3.2.5 Exercises

1] Find the general solution of $y'' + 4y' + 3y = 3e^{t}$

Solution:

a] Solve the associated homogeneous equation:

$y'' + 4y' + 3y = 0$ (constant coeff. eqn!)

$r^2 + 4r + 3 = 0$ (char. eqn!)

$\Rightarrow r_1 = -1, \; r_2 = -3$

Thus $\boxed{y_1(t) = e^{-t}}$, $\boxed{y_2(t) = e^{-3t}}$ form a fundam. set of solutions (basis).

b] Find a particular solution (using variation of param.)

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-t} & e^{-3t} \\ -e^{-t} & -3e^{-3t} \end{vmatrix} = -3e^{-4t} + e^{-4t} = -2e^{-4t}$$

$$u_1(t) = -\int \frac{e^{-3t} \cdot 3e^{t}}{-2e^{-4t}} dt = \frac{3}{2} \int e^{2t} dt = \frac{3}{4} e^{2t}$$

$$u_2(t) = \int \frac{e^{-t} \cdot 3e^{t}}{-2e^{-4t}} dt = -\frac{3}{2} \int e^{4t} dt = -\frac{3}{8} e^{4t}$$

Thus

$$\boxed{y_p(t)} = \left(\frac{3}{4} e^{2t}\right) e^{-t} + \left(-\frac{3}{8} e^{4t}\right) e^{-3t}$$

$$= \frac{3}{4} e^{t} - \frac{3}{8} e^{t} = \boxed{\frac{3}{8} e^{t}}$$

and the general solution

$$\boxed{y(t) = \frac{3}{8}e^{t} + c_1 e^{-t} + c_2 e^{-3t}}$$

4] Find the general solution of $y'' - 5y' + 4y = \sinh(t)$

Solution:

a] Solve the associated homogeneous equation:

$$y'' - 5y' + 4y = 0$$

$$\underbrace{r^2 - 5r + 4}_{(r-4)(r-1)} = 0 \Rightarrow r_1 = 1,\ r_2 = 4$$

Thus $\boxed{y_1(t) = e^{t},\ y_2(t) = e^{4t}}$ form a fundamental set of solutions (basis)..

b] Find a particular solution (using var. of par.)

$$y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$$

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{t} & e^{4t} \\ e^{t} & 4e^{4t} \end{vmatrix} = 4e^{5t} - e^{5t} = 3e^{5t}$$

$$u_1(t) = -\int \frac{e^{4t}\sinh(t)}{3e^{5t}}\,dt = -\frac{1}{6}\int \frac{e^{4t}(e^{t} - e^{-t})}{e^{5t}}\,dt$$

$$= -\frac{1}{6}\int (1 - e^{-2t})\,dt = -\frac{1}{6}\left(t + \frac{1}{2}e^{-2t}\right) = -\frac{t}{6} - \frac{1}{12}e^{-2t}$$

$$u_2(t) = \int \frac{e^t \sinh(t)}{3e^{5t}}\,dt = \frac{1}{6}\int e^{-4t}(e^t - e^{-t})\,dt$$

$$= \frac{1}{6}\int (e^{-3t} - e^{-5t})\,dt = \frac{1}{6}\left(-\frac{1}{3}e^{-3t} + \frac{1}{5}e^{-5t}\right)$$

$$= -\frac{1}{18}e^{-3t} + \frac{1}{30}e^{-5t}$$

Thus

$$\boxed{y_p(t)} = \left(-\frac{1}{6} - \frac{1}{12}e^{-2t}\right)e^t + \left(-\frac{1}{18}e^{-3t} + \frac{1}{30}e^{-5t}\right)e^{4t}$$

$$= -\frac{1}{6}e^t - \frac{1}{12}e^{-t} - \frac{1}{18}e^t + \frac{1}{30}e^{-t}$$

$$= \boxed{-\frac{2}{9}e^t - \frac{1}{20}e^{-t}}$$

and the general solution

$$\boxed{y(t) = \left(-\frac{2}{9}e^t - \frac{1}{20}e^{-t}\right) + C_1e^t + C_2e^{4t}}$$

5] Find the general solution of $y'' + 4y' + 13y = \sin t$

<u>Solution</u>:

a] Solve the associated homogeneous equation.

$$y'' + 4y' + 13y = 0$$

$$\underline{r^2 + 4r + 13} = 0$$

$$(r+2)^2 - 4 + 13$$

$$\Leftrightarrow \quad (r+2)^2 = -9$$

$$\Rightarrow \quad r+2 = \pm\sqrt{-9} = \pm 3i$$

hence $r_1 = -2+3i, \quad r_2 = \overline{r_1} = -2-3i$

and $\alpha = -2, \quad \beta = 3$

Thus $\boxed{y_1(t) = e^{-2t}\cos(3t), \quad y_2(t) = e^{-2t}\sin(3t)}$

b] Find a particular solution

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$W(y_1,y_2)(t) = \begin{vmatrix} e^{-2t}\cos(3t) & e^{-2t}\sin(3t) \\ -2e^{-2t}\cos(3t) - 3e^{-2t}\sin(3t) & -2e^{-2t}\sin(3t) + 3e^{-2t}\cos(3t) \end{vmatrix}$$

$$= -2e^{-4t}\sin(3t)\cos(3t) + 3e^{-4t}\cos^2(3t) + $$
$$+ 2e^{-4t}\sin(3t)\cos(3t) + 3e^{-4t}\sin^2(3t)$$
$$= 3e^{-4t}$$

$$u_1(t) = -\int \frac{e^{-2t}\sin(3t)\sin(t)}{3e^{-4t}}\,dt = -\frac{1}{3}\int e^{2t}\underbrace{\sin(3t)\sin(t)}_{\frac{1}{2}(\cos(2t)-\cos(4t))}\,dt$$

Use Appendix I: use $a=2,\ b=2$ $\qquad a=2,\ b=4$

$$= -\frac{1}{6}\int e^{2t}\cos(2t) + \frac{1}{6}\int e^{2t}\cos(4t) =$$

$$= -\frac{1}{6}\left(\frac{e^{2t}}{\cancel{8}_{4}}(\cancel{2}\cos(2t)+\cancel{2}\sin(2t))\right) + \frac{1}{6}\left(\frac{e^{2t}}{\cancel{20}_{10}}(\cancel{2}\cos(4t) + \cancel{4}^{2}\sin(4t))\right)$$

$$= -\frac{e^{2t}}{24}(\cos(2t)+\sin(2t)) + \frac{e^{2t}}{60}(\cos(4t)+2\sin(4t))$$

$$= -\frac{1}{120} e^{2t}\left[5\cos(2t) - 2\cos(4t) + 5\sin(2t) - 4\sin(4t)\right]$$

$$u_2(t) = \int \frac{e^{-2t}\cos(3t)\sin(t)}{3e^{-4t}}\,dt = \frac{1}{3}\int e^{2t}\underbrace{\sin(t)\cos(3t)}_{\frac{1}{2}\left(\sin(-2t)+\sin(4t)\right)}\,dt$$

$$= -\frac{1}{6}\int e^{2t}\sin(2t)\,dt + \frac{1}{6}\int e^{2t}\sin(4t)\,dt$$

$a = 2,\ b = 2$ $\qquad$ $a = 2,\ b = 4$

Use Appendix II, with

$$= -\frac{1}{6}\left[\frac{e^{2t}}{\cancel{8}_{\,4}}\left(\cancel{2}\sin(2t) - \cancel{2}\cos(2t)\right)\right] + \frac{1}{6}\left[\frac{e^{2t}}{\cancel{20}_{\,10}}\left(\cancel{2}\sin(4t) - \cancel{4}^{\,2}\cos(4t)\right)\right]$$

$$= -\frac{e^{2t}}{24}\left[\sin(2t) - \cos(2t)\right] + \frac{e^{2t}}{60}\left[\sin(4t) - 2\cos(4t)\right]$$

$$= -\frac{1}{120} e^{2t}\left[5\sin(2t) - 2\sin(4t) - 5\cos(2t) + 4\cos(4t)\right]$$

Hence

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

and

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t)$$

with y_p, u_1, u_2, y_1, y_2 as computed above.

6] The simple spring-mass system ...

$$my'' + \gamma y' + ky = f(t)$$

here $m = 1$, $\gamma = 0$, $k = 9$.

Thus

$$y'' + 9y = f(t)$$

a] Evaluate a set of system basis vectors y_1, y_2 and evaluate the 2nd-order Green's function $G_2(t,s)$.

<u>Solution:</u>

i] Solve the associated homogeneous solution

$$y'' + 9y = 0$$

$$r^2 + 9 = 0 \quad \Rightarrow \quad r_1 = 3i, \; r_2 = \bar{r}_1$$

$$\alpha = 0, \; \beta = 3$$

$$\boxed{y_1(t) = \cos(3t)}, \; \boxed{y_2(t) = \sin(3t)}$$

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix}$$

$$= 3$$

ii] Green's function

$$\boxed{G_2(t,s) = \frac{\sin(3t)\cos(3s) - \cos(3t)\sin(3s)}{3}}$$

b] If the initial conditions are $y(0) = 2$ [m]
$v(0) = 10$ [m/sec]
and the driving force $f(t) = 10\sin t$ [N], evaluate the time dependent solution $y(t)$ for the problem.

Solution:

$$y_p(t) = \int G_2(t,s) f(s)\,ds = \frac{10}{3}\int \big(\sin(3t)\cos(3s) - \cos(3t)\sin(3s)\big)\cdot \sin(s)\,ds$$

$$= \frac{10}{3}\sin(3t)\int \underbrace{\sin(s)\cos(3s)}_{\frac{1}{2}(\sin(-2s)+\sin(4s))}ds - \frac{10}{3}\cos(3t)\int \underbrace{\sin(3s)\sin(s)}_{\frac{1}{2}(\cos(2s)-\cos(4s))}ds$$

$$= \frac{5}{3}\sin(3t)\left[\frac{1}{2}\cos(2s) - \frac{1}{4}\cos(4s)\right]_{s=t}$$

$$- \frac{5}{3}\cos(3t)\left[\frac{1}{2}\sin(2s) - \frac{1}{4}\sin(4s)\right]_{s=t}$$

$$= \frac{5}{6}\sin(3t)\cos(2t) - \frac{5}{12}\sin(3t)\cos(4t) - \frac{5}{6}\sin(2t)\cos(3t) + \frac{5}{12}\sin(4t)\cos(3t)$$

$$= \frac{5}{6}\sin(3t-2t) + \frac{5}{12}\sin(4t-3t)$$

$$= \frac{5}{4}\sin(t)$$

$$\boxed{y_p(t) = \frac{5}{4}\sin(t)}$$

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t)$$

$$= \frac{5}{4}\sin(t) + C_1\cos(3t) + C_2\sin(3t)$$

$$y'(t) = \frac{5}{4}\cos(t) - 3C_1\sin(3t) + 3C_2\cos(3t)$$

$$y(0) = 2 \Rightarrow 2 = C_1$$

$$y'(0) = 10 \Rightarrow 10 = \frac{5}{4} + 3C_2 \Rightarrow C_2 = \frac{35}{12}$$

Hence $\boxed{y_p(t) = \frac{5}{4}\sin(t) + 2\cos(3t) + \frac{35}{12}\sin(3t)}$

Appendix

I.
$$\int \underbrace{e^{at}}_{u'} \underbrace{\cos(bt)}_{v}\, dt = \frac{1}{a} e^{at} \cos(bt) + \frac{b}{a} \int \underbrace{e^{at}}_{u'} \underbrace{\sin(bt)}_{v}\, dt =$$

$$= \frac{1}{a} e^{ut} \cos(bt) + \frac{b}{a}\left[\frac{1}{a} e^{at} \sin(bt) - \frac{b}{a}\int e^{at} \cos(bt)\, dt\right] =$$

$$= \frac{1}{a} e^{at} \cos(bt) + \frac{b}{a^2} e^{at} \sin(bt) - \frac{b^2}{a^2} \int e^{at} \cos(bt)\, dt$$

Thus
$$\underbrace{\left(1 + \frac{b^2}{a^2}\right)}_{\frac{a^2+b^2}{a^2}} \int e^{at} \cos(bt)\, dt = e^{at}\left(\frac{1}{a}\cos(bt) + \frac{b}{a^2}\sin(bt)\right)$$

and
$$\boxed{\int e^{at} \cos(bt)\, dt = \frac{e^{at}}{a^2+b^2}\left[a \cos(bt) + b \sin(bt)\right] + C}$$

II.
$$\int \underbrace{e^{at}}_{u'} \underbrace{\sin(bt)}_{v}\, dt = \frac{1}{a} e^{at} \sin(bt) - \frac{b}{a} \int \underbrace{e^{at}}_{u'} \underbrace{\cos(bt)}_{v}\, dt =$$

$$= \frac{1}{a} e^{at} \sin(bt) - \frac{b}{a}\left[\frac{1}{a} e^{at} \cos(bt) + \frac{b}{a}\int e^{at} \sin(bt)\, dt\right] =$$

$$= \frac{1}{a} e^{at} \sin(bt) - \frac{b}{a^2} e^{at} \cos(bt) - \frac{b^2}{a^2} \int e^{at} \sin(bt)\, dt$$

Thus
$$\underbrace{\left(1 + \frac{b^2}{a^2}\right)}_{\frac{a^2+b^2}{a^2}} \int e^{at} \sin(bt)\, dt = \frac{1}{a} e^{at} \sin(bt) - \frac{b}{a^2} e^{at} \cos(bt)$$

and
$$\boxed{\int e^{at} \sin(bt)\, dt = \frac{e^{at}}{a^2+b^2}\left[a \sin(bt) - b \cos(bt)\right] + C}$$