3.1.1 Exercises

1] Solve $ty' + 2y = e^t + \ln t$

Solution: Note this equation is only defined for $t \in (0, \infty)$, thus $t \neq 0$. We can therefore divide by t to cast the eqtn into standard form:

$$y' + \underbrace{\frac{2}{t}}_{p(t)} y = \underbrace{\frac{e^t}{t} + \frac{\ln t}{t}}_{g(t)}, \qquad t \in (0, \infty)$$

Thus, by Thm 3.1,

$$\boxed{\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = \left[e^{\ln t}\right]^2 = t^2}$$

and

$$\boxed{y(t)} = \frac{1}{t^2}\left(\int t^2\left(\frac{e^t}{t} + \frac{\ln t}{t}\right) dt + C\right) =$$

$$= \frac{1}{t^2} \int \left(te^t + t\ln t\right) dt + \frac{C}{t^2} = \frac{1}{t^2}\left(te^t - e^t + \tfrac{1}{2}t^2 \ln(t) - \tfrac{1}{4}t^2\right)$$

$$= \frac{1}{t^2}\left(te^t - e^t + \tfrac{1}{2}t^2 \ln t - \tfrac{1}{4}t^2\right) + \frac{C}{t^2}$$

$$\boxed{= \left(\frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{2}\ln t - \frac{1}{4}\right) + \frac{C}{t^2}}$$

2] Solve $y' + \tan(t) y = \sec(t)$ use Green's function

Solution: $\mu(t) = e^{\int \tan t \, dt} = e^{-\ln|\cos t|} = e^{\ln|\sec t|}$

$= \sec t \quad$ for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, $\boxed{y_h(t) = \frac{1}{\mu(t)} = \cos t}$ is a basis for the homogeneous equation.

Therefore, the Green's function for this equation is

$$G_1(t,s) = \frac{\cos(t)}{\cos(s)}, \quad t, s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

and thus

$$\boxed{y_p(t) =} \int_{t_0}^{t} G_1(t,s)\, g(s)\, ds = \int_{t_0}^{t} \frac{\cos(t)\sec(s)}{\cos(s)}\, ds$$

$$= \cos(t) \int_{t_0}^{t} \sec^2(s)\, ds = \cos(t) \tan(s) \Big|_{t_0}^{t}$$

$$= \cos(t)\left[\tan(t) - \tan(t_0)\right]$$

we pick $t_0 = 0$, then

$$= \cos(t)\tan(t) = \boxed{\sin(t)}$$

The general solution is

$$\boxed{y(t) =} C\, y_h(t) + y_p(t)$$

$$= \boxed{C \cdot \cos(t) + \sin(t)}$$

3] Solve the IVP $y' + 2y = t + \sin(t)$

$$y(0) = 4$$

<u>Solution</u>: $\boxed{\mu(t) =} e^{\int 2\,dt} = \boxed{e^{2t}}$, thus

$$y(t) = e^{-2t}\left(\int_0^t e^{2s}(s + \sin(s))\, ds + C \right) =$$

$$= e^{-2t}\left(\tfrac{3}{2} + te^{t} - e^{t} - \tfrac{1}{2}\cos(t) + \tfrac{1}{2}e^{t}\sin(t) + C\right)$$

$$= \tfrac{3}{2}e^{-2t} + te^{-t} - e^{-t} - \tfrac{1}{2}e^{-2t}\cos(t) + \tfrac{1}{2}e^{-t}\sin(t) + Ce^{-2t}$$

To satisfy the initial condition, solve $y(0) = 4$

$$\Leftrightarrow \quad 4 = \underbrace{\tfrac{3}{2} - 1 - \tfrac{1}{2}}_{0} + C \quad \Rightarrow C = 4$$

Hence,

$$\boxed{y(t) = \tfrac{3}{2}e^{-2t} + te^{-t} - e^{-t} - \tfrac{1}{2}e^{-2t}\cos(t) + \tfrac{1}{2}e^{-t}\sin(t) + 4e^{-2t}}$$

$$\boxed{= \left(\tfrac{11}{2} - \tfrac{1}{2}\cos(t)\right)e^{-2t} + \left(t - 1 + \tfrac{1}{2}\sin(t)\right)e^{-t}}$$

4) Solve the IVP $(2t^2+1)y' - 10ty = 4t$, $y(0) = 7$

Solution: We first put the equation in standard form

$$y' \underbrace{- \frac{10t}{2t^2+1}}_{p(t)} y = \frac{4t}{2t^2+1}, \quad y(0) = 7$$

$$\boxed{\mu(t) = e^{\int p(t)dt}} = e^{-\frac{5}{2}\int \frac{4t}{2t^2+1}dt} = e^{-\frac{5}{2}\ln(2t^2+1)}$$

$$\boxed{= \frac{1}{(2t^2+1)^{\frac{5}{2}}}}$$

We solve the equation "by hand" and first multiply by $\mu(t)$:

$$\left[\frac{1}{(2t^2+1)^{\frac{5}{2}}}y\right]' = \frac{4t}{[2t^2+1]^{\frac{7}{2}}} \quad \Big| \int dt$$

$$\frac{y}{(2t^2+1)^{\frac{5}{2}}} = \int \frac{4t}{(2t^2+1)^{7/2}}\,dt = -\frac{2}{5}(2t^2+1)^{-\frac{5}{2}} + C$$

$$\Rightarrow \quad y(t) = -\frac{2}{5} + C\,(2t^2+1)^{\frac{5}{2}}$$

In order to satisfy the initial condition, solve

$$y(0) = 7$$

$$\Leftrightarrow \quad 7 = -\frac{2}{5} + C \quad \Rightarrow C = \frac{37}{5}$$

Hence,

$$\boxed{y(t) = -\frac{2}{5} + \frac{37}{5}(2t^2+1)^{\frac{5}{2}}}$$