3.6.4 Exercises

1] Verify the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Solution:

a] We first observe that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \lim_{a\to\infty}\int_{-a}^{a}\int_{-a}^{a} e^{-x^2-y^2}\,dx\,dy$$

$$= \lim_{r\to\infty}\iint_{D} e^{-x^2-y^2}\,dx\,dy$$

where $D_\rho = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \leq \rho^2\}$

b] Next observe that

$$\int_{-a}^{a}\int_{-a}^{a} \underbrace{e^{-x^2-y^2}}_{e^{-x^2}e^{-y^2}}\,dx\,dy = \int_{-a}^{a} e^{-x^2}\,dx \int_{-a}^{a} e^{-y^2}\,dy$$

by Cor. 2.3 (p. 80).

Hence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2$$

c] We evaluate the integral $\lim_{\rho\to\infty}\iint_{D_\rho} e^{-x^2-y^2}\,dx\,dy$ using cylindrical coordinates (p. 89, 90)

$$\iint_{D_r} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\int_0^{\rho} e^{-r^2}\,r\,dr\,d\theta =$$

with $x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$

$$= \int_0^{2\pi} \frac{1}{2} \int_0^{\rho^2} e^{-u}\,du\,d\theta = \frac{1}{2}\int_0^{2\pi} \left[-e^{-u}\right]_0^{\rho^2} d\theta = \frac{1}{2}\int_0^{2\pi} (1-e^{-\rho^2})\,d\theta$$

sub $u = r^2 \Rightarrow du = 2r\,dr \Rightarrow r\,dr = \frac{1}{2}du$

$u(0)=0,\ u(r)=\rho^2$

$$= \pi\,(1-e^{-\rho^2})$$

Hence

$$\lim_{\rho\to\infty} \iint_{D_\rho} e^{-x^2-y^2}\,dx\,dy = \lim_{\rho\to\infty} \pi(1-e^{-\rho^2}) = \pi$$

Thus

$$\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2$$

and therefore

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

2) Compute $\Gamma(\frac{3}{2})$

Solution: By Example 3.26 $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Thus, by Thm 3.44 (1):

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

3| Compute $\Gamma\left(\frac{2n+1}{2}\right)$

<u>Solution</u>:

$n=1$: $\quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

$n=2$: $\quad \Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1\cdot 3\sqrt{\pi}}{2^2}$

$n=3$: $\quad \Gamma\left(\overset{2\cdot 3+1}{\frac{7}{2}}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{1\cdot 3\cdot 5\sqrt{\pi}}{2^3}$

$n=4$: $\quad \Gamma\left(\overset{2\cdot 4+1}{\frac{9}{2}}\right) = \Gamma\left(\frac{7}{2}+1\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{1\cdot 3\cdot 5\cdot 7}{2^4}\sqrt{\pi}$

Hence

$$\boxed{\Gamma\left(\frac{2n+1}{2}\right)} = \frac{\prod_{k=0}^{n-1}(2k+1)}{2^n}\sqrt{\pi}$$

$$= \frac{1 \overset{2\cdot 1}{\cdot} 3 \overset{2\cdot 2}{\cdot} 5 \overset{2\cdot 3}{\cdot} \cdots \overset{2\cdot(n-1)}{\cdot} (2n-1)}{2^n \cdot 2^{n-1}(n-1)!}\sqrt{\pi}$$

$$= \boxed{\frac{(2n-1)!\sqrt{\pi}}{2^{2n-1}(n-1)!}}$$

4| Using the ratio test, show that J_ν converges for all $x \in \mathbb{R}$, $\nu > 0$.

<u>Solution</u>: $\quad J_\nu(x) = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1+\nu)}\left(\frac{x}{2}\right)^{2k}$

By Thm 3.20(2), it suffices to compute the radius of convergence of

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1+\nu)}\left(\frac{x}{2}\right)^{2k}$$

To this end, let $y := \left(\frac{x}{2}\right)^2$, then $y^k = \left(\frac{x}{2}\right)^{2k}$, and

$$f(y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1+\nu)}\, y^k$$

We now use, Thm 3.31 (1) on page 145. Note that $b_k = \dfrac{(-1)^k}{k!\,\Gamma(k+1+\nu)}$. Then

$$R = \lim_{k\to\infty} \left| \frac{b_k}{b_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{\overset{1}{\cancel{(-1)^k}}\ \overset{k+1}{\cancel{(k+1)!}}\ \overset{k+1+\nu}{\cancel{\Gamma(k+2+\nu)}}}{\underset{1}{\cancel{k!}}\ \cancel{\Gamma(k+1+\nu)}\ \underset{(-1)}{\cancel{(-1)^{k+1}}}} \right|$$

(recall $\Gamma(x+1) = x\,\Gamma(x)$)

$$= \lim_{k\to\infty} \left[(k+1)(k+1+\nu) \right] = \infty$$

Hence, $f(y)$ converges for all y with $|y| < \infty$ $\Leftrightarrow$ $|x| < \infty$.

5] By termwise differentiation, show that $J_0' = -J_1$

Solution:

$$J_0'(x) = \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} \right]' =$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k\ \cancel{2k}}{\underset{(k-1)!}{\cancel{k!}}\ \Gamma(k+1)\cdot\cancel{2}} \left(\frac{x}{2}\right)^{2k-1}$$

sub $\ell = k-1$, $k = \ell + 1$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{\ell!\ \Gamma(\ell+1+1)} \left(\frac{x}{2}\right)^{2\ell+1}$$

$$= - \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!\ \Gamma(\ell+1+1)} \left(\frac{x}{2}\right)^{2\ell+1}$$

$$= - J_1(x)$$

6] Solve the ivp $xy''+y'+xy=0$, $y(0)=13$, $y'(0)=0$

Solution:

a] By multiplying the ode by $x\ (\neq 0)$, we obtain

$$x^2y''+xy'+(x^2-0)y=0$$

which is the BE of order $\nu=0$. It's general solution is (Thm 3.49, p. 192) is

$$y(x)=C_1J_0(x)+C_2Y_0(x)$$

(Note that Bessel functions of the 2nd kind are also defined for $\nu\in\mathbb{Z}$, although not by Def 3.32! Also note that Y_ν is unbounded at $x_0=0$! see Rem. 3.18 p. 192)

b] apply the initial conditions:

$$13=y(0)=C_1\underbrace{J_0(0)}_{=1}+C_2Y_0(0)$$

implies $C_2=0$ and $C_1=13$

$$y'(x)=[13J_0(x)]'=-13J_1(x)$$

$$y'(0)=-13\underbrace{J_1(0)}_{=0}=0$$

Remark $\quad J_0(0)=\left.\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k}\right|_{x=0}$

$$=\frac{(-1)^0}{0!\,\Gamma(1)}=1 \quad \text{constant term!}$$

$$J_1(0) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1+1)} \left(\frac{x}{2}\right)^{2k+1} \Bigg|_{x=0}$$

$$\doteq x \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1+1)} \left(\frac{x}{2}\right)^{2k} \Bigg|_{x=0}$$

$$= 0$$