dots$$

$$\boxed{A_{mn}} = -\frac{\int_0^M\int_0^L F(x,y)\sin\left(\frac{(m+1)\pi}{L}x\right)\cos\left(\frac{n\pi}{M}y\right)dx\,dy}{\underbrace{\left[\frac{(m+1)^2\pi^2}{L^2}+\frac{n^2\pi^2}{M^2}\right]}_{\frac{(M^2(m+1)+L^2n^2)\pi^2}{L^2M^2}}\underbrace{\int_0^L \sin\left(\frac{(m+1)\pi}{L}x\right)dx}_{=\frac{L}{2}}\underbrace{\int_0^M \cos\left(\frac{n\pi}{M}y\right)dy}_{=\frac{M}{2}}}$$

$$= \boxed{-\frac{4LM}{(M^2(m+1)^2+L^2n^2)\pi^2}\int_0^M\int_0^L F(x,y)\sin\left(\frac{(m+1)\pi}{L}x\right)\cos\left(\frac{n\pi}{M}y\right)dx\,dy}$$

B. Laplace eqn

Since this eqn has 2 nonhom. bcs., we have to decompose it further:

B1: $w^{(1)}_{xx} + w^{(1)}_{yy} = 0$

$w^{(1)}(0,y) = f(y)$

$w^{(1)}(L,y) = 0$

$w^{(1)}_y(x,0) = 0$

$w^{(1)}_y(x,M) = 0$

B2: $w^{(2)}_{xx} + w^{(2)}_{yy} = 0$

$w^{(2)}(0,y) = 0$

$w^{(2)}(L,y) = g(y)$

$w^{(2)}_y(x,0) = 0$

$w^{(2)}_y(x,M) = 0$

B1: Laplace Eqn

Note, this eqn was solved in Example 4.8 (p 258ff)

$$\boxed{w^{(1)}(x,y) = B_0(x-L) + \sum_{n=1}^{\infty} B_n \sinh\left(\tfrac{n\pi}{M}(x-L)\right)\cos\left(\tfrac{n\pi}{M}y\right)}$$

with
$$\boxed{B_0 = -\frac{1}{ML}\int_0^M f(y)\,dy}$$

$$\boxed{B_n = -\frac{2}{M\sinh\left(\frac{n\pi}{M}L\right)}\int_0^M f(y)\cos\left(\tfrac{n\pi}{M}y\right)dy}$$

for $n=1,2,3,\dots$

B2: Laplace Eqn

i) sep. vars. $\quad w^{(2)}(x,y) = X(x)\,Y(y)$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \mu \Rightarrow \begin{cases} X''(x) - \mu X(x) = 0 \\ Y''(y) + \mu Y(y) = 0 \end{cases}$$

ii) transl. bcs: $X(0)=0$, $Y'(0)=0$, $Y'(M)=0$

iii) solve SL probl.:

$$\left.\begin{array}{l} Y''(y) + \mu Y(y) = 0 \\ Y'(0) = 0 \\ Y'(M) = 0 \end{array}\right\} \Rightarrow \begin{cases} \boxed{\mu_n = \left[\frac{n\pi}{M}\right]^2} \\ \boxed{Y_n(y) = \begin{cases} 1, & n=0 \\ \cos\left(\frac{n\pi}{M}y\right), & n=1,2,\dots \end{cases}} \end{cases}$$

iv) solve remaining eqn

$$X''(x) - \mu_n X(x) = 0, \quad \mu_n \geq 0 \ !$$

$$X(0) = 0$$

$$X_n(x) = \begin{cases} C_1 + C_2 x, & n=0 \\ C_1\cosh\left(\frac{n\pi}{M}x\right) + C_2\sinh\left(\frac{n\pi}{M}x\right), & n=1,2,\dots \end{cases}$$

$$0 = X_n(0) = \begin{cases} c_1, & n=0 \\ c_1, & n=1,2,\dots \end{cases} \Rightarrow c_1=0,\ c_2 \neq 0$$

hence

$$\boxed{X_n(x) = \begin{cases} x, & n=0 \\ \sinh\left(\frac{n\pi}{M}x\right), & n=1,2,\dots \end{cases}}$$

v] gen. solution

$$w_n^{(2)}(x,y) = X_n(x)\, Y_n(y) = \begin{cases} x, & n=0 \\ \sinh\left(\frac{n\pi}{M}x\right)\cos\left(\frac{n\pi}{M}y\right), & n=1,2,\dots \end{cases}$$

$$\boxed{w^{(2)}(x,y) = D_0 x + \sum_{n=1}^{\infty} D_n \sinh\left(\frac{n\pi}{M}x\right)\cos\left(\frac{n\pi}{M}y\right)}$$

vi] apply nonhom. bc:

$$g(y) = w^{(2)}(L,y) = D_0 L + \sum_{n=1}^{\infty}\left[D_n \sinh\left(\frac{n\pi}{M}L\right)\right]\cos\left(\frac{n\pi}{M}y\right)$$

with

$$D_0 L = \frac{\int_0^M g(y)\,dy}{\int_0^M dy} = \frac{1}{M}\int_0^M g(y)\,dy$$

$$D_n \sinh\left(\frac{n\pi}{M}L\right) = \frac{\int_0^M g(y)\cos\left(\frac{n\pi}{M}y\right)dy}{\underbrace{\int_0^M \cos^2\left(\frac{n\pi}{M}y\right)dy}_{=\frac{M}{2}}}$$

and therefore

$$\boxed{D_0 = \frac{1}{LM} \int_0^M g(y)\,dy}$$

$$\boxed{D_n = \frac{2}{M \sinh\left(\frac{n\pi}{M} L\right)} \int_0^M g(y) \cos\left(\frac{n\pi}{M} y\right) dy}$$

for $n = 1, 2, \ldots$

c) Combine solution of subproblem

$$\boxed{\begin{aligned} u(x,y) &= v(x,y) + w^{(1)}(x,y) + w^{(2)}(x,y) = \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \sin\left(\frac{(m+1)\pi}{L} x\right) \cos\left(\frac{n\pi}{M} y\right) + \\ &\quad + B_0 (x-L) + \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{M}(x-L)\right) \cos\left(\frac{n\pi}{M} y\right) \\ &\quad + D_0 x + \sum_{n=1}^{\infty} D_n \sinh\left(\frac{n\pi}{M} x\right) \cos\left(\frac{n\pi}{M} y\right) \end{aligned}}$$

where

$$\boxed{A_{m0} = -\frac{2L}{(m+1)^2 \pi^2 M} \int_0^M \int_0^L F(x,y) \sin\left(\frac{(m+1)\pi}{L} x\right) dx\,dy}$$

$$\boxed{\begin{aligned} A_{mn} = -\frac{4LM}{(M^2 (m+1)^2 + L^2 n^2)\pi^2} \int_0^M \int_0^L F(x,y) \sin\left(\frac{(m+1)\pi}{L} x\right) \cdot \\ \cdot \cos\left(\frac{n\pi}{M} y\right) dx\,dy \end{aligned}}$$

for $m, n = 0, 1, 2, \ldots$

$$\boxed{B_0 = -\frac{1}{ML}\int_0^M f(y)\,dy}$$

$$\boxed{B_n = -\frac{2}{M\sinh\left(\frac{n\pi}{M}L\right)}\int_0^M f(y)\cos\left(\frac{n\pi}{M}y\right)dy}$$

$$\boxed{D_0 = \frac{1}{LM}\int_0^M g(y)\,dy}$$

$$\boxed{D_n = \frac{2}{M\sinh\left(\frac{n\pi}{M}L\right)}\int_0^M g(y)\cos\left(\frac{n\pi}{M}y\right)dy}$$

for $n=1,2,3,\ldots$