3.4.4 Exercises

1) Prove that for $1 \leq k \leq n$:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

proof: $$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} =$$

$$= \frac{n!\,(n-k+1)}{k!(n-k)!\,(n-k+1)} + \frac{k \cdot n!}{(k-1)!\,k\,(n-k+1)!}$$

$$= \frac{n!(n+1) - \cancel{n!k} + \cancel{n!k}}{k!\,(n-k+1)!}$$

$$= \frac{(n+1)!}{k!\,(n-k+1)!} = \binom{n+1}{k}$$

2) Evaluate

$$\sum_{k=0}^{n} \binom{2n}{2k} - \sum_{k=0}^{n-1} \binom{2n}{2k+1} = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k$$

$$= (1+(-1))^{2n} = 0$$

4) Give the power series expansion of

a) $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{x}{x-1}$

in x and determine their radius of convergence.

Solution: We modify the geometric series expansion

$$G(x) := \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for } |x| < 1$$

a] $f(x) = -G(x), \quad |x| < 1$

hence

$$f(x) = -\sum_{k=0}^{\infty} x^k, \quad |x| < 1$$

radius of convergence $R_f = 1$

b] $g(x) = -x\,G(x), \quad |x| < 1$

hence

$$g(x) = -\sum_{k=0}^{\infty} x^{k+1}, \quad |x| < 1$$

radius of convergence $R_g = 1$

6] Find the radius of convergence

a] $\sum_{n=0}^{\infty} \frac{x^n}{(n+3)^2}$, b] $\sum_{n=0}^{\infty} \frac{n^2}{2^n}(x-1)^n$, c] $\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n$

<u>Solution</u> a] $b_n = \frac{1}{(n+3)^2}$, we use Thm 3.31 (p 145)

$$\boxed{R} = \lim_{n\to\infty} \left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{1}{(n+3)^2}}{\frac{1}{(n+4)^2}} = \lim_{n\to\infty} \left[\frac{n+4}{n+3}\right]^2 = \text{(by Thm 3.13) (p. 138)}$$

$$= \left[\lim_{n\to\infty} \frac{n+4}{n+3}\right]^2 = \left[\lim_{n\to\infty} \frac{(n+4)\frac{1}{n}}{(n+3)\frac{1}{n}}\right]^2 = \left[\lim_{n\to\infty} \frac{1+\frac{4}{n}}{1+\frac{3}{n}}\right]^2 =$$

$$= \left[\frac{1+0}{1+0}\right]^2 = \boxed{1}$$

b) $b_n = \frac{n^2}{2^n}$, we use Thm 3.31 (p 145)

$$\boxed{R = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty}\frac{\frac{n^2}{2^n}}{\frac{(n+1)^2}{2^{n+1}}} = \lim_{n\to\infty}\frac{n^2\,\cancelto{2}{2^{n+1}}}{\cancelto{1}{2^n}\,(n+1)^2}}$$

$$= \lim_{n\to\infty}\left(2\left(\frac{n}{n+1}\right)^2\right) \underset{\text{Thm 3.12, 2}}{=\!=\!=} 2\lim_{n\to\infty}\left[\frac{n}{n+1}\right]^2 \underset{\text{Thm 3.13}}{=\!=\!=}$$

$$= 2\left[\lim_{n\to\infty}\frac{n}{n+1}\right]^2 = 2\cdot 1^2 = \boxed{2}$$

c) $b_n = \frac{n!}{n^n}$, we use Thm 3.31 (p 145)

$$\boxed{R = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty}\frac{\frac{n!}{n^n}}{\frac{(n+1)!}{(n+1)^{n+1}}} = \lim_{n\to\infty}\frac{\cancelto{1}{n!}\,\cancelto{(n+1)^n}{(n+1)^{n+1}}}{n^n\,\cancelto{\cancelto{1}{(n+1)}}{(n+1)!}}}$$

$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1 = \boxed{e}$$

by Thm 3.19, 8 on page 139.

7) Show that $(x-1)p(x) = \frac{2(x-1)}{x}$ is analytic at $x_0=1$ by computing its Taylor series expansion in $x-1$.

Solution:

$$\frac{2(x-1)}{x} = 2 - 2\cdot\frac{1}{x}$$

In Example 3.18 (p. 153), we have

$$\frac{1}{x} = \sum_{k=0}^{\infty}(-1)^k(x-1)^k, \quad |x-1|<1$$

Hence,

$$\frac{2(x-1)}{x} = 2 - 2\cdot\frac{1}{x} = 2 - 2\sum_{k=0}^{\infty}(-1)^k(x-1)^k, \quad |x-1|<1$$

is the Taylor series expansion for $(x-1)p(x)$.

8) Show that $x^2q(x) = \dfrac{3x}{(x-1)^3}$ is analytic at $x_0 = 0$ by computing its Taylor series expansion in x.

Solution: Consider the geom. series expansion

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

By, Thm 3.32 (p. 146)

$$\underbrace{\frac{d}{dx}\frac{1}{1-x}} = \sum_{n=0}^{\infty}\frac{d}{dx}x^n$$

$$(1-x)^{-2} = \sum_{n=1}^{\infty} n x^{n-1}, \quad |x|<1$$

and

$$\underbrace{2(1-x)^{-3}} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \quad |x|<1$$

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \quad |x|<1$$

Hence

$$\frac{3x}{(x-1)^3} = -\frac{3}{2}x\cdot\frac{2}{(1-x)^3} = -\frac{3}{2}x\cdot\sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$$= -\sum_{n=2}^{\infty}\frac{3n(n-1)}{2}x^{n-1}, \quad |x|<1$$

is the Taylor series expansion of $x^2q(x)$ in x.

9] Using the power series method, find complete solution around the origin for the following differential equation.

a] $y' - \lambda y = 0$

b] $y'' - \lambda^2 y = 0$

c] $y'' + x y' + 2y = 0$

<u>Solution</u>: Note: $x_0 = 0$ is an ordinary point for all three equation. Since $a_2(x) = 1$ does not have any zeros, all power series solution for a, b, c have radius of convergence $R = \infty$.

a] $y' - \lambda y = 0$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

hence

$$0 = \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} \lambda c_n x^n = \sum_{m=0}^{\infty} (m+1) c_{m+1} x^m - \sum_{n=0}^{\infty} \lambda c_n x^n$$

sub $m = n-1$

$n = m+1$

$$= \sum_{n=0}^{\infty} \left[(n+1) c_{n+1} - \lambda c_n \right] x^n$$

By the Identity Thm (Thm 3.33, p. 147) follows

$$(n+1) c_{n+1} - \lambda c_n = 0 \quad \text{for } n = 0, 1, 2, \ldots$$

Hence,

$$\boxed{c_{n+1} = \frac{\lambda c_n}{n+1}, \quad n = 0, 1, 2, \ldots}$$

and therefore

$$c_0 = \text{arbitrary}$$

$$c_1 = \frac{\lambda c_0}{1}$$

$$C_2 = \frac{\lambda C_1}{2} = \frac{\lambda^2 C_0}{1\cdot 2}$$

$$C_3 = \frac{\lambda\cdot C_2}{3} = \frac{\lambda^3\cdot C_0}{1\cdot 2\cdot 3}$$

we conclude $\quad C_n = \dfrac{\lambda^n\cdot C_0}{n!}$

and $\quad y(x) = \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} C_0 \dfrac{\lambda^n x^n}{n!}$

$$\boxed{y(x) = C_0 \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!} = e^{\lambda x}\cdot C_0} \qquad |x| < \infty$$

b) $y'' - \lambda^2 y = 0$

$$y(x) = \sum_{n=0}^{\infty} C_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n C_n x^{n-1}; \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

hence,

$$0 = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=0}^{\infty} \lambda^2 C_n x^n = \sum_{m=0}^{\infty} (m+2)(m+1) C_{m+2} x^m - \sum_{n=0}^{\infty} \lambda^2 C_n x^n$$

sub $m = n-2$
$\Rightarrow n = m+2$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) C_{n+2} - \lambda^2 C_n\right] x^n$$

By the Identity Thm (Thm 3.33, p 147), follows

$$(n+2)(n+1) C_{n+2} - \lambda^2 C_n = 0 \qquad \text{for } n = 0, 1, 2, \ldots$$

Hence

$$\boxed{C_{n+2} = \frac{\lambda^2 C_n}{(n+1)(n+2)}, \qquad n = 0, 1, 2, \ldots}$$

$c_0 =$ arbitrary

$c_1 =$ — " —

$$c_2 = \frac{\lambda^2 c_0}{1\cdot 2}$$

$$c_3 = \frac{\lambda^2 \cdot c_1}{2\cdot 3}$$

$$c_4 = \frac{\lambda^2 \cdot c_2}{3\cdot 4} = \frac{\lambda^4 \cdot c_0}{1\cdot 2\cdot 3\cdot 4}$$

$$c_5 = \frac{\lambda^2 \cdot c_3}{4\cdot 5} = \frac{\lambda^4 \cdot c_1}{2\cdot 3\cdot 4\cdot 5}$$

$$c_6 = \frac{\lambda^2 \cdot c_4}{5\cdot 6} = \frac{\lambda^6 \cdot c_0}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$$

$$c_7 = \frac{\lambda^2 \cdot c_5}{6\cdot 7} = \frac{\lambda^6 \cdot c_1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$$

hence $c_{2k} = \dfrac{\lambda^{2k} \cdot c_0}{(2k)!}$, $c_{2k+1} = \dfrac{\lambda^{2k} \cdot c_1}{(2k+1)!}$, $k = 0, 1, 2, \ldots$

and $y(x) = \displaystyle\sum_{k=0}^{\infty} \left(c_{2k} x^{2k} + c_{2k+1} x^{2k+1} \right)$

$$\boxed{\begin{aligned} y(x) &= c_0 \sum_{k=0}^{\infty} \frac{(\lambda x)^{2k}}{(2k)!} + \frac{c_1}{\lambda} \sum_{k=0}^{\infty} \frac{(\lambda x)^{2k+1}}{(2k+1)!} \\ &= c_0 \cosh(\lambda x) + \frac{c_1}{\lambda} \sinh(\lambda x) \end{aligned}}$$

e) $y'' + xy' + 2y = 0$

$$y(x) = \sum_{n=0}^{\infty} C_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n C_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

hence

$$0 = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=1}^{\infty} n C_n x^n + \sum_{n=0}^{\infty} 2 C_n x^n = \sum_{m=0}^{\infty} (m+2)(m+1) C_{m+2} x^m +$$

sub $m = n-2$
$\Rightarrow n = m+2$

$$\sum_{n=1}^{\infty} n C_n x^n + \sum_{n=0}^{\infty} 2 C_n x^n$$

$$= (2C_2 + 2C_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) C_{n+2} + (n+2) C_n\right] x^n$$

By the identity Thm (Thm 3.33, p.147) follows

$$2C_2 + 2C_0 = 0$$

$$(n+2)(n+1) C_{n+2} + (n+2) C_n = 0 \quad \text{for } n = 1, 2, 3, \ldots$$

hence

$$\boxed{\begin{aligned} C_2 &= -C_0 \\ C_{n+2} &= -\frac{C_n}{n+1} \quad \text{for } n = 1, 2, 3, \ldots \end{aligned}}$$

C_0 = arbitrary

C_1 = — " —

$$C_2 = -C_0 = (-1)^1 \frac{C_0}{1}$$

$$C_3 = -\frac{C_1}{2} = (-1)^1 \frac{C_1}{2}$$

$$C_4 = -\frac{C_2}{1 \cdot 3} = (-1)^2 \frac{C_0}{1 \cdot 3}$$

$$C_5 = -\frac{C_3}{4} = (-1)^2 \frac{C_1}{2 \cdot 4}$$

$$C_6 = -\frac{C_4}{5} = (-1)^3 \frac{C_0}{3 \cdot 5}$$

$$C_7 = -\frac{C_5}{6} = (-1)^3 \frac{C_1}{2 \cdot 4 \cdot 6}$$

c_0 arbitrary

c_1 — " —

$$c_{2k} = \frac{(-1)^k \cdot c_0}{\underset{2\cdot 1 \quad 2\cdot 2 \quad 2\cdot 3 \quad \cdots \quad 2\cdot(k-1)}{1\cdot 3\cdot 5 \cdots (2k-1)}} = \frac{(-1)^k \cdot 2^{k-1} \cdot (k-1)!}{(2k-1)!} c_0, \quad k=1,2,\ldots$$

$$c_{2k-1} = \frac{(-1)^k \cdot c_1}{\underbrace{2\cdot 4\cdot 6 \cdots 2k}_{2\cdot 1 \cdot 2\cdot 2 \cdot 2\cdot 3 \cdots 2\cdot k}} = \frac{(-1)^k c_1}{2^k \cdot k!}, \quad k=1,2,\ldots$$

Hence,
$$y(x) = c_0 + c_1 x + \sum_{k=1}^{\infty} \left(c_{2k-1} x^{2k-1} + c_{2k} x^{2k} \right)$$

$$= c_0 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{k-1} (k-1)!}{(2k-1)!} x^{2k} \right) + c_1 \left(x + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k-1} \right)$$