3.7.4 Exercises

1) Verify the identities

a) $\left[x^{-\nu} y_\nu(x)\right]' = -x^{-\nu} y_{\nu+1}(x)$ for $\nu \notin \mathbb{Z}$

Solution:

$$\left[x^{-\nu} y_\nu(x)\right]' = \left[\frac{\cos(\nu\pi)}{\sin(\nu\pi)}\left(x^{-\nu} J_\nu(x)\right) - \frac{1}{\sin(\nu\pi)}\left(x^{-\nu} J_{-\nu}(x)\right)\right]'$$

$$= \frac{\cos(\nu\pi)}{\sin(\nu\pi)}\left(-x^{-\nu} J_{\nu+1}(x)\right) - \frac{1}{\sin(\nu\pi)} x^{-\nu} J_{-\nu-1}(x)$$

$$= -\frac{\cos((\nu+1)\pi)}{\sin((\nu+1)\pi)} x^{-\nu} J_{\nu+1}(x) + \frac{1}{\sin((\nu+1)\pi)} x^{-\nu} J_{-(\nu+1)}(x)$$

$$= -x^{-\nu}\left(\frac{\cos((\nu+1)\pi)}{\sin((\nu+1)\pi)} J_{\nu+1}(x) - \frac{1}{\sin((\nu+1)\pi)} J_{-(\nu+1)}(x)\right)$$

$$= -x^{-\nu} y_{\nu+1}(x)$$

b) $\frac{2\nu}{x} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x)$

Solution: By Cor. 3.6, we have

$$x J_\nu'(x) + \nu J_\nu(x) = x J_{\nu-1}(x)$$

$$x J_\nu'(x) - \nu J_\nu(x) = -x J_{\nu+1}(x)$$

Subtraction the 2nd eqn from the first yields

$$2\nu J_\nu(x) = x\left(J_{\nu-1}(x) + J_{\nu+1}(x)\right)$$

and then $\frac{2\nu}{x} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x)$

c) $J_\nu'(x) = \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right]$

Solution: By Cor. 3.6, we have

$$x J_\nu'(x) + \nu J_\nu(x) = x J_{\nu-1}(x)$$
$$x J_\nu'(x) - \nu J_\nu(x) = -x J_{\nu+1}(x)$$

Adding the equations, we obtain

$$2x J_\nu'(x) = x\left(J_{\nu-1}(x) - J_{\nu+1}(x)\right)$$

and then

$$J_\nu'(x) = \frac{1}{2}\left(J_{\nu-1}(x) - J_{\nu+1}(x)\right).$$

2) Express $J_4(ax)$ in terms of $J_0(ax)$ and $J_1(ax)$.

Solution: We use the recurrence formula of Cor. 3.7 (1): $\boxed{J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)}$

$$J_4(ax) = \frac{6}{ax} J_3(ax) - J_2(ax) =$$
$$= \frac{6}{ax}\left[\frac{4}{ax} J_2(ax) - J_1(ax)\right] - J_2(ax) =$$
$$= \left[\frac{24}{a^2x^2} - 1\right] J_2(ax) - \frac{6}{ax} J_1(ax)$$
$$= \left[\frac{24}{a^2x^2} - 1\right]\left(\frac{2}{ax} J_1(ax) - J_0(ax)\right) - \frac{6}{ax} J_1(ax)$$
$$= \left[\frac{48}{a^3x^3} - \frac{8}{ax}\right] J_1(ax) - \left[\frac{24}{a^2x^2} - 1\right] J_0(ax)$$

3] Show that

$$\left(x J_\nu(x) J_{\nu+1}(x)\right)' = x\left[J_\nu^2(x) - J_{\nu+1}^2(x)\right]$$

Solution: Note that $x = \dfrac{x^{-\nu+(\nu+1)}}{1}$ hence

$$\left(x J_\nu(x) J_{\nu+1}(x)\right)' = \left(\left[x^{-\nu} J_\nu(x)\right]\left[x^{\nu+1} J_{\nu+1}(x)\right]\right)' =$$

$$= -x^{-\nu} J_{\nu+1}(x)\, x^{\nu+1} J_{\nu+1}(x) + x^{-\nu} J_\nu(x)\, x^{\nu+1} J_\nu(x)$$

$$= x\left[J_\nu^2(x) - J_{\nu+1}^2(x)\right]$$

4] Solve the integrals

a] $\int J_1(x)\,dx$

Solution: By Cor. 3.8 (4), p. 196,

$$\int J_1(x)\,dx = -J_0(x) + C$$

b] $\int x J_0(x)\,dx$

Solution: By Cor. 3.8 (3), p. 196,

$$\int x J_0(x)\,dx = x J_1(x) + C$$

c) $\int x^3 J_0(x)\,dx$

Solution:

$$\int x^3 J_0(x)\,dx = \int \underbrace{x J_0(x)}_{u'}\, \underbrace{x^2}_{v}\, dx = x J_1(x)\, x^2 - 2\int x J_1(x)\, x\, dx$$

$$= x^3 J_1(x) - 2\int \underbrace{J_1(x)}_{u'}\, \underbrace{x^2}_{v} dx =$$

$$= x^3 J_1(x) - 2\left[-J_0(x)x^2 + 2\int J_0(x)\, x\, dx\right]$$

$$= x^3 J_1(x) + 2x^2 J_0(x) - 4\underbrace{\int x J_0(x)\,dx}_{= x J_1(x)}$$

$$\boxed{= (x^3 - 4x) J_1(x) + 2x^2 J_0(x) + C}$$

d) $\int \frac{J_2(5x)}{x^3}\, dx$

Solution:

$$\int \frac{J_2(5x)}{x^3}\, dx = 25\int \frac{J_2(y)}{y^3}\, dy =$$

sub $y = 5x$, $dy = 5\,dx$, $dx = \frac{1}{5}dy$, $x = \frac{1}{5}y$

$$= 25 \int \underbrace{y^{-3}}_{u'}\, \underbrace{J_2(y)}_{v}\, dy = 25\left[-\tfrac{1}{2} y^{-2} J_2(y) + \tfrac{1}{2}\int y^{-2}\, \underbrace{J_2'(y)}_{\frac{1}{2}(J_1(y) - J_3(y))}\, dy\right]$$

$$= -\frac{25}{2} y^{-2} J_2(y) + \frac{25}{4}\int y^{-2} J_1(y)\, dy - \frac{25}{4}\underbrace{\int y^{-2} J_3(y)\, dy}_{-y^{-2} J_2(y)}$$

$$= \underbrace{\left[-\frac{25}{2}+\frac{25}{4}\right]}_{-\frac{25}{4}} y^{-2} J_2(y) + \frac{25}{4}\int \underbrace{y^{-2}}_{u'}\, \underbrace{J_1(y)}_{v}\, dy =$$

$$= -\frac{25}{4} y^{-2} J_2(y) + \frac{25}{4}\left[-y^{-1}J_1(y) + \int y^{-1}\underbrace{J_1'(y)}_{\frac{1}{2}(J_0(y)-J_2(y))}\, dy\right]$$

$$= -\frac{25}{4} y^{-2} J_2(y) - \frac{25}{4} y^{-1} J_1(y) - \frac{25}{8}\underbrace{\int y^{-1} J_2(y)\, dy}_{-y^{-1}J_1(y)} + \frac{25}{8}\int y^{-1} J_0(y)\, dy$$

$$= -\frac{25}{4} y^{-2} \underbrace{J_2(y)}_{= \frac{2}{y}J_1(y) - J_0(y)} - \frac{25}{8} y^{-1} J_1(y) + \frac{25}{8}\int y^{-1} J_0(y)\, dy$$

$$= -\frac{25}{2} y^{-3} J_1(y) + \frac{25}{4} J_0(y) - \frac{25}{8} y^{-1} J_1(y) + \frac{25}{8}\int y^{-1} J_0(y)\, dy$$

$$= -\frac{25}{8}\left[4y^{-3} + y^{-1}\right] J_1(y) + \frac{25}{4} J_0(y) + \frac{25}{8}\int y^{-1} J_0(y)\, dy$$

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$$= -\frac{25}{8}\left[4(5x)^{-3} + (5x)^{-1}\right] J_1(5x) + \frac{25}{4} J_0(5x) + \frac{25}{8}\int (5x)^{-1} J_0(5x)\, dx$$

$$\boxed{= -\frac{1}{8}\left[\frac{4}{x^3} + \frac{5}{x}\right] J_1(5x) + \frac{25}{4} J_0(5x) + \frac{5}{8}\int \frac{J_0(5x)}{x}\, dx}$$