4.1.5 Exercises

1] $u_t = k\, u_{xx}, \quad x \in [0,1]$

bcs: $u(0,t) = 0$

$u_x(1,t) = 0$

ic: $u(x,0) = f(x), \quad x \in [0,1]$

i] $f_1(x) = x^2$, ii] $f_2(x) = x(1 - \frac{1}{2}x)$

Solution: a] sep. variables $u(x,t) = X(x)T(t)$

$\Rightarrow X(x)T'(t) = k\, X''(x)T(t) \quad \Big| \cdot \frac{1}{kX(x)T(t)}$

$\Rightarrow \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\mu \Rightarrow \begin{cases} T'(t) + k\mu T(t) = 0 \\ X''(x) + \mu X(x) = 0 \end{cases}$

b] transl. bcs: $X(0)T(t) = 0 \ \forall t \Rightarrow X(0) = 0$

$X'(1)T(t) = 0 \ \forall t \Rightarrow X'(1) = 0$

c] solve SLP: $\mu = \lambda^2$, $1 = L$, use SL table for Euler operator:

$\lambda_n = \frac{(2n+1)\pi}{2 \cdot 1} = (2n+1)\frac{\pi}{2}, \quad n = 0,1,2,\ldots$

$\boxed{X_n(x) = \sin\left(\frac{(2n+1)\pi x}{2}\right),} \quad n = 0,1,2,\ldots$

d] solve time eqn:

$T'(t) + k\mu_n T(t) = 0, \quad \mu_n = \lambda_n^2 = \frac{(2n+1)\pi^2}{4}$

$\boxed{T_n(t) = e^{-\frac{(2n+1)^2\pi^2 k t}{4}},} \quad n = 0,1,2,\ldots$

e] gen. sol.

$u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{(2n+1)\pi x}{2}\right) e^{-\frac{(2n+1)^2\pi^2 k t}{4}}$

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} a_n u_n(x,t)}$$

f] apply i.c.:

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} a_n u_n(x,0) = \sum_{n=0}^{\infty} a_n \sin\left(\frac{(2n+1)\pi x}{2}\right)$$

where

$$\boxed{a_n = \frac{\int_0^1 f(x)\sin\left(\frac{(2n+1)\pi x}{2}\right)dx}{\int_0^1 \sin^2\left(\frac{(2n+1)\pi x}{2}\right)dx}}$$

i] $f_1(x) = x^2$

$$a_n = \frac{\int_0^1 x^2\sin\left(\frac{(2n+1)\pi x}{2}\right)dx}{\int_0^1 \sin^2\left(\frac{(2n+1)\pi x}{2}\right)dx}$$

ii] $f_2(x) = x(1-\frac{1}{2}x)$ see Exercise_4-1-5-1

2] $u_t = k u_{xx}$, $x \in [0,1]$

bcs: $u_x(0,t) = 0$

$u_x(1,t) = 0$

ic: $u(x,0) = f(x)$

i] $f_1(x) = 1 - x^2$ ii] $f_2(x) = x^2(1 - \frac{2}{3}x)$

<u>Solution</u> a] sep of variables: $u(x,t) = X(x)T(t)$, as above, leads to $T'(t) + k\mu T(t) = 0$, $X''(x) + \mu X(x) = 0$

b] transl. bcs: $X'(0) = 0$, $X'(1) = 0$

c] solve SLP: $\mu = \lambda^2$, $1 = L$, use SL-table Euler operator

$\boxed{\lambda_n = \frac{n\pi}{1} = n\pi,}$ $\boxed{X_n(x) = \begin{cases} 1, & n=0 \\ \cos(n\pi x), & n>0 \end{cases}}$

$n = 0, 1, 2, \ldots$

d] solve time eqn:

$T'(t) + k\mu_n T(t) = 0 \Rightarrow \boxed{T_n(t) = \begin{cases} 1, & n=0 \\ e^{-k(n\pi)t}, & n>0 \end{cases}}$

e] gen. solution:

$$u_n(x,t) = X_n(x)T_n(t) = \begin{cases} 1, & n=0 \\ \cos(n\pi x)\, e^{-kn\pi t}, & n>0 \end{cases}$$

$$\boxed{u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n u_n(x,t)}$$

f] apply ic:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n u_n(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

where
$$\boxed{a_0 = \frac{\int_0^1 f(x)\,dx}{\underbrace{\int_0^1 1\,dx}_{=1}}, \quad a_n = \frac{\int_0^1 f(x)\cos(n\pi x)\,dx}{\underbrace{\int_0^1 \cos^2(n\pi x)\,dx}_{\frac{1}{2}}}}$$

i] $f_1(x) = 1 - x^2$

$$a_0 = \int_0^1 (1-x^2)\,dx, \quad a_n = 2\int_0^1 (1-x^2)\cos(n\pi x)\,dx$$

$f_2(x) = x^2(1-\frac{2}{3}x)$ see Exercise_4-1-5-2

$$a_0 = \int_0^1 x^2(1-\tfrac{2}{3}x)\,dx, \quad a_n = 2\int_0^1 x^2(1-\tfrac{2}{3}x)\cos(n\pi x)\,dx$$

3] $u_t = k u_{xx}, \quad x \in [0,1]$

bcs: $u(0,t) = 0$

$u(1,t) + u_x(1,t) = 0$

ic: $u(x,0) = f(x)$

i] $f_1(x) = 1$, ii] $f_2(x) = x(1-\frac{2}{3}x)$

Solution: a) sep. of variables: $u(x,t) = X(x)T(t)$, as above, leads to $T'(t) + k\mu T(t) = 0$, $X''(x) + \mu X(x) = 0$

b) translate bcs:

$X(0)T(t) = 0 \ \forall t \Rightarrow X(0) = 0$

$\underbrace{X(1)T(t) + X'(1)T(t)}_{(X(1)+X'(1))T(t)} = 0 \ \forall t \Rightarrow X(1) + X'(1) = 0$

c) solve SLP: $\mu = \lambda^2$, $l = L$, use SL table Euler operator $\alpha_2 = \beta_2 = 1$

$X''(x) + \mu X(x) = 0$

$X(0) = 0$

$X(1) + X'(1) = 0$

$\boxed{\lambda_n > 0 \quad \text{pos. sol. of } \sin(\lambda) + \lambda\cos(\lambda) = 0}$

$\boxed{X_n(x) = \sin(\lambda_n x)}$, $n = 0, 1, 2, \ldots$

d) solve time eqn

$T'(t) + k\underbrace{\mu_n}_{\lambda_n^2} T(t) = 0$, $\boxed{T_n(t) = e^{-k\lambda_n^2 t}}$

e) gen. sol.

$u_n(x,t) = X_n(x)T_n(t) = \sin(\lambda_n x)\, e^{-k\lambda_n^2 t}$

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} a_n u_n(x,t)}$$

f) apply bc:

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} a_n u_n(x,0) = \sum_{n=0}^{\infty} a_n \sin(\lambda_n x)$$

where $\boxed{a_n = \dfrac{\int_0^1 f(x)\sin(\lambda_n x)\,dx}{\int_0^1 \sin^2(\lambda_n x)\,dx}}$ $n = 0, 1, 2, \ldots$

i] $a_n = \dfrac{\int_0^1 \sin(\lambda_n x)\,dx}{\int_0^1 \sin^2(\lambda_n)\,dx}$

ii] $a_n = \dfrac{\int_0^1 x(1-\frac{2}{3}x)\sin(\lambda_n x)\,dx}{\int_0^1 \sin^2(\lambda_n)\,dx}$

see Exercise_4-1-5—3

5] $u_t = \dfrac{k\left(u_r(r,t) + r u_{rr}(r,t)\right)}{r}$, $r \in [0,1]$

bc: $|u(0,t)| < \infty$

$u(1,t) = 0$

ic $u(r,0) = f(r)$

i] $f_1(r) = r^2$, ii] $f_2(r) = 1$

Solution a] sep. of variables: $u(r,t) = R(r)\,T(t)$

$R(t)\,T'(t) = \dfrac{k(R'(r)T(t) + rR''(r)T(t))}{r}$ $\Big|\cdot \dfrac{1}{k\,R(r)T(t)}$

$\Rightarrow \dfrac{T'(t)}{kT(t)} = \dfrac{R'(r) + rR''(r)}{rR(r)} = -\lambda^2$

$\Rightarrow T'(t) + \lambda^2 k T(t) = 0$

$r^2R''(r) + r'R(r) + \lambda^2 r^2 R(r) = 0$

b] translate bcs

$|R(0)T(t)| < \infty \;\forall t \Rightarrow |R(0)| < \infty$

$|R(1)T(t)| = 0 \;\forall t \Rightarrow R(1) = 0$

c] solve SL problem

$\boxed{w(r) = r}$

$$\left.\begin{array}{l} r^2R''(r) + r'R(r) + \lambda^2 r^2 R(r) = 0 \\ |R(0)| < \infty \\ R(1) = 0 \end{array}\right\} \xrightarrow[\substack{\nu=0\\ L=1}]{\text{SL-table p231}}$$

$\boxed{\lambda_n > 0 \text{ sol of } J_0(\lambda) = 0}$

$\boxed{R_n(r) = J_0(\lambda_n r)}$

$n = 0, 1, 2, \ldots$

d] solve time eqn:

$$T'(t) + \lambda_n^2 k T(t) = 0$$

$$\boxed{T_n(t) = e^{-k\lambda_n^2 t}}$$

e] gen. solution

$$u_n(r,t) = R_n(r) T_n(t)$$

$$\boxed{u(r,t) = \sum_{n=0}^{\infty} C_n u_n(r,t) = \sum_{n=0}^{\infty} C_n R_n(r) T_n(t)}$$

f] apply I.C:

$$f(r) = u(r,0) = \sum_{n=0}^{\infty} C_n R_n(r) \quad , \text{ note } T_n(0) = 1$$

$$= \sum_{n=0}^{\infty} C_n J_0(\lambda_n r)$$

write

$$\boxed{C_n = \frac{\int_0^1 f(r) J_0(\lambda_n r) r \, dr}{\int_0^1 J_0^2(\lambda_n r) r \, dr} \quad ,} \quad n = 0, 1, 2, \ldots$$

i] $$C_n = \frac{\int_0^1 r^3 J_0(\lambda_n r) dr}{\int_0^1 J_0^2(\lambda_n r) r \, dr}$$

ii] $$C_n = \frac{\int_0^1 J_0(\lambda_n r) r \, dr}{\int_0^1 J_0^2(\lambda_n r) r \, dr}$$

see Exercise_4-1-5-5

6] $$u_t = \frac{k(u_r(r,t) + r u_{rr}(r,t))}{r} \quad , \quad r \in [0,1]$$

bcs: $|u(0,t)| < \infty$

$u_r(1,t) = 0$

ic: $u(r,0) = f(r)$

i) $f_1(r) = r^2$,   ii) $f_2(r) = 1$

<u>Solution</u>: a) sep. of variables: $u(r,t) = R(r)\,T(t)$

$$R(t)\,T'(t) = \frac{k\left(R'(r)T(t) + r R''(r)\,T(t)\right)}{r} \qquad \Big|\cdot \frac{1}{kR(r)T(t)}$$

$$\Rightarrow \frac{T'(t)}{kT(t)} = \frac{R'(r) + rR''(r)}{rR(r)} = -\lambda^2$$

$$\Rightarrow \quad T'(t) + \lambda^2 k T(t) = 0$$

$$r^2R''(r) + rR'(r) + \lambda^2 r^2 R(r) = 0$$

b) translate bcs

$$|R(0)T(t)| < \infty \ \forall t \Rightarrow \quad |R(0)| < \infty$$

$$R'(1)T(t) = 1 \ \forall t \Rightarrow \quad R'(1) = 0$$

c) solve SL problem

$$\left.\begin{aligned} &r^2R''(r) + rR'(r) + \lambda^2 r^2 R(r) = 0 \\ &|R(0)| < \infty \\ &R'(1) = 0 \end{aligned}\right\} \xrightarrow[\substack{\nu = 0 \\ L = 1}]{\boxed{w(r) = r} \ \text{SL-table p 231}} \boxed{\begin{aligned} &\lambda_n \geq 0 \\ &\text{sol. of} \\ &J_0'(\lambda) = 0 \end{aligned}}$$

$\lambda_0 = 0$ is Eigenvalue!

$$\boxed{R_n(r) = \begin{cases} 1, & n = 0 \\ J_0(\lambda_n r), & n = 1, 2, \dots \end{cases}}$$

d) solve time eqn

$$T'(t) + \lambda_n^2 k T(t) = 0$$

$$\boxed{T_n(t) = \begin{cases} 1, & n = 0 \\ e^{-\lambda_n^2 k t}, & n = 1, 2, \dots \end{cases}}$$

e] general solution

$$u_n(r,t) = R_n(r)\,T_n(t)$$

$$u(r,t) = \sum_{n=0}^{\infty} C_n u_n(r,t) = C_0 + \sum_{n=1}^{\infty} C_n J_0(\lambda_n r)\, e^{-\lambda_n^2 k t}$$

f] apply i.c.

$$f(r) = u(r,0) = C_0 + \sum_{n=1}^{\infty} C_n J_0(\lambda_n r)$$

with

$$\boxed{C_0 = \frac{\int_0^1 f(r)\, r\,dr}{\int_0^1 r\,dr}} \quad \frac{1}{2}$$

$$\boxed{C_n = \frac{\int_0^1 f(r)\, J_0(\lambda_n r)\, r\,dr}{\int_0^1 J_0^2(\lambda_n r)\, r\,dr} \qquad n = 1, 2, \ldots}$$

i] $C_0 = 2\int_0^1 r^3\,dr = \frac{1}{2}$, $\quad C_n = \dfrac{\int_0^1 r^3 J_0(\lambda_n r)\,dr}{\int_0^1 J_0^2(\lambda_n r)\, r\,dr}$

ii] $C_0 = 2\int_0^1 r\,dr = 1$, $\quad C_n = \dfrac{\int_0^1 J_0(\lambda_n r)\, r\,dr}{\int_0^1 J_0^2(\lambda_n r)\, r\,dr}$

see Exercise_4-1-5--6