The Effectiveness of Subsidies and Taxes in Atomic Congestion Games

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***Abstract*— Are subsidies or taxes more effective at influencing user behavior? To answer this question, we focus on the well-studied framework of atomic congestion games which model resource allocation problems in non-cooperative environments. Examples of such resource allocation problems include transportation networks, task assignment, content distribution systems, among others. Monetary incentives, in the form of taxes or subsidies, are commonly employed in such systems to influence self-interested behavior and improve system efficiency. Our first result demonstrates that subsidies can provide strong improvement guarantees when compared to taxes of a similar magnitude. While interesting, our second result demonstrates that this improvements come at the expense of robustness. In particular, taxes provide greater robustness guarantees to mischaracterizations in the societal response when compared to subsidies. Hence, whether a system operator should employ taxes or subsidies depends intimately on the knowledge of the user population.**

***Index Terms*— Game theory, Agents-based systems**

I. Introduction

THE performance of many real-world systems is often heavily influenced by the choices of the system's users, e.g., commerce and transport [2], traffic networks [3], [4], ride sharing [5], and content distribution [6]. It is often the case that these self-interested choices lead to sub-optimal system behavior, and this inefficiency is often measured via the *price of anarchy* [7], [8]. Informally, the price of anarchy is the ratio between the system welfare that occurs from users' self interested decision making and the optimal system welfare.

One encouraging method to improve performance in these systems is to introduce monetary *incentives* in order to alter users decision making process and promote more desirable group behavior [9], [10]. A well studied form of incentive is to institute *taxes* that increase a user's perceived cost when taking actions that are undesirable from system level perspective [11], [12]. Taxes have been shown to be effective at reducing the price of anarchy ratio [13], [14] and have been studied in several areas including tolls in transportation [15]–[17]. Another suitable method for incentivizing users is to *subsidize*, or monetarily reward, choices that are beneficial to the overall system performance. Subsidies have also been studied as a tool to influence users in traffic and transportation [18] and have additionally been used to influence drivers in ride sharing [5] and providers in digital distribution [6]. Though both incentives can be effective at influencing group behavior, in this work we show that there are inherent tradeoffs between taxes and subsidies as it pertains to performance and robustness.

This work focuses on understanding these tradeoffs by investigating the framework of atomic congestion games. In such environments, there exists a finite number of users that make use of a shared group of resources where the quality of service associated with each resource depends on its level of utilization. A classic example of a congestion game centers on transportation networks [19], where the resources are represented by roads and the quality of service is captured by the congestion on that road. An alternative example is a networked content distribution system where servers host shared data that can be requested by end users [6]. Regardless of the specific setting, the emergent behavior in such systems is typically characterized by a *Nash equilibrium* which captures a form of stable collective behavior where no single entity can improve its performance through unilateral deviations. Accordingly, the price of anarchy is defined as the ratio between the aggregate user cost in a Nash equilibrium and the minimum aggregate cost. To mitigate the inefficiency caused by users' self-interested decision making, a system operator can introduce incentives to alter the users' preferences and promote actions that lead to equilibria with lower system cost. The performance of an incentive can be measured by how much it reduces the price of anarchy ratio relative to the non-incentivized case.

The use of taxes, subsidies, or a combination of the two have all been studied as tools that can be effective at improving efficiency in atomic congestion games [11]–[14]. Independently, the authors of [4], [17], [20], [21] study settings where there are constraints on the magnitude of incentives and/or users are heterogeneous in their response to incentives. The authors of [22] consider the use of a mix of subsidies and taxes that are budget balanced; in this work, we focus on the relative performance guarantees of using either subsidies or taxes exclusively, and show the relative effectiveness of each incentive at reducing the inefficiency of worst-case equilibria.

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The main results of this work are a generalization of [1] to atomic congestion games.

In atomic congestion games, both subsidies and taxes can be effective at reducing the price of anarchy. In this work, we compare the performance and robustness of subsidies and taxes under budgetary constraints in atomic congestion games. Our contributions are outlined as follows:

Performance: First, we focus on the ability of the optimally designed subsidy and tax to incentivize more efficient social behavior when users are homogeneous, with a known sensitivity to incentives. Specifically, we ask if either incentive offers superior performance guarantees when designed under budgetary constraints. In Theorem 1, we show that under similar budgetary constraints subsidies are capable of offering better price of anarchy guarantees than taxes.

Robustness: Next, we seek to understand the robustness of each incentive when users' responses to these monetary transactions are uncertain. In contrast to their capabilities in the nominal setting, taxes are more robust to unknown user heterogeneity than subsidies. Theorem 2 shows that user heterogeneity will have a larger impact on the performance guarantees of subsidies than taxes.

Additionally, in Section III-B, we use recent results in optimal incentive design to simulate and illustrate the main results of this work.

II. Preliminaries

A. Atomic Congestion Game Model

The structure of an atomic congestion game can formally be described by a set of resources $\mathcal{E}$ and a set of users $N = \{1, \dots, n\}$. Each user $i \in N$ selects a subset of the possible resources $a_i \in \mathcal{A}_i \subset 2^{\mathcal{E}}$, where $2^{\mathcal{E}}$ is the power set of $\mathcal{E}$. An allocation of users' actions is denoted by the tuple $a = (a_1, \dots, a_n) \in \mathcal{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$.

To model the increase in cost from the presence of multiple users, each resource $e \in \mathcal{E}$ has a non-decreasing congestion function $c_e : \{0, 1, \dots, n\} \to \mathbb{R}_{\geq 0}$ that maps the number of users sharing a resource to a non-negative cost. These functions capture the phenomenon that larger congestion leads to higher costs and lower quality of service. In an allocation a, the system cost, or *total congestion*, is

$$C(a) := \sum_{e \in \mathcal{E}} |a|_e c_e(|a|_e), \tag{1}$$

where $|a|_e$ denotes the number of users sharing a resource e in an allocation $|a|_e$, or the cardinality of the set $\{i \in N | e \in a_i\}$. An instance of an atomic congestion game can be specified by the tuple $G = (\mathcal{E}, N, \mathcal{A}, \{c_e\}_{e \in \mathcal{E}})$.

For minimizing the total congestion, an optimal allocation satisfies $a^{\text{opt}} \in \arg\min_{a \in \mathcal{A}} C(a)$. Though such allocations may be desirable, in this work we are interested in the allocations that emerge from users' self-interested decision making. When each user seeks to minimize their own observed cost, it is well known that self-interested decision making leads to sub-optimal group behavior [7], [8]. As a means to promote more efficient group behavior, a system designer can implement incentives to alter the users' costs. Let $\tau_e(|a|_e)$ denote an incentive function assigned to a resource $e \in \mathcal{E}$ that can vary with the number of users sharing the resource[1]. In an allocation $a \in \mathcal{A}$, a player $i \in N$ experiences the sum of congestion and monetary costs for each resource they use,

$$J_i(a) = \sum_{e \in a_i} c_e(|a|_e) + \tau_e(|a|_e). \tag{2}$$

We use the notion of the *Nash equilibrium* to describe the plausible emergent behavior in the system when users seek to minimize their own cost. An allocation $a^{\text{NE}} \in \mathcal{A}$ is a Nash equilibrium if

$$J_i(a_i^{\text{NE}}, a_{-i}^{\text{NE}}) \leq J_i(a_i', a_{-i}^{\text{NE}}), \quad \forall a_i' \in \mathcal{A}_i,\ i \in N, \tag{3}$$

where a_{-i}^{NE} denotes the actions of every user other than user i. The set of all Nash equilibria in a congestion game G with incentive functions $\{\tau_e\}_{e \in \mathcal{E}}$ is denoted $\text{NE}(G, \{\tau_e\}_{e \in \mathcal{E}})$. By selecting these incentive functions intelligently, a system designer can incentivize the users of the system to reach more desirable states as Nash equilibria. In [8], it is shown that a Nash equilibrium always exists in an atomic congestion game, though need not be unique.

To assess the performance of an incentive mechanism, we define the price of anarchy as the worst-case ratio between the equilibrium system cost and the optimal system cost. In a congestion game G with incentive functions $\{\tau_e\}_{e \in \mathcal{E}}$, the price of anarchy is

$$\text{PoA}(G, \{\tau_e\}_{e \in \mathcal{E}}) = \max_{a^{\text{NE}} \in \text{NE}(G, \{\tau_e\}_{e \in \mathcal{E}})} \frac{C(a^{\text{NE}})}{C(a^{\text{opt}})}. \tag{4}$$

This ratio quantifies the worst-case performance guarantee in a Nash equilibrium; in this work, a system designer will design incentives to reduce the price of anarchy ratio.

B. Incentive Mechanisms

To determine how incentive functions are assigned to resources, we will investigate the use of *incentive mechanisms*. In a congestion game G, consider the set

$$\mathcal{C}(G) = \{(c_e, e, G)\}_{e \in \mathcal{E}}, \tag{5}$$

whose elements are tuples of the congestion function, resource index, and congestion game for each resource. Further, for a set of problems $\mathcal{G}$, let $\mathcal{C}(\mathcal{G}) = \cup_{G \in \mathcal{G}} \mathcal{C}(G)$ denote the set of all resources in the family of problems $\mathcal{G}$.

For each resource e in the congestion game G with congestion function c_e, an incentive mechanism T assigns an incentive $T(c_e; e, G)$, i.e., the incentive function $\tau_e(x) = T(c_e; e, G)[x]$, where $T(c_e; e, G)[x]$ is the incentive evaluated at x users. This mapping is denoted $T : \mathcal{C}(\mathcal{G}) \to \mathcal{T}$, where $\mathcal{T}$ denotes some set of allowable incentive functions. For brevity, we will often write $T(c_e)$ to denote the incentive applied to a resource e with congestion function c_e, but it is assumed unless otherwise stated that the incentive designer has full knowledge of the exact resource and congestion game structure.

[1]These incentives are often termed 'flow-varying' in the transportation setting in contrast to constant incentives that are fixed.

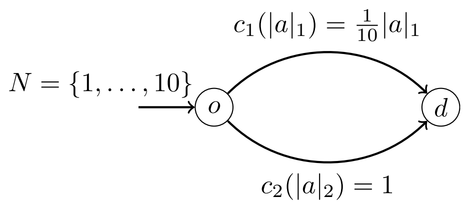


Fig. 1: Two link parallel congestion game. One edge possesses a linear congestion function, the other a constant congestion function. 10 users choose one of two resources: an upper path and lower path to traverse the network, i.e., $\mathcal{A}_1 = \cdots = \mathcal{A}_{10} = \{1, 2\}$.

C. Taxes, Subsidies, & Budgetary Constraints

To quantify the ability of incentives to reduce worst-case inefficiency over a class of instances, we extend the price of anarchy definition to include sets of congestion games $\mathcal{G}$, i.e., $\mathrm{PoA}(\mathcal{G}, T) = \max_{G \in \mathcal{G}} \mathrm{PoA}(G, T)$, where the incentive mechanism T is used in each congestion game $G \in \mathcal{G}$. The price of anarchy now bounds the worst case performance ratio over the set of instances. A system designer's objective is to reduce this inefficiency; an optimal incentive mechanism is therefore one that minimizes the price of anarchy. In this work, we seek to compare the effectiveness of subsidy and tax mechanisms in minimizing the price of anarchy.

We differentiate between two forms of incentive: let $\tau_e^+ : \{0, \dots, n\} \to \mathbb{R}_{\geq 0}$ denote a *tax* that increases the cost of a resource e to dissuade users from utilizing it and let $\tau_e^- : \{0, \dots, n\} \to \mathbb{R}_{\leq 0}$ denote a *subsidy* that reduces the cost of a resource e to persuade users to utilize it. A taxation mechanism is an incentive mechanism that only applies taxing functions to resources, i.e., $T^+ : \mathcal{C}(\mathcal{G}) \to \mathcal{T}^+$, where $\mathcal{T}^+$ is the set of non-negative functions defined on the integers from 0 to n. A subsidy mechanism is defined analogously for an incentive mechanism that maps to only subsidizing functions. An optimal taxation or subsidy mechanism is one that minimizes the price of anarchy.

To highlight the use of subsidies and taxes in improving efficiency, we offer the following example.

Example 1: Consider the network congestion game depicted in Fig. 1. When there is no incentive mechanism, in the worst-case Nash equilibrium, each of the 10 users take the upper path and observe a cost of $J_i(a^{\mathrm{NE}}) = 1$, and the total congestion of the Nash equilibrium is $C(a^{\mathrm{NE}}) = 10$. However, the optimal allocation, in which half the users take the upper path and half take the lower, the total congestion is $C(a^{\mathrm{opt}}) = 15/2$ leading to a price of anarchy of $\mathrm{PoA}(G, \emptyset) = 4/3$.

Now, consider that a tax $\tau_1(x) = x/5 - \epsilon$ and $\tau_2(x) = 0$ is applied to the problem, where $\epsilon \in (0, 1/10)$. Now, in a Nash equilibrium, 5 users use the upper path and 5 use the lower giving a price of anarchy of $\mathrm{PoA}(G, T^{\mathrm{mc}}) = 1$. Similarly, if a subsidizing mechanism assigns incentives $\tau_1(x) = 0$ and $\tau_2(x) = -\frac{1}{2} + \epsilon$, the price of anarchy will be reduced to 1.

This example highlights how both subsidies and taxes can reduce the price of anarchy ratio in atomic congestion games.

In settings where either subsidies or taxes are used to influence user behavior, budgetary constraints can be used to limit the financial obligations of the system operator or the costs incurred by system users. To formalize the notion of a budgetary constraint, a bounded taxing function for a resource $e \in \mathcal{E}$ must satisfy $\tau_e^+(x) \in [0, \beta \cdot c_e(x)]$ for each $x \in \{0, \dots, n\}$, where $\beta \geq 0$ serves as a bounding coefficient and a smaller value of β implies a more strict budgetary constraint. A bounded taxation mechanism only applies appropriately bounded taxes, denoted by $T^+(c_e; \beta)$. Similarly, a bounded subsidy satisfies $\tau_e^-(x) \in [-\beta \cdot c_e(x), 0]$ for each $x \in \{0, \dots, n\}$, where now the non-positive incentive is bounded below by a similar constraint, and a bounded subsidizing mechanism is denoted $T^-(c_e; \beta)$. This form of budgetary constraint is similar to conditions in [1], [23]. Though many forms of budgetary constraint can be considered, this form is chosen as it can be applied to local and global incentive mechanisms, captures the idea that larger congestion can be incentivized more significantly, and avoids trivialities caused by arbitrarily large cost functions. Additionally, these constraints can be represented as the total incentive being within a multiplicative factor β of the total congestion, i.e., $\sum_{e \in \mathcal{E}} |a|_e \cdot |\tau_e(|a|_e)| \leq \beta \cdot C(a)$.

Let $\mathcal{T}_\beta^+$ denote the set of all taxation mechanisms bounded by β, i.e., $\mathcal{T}_\beta^+ = \{T : \mathcal{C}(\mathcal{G}) \to \mathcal{T}^+(\beta)\}$ where $\mathcal{T}^+(\beta) = \{\tau_e^+ \in \mathcal{T}^+ | \tau_e^+(x) \in [0, \beta \cdot c_e(x)]\}$ is the set of taxing functions bounded by β. An optimal bounded taxation mechanism is thus the element of this set that minimizes the price of anarchy, i.e., $T^{\mathrm{opt}+}(\beta) \in \arg\min_{T^+ \in \mathcal{T}_\beta^+} \mathrm{PoA}(\mathcal{G}, T^+)$. The set of bounded subsidizing mechanisms $\mathcal{T}_\beta^-$ and optimal subsidizing mechanism $T^{\mathrm{opt}-}(\beta)$ are defined analogously. Comparing the price of anarchy guarantee of the optimal taxation mechanism and optimal subsidizing mechanism will show which form of incentive can achieve better performance under budgetary constraints.

Though we consider any incentive bound $\beta \geq 0$, we offer the following definition to differentiate from trivial cases.

Definition 1: A tax (subsidy) is *tightly bounded* if $\tau(x) = \beta c(x)$, (if $\tau(x) = -\beta c(x)$) for some $x \in \{1, \dots, n\}$. When an optimal incentive is tightly bounded, the budgetary constraint is active.

III. Our Contributions

A. Performance of Bounded Incentives

In this section, we consider the performance capabilities of the optimal subsidy and tax designed under budgetary constraints. We evaluate a general comparison between the effectiveness of bounded taxes and subsidies in atomic congestion games with general problem structures and congestion functions. We show in Theorem 1 that, under similar budgetary constraints, subsidies are more effective at mitigating the inefficiency caused by selfish decision making, measured by the price of anarchy ratio. Further, when the budgetary constraint is active, the performance of subsidies is strictly better.

Theorem 1: For an atomic congestion game G, under a bounding factor $\beta \geq 0$ the optimal subsidy mechanism $T^{\mathrm{opt}-}(\beta)$ has no greater price of anarchy than the optimal taxation mechanism $T^{\mathrm{opt}+}(\beta)$, i.e.,

$$\mathrm{PoA}\left(G, T^{\mathrm{opt}+}(\beta)\right) \geq \mathrm{PoA}\left(G, T^{\mathrm{opt}-}(\beta)\right) \geq 1. \quad (6)$$

Additionally, if every optimal subsidy is tightly bounded with bounding factor $\beta > 0$ for each $c_e \in \mathcal{C}(\mathcal{G})$, then the first inequality in (6) is strict.

The proof of Theorem 1 makes use of the following lemma on an equivalence transformation on incentive mechanisms.

Lemma 1: Let $T : \mathcal{C}(\mathcal{G}) \to \mathcal{T}$ be an incentive mechanism over the family of atomic congestion games $\mathcal{G}$. If another influencing mechanism is defined as $T_\lambda(c_e) = \lambda T(c_e) + (\lambda - 1)c_e$ for any $\lambda > 0$, then $\mathrm{PoA}(\mathcal{G}, T) = \mathrm{PoA}(\mathcal{G}, T_\lambda)$.

Proof: Let a^{NE} be a Nash equilibrium for a game $G \in \mathcal{G}$ under influencing mechanism T. User $i \in N$ observes cost

$$J_i(a^{\mathrm{NE}}) = \sum_{e \in a_i^{\mathrm{NE}}} c_e(|a^{\mathrm{NE}}|_e) + \tau_e(|a^{\mathrm{NE}}|_e), \tag{7}$$

and by the definition of Nash equilibrium, will have preferences satisfying the inequality in (3). In the same allocation a^{NE}, but now under influencing mechanism $\hat{T}(c_e) = \lambda T(c_e) + (\lambda - 1)c_e$ where $\lambda > 0$, user i observes cost

$$\begin{aligned}\hat{J}_i(a^{\mathrm{NE}}) &= \sum_{e \in a_i^{\mathrm{NE}}} c_e(|a^{\mathrm{NE}}|_e) + \lambda\tau_e(|a^{\mathrm{NE}}|_e) + (\lambda - 1)c_e(|a^{\mathrm{NE}}|_e) \\ &= \sum_{e \in a_i^{\mathrm{NE}}} \lambda\left(\tau_e(|a^{\mathrm{NE}}|_e) + c_e(|a^{\mathrm{NE}}|_e)\right) = \lambda J_i(a^{\mathrm{NE}}).\end{aligned}$$

Observe that through the same process, it can be shown that $\hat{J}_i(a) = \lambda J_i(a)$ for every $a \in \mathcal{A}$ and $i \in N$. From (3),

$$\begin{aligned}(1/\lambda)\hat{J}_i(a^{\mathrm{NE}}) &\le (1/\lambda)\hat{J}_i(a_i', a_{-i}^{\mathrm{NE}}), \quad \forall a_i' \in \mathcal{A}_i \\ \hat{J}_i(a^{\mathrm{NE}}) &\le \hat{J}_i(a_i', a_{-i}^{\mathrm{NE}}), \quad \forall a_i' \in \mathcal{A}_i.\end{aligned} \tag{8}$$

(8) holds for all $i \in N$, satisfying that a^{NE} is a Nash equilibrium in G under $\hat{T}$. It is therefore the case that any equilibrium in any game $G \in \mathcal{G}$ under T is also an equilibrium under $\hat{T}$. Because this approach holds for every equilibrium, it is the case that $\mathrm{NE}(G, T) = \mathrm{NE}(G, \hat{T})$. This holds for every game $G \in \mathcal{G}$, so it holds for the price of anarchy maximizer, which is the same as in Lemma 1 by definition. ■

Proof of Theorem 1: First, observe that if $\beta = 0$ the only permissible incentive function for taxes or subsidies is $\tau_e^+(x) = \tau_e^-(x) = 0$ for all $x \in \{1, \ldots, n\}$, i.e., there is no incentive. Therefore, the left and right hand side of (6) equate to the unincentivized case and (6) holds with equality.

Let $j_e(x) = c_e(x) + \tau_e(x)$ denote the cost a player observes for utilizing a resource e when x users are utilizing it. In an allocation $a \in \mathcal{A}$, the observed cost of a player $i \in N$ can be rewritten as $J_i(a) = \sum_{e \in a_i} j_e(|a|_e)$. In the case where $\beta > 0$, a bounded taxing function for a resource must exist between $\tau_e^+(x) \in [0, \beta \cdot c_e(x)]$, and the resources observed cost satisfies $j_e^+(x) \in [c_e(x), (1 + \beta) \cdot c_e(x)]$. Similarly, a subsidy on a resource must exist between $\tau_e^-(x) \in [-\beta \cdot c(x), 0]$, and the observed cost satisfies $j_e^-(x) \in [(1 - \beta) \cdot c_e(x), c_e(x)]$.

Let $T^+(c_e; \beta)$ be a bounded taxation mechanism with resource costs of $j_e^+(x)$. Now, define $T_\lambda(c_e) = \lambda T^+(c_e; \beta) + (\lambda - 1)c_e$; from Lemma 1, T^+ and T_λ have the same price of anarchy for any $\lambda > 0$. Let $\hat{j}_e$ be the resource cost under influencing mechanism T_λ, from the construction of T_λ

$$\hat{j}_e = c_e + T_\lambda(c_e) = c_e + \lambda T^+(c_e; \beta) + (\lambda - 1)c_e = \lambda j_e^+. \tag{9}$$

We now look at the cases where $\beta \in (0, 1)$ and $\beta \ge 1$ respectively. When $\beta \in (0, 1)$, let $\lambda = (1 - \beta)$. Now, the incentive mechanism T_λ assigns to a resource e a cost

$$\begin{aligned}\hat{j}_e(x) = (1 - \beta)j_e^+(x) &\in [(1 - \beta)c_e(x), (1 - \beta^2)c_e(x)] \\ &\subset [(1 - \beta)c_e(x), c_e(x)],\end{aligned}$$

thus the resource cost exists in a set that implies the incentive applied to that resource is a subsidy bounded by β. Because this is true for all resources while using this incentive mechanism, T_λ is a permissible subsidy mechanism bounded by β with the same price of anarchy as T^+. If $\beta \ge 1$ let $\lambda = 1/(1 + \beta)$ and get

$$\begin{aligned}\hat{j}_e(x) = \frac{1}{(1 + \beta)} j_e^+(x) &\in \left[\frac{1}{(1 + \beta)} c_e(x), c_e(x)\right] \\ &\subset [(1 - \beta)c_e(x), c_e(x)],\end{aligned}$$

and again, the resources cost is restricted to a set that implies T_λ is a permissible subsidy mechanism bounded by β. By letting $T^+ = T^{\mathrm{opt}+}$ we obtain (6).

We have proven that, for $\beta > 0$, if $\mathrm{PoA}(\mathcal{G}, T^{\mathrm{opt}-}(\beta)) = \mathrm{PoA}(\mathcal{G}, T^{\mathrm{opt}+}(\beta))$, then there exists a $T^{\mathrm{opt}-}(\beta)$ that is not tightly bounded (i.e., the budgetary constraint is not active). The contrapositive of this is that if every optimal subsidy is tightly bounded, the price of anarchy guarantees are not equal. In this case, the optimal subsidies are each tightly bounded and $\mathrm{PoA}(\mathcal{G}, T^{\mathrm{opt}-}(\beta)) < \mathrm{PoA}(\mathcal{G}, T^{\mathrm{opt}+}(\beta))$, proving the final part of Theorem 1. ■

B. Robustness to User Heterogeneity

In this section, we study the case where users vary in their response to incentives. Specifically, each user $i \in N$ has a price sensitivity $s_i > 0$ that quantifies how a user relates monetary costs and congestion cost and is the reciprocal of value of time; the user's cost in an allocation a, with incentives $\{\tau_e\}_{e \in \mathcal{E}}$, becomes

$$J_i(a) = \sum_{e \in a_i} c_e(|a|_e) + s_i \cdot \tau_e(|a|_e). \tag{10}$$

A population of users is denoted by the distribution $s : N \to [S_{\mathrm{L}}, S_{\mathrm{U}}]$ where $S_{\mathrm{U}} \ge S_{\mathrm{L}} > 0$ are some known lower and upper bounds. The Nash equilibrium of a congestion game G with population s and incentive mechanism T is denoted by $\mathrm{NE}(G, s, T)$ and the price of anarchy $\mathrm{PoA}(G, s, T)$ is now the worst case performance ratio of total congestion between this set of equilibria and the optimal allocation.

In this work, we are particularly interested in the effect of unknown user price heterogeneity on the effectiveness of subsidies and taxes. Consider a set of possible price-sensitivity distributions $\mathcal{S} = \{s : N \to [S_{\mathrm{L}}, S_{\mathrm{U}}]\}$ where the system operator knows only the support of users' possible price sensitivities. The price of anarchy bound over a set of congestion games $\mathcal{G}$ with possible populations $\mathcal{S}$ and incentive mechanism T is denoted by

$$\mathrm{PoA}(\mathcal{G}, \mathcal{S}, T) = \max_{G \in \mathcal{G}} \max_{s \in \mathcal{S}} \mathrm{PoA}(G, s, T). \tag{11}$$

An optimal bounded and robust taxation mechanism $T^{\mathrm{opt}+}(\beta, \mathcal{S}) \in \mathcal{T}_\beta^+$ or subsidizing mechanism $T^{\mathrm{opt}-}(\beta, \mathcal{S}) \in \mathcal{T}_\beta^-$ is one that minimizes (11).

We seek to understand how robust subsidies and taxes are to unknown user heterogeneity. Here, the ratio $S_{\mathrm{U}}/S_{\mathrm{L}}$ is representative of the possible heterogeneity that can exist in the population. The following definition describes classes of congestion games in which the effectiveness of incentives declines with increase in user heterogeneity.

Definition 2: A class of congestion games is *responsive to player heterogeneity* if $\mathrm{PoA}(\mathcal{G}, \mathcal{S}, T^*)$ is strictly increasing with $S_{\mathrm{U}}/S_{\mathrm{L}} > 1$ for an optimal bounded incentive mechanism $T^* \in \arg\min_T \mathrm{PoA}(\mathcal{G}, \mathcal{S}, T)$.
These classes of games are those that have a degradation in performance from increased player heterogeneity, even while the optimal incentive mechanism is in use; many classes of well studied congestion games possess this property [1].

In the following theorem, we show that when bounded subsidies and taxes perform similarly in the nominal, homogeneous setting (i.e., $s_i = S$ for all $i \in N$ for some known value $S > 0$), taxes prove to be more robust than subsidies to the introduction of user heterogeneity.

Theorem 2: For a class of congestion games $\mathcal{G}$, define a tax bound β^+ and a subsidy bound β^- such that the respective, optimal incentives offer the same performance in the homogeneous setting, i.e.,

$$\mathrm{PoA}\left(\mathcal{G}, T^{\mathrm{opt}-}(\beta^-)\right) = \mathrm{PoA}\left(\mathcal{G}, T^{\mathrm{opt}+}(\beta^+)\right), \tag{12}$$

then at the introduction of player heterogeneity,

$$\mathrm{PoA}\left(\mathcal{G}, \mathcal{S}, T^{\mathrm{opt}-}(\beta^-, \mathcal{S})\right) \geq \mathrm{PoA}\left(\mathcal{G}, \mathcal{S}, T^{\mathrm{opt}+}(\beta^+, \mathcal{S})\right) \geq 1. \tag{13}$$

Additionally, each inequality in (13) is strict if $\mathcal{G}$ is responsive to player heterogeneity and $S_{\mathrm{L}} < S_{\mathrm{U}}$.

Proof of Theorem 2: To prove the claim, we start by showing that, for a class of congestion games $\mathcal{G}$ and incentive mechanism T, if $T_\lambda(c) = (\lambda - 1)c + \lambda T(c)$, then $\mathrm{PoA}(\mathcal{G}, \mathcal{S}, T_\lambda)$ is non-increasing with λ and strictly decreasing if $\mathcal{G}$ is responsive to user heterogeneity and $S_{\mathrm{L}} < S_{\mathrm{U}}$.

First, we assume without loss of generality, that $S_{\mathrm{L}} = 1$. To see this, we make an equivalent problem where this is true and show the same price of anarchy bound holds. Let T be any incentive mechanism bounded by β and $\mathcal{S}$ be a family of sensitivity distributions with lower bound S_{L} and upper bound S_{U}. In any game $G \in \mathcal{G}$, a player $i \in N$ observes costs as expressed in (10). Observe that if we normalize every sensitivity distribution $s \in \mathcal{S}$ by multiplying by $1/S_{\mathrm{L}}$ and correspondingly scale the incentive by S_{L} the incentive will be bounded by $S_{\mathrm{L}} \cdot \beta > 0$ and the player cost remains unchanged. It is therefore the case that any equilibrium is preserved and unchanged, enforcing that $\mathrm{PoA}(\mathcal{G}, \mathcal{S}, T) = \mathrm{PoA}\left(\mathcal{G}, \mathcal{S}/S_{\mathrm{L}}, S_{\mathrm{L}} \cdot T\right)$. Accordingly, we will consider that $S_{\mathrm{L}} = 1$ throughout.

Let a be an allocation in $G \in \mathcal{G}$ induced by sensitivity distribution $s \in \mathcal{S}$, and let T be an incentive mechanism that assigns taxes τ_e^+. From Lemma 1 a nominally equivalent incentive mechanism can be found by using the transformation $\hat{T}(c_e; \lambda) = (\lambda - 1)c_e + \lambda T(c_e)$, where choosing λ sufficiently close to zero causes $\hat{T}$ to be a subsidy mechanism. We will show that for any $\lambda \in (0, 1)$, the incentive mechanism $\hat{T}$ performs worse than T with user heterogeneity.

Let $\hat{s}$ be a new sensitivity distribution such that

$$\hat{s}_i = g(s_i, \lambda) = \frac{s_i}{\lambda + s_i - s_i \lambda}, \tag{14}$$

for all $i \in N$. Now, consider an agent's cost in an allocation a with sensitivity $\hat{s}$ under incentive mechanism $\hat{T}$. An agent $i \in N$ utilizing action a_i in allocation a experiences cost,

$$\begin{aligned} \hat{J}_i(a) &= \sum_{e \in a_i} c_e(|a|_e) + \hat{s}_i \hat{T}(c_e(|a|_e); \lambda) \\ &= \sum_{e \in a_i} c_e(|a|_e) + \hat{s}_i [(\lambda - 1)c_e + \lambda \tau_e^+(|a|_e)] \\ &= \frac{\lambda}{\lambda + s_i - s_i \lambda} \sum_{e \in a_i} (c_e(|a|_e) + s_i \tau_e^+(|a|_e)), \end{aligned}$$

which is proportional to $J_i(a)$. By observing proportional costs, players preserve the same preferences over paths, preserving the same Nash equilibria.

Finally, we show that $\hat{s}$ is a feasible sensitivity distribution in $\mathcal{S}$. From the original bounds S_{L} and S_{U}, any generated distribution $\hat{s}$ exists between $g(S_{\mathrm{L}}, \lambda)$ and $g(S_{\mathrm{U}}, \lambda)$. From before, $S_{\mathrm{L}} = 1$, thus from (14), $g(S_{\mathrm{L}} = 1, \lambda) = 1 = S_{\mathrm{L}}$, for any $\lambda \in (0, 1)$. Now, any generated distribution satisfies $g(S_{\mathrm{U}}, \lambda) = \frac{S_{\mathrm{U}}}{\lambda + S_{\mathrm{U}} - S_{\mathrm{U}} \lambda} \leq S_{\mathrm{U}}$, for any $\lambda \in (0, 1)$. Thus any generated distribution $\hat{s}$ is appropriately bounded by S_{L} and S_{U} and is a feasible distribution in $\mathcal{S}$. By choosing a to be a Nash equilibrium, we can see that any Nash equilibrium that can be induced by some $s \in \mathcal{S}$ while using T can similarly be induced by $\hat{s} \in \mathcal{S}$ while using $\hat{T}$ that are constructed as described using some $\lambda \in (0, 1)$. Because the same equilibria can emerge while using T_λ with populations from $\mathcal{S}$, the price of anarchy with user heterogeneity is non-decreasing as λ decreases, showing the monotonicity. Further, if $S_{\mathrm{L}} < S_{\mathrm{U}}$, then $S_{\mathrm{L}} \leq g(S_{\mathrm{L}}, \lambda) \leq g(S_{\mathrm{U}}, \lambda) < S_{\mathrm{U}}$, showing the new incentive T_λ is equally affected by a smaller amount of heterogeneity, and if $\mathcal{G}$ is responsive to user heterogeneity, the price of anarchy is strictly increasing with λ.

The theorem follows closely from Lemma 1 and the previous observation. First, suppose $T^{\mathrm{opt}+}(\beta^+)$ is an optimal taxation mechanism bounded by β^+. From Lemma 1 there exists a nominally equivalent subsidy T_λ^-. If $T_\lambda^- \notin \mathcal{T}_{\beta^-}^-$, then there must exist a $T_\lambda^+ \in \mathcal{T}_{\beta^+}^+$ that is nominally equivalent to $T^{\mathrm{opt}-}(\beta^-)$ from the monotonicity and invertability of the transformation in Lemma 1. From (12), this implies there exists a nominally equivalent $T^{\mathrm{opt}+}(\beta^+)$ and $T^{\mathrm{opt}-}(\beta^-)$.

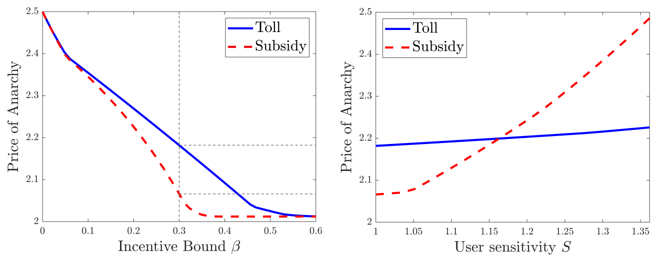
Now, let $T^{\mathrm{opt}-}(\beta^-, \mathcal{S})$ be the optimal subsidy with player heterogeneity bounded by β^-. From the fact before, we know there exists a tax T^+ that is nominally equivalent to $T^{\mathrm{opt}-}(\beta^-, \mathcal{S})$ and bounded by β^+. From the monotonicity of the price of anarchy from the first part of this proof, we obtain that

$$\mathrm{PoA}(\mathcal{G}, \mathcal{S}, T^+) \leq \mathrm{PoA}(\mathcal{G}, \mathcal{S}, T^{\mathrm{opt}-}(\beta^-, \mathcal{S})), \tag{15}$$

and by the definition of $T^{\mathrm{opt}+}(\beta^+, \mathcal{S})$, we get

$$\mathrm{PoA}(\mathcal{G}, \mathcal{S}, T^{\mathrm{opt}+}(\beta^+, \mathcal{S})) \leq \mathrm{PoA}(\mathcal{G}, \mathcal{S}, T^+). \tag{16}$$

Combining (15) and (16) gives (13). If the class of games is responsive to player heterogeneity, then $\mathrm{PoA}(\mathcal{G}, \mathcal{S}, T_\lambda)$ is strictly decreasing with λ and the relationship is strict. ■



(a) Price of anarchy with bounded incentives

(b) Price of anarchy with player heterogeneity

Fig. 2: Price of Anarchy bounds for comparable taxes and subsidies in affine congestion games. (Left) Price of Anarchy under optimal tax and subsidy respectively bounded by a factor β. (Right) Price of Anarchy of the optimal incentives bounded by $\beta = 0.3$ (highlighted by dashed lines on left) while each player has sensitivity $S \geq 1$.

IV. Numerical Evaluation

In [13], the authors introduce a tractable linear program whose solution is the optimal, local incentive scheme that minimizes the worst case price of anarchy over a set of problem instances. Such incentive mechanisms have the added constraint that resources with the same congestion function must be assigned the same incentive function. These forms of incentives are desirable in settings with uncertainty or where the problem structure is frequently changing, where partial changes to the congestion function or problem structure does not require a global redesign of the incentive mechanism. Because these incentives are a subset of the mechanisms considered in this work, the main results hold.

For illustrative purposes, we consider the class of all atomic congestion games with affine congestion functions $\mathcal{G}$, i.e., $c_e(x) = m_e x + b_e$ for every $e \in \mathcal{E}$. In Fig. 2a, the price of anarchy of the optimal subsidy mechanism and optimal taxation mechanism are shown for varying values of β. As stated in Theorem 1, the price of anarchy guarantee is better for subsidies than taxes for every value of $\beta > 0$. This figure shows that the difference in performance can be significant.

To understand the robustness of each of these incentives, consider the optimal subsidy and tax designed for a population of users each with sensitivity $S_i = 1$ bounded by a factor of $\beta = 0.3$. In Fig. 2b, the price of anarchy for each incentive is shown when each user actually has a sensitivity $S_i = S \geq 1$. When the population's sensitivity is sufficiently close to 1, the users response is close to the case which the incentives were designed for, and subsidies outperform taxes. However, when the population's sensitivity differs from what was anticipated, we see the price of anarchy guarantee of subsidies degrades much more quickly than that of taxes. This is reminiscent of Theorem 2 where taxes prove to be more robust to unknown user heterogeneity than subsidies.

V. Conclusion

This work generalized the results of [1] to the setting of atomic congestion games. Specifically, it is shown that in the model of congestion games with a finite number of users, subsidies offer better performance guarantees under budgetary constraints than taxes, but taxes are more robust to user heterogeneity than subsidies.

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