Incentive Design for Congestion Games with Unincentivizable Users

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***Abstract*— Incentives are an effective tool to alter user preferences to promote more efficient group behavior. It is often assumed that these incentives can be levied on any and every user in the system; in many settings, this is not the case. Accordingly, how should a system operator design incentives that only affect a fraction of the users? The network routing literature contains many results showing the effectiveness of monetary taxes to influence self-interested users' behavior and improve system efficiency. These results typically assume that all users in the network are influenced by incentives; however, this need not be the case if incentives are opted into or if some users do not experience or are unaffected by monetary fees. In this work, we address the problem of designing incentives for populations of users where a fraction of the population is not influenced by incentives in their decision-making process. By focusing on the setting of parallel-network selfish routing problems, surprisingly we find that the tolls that are optimal when the full population is incentivizable remain optimal when only a fraction of the population is incentivizable, though at reduced effectiveness. To measure the impact that the unincentivizable users have on the efficacy of the optimal tolling scheme, we derive worst-case performance bounds in a simple class of networks when only a fraction of the users can be incentivized.**

I. Introduction

In many large-scale social systems, the self-interested decisions of individual users shape the behavior of the system as a whole. Though users may make decisions rationally, emergent collective behavior may be sub-optimal from a system-level perspective. This phenomenon is present in traffic and transportation [1], [2], energy production/consumption [3], markets and commerce [4], and many other settings. A well-studied way to quantify this inefficiency is the *price of anarchy*, which measures the worst-case ratio between the system cost when users act in their own self-interest and the optimal system cost.

One well-studied method of incentivizing users is to levy taxes that charge monetary fees to users for making decisions that are unfavorable for the systems performance [5]–[8]. Incentives of this form have been shown to be effective at mitigating system-level inefficiency and reducing the price of anarchy [9]–[11]. Though the use of incentives is encouraging, many of the proposed techniques require that incentives can be levied on any and every user in the system and that each user will elicit a response. The fact that this need not be the case in many settings is, to the best of the authors' knowledge, not considered in the literature; the questions of how to design incentives and how incentives perform when only a fraction of the population is incentivizable is thus unknown.

In this work, we consider a network routing problem where users are routed through a network with congestible edges. A user that takes a path in the network will experience a cost that grows with the number of users taking the same path; the path's congestion cost is called the latency in reference to the model's common use to capture transit delay in traffic networks. Finding a path for each user that minimizes the aggregate transit delay is easy if the system designer has direct control of each user's action. However, when users select their routes to minimize their own experienced delay, the emergent system behavior can be far from optimal [12], [13]. Many works have studied the use of incentives in this setting by designing mechanisms that assign tolls to each path, adding a monetary fee to any user taking that path [5]–[11]; each finds that tolls can be used effectively to reduce inefficiency.

Nonetheless, incentives are only effective if users perceive them; in many settings, this may not be the case. For example, consider a bus system that incentivizes patient riders to wait for a later bus by giving out coupons from an app. The incentives are only visible to users who have this app, not all riders [14]. Alternatively, a road toll designer may want to implement a taxation scheme that users must opt in to; in [15], the authors offered incentives to drivers in India to reduce congestion by giving each user a sum of money that is decreased when users make travel decisions that increase congestion. These decision-influencing incentives could only be charged to users who opted in to the experiment. Even still, some users may be so insensitive to monetary fees or place such a large value on their time that the incentives will not alter their decision-making process [16], [17]. To better understand the effectiveness of incentives with these types of users, we investigate how to design incentives when they are only applied to a portion of the population.

Previous works have considered various constraints on the implementation of incentives. In both [18], [19], the authors consider various budgetary constraints on an incentive scheme: [18] considers that the magnitude of incentives must be bounded while [19] studies incentive design when the net expenditure of the system operator must be zero. In both settings, incentives exist that improve system performance. Another well-studied setting is when users are heterogeneous in their response to monetary incentives [5], [6], [9]. Though the incentive design problem becomes more difficult, results in the literature show that incentives can reduce system-

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level inefficiency. In [20], the author investigates system inefficiency when some users are selfish while others are altruistic, and finds that partial altruism can cause worse system performance than if each user were selfish; however, the opportunity of designing incentives for agents is not considered. These works collectively cover settings where incentives are constrained and users have different objectives and incentive responses, but none of these works consider that there may exist users in the system that do not (or can not) respond to incentives at all.

Naturally, effective implementation of any incentive scheme is heavily dependent on how users respond to incentives, if at all. In this work, we find the optimal taxation mechanism in parallel network routing problems when only a fraction of the population observes and responds to the incentives. Interestingly, we find that the taxation mechanism that is optimal in the nominal setting remains optimal when only observed by a fraction of users. To quantify what capabilities a system designer has in incentivizing these types of populations, we find the price of anarchy bound in two-link congestion networks while using the optimal taxation mechanism. Our results are outlined as follows:

Theorem 1: When only a fraction of the population can observe and respond to incentives, we show that the marginal cost toll is the optimal taxation mechanism and minimizes the worst-case price of anarchy in parallel networks. Interestingly, this mechanism is also optimal when all users can be incentivized.

Theorem 2: While using the marginal-cost incentive mechanism, the system operator experiences some performance loss from the presence of unincentivizable users. We characterize the attainable performance guarantees in the specific context of two-link affine-cost congestion games. Our results imply that performance gradually improves as more people observe incentives.

II. Model

A. Routing Problem

To define a network routing problem, let (V, E) be a graph containing a set of vertices V and a set of directed edges E between those vertices $E \subseteq \{(u, v) \mid u, v \in V\}$. In the case of parallel-network routing problems, a unit mass of traffic is directed from a source s to a destination d using a set of parallel edges E that connect them, i.e., $V = (s, d)$. A *feasible flow* f over the network is an assignment of traffic over the edges such that $f = \{f_e\}_{e \in E} \in \Delta(E)$, where $\Delta(E)$ denotes the standard probability simplex over the set E, i.e., each edge flow $f_e \geq 0$ and $\sum_{e \in E} f_e = 1$. Each user that utilizes an edge $e \in E$ experiences a cost which grows with the amount of congestion on that edge. In the area of transportation, this cost is expressed as transit delay experienced by the users on the edge. We model the cost of utilizing edges via a non-decreasing function of the mass of traffic on that edge; we term these functions latency functions, where for each edge $e \in E$ the latency (or experienced cost) is the output of a mapping $\ell_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

A system designer is interested in minimizing the aggregate delay experienced by the users in the system. This system cost is modeled by the *total latency*:

$$\mathcal{L}(f) = \sum_{e \in E} f_e \cdot \ell_e(f_e), \tag{1}$$

An *optimal flow* $f^{\mathrm{opt}} \in \arg\min_{f \in \Delta E} \mathcal{L}(f)$ is a feasible flow that minimizes the total latency. A parallel routing problem instance can be described by the tuple $G = (E, \{\ell_e\}_{e \in E})$.

B. Taxation Mechanism & Performance Metrics

Though a system designer may desire to minimize the total latency in a network routing problem, they may lack the ability to enforce these assignments when users are able to choose their own routes. When users choose their routes to minimize their own experienced latency, we model this selfish routing problem as a non-atomic congestion game. Let $N = [0, 1]$ denote the set of infinitely many, infinitesimal agents, and $e(x)$ denote the edge chosen by user $x \in N = [0, 1]$. In a feasible flow f, user $x \in [0, 1]$ on edge $e(x) \in E$ experiences a cost

$$J_x(f) = \ell_{e(x)}(f_{e(x)}). \tag{2}$$

When users select edges to minimize their own perceived cost, the emergent group behavior need not be the same as the system optimal. A feasible flow f^{nf} is a *Nash flow* if it emerges from every user picking the edge that minimizes their experienced cost, i.e.,

$$J_x(f) = \min_{e \in E}\{\ell_e(f_e)\}, \quad \forall\, x \in [0, 1]. \tag{3}$$

We call this a Nash flow because it emerges as a Nash equilibrium from the game formed between users whose costs are their experienced latency; this is identical to the notion of Wardrop equilibrium where every user takes a path of minimal cost.

It is well known in this setting that the system cost in a Nash flow can be much worse than the system optimal [13]. To alleviate this inefficiency, many works have studied the use of incentives to promote more desirable system operation [9], [10], [18]. This work studies the use of incentives in the form of taxes levied on users based on what edge they use in the network. These taxes are implemented by designing flow-varying tolling functions $\tau_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ to each edge $e \in E$. Users who are assessed these incentives now experience both monetary and temporal costs. A user $x \in [0, 1]$ that utilizes the edge $e(x)$ will have a tolled cost

$$J_x(f) = \ell_{e(x)}(f_{e(x)}) + \tau_{e(x)}(f_{e(x)}). \tag{4}$$

Under these incentives, a tolled Nash equilibrium f emerges when each user now seeks to minimize their tolled plus latency cost, i.e.,

$$J_x(f) = \min_{e \in E}\{\ell_e(f_e) + \tau_e(f_e)\}, \quad \forall\, x \in [0, 1]. \tag{5}$$

These taxes (or tolls) are assigned to edges by way of a *taxation mechanism*. Different taxation mechanisms have been proposed to address the inefficiency of selfish routing.

A taxation mechanism T takes as input the routing problem G, the edge to which an incentive is assigned e, and the latency function on that edge ℓ_e. The output of the function $T(G, e, \ell_e)$ is the tolling function τ_e. An example of a well-studied taxation mechanism is the marginal-cost tax, where the tolling functions are of the form

$$T^{\mathrm{mc}}(G, e, \ell_e)[f_e] = \tau_e^{\mathrm{mc}}(f_e) = f_e \cdot \frac{d}{df_e}\ell_e(f_e). \quad (6)$$

These tolls have been shown to incentivize optimal routing in the classic setting of non-atomic congestion games where latency functions are convex, non-decreasing, and continuously differentiable.

Though incentives can be useful, not every incentive mechanism is completely effective and various system constraints or uncertainties can harm system performance. The performance of a taxation mechanism can be evaluated using the concept of *price of anarchy*. Let $\mathcal{L}^{\mathrm{nf}}(G, T)$ denote the total latency of a Nash flow f^{nf} on network G, when tolls are assigned using taxation mechanism T. Let $\mathcal{L}^{\mathrm{opt}}(G)$ denotes the total latency of the optimal flow f^{opt}. Notice that $\mathcal{L}^{\mathrm{opt}}(G)$ only depends on the network G itself. The price of anarchy compares the system cost under a Nash flow with that under the optimal flow, and is defined as:

$$\mathrm{PoA}(G, T) = \frac{\mathcal{L}^{\mathrm{nf}}(G, T)}{\mathcal{L}^{\mathrm{opt}}(G)} \geq 1. \quad (7)$$

A system designer thus seeks to design incentives with the objective of minimizing the price of anarchy ratio. In the classical setting, where any and every user can be assigned incentives, the marginal cost taxation mechanism defined in (6) achieves a price of anarchy of 1 and is hence optimal.

C. Unincentivizable Users

In this work, we are interested in understanding how to design taxation mechanisms when some users do not experience or are not influenced by incentives, e.g., app based incentive programs [15]. Additionally, we are interested in understanding what capabilities an incentive designer has in reducing the price of anarchy in this unincentivizable population setting.

To that end, we seek to design a taxation mechanism for populations of users that are a mix of incentivizable and unincentivizable users. The incentivizable users pick the edge that minimizes the sum of the toll and the edge latency, while the unincentivizable users pick the edge that minimizes only the edge latency. When only a fraction $\beta \in [0, 1]$ of the users observe the applied incentive, a Nash flow is defined as

$$J_x(f^{\mathrm{nf}}) = \min_{e \in E}\{\ell_e(f_e) + \tau_e(f_e)\}, \quad \forall\, x \in [0, \beta]. \quad (8)$$

$$J_x(f^{\mathrm{nf}}) = \min_{e \in E}\{\ell_e(f_e)\}, \quad \forall\, x \in (\beta, 1]. \quad (9)$$

Because the fraction of users that respond to incentives affects the Nash flow of the congestion game, the price of anarchy is now dependent on β, thus we extend the original definition to

$$\mathrm{PoA}(G, \beta, T) = \frac{\mathcal{L}^{\mathrm{nf}}(G, \beta, T)}{\mathcal{L}^{\mathrm{opt}}(G)} \geq 1, \quad (10)$$

where $\mathcal{L}^{\mathrm{nf}}(G, \beta, T)$ is the total latency in the Nash flow in the routing problem G with incentive mechanism T that is observed by β fraction of the population. The new congestion game with incentivizable and unincentivizable remains a stable population game, and a Nash equilibrium is guaranteed to exist [21]

An optimal taxation mechanism is one that minimizes the price of anarchy when only a fraction β of the population is incentivizable. The best taxation mechanism for a system designer minimizes worst-case system inefficiency while the network and/or the fraction of users that are incentivizable are unknown. Therefore, when only a fraction β of the population is incentivizable, an optimal taxation mechanism is

$$T^{\mathrm{opt}}(\beta) \in \arg\inf_{T}\ \mathrm{PoA}(G, \beta, T).$$

We investigate what the optimal taxation mechanism is and how effective the system designer is in reducing the price of anarchy when there is an unincentivizable group of users in the population.

III. Main Results

Our first results focuses on the design of optimal incentives for routing problems with both incentivizable and unincentivizable users. Interestingly, our first theorem demonstrates that the structure of the optimal taxation mechanism is not impacted by the presence of unincentivizable users. That is, the marginal-cost toll defined (6) is the optimal incentive mechanism irrespective of the fraction of unincentivizable users.

Theorem 1. *In any parallel network G, in which each edge $e \in E$ posses a non-decreasing, convex, continuously-differentiable latency function ℓ_e, if only a fraction $\beta \in [0, 1]$ of the the population observe incentives, a taxation mechanism that minimizes the price of anarchy (i.e., satisfies* (II-C)*) is the marginal cost taxation mechanism $T^{\mathrm{mc}}(\ell)[x] = \ell_e(x) + x \cdot \frac{d\ell}{dx}(x)$.*

Proof: Before considering the problem of designing a tolling scheme, we focus on the problem in which the fraction of users $\beta \in [0, 1]$ can be routed directly while the remaining users minimize their observed latency. Let $d \in (1-\beta)\Delta(E)$ denote the network flow of self-routing users and $c \in \beta\Delta(E)$ denote the flow of the directly controlled users. For an edge $e \in E$ with latency function ℓ_e and flow x, we define the marginal cost of the edge as

$$\ell_e^{\mathrm{mc}}(x) = \ell_e(x) + x \cdot \frac{d\ell_e}{dx}(x).$$

In a flow of users $f = c+d$, the total latency can be expressed using the marginal cost function as

$$\mathcal{L}(f) = \sum_{e \in E} f_e \cdot \ell_e(f_e) = \sum_{e \in E} \int_0^{f_e} \ell_e^{\mathrm{mc}}(x)dx.$$

By observing that the partial derivative $\partial\mathcal{L}(f)/\partial f_e = \ell_e^{\mathrm{mc}}(f_e) \geq 0$, we can note that the marginal cost of an edge quantifies the rate of increase in total latency for adding users to the edge e.

Now, consider the problem where the unincentivizable group of users d chooses routes to minimize their observed latency. A system designer that seeks to add the traffic c to minimize the total latency will sequentially add them to the edge with the least marginal cost $\ell_e^{\mathrm{mc}}(c_e + d_e)$. If the addition of this traffic to an edge e causes an unincentivizable user to change paths, they will only choose another utilized edge e' to satisfy that both edges have minimal latency; this is the case if traffic were added to any such utilized edge. The unincentivizable users will never leave for a non-utilized edge e' because

$$\ell_{e'}^{\mathrm{mc}}(0) = \ell_{e'}(0) < \ell_e(c_e + d_e) \le \ell_e^{\mathrm{mc}}(c_e + d_e)$$

would contradict that the controlled traffic is added to the edge of least marginal cost. Because adding traffic to any utilized edge will have the same effect on the unincentivizable users, and thus overall network flow, the optimal option remains to add traffic to the minimum marginal cost edge. If the added traffic does not alter the actions of the unincentivizable traffic, then adding it to the edge of minimal marginal cost is certainly best.

The added traffic can be made to minimize the marginal cost by making each user route selfishly with respect to the marginal cost of the edge. This can be achieved by instituting the marginal cost taxation mechanism $T^{\mathrm{mc}}(\ell)[x] = \ell_e(x) + x \cdot \frac{d\ell}{dx}(x)$ to each edge for the incentivizable users. □

Now that we know the marginal cost taxation mechanism is the best incentives mechanism for any parallel networks with non-decreasing, convex, continuously-differentiable latency function, we look into the specific context of 2-link parallel networks with affine latency function and characterize the attainable performance guarantees.

Theorem 2. *In any two-link, parallel network congestion game with affine latency functions G_2, if $\beta \in [0, 1]$ fraction of the users experience a marginal cost toll, then the price of anarchy satisfies*

$$\mathrm{PoA}(G_2, \beta, T^{\mathrm{mc}}) \le \frac{4}{3 + \beta}. \quad (11)$$

Proof: Let G be a two-link parallel network with latency functions $\ell_1(f_1) = a_1 f_1 + b_1$ and $\ell_2(f_2) = a_2 f_2 + b_2$. Without loss of generality, we order the edges of G in increasing order of the constant; that is $b_1 \le b_2$. In our search for a network that has the worst price of anarchy, we want to narrow down our search space to networks where $a_1 = 1$. We can do that because scaling all latency functions by a positive constant does not change the Nash flow or Opt flow, nor does it change the price of anarchy. We multiply a_1, a_2, b_1, b_2 of G by $\frac{1}{a_1}$. The latency function of G is now $\ell_1(f_1) = f_1 + b_1$ and $\ell_2(f_2) = a_2 f_2 + b_2$. Notice that the a_2, b_2 here is technically a_2/a_1, b_2/a_1; However, a_2, b_2 can be any value, and we keep the notation for convenience.

Step 1: Identifying relevant cases

First, we characterize Nash flows that give the non-trivial price of anarchy values. In a Nash flow, the unincentivizable users have three possible actions.

1) They only use edge 1, because the cost of using edge 1 is strictly smaller than the cost of using edge 2.
2) They only use edge 2, because the cost of using edge 2 is strictly smaller than the cost of using edge 1.
3) They are indifferent because the cost of using edge 1 and edge 2 is equal.

The same three possible actions hold for the incentivizable users as well, resulting in a total of 9 possible cases. To identify relevant Nash flow cases for the worst-case price of anarchy, we first prove certain cases are impossible through contradiction, then we prove that certain cases always have the price of anarchy of 1, and are thus trivial for our purposes. In the end, only two relevant cases remain.

The following cases are impossible. It is impossible for both population to only use edge 2, which would imply $\ell_1(0) = b_1 > a_2 + b_2 = \ell_2(1)$, given $b_1 \le b_2$.

It is impossible for the incentivizable population only uses edge 1, while the unincentivizable population only use edge 2 or remains indifferent. Using (8) and from each populations described preferences, we know that

$$\begin{aligned} 2f_1^{\mathrm{nf}} + b_1 &< 2a_2 f_2^{\mathrm{nf}} + b_2, \\ f_1^{\mathrm{nf}} + b_1 &\ge a_2 f_2^{\mathrm{nf}} + b_2, \end{aligned}$$

which can be rewritten as

$$b_2 - b_1 > 2(f_1^{\mathrm{nf}} - a_2 f_2^{\mathrm{nf}}) \quad (12)$$

$$b_2 - b_1 \le f_1^{\mathrm{nf}} - a_2 f_2^{\mathrm{nf}} \quad (13)$$

Given $b_2 - b_1 \ge 0$, (12) contradicts (13).

The following cases are trivial. If both types of users only use edge 1, then

$$\ell_1(1) = 1 + b_1 < b_2 = \ell_2(0), \quad (14)$$

$$\ell_1(1) + \tau_1(1) = 2 + b_1 < b_2 = \ell_2(0) + \tau_2(0). \quad (15)$$

Because of (15), the optimal flow also routes both types of users to edge 1. Same Nash and optimal flow give a price of anarchy of 1.

If the incentivizable population is indifferent, then

$$2f_1^{\mathrm{nf}} + b_1 = 2a_2(1 - f_1^{\mathrm{nf}}) + b_2,$$

which can be rewritten as

$$f_1^{\mathrm{nf}} = \frac{2a_2 + b_2 - b_1}{2 + 2a_2}$$

Substitute $f_2 = 1 - f_1$ and solve for the minimum of the quadratic function (1) in f_1, we obtain the optimal flow on edge 1 as

$$f_1^{\mathrm{opt}} = \frac{2a_2 + b_2 - b_1}{2 + 2a_2} \quad (16)$$

Notice that the optimal flow of a network G does not depend on β or the Nash flow behavior. Therefore, this expression for calculating the optimal flow works for the other cases as well. Since the Nash flow equals the optimal flow, the price of anarchy of G is always 1.

The only two relevant cases left are when the incentivizable population only uses edge 2, and the unincentivizable population either only uses edge 1 or remains indifferent.

Step 2: Reducing worst-case search

For any congestion game G, we can find a new game $\hat{G}$ that has the same or higher price of anarchy through a series of transformations. This allows us to search for the price of anarchy guarantees to only networks that are possible after the transformations.

Consider the case where the incentivizable population only uses edge 2, and the unincentivizable population only uses edge 1. G satisfies the following inequality:

$$(1-\beta)+b_1 < a_2\beta + b_2, \tag{17}$$

$$2(1-\beta)+b_1 > 2a_2\beta + b_2. \tag{18}$$

In the first transformation, we demonstrate that removing b_1 increases the price of anarchy.

Let $\hat{G}$ be a two-link parallel network with latency function $\ell_1(f_1) = f_1$ and $\ell_2(f_2) = a_2 f_2 + \hat{b_2} = a_2 f_2 + b_2 - b_1$. From (17) and (18), it is clear that $(1-\beta) < a_2\beta + b_2 - b_1$ and $2(1-\beta) > 2a_2\beta + b_2 - b_1$, which implies that the incentivizable population of $\hat{G}$ also only uses edge 2, and the unincentivizable population of $\hat{G}$ also only uses edge 1 during Nash flow. Thus the Nash flow of G and $\hat{G}$ are both $1-\beta$. Substitute $f_1^{\text{nf}} = \hat{f_1}^{\text{nf}} = 1-\beta$ into (1), we observe that

$$\mathcal{L}^{\text{nf}}(G,\beta,T^{\text{mc}}) = \mathcal{L}^{\text{nf}}(\hat{G},\beta,T^{\text{mc}}) + b_1$$

Substitute (16) into (1), we observe that

$$\mathcal{L}^{\text{opt}}(G) = \mathcal{L}^{\text{opt}}(\hat{G}) + b_1$$

Comparing the Price of Anarchy of G and $\hat{G}$ as follows:

$$\begin{aligned}\text{PoA}(G,\beta,T^{\text{mc}}) &= \frac{\mathcal{L}^{\text{nf}}(\hat{G},\beta,T^{\text{mc}}) + b_1}{\mathcal{L}^{\text{opt}}(\hat{G}) + b_1} \\ &\leq \frac{\mathcal{L}^{\text{nf}}(\hat{G},\beta,T^{\text{mc}})}{\mathcal{L}^{\text{opt}}(\hat{G})} = \text{PoA}(\hat{G},\beta,T^{\text{mc}})\end{aligned}$$

Since $\hat{G}$ has the same or higher PoA than G, we consider b_1 of G is 0 for the rest of this proof.

In the second transformation, we show that removing a_2 increases the price of anarchy.

Let $\hat{G}$ has latency functions $l_1(\hat{f_1}) = \hat{f_1}$ and $l_2(\hat{f_2}) = \hat{b_2} = b_2 + a_2\beta$. We first show the Nash flow of $\hat{G}$ also follows the pattern that the incentivizable population only uses edge 2 and the unincentivizable population only uses edge 1. This is true if the following inequality holds.

$$\hat{f_1}^{\text{nf}} < \hat{b_2} < 2\hat{f_1}^{\text{nf}} \tag{19}$$

Since both $\hat{b_2}$ and $\hat{f_1}^{\text{nf}}$ are positive, the inequality is clearly true. Thus the Nash flow of G and $\hat{G}$ are the same. Substitute $\hat{f_1}^{\text{nf}} = f_1^{\text{nf}} = 1-\beta$ into (1), we observe that $\mathcal{L}^{\text{nf}}(\hat{G}) = \mathcal{L}^{\text{nf}}(G)$.

The optimal flow of G and $\hat{G}$ is obtained by (16). Substitute $\hat{f_1}^{\text{opt}} = (a_2\beta + b_2)/2$ and $f_1^{\text{opt}} = (2a_2 + b_2)/(2a_2+2)$ into (1), we observe that

$$\begin{aligned}&\mathcal{L}^{\text{opt}}(G,\beta,T^{\text{mc}}) - \mathcal{L}^{\text{opt}}(\hat{G},\beta,T^{\text{mc}}) \\ &= \frac{a_2(a_2^2\beta^2 + a_2\beta^2 + (2-b_2)(2-2\beta-2\beta a_2 - b_2))}{4(a_2+1)}.\end{aligned} \tag{20}$$

Since unincentivizable population of G prefers edge 2, we know that $2-2\beta > 2\beta a_2 + b_2$. Therefore, $2-2\beta-2\beta a_2 - b_2 > 0$ and $2 - b_2 > 0$, (20) is non-negative, and $\text{PoA}(G) \leq \text{PoA}(\hat{G})$.

Finally, we substitute $f_1^{\text{nf}} = 1-\beta$ and $f_1^{\text{opt}} = \hat{b_2}/2$ into (10), and calculate that

$$\text{PoA}(\hat{G},\beta,T^{\text{mc}}) = \frac{(1-\beta)^2 + \beta\hat{b_2}}{\hat{b_2} - \frac{\hat{b_2}^2}{4}}. \tag{21}$$

Consider the case where the incentivizable population only uses edge 2, and the unincentivizable population is indifferent. G satisfies the following inequality:

$$2f_1 + b_1 > 2a_2 f_2 + b_2, \tag{22}$$

$$f_1^{\text{nf}} = \frac{a_2 + b_2 - b_1}{1 + a_2} \leq 1 - \beta \tag{23}$$

We again show that removing b_1, a_2 increase the price of anarchy. Let $\hat{G}$ be a two-link parallel network with latency function $\ell_1(f_1) = a_1 f_1$ and $\ell_2(f_2) = a_2 f_2 + \hat{b_2} = a_2 f_2 + b_2 - b_1$. From (22) and (23), it is clear that $2a_1 f_1 > 2a_2 f_2 + b_2 - b_1$ and $f_1^{\text{nf}} = \frac{a_2+b_2-b_1}{a_1+a_2} \leq 1-\beta$, which implies that the incentivizable population of $\hat{G}$ prefers edge 2, and the unincentivizable population of $\hat{G}$ is indifferent during Nash flow. Thus the Nash flow of G and $\hat{G}$ are the same. Substitute (23) into (1), we observe that

$$\mathcal{L}^{\text{nf}}(G,\beta,T^{\text{mc}}) = \mathcal{L}^{\text{nf}}(\hat{G},\beta,T^{\text{mc}}) + b_1$$

Using (16), we find optimal flow of G and $\hat{G}$ to be the same. Substitute $f_1^{\text{opt}} = \hat{f_1}^{\text{opt}} = (2a_2 + b_2 - b_1)/(2a_1 + 2a_2)$ into (1), we observe that

$$\mathcal{L}^{\text{opt}}(G) = \mathcal{L}^{\text{opt}}(\hat{G}) + b_1$$

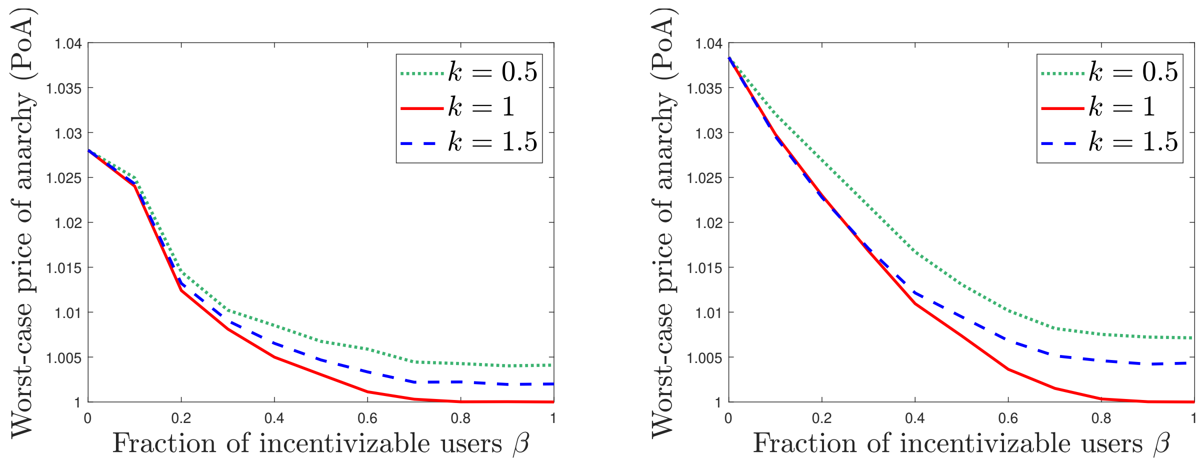
Comparing the Price of Anarchy of G and $\hat{G}$ as follows:

$$\begin{aligned}\text{PoA}(G,\beta,T^{\text{mc}}) &= \frac{\mathcal{L}^{\text{nf}}(\hat{G},\beta,T^{\text{mc}}) + b_1}{\mathcal{L}^{\text{opt}}(\hat{G}) + b_1} \\ &\leq \frac{\mathcal{L}^{\text{nf}}(\hat{G},\beta,T^{\text{mc}})}{\mathcal{L}^{\text{opt}}(\hat{G})} = \text{PoA}(\hat{G},\beta,T^{\text{mc}})\end{aligned}$$

Therefore, removing b_1 in this case also does not decrease the price of anarchy.

Then we prove that removing a_2 does not decrease price of anarchy. First, we verify the Nash flow of $\hat{G}$ also follows the pattern that the incentivizable population only uses edge 2 and the unincentivizable population is indifferent, by showing that $2\hat{f_1}^{\text{nf}} > \hat{b_2}$. Since $\hat{f_1}^{\text{nf}} = \hat{b_2}$, it is true.

Use (23) to calculate Nash flow, and (16) to calculate optimal flow, we find that $f_1^{\text{nf}} = \frac{a_2+b_2}{a_2+1}$, $f_1^{\text{opt}} = \frac{2a_2+b_2}{2a_2+2}$,



(a) Average PoA of 2-link parallel networks

(b) Average PoA of 5-link parallel networks

Fig. 1: Average price of anarchy over randomly generated routing problem instances for three taxation mechanisms: marginal cost tolls, more aggressive tolls, and less aggressive tolls. The more and less aggressive tolls are realized by scaling the marginal cost toll by a factor of 1.5 and 0.5 respectively. Though one may think that charging users more or less would be helpful when not all users are incentivizable, the marginal cost toll performs best, as predicted by Theorem 1. The left plot shows the average price of anarchy in 1000 randomly generated 2-link networks and the right shows the average price of anarchy in 1000 randomly generated 5 link networks. The similarity between the two shows motivates that the worst-case performance bound in Theorem 2 generalizes to all parallel networks.

$\hat{f}_1^{\,\text{nf}} = \beta a_2 + b_2$, and $\hat{f}_1^{\,\text{opt}} = (\beta a_2 + b_2)/2$. Substitute these value into (1) and (10), we obtain that

$$\begin{aligned} &\text{PoA}(\hat{G}, \beta, T^{\text{mc}}) - \text{PoA}(G, \beta, T^{\text{mc}}) \\ &\qquad = \frac{-4a_2(b_2 + \beta a_2 + \beta b_2)}{(-b_2^2 + 4b_2 + 4a_2)(a_2 + \beta b_2 - 4)} \end{aligned} \tag{24}$$

Since the incentivizable population of $\hat{G}$ strictly only uses edge 2, $\hat{f}_1^{\,\text{nf}} = \beta a_2 + b_2 \leq 1 - \beta$. Thus $\beta a_2 + b_2 < 4$. This means $a_2 + \beta b_2 - 4 < 0$, $-b_2^2 + 4b_2 > 0$, expression (24) is non-negative, and $\text{PoA}(G) \leq \text{PoA}(\hat{G})$.

Finally, we substitute $\hat{f}_1^{\,\text{nf}} = \hat{b_2}$ and $\hat{f}_1^{\,\text{opt}} = \hat{b_2}/2$ into (10) and calculate that

$$\text{PoA}(\hat{G}, \beta, T^{\text{mc}}) = \frac{\hat{b_2}}{\hat{b_2} - \frac{\hat{b_2}^{\,2}}{4}}. \tag{25}$$

Step 3: Finding worst-case price of anarchy

Now that we find the worst price of anarchy in the remaining two cases, we compare these two values, (21) and (25), to find the worst-case price of anarchy. Notice that $\hat{b_2}$ in equation (21) and (25) have different bounds. To distinguish them, we will denote the b_2 from equation (21) $\hat{b_2}^{\,\beta}$.

Remember that (21) comes from the case when the incentivizable population only uses edge 2, and the unincentivizable population only uses edge 1, whose bound is given by (17) and (18). We substitute $\hat{b_1} = 0$ and $\hat{a_2} = 0$ from the network $\hat{G}$ into (17) and (18) and observe that

$$1 - \beta \leq \hat{b_2}^{\,\beta} \leq 2(1 - \beta). \tag{26}$$

Remember that (25) comes from the case when the incentivizable population only uses edge 2, and the unincentivizable population is indifferent, whose bound is given by (22) and (23). We substitute $\hat{b_1} = 0$ and $\hat{a_2}$ from the network $\hat{G}$ into (23) and observe that

$$\hat{b_2} \leq 1 - \beta. \tag{27}$$

From (26) and (27), we observe that $\hat{b_2} \leq (1 - \beta) = (1 - \beta)^2 + \beta(1 - \beta) \leq (1 - \beta)^2 + \beta b_2^{\beta}$, which tells us that the value of (21) is always bigger than that of (25).

We find that $\hat{b_2} = 1 - \beta$ maximizes (21), a quadratic function of β. Substitute $\hat{b_2} = 1 - \beta$ into (21), the equation becomes $\frac{4}{3+\beta}$. Thus $\text{PoA}(G_2, \beta, T^{\text{mc}}) \leq \frac{4}{3+\beta}$. □

Notice that when no incentives are present ($\beta = 0$), $\text{PoA}(\mathcal{G}) = \frac{4}{3}$, which corresponds to the price of anarchy found in [13]; when all users are incentivized ($\beta = 1$), $\text{PoA}(\mathcal{G}) = 1$, which corresponds to the result from [12] that marginal-cost tolls can make Nash flow optimal for parallel networks.

$\frac{4}{3+\beta}$ decreases at a rate faster than linear on $\beta \in [0, 1]$, and its rate of decline maximizes at $\beta = 0.464$. This implies that when we are already incentivizing a large fraction of the population, losing some has little to no effect on the price of anarchy. However, when we are just incentivizing a very few (< 0.464) portion of the population, adding on some more incentivized users can sharply improve the system efficiency.

IV. Simulation

To highlight the main results of this work, and to motivate our future directions, we provide a Monte-Carlo simulation on the price of anarchy of various incentives. While Theorem 1 shows that marginal cost tolls are an optimal incentive mechanism, we do not discuss how much better it performs than if users had been tolled more or less aggressively. To show this, we randomly generate both 2-link and 5-link networks and compute the average price of anarchy under marginal cost tolls, a more aggressive toll, and a less aggressive toll.

To formally describe taxation mechanisms with varying levels of aggressiveness, we define a scaled marginal cost toll as $\tau^{\mathrm{smc}}(f) = k\tau^{\mathrm{mc}}(f)$ for all $f \geq 0$. We examine three cases: $k = 1$ which is equivalent to the marginal cost tolls, $k = 0.5$ for a less aggressive toll, and $k = 1.5$ for a more aggressive toll. It would be reasonable to believe that charging users more or less may be useful when only a fraction of the population can be incentivized, however, our results show otherwise.

For our simulation, we randomly generate 1000 2-link and 1000 5-link networks with affine latency functions by sampling the parameters a_e and b_e uniformly from 0 to 1 for each edge. In each, we solve for the Nash flow under each of the three proposed taxation mechanisms when only a fraction β of the population can be incentivized and compute the price of anarchy. We then compute the average price of anarchy over the 1000 instances for each size of the network and use this as a comparison for the performance of each tolling scheme.

Figure 1a shows the average price of anarchy for 2-link networks for each tolling scheme; it illustrates that the average price of anarchy is lower using marginal-cost tolls than either the more or less aggressive tolling scheme. We can see that the average performance is much better than the worst-case performance, and stays relatively close to 1. Figure 1b shows a similar relationship between the marginal cost toll and the more and less aggressive tolling schemes as in Fig. 1a, indicating that similar gains in performance are capable in the 5-link setting. This leads us to believe that Theorem 2 will hold for any parallel network with affine latency functions. Proving this bound is the subject of ongoing work.

V. CONCLUSION

Though incentives, specifically tolls in the form of taxes, have been proven to alter sub-optimal emergent behavior caused by selfish user behavior, it is not always the case that all users can observe and respond to incentives. Intuitively, we might consider taxing the rest more since we lost a fraction of the population to incentivize. However, this work shows that the optimal taxation mechanism in this setting is still marginal cost tolls. Further, we provide a performance guarantee after losing a fraction of the population to incentivize in the context of two-link parallel networks. There is a trade-off between incentivizing more people and having a smaller system cost. A system designer can pick a sweet point that is right for his/her problem. Future work should investigate performance bounds for more general classes of networks.

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