2.2.1 Exercises

1] $a_i \rightsquigarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $a_j \rightsquigarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ (1st order tensors)

$$A_{ij} \rightsquigarrow \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = (A_{ij})_{i,j=1,2,3}$$ (2nd order tensor)

$$A_{ijk} \rightsquigarrow (A_{ijk})_{i,j,k=1,2,3}$$ (3rd order tensor)

2] Given the coordinate transformation matrix

$$A = \begin{bmatrix} \frac{12}{25} & -\frac{9}{25} & \frac{4}{5} \\ \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{16}{25} & \frac{12}{25} & \frac{3}{5} \end{bmatrix}$$

a] show A is orthogonal, i.e $AA^t = I$

$$AA^t = \begin{bmatrix} \frac{12}{25} & -\frac{9}{25} & \frac{4}{5} \\ \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{16}{25} & \frac{12}{25} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{12}{25} & \frac{3}{5} & -\frac{16}{25} \\ -\frac{9}{25} & \frac{4}{5} & \frac{12}{25} \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b]

$$A\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{bmatrix} \frac{12}{25} & -\frac{9}{25} & \frac{4}{5} \\ \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{16}{25} & \frac{12}{25} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{29}{25} \\ \frac{4}{5} \\ -\frac{3}{25} \end{bmatrix}$$

c] Note the plane in the x_i-system has normal vector $\begin{pmatrix} 2 \\ -\frac{1}{3} \\ 1 \end{pmatrix}$, while the plane in the x_i'-system has normal vector $\begin{pmatrix} \frac{47}{25} \\ \frac{14}{15} \\ -\frac{21}{25} \end{pmatrix}$.

However, the x_i'-coordinates of $\begin{pmatrix} 2 \\ -\frac{1}{3} \\ 1 \end{pmatrix}$ are

$$A\begin{pmatrix} 2 \\ -\frac{1}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{47}{25} \\ \frac{14}{15} \\ -\frac{21}{25} \end{pmatrix}$$, hence the two normal vectors are equal.

i.e. the two planes are parallel. To show that these planes are equal, we pick a point in the x_i-system on the first plane, say

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

and check whether its x_i'-coordinates satisfy the 2nd plane equation:

$$A\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{6}{25} \\ \frac{3}{10} \\ -\frac{8}{25} \end{pmatrix}$$

Indeed,

$$\frac{47}{25} \cdot \frac{6}{25} + \frac{14}{15} \cdot \frac{3}{10} - \frac{21}{25}\left(-\frac{8}{25}\right) = 1$$

Hence, the two planes coincide.