3.8.9 Exercises

1] Write in Sturm-Liouville form:

a] $y'' + \lambda y = 0$

Solution: $\mathcal{L}(y) := y''$ is already in SL-form.

b] $x^2 y'' + xy' + \lambda y = 0$

Solution: $\mathcal{L}(y) := x^2 y'' + xy'$ is not in SL-form (since $(x^2)' \neq x$). We have to multiply the equation by the integrating factor

$$r(x) := \frac{e^{\int \frac{a_1(x)}{a_2(x)} dx}}{a_2(x)} = \frac{e^{\int \frac{x}{x^2} dx}}{x^2} = \frac{1}{x}$$

which yields

$$\boxed{x y'' + y' + \lambda \left(\frac{1}{x}\right) y = 0}$$

Note $w(x) = \frac{1}{x}$ is the weight function of this equation!

c] $xy'' + (1-x)y' + \lambda y = 0$

Solution: $\mathcal{L}(y) = xy'' + (1-x)y'$ is not in SL-form. We have to multiply the equation by

$$r(x) = \frac{e^{\int \frac{a_1(x)}{a_2(x)} dx}}{a_2(x)} = \frac{e^{\int \frac{1-x}{x} dx}}{x} = \frac{e^{\int (\frac{1}{x} - 1) dx}}{x} =$$

$$= e^{-x}$$

which yields

$$\boxed{x e^{-x} y'' + (e^{-x} - x e^{-x}) y' + \lambda \underbrace{e^{-x}}_{w(x)} y = 0}$$

2] Determine the Eigenfunctions of

a] $y'' + \lambda y = 0, \quad y(0) = 0, \; y(1) = 0$

<u>Solution</u> :

i] <u>gen. sol</u> :

$$y(x) = \begin{cases} C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x), & \lambda > 0 \\ C_1 + C_2 x, & \lambda = 0 \end{cases}$$

ii] <u>Apply bvs</u> :

$$0 = y(0) = \begin{cases} C_1, & \lambda > 0 \\ C_1, & \lambda = 0 \end{cases} \Rightarrow C_1 = 0, \; C_2 \neq 0$$

$$y(x) = \begin{cases} C_2 \sin(\sqrt{\lambda} x), & \lambda > 0 \\ C_2 x, & \lambda = 0 \end{cases}$$

$$0 = y(1) = \begin{cases} C_2 \sin(\sqrt{\lambda}), & \lambda > 0 \\ C_2, & \lambda = 0 \end{cases}$$

$$\Leftrightarrow \quad \sin(\sqrt{\lambda}) = 0, \; \lambda > 0 \quad \Leftrightarrow \quad \sqrt{\lambda} = n\pi, \; n = 1, 2, \ldots$$

hence $\quad \sqrt{\lambda_n} = (n+1)\pi, \; n = 0, 1, 2, \ldots$

$$\boxed{y_n(x) = \sin\left((n+1)\pi x\right), \quad n = 0, 1, 2, \ldots}$$

b] $y'' + \lambda y = 0, \quad y'(0) = 0, \; y(1) = 0$

<u>Solution</u> :

i] <u>gen. sol</u> :

$$y(x) = \begin{cases} C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x), & \lambda > 0 \\ C_1 + C_2 x, & \lambda = 0 \end{cases}$$

$$y'(x) = \begin{cases} -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x), & \lambda > 0 \\ C_2, & \lambda = 0 \end{cases}$$

ii) Apply bcs

$$0 = y'(0) = \begin{cases} C_2\sqrt{\lambda}, & \lambda > 0 \\ C_2 & \lambda = 0 \end{cases} \Rightarrow C_2 = 0, \ C_1 \neq 0$$

hence

$$y(x) = \begin{cases} C_1 \cos(\sqrt{\lambda} x), & \lambda > 0 \\ C_1 & \lambda = 0 \end{cases}$$

$$0 = y(1) = \begin{cases} C_1 \cos(\sqrt{\lambda}), & \lambda > 0 \\ \underset{\neq 0}{C_1} & \lambda = 0 \end{cases}$$

$$\Leftrightarrow \quad \cos(\sqrt{\lambda}) = 0, \ \lambda > 0 \quad \Leftrightarrow \quad \sqrt{\lambda} = (2n+1)\frac{\pi}{2}, \ n = 0, 1, 2,$$

Hence,

$$\sqrt{\lambda_n} = (2n+1)\frac{\pi}{2}, \quad n = 0, 1, 2, -$$

$$\boxed{y_n(x) = \cos\left((2n+1)\frac{\pi}{2}x\right), \quad n = 0, 1, 2, \ldots}$$

c) $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) + y'(1) = 0$

Solution:

i) gen. sol

$$y(x) = \begin{cases} C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x), & \lambda > 0 \\ C_1 + C_2 x, & \lambda = 0 \end{cases}$$

$$y'(x) = \begin{cases} -C_1\sqrt{\lambda}\sin(\sqrt{\lambda} x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda} x), & \lambda > 0 \\ C_2, & \lambda = 0 \end{cases}$$

ii] Apply bcs

$$0 = y'(0) = \left.\begin{cases} C_2\sqrt{\lambda}, & \lambda > 0 \\ C_2, & \lambda = 0 \end{cases}\right\} \Rightarrow C_2 = 0,\ C_1 \neq 0$$

hence

$$y(x) = \begin{cases} C_1 \cos(\sqrt{\lambda}x), & \lambda > 0 \\ C_1 & \lambda = 0 \end{cases}$$

$$y'(x) = \begin{cases} -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x), & \lambda > 0 \\ 0, & \lambda = 0 \end{cases}$$

$$0 = y(1) + y'(1) = \begin{cases} C_1\cos(\sqrt{\lambda}) - C_1\sqrt{\lambda}\sin(\sqrt{\lambda}), & \lambda > 0 \\ \underset{\neq 0}{C_1} & \lambda = 0 \end{cases}$$

$$\Leftrightarrow \qquad \cos(\sqrt{\lambda}) - \sqrt{\lambda}\sin(\sqrt{\lambda}) = 0, \quad \lambda > 0$$

Hence, the Eigenvalue are the positive solutions

$$0 < \mu_0 < \mu_1 < \mu_2 < \cdots$$

of the Eigenvalue equation $w(\mu) = \cos(\mu) - \mu\sin(\mu) = 0$

and the Eigenfunction

$$\boxed{y_n(x) = \cos(\mu_n x), \quad n = 0, 1, 2, \cdots}$$

$\mu_0 = 0.860334$

$\mu_1 = 3.425618$

$\mu_2 = 6.437298$

$\mu_3 = 9.529334$

$\mu_4 = 12.645287$

$\mu_5 = 15.771285$

$\vdots$

3) Find the Eigenfunction expansion of

$$f(x) := \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}$$

using the Eigenfunction determined in Exerc. 2

<u>Solution</u>

a) $y_n(x) = \sin((n+1)\pi x), \quad n = 0,1,2,\ldots$

$$a_n = \frac{\int_0^1 f(x)\, y_n(x)\, dx}{\underbrace{\int_0^1 y_n^2(x)\, dx}_{=\frac{1}{2}}} = \frac{2\left[\pi(-1)^n n + \pi(-1)^n + 2\cos\left(\frac{n\pi}{2}\right)\right]}{n^2(n^2+2n+1)}$$

$$Sfa(x) = \sum_{n=0}^{\infty} a_n \sin((n+1)\pi x)$$

b) $y_n(x) = \cos\left((2n+1)\frac{\pi}{2} x\right), \quad n = 0,1,2,\ldots$

$$a_n = \frac{\int_0^1 f(x)\, y_n(x)\, dx}{\underbrace{\int_0^1 y_n^2(x)\, dx}_{=\frac{1}{2}}} = \frac{4\left[2\pi(-1)^n n + \pi(-1)^n - 4 + 2\cos\left(\frac{n\pi}{2}\right)\sqrt{2} - 2\sin\left(\frac{n\pi}{2}\right)\sqrt{2}\right]}{\pi^2(4n^2+4n+1)}$$

c) $\varrho_n > 0$ solution of $w(\varrho) = \cos(\varrho) - \varrho\sin(\varrho) = 0$

...

see Maple Worksheet Exercise_3-8-9-3.mw

4) verify SL table p 215 ND

<u>Solution</u>: $X'' + \lambda^2 X = 0, \quad x \in [0, L]$

$X'(0) = 0$

$X(L) = 0$

a) Note the Euler operator is in SL-form, by Thm 3.53 (p 205) $\lambda \geq 0$, therefore, we only need to consider the non-neg. solution of the characteristic eqn

$$r^2 + \lambda^2 = 0$$

and obtain the general solution:

$$X(x) = \begin{cases} C_1 \cos(\lambda x) + C_2 \sin(\lambda x), & \lambda > 0 \\ C_1 + C_2 x, & \lambda = 0 \end{cases}$$

$$X'(x) = \begin{cases} -C_1 \lambda \sin(\lambda x) + C_2 \lambda \cos(\lambda x), & \lambda > 0 \\ C_2, & \lambda = 0 \end{cases}$$

b) apply the BC:

$$0 = X'(0) = \left.\begin{cases} C_2 \lambda, & \lambda > 0 \\ C_2, & \lambda = 0 \end{cases}\right\} \Rightarrow C_2 = 0, \quad C_1 \neq 0$$

Thus

$$X(x) = \begin{cases} C_1 \cos(\lambda x), & \lambda > 0 \\ C_1, & \lambda = 0 \end{cases}$$

$$0 = X(L) = \begin{cases} C_1 \cos(\lambda L), & \lambda > 0 \\ C_1 & \lambda = 0 \end{cases}$$

Since $C_1 \neq 0$, $\lambda > 0$ and $\cos(\lambda L) = 0$

$\Leftrightarrow \quad \lambda L = (2n+1)\frac{\pi}{2}$ for $n = 0, 1, 2, \cdots$

i.e.

$$\lambda_n = \frac{(2n+1)\pi}{2L} \quad \text{for } n = 0,1,2,\ldots$$

$$X_n(x) = \cos\left(\frac{(2n+1)\pi}{2L}x\right), \quad \text{for } n = 0,1,2,\ldots$$

6] $x^2y'' + xy' + \lambda y = 0, \quad x \in [1,2]$

$y'(1) = 0$

$y'(2) = 0$

Find the Eigenvalues, -functions, and evaluate the Eigen-function expansion

a] $f_1(x) = x - 1$, b] $f_2(x) = x(2-x)$, c] $f_3(x) = -\frac{2}{3}x^3 + 3x^2 - 4x + \frac{5}{3}$

Solution:

i] solve SLP

$x^2y'' + xy' + \lambda y = 0$ is not in SL-form, we multiply by the integrating factor

$$r(x) = \frac{e^{\int \frac{a_1(x)}{a_2(x)}dx}}{a_2(x)} = \frac{1}{x}, \quad a_2(x) = x^2, \; a_1(x) = x$$

and obtain

$$xy'' + y' + \lambda\left(\frac{1}{x}\right)y = 0$$

Note, the weight function for this SL problem is

$\boxed{w(x) = \frac{1}{x}}$ by Thm 3.53, p. 205, $\lambda \geq 0$

a] gen. solution

$$y(x) = x^r, \; y'(x) = rx^{r-1}; \; y''(x) = r(r-1)x^{r-2}, \quad x > 0$$

$$x^r\underbrace{(r(r-1) + r + \lambda)}_{r^2+\lambda} = 0 \quad \Big| \cdot x^{-r}$$

char. eqtn: $r^2 + \lambda = 0$, $\lambda \geq 0$, hence

$$y(x) = \begin{cases} C_1 \cos(\sqrt{\lambda}\ln|x|) + C_2 \sin(\sqrt{\lambda}\ln|x|), & \lambda > 0 \\ C_1 + C_2 \ln|x|, & \lambda = 0 \end{cases}$$

$$y'(x) = \begin{cases} -C_1 \frac{\sqrt{\lambda}}{x} \sin(\sqrt{\lambda}\ln|x|) + C_2 \frac{\sqrt{\lambda}}{x} \cos(\sqrt{\lambda}\ln|x|), & \lambda > 0 \\ \frac{C_2}{x}, & \lambda = 0 \end{cases}$$

B] apply bcs

$$0 = y'(1) = \left.\begin{cases} C_2\sqrt{\lambda}, & \lambda > 0 \\ C_2, & \lambda = 0 \end{cases}\right\} \Rightarrow C_2 = 0, \; C_1 \neq 0$$

hence
$$y(x) = \begin{cases} C_1 \cos(\sqrt{\lambda}\ln|x|), & \lambda > 0 \\ C_1, & \lambda = 0 \end{cases}$$

$$y'(x) = \begin{cases} -C_1 \frac{\sqrt{\lambda}}{x} \sin(\sqrt{\lambda}\ln|x|), & \lambda > 0 \\ 0, & \lambda = 0 \end{cases}$$

$$0 = y'(2) = \begin{cases} -C_1 \frac{\sqrt{\lambda}}{2} \sin(\sqrt{\lambda}\ln 2), & \lambda > 0 \\ 0, & \lambda = 0 \end{cases}$$

Thus, $\lambda = 0$ or $\left(\sin(\sqrt{\lambda}\ln 2) = 0 \text{ and } \lambda > 0\right)$

$\Leftrightarrow$ $\lambda = 0$ or $\left(\sqrt{\lambda}\ln 2 = n\pi, \; n = 1, 2, \ldots\right)$

$\Leftrightarrow$ $$\boxed{\lambda_n = \frac{n^2\pi^2}{(\ln 2)^2} \quad \text{for } n = 0, 1, 2, \ldots}$$

$$\boxed{y_n(x) = \begin{cases} \cos\left(\frac{n\pi}{\ln 2} x\right), & n = 1, 2, \ldots \\ 1, & n = 0 \end{cases}}$$

ii] compute Eigenfunction expansion

$$f(x) = a_0 \cdot 1 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{\ln 2} x\right)$$

where

$$a_0 = \frac{\int_1^2 f(x) \cdot \frac{1}{x} dx}{\int_1^2 1\, dx} = \int_1^2 f(x) \cdot \frac{1}{x} dx$$

$$a_n = \frac{\int_1^2 f(x) \cos\left(\frac{n\pi}{\ln 2} x\right) \cdot \frac{1}{x} dx}{\int_1^2 \cos^2\left(\frac{n\pi}{\ln 2} x\right) \cdot \frac{1}{x} dx} \quad \text{for } n = 1, 2, \ldots$$

See Maple worksheet $\boxed{\text{Exercise\_3-8-9-6.mw}}$

7] verify SL table p 221 D-D

Solution: $x^2 X'' + x X' + (\lambda^2 x^2 - \nu^2) X = 0, \quad x \in [a,b], \; 0 < a < b$

$X(a) = 0$
$X(b) = 0$

a] $x^2 X'' + x X' + (\lambda^2 x^2 - \nu^2) X = 0$ is not in SL form, we multiply by the integrating factor

$$r(x) = \frac{e^{\int \frac{a_1(x)}{a_2(x)} dx}}{a_2(x)} = \frac{1}{x}, \quad a_2(x) = x^2, \; a_1(x) = x$$

and obtain

$$\underbrace{x X'' + X' - \frac{\nu^2}{x} X}_{\text{SL-operator}} + \lambda^2 \underbrace{x}_{w(x)} X = 0$$

Note, the weight function for this SL problem is $\boxed{w(x) = x}$

b] general solution

$$X(x) = \begin{cases} C_1 x^{\nu} + C_2 x^{-\nu}, & \lambda = 0,\ \nu > 0 \\ C_1 + C_2 \ln x, & \lambda = 0,\ \nu = 0 \\ C_1 J_\nu(\lambda x) + C_2 Y_\nu(\lambda x), & \lambda > 0 \end{cases}$$

c] apply bcs.

i] $\lambda = 0,\ \nu > 0$

$$0 = X(a) = C_1 a^{\nu} + C_2 a^{-\nu}$$

$$0 = X(b) = C_1 b^{\nu} + C_2 b^{-\nu}$$

This is a homogeneous system of linear equations in C_1, C_2 with coefficient determinant

$$\begin{vmatrix} a^{\nu} & a^{-\nu} \\ b^{\nu} & b^{-\nu} \end{vmatrix} = a^{-\nu} b^{-\nu} \begin{vmatrix} a^{2\nu} & 1 \\ b^{2\nu} & 1 \end{vmatrix} = (ab)^{-\nu}(a^{2\nu} - b^{2\nu})$$

$$= (ab)^{-\nu}(a^{\nu} - b^{\nu})(a^{\nu} + b^{\nu}) \neq 0$$

since $0 < a < b$. Hence, the system has only the trivial solution $C_1 = C_2 = 0$, i.e. $X(x) = 0$, which is not an Eigenfunction

ii] $\lambda = 0,\ \nu = 0$

$$0 = X(a) = C_1 + C_2 \ln a$$

$$0 = X(b) = C_1 + C_2 \ln b$$

As above, consider the coefficient determinant

$$\begin{vmatrix} 1 & \ln a \\ 1 & \ln b \end{vmatrix} = \ln b - \ln a = \ln \frac{b}{a} \neq 0, \text{ since } b \neq a$$

Hence, the hom. system of linear equations has only the trivial solution $c_1 = c_2$, i.e $X(x) = 0$ $\forall$.

Therefore $\boxed{\lambda > 0}$ and

$$0 = c_1 J_\nu(\lambda a) + c_2 Y_\nu(\lambda a)$$

$$0 = c_1 J_\nu(\lambda b) + c_2 Y_\nu(\lambda b)$$

In order for this system to have a non-trivial solution (c_1, c_2), the coefficient determinant

$$\begin{vmatrix} J_\nu(\lambda a) & Y_\nu(\lambda a) \\ J_\nu(\lambda b) & Y_\nu(\lambda b) \end{vmatrix} = 0$$

$$\Leftrightarrow \quad \boxed{\begin{array}{l} J_\nu(\lambda a) Y_\nu(\lambda b) - Y_\nu(\lambda a) J_\nu(\lambda b) = 0 \\ \lambda > 0 \end{array}}$$

which by Thm 3.53 p 205 has countably many solutions $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$

$$y(x) = c_1 J_\nu(\lambda_n x) + c_2 Y_\nu(\lambda_n x)$$

$$= c_1 J_\nu(\lambda_n x) - \frac{c_1 J_\nu(\lambda a)}{Y_\nu(\lambda a)} Y_\nu(\lambda_n x)$$

$$c_2 = \frac{-c_1 J_\nu(\lambda a)}{Y_\nu(\lambda a)} \quad \text{write} \quad c_1 = Y_\nu(\lambda a)$$

$$= Y_\nu(\lambda a) J_\nu(\lambda_n x) - J_\nu(\lambda_n a) Y_\nu(\lambda_n x)$$

8] Verify the SL table p 231 for N

Solution $x^2X''+xX'+(\lambda^2x^2-\nu^2)X=0, \quad x\in[0,L]$

$|X(0)|<\infty$

$X'(L)=0$

a] see Example 3.38, sing. problem, $w(x)=x$

b] general solution

$$X(x)=\begin{cases} C_1x^{\nu}+C_2x^{-\nu}, & \lambda=0, \nu>0 \\ C_1+C_2\ln x, & \lambda=0, \nu=0 \\ C_1J_\nu(\lambda x)+C_2Y_\nu(\lambda x), & \lambda>0 \end{cases}$$

$$X'(x)=\begin{cases} C_1\nu x^{\nu-1}-C_2\nu x^{-\nu-1}, & \lambda=0, \nu>0 \\ \frac{C_2}{x}, & \lambda=0, \nu=0 \\ C_1\lambda J_\nu'(\lambda x)+C_2\lambda Y_\nu'(\lambda x), & \lambda>0 \end{cases}$$

c] apply bcs.

i] $\lambda=0$.

since, $y=x^{-\nu}$, $y=\ln x$, Y_ν are unbounded at $x=0$,
$|x(0)|<\infty$ implies, $C_2=0$

Hence

$$X(x)=\begin{cases} C_1x^{\nu}, & \lambda=0, \nu>0 \\ C_1 & \lambda=0, \nu=0 \\ C_1J_\nu(\lambda x), & \lambda>0 \end{cases}$$

$$X'(x)=\begin{cases} C_1\nu x^{\nu-1}, & \lambda=0, \nu>0 \\ 0, & \lambda=0, \nu=0 \\ C_1\lambda J_\nu'(\lambda x), & \lambda>0 \end{cases}$$

$$0 = X'(L) = \begin{cases} \overbrace{c_1 \nu L^{\nu-1}}^{\neq 0}, & \lambda = 0, \nu > 0 \\ 0, & \lambda = 0, \nu = 0 \\ c_1 \lambda J_\nu'(\lambda L), & \lambda > 0 \end{cases}$$

Hence

<u>case</u> $\nu = 0$: $\quad \lambda_0 = 0$

<u>case $\nu > 0$</u>: $\quad J_\nu'(\lambda L) = 0$ positive solution

$$0 < \lambda_0 < \lambda_1 < \cdots$$

Thus,

<u>If $\nu = 0$</u>: solution $\quad 0 = \lambda_0 < \lambda_1 < \cdots$ of $J_0'(\lambda L) = 0$

$$X_n(x) = \begin{cases} 1, & n = 0 \\ J_0(\lambda_n x), & n = 1, 2, \cdots \end{cases}$$

If $\nu > 0$: solution $\quad 0 < \lambda_0 < \lambda_1 < \cdots$ of $J_\nu'(\lambda L) = 0$

$$X_n(x) = J_\nu(\lambda_n x), \quad n = 0, 1, 2, \cdots$$

9] $x^2 y'' + x y' + \lambda^2 x^2 y = 0, \quad x \in [0, 1]$

$|y(0)| < \infty$

$y'(1) = 0$

Evaluate the Eigenfunction expansion for

a] $f_1(x) = x^2$, b] $f_2(x) = 1$, c] $f_3(x) = 1 - x^2$

<u>Solution</u>: (i) $x^2 y'' + x y' + \lambda^2 x^2 y = 0$ is not in SL form,

we multiply by the integrating factor

$$r(x) = \frac{e^{\int \frac{a_1(x)}{a_2(x)} dx}}{a_2(x)} = \frac{1}{x}, \qquad a_2(x) = x^2, \; a_1(x) = x$$

and obtain

$$\underbrace{xy''+y'}_{\text{SL-operator}} + \lambda^2 \underbrace{x}_{w(x)\ \text{weight function}} y = 0$$

$$\boxed{w(x) = x}$$

(ii) general solution

$$y(x) = \begin{cases} c_1 + c_2 \ln|x|, & \lambda = 0 \\ c_1 J_0(\lambda x) + c_2 Y_0(\lambda x), & \lambda > 0 \end{cases} \qquad (\text{CE}: r^2 = 0\ !)$$

(iii) apply bc's

$\lambda = 0$: $\quad |y(0)| < \infty \Rightarrow c_2 = 0$

$\lambda > 0$: $\quad |y(0)| < \infty \Rightarrow c_2 = 0$

hence

$$y(x) = \begin{cases} c_1, & \lambda = 0 \\ c_1 J_0(\lambda x), & \lambda > 0 \end{cases}$$

$$y'(x) = \begin{cases} 0, & \lambda = 0 \\ c_1 \lambda J_0'(\lambda x), & \lambda > 0 \end{cases}$$

$$0 = y'(1) = \begin{cases} 0, & \lambda = 0 \\ c_1 \lambda J_0'(\lambda), & \lambda > 0 \end{cases}$$

Thus $\lambda_0 = 0$ and for $n > 0$, the λ_n are the positive zeros of $\boxed{\lambda J_0'(\lambda) = 0}$

$$y_n(x) = \begin{cases} 1, & n = 0 \\ J_0(\lambda_n x), & n = 1, 2, 3, \dots \end{cases}$$

$$f(x) = a_0 1 + \sum_{n=1}^{\infty} a_n J_0(\lambda_n x)$$

where
$$a_0 = \frac{\int_0^1 f(x)\, x\, dx}{\int_0^1 1 \cdot x\, dx}$$

$$a_n = \frac{\int_0^1 f(x)\, J_0(\lambda_n x)\, x\, dx}{\int_0^1 J_0^2(\lambda_n x)\, x\, dx} \qquad n = 1, 2, \ldots$$

see $\boxed{\text{Exercise\_3-8-9—9.mw}}$

10] Verify Cor. 3.9 on page 204

Solution: (1) see p. 204

(2) suppose Φ_1 and Φ_2 are two Eigenfunctions relative to the distinct Eigenvalues λ_1 and λ_2, then

$$\lambda_1 \langle \Phi_1, \Phi_2 \rangle_w = \lambda_1 \langle w\Phi_1, \Phi_2 \rangle = \langle \lambda_1 w \Phi_1, \Phi_2 \rangle =$$

$$= -\langle \mathcal{L}(\Phi_1), \Phi_2 \rangle \overset{\text{Thm. 3.52}}{\underset{\text{p 203}}{=\!=}} -\langle \Phi_1, \mathcal{L}(\Phi_2) \rangle =$$

$$= \langle \Phi_1, \lambda_2 w \Phi_2 \rangle = \lambda_2 \langle \Phi_1, w\Phi_2 \rangle =$$

$$= \lambda_2 \langle \Phi_1, \Phi_2 \rangle_w$$

$$\Leftrightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \langle \Phi_1, \Phi_2 \rangle_w = 0 \qquad \Rightarrow \langle \Phi_1, \Phi_2 \rangle_w = 0$$

i.e $\Phi_1 \perp_w \Phi_2$.