3.3.1 Exercises

1] Find the general solution of $t^2y''+ty'-y=0$

Solution: $y(t)=t^r$, $t>0$ (the case $t<0$ is analogous!)
$y'(t)=rt^{r-1}$; $y''(t)=(r-1)rt^{r-2}$

then, $(r-1)rt^r+rt^r-t^r=0$, and dividing by t^r, we obtain

$$r^2-r+r-1=0$$

$$r^2-1=0$$

hence, $r_1=1$, $r_2=-1$, and then

$$\boxed{y(t)=C_1t+C_2t^{-1}}\quad, t<0 \text{ or } t>0$$

2] Find the general solution of $t^2y''-ty'+2y=1+\ln^2|t|$

Solution: a] solve the associated homogeneous equation:

$t^2y''-ty'+2y=0$

$y(t)=t^r$, $t>0$ (the case $t<0$ is analogue!)
$y'(t)=rt^{r-1}$; $y''(t)=(r-1)rt^{r-2}$

$(r-1)rt^r-rt^r+2t^r=0 \Rightarrow r^2-r-r+2=0$

$$\left.\begin{array}{l} r^2-2r+2=0 \\ (r-1)^2-1+2=0 \end{array}\right\} \Rightarrow \begin{array}{l} (r-1)^2=-1 \\ |r-1|=i \end{array} \quad \Big|\ \Gamma$$

$\Rightarrow r_1=1+i$, $r_2=\bar{r}_1$

$\alpha=1$, $\beta=1$

$$\boxed{y_1(t)=t\cos(\ln t),\quad y_2=t\sin(\ln t)}$$

b) Find a particular solution

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$W(y_1, y_2)(t) = \begin{vmatrix} t\cos(\ln t) & t\sin(\ln t) \\ \cos(\ln t) - \sin(\ln t) & \sin(\ln t) + \cos(\ln t) \end{vmatrix}$$

$$\underline{\underline{R_2 \to R_2 - \tfrac{1}{t}R_1}} \begin{vmatrix} t\cos(\ln t) & t\sin(\ln t) \\ -\sin(\ln t) & \cos(\ln t) \end{vmatrix} =$$

$$= t\left[\cos^2(\ln t) + \sin^2(\ln t)\right] = t$$

Before we apply Thm 3.7, we have to put the CE-equation into normal form!

$$y'' - \frac{1}{t}y' + \frac{2}{t^2}y = \frac{1 + \ln^2 t}{t^2}$$

$$u_1(t) = -\int \frac{t\sin(\ln t) \cdot \frac{1 + \ln^2 t}{t^2}}{t}\,dt$$

$$= -\int \frac{\sin(\ln t)(1 + \ln^2 t)}{t^2}\,dt =$$

sub $s = \ln t$, $ds = \frac{1}{t}dt$, $t = e^s$

see Appendix

$$= -\int e^{-s}\sin(s)\,ds - \int s^2 e^{-s}\,ds$$

$$= \frac{1}{2}e^{-s}(\sin(s) + \cos(s)) + (s^2 + 2s + 1)e^{-s}$$

$$= \frac{1}{2}\cdot\frac{1}{t}(\sin(\ln t) + \cos(\ln t)) + \frac{1}{t}(\ln^2 t + 2\ln t + 1)$$

$$u_2(t) = \int \frac{t\cos(\ln t)\cdot\frac{1+\ln^2 t}{t^2}}{t}\,dt$$

$$= \int \frac{\cos(\ln t)(1+\ln^2 t)}{t^2}\,dt$$

sub $s=\ln t$, $ds=\frac{1}{t}dt$, $t=e^s$

see Appendix

$$= \int e^{-s}\cos(s)\,ds + \int s^2 e^{-s}\,ds$$

$$= \frac{1}{2}e^{-s}(\sin(s)-\cos(s)) - e^{-s}(s^2+2s+1)$$

$$= \frac{1}{2}\cdot\frac{1}{t}\left(\sin(\ln t)-\cos(\ln t)\right) - \frac{1}{t}\left(\ln^2 t+2\ln t+1\right)$$

Then

$$y_p(t) = \left[\frac{1}{2}(\sin(\ln t)+\cos(\ln t)) + \ln^2 t+2\ln t+1\right]\cos(\ln t)$$
$$+ \left[\frac{1}{2}(\sin(\ln t)-\cos(\ln t)) - \ln^2 t - 2\ln t - 1\right]\sin(\ln t)$$

$$= \frac{1}{2}\sin(\ln t)\cos(\ln t) + \frac{1}{2}\cos^2(\ln t) + \ln^2 t\cos(\ln t) +$$
$$+ 2\ln t\cos(\ln t) + \cos(\ln t) + \frac{1}{2}\sin^2(\ln t) -$$
$$- \frac{1}{2}\sin(\ln t)\cos(\ln t) - \ln^2 t\sin(\ln t) - 2\ln t\sin(\ln t)$$
$$- \sin(\ln t)$$

$$= \frac{1}{2} + \ln^2 t\left(\cos(\ln t) - \sin(\ln t)\right) +$$
$$+ 2\ln t\left(\cos(\ln t) - \sin(\ln t)\right) + \cos(\ln t) - \sin(\ln t)$$

$$= \frac{1}{2} + \left(\ln^2 t + 2\ln t + 1\right)\left(\cos(\ln t) - \sin(\ln t)\right)$$

$$\boxed{y_p(t) = \frac{1}{2} + (\ln^2|t| + 2\ln|t| + 1)(\cos(\ln|t|) - \sin(\ln|t|))}$$

and

$$\boxed{\begin{aligned} y(t) &= u_p(t) + C_1 y_1(t) + C_2 y_2(t) \\ &= \frac{1}{2} + (\ln^2|t| + 2\ln|t| + 1)(\cos(\ln|t|) - \sin(\ln|t|)) \\ &\quad + C_1 t\cos(\ln|t|) + C_2 t\sin(\ln|t|) \end{aligned}}$$

3] Solve the IVP $t^2y'' - 3ty' + 3y = \ln t$, $y(1) = 1$, $y'(1) = 2$

<u>Solution</u> :

a] solve the associated hom. eqn :

$$t^2y'' - 3ty' + 3y = 0$$

$y(t) = t^r$, $t > 0$ (the case $t < 0$ is analogue !)

$y'(t) = rt^{r-1}$, $y''(t) = (r-1)rt^{r-1}$

$$(r-1)rt^r - 3rt^r + 3t^r = 0 \Rightarrow r^2 - r - 3r + 3 = 0$$

$$\underbrace{r^2 - 4r + 3}_{(r-3)(r-1)} = 0 \Rightarrow r_1 = 1, \ r_2 = 3$$

$$\boxed{y_1(t) = t, \ y_2(t) = t^3}$$

b] Find a particular solution :

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$W(y_1,y_2)(t) = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} = 2t^3$$

Before applying Thm 3.7, transform the CE eqn into normal form!

$$y'' - \frac{3}{t}y' + \frac{3}{t^2}y = \frac{\ln t}{t^2}$$

$$u_1(t) = -\int \frac{t^3 \cdot \frac{\ln t}{t^2}}{2t^3}\,dt = -\frac{1}{2}\int \frac{\ln t}{t^2}\,dt$$

sub $s = \ln t$, $ds = \frac{1}{t}dt$, $t = e^s$

$$= -\frac{1}{2}\int \underbrace{s}_{u}\,\underbrace{e^{-s}}_{v'}\,ds = -\frac{1}{2}\left[-se^{-s} + \int e^{-s}ds\right]$$

$$= -\frac{1}{2}\left[-se^{-s} - e^{-s}\right] = -\frac{1}{2}\left[-\frac{\ln t}{t} - \frac{1}{t}\right]$$

$$= \frac{1}{2t}\cdot(\ln t + 1)$$

$$u_2(t) = \int \frac{t\cdot\frac{\ln t}{t^2}}{2t^3}\,dt = \frac{1}{2}\int \frac{\ln t}{t^4}\,dt =$$

sub $s = \ln t$, $ds = \frac{1}{t}dt$, $t = e^s$

$$= \frac{1}{2}\int \underbrace{s}_{u}\,\underbrace{e^{-3s}}_{v'}\,ds = \frac{1}{2}\left[-\frac{s}{3}e^{-3s} + \frac{1}{3}\int e^{-3s}ds\right]$$

$$= -\frac{s}{6}e^{-3s} - \frac{1}{18}e^{-3s} = -\frac{1}{6}(\ln t)\frac{1}{t^3} - \frac{1}{18}\cdot\frac{1}{t^3}$$

$$= -\frac{1}{18t^3}(3\ln t + 1)$$

$$y_p(t) = \frac{1}{2t}(\ln t + 1)\cdot t - \frac{1}{18t^3}(3\ln t + 1)\cdot t^3$$

$$= \frac{1}{2}(\ln t + 1) - \frac{1}{18}(3\ln t + 1)$$

$$= \frac{1}{2}\ln t + \frac{1}{2} - \frac{1}{6}\ln t - \frac{1}{18}$$

$$= \frac{1}{3}\ln t + \frac{4}{9}$$

$$\boxed{y_p(t) = \frac{1}{3}\ln|t| + \frac{4}{9}}$$

$$\boxed{y(t) = \left(\frac{1}{3}\ln|t| + \frac{4}{9}\right) + C_1 t + C_2 t^3}$$

Apply initial condition.

$$y'(t) = \frac{1}{3t} + C_1 + 3C_2 t^2$$

$$y(1) = 1 \Rightarrow 1 = \frac{4}{9} + C_1 + C_2$$

$$y'(1) = 2 \Rightarrow 2 = \frac{1}{3} + C_1 + 3C_2$$

i.e.

$$\left.\begin{aligned} C_1 + C_2 &= \frac{5}{9} \\ C_1 + 3C_2 &= \frac{5}{6} \end{aligned}\right\} \Rightarrow 2C_2 = \frac{5}{6} - \frac{5}{9} = \frac{15-10}{18}$$

$$C_2 = \frac{5}{36}, \quad C_1 = \frac{5}{9} - \frac{5}{36} = \frac{20-5}{36} = \frac{15}{36} = \frac{5}{12}$$

$$\boxed{y(t) = \left(\frac{1}{3}\ln|t| + \frac{4}{9}\right) + \frac{15}{36}t + \frac{5}{12}t^3}$$

4] Find the green's function for $t^2y'' + 7ty' + 9y = f(t)$

<u>Solution:</u>

a) Solve the associated homog. eqtn:

$$t^2y'' + 7ty' + 9y = 0$$

$y(t) = t^r$, $t > 0$ (the case $t < 0$ is analogue!)
$y'(t) = rt^{r-1}$, $y''(t) = (r-1)rt^{r-2}$

$$(r-1)rt^r + 7rt^r + 9t^r = 0 \Rightarrow r^2 - r + 7r + 9 = 0$$

$$\left.\underbrace{r^2 + 6r + 9}_{(r+3)^2} = 0\right\} \Rightarrow r = r_1 = r_2 = -3$$

$\boxed{y_1(t) = |t|^{-3},}$ $\boxed{y_2(t) = |t|^{-3}\ln t}$

b) Compute Green's function

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{-3} & t^{-3}\ln t \\ -3t^{-4} & -3t^{-4}\ln t + t^{-4} \end{vmatrix} =$$

$$= t^{-3}t^{-4}\begin{vmatrix} 1 & \ln t \\ -3 & -3\ln t + 1 \end{vmatrix} =$$

$$= t^{-7}\left[-3\ln t + 1 + 3\ln t\right] = t^{-7}$$

Then,

$$\boxed{G_2(t,s) = \frac{(t^{-3}\ln t)s^{-3} - t^{-3}s^{-3}\ln s}{s^{-7}}} = \boxed{\frac{s^4(\ln t - \ln s}{t^3}}$$

6] Given one homogeneous solution $y_1(t) = t^3$, find the second-order Green's function $G_2(t,s)$ of

$$t^2 y'' - 6y = f(t)$$

<u>Solution</u>

a] Find a second solution for the associated hom. eqn

i] solve $t^2 y'' - 6y = 0$ $\quad (*)$

$y(t) = t^r, \quad y'(t) = r t^{r-1}, \quad y''(t) = r(r-1) t^{r-2}$

$r(r-1)t^r - 6t^r = 0, \qquad (r > 0)$

$\Rightarrow \underbrace{r^2 - r - 6}_{} = 0$

$(r-3)(r+2) = 0$

$\Rightarrow r_1 = 3, \quad r_2 = -2$

$\boxed{y_2(t) = t^{-2}}$

or we could also use

ii] by reduction of order

$$\tilde{y}_2(t) = y_1(t) \cdot \int \frac{e^{-\int p(t)dt}}{[y_1(t)]^2}$$

<u>Note</u>: First transform eqn $(*)$ into S.f:

$$y'' - \frac{6}{t^2} y = 0, \quad p(t) = 0, \quad q(t) = -\frac{6}{t^2}$$

$$= t^3 \int \frac{1}{t^6} dt = t^3 \int t^{-6} dt = t^3 \cdot \frac{t^{-5}}{-5}$$

$$= -\frac{1}{5} t^{-2}$$

$y_2(t) = t^{-2}$

b] compute the green's function

i] $W(y_1,y_2)(t) = \begin{vmatrix} t^3 & t^{-2} \\ 3t^2 & -2t^{-3} \end{vmatrix} = -2-3=-5$

ii] $\boxed{G_2(t,s)} = \dfrac{t^{-2}s^3 - t^3 s^{-2}}{-5} =$

$= \boxed{\dfrac{1}{5}\left(\dfrac{t^3}{s^2} - \dfrac{s^3}{t^2}\right)}$

Appendix:

I.

$$\int \underbrace{e^{-s}}_{u'}\underbrace{\sin(s)}_{v}\,ds = -e^{-s}\sin(s) + \int \underbrace{e^{-s}}_{u'}\underbrace{\cos(s)}_{v}\,ds$$

$$= -e^{-s}\sin(s) + \left(-e^{-s}\cos(s) - \int e^{-s}\sin(s)\,ds\right)$$

$$\Rightarrow \int e^{-s}\sin(s)\,ds = -\frac{1}{2}e^{-s}\left(\sin(s)+\cos(s)\right) + C$$

III.

$$\int \underbrace{s^2}_{u}\underbrace{e^{-s}}_{v'}\,ds = -s^2e^{-s} + 2\int \underbrace{s}_{u}\underbrace{e^{-s}}_{v'}\,ds = -s^2e^{-s} +$$

$$+2\left(-se^{-s} + \int e^{-s}\,ds\right) =$$

$$= -s^2e^{-s} - 2se^{-s} - e^{-s} + C$$

II.

$$\int \underbrace{e^{-s}}_{u'}\underbrace{\cos(s)}_{v}\,ds = -e^{-s}\cos(s) - \int \underbrace{e^{-s}}_{u'}\underbrace{\sin(s)}_{v}\,ds$$

$$= -e^{-s}\cos(s) - \left(-e^{-s}\sin(s) + \int e^{-s}\cos(s)\,ds\right)$$

$$= -e^{-s}\cos(s) + e^{-s}\sin(s) - \int e^{-s}\cos(s)\,ds$$

$$\Rightarrow \int e^{-s}\cos(s)\,ds = \frac{1}{2}e^{-s}\left(\sin(s)-\cos(s)\right) + C$$