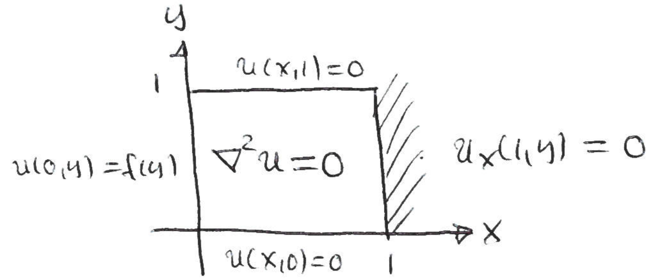
4.2.4 Exercises

1] $u_{xx}(x,y) + u_{yy}(x,y) = 0$, $\quad (x,y) \in (0,1) \times (0,1)$

bcs
$$u(0,y) = f(y) = y(1-y)$$
$$u_x(1,y) = 0$$
$$u(x,0) = 0$$
$$u(x,1) = 0$$



<u>solution</u>: a] sep. variables

$$u(x,y) = X(x)\,Y(y) \Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0 \quad \Big|\cdot \frac{1}{X(x)Y(y)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \mu$$

$$\Rightarrow \quad X''(x) - \mu X(x) = 0$$
$$Y''(y) + \mu Y(y) = 0$$

b] translate bcs

$$X'(1)Y(y) = 0 \;\forall y \Rightarrow X'(1) = 0$$
$$X(x)Y(0) = 0 \;\forall x \Rightarrow Y(0) = 0$$
$$X(x)Y(1) = 0 \;\forall x \Rightarrow Y(1) = 0$$

2 hom. bc. on y !

c] solve SL problem

$$\left.\begin{aligned} &Y''(y) + \mu Y(y) = 0 \\ &Y(0) = 0 \\ &Y(1) = 0 \end{aligned}\right\} \xrightarrow[\mu=\lambda^2,\ L=1]{\text{SL table p 215}} \quad \boxed{\lambda_n = (n+1)\pi} \quad \boxed{Y_n(y) = \sin\big((n+1)\pi y} \quad n = 0,1,2,\ldots$$

d] solve remaining eqn

$$X''(x) - \mu_n X(x) = 0, \qquad \mu_n = (n+1)^2\pi^2 > 0$$
$$X'(1) = 0 \qquad \text{char. eqn} \quad r^2 = (n+1)^2\pi^2$$

Note: boundary cond. on non-zero boundary, use shifted basis funct.!

$$X_n(x) = C_1^{(n)}\cosh\big((n+1)\pi(x-1)\big) + C_2^{(n)}\sinh\big((n+1)\pi(x-1)\big)$$
$$X_n'(x) = C_1^{(n)}(n+1)\pi\sinh\big((n+1)\pi(x-1)\big) + C_2^{(n)}(n+1)\pi\cosh\big((n+1)\pi(x-1)\big)$$

e] apply bc

$$0 = X_n'(1) = C_2^{(n)}(n+1)\pi \quad \Rightarrow \quad C_2^{(n)} = 0 \,, \quad C_1^{(n)} \neq 0$$

hence

$$\boxed{X_n(x) = \cosh\left((n+1)\pi(x-1)\right)} \quad n = 0, 1, 2, \ldots$$

f] gen. solution

$$u_n(x,y) = X_n(x)\, Y_n(y) = \cosh\left((n+1)\pi x\right) \sin\left((n+1)\pi y\right)$$

$$\boxed{u(x,y) = \sum_{n=0}^{\infty} C_n X_n(x) Y_n(y)}$$

g] apply nonhom. bc :

$$f(y) = u(0,y) = \sum_{n=0}^{\infty} C_n \underbrace{\cosh\left((n+1)\pi(0-1)\right)}_{\cosh((n+1)\pi)} \sin\left((n+1)\pi y\right)$$

$$= \sum_{n=0}^{\infty} C_n \cosh((n+1)\pi) \sin\left((n+1)\pi y\right)$$

write

$$\boxed{C_n = \frac{\int_0^1 f(y) \sin\left((n+1)\pi y\right) dy}{\cosh((n+1)\pi) \int_0^1 \sin^2\left((n+1)\pi y\right) dy}} \Rightarrow \frac{1}{2}$$

Note $$\int_0^L \underbrace{\sin^2\left(\frac{(n+1)\pi y}{L}\right)}_{\frac{1}{2}\left(1 - \cos\left(\frac{2(n+1)\pi y}{L}\right)\right)} dy = \frac{1}{2}\left[y - \frac{L}{2(n+1)\pi} \sin\left(\frac{2(n+1)\pi y}{L}\right)\right]_0^L$$

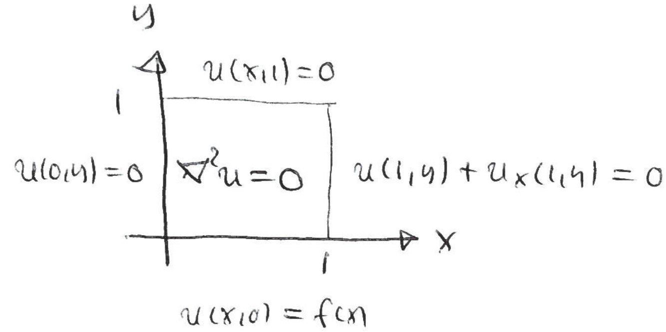
$$= \frac{1}{2} L = \frac{L}{2}$$

~~for~~ $f(y) = y(1-y)$

$$C_n = 2 \int_0^1 y(1-y) \sin\left((n+1)\pi y\right) dy$$

2) $u_{xx}(x,y) + u_{yy}(x,y) = 0$

bcs
$$\left.\begin{aligned} u(0,y) &= 0 \\ u(1,y) + u_x(1,y) &= 0 \end{aligned}\right] \text{hom bcs!}$$
$$u(x,1) = 0$$
$$u(x,0) = f(x) = x\left(1-\tfrac{2}{3}x\right)$$



Solution

a) sep. of var.: $u(x,y) = X(x)\,Y(y) \Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0 \;\Big|\, \frac{1}{X(x)Y(y)}$

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\mu$$

$$\Rightarrow \begin{aligned} X''(x) + \mu X(x) &= 0 \\ Y''(y) - \mu Y(y) &= 0 \end{aligned}$$

b) transl. bcs:

$X(0)Y(y) = 0 \;\forall y \Rightarrow X(0) = 0$

$[X(1) + X'(1)]Y(y) = 0 \;\forall y \Rightarrow X(1) + X'(1) = 0$

$X(x)Y(1) = 0 \Rightarrow Y(1) = 0$

c) solve SL problem

$$\left.\begin{aligned} X''(x) + \mu X(x) &= 0 \\ X(0) &= 0 \\ X(1) + X'(1) &= 0 \end{aligned}\right\} \underset{\substack{p.215 \\ \mu=\lambda^2 \\ L=1 \\ \alpha_2=\beta_2=1}}{\overset{\text{SL-table}}{\Longrightarrow}} \boxed{\begin{aligned} &\lambda_n > 0 \text{ pos. sol. of} \\ &\sin(\lambda) + \lambda\cos(\lambda) = 0 \end{aligned}}$$

$$\boxed{X_n(x) = \sin(\lambda_n x)}$$

$n = 0, 1, 2, \ldots$

$\mu_n = \lambda_n^2$

d) solve remaining eqn

$$Y''(y) - \mu_n Y(y) = 0$$
$$Y(1) = 0$$

Note bc on nonzero-boundary use shifted basis functions!

char. eqn $r^2 = \lambda_n^2$, $\lambda_n > 0$

$$Y_n(y) = C_1^{(n)} \cosh\left(\lambda_n(y-1)\right) + C_2^{(n)} \sinh\left(\lambda_n(y-1)\right)$$

$$0 = y(1) = C_1^{(n)} \quad \Rightarrow \quad C_1^{(n)} = 0, \quad C_2^{(n)} \neq 0$$

Hence

$$\boxed{Y_n(y) = \sinh\left(\lambda_n(y-1)\right)} \qquad n = 0, 1, 2, \cdots$$

e) gen. sol.

$$u_n(x,y) = X_n(x) Y_n(y) = \sin(\lambda_n x) \sinh\left(\lambda_n(y-1)\right)$$

$$\boxed{u_n(x,y) = \sum_{n=0}^{\infty} C_n u_n(x,y)}$$

f) apply nonhom. bc.

note sinh is odd!

$$f(x) = u_n(x,0) = \sum_{n=0}^{\infty} C_n \sin(\lambda_n x) \sinh\left(\lambda_n(0-1)\right)$$

$$= \sum_{n=0}^{\infty} \left[-C_n \sinh(\lambda_n)\right] \sin(\lambda_n x)$$

where

$$-C_n \sinh(\lambda_n) = \frac{\int_0^1 f(x) \sin(\lambda_n x)\, dx}{\int_0^1 \sin^2(\lambda_n x)\, dx}$$

i.e.

$$\boxed{C_n = -\frac{\int_0^1 f(x) \sin(\lambda_n x)\, dx}{\sinh(\lambda_n) \int_0^1 \sin^2(\lambda_n x)\, dx}}$$

for $f(x) = x\left(1 - \frac{2}{3}x\right)$

$$C_n = -\frac{\int_0^1 x\left(1-\frac{2}{3}x\right) \sin(\lambda_n x)\, dx}{\sinh(\lambda_n) \int_0^1 \sin^2(\lambda_n x)\, dx}$$

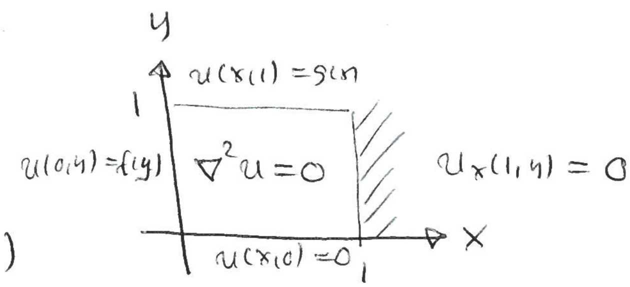
3] $u_{xx}(x,y) + u_{yy}(x,y) = 0$

bcs: $u_x(1,y) = 0$

$u(0,y) = f(y) = y(1-y)$

$u(x,0) = 0$

$u(x,1) = g(x) = x(1-\frac{1}{2}x)$



Solution:

I. Partition solution $\boxed{u = v + w}$

Subproblem (A)

$v_{xx}(x,y) + v_{yy}(x,y) = 0$

$v_x(1,y) = 0$

$v(0,y) = 0$] 2 hom. bcs in x!

$v(x,0) = 0$

$v(x,1) = g(x)$

Subproblem (B)

$w_{xx}(x,y) + w_{yy}(x,y) = 0$

$w_x(1,y) = 0$

$w(0,y) = f(y)$

$w(x,0) = 0$

$w(x,1) = 0$] 2 hom. bcs in y!

II. Solve subproblem

A. a] sep. var. $v(x,y) = X(x)Y(y) \Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0$

$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\mu \Rightarrow X''(x) + \mu X(x) = 0,\ Y''(y) - \mu Y(y) = 0$

b] transf. bcs: $X'(1) = 0,\ X(0) = 0,\ Y(0) = 0,$

c] solve SLP: $X''(x) + \mu X(x) = 0,\ X(0) = 0,\ X'(1) = 0$

by SL table p215 with $\mu = \lambda^2,\ L = 1$: $\boxed{\mu_m^{(A)} = \left[(2m+1)\frac{\pi}{2}\right]^2},$

$\boxed{X_m^{(A)}(x) = \sin\left(\frac{(2m+1)\pi}{2}x\right)},$ for $m = 0, 1, 2, \ldots$

d] solve rem. eqn: $y''(y) - \mu_m^{(A)} y(y) = 0, \quad \mu_m^{(A)} > 0, \quad m = 0,1,2,\ldots$

$$y_m(y) = C_1 \cosh\left(\frac{(2m+1)\pi}{2} y\right) + C_2 \sinh\left(\frac{(2m+1)\pi}{2} y\right)$$

$$0 = y_m(0) = C_1 \Rightarrow C_1 = 0, \; C_2 \neq 0, \text{ hence}$$

$$\boxed{y_m^{(A)}(y) = \sinh\left(\frac{(2m+1)\pi}{2} y\right)}, \quad m = 0,1,2,\ldots$$

e] gen. solution: $v_m(x,y) = X_m^{(A)}(x)\, y_m^{(A)}(y) = \sin\left(\frac{(2m+1)\pi}{2} x\right) \sinh\left(\frac{(2m+1)\pi}{2} y\right)$

$$\boxed{v(x,y) = \sum_{m=0}^{\infty} A_m v_m(x,y) = \sum_{m=0}^{\infty} A_m X_m^{(A)}(x)\, y_m^{(A)}(y)}$$

apply nonhom. bc:

$$g(x) = v(x,1) = \sum_{m=0}^{\infty} A_m X_m^{(A)} y_m^{(A)}(1) =$$

$$= \sum_{m=0}^{\infty} \left[A_m \sinh\left(\frac{(2m+1)\pi}{2}\right)\right] \sin\left(\frac{(2m+1)\pi}{2} x\right)$$

hence

$$A_m \sinh\left(\frac{(2m+1)\pi}{2}\right) = \frac{\int_0^1 g(x) \sin\left(\frac{(2m+1)\pi}{2} x\right) dx}{\underbrace{\int_0^1 \sin^2\left(\frac{(2m+1)\pi}{2} x\right) dx}_{=\frac{1}{2}}}$$

and thus

$$\boxed{A_m = \frac{2}{\sinh\left(\frac{(2m+1)\pi}{2}\right)} \int_0^1 g(x) \sin\left(\frac{(2m+1)\pi}{2} x\right) dx}$$

for $m = 0,1,2,\ldots$

B. a] sep. var. $w(x,y) = X(x)\, y(y) \Rightarrow X''(x)\, y(y) + X(x)\, y''(y) = 0$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{y''(y)}{y(y)} = \mu \Rightarrow X''(x) - \mu X(x) = 0, \quad y''(y) + \mu y(y) = 0$$

b] transl. bcs: $X'(1) = 0, \; y(0) = 0, \; y(1) = 0$

c] solve SL problem: $y''(y) + \mu y(y) = 0, \; y(0) = 0, \; y(1) = 0$

by SL table p215 with $\mu = \lambda^2, L = 1$: $\boxed{\mu_n^{(B)} = \left[(n+1)\pi\right]^2},$

$\boxed{y_n^{(B)}(y) = \sin((n+1)\pi y)},$ for $n = 0,1,2,\ldots$

d] solve rem. eqn: $X''(x) - \mu_n^{(B)} X(x) = 0$, $\mu_n^{(B)} > 0$, $n = 0,1,2,\cdots$
with $X'(1) = 0$ Note bc is on non-zero boundary!

$$X_n(x) = C_1 \cosh\left((n+1)\pi(x-1)\right) + C_2 \sinh\left((n+1)\pi(x-1)\right)$$

$$X_n'(x) = C_1 (n+1)\pi \sinh\left((n+1)\pi(x-1)\right) + C_2 (n+1)\pi \cosh\left((n+1)\pi(x-1)\right)$$

$0 = X_n'(1) = C_2(n+1)\pi \Rightarrow C_2 = 0$, $C_1 \neq 0$, hence

$$\boxed{X_n^{(B)}(x) = \cosh\left((n+1)\pi(x-1)\right),} \quad n = 0,1,2,\ldots$$

e] gen. solution: $w_n(x,y) = X_n^{(B)}(x)\, Y_n^{(B)}(y) = \cosh\left((n+1)\pi(x-1)\right)\sin\left((n+1)\pi y\right)$

$$\boxed{w(x,y) = \sum_{n=0}^{\infty} B_n w_n(x,y) = \sum_{n=0}^{\infty} B_n X_n^{(B)}(x)\, Y_n^{(B)}(y)}$$

apply nonh. bc:

$$f(y) = w(0,y) = \sum_{n=0}^{\infty} B_n X_n^{(B)}(0)\, Y_n^{(B)}(y) =$$

$$= \sum_{n=0}^{\infty}\Big[B_n \underbrace{\cosh\left((n+1)\pi(-1)\right)}_{=\cosh((n+1)\pi)}\Big]\sin\left((n+1)\pi y\right)$$ — note: cosh is even!

hence

$$B_n \cosh((n+1)\pi) = \frac{\int_0^1 f(y)\sin((n+1)\pi y)\,dy}{\underbrace{\int_0^1 \sin^2((n+1)\pi y)\,dy}_{=\frac{1}{2}}}$$

and thus

$$\boxed{B_n = \frac{2}{\cosh((n+1)\pi)} \int_0^1 f(y)\sin\left((n+1)\pi y\right)dy}$$

for $n = 0,1,2,\cdots$

III. Combine solutions of subproblems

$$\boxed{\begin{aligned} u(x,y) &= v(x,y) + w(x,y) \\ &= \sum_{m=0}^{\infty} A_m \sin\left(\frac{(2m+1)\pi}{2}x\right)\sinh\left(\frac{(2m+1)\pi}{2}y\right) \\ &\quad + \sum_{n=0}^{\infty} B_n \sin\left((n+1)\pi y\right)\cosh\left((n+1)\pi(x-1)\right) \end{aligned}}$$

with

$$\boxed{A_m = \frac{2}{\sinh\left(\frac{(2m+1)\pi}{2}\right)} \int_0^1 g(x)\sin\left(\frac{(2m+1)\pi}{2}x\right)dx}$$

$$\boxed{B_n = \frac{2}{\cosh((n+1)\pi)} \int_0^1 f(y)\sin\left((n+1)\pi y\right)dy}$$

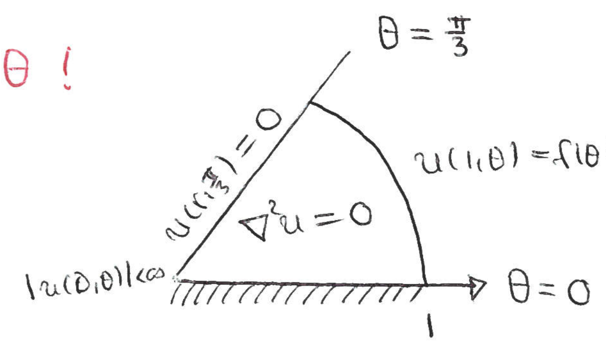
for $m,n = 0,1,2,\cdots$

4) $$\frac{u_r(r,\theta) + r\,u_{rr}(r,\theta)}{r} + \frac{u_{\theta\theta}(r,\theta)}{r^2} = 0\,, \quad (r,\theta)\in(0,1)\times\left(0,\tfrac{\pi}{3}\right)$$

$$\left.\begin{aligned} &u_\theta(r,0) = 0 \\ &u\left(r,\tfrac{\pi}{3}\right) = 0 \end{aligned}\right] \text{ hom. bcs. in } \theta\,!$$

$$|u(0,\theta)| < \infty$$

$$u(1,\theta) = f(\theta) = 1 - \frac{9}{\pi^2}\theta^2$$



Solution:

a) sep. variables: $u(r,\theta) = R(r)\,\Theta(\theta)$

$$\frac{R'(r)\,\Theta(\theta) + r\,R''(r)\,\Theta(\theta)}{r} + \frac{R(r)\,\Theta''(\theta)}{r^2} = 0 \qquad \Bigg|\; \frac{r^2}{R(r)\,\Theta(\theta)}$$

$$\frac{rR'(r) + r^2R''(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda^2$$

$$\Rightarrow r^2R''(r) + rR'(r) - \lambda^2 R(r) = 0$$

$$\Theta''(\theta) + \lambda^2 \Theta(\theta) = 0$$

b] transl. bcs

$$\Theta'(0) = 0, \quad \Theta\left(\frac{\pi}{3}\right) = 0, \quad |R(0)| < \infty$$

c] solve SL problem:

$$\left.\begin{array}{l} \Theta''(\theta) + \lambda^2 \Theta(\theta) = 0 \\ \Theta'(0) = 0, \quad \Theta\left(\frac{\pi}{3}\right) = 0 \end{array}\right\} \xrightarrow[L=\frac{\pi}{3}]{\text{SL table p215}} \begin{array}{l} \boxed{\lambda_n = \frac{(2n+1)\pi}{2\cdot\frac{\pi}{3}} = \frac{3}{2}(2n+1)} \\ \boxed{\Theta_n(\theta) = \cos\left(\frac{3}{2}(2n+1)\theta\right)} \end{array}$$

for $n = 0, 1, 2, \ldots$

d] solve remaining eqn:

$$r^2R''(r) + rR'(r) - \lambda_n^2 R(r) = 0, \quad \lambda_n > 0$$

$|R(0)| < \infty$ $\quad$ ($\Rightarrow$ char eqn $r(r-1) + r - \lambda_n^2 = 0$ $\Leftrightarrow$ $r^2 = \lambda_n^2$)

$$R_n(r) = C_1^{(n)} r^{\frac{3}{2}(2n+1)} + C_2^{(n)} r^{-\frac{3}{2}(2n+1)}$$

since $|R(0)| < \infty$ $\Rightarrow$ $C_2^{(n)} = 0$, $C_1^{(n)} \neq 0$

hence $\boxed{R_n(r) = r^{\frac{3}{2}(2n+1)},}$ $\quad n = 0, 1, 2, \ldots$

e] gen. solution: $\quad u_n(r,\theta) = R_n(r)\,\Theta_n(\theta)$

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n u_n(r,\theta) = \sum_{n=0}^{\infty} A_n R_n(r)\,\Theta_n(\theta)$$

f] apply nonhom. bc:

$$f(\theta) = u(1,\theta) = \sum_{n=0}^{\infty} A_n R_n(1)\,\Theta(\theta) = \sum_{n=0}^{\infty} A_n \cos\left(\tfrac{3}{2}(2n+1)\theta\right)$$

hence

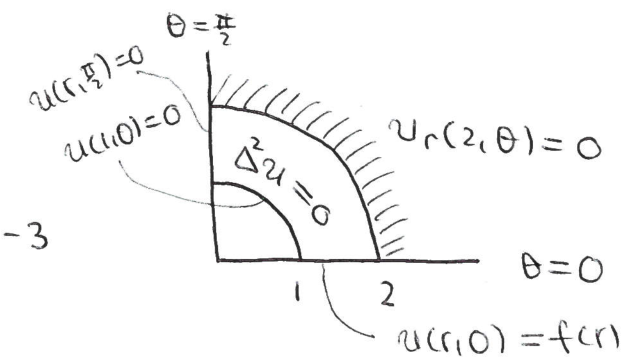
$$\boxed{A_n = \frac{\int_0^{\frac{\pi}{3}} f(\theta)\cos\left(\frac{3}{2}(2n+1)\theta\right)d\theta}{\underbrace{\int_0^{\frac{\pi}{3}} \cos^2\left(\frac{3}{2}(2n+1)\theta\right)d\theta}_{=\frac{\pi}{6}}}}$$

$$\boxed{= \frac{6}{\pi}\int_0^{\frac{\pi}{3}} f(\theta)\cos\left(\frac{3}{2}(2n+1)\theta\right)d\theta}$$

5]

$$\frac{u_r(r,\theta) + r\,u_{rr}(r,\theta)}{r} + \frac{u_{\theta\theta}(r,\theta)}{r^2} = 0, \qquad (r,\theta)\in(1,2)\times(0,\tfrac{\pi}{2})$$

$u(1,\theta) = 0$
$u_r(2,\theta) = 0$] 2 hom. bc in r !
$u(r,\frac{\pi}{2}) = 0$
$u(r,0) = f(r) = -r^2+4r-3$



Solution :

a] sep var. : $u(r,\theta) = R(r)\,\Theta(\theta)$

$$\frac{R'(r)\,\Theta(\theta) + r\,R''(r)\,\Theta(\theta)}{r} + \frac{R(r)\,\Theta''(\theta)}{r^2} = 0 \qquad \Big| \; \frac{r^2}{R(r)\,\Theta(\theta)}$$

$$\Rightarrow \frac{r R'(r) + r^2 R''(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda^2$$

$$\Rightarrow r^2R''(r) + rR'(r) + \lambda^2 R(r) = 0$$

$$\Theta''(\theta) - \lambda^2\Theta(\theta) = 0$$

b] transl. bcs

$R(1) = 0,\ R'(2) = 0,\ \Theta(\frac{\pi}{2}) = 0$

c) solve SL problem

$$\left.\begin{array}{l} r^2R''(r)+rR'(r)+\lambda^2R(r)=0 \\ R(1)=0,\quad R'(2)=0 \end{array}\right\} \xrightarrow[L=2,\ w(r)=\frac{1}{r}]{\text{SL table, p 219}} \boxed{\lambda_n=\frac{(2n+1)\pi}{2\ln 2}}$$

$$\boxed{R_n(r)=\sin\left(\frac{(2n+1)\pi}{2\ln 2}\ln(r)\right)}$$

d) solve remaining eqn

$\Theta''(\theta)-\lambda_n^2\Theta(\theta)=0,\qquad$ char. eqn $\quad \rho^2=\lambda_n^2,\ \lambda_n>0$

$\Theta(\frac{\pi}{2})=0\qquad$ Note: non-zero boundary!

$$\Theta_n(\theta)=C_1\cosh\left(\frac{(2n+1)\pi}{2\ln 2}(\theta-\tfrac{\pi}{2})\right)+C_2\sinh\left(\frac{(2n+1)\pi}{2\ln 2}(\theta-\tfrac{\pi}{2})\right)$$

$$0=\Theta(\tfrac{\pi}{2})=C_1 \Rightarrow C_1=0,\ C_2\neq 0$$

hence

$$\boxed{\Theta_n(\theta)=\sinh\left(\frac{(2n+1)\pi}{2\ln 2}(\theta-\tfrac{\pi}{2})\right)}\qquad \text{for } n=0,1,2,\ldots$$

e) gen. sol.

$$u_n(r,\theta)=R_n(r)\,\Theta_n(\theta)=\sin\left(\frac{(2n+1)\pi}{2\ln 2}\ln(r)\right)\sinh\left(\frac{(2n+1)\pi}{2\ln 2}(\theta-\tfrac{\pi}{2})\right)$$

$$u(r,\theta)=\sum_{n=0}^{\infty}A_nu_n(r,\theta)=\sum_{n=0}^{\infty}A_nR_n(r)\,\Theta_n(\theta)$$

f) apply non-hom. bc.

$$f(r)=u(r,0)=\sum_{n=0}^{\infty}A_n\sin\left(\frac{(2n+1)\pi}{2\ln 2}\ln(r)\right)\underbrace{\sinh\left(\frac{(2n+1)\pi}{2\ln 2}(0-\tfrac{\pi}{2})\right)}_{=-\sinh\left(\frac{(2n+1)\pi^2}{4\ln 2}\right)}$$

$$=\sum_{n=0}^{\infty}\left[-A_n\sinh\left(\frac{(2n+1)\pi^2}{4\ln 2}\right)\right]\sin\left(\frac{(2n+1)\pi}{2\ln 2}\ln(r)\right)$$

hence

$$-A_n \sinh\left(\frac{(2n+1)\pi^2}{4\ln 2}\right) = \frac{\displaystyle\int_1^2 f(r)\sin\left(\frac{(2n+1)\pi}{2\ln 2}\ln(r)\right)\cdot\frac{1}{r}\,dr}{\underbrace{\displaystyle\int_1^2 \sin^2\left(\frac{(2n+1)\pi}{2\ln 2}\ln(r)\right)\cdot\frac{1}{r}\,dr}_{=\frac{\ln 2}{2}}}$$

and therefore

$$\boxed{A_n = -\frac{2}{\ln 2\ \sin\left(\frac{(2n+1)\pi^2}{4\ln 2}\right)}\int_1^2 f(r)\sin\left(\frac{(2n+1)\pi}{2\ln 2}\ln(r)\right)\frac{1}{r}\,dr}$$

for $n=0,1,2,\ldots$