Optimal Taxes in Atomic Congestion Games

DARIO PACCAGNAN, Imperial College London, UK
RAHUL CHANDAN, BRYCE L. FERGUSON, and JASON R. MARDEN,
University of California at Santa Barbara, USA

How can we design mechanisms to promote efficient use of shared resources? Here, we answer this question in relation to the well-studied class of atomic congestion games, used to model a variety of problems, including traffic routing. Within this context, a methodology for designing tolling mechanisms that minimize the system inefficiency (price of anarchy) exploiting solely local information is so far missing in spite of the scientific interest. In this article, we resolve this problem through a tractable linear programming formulation that applies to and beyond polynomial congestion games. When specializing our approach to the polynomial case, we obtain tight values for the optimal price of anarchy and corresponding tolls, uncovering an unexpected link with load balancing games. We also derive optimal tolling mechanisms that are constant with the congestion level, generalizing the results of Caragiannis et al. [8] to polynomial congestion games and beyond. Finally, we apply our techniques to compute the efficiency of the marginal cost mechanism. Surprisingly, optimal tolling mechanism using only local information perform closely to existing mechanism that utilize global information, e.g., Bilò and Vinci [6], while the marginal cost mechanism, known to be optimal in the continuous-flow model, has lower efficiency than that encountered levying *no* toll. All results are tight for pure Nash equilibria and extend to coarse correlated equilibria.

CCS Concepts: • **Theory of computation** → **Algorithmic game theory**; **Quality of equilibria**; **Network games**;

Additional Key Words and Phrases: Congestion games, optimal taxation mechanisms, price of anarchy, Nash equilibrium, coarse correlated equilibrium, selfish routing, approximation algorithms

Authors' addresses: D. Paccagnan, Department of Computing, Imperial College London, South Kensington SW7 2AZ, UK; email: d.paccagnan@imperial.ac.uk; R. Chandan, B. L. Ferguson, and J. R. Marden, University of California at Santa Barbara, Center for Control, Dynamical Systems and Computation, Santa Barbara, CA, 93105, USA; emails: {rchandan, blf, jrmarden}@ucsb.edu.

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1 INTRODUCTION

Modern society is based on large-scale systems often at the service of end-users, e.g., transportation and communication networks. As the performance of such systems heavily depends on the interaction between the users' individual behaviour and the underlying infrastructure (e.g., drivers on a road-traffic network), the operation of such systems requires interdisciplinary considerations at the confluence between economics, engineering, and computer science.

A common issue arising in these settings is the performance degradation incurred when the users' individual objectives are not aligned to the "greater good". A prime example of how users' behaviour degrades the performance is provided by road-traffic routing: When drivers choose routes that minimize their individual travel time, the aggregate congestion could be much higher compared to that of a centrally imposed routing. A fruitful paradigm to tackle this issue is to employ appropriately designed tolling mechanisms, as widely acknowledged in the economic and computer science literature [5, 22, 26].

Pursuing a similar line of research, this article focuses on the design of tolling mechanisms that maximize the system efficiency associated with self-interested decision making in atomic *congestion games*, i.e., that minimize the resulting *price of anarchy* [21]. Within this context, we develop a methodology to compute the most efficient *local* tolling mechanism, i.e., the most efficient mechanism whose tolls levied on each resource depend only on the local properties of that resource. We do so for both the congestion-aware and congestion-independent settings, whereby tolls have or do not have access to the current congestion levels. A summary of our results, including a comparison with the literature, can be found in Table 1. Perhaps surprisingly, optimal local tolls deliver a price of anarchy close to that of existing tolls designed using global information [6].

1.1 Congestion Games, Local and Global Mechanisms

Congestion games were introduced in 1973 by Rosenthal [31] and since then have found applications in diverse fields, such as energy markets [29], machine scheduling [35], wireless data networks [36], sensor allocation [23], network design [2], and many more. While our results can be applied to a variety of problems, we consider traffic routing as our prime application.

In a congestion game, we are given a set of users $N = \{1, \ldots, n\}$ and a set of resources $\mathcal{E}$. Each user can choose a subset of the set of resources that she intends to use. We list all feasible choices for user $i \in N$ in the set $\mathcal{A}_i \subseteq 2^{\mathcal{E}}$. The cost for using each resource $e \in \mathcal{E}$ depends only on the total number of users concurrently selecting that resource and is denoted with $\ell_e : \mathbb{N} \to \mathbb{R}_{\geq 0}$. Once all users have made a choice $a_i \in \mathcal{A}_i$, each user incurs a cost obtained by summing the costs of all resources she selected. Finally, the system cost represents the cost incurred by *all* users,

$$\mathrm{SC}(a) = \sum_{i \in N} \sum_{e \in a_i} \ell_e(|a|_e), \tag{1}$$

where $|a|_e$ is the number of users selecting resource e in allocation $a = (a_1, \ldots, a_n)$. We denote with $\mathcal{G}$ the set containing all congestion games with a maximum of n agents and where all resource costs $\{\ell_e\}_{e \in \mathcal{E}}$ belong to a common set of functions $\mathcal{L}$.

Local and global tolling mechanisms. We assume users to be self-interested, and observe that self-interested decision making often results in a highly suboptimal system cost [30]. Consequently, there has been considerable interest in the application of tolling mechanisms to influence the resulting outcome [5, 6, 8, 14, 18, 26]. Formally, a tolling mechanism $T : G \times e \mapsto \tau_e$ is a map that associates an instance $G \in \mathcal{G}$ and a resource $e \in \mathcal{E}$ to the corresponding toll τ_e. Here $\tau_e : \{1, \ldots, n\} \to \mathbb{R}$ is a congestion-dependent toll, i.e., τ_e maps the number of users in resource e to the corresponding

toll. As a result, user $i \in N$ incurs a cost factoring both the cost of the resources and the tolls, i.e.,

$$C_i(a) = \sum_{e \in a_i} [\ell_e(|a|_e) + \tau_e(|a|_e)]. \tag{2}$$

Designing tolling mechanisms that utilize global information (such as knowledge of all resource costs, or knowledge of the feasible sets $\{\mathcal{A}_i\}_{i=1}^n$) is often difficult, as that might require the central planner to access private users information, in addition to the associated computational burden. Within the context of traffic routing (see Example 1 below), a global mechanism might produce a toll on edge e that depends on the structure of the network, on the travel time over all edges, as well as on the exact origin and destination of every user. On the contrary, local tolling mechanisms require much less information, are scalable, accommodate resources that are dynamically added or removed, and are robust against a number of variations, e.g., the common scenario where drivers modify their destination. Therefore, a significant portion of the literature has focused on designing tolls that exploit only local information. Formally, we say that a tolling mechanism T is *local* if the toll on each resource only uses information on the cost function ℓ_e of the same resource e, and no other information. In this case, we write $\tau_e = T(\ell_e)$ with slight abuse of notation. On the contrary, if the mechanism utilizes additional information, then we say it is *global*.

Example 1 (traffic routing). Within the context of traffic routing, $\mathcal{E}$ represents the set of edges defining the underlying road network over which users wish to travel. For this purpose, each user $i \in N$ can select any path connecting her origin to her destination node, thus producing a list of feasible paths $\mathcal{A}_i$. The travel time incurred by a user transiting on edge $e \in \mathcal{E}$ is captured by the function ℓ_e, and depends only on the number of users traveling through that very edge. In this context, the function ℓ_e takes into consideration geometric properties of the edge, such as its length, the number of lanes and speed limit [37]. The system cost in Equation (1) represents the time spent on the network by *all* users, and tolls are imposed on the edges to incentivize users in selecting paths that minimize the total travel time (1).

1.2 Performance Metrics

The performance of a tolling mechanism is typically measured by the ratio between the system cost incurred at the worst-performing emergent allocation and the minimum system cost. As users are assumed to be self-interested, the emergent allocation is described by any of the following equilibrium notions: pure or mixed Nash equilibrium, correlated or coarse correlated equilibrium—each being a superset of the previous [32].[1] When considering pure Nash equilibria, the performance of a mechanism T, referred to as the *price of anarchy* [21], is defined as

$$\text{PoA}(T) = \sup_{G \in \mathcal{G}} \frac{\text{NECost}(G, T)}{\text{MinCost}(G)}, \tag{3}$$

where $\text{MinCost}(G)$ is the minimum social cost for instance G as defined in Equation (1), and $\text{NECost}(G, T)$ denotes the highest social cost at a Nash equilibrium obtained when employing the mechanism T on the game G. Similarly, it is possible to define the price of anarchy for mixed Nash, correlated, and coarse correlated equilibria. While these different metrics need not be equal in general, they do *coincide* within our setting, as we will later clarify. Therefore, in the following, we will use $\text{PoA}(T)$ to refer to the efficiency values of any and all these equilibrium classes.

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[1]For a congestion game, existence of pure Nash equilibria (and thus of all other equilibrium notions mentioned above) is guaranteed even in the presence of tolls, due to the fact that the resulting game is potential.

1.3 Related Work

The interest in the design of tolls dates back to the early 1900s [30]. Since then, a large body of literature in the areas of transportation, economics, and computer science has investigated this approach [5, 14, 18, 26, 33]. Designing tolling mechanisms that optimize the efficiency is particularly challenging in the context of (atomic) congestion games, as observed, e.g., by Reference [16], in part due to the multiplicity of the equilibria. While most of the research [1, 3, 13, 32] has focused on providing efficiency bounds for given schemes or in the un-tolled case, much less is known regarding the *design* question. Owing to the technical difficulties, only partial results are available for global tolling mechanisms [6, 8, 15], while results for local mechanisms are limited to affine congestion games [8].

Reference [8] initiated the study of tolling mechanisms in the context of congestion games, restricting their attention to affine resource costs. Relative to this setting, they show how to compute a congestion-independent global toll yielding a (tight) price of anarchy of 2 for mixed Nash equilibria, in addition to a congestion-independent local toll yielding a (tight) price of anarchy of $1 + 2/\sqrt{3} \approx 2.155$ for pure Nash equilibria. Our result pertaining to the design of optimal *congestion-independent local* tolls generalizes this finding to any polynomial (and non polynomial) congestion game and holds tightly for both pure Nash and coarse correlated equilibria.

More recently, Reference [6] studies congestion-dependent global tolls for pure Nash and coarse correlated equilibria, in addition to one-round walks from the empty state. Relative to unweighted congestion games, they derive tolling mechanisms yielding a price of anarchy for coarse correlated equilibria equal to 2 for affine resource costs and similarly for higher-order polynomials. While the latter work provides a number of interesting insights (e.g., some closed form price of anarchy expressions), all the derived tolling schemes require global information such as network and user knowledge—an often impractical scenario. In contrast, our results on *optimal local tolls* focuses on the design of optimal mechanisms that exploit only local information, and thus are more widely applicable. Even if utilizing much less information, optimal local mechanisms are still competitive. For example, congestion-dependent optimal local tolls yield a price of anarchy of 2.012 for coarse correlated equilibria and affine congestion games (to be compared with a value of 2).

Related works have also explored modifications of the setup considered here: Reference [8] also studies singleton congestion games, Reference [15] focuses on symmetric network congestion games, Reference [25] on altruistic congestion games, while References [17, 19, 20] consider tolling a subset of the resources. Reference [34] focuses on the computation of approximate Nash equilibria in atomic congestion games. Therein, the authors design modified resource costs leveraging a methodology similarly to that developed in References [9, 12, 27].

Finally, we note that the design of tolling mechanisms is far better understood when the original congestion game is replaced by its continuous-flow approximation, as uniqueness of the Nash equilibrium is guaranteed. Limited to this setting, marginal cost tolls produce an equilibrium that is always optimal [4]. Within the atomic setting, our work demonstrates that marginal cost tolls do not improve—and instead significantly deteriorate—the resulting system efficiency.

1.4 Preview of Our Contributions

The core of our work is centred on designing optimal local tolling mechanisms and on deriving their corresponding prices of anarchy for various classes of congestion games. Our work develops on parallel directions, and the contributions include the following:

  (i) The design of optimal local tolling mechanisms;
  (ii) The design of optimal congestion-independent local tolling mechanisms;
  (iii) The study of marginal cost tolls and their inefficiency.

Table 1. Price of Anarchy Values for Congestion Games with Resource Costs of Degree at Most d

d	No toll [1]	Global toll from [6, 8]	Optimal local toll (this work)	Optimal constant local toll (this work)	Marginal cost toll (this work)
1	2.50	2	2.012	2.15	3.00
2	9.58	5	5.101	5.33	13.00
3	41.54	15	15.551	18.36	57.36
4	267.64	52	55.452	89.41	391.00
5	1513.57	203	220.401	469.74	2124.21
6	12345.20	877	967.533	3325.58	21337.00

All results are tight for pure Nash and also hold for coarse correlated equilibria. The columns feature the price of anarchy with no tolls, with global tolls from References [6, 8], with optimal local tolls, with optimal constant (i.e., congestion-independent) local tolls, and with marginal cost tolls, respectively. Columns four, five, and six, are composed of entirely novel results, except for the case of constant tolls with $d = 1$, which recovers [8]. Note that (i) optimal tolls relying only on local information perform closely to optimal tolls designed using global information, with a difference in performance below 1% for $d = 1$; (ii) congestion-independent tolls yield a price of anarchy comparable to that obtained using congestion-aware local tolls for polynomials of low degree. The code used to generate this table can be found in Reference [10].

Table 1 highlights the impact of our results on congestion games with polynomial cost functions of varying degree, though our methodology extends further. The following paragraphs describe our contributions in more details, while supporting code can be found in Reference [10].

Optimal local tolls. In Theorems 1 and 2 we provide a methodology for computing optimal local tolling mechanisms for congestion games. The resulting prices of anarchy for the case of polynomial congestion games are presented in Table 1 (fourth column), where we provide a comparison with those derived in References [6, 8] (third column), which instead make use of global information, for example, by letting the tolling function on resource e depend on *all* the other resource costs. Perhaps unexpectedly, the efficiency of optimal tolls designed using only local information almost matches that of existing tolls using global information [6, 8]. In addition to providing similar performances by means of less information, local tolls are robust against uncertain scenarios (e.g., modifications in the origin/destination pairs) and can be computed efficiently.

Extensive work has focused on quantifying the price of anarchy for load balancing games, that is congestion games where all action sets are singletons, i.e., $\mathcal{A}_i \subseteq \mathcal{E}$. Within this setting, and when all resource costs are affine and identical, the price of anarchy is ≈ 2.012 [7, 35]. Our results connect with this line of work, demonstrating that optimally tolled affine congestion games have a price of anarchy matching this value, and are tight already within this class. Stated differently, the price of anarchy of un-tolled and optimally tolled affine load balancing games with identical resources is the same. We believe such statement holds true more generally.

Optimal congestion-independent local tolls. Our methodology can also be exploited to derive optimal local mechanisms under more stringent structural constraints. One such constraint, studied in numerous settings, consists in the use of congestion-independent mechanisms, which are attractive because of their simplicity. A linear program to compute optimal congestion-independent local mechanisms is presented in Theorem 3, while the corresponding optimal prices of anarchy for the case of polynomial congestion games are displayed in Table 1 (fifth column) and derived in Corollary 1 as well as Section 5. All the results are novel, except for the case of $d = 1$, which recovers [8]. We observe that the performance of congestion-independent mechanisms is comparable with that of congestion-aware mechanisms for polynomials of low degree ($d \leq 3$) and still a good improvement over the un-tolled setup for high degree polynomials. In these cases, congestion-independent mechanisms are not only robust and simple to implement but also relatively efficient.

Marginal cost tolls are worse than no tolls. In non-atomic congestion games, any Nash equilibrium resulting from the application of the marginal contribution mechanism is optimal, i.e., it has a price of anarchy equal to 1. Corollary 2 shows how to utilize our approach to compute the efficiency of the marginal cost mechanism in the *atomic* setup. The resulting values of the price of anarchy are presented in the last column of Table 1 for polynomial congestion games. While the marginal cost mechanism ensures that a Nash equilibrium is optimal (i.e., its price of stability is one), utilizing the marginal cost mechanism on the atomic model yields a price of anarchy that is *worse* than that experienced levying no toll at all (compare the second and last column in Table 1). In other words, the design principle derived from the continuous-flow model does not carry over to the original atomic setup. This phenomenon manifests itself already in very simple settings, as we demonstrate in Figure 3. We conclude observing that our result differs significantly from that in Reference [24], where the authors show that, as the number of agents grow, marginal cost tolls become optimal. This difference stems from the fact that, in Reference [24], the network is *fixed* as the number of agents grows while here the worst case instance is a function of the number of agents.

1.5 Techniques and High-level Ideas

Underlying the developments presented above are a number of technical results, which stem from the observation that, in the majority of the existing literature, the set $\mathcal{L}$ contains all resource costs of the form $\ell(x) = \sum_{j=1}^{m} \alpha_j b_j(x)$ with $\alpha_j \geq 0$, and given basis functions $\{b_1, \ldots, b_m\}$. This describes the fact that each resource cost belongs to a known class of functions. For example, in polynomial congestion games of maximum degree d, each resource is associated to a cost of the form $\alpha_1 + \alpha_2 x + \cdots + \alpha_{d+1} x^d$, corresponding to the choice of basis functions $\{1, \ldots, x^d\}$. More generally, the decomposition $\ell(x) = \sum_{j=1}^{m} \alpha_j b_j(x)$ allows us to leverage a common framework to study different classes of problems not limited to polynomial congestion games.

In this context, we first show in Theorem 1 that an optimal local tolling mechanism is a linear map from the set of resource costs to the set of tolls. More precisely, we show that, for every resource cost $\ell(x) = \sum_{j=1}^{m} \alpha_j b_j(x)$, there exists an optimal local mechanism satisfying $T^{\text{opt}}(\ell) = T^{\text{opt}}(\sum_{j=1}^{m} \alpha_j b_j) = \sum_{j=1}^{m} \alpha_j T^{\text{opt}}(b_j)$, where the mechanism is obtained as a linear combination of $T^{\text{opt}}(b_j)$, with the *same* coefficients α_j used to define ℓ.[2] It is worth noting that this first result allows for a decoupling argument, whereby an optimal tolling function $T^{\text{opt}}(b_j)$ can be separately computed for each of the basis b_j. The key idea underpinning the result on linearity of optimal tolls lies in observing that any congestion game utilizing resource cost functions with coefficients $\alpha_j \in \mathbb{R}_{>0}$ and a possibly non-linear tolling mechanism T, can be mapped to a corresponding congestion game where (i) all coefficients α_j are identical to one, (ii) only the linear part of the tolling mechanism T is used, and (iii) the price of anarchy is identical to that of the original game (as the number of resources grows). Complementary to this, our second result in Theorem 1 reduces the problem of designing optimal basis tolls $\{T^{\text{opt}}(b_j)\}_{j=1}^{m}$ to a polynomially solvable linear program that also returns the tight value of the optimal price of anarchy. We do so by building upon the results in References [11, 28], which allow us to determine the performance of a tolling mechanism through the solution of a linear program (see Equation (11)), in a similar spirit to Reference [6], although with a provably tight characterization for *any* number of agents and tolling function. We exploit this result and construct a polynomially sized linear program that, for a given basis b_j, searches over all linear tolls to find $T^{\text{opt}}(b_j)$.

When the basis functions are convex and increasing (e.g., in the well-studied polynomial case), we are able to explicitly solve the linear program and provide an analytic expression for the

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[2]As word of caution, we remark that linearity of the optimal mechanism in the sense clarified above does *not* mean that the corresponding tolls are linear in the congestion level, i.e., does *not* mean that $\tau_e(x) = a_e x + b_e$ for some a_e, b_e.

optimal tolling function, as well as a semi-analytic expression for optimal price of anarchy (Theorem 2). The fundamental idea consists in showing that the set of active constraints at the solution gives rise to a telescopic recursion, whereby the optimal toll to be levied when $u + 1$ agents are selecting a resource can be written as a function of the optimal toll to be levied when only u agents are present. This is the most technical part of the manuscript, and the expression of the optimal price of anarchy reveals an unexpected connection with that for un-tolled load balancing games on identical machines [7, 35]. While the sizes of the programs in Theorems 1 and 2 grow (polynomially) with the number of agents n, in Section 4 we show how to design optimal tolling mechanisms that apply to any n (possibly infinite). Our approach consists in two steps: we first solve a linear program of finite dimension, and then extend its solution to arbitrary n.

Congestion-independent optimal local mechanisms as well as the efficiency of the marginal cost mechanism can also be computed through linear programs (Theorem 3 and Corollary 2). In the first case, admissible tolls basis $\{T^{\mathrm{opt}}(b_j)\}_{j=1}^m$ are constrained to be constant with the congestion, while in the second case the expression for marginal cost tolls is utilized. We provide analytical solutions to these two programs for the case of polynomial congestion games.

1.6   Organization

In Section 2, we derive linear programs to compute optimal local tolling mechanisms. We also provide optimal price of anarchy values for polynomial congestion games. In Section 3, we obtain an explicit solution to these programs that applies when resource costs are convex and increasing. Section 4 generalizes the previous results to arbitrarily large number of agents. In Section 5 and Section 6, we derive congestion independent tolling mechanism and evaluate the efficiency of the marginal cost mechanism. In these sections we also specialize the results to the polynomial case.

2   OPTIMAL TOLLING MECHANISMS

In this section, we develop a methodology to compute optimal local tolling mechanisms through the solution of tractable linear programs. To ease the notation, we define the set of triplets $I = \{(x, y, z) \in \mathbb{Z}_{\geq 0}^3 \text{ s.t. } 1 \leq x + y + z \leq n \text{ and either } xyz = 0 \text{ or } x + y + z = n\}$, for given $n \in \mathbb{N}$.

THEOREM 1. *A local mechanism minimizing the price of anarchy over congestion games with n agents, resource costs $\ell(x) = \sum_{j=1}^m \alpha_j b_j(x)$, $\alpha_j \geq 0$, and basis functions $\{b_1, \ldots, b_m\}$ is given by*

$$T^{\mathrm{opt}}(\ell) = \sum_{j=1}^m \alpha_j \cdot \tau_j^{\mathrm{opt}}, \quad \text{where} \quad \tau_j^{\mathrm{opt}} : \{1, \ldots, n\} \to \mathbb{R}, \quad \tau_j^{\mathrm{opt}}(x) = f_j^{\mathrm{opt}}(x) - b_j(x) \qquad (4)$$

and $\rho_j^{\mathrm{opt}} \in \mathbb{R}$, $f_j^{\mathrm{opt}} : \{1, \ldots, n\} \to \mathbb{R}$ solve the following linear programs (one per each b_j):

$$\max_{f \in \mathbb{R}^n, \, \rho \in \mathbb{R}} \rho$$
$$\text{s.t.} \quad b_j(x + z)(x + z) - \rho b_j(x + y)(x + y) + f(x + y)y - f(x + y + 1)z \geq 0 \quad \forall (x, y, z) \in I, \qquad (5)$$

where we define $b_j(0) = f(0) = f(n + 1) = 0$. Correspondingly, $\mathrm{PoA}(T^{\mathrm{opt}}) = \max_j \{1/\rho_j^{\mathrm{opt}}\}$.[3] These results are tight for pure Nash equilibria and extend to coarse correlated equilibria.

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[3]If we require non-negative tolls, an optimal mechanism is as in Equation (4), where we set $\tau_j^{\mathrm{opt}}(x) = f_j^{\mathrm{opt}}(x) \cdot \mathrm{PoA}^{\mathrm{opt}} - b_j(x)$.
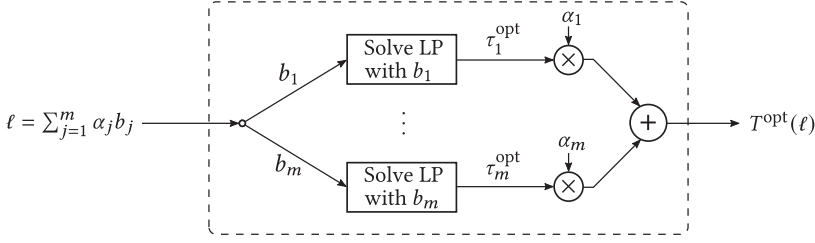
Fig. 1. Graphical illustration of the main result on the design of optimal tolls. The input consists of a resource cost $\ell(x)$ expressed as a combination of bases $b_j(x)$ with coefficients α_j. For each basis, we compute the associated toll optimal toll $\tau_j^{\text{opt}}(x) = f_j^{\text{opt}}(x) - b_j(x)$ by solving the linear program (LP) appearing in Equation (5). The resulting optimal toll is obtained as the linear combination of $\tau_j^{\text{opt}}(x)$ with the same coefficients α_j. The quantities $\tau_j^{\text{opt}}(x)$ can be precomputed and stored in a library, offloading the solution of the linear programs.

The above statement contains two fundamental results. The first part of the statement shows that an optimal tolling mechanism applied to the function $\ell(x) = \sum_{j=1}^m \alpha_j b_j(x)$ can be obtained as the linear combination of $\tau_j^{\text{opt}}(x)$, with the *same* coefficients α_j used to define ℓ. Complementary to this, the second part of the statement provides a practical technique to compute $\tau_j^{\text{opt}}(x)$ for each of the basis $b_j(x)$ as the solution of a tractable linear program. A graphical representation is included in Figure 1, while Python/Matlab code to design optimal tolls can be found in Reference [10].

We solved the latter linear programs for $n = 100$ and polynomials of degree $1 \le d \le 6$. The corresponding results are displayed in Table 1, while Section 4 shows that these results hold identically for arbitrarily large n. In the case of $d = 1$, the optimal price of anarchy is approximately 2.012, matching that of un-tolled load balancing games on identical machines [7, 35]. We observe that, in this restricted setting, the price of anarchy cannot be improved *at all* through local tolling mechanisms. In fact, no matter what non-negative tolling we choose, we can always construct a load balancing game on identical machines with a price of anarchy no lower than 2.012.[4]

We conclude observing that the decomposition of resource costs as linear combination of basis functions is, strictly speaking, not required for Theorem 1 to hold. Nevertheless, pursuing this approach would require to solve a linear program for each function in $\mathcal{L}$, a task that becomes daunting when $\mathcal{L}$ contains infinitely many functions, e.g., in the case of polynomial congestion games. In this case, Theorem 1 allows to compute optimal tolls by solving *finitely* many programs.

PROOF. We divide the proof in two parts to ease the exposition.

Part 1. We show that any local mechanism minimizing the price of anarchy over all linear local mechanisms, does so also over all linear and non-linear local mechanisms. We let T^{opt} be a mechanism that minimizes the price of anarchy over all linear local mechanisms, i.e., over all T satisfying

$$T\left(\sum_{j=1}^m \alpha_j b_j\right) = \sum_{j=1}^m \alpha_j T(b_j),$$

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[4]To do so, it is sufficient to utilize the instance in Reference [35, Thm 3.4], where the the resource cost x used therein is replaced with $x + \tau(x)$. The Nash equilibrium and the optimal allocation will remain unchanged, yielding the same price of anarchy value.

for all $\alpha_j \geq 0$. We intend to show that $\mathrm{PoA}(T^{\mathrm{opt}}) \leq \mathrm{PoA}(T)$ for any possible T (linear or non-linear). Toward this goal, assume, for a contradiction, that there exists a tolling mechanism $\hat{T}$ such that

$$\mathrm{PoA}(T^{\mathrm{opt}}) > \mathrm{PoA}(\hat{T}). \tag{6}$$

Let $\mathcal{G}_b$ be the class of games in which any resource e can only utilize a resource cost $\ell_e \in \{b_1, \ldots, b_m\}$. Since $\mathcal{G}_b \subset \mathcal{G}$, we have

$$\mathrm{PoA}(\hat{T}) \geq \sup_{G \in \mathcal{G}_b} \frac{\mathrm{NECost}(G, \hat{T})}{\mathrm{MinCost}(G)}. \tag{7}$$

Additionally, let $\mathcal{G}(\mathbb{Z}_{\geq 0}) \subset \mathcal{G}$ be the class of games with $\alpha_j \in \mathbb{Z}_{\geq 0}$ for all $j \in \{1, \ldots, m\}$, for all resources in $\mathcal{E}$. Construct the mechanism $\bar{T}$ by "linearizing" the mechanism $\hat{T}$, i.e., as

$$\bar{T}(\ell) = \bar{T}\left(\sum_{j=1}^{m} \alpha_j b_j\right) = \sum_{j=1}^{m} \alpha_j \hat{T}(b_j).$$

We observe that the efficiency of any instance $G \in \mathcal{G}_b$ to which the tolling mechanism $\hat{T}$ is applied, coincides with that of an instance $G \in \mathcal{G}(\mathbb{Z}_{\geq 0})$ to which $\bar{T}$ is applied, and vice versa. Thus,

$$\sup_{G \in \mathcal{G}_b} \frac{\mathrm{NECost}(G, \hat{T})}{\mathrm{MinCost}(G)} = \sup_{G \in \mathcal{G}(\mathbb{Z}_{\geq 0})} \frac{\mathrm{NECost}(G, \bar{T})}{\mathrm{MinCost}(G)} = \mathrm{PoA}(\bar{T}), \tag{8}$$

where the last equality holds due to Lemma 1 in Appendix A.1. Putting together Equations (6)–(8) gives

$$\mathrm{PoA}(T^{\mathrm{opt}}) > \mathrm{PoA}(\bar{T}). \tag{9}$$

Since T^{opt} minimizes the price of anarchy over all linear mechanisms, and since $\bar{T}$ is linear by construction, it must be $\mathrm{PoA}(T^{\mathrm{opt}}) \leq \mathrm{PoA}(\bar{T})$, a contradiction of Equation (9). Thus, T^{opt} minimizes the price of anarchy over any mechanism.

Part 2. We will derive a linear program to design optimal linear mechanisms. Putting this together with the claim in Part 1 will conclude the proof. Toward this goal, we will prove that any mechanism of the form

$$T(\ell) = \sum_{j=1}^{m} \alpha_j \tau_j^{\mathrm{opt}} \quad \text{with} \quad \tau_j^{\mathrm{opt}}(x) = \lambda \cdot f_j^{\mathrm{opt}}(x) - b_j(x) \tag{10}$$

is optimal, regardless of the value of $\lambda \in \mathbb{R}_{>0}$. While this is slightly more general than needed, setting $\lambda = 1$ will give the first claim. Additionally, setting $\lambda = \mathrm{PoA}^{\mathrm{opt}}$ will give the second claim, as this choice will ensure non-negativity of the tolls.

Before turning to the proof, we recall a result from Reference [11] that allows us to compute the price of anarchy for *given* linear tolling mechanism $T(\ell) = \sum_{j=1}^{m} \alpha_j \tau_j$. Upon defining $f_j(x) = b_j(x) + \tau_j(x)$ for all $1 \leq x \leq n$ and $j \in \{1, \ldots, m\}$, the authors show that the price of anarchy of T computed over congestion games $\mathcal{G}$ is identical for pure Nash and coarse correlated equilibria and is given by $\mathrm{PoA}(T) = 1/\rho^{\mathrm{opt}}$, where ρ^{opt} is the value of the following program:

$$\max_{f \in \mathbb{R}^n, \rho \in \mathbb{R}} \rho$$
$$\text{s.t.} \; b_j(x+z)(x+z) - \rho b_j(x+y)(x+y) + v[f_j(x+y)y - f_j(x+y+1)z] \geq 0 \; \forall (x, y, z) \in \mathcal{I},$$
$$\forall j \in \{1, \ldots, m\}. \tag{11}$$

We also remark that, when all functions $\{f_j\}_{j=1}^m$ are non-decreasing, it is sufficient to only consider a reduced set of constraints, following a similar argument to that in Reference [28, Cor. 1]. In this case, the linear program simplifies to

$$\max_{\rho \in \mathbb{R}, \, v \in \mathbb{R}_{\geq 0}} \rho$$

$$\text{s.t.} \quad b_j(v)v - \rho b_j(u)u + v[f_j(u)u - f_j(u+1)v] \geq 0$$

$$\forall u, v \in \{0, \ldots, n\} \quad u + v \leq n, \quad \forall j \in \{1, \ldots, m\}, \quad (12)$$

$$b_j(v)v - \rho b_j(u)u + v[f_j(u)(n-v) - f_j(u+1)(n-u)] \geq 0$$

$$\forall u, v \in \{0, \ldots, n\} \quad u + v > n, \quad \forall j \in \{1, \ldots, m\}.$$

We now leverage Equation (11) to prove that any mechanism in Equation (10) is optimal, as required. Toward this goal, we begin by observing that the optimal price of anarchy obtained when the resource costs are generated using all the basis functions $\{b_1, \ldots, b_m\}$ is no smaller than the optimal price of anarchy obtained when the resource costs are generated using a single basis function $\{b_j\}$ at a time (and therefore is no smaller than the highest of these optimal price of anarchy values). This follows readily, since the former class of games is a superset of the latter. Additionally, observe that a set of tolls minimizing the price of anarchy over the games generated using a single basis function $\{b_j\}$ is precisely that in Equation (10). This is because minimizing the price of anarchy amounts to designing f_j to maximize ρ in Equation (11), i.e., to solving the following program:

$$\max_{f \in \mathbb{R}^n} \max_{\rho \in \mathbb{R}, \, v \in \mathbb{R}_{\geq 0}} \rho$$

$$\text{s.t.} \; b_j(x+z)(x+z) - \rho b_j(x+y)(x+y) + v[f(x+y)y - f(x+y+1)z] \geq 0 \quad \forall (x, y, z) \in \mathcal{I},$$

which can be equivalently written as

$$\max_{\tilde{f} \in \mathbb{R}^n, \, \rho \in \mathbb{R}} \rho$$

$$\text{s.t.} \quad b_j(x+z)(x+z) - \rho b_j(x+y)(x+y) + \tilde{f}(x+y)y - \tilde{f}(x+y+1)z \geq 0 \quad \forall (x, y, z) \in \mathcal{I},$$

where we defined $\tilde{f} = v \cdot f$. While f_j^{opt} is defined in Equation (5) precisely as the solution of this last program, resulting in a price of anarchy of $1/\rho_j^{\text{opt}}$, note that $\lambda \cdot f_j^{\text{opt}}$ is also a solution, since its price of anarchy matches $1/\rho_j^{\text{opt}}$ (in fact, it can be computed using Equation (11) for which $(\rho, v) = (\rho_j^{\text{opt}}, 1/\lambda)$ are feasible).

The above reasoning shows that the optimal price of anarchy for a game with resource costs generated by $\{b_1, \ldots, b_m\}$ must be no smaller than $\max_j\{1/\rho_j^{\text{opt}}\}$. We now show that this holds with equality. Toward this goal, we note, thanks to Equation (11), that utilizing tolls as in Equation (6) for a game generated by $\{b_1, \ldots, b_m\}$ results in a price of anarchy of precisely $\max_j\{1/\rho_j^{\text{opt}}\}$. This follows as $(\min_j\{\rho_j^{\text{opt}}\}, 1/\lambda)$ is feasible for this program for any choice of $\lambda > 0$. This proves, as requested, that any tolling mechanism defined in Equation (10) is optimal.

We now verify that the choice $\lambda = \text{PoA}^{\text{opt}} = \max_j\{1/\rho_j^{\text{opt}}\}$ ensures positivity of the tolls, which is equivalent to $f_j^{\text{opt}}(x) - b_j(x)/\lambda \geq 0$ for all $x \in \{1, \ldots, n\}$. This follows readily, as setting $x = z = 0$ in Equation (5) results in the constraint $f(y) - \rho b_j(y) \geq 0$ for all $y \in \{1, \ldots, n\}$. Since f_j^{opt} and ρ_j^{opt} must be feasible for this constraint, we have $f_j^{\text{opt}}(y) - \rho_j^{\text{opt}} b_j(y) \geq 0$. One concludes observing that $f_j^{\text{opt}}(y) - b_j(y)/\lambda \geq f_j^{\text{opt}}(y) - \rho_j^{\text{opt}} b_j(y) \geq 0$, since $\lambda \geq 1/\rho_j^{\text{opt}}$. We observe that all results hold for both Nash and coarse correlated equilibria, as they were derived from Equation (11). □

3 EXPLICIT SOLUTION AND SIMPLIFIED LINEAR PROGRAM

In this section, we derive a simplified linear program as well as an analytical solution to the problem of designing optimal tolling mechanisms. We do so under the assumption that all basis functions are positive, increasing, and convex in the discrete sense.[5]

THEOREM 2. *Consider congestion games with n agents, where resource costs take the form* $\ell(x) = \sum_{j=1}^{m} \alpha_j b_j(x)$, $\alpha_j \geq 0$, *and basis* $b_j : \{1, \ldots, n\} \to \mathbb{R}$ *are positive, convex, strictly increasing.*[6]

(i) *A tolling mechanism minimizing the price of anarchy is as in Equation* (4), *where each* $f_j^{\mathrm{opt}} : \{1 \ldots, n\} \to \mathbb{R}$ *solves the following simplified linear program:*

$$\rho_j^{\mathrm{opt}} = \max_{f \in \mathbb{R}^n, \, \rho \in \mathbb{R}} \rho$$

$$\text{s.t.} \quad b_j(v)v - \rho b_j(u)u + f(u)u - f(u+1)v \geq 0 \quad \forall u, v \in \{0, \ldots, n\} \quad u + v \leq n,$$

$$b_j(v)v - \rho b_j(u)u + f(u)(n-u) - f(u+1)(n-u) \geq 0 \quad \forall u, v \in \{0, \ldots, n\} \quad u + v > n,$$

$$(13)$$

with $f(0) = f(n+1) = 0$. *The corresponding optimal price of anarchy is* $\max_j \{1/\rho_j^{\mathrm{opt}}\}$.

(ii) *An explicit expression for each* f_j^{opt} *is given by the following recursion, where* $f_j^{\mathrm{opt}}(1) = b_j(1)$,

$$f_j^{\mathrm{opt}}(u+1) = \min_{v \in \{1, \ldots, n\}} \beta(u, v) f_j^{\mathrm{opt}}(u) + \gamma(u, v) - \delta(u, v) \rho_j^{\mathrm{opt}},$$

$$\beta(u, v) = \frac{\min\{u, n-v\}}{\min\{v, n-u\}}, \quad \gamma(u, v) = \frac{b(v)v}{\min\{v, n-u\}}, \quad \delta(u, v) = \frac{b(u)u}{\min\{v, n-u\}},$$

$$(14)$$

$$\rho_j^{\mathrm{opt}} = \min_{(v_1, \ldots, v_n) \in \{1, \ldots, n\}^{n-1} \times \{0, \ldots, n\}} \frac{(n-v_n)\left(\prod_{u=1}^{n-1} \beta_u b_j(1) + \sum_{u=1}^{n-2}\left(\prod_{i=u+1}^{n-1} \beta_i\right)\gamma_u + \gamma_{n-1}\right) + b(v_n)v_n}{(n-v_n)\left(\sum_{u=1}^{n-2}\left(\prod_{i=u+1}^{n-1} \beta_i\right)\delta_u + \delta_{n-1}\right) + b(n)n},$$

$$(15)$$

where we use the short-hand notation β_u *instead of* $\beta(u, v_u)$ *and similarly for* γ_u *and* δ_u.

Before delving into the proof, we observe that the key difficulty in designing optimal tolls resides in the expressions of ρ_j^{opt} arising from Equation (15). Nevertheless, for any possible choice of $\bar{\rho}_j$ that approximates ρ_j^{opt} from below, i.e., $\bar{\rho}_j \leq \rho_j^{\mathrm{opt}}$, one can directly utilize the recursion in Equation (14) to design a valid tolling mechanism. The resulting price of anarchy would then amount to $\max_j\{1/\bar{\rho}_j\} > \max_j\{1/\rho_j^{\mathrm{opt}}\}$. This follows from the ensuing proof.

PROOF. As shown in Theorem 1, computing an optimal tolling mechanism amounts to utilizing Equation (4), where each τ_j^{opt} has been designed through the solution of the program in Equation (5). In light of this, we prove the theorem as follows: first, we consider a simplified linear program, where only a subset of the constraints enforced in Equation (5) are considered. Second, we show that a solution of this simplified program is given by $(\rho_j^{\mathrm{opt}}, f_j^{\mathrm{opt}})$ as defined above. Third, we show that f_j^{opt} is non-decreasing, thus ensuring that $(\rho_j^{\mathrm{opt}}, f_j^{\mathrm{opt}})$ is also feasible for the original over constrained program in Equation (5). From this we conclude that $(\rho_j^{\mathrm{opt}}, f_j^{\mathrm{opt}})$ must also be a solution of Equation (5), i.e., the second claim in the Theorem. We conclude with some cosmetics, and transform the simplified linear program whose solution is given by $(\rho_j^{\mathrm{opt}}, f_j^{\mathrm{opt}})$ in Equation

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[5]We say that a function $f : \{1, \ldots, n\} \to \mathbb{R}$ is convex if $f(x+1) - f(x)$ is non-decreasing in its domain.
[6]The result also holds if convexity and strict increasingness of $b_j(x)$ are weakened to strict convexity of $b_j(x)x$ and $b_j(n) > b_j(n-1)$. One such example is that of $b_j(x) = \sqrt{x}$.

(13), thus obtaining the first claim. Throughout the proof, we drop the index j from b_j as the proof can be repeated for each basis separately.   □

Simplified linear program. We begin rewriting the program in Equation (5), where instead of the indices (x, y, z), we use the corresponding indices (u, v, x) defined as $u = x + y$, $v = x + z$. The constraint indexed by (u, v, x) reads as $b(v)v - \rho b(u)u + f(u)(u - x) - f(u + 1)(v - x) \geq 0$. We now consider only the constraints where x is set to $x = \min\{0, u + v - n\}$, and $u, v \in \{0, \dots, n\}$, Such constraints read as $b(v)v - \rho b(u)u + \min\{u, n - v\}f(u) - \min\{v, n - u\}f(u + 1) \geq 0$.[7] Finally, we exclude the constraints with $v = 0$, $u \in \{1, \dots, n - 1\}$ and obtain the following *simplified linear program:*

$$\max_{f \in \mathbb{R}^n, \rho \in \mathbb{R}} \quad \rho$$
$$\text{s.t.} \quad b(v)v - \rho b(u)u + \min\{u, n - v\}f(u) - \min\{v, n - u\}f(u + 1) \geq 0$$
$$\forall (u, v) \in \{0, \dots, n\} \times \{1, \dots, n\} \cup (n, 0). \tag{16}$$

Proof that $(\rho^{\text{opt}}, f^{\text{opt}})$ solve (16). Toward the stated goal, we begin by observing that $(\rho^{\text{opt}}, f^{\text{opt}})$ is feasible by construction. For $u = 0$ this follows as the tightest constraints in Equation (16) read as $f^{\text{opt}}(1) \geq b(1)$ and we selected $f^{\text{opt}}(1) = b(1)$. Feasibility is immediate to verify for $u \in \{1, \dots, n - 1\}$, $v \in \{1, \dots, n\}$ as applying its definition gives $f^{\text{opt}}(u + 1) \leq \beta(u, v)f^{\text{opt}}(u) + \gamma(u, v) - \delta(u, v)\rho^{\text{opt}}$. Using the expressions of β, γ, δ, and rearranging gives exactly the constraint (u, v) in Equation (16). The only element of difficulty consists in showing that also the constraints with $u = n$, $v \in \{0, \dots, n\}$ are satisfied. Toward this goal, we observe that utilizing the recursive definition of f^{opt} we obtain an expression for $f^{\text{opt}}(n)$ as a function of ρ^{opt} with a nested succession of minimizations, which can be jointly extracted as follows:

$$f^{\text{opt}}(n) = \min_{v_{n-1}}\left\{\cdots + \min_{v_{n-2}}\left\{\cdots + \min_{v_1}\{\dots\}\right\}\right\} = \min_{v_{n-1}}\min_{v_{n-2}}\dots\min_{v_1}\{\dots\}.$$

This holds as $f^{\text{opt}}(u + 1) = \min_{v_u}[\beta_u \min_{v_{u-1}}(\beta_{u-1}f^{\text{opt}}(u - 1) - \delta_{u-1}\rho^{\text{opt}} + \gamma_{u-1}) - \delta_u\rho^{\text{opt}} + \gamma_u]$, and since $\beta_u \geq 0$, the latter simplifies to $f^{\text{opt}}(u + 1) = \min_{v_u}\min_{v_{u-1}}\beta_u\beta_{u-1}f^{\text{opt}}(u - 1) - (\beta_u\delta_{u-1} + \delta_u)\rho^{\text{opt}} + \beta_u\gamma_{u-1} + \gamma_u$. Repeating the argument recursively gives the desired expression. Hence,

$$f^{\text{opt}}(n) = \min_{(v_1, \dots, v_{n-1}) \in \{1, \dots, n\}^{n-1}} \prod_{u=1}^{n-1}\beta_u b_j(1) + \sum_{u=1}^{n-2}\left(\prod_{i=u+1}^{n-1}\beta_i\right)(\gamma_u - \delta_u\rho^{\text{opt}}) + (\gamma_{n-1} - \delta_{n-1}\rho^{\text{opt}})$$
$$\doteq \min_{(v_1, \dots, v_{n-1}) \in \{1, \dots, n\}^{n-1}} q(v_1, \dots, v_{n-1}; \rho^{\text{opt}}),$$

where we implicitly defined $q(v_1, \dots, v_{n-1}; \rho^{\text{opt}})$. The constraints we intend to verify read as $b(v)v - \rho b(n)n + (n - v)f^{\text{opt}}(n) \geq 0$ for all $v \in \{0, \dots, n\}$ and can be equivalently written as $\min_{v_n \in \{0, \dots, n\}}[b(v_n)v_n - \rho b(n)n + (n - v_n)f^{\text{opt}}(n)] \geq 0$. We substitute the resulting expression of $f^{\text{opt}}(n)$, extract the minimization over v_n as in the above, and are therefore left with $\min_{(v_1, \dots, v_{n-1}, v_n) \in \{1, \dots, n\}^{n-1} \times \{0, \dots, n\}}[b(v_n)v_n - \rho b(n)n + (n - v)q(v_1, \dots, v_{n-1}; \rho^{\text{opt}})] \geq 0$, which holds if and only if $b(v_n)v_n - \rho b(n)n + (n - v_n)q(v_1, \dots, v_{n-1}; \rho^{\text{opt}}) \geq 0$ for all possible tuples $(v_1, \dots, v_n)$. Rearranging these constraints and solving for ρ^{opt} will result in a set of inequalities on ρ^{opt} (one inequality for each tuple). Our choice of ρ^{opt} in Equation (15) is precisely obtained

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[7]Note that considering all these constraints with $u, v \in \{0, \dots, n\}$ results precisely in Equation (13). To see this, simply distinguish the cases based on whether $u + v \leq n$ or $u + v > n$.

by turning the most binding of these into an equality. This ensures that $(\rho^{\mathrm{opt}}, f^{\mathrm{opt}})$ are feasible also when $u = n$.

We now prove, by contradiction, that $(\rho^{\mathrm{opt}}, f^{\mathrm{opt}})$ is optimal. To do so, we assume that there exists $\hat{f}$, that is feasible and achieves a higher value $\hat{\rho} > \rho^{\mathrm{opt}}$. Since $(\hat{f}, \hat{\rho})$ is feasible, using the constraint with $u = 0$, $v = 1$, we have $\hat{f}(1) \le b(1) = f^{\mathrm{opt}}(1)$. Observing that $\min\{v, n - u\} > 0$ due to $v > 0$, $u < n$ and leveraging the constraints with $u = 1$ as well as the corresponding specific choice of $v = v_1^*$ (for given $u \in \{1, \dots, n-1\}$, we let v_u^* be an index $v \in \{1, \dots, n\}$ where the minimum in Equation (14) is attained), it must be that $\hat{f}(2)$ satisfies

$$\hat{f}(2) \le \frac{b(v_1^*)v_1^* - \hat{\rho} b(1) + \min\{1, n - v_1^*\} \hat{f}(1)}{\min\{v_1^*, n - 1\}} < \frac{b(v_1^*)v_1^* - \rho^{\mathrm{opt}} b(1) + \min\{1, n - v_1^*\} f^{\mathrm{opt}}(1)}{\min\{v_1^*, n - 1\}} = f^{\mathrm{opt}}(2).$$

Here the first inequality follows by feasibility of $\hat{f}$, the second is due to $\hat{\rho} > \rho^{\mathrm{opt}}$ and $\hat{f}(1) \le f^{\mathrm{opt}}(1)$. The final equality follows due to the definition of $f^{\mathrm{opt}}(2)$. Hence we have shown that $\hat{f}(2) < f^{\mathrm{opt}}(2)$. Noting that the only information we used to move from level u to $u + 1$ is that $\hat{\rho} > \rho^{\mathrm{opt}}$ and $\hat{f}(u) \le f^{\mathrm{opt}}(u)$, one can apply this argument recursively up until $u = n - 1$, and thus obtain $\hat{f}(n) < f^{\mathrm{opt}}(n)$. Nevertheless, leveraging the constraints with $u = n$ and $v = v_n^*$ gives $b(v_n^*)v_n^* - \hat{\rho} b(n)n + (n - v_n^*)\hat{f}(n) \ge 0$, or equivalently $\hat{\rho} \le (b(v_n^*)v_n^* + (n - v_n^*)\hat{f}(n))/(b(n)n)$. Thus,

$$\hat{\rho} \le \frac{b(v_n^*)v_n^* + (n - v_n^*)\hat{f}(n)}{b(n)n} \le \frac{b(v_n^*)v_n^* + (n - v_n^*)f^{\mathrm{opt}}(n)}{b(n)n} = \rho^{\mathrm{opt}},$$

where we used the fact that $n - v_n^* \ge 0$ and $\hat{f}(n) < f^{\mathrm{opt}}(n)$. Note that $\hat{\rho} \le \rho^{\mathrm{opt}}$ contradicts the assumption $\hat{\rho} > \rho^{\mathrm{opt}}$, thus concluding this part of the proof.

Proof that f^{opt} is non-decreasing. By contradiction, let us assume f^{opt} is decreasing at some index. Lemma 2 in the Appendix shows that, if this is the case, then f^{opt} continues to decrease, so that $f^{\mathrm{opt}}(n) \le f^{\mathrm{opt}}(n - 1)$. Note that it must be $f^{\mathrm{opt}}(n) > 0$, as if it were $f^{\mathrm{opt}}(n) \le 0$, then by definition of ρ^{opt} we would have

$$\rho^{\mathrm{opt}} = \min_{v \in \{0, \dots, n\}} \frac{b(v)v + (n - v)f^{\mathrm{opt}}(n)}{nb(n)} = \frac{0 + f^{\mathrm{opt}}(n)}{b(n)} \le 0,$$

since the minimum is attained at the lowest feasible v due to $b(v)v$ and $-vf^{\mathrm{opt}}(n)$ non-decreasing and increasing, respectively. This is a contradiction as the price of anarchy is bounded already in the un-tolled setup.[8] It must therefore be that the price of anarchy is bounded also when optimal tolls are used. Additionally, as we have removed a number of constraints from the linear program, the corresponding price of anarchy will be even lower. Therefore, it must be that $1/\rho^{\mathrm{opt}}$ is non-negative and bounded, so that $\rho^{\mathrm{opt}} > 0$ contradicting the last equation.

Thus, in the following we proceed with the case of $f^{\mathrm{opt}}(n) > 0$. It must be that

$$\rho^{\mathrm{opt}} = \min_{v \in \{0, \dots, n\}} \frac{b(v)v + (n-v)f^{\mathrm{opt}}(n)}{nb(n)} \le \min_{v \in \{1, \dots, n\}} \frac{b(v)v + (n-v)f^{\mathrm{opt}}(n)}{nb(n)} \le \min_{v \in \{1, \dots, n\}} \frac{b(v)v + (n-v)f^{\mathrm{opt}}(n-1)}{nb(n)},$$

where the first inequality holds as we are restricting the domain of minimization, the second because $f^{\mathrm{opt}}(n) \le f^{\mathrm{opt}}(n-1)$ and $n - v \ge 0$. Let us observe that $f^{\mathrm{opt}}(n)$ is defined as $f(n) = \min_{v \in \{1, \dots, n\}} [b(v)v + (n-v)f^{\mathrm{opt}}(n-1)] - \rho^{\mathrm{opt}}(n-1)b(n-1)$. Substituting $\min_{v \in \{1, \dots, n\}} [b(v)v + (n-$

$v)f^{\mathrm{opt}}(n-1)] = f^{\mathrm{opt}}(n) + \rho^{\mathrm{opt}}(n-1)b(n-1)$ in the former bound on ρ^{opt}, we get

$$\rho^{\mathrm{opt}} \leq \frac{f^{\mathrm{opt}}(n) + \rho^{\mathrm{opt}}(n-1)b(n-1)}{nb(n)} \quad \Longrightarrow \quad \rho^{\mathrm{opt}} \leq \frac{f^{\mathrm{opt}}(n)}{nb(n) - (n-1)b(n-1)}.$$

We want to prove that this gives rise to a contradiction. To do so, we will show that

$$\frac{f^{\mathrm{opt}}(n)}{nb(n) - (n-1)b(n-1)} < \min_{v \in \{0,\dots,n\}} \frac{b(v)v + (n-v)f^{\mathrm{opt}}(n)}{nb(n)}. \tag{17}$$

As a matter of fact, if the latter inequality holds true, then the proof is immediately concluded as

$$\rho^{\mathrm{opt}} \leq \frac{f^{\mathrm{opt}}(n)}{nb(n) - (n-1)b(n-1)} < \min_{v \in \{0,\dots,n\}} \frac{b(v)v + (n-v)f^{\mathrm{opt}}(n)}{nb(n)} = \rho^{\mathrm{opt}} \quad \Longrightarrow \quad \rho^{\mathrm{opt}} < \rho^{\mathrm{opt}},$$

where the first inequality has been shown above, the second is what remains to be proved, and the latter equality is by definition. Therefore, we are left to show Equation (17), which holds if we can show that $\forall v \in \{0,\dots,n\}$ it is

$$g(v) \doteq \frac{h(v) + (n-v)f^{\mathrm{opt}}(n)}{h(n)} - \frac{f^{\mathrm{opt}}(n)}{h(n) - h(n-1)} > 0,$$

where $h : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a function such that $h(v) = b(v)v$ for $v \in \{0,\dots,n\}$. We choose h to be continuously differentiable, strictly increasing, and strictly convex; one such function always exists.[9] We first consider the point $v = 0$. Observe that $g(0) > 0$ when $n > 1$ as

$$g(0) = \frac{f^{\mathrm{opt}}(n)}{b(n)} - \frac{f^{\mathrm{opt}}(n)}{nb(n) - (n-1)b(n-1)} > 0 \quad \Longleftrightarrow \quad f^{\mathrm{opt}}(n)[(n-1)b(n) - (n-1)b(n-1)] > 0,$$

which holds as $f^{\mathrm{opt}}(n) > 0$, $n > 1$, and $b(n) > b(n-1)$ strictly.

If $g'(v) \geq 0$ at $v = 0$, then the proof is complete as g is convex, and due to $g'(0) \geq 0$ it is non-decreasing for any $v \geq 0$ so that the constraint will be satisfied for all $v \geq 0$.

If this is not the case, then $g'(0) < 0$, which we consider now. Note that, at the point $v = n-1$, the derivative $g'(n-1) = [h'(n-1) - f^{\mathrm{opt}}(n)]/h(n)$ satisfies

$$h(n)g'(n-1) = h'(n-1) - f^{\mathrm{opt}}(n) \geq h'(n-1) - (h(n-1) - h(n-2)) \geq 0.$$

where the last inequality is due to convexity, while the first inequality holds as $f^{\mathrm{opt}}(n) \leq h(n-1) - h(n-2)$ thanks to Lemma 2 and $n \geq 2$.[10] Therefore, since $g'(0) < 0$, $g'(n-1) \geq 0$ and g convex, there must exist an unconstrained minimizer $v^* \in (0, n-1]$. We will guarantee that $g(v^*) > 0$ so that for any (real and thus integer) $v \in [0, n]$ it is $g(v) > 0$. The unconstrained minimizer satisfies $f^{\mathrm{opt}}(n) = h'(v^*)$, which we substitute, and are thus left with proving the final inequality

$$\frac{h(v^*) + (n-v)h'(v^*)}{h(n)} - \frac{h'(v^*)}{h(n) - h(n-1)} > 0,$$

which is equivalent to

$$[h(n) - h(n-1)]h(v^*) > h'(v^*)[(n-v^*)(h(n-1) - h(n)) + h(n)],$$

where we recall $0 < v^* \leq n-1$. As the left-hand side is positive due to h increasing and $v^* > 0$, the inequality holds trivially if the right-hand side is less or equal to zero, i.e., if $h(n) \leq (n-v^*)(h(n) -$

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[9]Observe that the function $b(v)v$ is positive, strictly increasing, and strictly convex in the discrete sense in its domain due to the assumptions.

[10]In fact, either n is the first index starting from which f^{opt} decreases (i.e., $f^{\mathrm{opt}}(n) < f^{\mathrm{opt}}(n-1)$) in which case $f^{\mathrm{opt}}(n) \leq \rho^{\mathrm{opt}}[b(n-1)(n-1) - b(n-2)(n-2)] \leq b(n-1)(n-1) - b(n-2)(n-2)$ due to $\rho^{\mathrm{opt}} \leq 1$, or the function starts decreasing at a $u+1 < n$ in which case Lemma 2 also shows that

$$f^{\mathrm{opt}}(n) \leq \cdots \leq f^{\mathrm{opt}}(u+1) \leq \rho^{\mathrm{opt}}[b(u)u - b(u-1)(u-1)] \leq b(u)u - b(u-1)(u-1) \leq b(n-1)(n-1) - b(n-2)(n-2),$$

where the inequalities hold due to $\rho^{\mathrm{opt}} \leq 1$ and the convexity of $b(u)u$.

$h(n-1)$). In the other case, when $(n-v^*)(h(n-1)-h(n))+h(n) > 0$, we leverage the fact that $h'(v^*) < (h(n)-h(v^*))/(n-v^*)$ by strict convexity of $h(x)$ in $x = v^* > 0$, so that

$$h'(v^*)[(n-v^*)(h(n-1)-h(n))+h(n)] < \frac{h(n)-h(v^*)}{n-v^*}[(n-v^*)(h(n-1)-h(n))+h(n)]$$

$$= \frac{h(n)}{n-v^*}[h(n)-h(v^*)]+[h(n)-h(n-1)][h(v^*)-h(n)]$$

$$\leq [h(n)-h(n-1)]h(v^*),$$

where the last inequality follows, since $[h(n)-h(n-1)][h(v^*)-h(n)] \leq 0$ and from $\frac{h(n)-h(v^*)}{n-v^*} \leq h(n)-h(n-1)$, which holds for $0 < v^* \leq n-1$ by convexity. This concludes this part of the proof.

Proof that $(\rho^{\mathrm{opt}}, f^{\mathrm{opt}})$ *is feasible also for Equation* (5) *and final cosmetics.* Recall from the first part of the proof that the constraints in Equation (5) can be equivalently written as $b(v)v - \rho b(u)u + f(u)(u-x) - f(u+1)(v-x) \geq 0$. Since f^{opt} is non-decreasing, following the argument in Reference [28, Cor. 1] one verifies that the tightest constraints are obtained when $x = \min\{0, u+v-n\}$. These constraints are already included in our simplified program of Equation (16), with the exception of those with $v = 0$ and $u \in \{0, \ldots, n-1\}$, which we have removed. To show that also these hold, we note that the constraint with $v = 0$ reads as $uf^{\mathrm{opt}}(u) \geq \rho^{\mathrm{opt}}ub(u)$, and is trivially satisfied for $u = 0$. We now show that also the constraints with $v = 0$, $u > 0$ hold. To do so, we consider the constraint corresponding to $v = 1$,

$$b(1)1 - \rho b(u)u + uf^{\mathrm{opt}}(u) - f^{\mathrm{opt}}(u+1) \geq 0.$$

Since f^{opt} is non-decreasing as shown in previous point, then $f^{\mathrm{opt}}(u+1) \geq f^{\mathrm{opt}}(1) = b(1)$. Hence,

$$0 \leq b(1)1 - \rho b(u)u + uf^{\mathrm{opt}}(u) - f^{\mathrm{opt}}(u+1) \leq b(1)1 - \rho^{\mathrm{opt}}b(u)u + uf^{\mathrm{opt}}(u) - b(1).$$

Thus, from the left- and right-hand sides we obtain the desired result $uf^{\mathrm{opt}}(u) \geq \rho^{\mathrm{opt}}ub(u)$.

We conclude with some cosmetics: The simplified linear program in Equation (16) is almost identical to that in Equation (13), except for the constraints with $v = 0$ and $u \in \{0, \ldots, n-1\}$, which we have removed in Equation (16). Nevertheless, we have just verified that an optimal solution does satisfy these constraints, too. Hence, we simply add them back to obtain Equation (13). □

4 OPTIMAL TOLLING MECHANISMS FOR ARBITRARY NUMBER OF AGENTS

While the linear programming formulations introduced in Equation (5) and Equation (13) provide an optimal tolling mechanism and the corresponding optimal price of anarchy when the number of agents is upper-bounded by n (finite), in this section we show how to design optimal tolling mechanisms for polynomial congestion games that apply to any n (possibly infinite) by solving a linear program of fixed size. The resulting prices of anarchy are already displayed in Table 1.

For ease of exposition, we consider congestion games where the set of resource costs is produced by non-negative combinations of a *single* monomial x^d, $d \geq 1$ at a time. This is without loss of generality, as one can derive optimal tolling mechanisms for polynomial congestion games with *maximum degree d*, i.e., generated by $\{1, x, \ldots, x^d\}$, simply repeating the ensuing reasoning separately for all polynomials of degree higher than 1 and lower-equal to d. No toll need to be applied to polynomials of order zero as the corresponding price of anarchy is 1.

The idea we leverage is as follows: First, we solve a linear program of *fixed* size $\bar{n}$, from which we obtain a set of tolls that are then extended analytically to any number of agents. This produces a mechanism for which we are able to quantify the corresponding price of anarchy over games with possibly infinitely many agents. Such price of anarchy value is an upper bound on the true optimal

Table 2. Lower and Upper Bounds (LB and UB) on the Values of the Optimal Price
of Anarchy for Polynomial Congestion Games with Arbitrarily Large Number of
Agents and $d = 1, 2, 3$

$\bar{n}$	$d = 1$		$d = 2$		$d = 3$	
	LB	UB	LB	UB	LB	UB
10	2.011825	2.038237	5.097187	5.316382	15.530175	17.138429
20	2.012067	2.019844	5.100974	5.147543	15.550847	15.751993
30	2.012067	2.014335	5.100974	5.119149	15.550852	15.684195
40	2.012067	2.012067	5.100974	5.100974	15.550852	15.550859

The LB vs UB shows how the tolls derived from f^∞ defined in (19) are approximately optimal
up the fifth decimal digit when we select $\bar{n} = 40$.

price of anarchy over games with possibly infinitely many agents, as the mechanism we design is
not necessarily optimal. At the same time, we solve the linear program in Equation (13) and thus
obtain the optimal price of anarchy for games with a maximum of $\bar{n}$ agents. The latter is a lower
bound for the optimal price of anarchy over games with possibly infinitely many agents. Letting
$\bar{n}$ grow, the upper bound matches the lower bound already for small values of $\bar{n}$, see Table 2.

While the construction of the lower bound follows readily by solving the linear program in
Equation (13) with $\bar{n}$ agents, in the following we describe the procedure to derive the upper bound.
More specifically, we clarify (i) what program of dimension $\bar{n}$ we solve, (ii) how we extend its
solution from $\bar{n}$ to infinity, and (iii) how we compute the resulting price of anarchy over games
with possibly infinitely many agents. In this section, we will always select $\bar{n}$ finite and even.

As for the first point, we consider the following linear program:

$$\max_{f \in \mathbb{R}^{\bar{n}}, \rho \in \mathbb{R}} \rho$$

$$\text{s.t.} \quad v^{d+1} - \rho u^{d+1} + f(u)u - f(u+1)v \geq 0 \qquad \forall u, v \in \{0, \ldots, \bar{n}\} \quad u + v \leq \bar{n},$$

$$v^{d+1} - \rho u^{d+1} + f(u)(\bar{n} - v) - f(u+1)(\bar{n} - u) \geq 0 \qquad \forall u, v \in \{0, \ldots, \bar{n}\} \quad u + v > \bar{n},$$

$$f(u) \leq u^d \qquad \forall u \in \{1, \ldots, \bar{n}\}$$

$$f(u) \geq f(u-1) \qquad \forall u \in \{2, \ldots, \bar{n}\},$$

(18)

with the usual convention that $f(0) = f(\bar{n} + 1) = 0$. Note that the previous program is identical to that
in Equation (13) with $b(x) = x^d$, except that we have included two additional sets of constraints. We let
$(f^{\mathrm{opt}}, \rho^{\mathrm{opt}})$ be a solution of this program and utilize it to define $f^\infty : \mathbb{N} \to \mathbb{R}$ as follows:

$$f^\infty(x) = \begin{cases} f^{\mathrm{opt}}(x) & \text{for} \quad x \leq \bar{n}/2 \\ \beta \cdot x^d & \text{for} \quad x > \bar{n}/2 \end{cases}, \quad \text{where} \quad \beta = \frac{f^{\mathrm{opt}}(\bar{n}/2)}{(\bar{n}/2)^d}. \tag{19}$$

Informally, the idea is to extend $\tau^\infty(x) = f^\infty(x) - b(x)$ from $\bar{n}/2$ to infinity with a polynomial of the
same order of the original x^d. Note that $\beta \geq 0$ is chosen so that the two expressions defining f^∞ match
for $x = \bar{n}/2$.[11] While the expression of f^∞ and all forthcoming quantities depends on the choice of $\bar{n}$,
we do not make this explicit to ease the notation. Lemma 3 in the Appendix ensures that the price of
anarchy of f^∞ is identical for pure Nash and coarse correlated equilibria, and is upper bounded over
games with possibly infinitely many agents by $1/\rho^\infty$, where ρ^∞ is given by

$$\rho^\infty = \min \left\{ \rho^{\mathrm{opt}}, \ \beta - d \left(1 + \frac{2}{\bar{n}}\right)^{d+1} \left(\frac{\beta}{d+1}\right)^{1+\frac{1}{d}} \right\}.$$

---

[11]Observe that $f^\infty(1) \geq 0$, since having $f^\infty(1) < 0$ would always result in a lower performance, as shown in [28]. Therefore
$f^\infty(\bar{n}/2) \geq 0$ as it is feasible for Equation (18), which includes the constraint $f(x+1) \geq f(x)$. Hence, $\beta \geq 0$.

As clarified above this represents an upper bound on the optimal price of anarchy. The upper and lower bounds displayed in Table 2 have been computed according to the procedure just described, and demonstrate that, for a relatively small $\bar{n} = 40$, the mechanism obtained from f^∞ is approximately optimal up to the fifth decimal digit for polynomial congestion games with $d = 1, 2, 3$.

Finally, we observe that the tolling mechanism $T^\infty(\alpha \ell) = \alpha T^\infty(\ell) = \alpha \tau^\infty$, where $\tau^\infty(x) = f^\infty(x) - b(x)$ might not satisfy $\tau^\infty(x) \geq 0$ for all $x \in \mathbb{N}$ (i.e., they might be monetary incentives and not tolls). Nevertheless, multiplying f^∞ with a factor $\gamma > 0$ produces tolls $\gamma f^\infty(x) - b(x)$ with identical price of anarchy (the proof of Lemma 3 will hold with $\nu = 1/\gamma$ in place of $\nu = 1$). Therefore, one simply needs to consider tolls of the form $\gamma f^\infty(x) - b(x)$, where γ is chosen sufficiently large to ensure that $\gamma f^\infty(x) - b(x) \geq 0$ for all $x \in \mathbb{N}$; one such γ always exists. When multiple basis are present, we select γ as a common scaling factor to ensure non-negativity of all tolls basis.

5   CONGESTION-INDEPENDENT TOLLING MECHANISMS

In this section, we provide a general methodology to compute optimal congestion-independent local tolling mechanisms for games generated by $\{b_1, \ldots, b_m\}$. We also specialize the result to polynomial congestion games providing explicit expressions for the tolls and the prices of anarchy. In this section, we consider basis functions that are convex in the discrete sense (see Footnote 5).

THEOREM 3.   *A local congestion-independent mechanism minimizing the price of anarchy over congestion games with n agents, and resource costs* $\ell(x) = \sum_{j=1}^m \alpha_j b_j(x)$, $\alpha_j \geq 0$, *with convex positive non-decreasing basis functions* $\{b_1, \ldots, b_m\}$ *is given by*

$$T^{\mathrm{opt}}(\ell) = \sum_{j=1}^m \alpha_j \cdot \tau_j^{\mathrm{opt}}, \quad \text{where } \tau_j^{\mathrm{opt}} \in \mathbb{R}, \quad \tau_j^{\mathrm{opt}} = \left(\frac{1}{\nu^{\mathrm{opt}}} - 1\right) b_j(1) \quad (20)$$

and $\rho^{\mathrm{opt}} \in \mathbb{R}$, $\nu^{\mathrm{opt}} \in \mathbb{R}_{\geq 0}$ *solve the linear program*

$$\max_{\rho \in \mathbb{R}, \, \nu \in [0,1]} \rho$$

$$\text{s.t.} \qquad b_j(v)v - \rho b_j(u)u + v[b_j(u)u - b_j(u+1)v] + b_j(1)(1-v)(u-v) \geq 0$$

$$\forall u, v \in \{0, \ldots, n\} \quad u + v \leq n \quad u \geq v, \quad \forall j \in \{1, \ldots, m\},$$

$$b_j(v)v - \rho b_j(u)u + v[b_j(u)(n-v) - b_j(u+1)(n-u)] + b_j(1)(1-v)(u-v) \geq 0$$

$$\forall u, v \in \{0, \ldots, n\} \quad u + v > n \quad u \geq v, \quad \forall j \in \{1, \ldots, m\},$$

$$(21)$$

where we define $b_j(0) = b_j(n+1) = 0$. *Correspondingly,* $\mathrm{PoA}(T^{\mathrm{opt}}) = 1/\rho^{\mathrm{opt}}$, *and the optimal tolls are non-negative.*[12] *The result is tight for pure Nash equilibria and extends to coarse correlated equilibria.*

The optimal prices of anarchy arising from the solution of Equation (21) for polynomials of order at most $d = 1, 2, \ldots, 6$ and $n = 100$ are shown in the fifth column of Table 1. Before proceeding with proving the theorem, we specialize its result to polynomial congestion games with $d \geq 2$ and arbitrarily large n. This allows us to derive explicit expressions matching the values featured in Table 1 and holding for arbitrarily large n. We do not study the case of $d = 1$ as this has been analyzed in Reference [8], resulting in an optimal price of anarchy of $1 + 2/\sqrt{3} \approx 2.15$, which we also recover through the solution of the linear program above.

COROLLARY 1.   *Consider polynomial congestion games of maximum degree $d = 2$ and arbitrarily large number of agents, i.e., congestion game where the cost on resource e is* $\ell_e(x) = \alpha_e x^2 + \beta_e x + \gamma_e$,

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[12]The result also holds if convexity of $b_j(x)$ is weakened to convexity of $b_j(x)x$. One example is that of $b_j(x) = \sqrt{x}$.

with non-negative $\alpha_e, \beta_e, \gamma_e$. An optimal congestion-independent mechanism satisfies

$$T^{\text{opt}}(\ell_e) = 3\alpha_e, \qquad \text{PoA}(T^{\text{opt}}) = \frac{16}{3} \approx 5.33. \tag{22}$$

The result is tight for pure Nash equilibria and extends to coarse correlated equilibria.

Following a similar line of reasoning to that of Corollary 1 (see the next page for its proof), it is possible to derive an expression for the optimal price of anarchy with constant tolls also in the case of $3 \le d \le 6$, i.e.,

$$\text{PoA}(T^{\text{opt}}) = \frac{\bar{u}(\bar{u}+1)^{d+1} - \bar{u}^{d+1}[\bar{u} + (\bar{u}+2)^d] + (\bar{u}+1)^{2d+1} - (\bar{u}+1)^{d+1}}{\bar{u}(\bar{u}+1)((\bar{u}+1)^d - \bar{u}^d) + (\bar{u}+1)^{d+1} - \bar{u}(\bar{u}+2)^d - 1}, \tag{23}$$

where $\bar{u}$ is the floor of the unique real positive solution to $u^{d+1} + 1 = (u+1)^d + u$. For example,

$$d = 3 \quad \Longrightarrow \quad \bar{u} = 2, \quad \text{PoA}(T^{\text{opt}}) = \frac{2 \cdot 3^4 - 2^4 \cdot (2 + 4^3) + 3^7 - 3^4}{2 \cdot 3 \cdot (3^3 - 2^3) + 3^4 - 2 \cdot 4^3 - 1} = \frac{1212}{66} \approx 18.36.$$

Similarly, with $d = 4, \ldots, 6$, it is, respectively, $\text{PoA}(T^{\text{opt}}) = 111588/1248 \approx 89.41$, $\text{PoA}(T^{\text{opt}}) = 1922184/4092 \approx 469.74$, $\text{PoA}(T^{\text{opt}}) = 32963196/9912 \approx 3325.58$, matching the values in Table 1. While we do not formally prove the expression (23) in the interest of conciseness, the key idea consists in observing that the two most binding constraints appearing in the linear program of Theorem 3 are those obtained with $(u, v) = (\bar{u}, 1)$ and $(u, v) = (\bar{u} + 1, 1)$. Turning the corresponding inequalities into equalities and solving for ρ and v gives the result in Equation (23).

We now focus on proving Theorem 3, followed by Corollary 1.

PROOF OF THEOREM 3. The fact that optimal local congestion-independent tolls are linear in the sense that $T^{\text{opt}}(\ell) = T^{\text{opt}}(\sum_{j=1}^{m} \alpha_j b_j) = \sum_{j=1}^{m} \alpha_j T^{\text{opt}}(b_j)$ can be proven following the same steps of Theorem 1. Therefore it suffices to determine the best linear local congestion-independent toll. Toward this goal, we observe that the price of anarchy of a *given* linear local constant toll $T(\sum_{j=1}^{m} \alpha_j b_j) = \sum_{j=1}^{m} \alpha_j \tau_j$, $\tau_j \in \mathbb{R}_{\ge 0}$ can be determined as the solution of the following program, which applies, thanks to Equation (12), since $f_j(x) = b_j(x) + \tau_j$ is non-decreasing,

$$\max_{\rho \in \mathbb{R}, v \in \mathbb{R}_{\ge 0}} \rho$$

$$\text{s.t.} \qquad b_j(v)v - \rho b_j(u)u + v[(b_j(u) + \tau_j)u - (b_j(u+1) + \tau_j)v] \ge 0$$

$$\forall u, v \in \{0, \ldots, n\} \quad u + v \le n, \quad \forall j \in \{1, \ldots, m\}, \quad (24)$$

$$b_j(v)v - \rho b_j(u)u + v[(b_j(u) + \tau_j)(n - v) - (b_j(u+1) + \tau_j)(n - u)] \ge 0$$

$$\forall u, v \in \{0, \ldots, n\} \quad u + v > n, \quad \forall j \in \{1, \ldots, m\}.$$

We also recall that Equation (24) tightly characterizes the price of anarchy for pure Nash equilibria, and the corresponding bound extends to coarse correlated equilibria. Determining the best non-negative toll amounts to letting $(\tau_1, \ldots, \tau_m) \in \mathbb{R}_{\ge 0}^m$ be decision variables, over which we need to maximize. While this would result in a bi-linear program, we define $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{R}_{\ge 0}^m$ with $\sigma_j = v\tau_j$, and consider the following linear program:

$$\max_{\rho \in \mathbb{R}, \ v \in \mathbb{R}_{\ge 0}, \ \sigma \in \mathbb{R}_{\ge 0}^m} \rho$$

$$\text{s.t.} \qquad b_j(v)v - \rho b_j(u)u + v[b_j(u)u - b_j(u+1)v] + \sigma_j(u - v) \ge 0$$

$$\forall u, v \in \{0, \ldots, n\} \quad u + v \le n, \quad \forall j \in \{1, \ldots, m\},$$

$$b_j(v)v - \rho b_j(u)u + v[b_j(u)(n - v) - b_j(u+1)(n - u)] + \sigma_j(u - v) \ge 0$$

$$\forall u, v \in \{0, \ldots, n\} \quad u + v > n, \quad \forall j \in \{1, \ldots, m\},$$
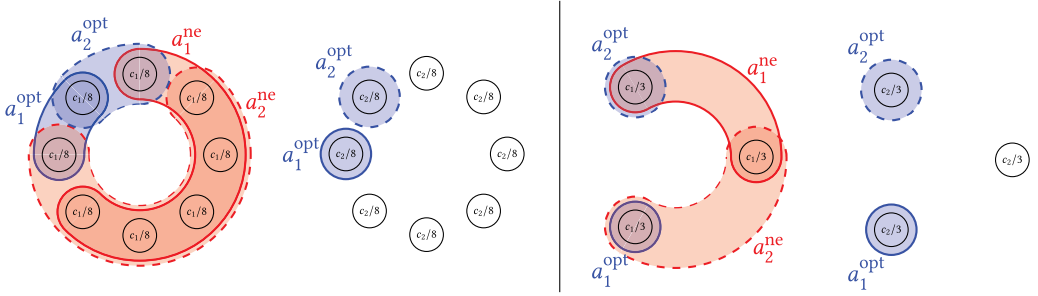$$\tag{25}$$

Fig. 2. Game construction used to lower bound the price of anarchy for quadratic congestion games with fixed tolls $\tau \geq 3$ (left) and $\tau < 3$ (right). On the left (respectively, right), the available actions of two of the eight (respectively, three) agents are shown. The solid red shape contains the resources utilized by the first user in the action a_1^{ne}, while the solid blue shape contains the resources utilized by the first user in the action a_1^{opt}. User 2 has similar actions but rotated clockwise on each circle by one resource. Each of the remaining agents' actions are defined similarly by rotating about the apparent "ring."

which is an exact reformulation of Equation (24), except for the fact that we are not including the (non-linear) constraint requiring $\sigma_j = 0$ whenever $v = 0$. We will rectify this at the end by showing that $v^{\text{opt}} > 0$.

Lemma 5 in the Appendix leverages the fact that the basis functions are convex, positive, non-decreasing by assumption, so that only the constraints with $u \geq v, u \geq 1$ and $(u, v) = (0, 1)$ need to be accounted for in Equation (25). Due to the fact that $u - v \geq 0$ for $u \geq 1$, to maximize ρ, we choose σ_j as large as possible. Observing that the only upper bound on σ_j arises from the choice of $(u, v) = (0, 1)$ and reads as $\sigma_j \leq (1 - v)b_j(1)$, we set $\sigma_j = (1 - v)b_j(1)$, and translate the constraint $\sigma_j \geq 0$ into $v \leq 1$, thus obtaining Equation (21). To conclude we are left to show that v^{opt} solving Equation (21) is non-zero. To do so, note that solving the program for fixed $v = 0$ results in $\rho = b(1)/b(n)$ (the tightest constraint is $(u, v) = (n, 0)$), while an arbitrarily small but positive v would give a strictly higher ρ.

Once v^{opt} is determined, the optimal tolls can be derived from $v^{\text{opt}}\tau_j^{\text{opt}} = (1 - v^{\text{opt}})b_j(1)$, recalling that $v^{\text{opt}} > 0$, thus yielding (20). Non-negativity of the tolls follow from the fact that we impose $v \leq 1$ so that $\tau_j^{\text{opt}} = (1/v^{\text{opt}} - 1) b_j(1) \geq 0$. □

We now focus on Corollary 1 and prove the result through a different approach other than applying Theorem 3, with the hope of providing the reader with an independent perspective.

PROOF OF COROLLARY 1. We prove the claim in two steps. First, we show that the price of anarchy for any constant toll and pure Nash equilibria is lower-bounded by 16/3. Second, we show that the price of anarchy of T^{opt} is upper-bounded by 16/3 for both Nash and coarse correlated equilibria.

For the lower bound, it suffices to consider resource costs of the form $\ell_e(x) = \alpha_e x^2$, whereby any constant linear tolling mechanisms takes the form $T(\ell_e) = \alpha_e \tau$, for some scalar $\tau \geq 0$. For any $\tau \geq 3$ we consider the following problem instance: There are 8 agents each with two actions a_i^{ne} and a_i^{opt}. In action a_i^{ne}, user i selects 6 of the available 8 resources, which are associated to $c_1 x^2/8$; in a_i^{opt} user i selects the remaining two resources with costs $c_1 x^2/8$, as well as one resource with cost $c_2 x^2/8$ (we will fix c_1 and c_2 at a later stage). Each player has a similar pair of actions, but each subsequent agent is offset by one from the prior user, as depicted in Figure 2.

In this game, the system and user costs can be computed as in the following $\text{SC}(a^{\text{ne}}) = \sum_e |a|_e \ell_e(|a|_e) = 8 \cdot 6\ell_e(6) = 8 \cdot 6c_1 b(6)/8$, $C_i(a^{\text{ne}}) = \sum_{e \in a_i}[\ell_e(|a|_e) + \alpha_e \tau] = 6 \cdot (c_1 b(6)/8 + c_1\tau/8) = 6c_1(b(6) + \tau)/8$, and similarly

$$\text{SC}(a^{\text{ne}}) = 6(c_1 b(6)) = 216c_1, \qquad\qquad \text{SC}(a^{\text{opt}}) = 2c_1 b(2) + c_2 b(1) = 8c_1 + c_2,$$

$$C_i(a^{\text{ne}}) = \frac{6}{8}c_1(b(6) + \tau) = \frac{1}{8}(216c_1 + 6c_1\tau), \qquad C_i(a_i^{\text{opt}}, a_{-i}^{\text{ne}}) = \frac{2}{8}c_1(49 + \tau) + \frac{1}{8}c_2(1 + \tau). \tag{26}$$

We normalize the costs in the game setting $\text{SC}(a^{\text{ne}}) = 1$, which results in $c_1 = 1/216$ from Equation (26). To ensure that the joint action a^{ne} is a Nash equilibrium (at least weakly), we impose that $C_i(a^{\text{ne}}) = C_i(a_i^{\text{opt}}, a_{-i}^{\text{ne}})$ for any player i. This condition is satisfied when $c_2 = (2\tau + 59)/(108(1 + \tau))$. Hence, the price of anarchy in this game is lower-bounded by $\text{SC}(a^{\text{ne}})/\text{SC}(a^{\text{opt}}) = 1/\text{SC}(a^{\text{opt}}) = 1/(8c_1 + c_2)$. This expression is no smaller than 16/3 for any choice of $\tau \geq 3$ (in particular, equal when we set $\tau = 3$), where we have utilized the values of c_2 from above and $c_1 = 1/216$.

For $\tau < 3$ we construct a game with similar features. There are 3 users each with actions a_i^{ne} and a_i^{opt}. In action a_i^{ne}, user i selects 2 of the 3 available resources featuring a cost $c_1 x^2/3$; in a_i^{opt} they select the remaining resource with cost $c_1 x^2/3$, as well as one resource with cost $c_2 x^3/3$. Each user has a similar pair of actions, but each subsequent agent is offset by one from the prior (see Figure 2). In this game, we obtain the following system and user costs:

$$\text{SC}(a^{\text{ne}}) = 2(c_1 b(2)) = 8c_1, \qquad\qquad \text{SC}(a^{\text{opt}}) = c_1 b(1) + c_2 b(1) = c_1 + c_2$$

$$C_i(a^{\text{ne}}) = \frac{2}{3}c_1(b(2) + \tau) = \frac{1}{3}(8c_1 + 2c_1\tau), \qquad C_i(a_i^{\text{opt}}, a_{-i}^{\text{ne}}) = \frac{c_1}{3}(9 + \tau) + \frac{c_2}{3}(1 + \tau).$$

As in the previous example, we set $\text{SC}(a^{\text{ne}}) = 1$ implying $c_1 = 1/8$. To ensure that the joint action a^{ne} is a Nash equilibrium, we impose that $C_i(a^{\text{ne}}) = C_i(a_i^{\text{opt}}, a_{-i}^{\text{ne}})$ for any player i, resulting in $c_2 = (\tau - 1)/(8(1 + \tau))$. The resulting price of anarchy is lower-bounded by $1/\text{SC}(a^{\text{opt}}) = 1/(c_1 + c_2)$. This quantity is no smaller than 16/3 for any choice of $\tau < 3$.

Finally, for the fixed toll $\tau = 3$, we upper-bound the price of anarchy at 16/3. For ease of presentation, we first consider the case where the cost on resource e is $\ell_e(x) = \alpha_e x^2$ for some $\alpha_e \geq 0$. We will show at the end how to extend this result to the case of $\ell_e(x) = \alpha_e x^2 + \beta_e x + \gamma_e$. Toward this goal, let a^{ne} (respectively, a^{opt}) be an equilibrium (respectively, optimum) allocation in a congestion game G, with n users, resources in $e \in \mathcal{E}$. The cost at equilibrium satisfies

$$\text{SC}(a^{\text{ne}}) \leq \sum_{i=1}^n C_i(a_i^{\text{opt}}, a_{-i}^{\text{ne}}) - \sum_{i=1}^n C_i(a^{\text{ne}}) + \text{SC}(a^{\text{ne}}), \tag{27}$$

$$= \sum_{e \in \mathcal{E}} \alpha_e \left[z_e((x_e + y_e + 1)^2 + \tau) - y_e((x_e + y_e)^2 + \tau) + (x_e + y_e)^3 \right], \tag{28}$$

$$\leq \sum_{e \in \mathcal{E}} \alpha_e [(x_e + z_e)((x_e + y_e + 1)^2 + 3) - (x_e + y_e)((x_e + y_e)^2 + 3) + (x_e + y_e)^3], \tag{29}$$

$$= \sum_{e \in \mathcal{E}} \alpha_e \left[3((x_e + z_e) - (x_e + y_e)) + (x_e + z_e)(x_e + y_e + 1)^2 \right]$$

$$\leq \sum_{e \in \mathcal{E}} \alpha_e \left[4(x_e + z_e)^3 + \frac{1}{4}(x_e + y_e)^3 \right] = 4\text{SC}(a^{\text{opt}}) + \frac{1}{4}\text{SC}(a^{\text{ne}}), \tag{30}$$

where $y_e = |a^{\text{ne}}|_e - x_e$, $z_e = |a^{\text{opt}}|_e - x_e$, and $x_e = |\{i \in N \text{ s.t. } e \in a_i^{\text{ne}} \cap a_i^{\text{opt}}\}|$. Observe that Equation (27) holds from the definition of Nash equilibrium, while Equation (28) follows from the parameterization introduced in Reference [28] and substituting $b_e(x) = x^2$. Equation (29) follows by replacing $\tau = 3$ and by $x_e \geq 0$. To see that Equation (30) holds for all integers $x_e, y_e \geq 0$, we define

$u = x_e + y_e \geq 0$, $v = x_e + z_e \geq 0$, and divide the argument in two parts depending on whether the integer tuple $(u, v) \in \{u \geq 22 \text{ or } v \geq 8\}$ or not. For the case of $(u, v) \in \{u \geq 22 \text{ or } v \geq 8\}$, we observe that $4v^3 + \frac{1}{4}u^3 - 3v + 3u - v(u + 1)^2 \geq 4v^3 + \frac{1}{4}u^3 - v(u^2 + 2u + 4) \geq 4v^3 + \frac{1}{4}u^3 - v(u + 2)^2$, and therefore we are left to prove

$$4v^3 + \frac{1}{4}u^3 - v(u + 2)^2 \geq 0. \tag{31}$$

For every fixed $u \geq 0$, differentiating with respect to v shows that the left-hand side of Equation (31) has a unique global minimum in the positive orthant at $v = (u + 2)/\sqrt{12}$. For any $u > 22$, this minimum satisfies Equation (31), thus for any $v \geq 0$ and $u > 22$ (31) is satisfied. Additionally observe that, when $v = 8$, Equation (31) is satisfied for each $u \in \{0, \dots, 22\}$. Further, for fixed $0 \leq u \leq 22$, the left-hand side of Equation (31) is increasing in v for $v \geq 8$. This implies that Equation (31) holds for every $v \geq 8$ as well. Therefore Equation (31) (and consequently Equation (30)) is satisfied for all $(u, v) \in \{u \geq 22 \text{ or } v \geq 8\}$. One can enumerate the finitely many non-negative integers (u, v) with $u < 22$, $v < 8$ and verify that Equation (30) holds.

The inequality in Equation (30) implies that the price of anarchy is upper bounded by $\frac{4}{1-1/4} = 16/3$ when resource costs take the form $\ell_e(x) = \alpha_e x^2$ for some $\alpha_e \geq 0$. Observe that this bound holds for arbitrarily large n and matches the solution of the linear program, stated in Table 1. We now generalize this result to $\ell_e(x) = \alpha_e x^2 + \beta_e x + \gamma_e$, where α_e, β_e, and γ_e are non-negative. To do so, we start from Equation (27) and note that Equation (28) now contains the sum of three contributions: contributions relative to α_e, contributions relative to β_e, and contributions relative to γ_e. Hence, it suffices to prove the following two additional inequalities to complete the reasoning, that is,

$$\sum_{e \in \mathcal{E}} \beta_e \left[z_e(x_e + y_e + 1) - y_e(x_e + y_e) + (x_e + y_e)^2 \right] \leq \sum_{e \in \mathcal{E}} \beta_e \left[4(x_e + z_e)^2 + \frac{1}{4}(x_e + y_e)^2 \right], \tag{32}$$

$$\sum_{e \in \mathcal{E}} \gamma_e \left[z_e - y_e + x_e + y_e \right] \leq \sum_{e \in \mathcal{E}} \gamma_e \left[4(x_e + z_e) + \frac{1}{4}(x_e + y_e) \right], \tag{33}$$

where we recall that no toll is associated to the presence of β_e or γ_e in Equation (22). Summing these two inequalities with the inequality from Equation (28) and Equation (30) will, in fact, yield the desired claim. While the proof of Equation (33) is immediate, the argument used to show that Equation (32) is similar to that following Equation (28). In fact, since $x_e \geq 0$, we have $z_e(x_e + y_e + 1) - y_e(x_e + y_e) + (x_e + y_e)^2 \leq (x_e + z_e)(x_e + y_e + 1) - (x_e + y_e)^2 + (x_e + y_e)^2$. Thus, we are left to show that for every non-negative integer u and v it is $4v^2 + \frac{1}{4}u^2 - v - uv \geq 0$, where we make use of the same coordinates introduced earlier. This inequality is satisfied by all non-negative integer points, since $4v^2 + \frac{1}{4}u^2 - v - uv \geq 0$ describes the region outside an ellipse located in the (u, v) plane entirely on the left of the line $u = 1$ and entirely south of the line $v = 1$, where the inequality is trivially satisfied for $u = v = 0$. Finally, we observe that the technique used to bound the price of anarchy extends to coarse correlated equilibria due to linearity of the expectation. □

6 (IN)EFFICIENCY OF THE MARGINAL COST MECHANISM

In this section, we study the efficiency of the marginal cost mechanism, whereby the toll imposed to each user corresponds to her marginal contribution to the system cost. In the atomic setup, the marginal cost mechanism takes the form $T^{\mathrm{p}}(\ell) = \tau^{\mathrm{p}}$, and the corresponding tolls read $\tau^{\mathrm{p}}(x) = (x - 1)(\ell(x) - \ell(x - 1))$, where we set $\ell(0) = 0$. We recall that T^{p} ensures that the best performing equilibrium is optimal, i.e., its price of stability is 1 [23]. The following Corollary shows how to compute $\mathrm{PoA}(T^{\mathrm{p}})$ through the solution of a linear program when bases are discrete convex (see

Nash routing, $\tau_e(x) = 0$ Nash routing, $\tau_e^{\mathrm{p}}(x) = x - 1$
(A) System Cost: 2   (B) System cost: 6
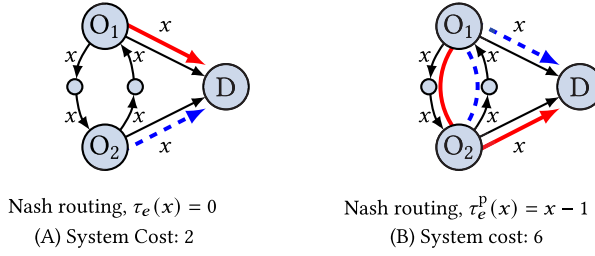
Fig. 3. Instance used to demonstrates that the price of anarchy associated to the marginal cost toll mechanism is at least 3 in affine congestion games. Two users travel from O_1/O_2 to D, where each edge features a cost $\ell_e(x) = x$. In the un-tolled case (A) the system cost at the worst Nash-equilibrium is 2. The situation worsens when using marginal cost tolls (B), as the worst Nash equilibrium gives a system cost of 6.

Footnote 5). We also provide the analytical expression of $\mathrm{PoA}(T^{\mathrm{p}})$ for polynomial congestion games with $d = 1$, and note that a similar argument carries over when $d \geq 1$.

COROLLARY 2. *The price of anarchy of the marginal cost mechanism $T^{\mathrm{p}}(\ell) = \tau^{\mathrm{p}}$, with $\tau^{\mathrm{p}}(x) = (x - 1)(\ell(x) - \ell(x - 1))$ over congestion games with n agents, and resource costs generated by a non-negative linear combination of convex basis functions $\{b_1, \ldots, b_m\}$ equals $1/\rho^{\mathrm{opt}}$, where ρ^{opt} solves the following linear program*

$$\max_{\rho \in \mathbb{R}, v \in \mathbb{R}_{\geq 0}} \rho$$

s.t.

$$b_j(v)v - \rho b_j(u)u + v[(u^2 - uv)b_j(u) - u(u - 1)b_j(u - 1) - v(u + 1)b_j(u + 1)] \geq 0$$
$$\forall u, v \in \{0, \ldots, n\} \quad u + v \leq n, \quad \forall j \in \{1, \ldots, m\},$$

$$b_j(v)v - \rho b_j(u)u + v[ub_j(u)(2n - u - v) + (u - 1)b_j(u - 1)(v - n) + (u + 1)b_j(u + 1)(u - n)] \geq 0$$
$$\forall u, v \in \{0, \ldots, n\} \quad u + v > n, \quad \forall j \in \{1, \ldots, m\},$$

where we set $b_j(-1) = b_j(0) = b_j(n + 1) = 0$.[13]
For affine congestion games with arbitrarily large number of agents, we have $\mathrm{PoA}(T^{\mathrm{p}}) = 3$. Both results are tight for pure Nash equilibria, and also hold for coarse correlated equilibria.

PROOF. We begin with the first claim, and observe that the marginal cost mechanism is linear, in the sense that $T^{\mathrm{p}}(\sum_{j=1}^{m} \alpha_j b_j) = \sum_{j=1}^{m} \alpha_j T^{\mathrm{p}}(b_j)$ for all non-negative α_j and for all basis functions. Additionally, the functions $f_j(x) = b_j(x) + (x - 1)(b_j(x) - b_j(x - 1))$ are non-decreasing in their domain. This is because

$$f_j(x + 1) - f_j(x) = b_j(x + 1) + x(b_j(x + 1) - b_j(x)) - b_j(x) - (x - 1)(b_j(x) - b_j(x - 1))$$
$$= (x + 1)b_j(x + 1) - xb_j(x) - (xb_j(x) - (x - 1)b_j(x - 1)) \geq 0,$$

for all $x \in \{1, \ldots, n - 1\}$, where the inequality holds as each function $b_j(x)x$ is convex (since each $b_j(x)$ is so), and thus its discrete derivative is non-decreasing. It follows that the price of anarchy can be computed using the linear program in Equation (12), which provides tight results for pure Nash equilibria that extend to coarse correlated equilibria. Substituting $f_j(x) = b_j(x) + (x - 1)(b_j(x) - b_j(x - 1))$ in Equation (12) we obtain the desired result.

We now focus on affine congestion games, and prove that $\mathrm{PoA}(T^{\mathrm{p}}) = 3$. Toward this goal, we observe that Figure 3 is an example of an affine congestion game using the marginal cost mechanism. Thus, we conclude that $\mathrm{PoA}(T^{\mathrm{p}}) \geq 3$ for affine congestion games and pure Nash equilibria.

---

[13]The result also holds under the weaker requirement that only $b_j(x)x$ are convex.

We now show that $\mathrm{PoA}(T^{\mathrm{p}}) \leq 3$ whenever each resource e is associated to a cost $\ell_e(x) = \alpha_e x + \beta_e$. In this case, we have $T^{\mathrm{p}}(\ell_e) = \tau^{\mathrm{p}}$, where $\tau^{\mathrm{p}}(x) = \alpha_e(x-1)$ is independent of β_e, thanks to its definition. For an equilibrium allocation $a^{\mathrm{ne}} \in \mathcal{A}$ and $a^{\mathrm{opt}} \in \mathcal{A}$ and optimal allocation, we have

$$
\begin{aligned}
\mathrm{SC}(a^{\mathrm{ne}}) &\leq \sum_{i=1}^{n} \mathrm{C}_i(a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}) - \sum_{i=1}^{n} \mathrm{C}_i(a^{\mathrm{ne}}) + \mathrm{SC}(a^{\mathrm{ne}}) \\
&= \sum_{e \in \mathcal{E}} \alpha_e \Big[z_e(2x_e + 2y_e + 1) - y_e(2x_e + 2y_e - 1) + (x_e + y_e)^2 \Big] + \beta_e [z_e - y_e + (x_e + y_e)] \\
&\leq \sum_{e \in \mathcal{E}} \alpha_e \Big[(x_e + y_e) - (x_e + y_e)^2 + 2(x_e + y_e)(x_e + z_e) + (x_e + z_e) \Big] + \beta_e [x_e + z_e] \\
&\leq \sum_{e \in \mathcal{E}} \alpha_e \Big[3(x_e + z_e)^2 \Big] + \beta_e \Big[3(x_e + z_e) \Big] = 3 \cdot \mathrm{SC}(a^{\mathrm{opt}}),
\end{aligned}
$$

where we utilize the notation $y_e = |a^{\mathrm{ne}}|_e - x_e$, $z_e = |a^{\mathrm{opt}}|_e - x_e$, and $x_e = |\{i \in N \text{ s.t. } e \in a_i^{\mathrm{ne}} \cap a_i^{\mathrm{opt}}\}|$. The first inequality holds by definition of Nash equilibrium, and the second holds due to non-negativity of x_e and α_e. One verifies that the last inequality holds, using $u = x_e + y_e \geq 0$, $v = x_e + z_e \geq 0$, and observing that the region $3v^2 + u^2 - u - 2uv - v \geq 0$ is the exterior of an ellipse containing all $(u, v) \in \mathbb{Z}_{\geq 0}^2$. We remark that the latter reasoning extends identical to coarse correlated equilibria exploiting linearity of the expectation. Rearranging the above inequality, we get $\mathrm{PoA}(T^{\mathrm{p}}) \leq \mathrm{SC}(a^{\mathrm{ne}})/\mathrm{SC}(a^{\mathrm{opt}}) \leq 3$ for pure Nash as well as coarse correlated equilibria. □

7 CONCLUSIONS AND OPEN PROBLEMS

This work derives optimal local tolling mechanisms and corresponding prices of anarchy for atomic congestion games. We do so for both the setup where tolls are congestion-aware and congestion-independent. Finally, we derive price of anarchy values for the marginal cost mechanism. Our results generalize those of Reference [8], and show that the efficiency of optimal tolls utilizing solely local information is comparable to that of existing tolls using global information [6]. Further, we show that utilizing the marginal cost mechanism on the atomic setup is worse than levying no toll.

Open questions. Our work leaves a number of open questions, two of which are discussed next.

—While we observed that the price of anarchy for optimally tolled affine congestion games matches that of affine load balancing games on identical machines, we conjecture such result holds more generally, at least for polynomial congestion games.
—In this manuscript we focused on the worst-case efficiency metric both with respect to the game instance, and the resulting equilibrium. It is currently unclear if, and to what extent, optimizing the price of anarchy impacts other more optimistic performance metrics.

A APPENDIX

A.1 Results used in Section 2

LEMMA 1. *Consider the class of congestion games $\mathcal{G}$. For any linear tolling mechanism T, it is*

$$
\mathrm{PoA}(T) = \sup_{G \in \mathcal{G}(\mathbb{Z}_{\geq 0})} \frac{\mathtt{NECost}(G, T)}{\mathtt{MinCost}(G)},
$$

where $\mathcal{G}(\mathbb{Z}_{\geq 0}) \subset \mathcal{G}$ is the subclass of games with $\alpha_j \in \mathbb{Z}_{\geq 0}$ for all $j \in \{1, \dots, m\}$, for all resources in $\mathcal{E}$.

PROOF. We divide the proof in two steps. First, we show that

$$\text{PoA}(T) = \sup_{G \in \mathcal{G}(\mathbb{Q}_{\geq 0})} \frac{\text{NECost}(G, T)}{\text{MinCost}(G)}, \tag{34}$$

where $\mathcal{G}(\mathbb{Q}_{\geq 0}) \subset \mathcal{G}$ is the subclass of games with $\alpha_j \in \mathbb{Q}_{\geq 0}$ for all $j \in \{1, \dots, m\}$, for all resources in $\mathcal{E}$. Toward this goal, observe that Equation (34) holds trivially with $\geq$ in place of the equality sign, as $\mathbb{R}_{\geq 0} \supset \mathbb{Q}_{\geq 0}$. To show that the converse inequality also holds, observe that the price of anarchy of a given linear mechanisms T (computed over all meaningful instances where $\text{NECost}(G, T) > 0$) can be computed utilizing the linear program reported in Equation (11). By strong duality, we have $\text{PoA}(T) = 1/C^*$, where C^* is the value of the dual program of Equation (11), i.e.,

$$C^* = \underset{\theta(x,y,z,j)}{\text{minimize}} \sum_{x,y,z,j} b_j(x+z)(x+z)\theta(x,y,z,j), \tag{35a}$$

$$\text{s.t.} \quad \sum_{x,y,z,j} \left[f_j(x+y)y - f_j(x+y+1)z \right] \theta(x,y,z,j) \leq 0, \tag{35b}$$

$$\sum_{x,y,z,j} b_j(x+y)(x+y)\theta(x,y,z,j) = 1, \tag{35c}$$

$$\theta(x,y,z,j) \geq 0 \quad \forall (x,y,z,j) \in \mathcal{I}, \tag{35d}$$

where we define $b_j(0) = f_j(0) = f_j(n+1) = 0$ for convenience, $\mathcal{I} = \{(x,y,z,j) \in \mathbb{Z}_{\geq 0}^4 \text{ s.t. } 1 \leq x + y + z \leq n, \ 1 \leq j \leq m\}$, and the minimum is intended over the entire tuple $\{\theta(x,y,z,j)\}_{(x,y,z,j) \in \mathcal{I}}$. Let $\{\theta^*(x,y,z,j)\}_{(x,y,z,j) \in \mathcal{I}}$ denote an optimal solution (which exists, due to the non-emptiness and boundedness of the constraint set, which can be proven using the same argument in Reference [28, Thm. 2]). If all $\theta^*(x,y,z,j)$ are rational, then consider the game G defined as follows. For every $i \in \{1, \dots, n\}$ and for every $(x,y,z,j) \in \mathcal{I}$, we create a resource identified with $e(x,y,z,j,i)$, and assign to it the resource cost $\alpha_j b_j$, where $\alpha_j = \theta^*(x,y,z,j)/n$. The game G features n players, where player $p \in \{1, \dots, n\}$ can either select the resources in the allocation a_p^{opt} or in a_p^{ne}, defined by

$$a_p^{\text{opt}} = \cup_{i=1}^n \cup_{j=1}^m \{e(x,y,z,j,i) : x + y \geq 1 + ((i-p) \bmod n)\},$$
$$a_p^{\text{ne}} = \cup_{i=1}^n \cup_{j=1}^m \{e(x,y,z,j,i) : x + z \geq 1 + ((i-p+z) \bmod n)\}.$$

Note that the above construction is an extension of that appearing in Reference [28] to the case of multiple basis functions. Since G has

$$\text{NECost}(G, T) = \sum_{x,y,z,j} b_j(x+y)(x+y)\theta^*(x,y,z,j) = 1,$$
$$\text{MinCost}(G) \leq \sum_{x,y,z,j} b_j(x+z)(x+z)\theta^*(x,y,z,j) = C^*,$$

(see Reference [28, Thm. 2] for this), its price of anarchy is no smaller than $1/C^*$. Observe that G features only non-negative rational resource costs' coefficients (i.e., $G \in \mathcal{G}(\mathbb{Q}_{\geq 0})$), therefore Equation (34) follows readily.

If at least one entry in the tuple $\{\theta^*(x,y,z,j)\}_{(x,y,z,j) \in \mathcal{I}}$ is not rational, then we will prove the existence of a sequence of games $G^k \in \mathcal{G}(\mathbb{Q}_{\geq 0})$ whose worst-case efficiency converges to $\text{PoA}(T)$ as

$k \to \infty$. This would imply that Equation (34) holds with $\leq$ in place of the equality sign, concluding the proof. To do so, let us consider the set

$$S = \{\{\theta(x, y, z, j)\}_{(x,y,z,j) \in I} \text{ s.t. (35b), and (35d) hold}\}.$$

Observe that S is non-empty, and that for any tuple belonging to S, we can find a sequence of non-negative rational tuples $\{\{\theta^k(a, x, b, j)\}_{(x,y,z,j) \in I}\}_{k=1}^{\infty}$ (i.e., $\theta^k(a, x, b, j) \in \mathbb{Q}_{\geq 0}$ for all a, x, b, j and k), that converges to it.

Let $\{\{\theta^k(a, x, b, j)\}_{(x,y,z,j) \in I}\}_{k=1}^{\infty}$ be a sequence of tuples converging to $\{\theta^*(a, x, b, j)\}_{(x,y,z,j) \in I}$, which belongs to S. For each tuple $\{\theta^k(a, x, b, j)\}_{(x,y,z,j) \in I}$ in the sequence, define the game G^k following the same construction introduced above with $\theta^k(a, x, b, j)$ in place of $\theta^*(a, x, b, j)$. Following the same reasoning as above, it is $\text{NECost}(G^k, T) = \sum_{x,y,z,j} b_j(x + y)(x + y)\theta^k(x, y, z, j)$, and $\text{MinCost}(G^k) \leq \sum_{x,y,z,j} b_j(x + z)(x + z)\theta^k(x, y, z, j)$. Therefore,

$$\text{PoA}^k = \frac{\text{NECost}(G^k, T)}{\text{MinCost}(G^k)} \geq \frac{\sum_{x,y,z,j} b_j(x + y)(x + y)\theta^k(x, y, z, j)}{\sum_{x,y,z,j} b_j(x + z)(x + z)\theta^k(x, y, z, j)},$$

from which we conclude that

$$\lim_{k \to \infty} \text{PoA}^k \geq \lim_{k \to \infty} \frac{\sum_{x,y,z,j} b_j(x + y)(x + y)\theta^k(x, y, z, j)}{\sum_{x,y,z,j} b_j(x + z)(x + z)\theta^k(x, y, z, j)} = \frac{\sum_{x,y,z,j} b_j(x + y)(x + y)\theta^*(x, y, z, j)}{\sum_{x,y,z,j} b_j(x + z)(x + z)\theta^*(x, y, z, j)} = \frac{1}{C^*},$$

as $\theta^k(a, x, b, j) \to \theta^*(a, x, b, j)$ for $k \to \infty$. This completes the first step.

The second and final step consist in showing that

$$\sup_{G \in \mathcal{G}(\mathbb{Q}_{\geq 0})} \frac{\text{NECost}(G, T)}{\text{MinCost}(G)} = \sup_{G \in \mathcal{G}(\mathbb{Z}_{\geq 0})} \frac{\text{NECost}(G, T)}{\text{MinCost}(G)}.$$

Toward this goal, for any given game from the above-defined sequence $G^k \in \mathcal{G}(\mathbb{Q}_{\geq 0})$, let d_{G^k} denote the lowest common denominator among the resource cost coefficients α_j, across all the resources of the game. Define $\hat{\alpha}_j = \alpha_j \cdot d_{G^k} \in \mathbb{Z}_{\geq 0}$ for all $j \in \{1, \ldots, m\}$, for all resources in $\mathcal{E}$. Since the tolling mechanisms T is linear by assumption, the equilibrium conditions are independent to uniform scaling of the resource costs and tolls by the coefficient d_{G^k}. Therefore any game in the sequence G^k with tolling mechanism T and resource cost coefficients $\{\alpha_j\}_{j=1}^m$ has the same worst-case equilibrium efficiency as a game $\hat{G}^k$, which is identical to G^k except that it has resource cost coefficients $\{\hat{\alpha}_j\}_{j=1}^m$. Observing that $\hat{G}^k$ belongs to $\mathcal{G}(\mathbb{Z}_{\geq 0})$ concludes the proof. □

A.2 Results used in Section 3

LEMMA 2. *Let $b : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a nondecreasing, convex function, and let $0 < \rho \leq 1$ be a given parameter. Further, define the function $f : \{1, \ldots, n\} \to \mathbb{R}$ such that $f(1) = b(1)$ and*

$$f(u + 1) \doteq \min_{v_u \in \{1, \ldots, n\}} \frac{\min\{u, n - v_u\} \cdot f(u) - b(u)u \cdot \rho + b(v_u)v_u}{\min\{v_u, n - u\}}, \tag{36}$$

for all $u \in \{1, \ldots, n - 1\}$. Then, for the lowest value $1 \leq \hat{u} \leq n - 1$ such that $f(\hat{u} + 1) < f(\hat{u})$, it must hold that $f(u + 1) < f(u)$ for all $u \in \{\hat{u}, \ldots, n - 1\}$.

Proof. The proof is presented in two parts as follows: in Part 1, we identify inequalities given that $f(\hat{u} + 1) < f(\hat{u})$, for $1 \leq \hat{u} \leq n - 1$ as defined in the claim; and, in Part 2, we use a recursive argument to prove that $f(u + 1) < f(u)$ holds for all $\hat{u} + 1 \leq u \leq n - 1$, using the inequalities derived in Part 1.

Part 1. We define v_u^* as one of the arguments that minimize the right-hand side of Equation (36) for each $u \in \{1, \ldots, n - 1\}$. By assumption, it must hold that $f(\hat{u} + 1) < f(\hat{u})$, which implies that

$$
\begin{aligned}
f(\hat{u}) &> \min_{v \in \{1, \ldots, n\}} \frac{\min\{\hat{u}, n - v\}}{\min\{v, n - \hat{u}\}} f(\hat{u}) - \frac{b(\hat{u})\hat{u}}{\min\{v, n - \hat{u}\}} \rho + \frac{b(v)v}{\min\{v, n - \hat{u}\}} \\
&= \min_{v \in \{1, \ldots, n\}} \frac{\min\{\hat{u} - 1, n - v\}}{\min\{v, n - \hat{u} + 1\}} f(\hat{u} - 1) - \frac{b(\hat{u} - 1)(\hat{u} - 1)}{\min\{v, n - \hat{u} + 1\}} \rho + \frac{b(v)v}{\min\{v, n - \hat{u} + 1\}} \\
&\quad + \frac{\min\{\hat{u}, n - v\}}{\min\{v, n - \hat{u}\}} f(\hat{u}) - \frac{\min\{\hat{u} - 1, n - v\}}{\min\{v, n - \hat{u} + 1\}} f(\hat{u} - 1) - \frac{b(\hat{u})\hat{u}}{\min\{v, n - \hat{u}\}} \rho \\
&\quad + \frac{b(\hat{u} - 1)(\hat{u} - 1)}{\min\{v, n - \hat{u} + 1\}} \rho + \frac{b(v)v}{\min\{v, n - \hat{u}\}} - \frac{b(v)v}{\min\{v, n - \hat{u} + 1\}},
\end{aligned}
$$

where the strict inequality holds by the definition of $f(\hat{u} + 1)$. Recall that

$$
f(\hat{u}) \doteq \min_{v \in \{1, \ldots, n\}} \frac{\min\{\hat{u} - 1, n - v\}}{\min\{v, n - \hat{u} + 1\}} f(\hat{u} - 1) - \frac{b(\hat{u} - 1)(\hat{u} - 1)}{\min\{v, n - \hat{u} + 1\}} \rho + \frac{b(v)v}{\min\{v, n - \hat{u} + 1\}}.
$$

Thus, if $v_{\hat{u}}^* \leq n - \hat{u}$, then the above strict inequality with $f(\hat{u})$ can only be satisfied if

$$
f(\hat{u} + 1) < f(\hat{u}) \leq \hat{u} f(\hat{u}) - (\hat{u} - 1) f(\hat{u} - 1) < [b(\hat{u})\hat{u} - b(\hat{u} - 1)(\hat{u} - 1)] \cdot \rho.
$$

Similarly, if $v_{\hat{u}}^* \geq n - \hat{u} + 1$, then it must hold that

$$
\begin{aligned}
(n - v_{\hat{u}}^*) \left[\frac{f(\hat{u})}{n - \hat{u}} - \frac{f(\hat{u} - 1)}{n - \hat{u} + 1} \right] + \left[\frac{1}{n - \hat{u}} - \frac{1}{n - \hat{u} + 1} \right] b(v_{\hat{u}}^*) v_{\hat{u}}^* &< \left[\frac{b(\hat{u})\hat{u}}{n - \hat{u}} - \frac{b(\hat{u} - 1)(\hat{u} - 1)}{n - \hat{u} + 1} \right] \cdot \rho \\
\implies \qquad \left[\frac{1}{n - \hat{u}} - \frac{1}{n - \hat{u} + 1} \right] [(n - v_{\hat{u}}^*) f(\hat{u}) + b(v_{\hat{u}}^*) v_{\hat{u}}^*] &< \left[\frac{b(\hat{u})\hat{u}}{n - \hat{u}} - \frac{b(\hat{u} - 1)(\hat{u} - 1)}{n - \hat{u} + 1} \right] \cdot \rho \\
\iff \qquad\qquad\qquad\qquad\qquad\qquad\qquad f(\hat{u} + 1) &< [b(\hat{u})\hat{u} - b(\hat{u} - 1)(\hat{u} - 1)] \rho,
\end{aligned}
$$

where the first line implies the second line, because $f(\hat{u}) \geq f(\hat{u} - 1)$, by the definition of $\hat{u}$ in the claim, and the second line is equivalent to the third by the definitions of $f(\hat{u} + 1)$ and $v_{\hat{u}}^*$. This concludes Part 1 of the proof.

Part 2. In this part of the proof, we show by recursion that if $f(\hat{u} + 1) < f(\hat{u})$, then $f(u + 1) < f(u)$ for all $u \in \{\hat{u} + 1, \ldots, n - 1\}$. We do so by showing that, if $f(u) < f(u - 1) < \ldots < f(\hat{u} + 1)$ for any $u \in \{\hat{u} + 1, \ldots, n - 1\}$, then it must hold that $f(u + 1) < f(u)$. Thus, in the following reasoning, we assume that $u \in \{\hat{u} + 1, \ldots, n - 1\}$, and that $f(u) < f(u - 1) < \ldots < f(\hat{u} + 1)$.

We begin with the case of $v_{u-1}^* < n - u + 1$, which gives us that $v_{u-1}^* \leq n - u$. Observe that

$$
f(u+1) \doteq \min_{v_u \in \{1,\ldots,n\}} \frac{\min\{u, n - v_u\}}{\min\{v_u, n - u\}} f(u+1) - \frac{b(u)u}{\min\{v_u, n - u\}} \rho + \frac{b(v_u)v_u}{\min\{v_u, n - u\}}
$$

$$
= \min_{v_u \in \{1,\ldots,n\}} \frac{\min\{u - 1, n - v_u\}}{\min\{v_u, n - u + 1\}} f(u-1) - \frac{b(u-1)(u-1)}{\min\{v_u, n - u + 1\}} \rho + \frac{b(v_u)v_u}{\min\{v_u, n - u + 1\}}
$$

$$
+ \frac{\min\{u, n - v_u\}}{\min\{v_u, n - u\}} f(u+1) - \frac{\min\{u - 1, n - v_u\}}{\min\{v_u, n - u + 1\}} f(u-1) - \frac{b(u)u}{\min\{v_u, n - u\}} \rho
$$

$$
+ \frac{b(u-1)(u-1)}{\min\{v_u, n - u + 1\}} \rho + \frac{b(v_u)v_u}{\min\{v_u, n - u\}} - \frac{b(v_u)v_u}{\min\{v_u, n - u + 1\}}
$$

$$
\leq f(u) + \frac{u}{v_{u-1}^*} f(u) - \frac{u - 1}{v_{u-1}^*} f(u-1) - \frac{b(u)u - b(u-1)(u-1)}{v_{u-1}^*} \rho
$$

$$
< f(u) + \frac{1}{v_{u-1}^*} f(\hat{u}+1) - \frac{1}{v_{u-1}^*} [b(u)u - b(u-1)(u-1)] \rho < f(u),
$$

where the first inequality holds by evaluating the minimization at $v_u = v_{u-1}^*$, the second inequality holds, because $f(u) < f(u-1)$ and $f(u) \leq f(\hat{u}+1)$, by assumption, and the final inequality holds by the result showed in Part 1 and because $b(\cdot)$ is nondecreasing and convex.

Next, we consider the scenario in which $v_{u-1}^* > n - u + 1$. Observe that

$$
f(u+1) \leq f(u) + (n - v_{u-1}^*) \left[\frac{f(u)}{n - u} - \frac{f(u-1)}{n - u + 1} \right] + \left[\frac{1}{n - u} - \frac{1}{n - u + 1} \right] b(v_{u-1}^*) v_{u-1}^*
$$

$$
- \frac{b(u)u}{n - u} \rho + \frac{b(u-1)(u-1)}{n - u + 1} \rho
$$

$$
< f(u) + \left[\frac{1}{n - u} - \frac{1}{n - u + 1} \right] [(n - v_{u-1}^*) f(u-1) + b(v_{u-1}^*) v_{u-1}^*]
$$

$$
- \frac{b(u)u}{n - u} \rho + \frac{b(u-1)(u-1)}{n - u + 1} \rho
$$

$$
= f(u) + \left[\frac{1}{n - u} - \frac{1}{n - u + 1} \right] [(n - u + 1) f(u) + b(u-1)(u-1)\rho]
$$

$$
- \frac{b(u)u}{n - u} \rho + \frac{b(u-1)(u-1)}{n - u + 1} \rho
$$

$$
\leq f(u) + \frac{1}{n - u} f(\hat{u}+1) - \frac{1}{n - u} [b(u)u - b(u-1)(u-1)] \rho
$$

$$
< f(u),
$$

where the first inequality holds by evaluating the minimization at $v_u = v_{u-1}^*$; the second inequality holds, because $f(u) < f(u-1)$, and by assumption, the equality holds by the definitions of $f(u)$ and v_{u-1}^*; the third inequality holds, because $f(u) \leq f(\hat{u}+1)$, by assumption; and the final inequality holds by the identity we showed in Part 1 and because b is nondecreasing and convex.

Finally, we consider the scenario in which $v_{u-1}^* = n - u + 1$. Observe that

$$f(u + 1) \le f(u) + \frac{u-1}{n-u}f(u) - \frac{u-1}{n-u+1}f(u-1) - \frac{b(u)u}{n-u}\rho + \frac{b(u-1)(u-1)}{n-u+1}\rho$$
$$+ \left[\frac{1}{n-u} - \frac{1}{n-u+1}\right]b(n-u+1)(n-u+1)$$

$$< f(u) + \left[\frac{1}{n-u} - \frac{1}{n-u+1}\right][(n - v_{u-1}^*)f(u-1) + b(v_{u-1}^*)v_{u-1}^*]$$
$$- \frac{b(u)u}{n-u}\rho + \frac{b(u-1)(u-1)}{n-u+1}\rho$$

$$< f(u),$$

where the first inequality holds by evaluating the minimization at $v_u = v_{u-1}^*$, the second inequality holds, because $f(u) < f(u-1)$, by assumption, and the final inequality holds by the same reasoning as for $v_{u-1}^* > n - u + 1$. □

A.3 Results used in Section 4

LEMMA 3. *For given $d \ge 1$, consider the linear program defined in Equation (18), and let $(f^{\mathrm{opt}}, \rho^{\mathrm{opt}})$ be a solution. Further let f^∞ be defined in Equation (19) utilizing $(f^{\mathrm{opt}}, \rho^{\mathrm{opt}})$. The price of anarchy of f^∞ over congestion games of degree d with possibly infinitely many agents is upper bounded by $1/\rho^\infty$, where*

$$\rho^\infty = \min\left\{\rho^{\mathrm{opt}}, \ \beta - d\left(1 + \frac{2}{\bar{n}}\right)^{d+1}\left(\frac{\beta}{d+1}\right)^{1+\frac{1}{d}}\right\}.$$

This result is tight for pure Nash equilibria and holds for coarse correlated equilibria, too.

PROOF. Since f^∞ is non-decreasing by construction, we characterize its performance over games with a maximum of n agents through the linear program in Equation (12) with $b(x) = x^d$, that is

$$\max_{\rho \in \mathbb{R}, v \in \mathbb{R}_{\ge 0}} \rho$$

s.t. $\qquad b(v)v - \rho b(u)u + v[f^\infty(u)u - f^\infty(u+1)v] \ge 0 \quad \forall u, v \in \{0, \dots, n\} \quad u + v \le n,$

$$b(v)v - \rho b(u)u + v[f^\infty(u)(n-v) - f^\infty(u+1)(n-u)] \ge 0 \quad \forall u, v \in \{0, \dots, n\} \quad u + v > n.$$
(37)

Upper bounding the price of anarchy of f^∞ amounts to finding a feasible solution to this linear program; the challenging task is that we intend to do so for n arbitrary large. We claim that $(\rho, v) = (\rho^\infty, 1)$ is feasible for any (possibly infinite) n, so that the claimed upper bound on the price of anarchy follows. To prove this, we divide the discussion in two parts as the expressions are different for $u + v \le n$ and $u + v > n$. Before doing so, we study the degenerate case of $u = 0$, for which the constraints reduce to $f^\infty(1) \le 1$, which holds as f^∞ is feasible for the linear program in Equation (18), which already includes this constraint.

—Case of $u + v \le n, u \ge 1$. The constraints read as

$$v^{d+1} - \rho u^{d+1} + v[uf^\infty(u) - vf^\infty(u+1)] \ge 0,$$
(38)

which we want to hold for any integers $u \ge 1$, $v \ge 0$ (the bound on the indices $u + v \le n$ can be dropped as we are interested in the case of arbitrary n).

• In the region where $0 \le v \le u, 1 \le u < \bar{n}/2$ (region A in Figure 4) these constraints certainly hold with $\rho \le \rho^{\mathrm{opt}}, v = 1$. This follows, because $f^\infty(u) = f^{\mathrm{opt}}(u)$ when $u \le \bar{n}/2$, and f^{opt} is feasible for the program in Equation (18), which includes these constraints.
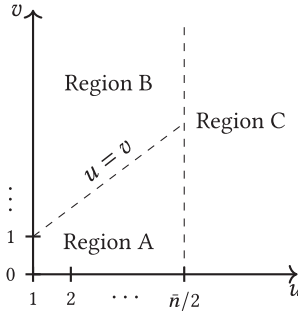
Fig. 4. Regions A, B, and C utilized in proof for the case $u + v \le n$.

- In the region where $v > u$, $1 \le u < \bar{n}/2$ (region B in Figure 4) these constraints also hold with $\rho \le \rho^{\text{opt}}$, $v = 1$ thanks to Lemma 4 part a).
- We are left with the region where $u \ge \bar{n}/2$, $v \ge 0$ (region C in Figure 4). In this case, $f^\infty(u) = \beta \cdot u^d$ by definition. With the choice of $v = 1$, the constraints in Equation (38) read

$$v^{d+1} - \rho u^{d+1} + \beta u^{d+1} - \beta v(u+1)^d \ge 0. \tag{39}$$

Observe that, for a fixed choice of integer $u \ge \bar{n}/2$, the left-hand side of Equation (39) is a convex function of v for $v \ge 0$. Its corresponding minimum value over the non-negative reals is

$$\left(\frac{\beta}{d+1}\right)^{1+\frac{1}{d}} (u+1)^{d+1} - \beta \left(\frac{\beta}{d+1}\right)^{\frac{1}{d}} (u+1)^{d+1} + (\beta - \rho)u^{d+1}.$$

Hence, Equation (39) is satisfied for a fixed choice of $u \ge \bar{n}/2$ and all integers $v \ge 0$ if the latter expression is non-negative. Simple algebra shows that this is the case if

$$\rho \le \beta - d\left(1 + \frac{1}{u}\right)^{d+1} \left(\frac{\beta}{d+1}\right)^{1+\frac{1}{d}}. \tag{40}$$

Since we would like Equation (39) to hold for all $u \ge \bar{n}/2$, and since the right-hand side in Equation (40) is increasing in n, it suffices to ask for Equation (40) to hold at the lowest admissible u, i.e., $u = \bar{n}/2$. Therefore, in order for Equation (39) to be satisfied for any $u \ge \bar{n}/2$, it suffices to select

$$\rho \le \beta - d\left(1 + \frac{2}{\bar{n}}\right)^{d+1} \left(\frac{\beta}{d+1}\right)^{1+\frac{1}{d}}. \tag{41}$$

— Case of $u + v > n$, $u \ge 1$. Lemma 4 part b) shows that the constraints corresponding to $u + v > n$ are satisfied for arbitrary n with the choice of $v = 1$, and ρ as in Equation (41) thanks to the fact that $f^\infty : \mathbb{N} \to \mathbb{R}$ is non-decreasing.

In conclusion, we verified that $(\rho, v) = (\rho^\infty, 1)$ is feasible for the program in Equation (37). It follows that the price of anarchy of f^∞ over games with arbitrarily large n is upper bounded as in the claim. □

LEMMA 4.

(a) Let $f : \{1, \ldots, n\} \to \mathbb{R}$, $\rho \ge 0$, and $f(x) \le x^d$ for all $1 \le x \le n$ and $d \ge 1$. The constraints $v^{d+1} - \rho u^{d+1} + u f(u) - v f(u+1) \ge 0$ obtained for any $v \in \mathbb{N}$, $v \ge u$, $u \in \{1, \ldots, n-1\}$ are satisfied if the same inequality holds for all $v = u \in \{1, \ldots, n-1\}$.

(b) Let $f : \mathbb{N} \to \mathbb{R}$ be non-decreasing, $d \geq 1$, $\rho \geq 0$. If the constraints $v^{d+1} - \rho u^{d+1} + uf(u) - vf(u + 1) \geq 0$ hold for all non-negative integers u, v, then $v^{d+1} - \rho u^{d+1} + (n - v)f(u) - (n - u)f(u + 1) \geq 0$ hold for all non-negative integers u, v with $u + v > n$, for any choice of $n \geq 1$ integer.

PROOF. We prove the two claims separately.

First claim. For $v = u$ the constraints of interest reduces to $\rho u^{d+1} \leq u^{d+1} + u(f(u) - f(u + 1))$, while for $v = u + p$ (with $p \geq 1$) the constraint reads as $\rho u^{d+1} \leq (u + p)^{d+1} + uf(u) - (u + p)f(u + 1)$. Proving the claim amounts to showing

$$u^{d+1} + u(f(u) - f(u + 1)) \leq (u + p)^{d+1} + uf(u) - (u + p)f(u + 1) \iff f(u + 1)$$
$$\leq \frac{(u + p)^{d+1} - u^{d+1}}{p}$$

for $p \geq 1$. The right-hand side is minimized at $p = 1$ (it describes the slope of the secant to the function u^{d+1} at abscissas u and $u + p$) due to the convexity of u^{d+1}. Therefore, it suffices to ensure that $f(u + 1) \leq (u + 1)^{d+1} - u^{d+1}$ for any choice of $u \in \{1, \ldots, n - 1\}$. By assumption, $f(u + 1) \leq (u + 1)^d$, so that we can equivalently prove $(u + 1)^d \leq (u + 1)^{d+1} - u^{d+1}$. The latter holds, as required, for all $u \in \{1, \ldots, n - 1\}$, since

$$(u + 1)^d \leq (u + 1)^{d+1} - u^{d+1} \iff (u + 1)^d \leq (u + 1)^d(u + 1) - u^{d+1} \iff u^d \leq (u + 1)^d.$$

Second claim. By assumption, for all non-negative integers u, v it is $\rho u^{d+1} \leq v^{d+1} + uf(u) - vf(u + 1)$, while we intend to show that for all non-negative integers u, v with $u + v > n$ it is $\rho u^{d+1} \leq v^{d+1} + (n - v)f(u) - (n - u)f(u + 1)$, regardless for the choice of $n \geq 1$ integer. Proving this is equivalent to showing

$$uf(u) - vf(u + 1) \leq (n - v)f(u) - (n - u)f(u + 1) \iff (u + v - n)(f(u) - f(u + 1)) \leq 0,$$

which holds, as required, due to the fact that $u + v - n > 0$ and $f(u) - f(u + 1) \leq 0$ due to f being non-decreasing.   □

A.4 Results used in Section 5

LEMMA 5. *Let $n \in \mathbb{N}$, and assume that the basis functions $\{b_1, \ldots, b_m\}$, $b_j : \mathbb{N} \to \mathbb{R}$ are convex (in the discrete sense), positive, and non-decreasing, for all $j = 1, \ldots, m$. Then, the constraints appearing in (25) with $u = 0$, $v = \{1, \ldots, n\}$ are satisfied if the constraint with $(u, v) = (0, 1)$ holds. Similarly, the constraints with $u \in \{1, \ldots, n\}$, $v \in \{0, \ldots, n\}$ and $u < v$ are satisfied if the constraints with $u \in \{1, \ldots, n\}$, $v \in \{0, \ldots, n\}$ and $u \geq v$ hold.*

PROOF. We begin with the constraints obtained for $u = 0$, which read as $b_j(v)v - vb_j(1)v - \sigma_j v \geq 0$. The constraint with $v = 0$ holds trivially, while the tightest constraint for $v > 0$ is obtained with $v = 1$ due to the fact that $b_j(v)v$ is increasing for $v > 0$. This shows that it is sufficient to consider the constraint with $(u, v) = (0, 1)$.

We now consider the constraints obtained for $u \geq 1$ and divide the proof in three parts, according to the regions in Figure 5.

- In the region where $v > u$ and $u + v \leq n$ (Region A in Figure 5), we show that if the constraint obtained with $v = u$ holds, then the constraints with $v > u$ also hold. Note that feasible values of v are upper bounded by $v \leq 1$. This is because the constraint with $(u, v) = (0, 1)$ reads as $(1 - v)b_j(1) \geq \sigma_j$, and since $\sigma_j \geq 0$, $b_j(1) > 0$, every feasible v must satisfy $1 - v \geq 0$. The constraint with $v = u + p$, $p \geq 1$ read as $b_j(u + p)(u + p) - \rho b_j(u)u + v[b_j(u)u - b_j(u + 1)(u + p)] - \sigma_j p \geq 0$, the tightest of which is obtained for the largest feasible value of σ_j, that is $\sigma_j = (1 - v)b_j(1)$. The constraint with $v = u$ reads as
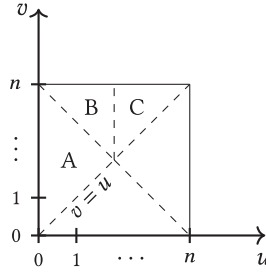
Fig. 5. Regions A, B, and C utilized in proof of Lemma 5.

$ub_j(u) - \rho u b_j(u) + vu(b_j(u) - b_j(u+1)) \geq 0$. Therefore, we intend to show that for every ρ and $0 \leq v \leq 1$ satisfying

$$ub_j(u) - \rho u b_j(u) + vu(b_j(u) - b_j(u+1)) \geq 0, \quad \text{it also holds for all } p \geq 1 \text{ that}$$

$$b_j(u+p)(u+p) - \rho b_j(u)u + v[b_j(u)u - b_j(u+1)(u+p)] - (1-v)b_j(1)p \geq 0.$$

Since both constraints describe straight lines in the plane (v, ρ), it suffices to verify that this is true at the extreme points $v = 0$ and $v = 1$.

When $v = 0$ the constraint with $v = u$ and $v = u + p$ read as $\rho b_j(u)u \leq b_j(u)u$, and $\rho b_j(u)u \leq b_j(u+p)(u+p) - b_j(1)p$. We therefore intend to show that

$$b_j(u)u \leq b_j(u+p)(u+p) - b_j(1)p \iff \frac{b_j(u+p)(u+p) - ub_j(u)}{p} \geq b_j(1), \qquad (42)$$

which holds, since $b_j(u)u$ is convex and $b_j(u)$ is non-decreasing so that $\frac{b_j(u+p)(u+p) - ub_j(u)}{p} \geq b_j(u+1)(u+1) - ub_j(u) \geq b_j(u+1) \geq b_j(1)$. When $v = 1$, following a similar reasoning, we are left to show that $ub_j(u) \leq b_j(u+p)(u+p) - pb_j(u+1)$, which holds thanks to the non-decreasingness of $b_j(u)$, indeed $b_j(u+p)(u+p) - pb_j(u+1) \geq b_j(u+p)u \geq b_j(u)u$.

- In the region where $v > u$, $u + v \geq n$ and $u \leq n/2$ (Region B in Figure 5), we intend to show that the following constraints hold $b_j(v)v - \rho b_j(u)u + v[b_j(u)(n-v) - b_j(u+1)(n-u)] + \sigma_j(u-v) \geq 0$. We do so by observing that the proof of the previous point did not require at all that $u + v \leq n$. Therefore, exploiting the same proof, we have $b_j(v)v - \rho b_j(u)u + v[b_j(u)u - b_j(u+1)v] + \sigma_j(u-v) \geq 0$ also for $u + v \geq n$, i.e., in the region B of interest. We exploit this to conclude, and, in particular, we show that the satisfaction of latter constraint implies the desired result. Toward this goal we need to show that, for $u + v > n$ it is

$$b_j(u)(n-v) - b_j(u+1)(n-u) \geq b_j(u)u - b_j(u+1)v \Leftrightarrow (b_j(u) - b_j(u+1))(n-u-v) \geq 0,$$

which holds due to the non-decreasingness of b_j and to $u + v > n$.

- In the region where $v > u$, $u + v \geq n$ and $u > n/2$ (Region C in Figure 5), we use the same approach of that in the first point. In particular, we intend to show that the when the constraints with $v = u$ hold, also the constraints with $v > u$ do so. Following a similar reasoning as in the above, this amount to showing for every ρ and $0 \leq v \leq 1$ satisfying

$$b_j(u)u - \rho b_j(u)u + v(n-u)(b_j(u) - b_j(u+1)) \geq 0, \quad \text{it also holds for all } p \geq 1 \text{ that}$$

$$b_j(u+p)(u+p) - \rho b_j(u)u + v[b_j(u)(n-u-p) - b_j(u+1)(n-u)] - (1-v)b_j(1)p \geq 0.$$

Since both constraints describe straight lines in the plane (v, ρ), it suffices to verify that this is true at the extreme points $v = 0$ and $v = 1$. When $v = 0$ we are left with $b_j(u)u \leq b_j(u+p)(u+p) - b_j(1)p$, which we already proved in Equation (42). When $v = 1$, we

need to show that $b_j(u + p)(u + p) - pb_j(u) \geq ub_j(u)$, which holds by non-decreasingness of b_j.   □

REFERENCES

[1] Sebastian Aland, Dominic Dumrauf, Martin Gairing, Burkhard Monien, and Florian Schoppmann. 2011. Exact price of anarchy for polynomial congestion games. *SIAM J. Comput.* 40, 5 (2011), 1211–1233.

[2] Elliot Anshelevich, Anirban Dasgupta, Jon Kleinberg, Eva Tardos, Tom Wexler, and Tim Roughgarden. 2008. The price of stability for network design with fair cost allocation. *SIAM J. Comput.* 38, 4 (2008), 1602–1623.

[3] Baruch Awerbuch, Yossi Azar, and Amir Epstein. 2013. The price of routing unsplittable flow. *SIAM J. Comput.* 42, 1 (2013), 160–177.

[4] Martin Beckmann, Charles B. McGuire, and Christopher B. Winsten. 1956. *Studies in the Economics of Transportation*. Technical Report.

[5] Pia Bergendorff, Donald W. Hearn, and Motakuri V. Ramana. 1997. Congestion toll pricing of traffic networks. In *Network Optimization*. Springer, 51–71.

[6] Vittorio Bilò and Cosimo Vinci. 2019. Dynamic taxes for polynomial congestion games. *ACM Trans. Econ. Comput.* 7, 3 (2019), 15:1–15:36.

[7] Ioannis Caragiannis, Michele Flammini, Christos Kaklamanis, Panagiotis Kanellopoulos, and Luca Moscardelli. 2011. Tight bounds for selfish and greedy load balancing. *Algorithmica* 61, 3 (2011), 606–637.

[8] Ioannis Caragiannis, Christos Kaklamanis, and Panagiotis Kanellopoulos. 2010. Taxes for linear atomic congestion games. *ACM Trans. Algor.* 7, 1 (2010), 13:1–13:31.

[9] Rahul Chandan, Dario Paccagnan, Bryce L. Ferguson, and Jason R. Marden. 2019. Computing optimal taxes in atomic congestion games. In *Proceedings of the 14th Workshop on the Economics of Networks, Systems and Computation (NetEcon'19)*. ACM, 2:1.

[10] Rahul Chandan, Dario Paccagnan, and Jason R. Marden. 2019. MATLAB and Python Packages to Compute and Optimize the Price of Anarchy. Retrieved from https://github.com/rahul-chandan/resalloc-poa.

[11] Rahul Chandan, Dario Paccagnan, and Jason R. Marden. 2019. Optimal mechanisms for distributed resource-allocation. *arXiv:1911.07823*. Retrieved from https://arxiv.org/abs/1911.07823.

[12] Rahul Chandan, Dario Paccagnan, and Jason R. Marden. 2019. When smoothness is not enough: Toward exact quantification and optimization of the price-of-anarchy. In *Proceedings of the 58th IEEE Conference on Decision and Control (CDC'19)*. IEEE, 4041–4046.

[13] George Christodoulou and Elias Koutsoupias. 2005. The price of anarchy of finite congestion games. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing, Baltimore*, Harold N. Gabow and Ronald Fagin (Eds.). ACM, 67–73.

[14] Richard Cole, Yevgeniy Dodis, and Tim Roughgarden. 2006. How much can taxes help selfish routing? *J. Comput. Syst. Sci.* 72, 3 (2006), 444–467.

[15] Dimitris Fotakis and Paul G. Spirakis. 2008. Cost-balancing tolls for atomic network congestion games. *Internet Math.* 5, 4 (2008), 343–363.

[16] Tobias Harks. 2019. *Pricing in Resource Allocation Games Based on Lagrangean Duality and Convexification*. *arXiv:1907.01976*. Retrieved from https://arxiv.org/abs/1907.01976.

[17] Tobias Harks, Ingo Kleinert, Max Klimm, and Rolf H. Möhring. 2015. Computing network tolls with support constraints. *Networks* 65, 3 (2015), 262–285.

[18] Donald W. Hearn and Motakuri V. Ramana. 1998. *Solving Congestion Toll Pricing Models*. Springer US, Boston, MA, 109–124.

[19] Martin Hoefer, Lars Olbrich, and Alexander Skopalik. 2008. Taxing subnetworks. In *Proceedings of the 4th International Workshop on Internet and Network Economics (WINE'08)*, Christos H. Papadimitriou and Shuzhong Zhang (Eds.), Lecture Notes in Computer Science, Vol. 5385. Springer, 286–294.

[20] Tomas Jelinek, Marcus Klaas, and Guido Schäfer. 2014. Computing optimal tolls with arc restrictions and heterogeneous players. In *Proceedings of the 31st International Symposium on Theoretical Aspects of Computer Science (STACS'14)*, Ernst W. Mayr and Natacha Portier (Eds.), LIPIcs,Vol. 25. Schloss Dagstuhl, Leibniz-Zentrum für Informatik, 433–444.

[21] Elias Koutsoupias and Christos H. Papadimitriou. 1999. Worst-case equilibria. In *Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science (STACS'99)*, Christoph Meinel and Sophie Tison (Eds.), Lecture Notes in Computer Science, Vol. 1563. Springer, 404–413.

[22] Jean-Jacques Laffont and David Martimort. 2002. *The Theory of Incentives: The Principal-Agent Model*. Princeton University Press.

[23] Jason R. Marden and Adam Wierman. 2013. Distributed welfare games. *Operat. Res.* 61, 1 (2013), 155–168.

[24] Reshef Meir and David C. Parkes. 2016. When are marginal congestion tolls optimal? In *Proceedings of the 9th International Workshop on Agents in Traffic and Transportation (ATT'16) co-located with the 25th International Joint Conference On Artificial Intelligence (IJCAI'16)*, Ana Lúcia C. Bazzan, Franziska Klügl, Sascha Ossowski, and Giuseppe Vizzari (Eds.), Vol. 1678.

[25] Igal Milchtaich. 2020. Internalization of social cost in congestion games. *Econ. Theory* (2020), 1–44.

[26] Steven A. Morrison. 1986. A survey of road pricing. *Transport. Res. A: Gen.* 20, 2 (1986), 87–97.

[27] Dario Paccagnan, Rahul Chandan, Bryce L. Ferguson, and Jason R. Marden. 2019. Optimal Taxes in Atomic Congestion Games. *arXiv:1911.09806*. Retrieved from https://arxiv.org/abs/1911.09806.

[28] Dario Paccagnan, Rahul Chandan, and Jason R. Marden. 2020. Utility design for distributed resource allocation—Part I: Characterizing and optimizing the exact price of anarchy. *IEEE Trans. Automat. Contr.* 65, 11 (2020), 4616–4631.

[29] Dario Paccagnan, Basilio Gentile, Francesca Parise, Maryam Kamgarpour, and John Lygeros. 2018. Nash and Wardrop equilibria in aggregative games with coupling constraints. *IEEE Trans. Automat. Contr.* 64, 4 (2018), 1373–1388.

[30] Arthur C. Pigou. 1920. *The Economics of Welfare*. Macmillan.

[31] Robert W. Rosenthal. 1973. A class of games possessing pure-strategy Nash equilibria. *Int. J. Game Theory* 2, 1 (1973), 65–67.

[32] Tim Roughgarden. 2015. Intrinsic robustness of the price of anarchy. *J. ACM* 62, 5 (2015), 32:1–32:42.

[33] William H. Sandholm. 2002. Evolutionary implementation and congestion pricing. *Rev. Econ. Stud.* 69, 3 (2002), 667–689.

[34] Alexander Skopalik and Vipin Ravindran Vijayalakshmi. 2020. Improving approximate pure Nash equilibria in congestion games. *arXiv:2007.15520*. Retrieved from https://arxiv.org/abs/2007.15520.

[35] Subhash Suri, Csaba D. Tóth, and Yunhong Zhou. 2007. Selfish load balancing and atomic congestion games. *Algorithmica* 47, 1 (2007), 79–96.

[36] Cem Tekin, Mingyan Liu, Richard Southwell, Jianwei Huang, and Sahand Haji Ali Ahmad. 2012. Atomic congestion games on graphs and their applications in networking. *IEEE/ACM Trans. Network.* 20, 5 (2012), 1541–1552.

[37] John Glen Wardrop. 1952. Some theoretical aspects of road traffic research. *Proc. Inst. Civil Eng.* 1, 3 (1952), 325–362.