2.5.2 Exercises

1] Find the directional derivative of $f(x_1,x_2,x_3) = x_1 e^{x_2} + x_2 e^{x_3} + x_3 e^{x_1}$ at $(0,0,0)$ in the direction $\vec{u} = 5\vec{e}_1 + \vec{e}_2 - 2\vec{e}_3$

Solution: a] compute the gradient of f at $(0,0,0)$:

$$\nabla f(x_1,x_2,x_3) = \begin{pmatrix} e^{x_2} + x_3 e^{x_1} \\ x_1 e^{x_2} + e^{x_3} \\ x_2 e^{x_3} + e^{x_1} \end{pmatrix} \Rightarrow \nabla f(0,0,0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

b] normalize direction vector:

$$\|\vec{u}\| = \sqrt{25+1+4} = \sqrt{30}$$

Let $$\vec{d} := \frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{30}} \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$

c] directional derivative

$$\boxed{D_d(0,0,0)} = \nabla f(0,0,0) \cdot \vec{d} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \left[\frac{1}{\sqrt{30}} \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} \right] =$$

$$= \frac{4}{\sqrt{30}} = \boxed{\frac{2}{15}\sqrt{30}}$$

2] Find the maximum rate of change of $f(x_1,x_2,x_3) := \sqrt{x_1^2+x_2^2+x_3^2}$ at $(3,6,-2)$.

Solution: We use Thm 2.17 and compute $\|\nabla f(3,6,-2)\|$:

$$\nabla f(x_1,x_2,x_3) = \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}} \end{pmatrix} = \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Thus

$$\nabla f(3,6,-2) = \frac{1}{\sqrt{9+36+4}} \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}$$

and therefore $\boxed{\|\nabla f(3,6,-2)\|} = \left\| \frac{1}{7} \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} \right\| = \boxed{1}$

3] Given $x_1^2 - 2x_2^2 + x_3^2 + x_2 x_3 = 2$

a] Find the equation of the tangent plane at $(2,1,-1)$

Solution: $\phi(x_1, x_2, x_3) = x_1^2 - 2x_2^2 + x_3^2 + x_2 x_3$

Note $\phi(2,1,-1) = 4 - 2 + 1 - 1 = 2$, hence $(2,1,-1)$ lies on the surface defined by the given equation.

By Def 2.19, $\nabla\phi(2,1,-1)$ is normal to surface at $(2,1,-1)$.

$$\nabla\phi(x_1,x_2,x_3) = \begin{pmatrix} 2x_1 \\ -4x_2 + x_3 \\ 2x_3 + x_2 \end{pmatrix}, \text{ hence } \nabla\phi(2,1,-1) = \begin{pmatrix} 4 \\ -5 \\ -1 \end{pmatrix}$$

Therefore, the tangent plane to the surface at $(2,1,-1)$ is given by

$$\begin{pmatrix} 4 \\ -5 \\ -1 \end{pmatrix} \cdot \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right] = 0$$

$$\Leftrightarrow \quad \boxed{4x_1 - 5x_2 - x_3 = 4}$$

b] Find the equation of the normal line to the surface at $(2,1,-1)$

Solution: The direction vector of the normal line is $\vec{d} := \nabla\phi(2,1,-1) = \begin{pmatrix} 4 \\ -5 \\ -1 \end{pmatrix}$, hence the line is given by

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ -1 \end{pmatrix}}$$

4] Write $\dfrac{\partial\phi}{\partial x_1} = b_1$, $\dfrac{\partial\phi}{\partial x_2} = b_2$, $\dfrac{\partial\phi}{\partial x_3} = b_3$ in indicial notation.

Solution $\quad \boxed{\phi_{,i} = b_i}$

5] Write

$$\frac{\partial^2\phi}{\partial x_1^2} - A_{11} = 0 \quad \frac{\partial^2\phi}{\partial x_1\partial x_2} - A_{12} = 0 \quad \frac{\partial^2\phi}{\partial x_1\partial x_3} - A_{13} = 0$$

$$\frac{\partial^2\phi}{\partial x_2\partial x_1} - A_{21} = 0 \quad \frac{\partial^2\phi}{\partial x_2\partial x_2} - A_{22} = 0 \quad \frac{\partial^2\phi}{\partial x_2\partial x_3} - A_{23} = 0$$

$$\frac{\partial^2\phi}{\partial x_3\partial x_1} - A_{31} = 0 \quad \frac{\partial^2\phi}{\partial x_3\partial x_2} - A_{32} = 0 \quad \frac{\partial^2\phi}{\partial x_3^2} - A_{33} = 0$$

in matrix and indicial notation.

a] in matrix notation

Note $\nabla^t\phi = (\phi_{,1}, \phi_{,2}, \phi_{,3})$, therefore (by abuse of notation)

$$\nabla\nabla^t\phi = (\nabla\phi_{,1}, \nabla\phi_{,2}, \nabla\phi_{,3})$$

$$= \begin{pmatrix} \phi_{,11} & \phi_{,21} & \phi_{,31} \\ \phi_{,12} & \phi_{,22} & \phi_{,32} \\ \phi_{,13} & \phi_{,23} & \phi_{,33} \end{pmatrix} =$$

$$= \begin{bmatrix} \frac{\partial^2\phi}{\partial x_1^2} & \frac{\partial^2\phi}{\partial x_1\partial x_2} & \frac{\partial^2\phi}{\partial x_1\partial x_3} \\ \frac{\partial^2\phi}{\partial x_2\partial x_1} & \frac{\partial^2\phi}{\partial x_2^2} & \frac{\partial^2\phi}{\partial x_2\partial x_3} \\ \frac{\partial^2\phi}{\partial x_3\partial x_1} & \frac{\partial^2\phi}{\partial x_3\partial x_2} & \frac{\partial^2\phi}{\partial x_3^2} \end{bmatrix}$$

Hence

$$\boxed{\nabla\nabla^t\phi - A = \Theta}$$

b] in indicial notation

$$\boxed{\phi_{,ji} - A_{ij} = 0}$$