4.5.2 Exercises

1) $u_t(x,t) = k\, u_{xx}(x,t) + h(x,t), \quad x \in (0,L),\ t > 0$

bcs: $u_x(0,t) = b_1(t)$

$u(L,t) = b_2(t)$

ic: $u(x,0) = f(x)$

Solution:

① Partition the solution

$$u(x,t) = s(x,t) + v(x,t)$$

with

[SSP] $s_{xx}(x,t) = 0$

$s_x(0,t) = b_1(t)$

$s(L,t) = b_2(t)$

[VP] $v_t(x,t) = k\, v_{xx}(x,t) + \underbrace{h(x,t) - s_t(x,t)}_{q(x,t)},$

$v_x(0,t) = 0$

$v(L,t) = 0$

$v(x,0) = f(x) - s(x,0)$

② Solve subproblem [SSP]

Since s is linear in x, s has the form

$$\boxed{s(x,t) = m(t)x + b(t)}$$

$$s_x(x,t) = m(t)$$

apply bcs:

$b_1(t) = s_x(0,t) = m(t)$

$b_2(t) = s(L,t) = m(t)L + b(t)$

hence

$$\boxed{\begin{aligned} m(t) &= b_1(t) \\ L\,m(t) + b(t) &= b_2(t) \end{aligned}}$$

$$\Rightarrow \quad \boxed{m(t) = b_1(t)} \qquad \boxed{b(t) = b_2(t) - L\, b_1(t)}$$

and therefore

$$\boxed{S(x,t) = b_1(t)\,x + b_2(t) - L\,b_1(t)}$$

$$\boxed{S_t(x,t) = b_1'(t)\,x + b_2'(t) - L\,b_1'(t)}$$

③ Solve subproblem [VP]:

a] compute Eigenvalues, -functions

$$v_t(x,t) = k\, v_{xx}(x,t)$$
$$v_x(0,t) = 0$$
$$v(L,t) = 0$$

i] separate variables: $v(x,t) = X(x)\,T(t)$

$$X(x)\,T'(t) = k\,X''(x)\,T(t) \qquad \Big| \cdot \frac{1}{k X(x) T(t)}$$

$$\frac{T'(t)}{k\,T(t)} = \frac{X''(x)}{X(x)} = -\mu^2$$

$$\Rightarrow \boxed{T'(t) + \mu^2 k\,T(t) = 0}$$

$$\boxed{X''(x) + \mu^2 X(x) = 0}$$

ii] transl. bcs: $X'(0) = 0, \; X(L) = 0$

iii] solve SL problem

$$\boxed{\begin{aligned} X''(x) + \mu^2 X(x) &= 0 \\ X'(0) &= 0 \\ X(L) &= 0 \end{aligned}} \Rightarrow \begin{cases} \boxed{\mu_n = \dfrac{(2n+1)\pi}{2L}} \\ \boxed{X_n(x) = \cos\left(\dfrac{(2n+1)\pi}{2L}\,x\right)} \end{cases}$$

Note: $$\boxed{X_n''(x) = -\underbrace{\frac{(2n+1)^2\pi^2}{4L^2}}_{\mu_n^2} X_n(x)}$$

b) Eigenfunction expansion of v and q:

$$\boxed{v(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x)}$$

$$\boxed{q(x,t) = \sum_{n=0}^{\infty} Q_n(t) X_n(x)}$$

where
$$\boxed{Q_n(t) = \frac{\int_0^L q(x,t) X_n(x)\, dx}{\int_0^L X_n^2(x)\, dx}}$$

for $n = 0, 1, 2, \dots$

determine $T_n(t)$:

$$v_t(x,t) = k\, v_{xx}(x,t) + q(x,t)$$

$$\sum_{n=0}^{\infty} T_n'(t) X_n(x) = k \sum_{n=0}^{\infty} T_n(t) \underbrace{X_n''(x)}_{= -\mu_n^2 X_n(x)} + \sum_{n=0}^{\infty} Q_n(t) X_n(x)$$

$$= \sum_{n=0}^{\infty} \left[-k\mu_n^2 T_n(t) \right] X_n(x) + \sum_{n=0}^{\infty} Q_n(t) X_n(x)$$

$$\Leftrightarrow \sum_{n=0}^{\infty} \left[T_n'(t) + k\mu_n^2 T_n(t) \right] X_n(x) = \sum_{n=0}^{\infty} Q_n(t) X_n(x)$$

Hence, since the functions X_n constitute a basis,

$$\boxed{T_n'(t) + k\mu_n^2 T_n(t) = Q_n(t)} \qquad n = 0, 1, 2, \dots$$

which is 1st order lode (with constant coeff.)

$$\boxed{T_n(t) = e^{-k\mu_n^2 t} \int_0^t e^{k\mu_n^2 \xi} Q_n(\xi)\, d\xi + C_n e^{-k\mu_n^2 t}}$$

c] apply i.c.

$$f(x) - s(x,0) = v(x,0) = \sum_{n=0}^{\infty} \underbrace{T_n(0)}_{\overset{\|}{C_n}} X_n(x)$$

hence

$$f(x) - s(x,0) = \sum_{n=0}^{\infty} C_n X_n(x)$$

$$\Rightarrow \quad \boxed{C_n =} \frac{\int_0^L (f(x) - s(x,0)) X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}$$

$$= \frac{\int_0^L (f(x) - s(x,0)) \cos\left(\frac{(2n+1)\pi}{2L} x\right) dx}{\underbrace{\int_0^L \cos^2\left(\frac{(2n+1)\pi}{2L} x\right) dx}_{= \frac{L}{2}}}$$

$$\boxed{= \frac{2}{L} \int_0^L (f(x) - s(x,0)) \cos\left(\frac{(2n+1)\pi}{2L} x\right) dx}$$

2] $u_t(x,t) = k u_{xx}(x,t) + h(x,t)$, $\quad x \in (0,L), t > 0$

$u(0,t) = b_1(t)$

$u_x(L,t) = b_2(t)$

$u(x,0) = f(x)$

Solution :

① Partition the solution

$$u(x,t) = s(x,t) + v(x,t)$$

with

[SSP] $S_{xx}(x,t) = 0$

$s(0,t) = b_1(t)$

$s_x(L,t) = b_2(t)$

[VP] $v_t(x,t) = k\, v_{xx}(x,t) + \underbrace{h(x,t) - s_t(x,t)}_{=:\, q(x,t)}$

$v(0,t) = 0$

$v_x(L,t) = 0$

$v(x,0) = f(x) - s(x,0)$

② Solve subproblem [SSP]:

Since s is linear in x, s has the form

$$\boxed{s(x,t) = m(t)\,x + b(t)}$$

$S_x(x,t) = m(t)$

apply bcs:

$b_1(t) = s(0,t) = b(t)$

$b_2(t) = s_x(L,t) = m(t)$

hence

$$\boxed{s(x,t) = b_2(t)\,x + b_1(t)}$$

$$\boxed{s_t(x,t) = b_2'(t)\,x + b_1'(t)}$$

③ Solve subproblem [VP]:

a) compute Eigenvalues, -functions

$v_t(x,t) = k\, v_{xx}(x,t)$

$v(0,t) = 0$

$v_x(L,t) = 0$

i) sep. var.: $v(x,t) = X(x)\,T(t)$

$$X(x)\,T'(t) = k\,X''(x)\,T(t) \quad \Big| \cdot \frac{1}{k\,X(x)\,T(t)}$$

$$\frac{T'(t)}{k\,T(t)} = \frac{X''(x)}{X(x)} = -\mu^2$$

$$\Rightarrow \boxed{T'(t) + \mu^2 k\,T(t) = 0}$$

$$\boxed{X''(x) + \mu^2 X(x) = 0}$$

ii) transl. bcs: $X(0) = 0,\ X'(L) = 0$

iii) solve SL-problem

$$\left.\begin{array}{l} X''(x) + \mu^2 X(x) = 0 \\ X(0) = 0 \\ X'(L) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \boxed{\mu_n = \dfrac{(2n+1)\pi}{2L}} \\ \boxed{X_n(x) = \sin\left(\dfrac{(2n+1)\pi}{2L}\,x\right)} \end{array}\right. \qquad n = 0,1,2,\dots$$

Note $$\boxed{X_n''(x) = -\mu_n^2 X_n(x) = -\frac{(2n+1)^2\pi^2}{4L^2}\,X_n(x)}$$

b) Eigenfunction expansion of v and q:

$$\boxed{v(x,t) = \sum_{n=0}^{\infty} T_n(t)\,X_n(x)}$$

$$\boxed{q(x,t) = \sum_{n=0}^{\infty} Q_n(t)\,X_n(x)}$$

$$\boxed{\text{where} \quad Q_n(t) = \frac{\int_0^L q(x,t)\,X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}} \qquad \text{for } n = 0,1,2,\dots$$

Determine $T_n(t)$!

as above (Ex 1), we find

$$\boxed{T_n'(t) + k\,\mu_n^2\,T(t) = Q_n(t)}, \qquad n = 0,1,2,\dots$$

$$\boxed{T_n(t) = e^{-k\mu_n^2 t} \int_0^t e^{k\mu_n^2 \xi}\,Q_n(\xi)\,d\xi + C_n e^{-k\mu_n^2 t}}$$

c) apply ic:

$$f(x)-s(x,0) = v(x,0) = \sum_{n=0}^{\infty} \underbrace{T_n(0)}_{= C_n} X_n(x)$$

Hence, $f(x)-s(x,0) = \sum_{n=0}^{\infty} C_n X_n(x)$

and therefore

$$\boxed{C_n = \frac{\int_0^L (f(x)-s(x,0)) X_n(x)\,dx}{\underbrace{\int_0^L X_n^2(x)\,dx}_{=\frac{L}{2}}}}$$

$$\boxed{= \frac{2}{L}\int_0^L (f(x)-s(x,0)) X_n(x)\,dx}$$

④ summary

$$\boxed{\begin{aligned} u(x,t) &= s(x,t) + v(x,t) \\ &= b_2(t)x + b_1(t) + \sum_{n=0}^{\infty} T_n(t) X_n(x) \end{aligned}}$$

where $X_n(x) = \sin\left(\frac{(2n+1)\pi}{2L}x\right)$, $\mu_n = \frac{(2n+1)\pi}{2L}$,

$$T_n(t) = e^{-k\mu_n^2 t}\int_0^t e^{k\mu_n^2 \xi} Q_n(\xi)\,d\xi + C_n e^{-k\mu_n^2 t}$$

$$Q_n(t) = \frac{2}{L}\int_0^L (h(x,t) - s_t(x,t)) X_n(x)\,dx$$

$$C_n = \frac{2}{L}\int_0^L (f(x)-s(x,0)) X_n(x)\,dx$$

3] $u_t(x,y,t) = k\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right) + h(x,y,t)$

bcs: $u(0,y,t) = b_1(y,t)$
$u(L,y,t) = b_2(y,t)$
$u(x,0,t) = 0$
$u(x,M,t) = 0$

ic: $u(x,y,0) = f(x,y)$

<u>Solution:</u>

① Partition the solution: $\boxed{u(x,y,t) = s(x,y,t) + v(x,y,t)}$

[SSP]: $s_{xx}(x,y,t) + s_{yy}(x,y,t) = 0$

bcs: $s(0,y,t) = b_1(y,t)$
$s(L,y,t) = b_2(y,t)$
$s(x,0,t) = 0$
$s(x,M,t) = 0$

} Laplace equation with "parameter t"

[VP]: $v_t(x,y,t) = k\left(v_{xx}(x,y,t) + v_{yy}(x,y,t)\right) + \underbrace{\left(h(x,y,t) - s_t(x,y,t)\right)}_{q(x,y,t):=}$

bcs: $v(0,y,t) = 0$
$v(L,y,t) = 0$
$s(x,0,t) = 0$
$s(x,M,t) = 0$

ic: $v(x,y,0) = f(x,y) - s(x,y,0)$

② Solve subproblem [SSP] Laplace eqn with 2 nonhom bcs:

a) partition solution $\boxed{s(x,y,t) = s^{(A)}(x,y,t) + s^{(B)}(x,y,t)}$

$s^{(A)}_{xx}(x,y,t) + s^{(A)}_{yy}(x,y,t) = 0$	$s^{(B)}_{xx}(x,y,t) + s^{(B)}_{yy}(x,y,t) = 0$
$s^{(A)}(0,y,t) = b_1(y,t)$	$s^{(B)}(0,y,t) = 0$
$s^{(A)}(L,y,t) = 0$	$s^{(B)}(L,y,t) = b_2(y,t)$
$s^{(A)}(x,0,t) = 0$] 2 hom	$s^{(B)}(x,0,t) = 0$] 2 hom
$s^{(A)}(x,M,t) = 0$] bcs in y !	$s^{(B)}(x,M,t) = 0$] bcs in y !

b) solve subproblem (A) and (B):

(A): i) sep. variables $S^{(A)}(x,y,t) = X(x)Y(y)T(t)$

$$X''(x)Y(y)T(t) + X(x)Y''(y)T(t) = 0 \qquad \Big| \cdot \frac{1}{X(x)Y(y)T(t)}$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \mu^2$$

$$\Rightarrow \boxed{X''(x) - \mu^2 X(x) = 0,} \quad \boxed{Y''(y) + \mu^2 Y(y) = 0}$$

ii) transl. bcs: $X(L) = 0$; $Y(0) = 0$, $Y(H) = 0$
bc on non-zero boundary!

iii) solve SL problem

$$\left.\begin{array}{l} Y''(y) + \mu^2 Y(y) = 0 \\ Y(0) = 0 \\ Y(H) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \boxed{\mu_n = \frac{(n+1)\pi}{H}}, \quad n = 0,1,2,\ldots \\ \boxed{Y_n(y) = \sin\left(\frac{(n+1)\pi}{H} y\right)} \end{array}\right.$$

iv) solve remaining eqn

$$X''(x) - \mu_n^2 X(x) = 0, \quad \mu_n^2 = \frac{(n+1)^2\pi^2}{H^2} > 0$$

$$\Rightarrow X_n(x) = C_1 \cosh\left(\frac{(n+1)\pi}{H}(x-L)\right) + C_2 \sinh\left(\frac{(n+1)\pi}{H}(x-L)\right)$$

apply hom. bc:

$$0 = X_n(L) = C_1 \Rightarrow C_1 = 0, \; C_2 \neq 0$$

hence

$$\boxed{X_n(x) = \sinh\left(\frac{(n+1)\pi}{H}(x-L)\right)}, \quad n = 0,1,2,\ldots$$

v) gen. solution

$$S_n^{(A)}(x,y,t) = X_n(x) Y_n(y) T_n^{(A)}(t)$$

$$= \sinh\left(\frac{(n+1)\pi}{H}(x-L)\right) \sin\left(\frac{(n+1)\pi}{H} y\right) T_n^{(A)}(t)$$

$$\boxed{S^{(A)}(x,y,t) = \sum_{n=0}^{\infty} T_n^{(A)}(t) \sinh\left(\frac{(n+1)\pi}{H}(x-L)\right) \sin\left(\frac{(n+1)\pi}{H} y\right)}$$

vi] apply nonhom. bc:

$$b_1(y,t) = S^{(A)}(0,y,t) = \sum_{n=0}^{\infty} T_n^{(A)}(t)\sinh\left(\frac{(n+1)\pi}{M}(-L)\right)\sin\left(\frac{(n+1)\pi}{M}y\right)$$

$$= \sum_{n=0}^{\infty}\left[-T_n^{(A)}(t)\sinh\left(\frac{(n+1)\pi L}{M}\right)\right]\sin\left(\frac{(n+1)\pi}{M}y\right)$$

hence

$$-T_n^{(A)}(t)\sinh\left(\frac{(n+1)\pi L}{M}\right) = \frac{\int_0^M b_1(y,t)\sin\left(\frac{(n+1)\pi}{M}y\right)dy}{\underbrace{\int_0^M \sin^2\left(\frac{(n+1)\pi}{M}y\right)dy}_{=\frac{M}{2}}}$$

and therefore

$$\boxed{T_n^{(A)}(t) = -\frac{2}{M\sinh\left(\frac{(n+1)\pi L}{M}\right)}\int_0^M b_1(y,t)\sin\left(\frac{(n+1)\pi}{M}y\right)dy}$$

for $n=0,1,2,\ldots$

(B) i] sep variables $S^{(B)}(x,y,t) = X(x)Y(y)T(t)$
as in (A) (note that in both cases the 2 hom. bcs are in y!)

$$\boxed{X''(x) - \mu^2 X(x) = 0, \quad Y''(y) + \mu^2 Y(y) = 0}$$

ii] transl. bcs: $X(0) = 0$, $Y(0)=0$, $Y(M) = 0$

bc on zero-bc!

iii] solve SL problem (as in (A))

$$\left.\begin{aligned} &Y''(y) + \mu^2 Y(y) = 0\\ &Y(0) = 0\\ &Y(M) = 0\end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &\boxed{\mu_n = \frac{(n+1)\pi}{M}}, \quad n=0,1,2,\ldots\\ &\boxed{Y_n(y) = \sin\left(\frac{(n+1)\pi}{M}y\right)}\end{aligned}\right.$$

iv] solve remaining eqn

$$X''(x) - \mu_n^2 X(x) = 0, \quad \mu_n^2 = \frac{(n+1)^2\pi^2}{M^2} > 0$$

hence

$$X_n(x) = C_1\cosh\left(\frac{(n+1)\pi}{M}x\right) + C_2\sinh\left(\frac{(n+1)\pi}{M}x\right)$$

apply hom. bc:

$$0 = X_n(0) = C_1 \Rightarrow C_1 = 0,\ C_2 \neq 0$$

hence

$$\boxed{X_n(x) = \sinh\left(\frac{(n+1)\pi}{M} x\right),} \qquad n = 0,1,2,\ldots$$

v) gen. solution

$$S_n^{(B)}(x,y,t) = X_n(x)\, Y_n(y)\, T_n^{(B)}(t)$$
$$= \sinh\left(\frac{(n+1)\pi}{M} x\right) \sin\left(\frac{(n+1)\pi}{M} y\right) T_n^{(B)}(t)$$

$$\boxed{S^{(B)}(x,y,t) = \sum_{n=0}^{\infty} T_n^{(B)}(t) \sinh\left(\frac{(n+1)\pi}{M} x\right) \sin\left(\frac{(n+1)\pi}{M} y\right)}$$

vi) apply nonhom. bc:

$$b_2(y,t) = S^{(B)}(L,y,t) = \sum_{n=0}^{\infty}\left[T_n^{(B)}(t) \sinh\left(\frac{(n+1)\pi L}{M}\right)\right] \sin\left(\frac{(n+1)\pi}{M} y\right)$$

hence

$$T_n^{(B)}(t) \sinh\left(\frac{(n+1)\pi L}{M}\right) = \frac{\int_0^M b_2(y,t) \sin\left(\frac{(n+1)\pi}{M} y\right) dy}{\underbrace{\int_0^M \sin^2\left(\frac{(n+1)\pi}{M} y\right) dy}_{= \frac{M}{2}}}$$

and therefore

$$\boxed{T_n^{(B)}(t) = \frac{2}{M \sinh\left(\frac{(n+1)\pi L}{M}\right)} \int_0^M b_2(y,t) \sin\left(\frac{(n+1)\pi}{M} y\right) dy}$$

for $n = 0,1,2,\ldots$

c) combine the solution of (A) and (B)

$$S(x,y,t) = S^{(A)}(x,y,t) + S^{(B)}(x,y,t)$$

$$= \sum_{n=0}^{\infty} T_n^{(A)}(t) \sinh\left(\frac{(n+1)\pi}{M}(x-L)\right) \sin\left(\frac{(n+1)\pi}{M} y\right)$$

$$+ \sum_{n=0}^{\infty} T_n^{(B)} \sinh\left(\frac{(n+1)\pi}{M} x\right) \sin\left(\frac{(n+1)\pi}{M} y\right)$$

$$= \sum_{n=0}^{\infty} \left[T_n^{(A)}(t) \sinh\left(\frac{(n+1)\pi}{M}(x-L)\right) + T_n^{(B)}(t) \sinh\left(\frac{(n+1)\pi}{M} x\right)\right] \sin\left(\frac{(n+1)\pi}{M} y\right)$$

where

$$T_n^{(A)}(t) = -\frac{2}{M \sinh\left(\frac{(n+1)\pi L}{M}\right)} \int_0^M b_1(y,t) \sin\left(\frac{(n+1)\pi}{M} y\right) dy$$

$$T_n^{(B)}(t) = \frac{2}{M \sinh\left(\frac{(n+1)\pi L}{M}\right)} \int_0^M b_2(y,t) \sin\left(\frac{(n+1)\pi}{M} y\right) dy$$

③ Solve subproblem [VP]

a) compute Eigenvalues, -functions

$$v_t(x,y,t) = k\left(v_{xx}(x,y,t) + v_{yy}(x,y,t)\right)$$

$$v(0,y,t) = 0$$
$$v(L,y,t) = 0$$
$$v(x,0,t) = 0$$
$$v(x,M,t) = 0$$

i) sep. var. $v(x,y,t) = X(x)Y(y)T(t)$
splits the pde into

$$T'(t) + \mu^2 k T(t) = 0$$
$$X''(x) + g^2 X(x) = 0$$
$$Y''(y) + \tau^2 Y(y) = 0$$
$$g^2 + \tau^2 = \mu^2$$

ii] transl. bcs:

$$X(0)=0,\ X(L)=0,\ Y(0)=0,\ Y(M)=0$$

iii] solve SL-problems:

$$X''(x)+\rho^2 X(x)=0 \qquad\qquad Y''(y)+\tau^2 Y(y)=0$$

$$X(0)=0 \qquad\qquad Y(0)=0$$

$$X(L)=0 \qquad\qquad Y(M)=0$$

$$\boxed{\rho_m=\frac{(m+1)\pi}{L}} \qquad\qquad \boxed{\tau_n=\frac{(n+1)\pi}{M}}$$

$$\boxed{X_m(x)=\sin\left(\frac{(m+1)\pi}{L}x\right)} \qquad\qquad \boxed{Y_n(x)=\sin\left(\frac{(n+1)\pi}{M}y\right)}$$

for $m,n=0,1,2,\ldots$

b] Eigenfunction expansion of v and q

$$\boxed{v(x,y,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}T_{mn}(t)X_m(x)Y_n(y)}$$

$$q(x,y,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}Q_{mn}(t)X_m(x)Y_n(y)$$

where

$$\boxed{Q_{mn}(t)}=\frac{\int_0^M\int_0^L q(x,y,t)X_m(x)Y_n(y)\,dx\,dy}{\underbrace{\int_0^L X_m^2(x)\,dx}_{=\frac{L}{2}}\ \underbrace{\int_0^M Y_n^2(y)\,dy}_{\frac{M}{2}}}$$

$$\boxed{=\frac{4}{LM}\int_0^M\int_0^L q(x,y,t)\sin\left(\frac{(m+1)\pi}{L}x\right)\sin\left(\frac{(n+1)\pi}{M}y\right)dx\,dy}$$

where

$$\boxed{T_{mn}(t) = e^{-k\mu_{mn}^2 t}\int_0^t e^{k\mu_{mn}^2 \xi} Q_{mn}(\xi)\,d\xi + C_{mn}e^{-k\mu_{mn}^2 t}}$$

with $\boxed{\mu_{mn}^2 = \rho_m^2 + \tau_n^2 = \pi^2\left(\frac{(m+1)^2}{L^2} + \frac{(n+1)^2}{M^2}\right)}$

and

$$\boxed{C_{mn}} = \frac{\int_0^M\int_0^L (f(x,y) - s(x,y,0))\,X_m(x)\,Y_n(y)\,dx\,dy}{\underbrace{\int_0^L X_m^2(x)\,dx}_{=\frac{L}{2}}\ \underbrace{\int_0^M Y_n^2(y)\,dy}_{=\frac{M}{2}}}$$

$$\boxed{= \frac{4}{LM}\int_0^M\int_0^L (f(x,y) - s(x,y,0))\sin\left(\frac{(m+1)\pi}{L}x\right)\sin\left(\frac{(n+1)\pi}{M}y\right)dx\,dy}$$

(4) Combine solution

$$u(x,y,t) = s(x,y,t) + v(x,y,t)$$

4] $u_t(x,y,t) = k(u_{xx}(x,y,t) + u_{yy}(x,y,t)) + h(x,y,t)$

bcs
$$u(0,y,t) = b_1(y,t)$$
$$u(L,y,t) = b_2(y,t)$$
$$u_y(x,0,t) = 0$$
$$u(x,M,t) = 0$$

ic.
$$u(x,y,0) = f(x,y)$$

Solution:

(1) Partition the solution: $\boxed{u(x,y,t) = s(x,y,t) + v(x,y,t)}$

[SSP] $s_{xx}(x,y,t) + s_{yy}(x,y,t) = 0$

bcs: $s(0,y,t) = b_1(y,t)$
$s(L,y,t) = b_2(y,t)$
$s_y(x,0,t) = 0$
$s(x,M,t) = 0$

[VP] $v_t(x,y,t) = k\left(v_{xx}(x,y,t) + v_{yy}(x,y,t)\right) + \dfrac{\left(h(x,y,t) - s_t(x,y,t)\right)}{q(x,y,t)}$

bcs: $v(0,y,t) = 0$
$v(L,y,t) = 0$
$v_y(x,0,t) = 0$
$v(x,M,t) = 0$

ic: $v(x,y,0) = f(x,y) - s(x,y,0)$

② solve subproblem [SSP]: Laplace eqn with 2 nonhom. bcs.

a) partition solution: $\boxed{s(x,y,t) = s^{(A)}(x,y,t) + s^{(B)}(x,y,t)}$

$s^{(A)}_{xx}(x,y,t) + s^{(A)}_{yy}(x,y,t) = 0$	$s^{(B)}_{xx}(x,y,t) + s^{(B)}_{yy}(x,y,t) = 0$
$s^{(A)}(0,y,t) = b_1(y,t)$	$s^{(B)}(0,y,t) = 0$
$s^{(A)}(L,y,t) = 0$	$s^{(B)}(L,y,t) = b_2(y,t)$
$s^{(A)}_y(x,0,t) = 0$	$s^{(B)}_y(x,0,t) = 0$
$s^{(A)}(x,M,t) = 0$] 2 hom bcs in y	$s^{(B)}(x,M,t) = 0$] 2 hom bcs in y

b) solve subproblem (A) and (B)

(A) i) sep var. $s^{(A)}(x,y,t) = X(x)Y(y)T(t)$

$X''(x)Y(y)T(t) + X(x)Y''(y)T(t) = 0 \quad \Big| \cdot \dfrac{1}{X(x)Y(y)T(t)}$

$\dfrac{X''(x)}{X(x)} + \dfrac{Y''(y)}{Y(y)} = 0 \Rightarrow \dfrac{X''(x)}{X(x)} = -\dfrac{Y''(y)}{Y(y)} = \mu^2$

$\Rightarrow \boxed{X''(x) - \mu^2 X(x) = 0} \quad \boxed{Y''(y) + \mu^2 Y(y) = 0}$

ii) transl. bcs: $X(L) = 0, \quad Y'(0) = 0, \quad Y(M) = 0$

bc on non-zero boundary!

iii] solve SL problem

$$\left.\begin{aligned} &Y''(y)+\mu^2 Y(y)=0\\ &Y'(0)=0\\ &Y(M)=0 \end{aligned}\right\} \Rightarrow \begin{aligned} &\boxed{\mu_n=\frac{(2n+1)\pi}{2M}}, \qquad n=0,1,2,\ldots\\ &\boxed{Y_n(y)=\cos\left(\frac{(2n+1)\pi}{2M}y\right)} \end{aligned}$$

iv] solve remaining eqn

$$X''(x)-\mu_n^2 X(x)=0, \qquad \mu_n^2=\frac{(2n+1)^2\pi^2}{4M^2}>0$$
$$X(L)=0$$

$$\Rightarrow \quad X_n(x)=C_1\cosh\left(\frac{(2n+1)\pi}{2M}(x-L)\right)+C_2\sinh\left(\frac{(2n+1)\pi}{2M}(x-L)\right)$$

apply hom. bc:

$$0=X(L)=C_1 \Rightarrow C_1=0, \quad C_2\neq 0$$

hence

$$\boxed{X_n(x)=\cosh\left(\frac{(2n+1)\pi}{2M}(x-L)\right)} \qquad n=0,1,2,\ldots$$

v] gen. solution

$$S_n^{(A)}(x,y,t)=X_n(x)Y_n(y)T_n^{(A)}(t)$$
$$=\cosh\left(\frac{(2n+1)\pi}{2M}(x-L)\right)\cos\left(\frac{(2n+1)\pi}{2M}y\right)T_n^{(A)}(t)$$

$$\boxed{S^{(A)}(x,y,t)=\sum_{n=0}^{\infty}T_n^{(A)}(t)\cos\left(\frac{(2n+1)\pi}{2M}(x-L)\right)\cos\left(\frac{(2n+1)\pi}{2M}y\right)}$$

vi] apply inh. bc:

$$b_1(y,t)=S^A(0,y,t)=\sum_{n=0}^{\infty}\underbrace{T_n^{(A)}(t)\cos\left(\frac{(2n+1)\pi}{2M}(-L)\right)}_{\left[T_n^{(A)}(t)\cos\left(\frac{(2n+1)\pi L}{2M}\right)\right]}\cos\left(\frac{(2n+1)\pi}{2}y\right)$$

hence

$$T_n^{(A)}(t)\cosh\left(\frac{(2n+1)\pi L}{2M}\right)=\frac{\int_0^M b_1(y,t)\cos\left(\frac{(2n+1)\pi}{2M}y\right)dy}{\underbrace{\int_0^M \cos^2\left(\frac{(2n+1)\pi}{2M}y\right)dy}_{=\frac{M}{2}}}$$

and therefore

$$\boxed{T_n^{(A)}(t) = \frac{2}{M\cosh\left(\frac{(2n+1)\pi L}{2M}\right)} \int_0^M b_1(y,t)\cos\left(\frac{(2n+1)\pi}{2M}y\right)dy}$$

for $n=0,1,2,\ldots$

(B) sep. var. – $S^{(B)}(x,y,t) = X(x)\,Y(y)\,T(t)$

Note that i] – iii] is analogue to (A), hence

$$\boxed{\mu_n = \frac{(2n+1)\pi}{2M}}, \quad \boxed{Y_n(y) = \cos\left(\frac{(2n+1)\pi}{2M}y\right)} \quad n=0,1,2,\ldots$$

iv] solve remaining eqn

$$X''(x) - \mu_n^2 X(x) = 0, \quad \mu_n^2 = \frac{(2n+1)^2\pi^2}{4M^2} > 0$$

$$X(0) = 0$$

$$\Rightarrow X_n(x) = C_1\cosh\left(\frac{(2n+1)\pi}{2M}x\right) + C_2\sin\left(\frac{(2n+1)\pi}{2M}x\right)$$

apply hom. bc,

$$0 = X(0) = C_1 \Rightarrow C_1 = 0, \; C_2 \neq 0$$

hence

$$\boxed{X_n(x) = \cosh\left(\frac{(2n+1)\pi}{2M}x\right)} \quad n=0,1,2,\ldots$$

v] gen. sol.

$$S_n^{(B)}(x,y,t) = X_n(x)\,Y_n(y)\,T_n(t)$$

$$= \cosh\left(\frac{(2n+1)\pi}{2M}x\right)\cos\left(\frac{(2n+1)\pi}{2M}y\right)T_n^{(B)}(t)$$

$$\boxed{S^{(B)}(x,y,t) = \sum_{n=0}^{\infty} T_n(t)\cosh\left(\frac{(2n+1)\pi}{2M}x\right)\cos\left(\frac{(2n+1)\pi}{2M}y\right)}$$

vi) apply nonhom bc:

$$b_2(y,t) = S^{(B)}(L,y,t) = \sum_{n=0}^{\infty}\left[T_n(t)\cosh\left(\frac{(2n+1)\pi}{2M}L\right)\right]\cos\left(\frac{(2n+1)\pi}{2M}y\right)$$

hence

$$T_n(t)\cosh\left(\frac{(2n+1)\pi L}{2M}\right) = \frac{\int_0^M b_2(y,t)\cos\left(\frac{(2n+1)\pi}{2M}y\right)dy}{\underbrace{\int_0^M \cos^2\left(\frac{(2n+1)\pi}{2M}y\right)dy}_{=\frac{M}{2}}}$$

and therefore

$$\boxed{T_n(t) = \frac{2}{M\cosh\left(\frac{(2n+1)\pi L}{2M}\right)}\int_0^M b_2(y,t)\cos\left(\frac{(2n+1)\pi}{2M}y\right)dy}$$

c) combine solution of (A) and (B)

$$\boxed{S(x,y,t)} = S^{(A)}(x,y,t) + S^{(B)}(x,y,t)$$

$$= \sum_{n=0}^{\infty} T_n^{(A)}(t)\cosh\left(\frac{(2n+1)\pi}{2M}(x-L)\right)\cos\left(\frac{(2n+1)\pi}{2M}y\right)$$

$$+ \sum_{n=0}^{\infty} T_n^{(B)}(t)\cosh\left(\frac{(2n+1)\pi}{2M}x\right)\cos\left(\frac{(2n+1)\pi}{2M}y\right)$$

$$\boxed{= \sum_{n=0}^{\infty}\left[T_n^{(A)}(t)\cos\left(\frac{(2n+1)\pi}{2M}(x-L)\right) + T_n^{(B)}(t)\cosh\left(\frac{(2n+1)\pi}{2M}x\right)\right]\cos\left(\frac{(2n+1)\pi}{2M}y\right)}$$

where

$$\boxed{T_n^{(A)}(t) = \frac{2}{M\cosh\left(\frac{(2n+1)\pi L}{2M}\right)}\int_0^M b_1(y,t)\cos\left(\frac{(2n+1)\pi}{2M}y\right)dy}$$

$$\boxed{T_n^{(B)}(t) = \frac{2}{M\cosh\left(\frac{(2n+1)\pi L}{2M}\right)}\int_0^M b_2(y,t)\cos\left(\frac{(2n+1)\pi}{2M}y\right)dy}$$

③ Solve subproblem [VP]

a) compute Eigenvalues, -functions

$$v_t(x,y,t) = k\,(v_{xx}(x,y,t) + v_{yy}(x,y,t))$$
$$v(0,y,t) = 0$$
$$v(L,y,t) = 0$$
$$v_y(x,0,t) = 0$$
$$v(x,M,t) = 0$$

i) sep. variables: $v(x,y,t) = X(x)\,Y(y)\,T(t)$
splits the pde into

$$\boxed{\begin{aligned} &T'(t) + \mu^2 k T(t) = 0 \\ &X''(x) + \rho^2 X(x) = 0 \\ &Y''(y) + \tau^2 Y(y) = 0 \\ &\rho^2 + \tau^2 = \mu^2 \end{aligned}}$$

ii) transl. bcs:
$X(0) = 0,\ X(L) = 0,\ Y'(0) = 0\ Y(M) = 0$

iii) solve SL-problem

$$\underbrace{\begin{aligned} &X''(x) + \rho^2 X(x) = 0 \\ &X(0) = 0 \\ &X(L) = 0 \end{aligned}}$$

$$\boxed{\rho_m = \frac{(m+1)\pi}{L}}$$

$$\boxed{X_m(x) = \sin\left(\frac{(m+1)\pi}{L}\,x\right)}$$

$$\underbrace{\begin{aligned} &Y''(y) + \tau^2 Y(y) = 0 \\ &Y'(0) = 0 \\ &Y(M) = 0 \end{aligned}}$$

$$\boxed{\tau_n = \frac{(2n+1)\pi}{2M}}$$

$$\boxed{Y_n(y) = \cos\left(\frac{(2n+1)\pi}{2M}\,y\right)}$$

for $m, n = 0, 1, 2, \ldots$

b) Eigenfunction expansion of v and q

$$\boxed{v(x,y,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} T_{mn}(t)\, X_m(x)\, Y_n(y)}$$

$$q(x,y,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Q_{mn}(t)\, X_m(x)\, Y_n(y)$$

where

$$\boxed{Q_{mn}(t) = \frac{\int_0^M \int_0^L q(x,y,t)\, X_m(x)\, Y_n(y)\, dx\, dy}{\underbrace{\int_0^L X_m^2(x)\, dx}_{\frac{L}{2}} \; \underbrace{\int_0^M Y_n^2(y)\, dy}_{\frac{M}{2}}}}$$

$$\boxed{= \frac{4}{LM} \int_0^M \int_0^L q(x,y,t)\, X_m(x)\, Y_n(y)\, dx\, dy}$$

and

$$T_{mn}(t) = e^{-k\mu_{mn}^2 t} \int_0^t e^{k\mu_{mn}^2 \xi}\, Q_{mn}(\xi)\, d\xi + C_{mn}\, e^{-k\mu_{mn}^2 t}$$

with

$$\mu_{mn}^2 = \sigma_m^2 + \tau_n^2 = \left[\frac{(m+1)^2}{L^2} + \frac{(2n+1)^2}{4M^2}\right]\pi^2$$

and

$$C_{mn} = \frac{\int_0^M \int_0^L \left(f(x,y) - s(x,y,0)\right) X_m(x)\, Y_n(y)\, dx\, dy}{\underbrace{\int_0^L X_m^2(x)\, dx}_{=\frac{L}{2}} \; \underbrace{\int_0^M Y_n^2(y)\, dy}_{\frac{M}{2}}}$$

$$\boxed{C_{mn} = \frac{4}{LM} \int_0^M \int_0^L \left(f(x,y) - s(x,y,0) \right) X_m(x)\, Y_n(y)\, dx\, dy}$$

④ Combine solution

$$u(x,y,t) = s(x,y,t) + v(x,y,t)$$

5] $u_t(x,y,t) = k\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right) + h(x,y,t)$

bcs: $u(0,y,t) = b_1(y,t)$

$u(L,y,t) = b_2(y,t)$

$u_y(x,0,t) = 0$

$u_y(x,M,t) = 0$

ic: $u(x,y,0) = f(x,y)$

<u>Solution</u>:

① Partition the solution: $\boxed{u(x,y,t) = s(x,y,t) + v(x,y,t)}$

[SSP] $s_{xx}(x,y,t) + s_{yy}(x,y,t) = 0$

bcs $s(0,y,t) = b_1(y,t)$

$s(L,y,t) = b_2(y,t)$

$s_y(x,0,t) = 0$

$s_y(x,M,t) = 0$

[VP] $v_t(x,y,t) = k\left(v_{xx}(x,y,t) + v_{yy}(x,y,t)\right) + \underbrace{\left(h(x,y,t) - s_t(x,y,t)\right)}_{g(x,y,t)}$

bcs: $v(0,y,t) = 0$

$v(L,y,t) = 0$

$v_y(x,0,t) = 0$

$v_y(x,M,t) = 0$

ics: $v(x,y,0) = f(x,y) - s(x,y,0)$

② solve subproblem [SSP]: Laplace eqn with 2 nonhom. bcs

a) partition solution $\boxed{S(x,y,t) = S^{(A)}(x,y,t) + S^{(B)}(x,y,t)}$

$$S^{(A)}_{xx}(x,y,t) + S^{(B)}_{yy} = 0$$
$$S^{(A)}(0,y,t) = b_1(y,t)$$
$$S^{(A)}(L,y,t) = 0$$
$$\left.\begin{array}{l} S^{(A)}_y(x,0,t) = 0 \\ S^{(A)}_y(x,M,t) = 0 \end{array}\right] \text{2 hom bcs in } y$$

$$S^{(B)}_{xx}(x,y,t) + S^{(B)}_{yy}(x,y,t) = 0$$
$$S^{(B)}(0,y,t) = 0$$
$$S^{(B)}(L,y,t) = b_2(y,t)$$
$$\left.\begin{array}{l} S^{(B)}_y(x,0,t) = 0 \\ S^{(B)}_y(x,M,t) = 0 \end{array}\right] \text{2 hom bcs in } y$$

b) solve subproblem (A) and (B)

(A) i) sep. var: $S^{(A)}(x,y,t) = X(x)\,Y(y)\,T(t)$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \mu^2 \Rightarrow \boxed{X''(x) - \mu^2 X(x) = 0}, \boxed{Y''(y) + \mu^2 Y(y) = 0}$$

ii) transl. bcs: $X(L) = 0$, $Y'(0) = 0$, $Y'(M) = 0$

bc on non-zero boundary!

iii) solve SL-problem

$$\left.\begin{array}{l} Y''(y) + \mu^2 Y(y) = 0 \\ Y'(0) = 0 \\ Y'(M) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \boxed{\mu_n = \frac{n\pi}{M}} \\ \boxed{Y_n(y) = \cos\left(\frac{n\pi}{M} y\right)} \end{array}\right. \quad n = 0,1,2,$$

note $\mu_0 = 0$!

iv) solve remaining eqn

$$X''(x) - \mu_n^2 X(x) = 0, \quad \mu_n^2 = \frac{n^2\pi^2}{M^2} \geq 0 \quad \left(r^2 = \frac{n^2\pi^2}{M^2},\ r = 0 \text{ if } n = 0\right)$$
$$X(L) = 0$$

$$X_n(x) = \begin{cases} C_1 + C_2(x - L), & n = 0 \\ C_1 \cosh\left(\frac{n\pi}{M}(x-L)\right) + C_2 \sinh\left(\frac{n\pi}{M}(x-L)\right), & n > 0 \end{cases}$$

$$0 = X_n(L) = \left.\begin{cases} C_1 & n = 0 \\ C_1 & n > 0 \end{cases}\right\} \Rightarrow C_1 = 0,\ C_2 \neq 0$$

hence

$$\boxed{X_n(x) = \begin{cases} x - L, & n = 0 \\ \sinh\left(\frac{n\pi}{M}(x-L)\right), & n > 0 \end{cases}}$$

v] gen. solution

$$S_n^{(A)}(x,y,t) = X_n(x)\, Y_n(y)\, T_n(t)$$
$$= \begin{cases} (x-L)\cdot 1 \cdot T_0^{(A)}(t), & n=0 \\ \sinh\left(\frac{n\pi}{M}(x-L)\right)\cos\left(\frac{n\pi}{M}y\right) T_n^{(A)}(t), & n>0 \end{cases}$$

$$\boxed{S^{(A)}(x,y,t) = T_0^{(A)}(t)(x-L) + \sum_{n=1}^{\infty} T_n^{(A)}(t)\sinh\left(\frac{n\pi}{M}(x-L)\right)\cos\left(\frac{n\pi}{M}y\right)}$$

vi] apply n.h. bc:

$$b_1(y,t) = S^{(A)}(0,y,t) = T_0^{(A)}(t)(-L) + \sum_{n=1}^{\infty} T_n^{(A)}(t)\underbrace{\sinh\left(\frac{n\pi}{M}(-L)\right)}_{= -\sinh\left(\frac{n\pi L}{M}\right)}\cos\left(\frac{n\pi}{M}y\right)$$

hence

$$-T_0^{(A)}(t)\cdot L = \frac{\int_0^M b_1(y,t)\,dy}{\underbrace{\int_0^M dy}_{= M}} \qquad (\;Y_0(y) = 1\;!\;)$$

$$\boxed{T_0^{(A)}(t) = -\frac{1}{ML}\int_0^M b_1(y,t)\,dy}$$

$$-T_n^{(A)}(t)\sinh\left(\frac{n\pi L}{M}\right) = \frac{\int_0^M b_1(y,t)\cos\left(\frac{n\pi}{M}y\right)dy}{\underbrace{\int_0^M \cos^2\left(\frac{n\pi}{M}y\right)dy}_{= \frac{M}{2}}}$$

$$\boxed{T_n^{(A)}(t) = -\frac{2}{M\sinh\left(\frac{n\pi L}{M}\right)}\int_0^M b_1(y,t)\cos\left(\frac{n\pi}{M}y\right)dy,} \quad n = 1, 2,

(B) i) sep. var. $S^{(B)}(x,y,t) = X(x)\,Y(y)\,T(t)$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \mu^2 \Rightarrow \boxed{X''(x) - \mu^2 X(x) = 0,}\ \boxed{Y''(y) + \mu^2 Y(y) = 0}$$

ii) transl. bcs: $X(0) = 0, \quad Y'(0) = 0, \quad Y'(M) = 0$

bc on zero-boundary!

iii) solve SL problem

$$\left.\begin{aligned} &Y''(y) + \mu^2 Y(y) = 0 \\ &Y'(0) = 0 \\ &Y'(M) = 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &\boxed{\mu_n = \frac{n\pi}{M},} \\ &\boxed{Y_n(y) = \cos\left(\frac{n\pi}{M} y\right)} \end{aligned}\right.$$

$\mu_0 = 0$!

$n = 0, 1, 2, \ldots$

iv) solve rem. prob.

$$X''(x) - \mu_n^2 X(x) = 0$$

$$X(0) = 0$$

$$X_n = \begin{cases} C_1 + C_2 x, & n = 0 \\ C_1 \cosh\left(\frac{n\pi}{M} x\right) + C_2 \sinh\left(\frac{n\pi}{M} x\right), & n > 0 \end{cases}$$

$$0 = X_n(0) = \left.\begin{cases} C_1 & n = 0 \\ C_1 & n > 0 \end{cases}\right\} \Rightarrow C_1 = 0, \quad C_2 \neq 0$$

hence

$$\boxed{X_n = \begin{cases} x, & n = 0 \\ \sinh\left(\frac{n\pi}{M} x\right), & n > 0 \end{cases}}$$

v) gen. solution

$$S_n^{(B)}(x,y,t) = X_n(x)\,Y_n(y)\,T_n(t)$$

$$= \begin{cases} x\,T_0(t), & n = 0 \\ \sinh\left(\frac{n\pi}{M} x\right) \cos\left(\frac{n\pi}{M} y\right) T_n(t), & n > 0 \end{cases}$$

$$\boxed{S^{(B)}(x,y,t) = T_0(t)\,x + \sum_{n=1}^{\infty} T_n(t)\sinh\left(\frac{n\pi}{M}x\right)\cos\left(\frac{n\pi}{M}y\right)}$$

vi] apply n.h. bcs

$$b_2(y,t) = S^{(B)}(L,y,t) = T_0(t)\,L + \sum_{n=1}^{\infty} T_n(t)\sinh\left(\frac{n\pi L}{M}\right)\cos\left(\frac{n\pi}{M}y\right)$$

hence

$$T_0(t)\,L = \frac{\int_0^M b_2(y,t)\,dy}{\underbrace{\int_0^M dy}_{=M}}$$

$$\boxed{T_0(t) = \frac{1}{LM}\int_0^M b_2(y,t)\,dy}$$

$$T_n(t)\sinh\left(\frac{n\pi L}{M}\right) = \frac{\int_0^M b_2(y,t)\cos\left(\frac{n\pi}{M}y\right)dy}{\underbrace{\int_0^M \cos^2\left(\frac{n\pi}{M}y\right)dy}_{=\frac{M}{2}}}, \quad n = 1, 2, \ldots$$

$$\boxed{T_n(t) = \frac{2}{M\sinh\left(\frac{n\pi L}{M}\right)}\int_0^M b_2(y,t)\cos\left(\frac{n\pi}{M}y\right)dy}$$

c] combine solution of (A) and (B)

$$S(x,y,t) = S^{(A)}(x,y,t) + S^{(B)}(x,y,t)$$

③ Solve subproblem [VP]

a] compute Eigenvalues, -function

$$v_t(x,y,t) = k\left(v_{xx}(x,y,t) + v_{yy}(x,y,t)\right)$$

$v(0,y,t) = 0$

$v(L,y,t) = 0$

$v_y(x,0,t) = 0$

$v_y(x,M,t) = 0$

i] sep. var. $v(x,y,t) = X(x)\,Y(y)\,T(t)$

spelt into

$$\boxed{\begin{aligned} &T'(t) + \mu^2 k T(t) = 0 \\ &X''(x) + \rho^2 X(x) = 0 \\ &Y''(y) + \tau^2 Y(y) = 0 \\ &\rho^2 + \tau^2 = \mu^2 \end{aligned}}$$

ii] transl. bcs:

$X(0) = 0,\ X(L) = 0,\ Y'(0) = 0,\ Y'(M) = 0$

iii] solve SL-problem

$$\underbrace{\begin{aligned} &X''(x) + \rho^2 X(x) = 0 \\ &X(0) = 0 \\ &X(L) = 0 \end{aligned}} \qquad \Bigg| \qquad \underbrace{\begin{aligned} &Y''(y) + \tau^2 Y(y) = 0 \\ &Y'(0) = 0 \\ &Y'(M) = 0 \end{aligned}}$$

$\Downarrow$

$$\boxed{\rho_m = \frac{(m+1)\pi}{L}} \qquad \boxed{\tau_n = \frac{n\pi}{M}}$$

$$\boxed{X_m(x) = \sin\left(\frac{(m+1)\pi}{L}x\right)} \qquad \boxed{Y_n(y) = \cos\left(\frac{n\pi}{M}y\right)}$$

$m, n = 0, 1, 2, \ldots$

b) Eigenfunction expansion

$$u(x,y,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} T_{mn}(t) X_m(x) Y_n(y)$$

$$= \sum_{m=0}^{\infty} T_{m0}(t) \sin\left(\frac{(m+1)\pi}{L} x\right) \cdot 1 +$$

$$+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} T_{mn}(t) \sin\left(\frac{(m+1)\pi}{L} x\right) \cos\left(\frac{n\pi}{M} y\right)$$

$$q(x,y,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Q_{mn}(t) X_m(x) Y_n(y)$$

$$= \sum_{m=0}^{\infty} Q_{m0}(t) \sin\left(\frac{(m+1)\pi}{L} x\right) \cdot 1 +$$

$$+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} Q_{mn}(t) \sin\left(\frac{(m+1)\pi}{L} x\right) \cos\left(\frac{n\pi}{M} y\right)$$

where
$$\boxed{Q_{m0}(t)} = \frac{\int_0^L \int_0^M q(x,y,t) \sin\left(\frac{(m+1)\pi}{L} x\right) dy\, dx}{\underbrace{\int_0^L \sin^2\left(\frac{(m+1)\pi}{L} x\right) dx}_{= \frac{L}{2}} \; \underbrace{\int_0^M 1\, dy}_{= M}}$$

$$\boxed{= \frac{2}{LM} \int_0^L \int_0^M q(x,y,t) \sin\left(\frac{(m+1)\pi}{L} x\right) dy\, dx}$$

$$Q_{mn}(t) = \frac{\int_0^L \int_0^M q(x,y,t) \sin\left(\frac{(m+1)\pi}{L} x\right) \cos\left(\frac{n\pi}{M} y\right) dy\, dx}{\underbrace{\int_0^L \sin^2\left(\frac{(m+1)\pi}{L} x\right) dx}_{= \frac{L}{2}} \; \underbrace{\int_0^M \cos\left(\frac{n\pi}{M} y\right) dy}_{= \frac{M}{2}}}$$

$$\boxed{= \frac{4}{LM} \int_0^L \int_0^M q(x,y,t) \sin\left(\frac{(m+1)\pi}{L} x\right) \cos\left(\frac{n\pi}{M} y\right) dy\, dy}$$

for $m = 0, 1, 2, \ldots$
$n = 1, 2, \ldots$

and

$$T_{mn}(t) = e^{-k\mu_{mn}^2 t} \int_0^t e^{k\mu_{mn}^2 \xi} Q_{mn}(\xi)\, d\xi + C_{mn} e^{-k\mu_{mn}^2 t}$$

with $\mu_{mn}^2 = \vartheta_m^2 + \tau_n^2 = \left[\frac{(m+1)^2}{L^2} + \frac{n^2}{M^2}\right]\pi^2$

and

$$C_{m0} = \frac{\int_0^M \int_0^L \left(f(x,y) - S(x,y,0)\right) \sin\left(\frac{(m+1)\pi}{L} x\right) dx\, dy}{\underbrace{\int_0^M dy}_{M} \underbrace{\int_0^L \sin^2\left(\frac{(m+1)\pi}{L} x\right) dx}_{L/2}}$$

$$\boxed{= \frac{2}{ML} \int_0^M \int_0^L \left(f(x,y) - S(x,y,0)\right) \sin\left(\frac{(m+1)\pi}{L} x\right) dx\, dy}$$

$$C_{mn} = \frac{\int_0^M \int_0^L \left(f(x,y) - S(x,y,0)\right) \sin\left(\frac{(m+1)\pi}{L} x\right) \cos\left(\frac{n\pi}{M} y\right) dx\, dy}{\int_0^M \cos^2\left(\frac{n\pi}{M} y\right) dy \int_0^L \sin^2\left(\frac{(m+1)\pi}{L} x\right) dx}$$

$$\boxed{= \frac{4}{LM} \int_0^M \int_0^L \left(f(x,y) - S(x,y,0)\right) \sin\left(\frac{(m+1)\pi}{L} x\right) \cos\left(\frac{n\pi}{M} y\right) dx\, dy}$$

for $m = 0, 1, 2, \ldots$ $n = 1, 2, \ldots$

④ combine solution

$$u(x,y,t) = s(x,y,t) + v(x,y,t)$$