3.5.4 Exercises

1) Find the singular points of each equation and determine whether they are regular or irregular.

a) $y'' + 2xy' + y = 0$

Solution: $a_2(x) = 1$ does not have any zeros, hence any $x_0 \in \mathbb{R}$ is an ordinary point.

b) $(1-x^2)y'' + y' - y = 0$

Solution: $a_2(x) = 1 - x^2 = 0 \Leftrightarrow x_0 = -1,\ x_1 = 1$ are the singular points of this equation.

$$y'' + \underbrace{\frac{1}{1-x^2}}_{p(x)} y' + \underbrace{\frac{-1}{1-x^2}}_{q(x)} y = 0$$

$x_0 = -1$: regular singular point

$(x-x_0)p(x) = (x+1)\frac{1}{1-x^2} = \frac{1}{1-x}$ is analytic at $x_0 = -1$

since $f(x) = 1-x$ is analytic at $x_0 = -1$, $f(x) \neq 0$ on $(-2,0)$, hence $\frac{1}{1-x} = \frac{1}{f(x)}$ is analytic at $x_0 = -1 \in (-2,0)$ by Thm 3.39 on page 153

$(x-x_0)^2 q(x) = (x+1)^2 \cdot \frac{-1}{1-x^2} = -\frac{x+1}{1-x}$ is analytic at $x_0 = -1$, since it is a quotient of functions that are analytic at $x_0 = -1$, and the denominator is different from 0 near $x_0 = -1$.

$x_1 = 1$: regular singular point

$(x-x_1)p(x) = (x-1)\frac{1}{1-x^2} = -\frac{1}{1+x}$ is analytic

at $x_1 = 1$ as the reciprocal of an analytic function that is different from zero near $x_0 = 1$. (Thm 3.39, p. 153)

$(x-x_1)^2 q(x) = (x-1)^2 \cdot \frac{-1}{1-x^2} = \frac{x-1}{1+x}$ is analytic

at $x_1 = 1$ as a product of analytic function at $x_1 = 1$; $1+x \neq 0$ near $x_1 = 1$. (Thm 3.39, p. 153)

c) $(x^4 - 3x^3 + 4x)y'' - x^2 y' + (x^3 - 2x - 1)y = 0$

Solution: $a_2(x) = x^4 - 3x^3 + 4x = x(x^3 - 3x^2 + 4) =$

$= x(x-1)(x^2+4x+4) = x(x-1)(x+2)^2$

Thus $x_0 = 0$, $x_1 = 1$, $x_2 = -2$ are the singular points of the equation.

$$y'' + \underbrace{\frac{-x^2}{(x^4 - 3x^3 + 4x)}}_{p(x)} y' + \underbrace{\frac{x^3 - 2x - 1}{(x^4 - 3x^3 + 4x)}}_{q(x)} y = 0$$

$x_0 = 0$ regular singular point

$$(x-x_0)p(x) = \frac{-x^3}{\underbrace{x^4 - 3x^2 + 4x}_{x(x-1)(x+2)^2}} = -\frac{x^2}{(x-1)(x+2)^2} \text{ is analytic}$$

at $x_0 = 0$ since rational functions are analytic in their domain

$$(x-x_0)^2 q(x) = \frac{\cancel{x^2}^{\,1}(x^3-2x-1)}{\underbrace{x^4-3x^2+4x}_{\cancel{x}(x-1)(x+2)^2}} = \frac{x^3-2x-1}{(x-1)(x+2)^2}$$

is analytic at $x_0=0$, since rational function are analytic on their domain.

$x_1 = 1$ regular singular point

$$(x-x_1)p(x) = (x-1)\frac{-x^2}{x(x-1)(x+2)^2} = \frac{-x}{x(x+2)^2}$$ is analytic at $x_1=1$

since rational function are analytic on their domain

$$(x-x_1)^2 q(x) = (x-x_1)^2 \frac{1}{x(x-1)(x+2)^2} = \frac{x-1}{x(x+2)^2}$$ is analytic at $x_1=1$

since rat. fu. are analytic on their domain.

$x_2 = -2$ not a regular singular point

$$(x-x_2)p(x) = (x+2)\frac{-x^2}{x(x-1)(x+2)^2} = \frac{-x^2}{x(x-1)(x+2)}$$ is not

analytic at $x_2=-2$, since $(x-x_2)p(x)$ is not finite!

We don't have to check $(x-x_2)^2 q(x)$!

e) $x^2(1-x)y'' + (1-x)y' + y = 0$

Solution $a_2(x) = x^2(1-x)$, then $x_0=0$ and $x_1=1$ are the singular points of the equation.

$$y'' + \underbrace{\frac{1}{x^2}}_{p(x)} y' + \underbrace{\frac{1}{x^2(1-x)}}_{q(x)} y = 0$$

$x_0 = 0$ is not a regular singular point

$(x - x_0)p(x) = x \cdot \frac{1}{x^2} = \frac{1}{x}$ is not analytic at $x_0 = 0$

since $(x - x_0)p(x)$ is not finite at $x_0 = 0$.

$x_1 = 1$ is a regular singular point

$(x - x_1)\,p(x) = (x-1)\cdot\frac{1}{x^2} = \frac{x-1}{x^2}$ is analytic at $x_1 = 1$

since rat. fn. are analytic on their domain

$(x - x_1)^2 q(x) = (x-1)^2 \cdot \frac{1}{x^2(1-x)} = \frac{x-1}{x^2}$ is analytic at $x_1 = 1$

since rat. fn. are analytic on their doma.

2) Find the indicial equation corresponding to each regular singular point

a) $3x^2y'' + xy' + (x-1)y = 0$

Solution:

i) singular points: $a_2(x) = 3x^2$, then $x_0 = 0$ is the only singular pt of the eqn.

$$y'' + \underbrace{\frac{1}{3x}}_{p(x)} y' + \underbrace{\frac{x-1}{3x^2}}_{q(x)} y = 0$$

$x_0 = 0$ is a regular singular pt of the eqn:

$(x - x_0)p(x) = x \cdot \frac{1}{3x} = \frac{1}{3}$ analytic

$(x - x_0)^2 q(x) = x^2 \cdot \frac{(x-1)}{3x^2} = \frac{1}{3}(x-1)$ analytic

ii) indicial eqn

$$p_0 = \lim_{x \to x_0} (x-x_0)p(x) = \lim_{x \to 0} \frac{1}{3} = \frac{1}{3}$$

$$q_0 = \lim_{x \to x_0} (x-x_0)^2 q(x) = \lim_{x \to 0} \frac{1}{3}(x-1) = -\frac{1}{3}$$

$$\boxed{h(r) = r^2 + (p_0 - 1)r + q_0} = \boxed{r^2 - \frac{2}{3}r - \frac{1}{3} = 0}$$

$$\Leftrightarrow \; 0 = 3r^2 - 2r - 1 = (3r+1)(r-1) \; \Rightarrow \boxed{r_1 = 1, \; r_2 = -\frac{1}{3}}$$

b) $(2x^4 + x^3 - x^2)y'' + y' - 6xy = 0$

Solution:

i) singular points: $a_2(x) = 2x^4 + x^3 - x^2 = x^2(2x^2 + x - 1) =$
$= x^2(2x-1)(x+1) = 0$, thus $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = -1$ are the singular points of the equation

ii) which singular points are regular

$$y'' + \underbrace{\frac{1}{x^2(2x-1)(x+1)}}_{p(x)} y' + \underbrace{\frac{-6}{x(2x-1)(x+1)}}_{q(x)} y = 0$$

$x_0 = 0$ is not regular

$$(x-x_0)p(x) = x \cdot \frac{1}{x^2(2x-1)(x+1)} = \frac{1}{x(2x-1)(x+1)} \text{ is not analytic}$$

at $x_0 = 0$, since $\lim_{x \to 0} \frac{1}{x(2x-1)(x+1)}$ does not exist!

$x_1 = \frac{1}{2}$ is regular

$$(x-x_1)p(x) = (x - \tfrac{1}{2}) \frac{1}{x^2(2x-1)(x+1)} = \frac{\cancel{(x-\frac{1}{2})}}{2x^2\cancel{(x-\frac{1}{2})}(x+1)} =$$

$$= \frac{1}{2x^2(x+1)} \text{ is analytic at } x_1 = \tfrac{1}{2},$$

since rat. fu. are analytic on their domains.

$$(x-x_1)^2 q(x) = (x-\tfrac{1}{2})^2 \frac{-6}{x(2x-1)(x+1)} = \frac{-6(x-\frac{1}{2})}{2x(x+1)}$$

$$= -\frac{3(x-\frac{1}{2})}{x(x+1)}$$ is analytic at $x_1=\frac{1}{2}$,

since rat. fu. are analytic on their domain.

<u>$x_2=-1$</u> is regular

$$(x-x_2)p(x) = (x+1)\frac{1}{x^2(2x-1)(x+1)} = \frac{1}{x^2(2x-1)}$$ is analytic

at $x_2=-1$, since rat. fu. are analytic on their domain

$$(x-x_2)^2 q(x) = (x+1)^2 \frac{-6}{x(2x-1)(x+1)} = \frac{-6(x+1)}{x(2x-1)}$$ is analytic

at $x_2=-1$, since rat. fu. are analytic on their domain

iii) indicial equation

<u>$x_1=\frac{1}{2}$:</u> $\quad p_0 = \lim\limits_{x\to\frac{1}{2}}(x-x_1)p(x) = \lim\limits_{x\to\frac{1}{2}} \frac{1}{2x^2(x+1)} = \frac{1}{2\cdot\frac{1}{4}\cdot\frac{3}{2}} =$

$= \frac{4}{3}$, $\quad q_0 = \lim\limits_{x\to\frac{1}{2}}(x-x_1)^2 q(x) = \lim\limits_{x\to\frac{1}{2}} \frac{-6(x-\frac{1}{2})}{2x(x+1)} = 0$

Thus,

$$\boxed{h(r) = r^2 + (p_0-1)r + q_0 = r^2 + \tfrac{1}{3}r = r\left(r+\tfrac{1}{3}\right) = 0}$$

hence, $\boxed{r_1 = 0,\ r_2 = -\frac{1}{3}}$

$\underline{x_2 = -1}$: $\quad p_0 = \lim_{x\to -1} (x - x_2)p(x) = \lim_{x\to -1} \frac{1}{x^2(2x-1)} = \frac{1}{1(-3)} = -\frac{1}{3}$

$$q_0 = \lim_{x\to -1} (x-x_2)^2 q(x) = \lim_{x\to -1} \frac{-6(x+1)}{x(2x-1)} = 0$$

Thus,

$$\boxed{h(r) = r^2 + (p_0 - 1)r + q_0 = r^2 - \frac{4}{3}r = r\left(r - \frac{4}{3}\right) = 0}$$

hence,
$$\boxed{r_1 = \frac{4}{3},\ r_2 = 0}$$

3| Find two linearly independent solutions near the origin and determine the values of x for which they are valid.

a| $2x^2y'' + (2x^2 + x)y' - y = 0$

i| $x_0 = 0$ is a regular singular pt:

$$y'' + \underbrace{\frac{2x+1}{2x}}_{p(x)} y' - \underbrace{\frac{1}{2x^2}}_{q(x)} y = 0$$

Note: $p(x) = \frac{1}{2} + x$, $q(x) = -\frac{1}{2}$ are power series with $R_p, R_q = \infty$!

$(x - x_0)p(x) = x \cdot \frac{2x+1}{2x} = x + \overbrace{\frac{1}{2}}^{p_0}$ is analytic at $x_0 = 0$

$(x - x_0)^2 q(x) = x^2\left(-\frac{1}{2x^2}\right) = \underbrace{-\frac{1}{2}}_{q_0}$ — " —

ii| indicial equation

$$\boxed{h(r)} = r^2 + (p_0 - 1)r + q_0 = r^2 - \frac{1}{2}r - \frac{1}{2} =$$

$$= \frac{1}{2}\left[2r^2 - r - 1\right] = \frac{1}{2}\left[(2r+1)(r-1)\right] = 0$$

$$\Rightarrow \boxed{r_1 = 1,\ r_2 = -\frac{1}{2}}$$

iii| recurrence

we seek a solution of the form $\quad y(x) = \sum_{j=0}^{\infty} c_j x^{j+r}$, $\quad c_0 \neq 0$, $x > 0$

$$y'(x)=\sum_{j=0}^{\infty}(j+r)c_j x^{j+r-1} \qquad y''(x)=\sum_{j=0}^{\infty}(j+r-1)(j+r)c_j x^{j+r-2}$$

Note, by Thm 3.42, the radius of convergence R of y is at least as large as $\min(R_p, R_q)=\infty$; i.e $R=\infty$!

$$0=2x^2y''+(2x^2+x)y'-y=$$

$$=\sum_{j=0}^{\infty}2(j+r-1)(j+r)c_j x^{j+r}+\sum_{j=0}^{\infty}2(j+r)c_j x^{j+r+1}+\sum_{j=0}^{\infty}(j+r)c_j x^{j+r}$$

$$-\sum_{j=0}^{\infty}c_j x^{j+r}$$

$$=\sum_{j=0}^{\infty}\left[2(j+r-1)(j+r)+(j+r)-1\right]c_j x^{j+r}+\sum_{j=0}^{\infty}2(j+r)c_j x^{j+r+1}$$

sub $k=j+1$, $j=k-1$

$$=\sum_{j=0}^{\infty}\left[2(j+r)^2-(j+r)-1\right]c_j x^{j+r}+\sum_{k=1}^{\infty}2(k-1+r)c_{k-1}x^{k+r}$$

$$=\left[2r^2-r-1\right]c_0x^r+\sum_{j=1}^{\infty}\left[\left(2(j+r)^2-(j+r)-1\right)c_j+2(j-1+r)c_{j-1}\right]x^{j+r}$$

Divide by x^r:

$$0=\left[2r^2-r-1\right]c_0+\sum_{j=1}^{\infty}\left[\left(2(j+r)^2-(j+r)-1\right)c_j+2(j-1+r)c_{j-1}\right]x^j$$

By the Identity Thm,

$$\left[2r^2-r-1\right]c_0=0, \quad c\neq 0$$

$$\underbrace{\left(2(j+r)^2-(j+r)-1\right)}_{2h(j+r)}c_j+2(j-1+r)c_{j-1}=0 \quad \text{for } j=1,2,\ldots$$

Thus

$$2r^2-r-1=0 \qquad \text{(indicial equation)}$$

(recurrence) $$\boxed{\left[2(j+r)^2-(j+r)-1\right]c_j=-2(j-1+r)c_{j-1} \quad \text{for } j=1,2,\ldots}$$

iv] First solution

$r_1 = 1: \quad \underbrace{\left[2(j+1)^2-(j+1)-1\right]}_{\underbrace{2j^2+4j+\cancel{2}-j\cancel{-1}\cancel{-1}}_{2j^2+3j}} c_j = -2j\,c_{j-1}, \quad \text{for } j=1,2,\dots$

Hence

$$\boxed{c_j = -\frac{2c_{j-1}}{2j+3} \quad \text{for } j=1,2,\dots}$$

$c_0 = 0$

$c_1 = -\frac{2c_0}{5}$

$c_2 = -\frac{2c_1}{7} = (-1)^2\frac{2^2 c_0}{5\cdot 7}$

$c_3 = -\frac{2c_2}{9} = (-1)^3\cdot\frac{2^3 c_0}{5\cdot 7\cdot 9}$

$c_4 = -\frac{2c_3}{11} = (-1)^4\cdot\frac{2^4 c_0}{5\cdot 7\cdot 9\cdot 11}$

$\frac{(-1)^4\cdot 2^4\cdot 3\cdot \overbrace{2\cdot 4\cdot 6\cdot 8\cdot 10}^{\overbrace{1\cdot 2\ \ 2\cdot 2\ \ 3\cdot 2\cdot 4\cdot 2\cdot 5\cdot 2}^{5!\cdot 2^5}}}{\underbrace{3\ 5\ 7\ 9\ 11\ 2\ 4\ 6\ 8\ 10}_{11!}} = \frac{(-1)^4\cdot 3\cdot 2^4\cdot 2^5\cdot 5!\,c_0}{11!}$

or

$$\boxed{c_j = \frac{(-1)^j\cdot 2^j\cdot c_0}{5\cdot 7\cdot\cdots\cdot(2j+3)} = \frac{(-1)^j\cdot 3\cdot 2^{2j+1}\cdot(j+1)!}{(2j+3)!}\,c_0 \quad j=0,1,2,\dots}$$

$$\boxed{y_1(x) = |x|\sum_{j=0}^{\infty}(-1)^j\,\frac{3\cdot 2^{2j+1}\cdot(j+1)!}{(2j+3)!}\,x^j, \qquad |x|<\infty}$$

v] Second solution

$r_2 = -\frac{1}{2}: \quad \underbrace{\left[2(j-\tfrac{1}{2})^2-(j-\tfrac{1}{2})-1\right]}_{\underbrace{2j^2-2j+\cancel{\tfrac{1}{2}}-j+\cancel{\tfrac{1}{2}}\cancel{-1}}_{(2j^2-3j)}} d_j = -2(j-\tfrac{3}{2})\,d_{j-1}, \quad j=1,2,\dots$

$(2j^2-3j)d_j = -(2j-3)d_{j-1}$

$$\underbrace{j}_{\neq 0}\underbrace{(2j-3)}_{\neq 0 \text{ for } j=1,2,\dots} d_j = -(2j-3)d_{j-1} \quad \text{for } j=1,2,3,\dots$$

hence, we can divide by $(2j-3)j$

hence

$$\boxed{d_j = -\frac{d_{j-1}}{j} \quad \text{for } j=1,2,3,\dots}$$

$d_0 \neq 0$

$d_1 = -\frac{d_0}{1}$

$d_2 = -\frac{d_1}{2} = (-1)^2 \frac{d_0}{1 \cdot 2}$

$d_3 = -\frac{d_2}{3} = (-1)^3 \frac{d_0}{1 \cdot 2 \cdot 3}$

hence

$d_0 \neq 0$

$d_j = \frac{(-1)^j d_0}{j!}$

and therefore

$$\boxed{y_2(x) = |x|^{-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^j}$$

b] $4x^2y'' + (4x+1)y = 0$

i] $x_0 = 0$ is a regular singular point:

$$y'' + \underbrace{\frac{0}{4x^2}}_{p(x)=0} y' + \underbrace{\frac{4x+1}{4x^2}}_{q(x)} y = 0$$

$(x-x_0)p(x) = 0 = p_0$ is analytic at $x_0=0$

$(x-x_0)^2 q(x) = x^2 \cdot \frac{4x+1}{4x^2} = \underbrace{x+\frac{1}{4}}_{= q_0}$ is analytic at $x_0=0$

Note $p(x) = 0$,
$q(x) = x + \frac{1}{4}$

are power series with radius of convergence $R_p = R_q = \infty$

ii] indicial equation

$$\boxed{h(r) = r^2 + (p_0-1)r + q_0 = r^2 - r + \frac{1}{4} = (r-\frac{1}{2})^2 = 0}$$

hence $\boxed{r_1 = r_2 = \frac{1}{2}}$

iii] recurrence

we seek a solution of the form $y(x) = \sum_{j=0}^{\infty} c_j x^{j+r}$, $c_0 \neq 0$, $x>0$

$y'(x) = \sum_{j=0}^{\infty} (j+r)c_k x^{j+r-1}$, $y''(x) = \sum_{j=0}^{\infty} (j+r)(j+r-1)c_j x^{j+r-2}$

Note: by Thm 3.42, the radius of convergence R of y is $\geq \min(R_p, R_q) = \infty$, hence $R = \infty$!

$$0 = 4x^2y'' + (4x+1)y = \sum_{j=0}^{\infty} 4(j+r)(j+r-1)c_j x^{j+r} +$$
$$+ \sum_{j=0}^{\infty} 4c_j x^{j+r+1} + \sum_{j=0}^{\infty} c_j x^{j+r}$$
$$= \sum_{j=0}^{\infty} [\underbrace{4(j+r)(j+r-1)+1}_{4(j+r)^2-4(j+r)+1}]c_j x^{j+r} + \sum_{j=0}^{\infty} 4c_j x^{j+r+1}$$

sub $k=j+1$, $j=k-1$

$$= \sum_{j=0}^{\infty} \left[4(j+r)^2 - 4(j+r)+1\right] c_j x^{j+r} + \sum_{k=1}^{\infty} 4 c_{k-1} x^{k+r}$$

$$= x^r \left[(4r^2 - 4r + 1) c_0 + \sum_{j=1}^{\infty} \left(\left(4(j+r)^2 - 4(j+r)+1\right) c_j + 4 c_{j-1} \right) x^j \right.$$

Divide by x^r, then by Identity Thm,

$$\underbrace{(4r^2 - 4r + 1)}_{4h(r)} \underset{\substack{\ne \\ 0}}{c_0} = 0$$

$$\underbrace{\left(4(j+r)^2 - 4(j+r)+1\right)}_{4h(j+r)} c_j = -4 c_{j-1} \quad \text{for } j = 1, 2, 3, \dots$$

Thus

$$\underbrace{4r^2 - 4r + 1}_{4h(r)} = 0 \qquad \text{(indicial eqn)}$$

$$\boxed{\left(4(j+r)^2 - 4(j+r)+1\right) c_j = -4 c_{j-1} \quad \text{for } j = 1, 2, 3 \dots}$$

iv) First solution

$r_1 = \frac{1}{2}$: $\quad \underbrace{\left[4\left(j+\tfrac{1}{2}\right)^2 - 4\left(j+\tfrac{1}{2}\right)+1\right]}_{4j^2} c_j = -4 c_{j-1}$

$$\boxed{c_j = -\frac{c_{j-1}}{j^2} \quad \text{for } j = 1, 2, 3, \dots}$$

$c_0 \neq 0$

$$c_1 = -\frac{c_0}{1} = -c_0$$

$$c_2 = -\frac{c_1}{2^2} = (-1)^2 \frac{c_0}{1^2 \cdot 2^2}$$

$$c_3 = -\frac{c_2}{3^2} = (-1)^3 \cdot \frac{c_0}{1^2 \cdot 2^2 \cdot 3^2}$$

Hence

$$c_j = \frac{(-1)^j c_0}{[j!]^2}$$

$$\boxed{y_1(x) = |x|^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j}{[j!]^2} x^j}$$

V) Second solution: we are in case 3 of Thm 3.42!

$r_1 = r_2 = \frac{1}{2}$

we seek a solution of the form

$$y(x) = \sum_{j=1}^{\infty} d_j x^{j+\frac{1}{2}} + y_1(x) \ln x, \qquad x > 0$$

of $4x^2 y'' + (4x+1) y = 0$

$$y'(x) = \sum_{j=1}^{\infty} (j+\tfrac{1}{2}) d_j x^{j-\frac{1}{2}} + y_1'(x) \ln x + \frac{1}{x} y_1(x)$$

$$y''(x) = \sum_{j=1}^{\infty} \underbrace{(j+\tfrac{1}{2})(j-\tfrac{1}{2})}_{j^2 - \frac{1}{4}} d_j x^{j-\frac{3}{2}} + y_1''(x) \ln x + \frac{2}{x} y_1'(x) - \frac{1}{x^2} y_1(x)$$

$$4x^2y''(x) = \sum_{j=1}^{\infty}(4j^2-1)d_j x^{j+\frac{1}{2}} + 4x^2y_1''(x)\ln x + 8xy_1'(x) - 4y_1(x)$$

$$4x\, y(x) = \sum_{j=1}^{\infty} 4 d_j x^{j+\frac{3}{2}} + 4xy_1(x)\ln x$$

$$y(x) = \sum_{j=1}^{\infty} d_j\, x^{j+\frac{1}{2}} + y_1(x)\ln x$$

Thus

$$0 = 4x^2y'' + (4x+1)y =$$

$$= \sum_{j=1}^{\infty}\Big[\underbrace{(4j^2-1)+1}_{4j^2}\Big]d_j x^{j+\frac{1}{2}} + 4x^2y_1''(x)\ln x + 8xy_1'(x) - 4y_1(x) + y_1(x)\ln x$$

$$+ \sum_{j=1}^{\infty} 4d_j x^{j+\frac{3}{2}} + 4xy_1(x)\ln x$$

$$= \sum_{j=1}^{\infty} 4j^2 d_j x^{j+\frac{1}{2}} + \sum_{j=1}^{\infty} 4d_j x^{j+\frac{3}{2}} + \overbrace{\left[4x^2y_1''(x) + (4x+1)y_1(x)\right]}^{0}\ln x$$

sub $k=j+1$, $j=k-1$

$$+ 8xy_1'(x) - 4y_1(x)$$

$$= \sum_{j=1}^{\infty} 4j^2 d_j x^{j+\frac{1}{2}} + \sum_{k=2}^{\infty} 4 d_{k-1} x^{k+\frac{1}{2}} + 8xy_1'(x) - 4y_1(x)$$

*)
$$= x^{\frac{1}{2}}\Bigg[(4d_1x + 4 - 12x - 4 + 4x) + \sum_{j=2}^{\infty} 4j^2 d_j x^j + \sum_{k=2}^{\infty} 4d_{k-1}x^k + \sum_{j=2}^{\infty}\frac{8(-1)^j(j+\frac{1}{2})}{[j!]^2}x^j - \sum_{j=2}^{\infty}\frac{4(-1)^j}{[j!]^2}x^j\Bigg]$$

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*) Since,

$$8xy_1'(x) = \sum_{j=0}^{\infty}\frac{8(-1)^j(j+\frac{1}{2})}{[j!]^2}x^{j+\frac{1}{2}}$$

$$-4y_1(x) = -\sum_{j=0}^{\infty}\frac{4(-1)^j}{[j!]^2}x^{j+\frac{1}{2}}$$

$$= x^{\frac{1}{2}}\left[(4d_1-8)x+\sum_{j=2}^{\infty}\left(4j^2d_j+4d_{j-1}+\frac{8(-1)^j}{(j-1)!\,j!}\right)x^j\right]$$

Divide by $x^{\frac{1}{2}}$, then by the Identity Thm,

$$4d_1-8=0$$

$$\cancel{4}^{1}j^2d_j+\cancel{4}^{1}d_{j-1}+\frac{\cancel{8}^{2}(-1)^j}{(j-1)!\,j!}=0 \quad \text{for } j=2,3,\cdots$$

Thus

$$\boxed{\begin{aligned} &d_1=2 \\ &d_j=-\frac{d_{j-1}+2(-1)^j}{j^2(j-1)!\,j!} \qquad j=2,3,\ldots \end{aligned}}$$

$$d_1=2$$

$$d_2=-\frac{2+2}{4\cdot 2}=-\frac{4}{8}=-\frac{1}{2}$$

$$d_3=-\frac{\left(-\frac{1}{2}-2\right)\cdot 2}{(9\cdot 2\cdot 6)\,2}=\frac{3}{216}$$

$$d_4=-\frac{\left(\frac{3}{216}+2\right)\cdot 216}{(16\cdot 6\cdot 24)\cdot 216}=\frac{435}{497664}$$

$$\vdots$$

$$\boxed{y_2(x)=|x|^{\frac{1}{2}}\sum_{j=1}^{\infty}d_jx^j}$$