Value of Information in Incentive Design: A Case Study in Simple Congestion Networks

Bryce L. Ferguson, *Graduate Student Member, IEEE*, Philip N. Brown, *Member, IEEE*, and Jason R. Marden, *Senior Member, IEEE*

***Abstract*— It is well-known that system performance can experience significant degradation from the self-interested choices of human users. Accordingly, in this article, we study the question of how a system operator can exploit system-level knowledge to derive incentives to influence societal behavior and improve system performance. Throughout, we focus on a simple class of routing games where the system operator has uncertainty regarding the network characteristics (i.e., latency functions) and population characteristics (i.e., sensitivity to monetary taxes). Specifically, we address the question of what information can be most effectively exploited in the design of taxation mechanisms to improve system performance. Our main results characterize an optimal marginal-cost taxation mechanism and associated performance guarantee for varying levels of network and population information. The value of a piece of information cannot be known a priori, so we adopt a worst-case interpretation of the value a piece of information is guaranteed to provide. Several interesting observations emerge about the relative value of information, including the fact that the value of population information saturates unless we also acquire more network knowledge.**

***Index Terms*— Algorithmic game theory, congestion games, incentives, value of information.**

Nomenclature

E	Edge set indexed by $e \in E$.
f_e	Flow on edge e.
$\ell_e(\cdot)$	Latency on edge e as a function of edge flow.
a_e	Linear coefficient of latency function on edge e.
b_e	Constant coefficient of latency function on edge e.
$\mathcal{L}(\cdot)$	Total latency as a function of the network flow.
$\mathcal{L}^{\mathrm{Nf}}(\cdot)$	Total latency in a Nash flow.
$\mathcal{L}^{\mathrm{opt}}(\cdot)$	Total latency in the optimal flow.
G	Congestion routing game problem instance.
$\mathcal{G}$	Family of congestion routing game problems.
$\tau_e(\cdot)$	Toll on edge e as a function of edge flow.
$s(\cdot)$	User price sensitivity as a function of user index.
$\mathcal{S}$	Set of population sensitivity distributions.
S_L, S_U	Lower and upper bounds on users' price sensitivity.
$\bar{s}$	Average user price sensitivity.
$\tau_e(\cdot)$	Toll applied to edge e as a function of edge flow.
$T(\cdot)$	Incentive mechanisms that maps edges to tolls.
k_{agn}	Scalar of network/sensitivity agnostic toll.
$k_{(\bar{s})}$	Scalar of network agnostic/mean aware toll.
$k_{(G)}$	Scalar of network aware/sensitivity agnostic toll.
$k_{(\bar{s},G)}$	Scalar of network/mean aware toll.
k^{gm}	Geometric mean scaling factor $1/\sqrt{S_L S_U}$.
$\mathrm{PoA}(\cdot)$	Price of anarchy as a function of a class of congestion. games, price sensitivity set, and incentive mechanisms.
$\mathrm{PoA}^{\star}(\cdot)$	Price of anarchy using optimal toll as a function of a class. of congestion games and price sensitivity set.

Common Abbreviations

opt	Optimal over respective domain.
Nf	Nash flow (often with respect to toll, population, and game).

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Bryce L. Ferguson and Jason R. Marden are with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106 USA (e-mail: blferguson@ece.ucsb.edu; jrmarden@ece.ucsb.edu).

Philip N. Brown is with the Department of Computer Science, University of Colorado at Colorado Springs, Colorado Springs, CO 80918 USA (e-mail: philip.brown@uccs.edu).

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I. Introduction

THE self-interested decision-making of system users can cause significant degradation in overall performance [2]. This emergent inefficiency caused by selfish behavior is commonly characterized by the ratio between the worst-case social welfare resulting from choices of self-interested users and the optimal social welfare; this quantity is often referred to as the *price of anarchy* [3] and is a well-studied metric of system-level inefficiency in the areas of resource allocation [4], distributed control [5], and transportation [6]. A common line of research studies how incentive mechanisms can be designed to influence users to make decisions more in line with the social optimal [7]. For the implementation of such incentives to be effective, a system designer must consider how the users will respond.

When designing an incentive scheme, a system designer is benefited by having more information about the problem, for example, a more accurate model of the system infrastructure or of the human users' behavior. This article seeks to understand how and what pieces of information aid in the incentive design task. Though increased understanding of the problem setting seems beneficial, in settings such as road traffic [8], power grids [9], supply chains [10], and advertising [11], among others, there is an abundance of potential information sources. Though we can devise ways to learn different pieces of information [12], there is some cost that must be invested in acquiring it. The central questions that we focus on in this article are as follows.

1) What are the incentive mechanisms that optimize the efficiency of the emergent collective behavior for a given level of informational awareness?
2) What type of information, that is, what specific information about the network or population, can be *best* exploited to improve the efficiency of the emergent collective behavior through an appropriately designed incentive mechanism?

We answer question 1) to understand how to effectively use the available information; however; the main message of this article revolves around the answers to question 2). In particular, is it better for a societal planner to invest in getting more detailed information regarding the infrastructure characteristics or population characteristics? How do the granularity and quality of the information impact the attainable performance guarantees?

We are particularly motivated by problems relating to congestion and traffic, where the system operator may wish to influence users away from more congestible roads. Recent work has studied this problem; in practice, Kreindler [13] performed experiments in Bangalore to see how users respond to incentives persuading them to alter their commute from the common, direct route to a less direct route. The author finds that without additional information, incentives provide limited opportunities and identified value-of-time/price-sensitivity as an important factor. Inspired by this and related studies [14], [15], we introduce a formal model where each user (driver) has to choose between two decisions (direct commute or long commute) and are influenced by their perceived cost (travel time) as well as an imposed tax. To capture the system designer's uncertainty, we consider that each user has their own price sensitivity, affecting how they relate temporal and monetary costs. We model this as a *congestion game* with two links and a population of heterogeneous users, where the behavior that emerges from users self-interested decisions is a *Nash flow*. It is widely known that the system-level behavior can be suboptimal and the degree of suboptimality is typically characterized by the price of anarchy [16]. Typically, this form of analysis is relegated to studying worst-case scenarios; we seek to extend this by considering how available information may alter what performance guarantees are attainable.

A. Related Works

Research has sought to explore the use of tolling or taxation mechanisms to improve the system cost in congestion games [17], [18], [19]. These monetary incentives alter users' preferences in a manner that reduces the price of anarchy; however, the majority of this work does not consider the effects of user heterogeneity.

In the works that do study heterogeneous users in congestion games, there are a number of positive and negative results pertaining to the effectiveness of taxation mechanisms [20], [21], [22], [23]. On the positive side, there always exists a taxation mechanism that can completely mitigate any efficiency loss [24], [25], [26]. On the negative side, this taxation mechanism intimately depends on the detailed information pertaining to both the network (i.e., topology, edge latency functions, etc.) as well as the population (i.e., demands, sensitivities, etc.), which significantly limits its applicability. Accordingly, recent work in [27] focuses on robust taxation mechanisms that do not require such extensive knowledge. While the derived taxation mechanism does not necessarily guarantee optimality on a network by network basis, it does provide strictly better performance guarantees than the uninfluenced behavior in broad classes of networks. Hence, these results hint at an apparent tradeoff between robustness and optimality.

B. Contributions

In this work, we seek to bridge the gap between optimal taxation mechanisms that require detailed information and robust tolls that require less information but may fail to perfectly optimize routing. We consider a case study in eight information domains and derive the tolling scheme that makes use of the available information *optimally* as well as the resulting price of anarchy bound.

Section II-C highlights these comparisons, and Section III provides formal proofs. Though the system model we consider is simple relative to the general class of congestion games, the observations of this work 1) provide lower bounds on the possible inefficiency that can occur more generally and 2) discover phenomena that, if can occur in simple settings, can occur more broadly and require consideration in future planning. Chief among these observations is that acquiring environmental knowledge (about the congestion rates of roads) proves more valuable than population knowledge (the exact price sensitivity of each user). Additional findings are discussed in the body of this article.

II. Model and Performance Metrics

A. Congestion Routing Game

Consider a population of users $N = [0, 1]$, represented by a closed interval. To model situations with a very large number of users, a player has infinitesimal mass and is indexed by a real number in $x \in [0, 1] = N$. To model a simple road traffic scenario, the users must traverse a graph from an origin o to a destination d by taking one of two routes, represented by parallel edges e_1 and e_2; this is designed to represent two commute options: a congestible, direct route and an open but indirect route. Let $E = \{e_1, e_2\}$ denote the set of edges. The function $\mathbf{e}: N \to E$ (assumed to be Lebesgue-integrable) captures the action of each user, that is, each user $x \in N$ takes an action by selecting a route $\mathbf{e}(x) \in E$. A flow on edge e is the mass of users taking that route as their action, or $f_e(\mathbf{e}) = \int_{x \in N} \mathbb{1}[\mathbf{e}(x) = e]dx$ where $\mathbb{1}[\cdot]$ is the indicator function. For notational convenience, we will omit the flow f reliance on $\mathbf{e}$ when clear from context. Let $f = (f_1, f_2) \in \Delta(E)$ denote a network flow, where $\Delta(E)$ denotes the standard probability simplex over the set E, that is, $\sum_{e \in E} f_e = 1$. To characterize transit delay, each edge $e \in E$ in the network has a latency function of the form

$$\ell_e(f_e) = a_e f_e + b_e \tag{1}$$

where $a_e \geq 0$ and $b_e \geq 0$ are coefficients used to model how transit delay on an edge grows with more traffic. The latency on an edge is thus a nondecreasing, nonnegative function of the flow on that edge. Though this model does not capture all the relevant features of traffic, this setting does capture the decision-making of a population of human users. Additionally, this simple model has been used to describe the driving patterns and congestion rates of commuters in real word traffic systems [13], proving useful in characterizing how incentives and users' decision-making affects global performance.

For a flow f, the system cost is characterized by the *total latency* in the network, defined as

$$\mathcal{L}(f) = \sum_{e \in E} f_e \cdot \ell_e(f_e) \tag{2}$$

and we denote the flow that minimizes this total latency as $f^{\text{opt}} \in \arg\min_{f \in \Delta E} \mathcal{L}(f)$. We specify a particular network by the tuple $G = (E, \{\ell_e\}_{e \in E})$.

This work examines taxation mechanisms as tools to influence the self-interested, price-sensitive user population to reach more efficient equilibria. We model this routing problem as a congestion game where each edge $e \in E$ is assigned a flow-dependent tolling function τ_e: $[0, 1] \to \mathbb{R}^+$. A user $x \in N$ has a price-sensitivity $s(x) > 0$; this price-sensitivity is subjective for each user and relates the user's cost from being tolled to their cost from experiencing delays and is the reciprocal of the user's value of time. Without loss of generality, we order players' indices by their individual price-sensitivity, that is, $s(x) \geq s(y)$ if $x \geq y$. The function s: $N \to \mathbb{R}_{\geq 0}$ thus captures the distribution of price-sensitivity over the users in population N. In a flow f, the cost function for a user x that is on an edge $\mathbf{e}(x) \in E$ can be expressed as

$$J_x(f) = \ell_{\mathbf{e}(x)}\big(f_{\mathbf{e}(x)}\big) + s(x)\tau_{\mathbf{e}(x)}\big(f_{\mathbf{e}(x)}\big). \tag{3}$$

Each user will choose to take the route that minimizes their own cost. When each user does so, the system reaches a *Nash equilibrium* $\mathbf{e}^{\text{Ne}}$, satisfying

$$\mathbf{e}^{\text{Ne}}(x) \in \arg\min_{e \in E}\{\ell_e(f_e(\mathbf{e})) + s(x)\tau_e(f_e(\mathbf{e}))\} \quad \forall x \in N.$$

In a Nash equilibrium, we will call the resulting network flow a *Nash flow* $f^{\text{Nf}} := f(\mathbf{e}^{\text{Ne}})$ (again, we typically omit the reliance on $\mathbf{e}$ for brevity), also known as a Wardrop equilibrium [28]. A game is therefore characterized by a network G, price-sensitivity distribution s: $[0, 1] \to \mathbb{R}^+$, and a set of tolling functions $\{\tau_e\}_{e \in E}$, denoted by the tuple $(G, s, \{\tau_e\}_{e \in E})$. It is shown in [29] that a Nash flow will always exist in a congestion game of this form, and the total latency of a Nash flow is unique for each s.

B. Taxation Mechanisms and Performance Metrics

To understand the robustness of a tolling scheme, we consider the performance over a class of networks and users' sensitivities. For a network G, we identify the latency functions which constitute the network by $L(G)$; furthermore, for a family of congestion games $\mathcal{G}$, let $L(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} L(G)$ be the set of all latency functions that exist in the games in $\mathcal{G}$.

A *taxation mechanism* T maps latency functions ℓ_e to tolling functions τ_e. For a family of networks $\mathcal{G}$, this mapping is denoted T: $L(\mathcal{G}) \to \mathcal{T}$, where $\mathcal{T}$ is the set of all admissible tolling functions on [0, 1]. In this work, we consider a form of tolling function that is linear with the flow on that edge known as *scaled marginal-cost tolls*. We parameterize the tolls by

$$\tau_e(f_e) = kf_e \cdot \frac{d\ell_e}{df_e}(f_e) = ka_e f_e \quad \forall\ e \in E \tag{4}$$

where k is a parameter set by the system designer and a_e is the linear component of the edge latency function. Though broader forms of tolling mechanisms can be used to effectively influence users, scaled marginal-cost tolls offer several properties useful for analysis and implementation. If one considers a setting where the toll designer is under no constraint (outside of the implied information constraints), then unbounded incentives can be designed that guarantee optimal performance in nearly every setting by using unbounded step functions or unbounded incentives as described in [22] and [27]. Because unbounded incentives are not reasonable in many settings, a toll designer has two options: they could choose to add a constraint bounding the magnitude of the incentives, or they could restrict their design to a class of tolling mechanisms that are intrinsically bounded. In this work, we focus on the latter by studying the design of optimal scaled marginal-cost tolls which are bounded whenever the latency is finite. Scaled marginal-cost tolls have been studied in congestion games with little available information on the network or users' price sensitivities [30]; furthermore, in [27], it is shown that in some low information settings, the optimal bounded tolls and associated performance guarantees can be found by solving for the optimal scaled marginal-cost toll; because of their desirable properties and connection to the literature, we consider scaled marginal-cost tolls throughout.

To formalize the notion of uncertainty in users' response, we consider families of sensitivity distributions that can occur when the system designer is only aware of the lower bound S_L and upper bound S_U on users' price sensitives. We define the set of possible sensitivity distributions as $\mathcal{S} = \{s$: $[0, 1] \to [S_L, S_U]\}$. When the average price sensitivity $\bar{s}$ of the users is introduced to the system designer, the set of possible distributions becomes $\mathcal{S}(\bar{s}) = \{s \in \mathcal{S} | \int_0^1 s(x)dx = \bar{s}\}$; it is clear that $\mathcal{S}(\bar{s}) \subseteq \mathcal{S}$. To evaluate the performance of a tolling mechanism, let $\mathcal{L}^{\text{Nf}}(G, s, T)$ be the total latency on a network G, with price sensitivity distribution s, in the Nash flow f^{Nf} when tolls are assigned according to taxation mechanism[1] T, and let $\mathcal{L}^{\text{opt}}(G)$ be the minimum total latency which occurs under the optimal flow f^{opt}. The *price of anarchy*

[1]The taxation mechanism is a mapping from latency functions to tolling functions. A game with taxation mechanism T is therefore denoted $(G, s, \{T(\ell_e)|\ell_e \in G\})$. For brevity, we simply denote this as (G, s, T).

TABLE I

PRICE OF ANARCHY BOUNDS UNDER OPTIMAL TAXATION MECHANISMS WITH VARYING AMOUNTS OF PARTIAL INFORMATION ($S_U/S_L = 10$)

	$\mathcal{S}_{>0}$ *sensitivity-agnostic*	$\mathcal{S}$ *bound-aware* $(S_{\mathrm{L}}, S_{\mathrm{U}})$	$\mathcal{S}(\overline{s})$ *mean-aware* $(S_{\mathrm{L}}, S_{\mathrm{U}}, \overline{s})$	$s(x)$ *distribution-aware*
$\mathcal{G}$ *network-agnostic*	$\mathrm{PoA}^*(\mathcal{G}, \mathcal{S}_{>0}) = 1.\overline{33}$ ([16], Prop. 3)	$\mathrm{PoA}^*(\mathcal{G}, \mathcal{S}) \approx 1.176$ (A, Thm. 1)	$\mathrm{PoA}^*(\mathcal{G}, \mathcal{S}(\overline{s})) \leq 1.1401$ (B, Thm. 2)	$\mathrm{PoA}^*(\mathcal{G}, s) \leq 1.1401$ (C, Thm. 3)
G *network-aware*	$\mathrm{PoA}^*(G, \mathcal{S}_{>0}) \leq 1.\overline{33}$ (D, Prop. 3)	$\mathrm{PoA}^*(G, \mathcal{S}) \leq 1.09$ (E, Thm. 4)	$\mathrm{PoA}^*(G, \mathcal{S}(\overline{s})) \leq 1.0494$ (F, Thm. 5)	$\mathrm{PoA}^*(G, s) = 1$ (G, Thm. 6), [25]

compares the Nash flow on a network with the optimal flow; this characterizes the inefficiency of the network and can be defined as

$$\mathrm{PoA}(G, s, T) = \frac{\mathcal{L}^{\mathrm{Nf}}(G, s, T)}{\mathcal{L}^{\mathrm{opt}}(G)} \geq 1. \tag{5}$$

We extend this definition to include families of networks and sensitivity distributions, that is,

$$\mathrm{PoA}(\mathcal{G}, \mathcal{S}, T) = \sup_{G \in \mathcal{G}} \sup_{s \in \mathcal{S}} \left\{ \frac{\mathcal{L}^{\mathrm{Nf}}(G, s, T)}{\mathcal{L}^{\mathrm{opt}}(G)} \right\} \tag{6}$$

such that the price of anarchy is now the worst-case inefficiency over possible networks and populations. Note that the same taxation mechanism T is applied to any realized instance.

C. Optimal Tolling and Our Contributions

The system designer's goal when designing a taxation mechanism is to minimize worst-case inefficiency given uncertainties over the network and/or user sensitivities. Thus, we define an optimal tolling mechanism as

$$T^* \in \operatorname*{arg\,inf}_{T: L(\mathcal{G}) \to \mathcal{T}} \mathrm{PoA}(\mathcal{G}, \mathcal{S}, T)$$

such that it is the taxation mechanism which minimizes the price of anarchy expressed in (6) for a given family of networks $\mathcal{G}$ and sensitivity distributions $\mathcal{S}$. To understand and compare the value of different pieces of information, we seek to quantify the performance guarantees under the optimal tolling mechanism in different information settings. Therefore, we define the price of anarchy bound under an optimal tolling mechanism as

$$\mathrm{PoA}^*(\mathcal{G}, \mathcal{S}) \triangleq \inf_{T: L(\mathcal{G}) \to \mathcal{T}} \mathrm{PoA}(\mathcal{G}, \mathcal{S}, T) \tag{7}$$

which will serve as the measure of how useful information is to the system designer.

In this article, we demonstrate the value of different pieces of information to a system designer by comparing the price of anarchy bounds of the optimal incentive in different information settings as shown in Table I. We consider these questions in the class of two link parallel networks as these networks often display worst-case inefficiency over larger classes of networks and allow us to analyze the benefit of these partially informed tolls [31]. Many of the results generalize to parallel and more general networks; for those that do not, these results provide lower bounds on the price of anarchy. In Section IV, we discuss this context in more detail. For uniformity of presentation, all results are expressed for two link networks. Additionally, the purpose of this work is to identify information factors that affect the incentive design task. If interesting observations can occur in simple problems, then certainly they can occur more generally.

Table I depicts a snap-shot of the theoretical results for $S_U/S_L = 10$. In the top left, the toll designer possesses no information about the network or users' price sensitivities, making the zero toll ($\tau_e = 0\ \forall e \in E$) optimal and recovering the price of anarchy bound for this class of networks of $4/3 = 1.\overline{33}$ [16]; we show this formally in Proposition 3. As the toll designer acquires more information, their performance improvements are captured by moving down and to the left. The information available to the system designer is encoded in the arguments of the price of anarchy expression defined in (7). With regard to network information, we consider two possible cases of information available to the system designer.

1) *Network-Agnostic* $\mathit{PoA}^*(\mathcal{G}, \cdot)$*:* The system designer is unaware of the specific problem instance and only knows the class of possible networks and must choose a taxation mechanism that is applied to each.[2]
2) *Network-Aware* $\mathit{PoA}^*(G, \cdot)$*:* The system designer is aware of the specific network and may design tolls for that specific instance.[3]

When the system designer is aware of the exact network characteristics, they will be able to design tolls more effectively. From this fact, we expect a network-aware toll to perform no worse than a network-agnostic one. The benefit of this information for different settings can be seen by comparing the two rows of Table I.

Similarly, we consider several settings for the system designer's knowledge of the user-sensitivity distribution.

1) *Sensitivity-Agnostic* $\mathit{PoA}^*(\cdot, \mathcal{S}_{>0})$*:* The system designer knows nothing about the users' sensitivities except they are bounded away from zero.
2) *Bound-Aware* $\mathit{PoA}^*(\cdot, \mathcal{S})$*:* The system designer knows the lower-bound S_L and upper-bound S_U, on users' possible sensitivities.
3) *Mean-Aware* $\mathit{PoA}^*(\cdot, \mathcal{S}(\overline{s}))$*:* The system designer knows the lower-bound S_L and upper-bound S_U as well as the mean $\overline{s}$ of users' sensitivities.

[2]Though it was assumed prior that the demand in the network is always of unit size, when the system designer is network-agnostic it is without loss of generality that they are also unaware of the demand in the network. We thus consider demand as an implied piece of network information.

[3]Though the network structure is consistent throughout each routing problem, we use the nomenclature of network-aware to match the literature, where network-agnostic tolls must be assigned with only local edge latency characteristics while network-aware tolls are designed with information of each edge's latency function.

4) *Distribution-Aware PoA*$^*(\cdot, s)$*:* The system designer knows the exact distribution on user sensitivities.

The sensitivity distribution serves as a model for the population's behavior. Refining the set of possible distributions reduces the designer's uncertainty and allows them to design more effective tolls. The benefit of increasing the information available to the system designer can be seen by comparing the columns of Table I.

The main focus of this work is to investigate which pieces of information give the greatest gains in the performance of tolls with respect to the price of anarchy ratio. Though it is clear that additional information will help tolls provide better guarantees, it is not obvious what will provide a better gain in performance when introduced to an uninformed system designer: network-awareness or population-awareness. Furthermore, the value of a piece of information is highly contextual, and it is impossible to know a priori what value new information provides, thus we adopt a worst-case approach for comparing performance. Comparing the worst-case performance bounds of each setting (such as those demonstrated in Table I) shows as follows.

1) Comparing elements (B) and (C) shows that the full distribution of users' price sensitivities need not be any more helpful than the mean alone, thus the value of population information saturates.
2) Comparing elements (D) and (G) shows that, in the absence of any population information, network-awareness may be of no help, however, in the presence of full population information, network-awareness allows for tolls that can always incentivize optimal self-routing.
3) Comparing elements (B) and (E) shows that the guaranteed value of information about network characteristics is more valuable than the guaranteed value of the mean of users' price sensitivities.

To further illustrate the results and highlight that the relationships between information settings hold more generally, we provide a plot of each price of anarchy bound for varying levels of population heterogeneity. Fig. 1 shows the best attainable price of anarchy bounds under scaled marginal cost tolling in the previously described information settings for each $S_L/S_U \in [0, 1]$. As the S_L/S_U approaches 1, there is less discrepancy between the different users' price sensitivities and all tolls can optimize performance; as S_L/S_U approaches zero, the differences between users' responses can be arbitrarily large and no toll is effective. For values of S_L/S_U in between, we see that the previously described relationships hold, and network information proves more valuable than population information. These findings illuminate several important considerations for incentive designers. In problems closely related to our described setting (such as the tolling experiment in [13]), our results inform what information the toll designer should invest in acquiring. Further work is needed to understand if identical conclusions are true in other settings; however, identifying them in our setting highlights that these comparisons need to be considered more broadly.

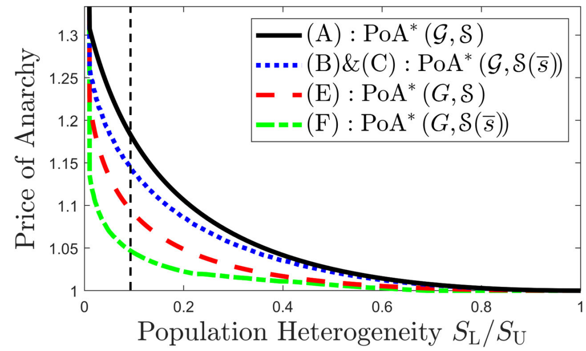

Fig. 1. Worst-case price of anarchy in each information setting over levels of user heterogeneity $S_L/S_U \in [0, 1]$. Each plot represents the best achievable price of anarchy bound using scaled marginal cost tolls for one of the information settings: (A) network-agnostic, mean-agnostic, (B) and (C) network-agnostic, mean (or full distribution)-aware, (E) network-aware, mean-agnostic, and (F) network-aware, mean-aware. Each toll guarantees superior performance than the untolled price of anarchy of 4/3. The values of each line at $S_L/S_U = 0.1$ are those presented in Table I. By varying the value of S_L/S_U, we can see that the relationship between settings holds when looking at worst-case performance guarantees.

III. Main Results

In each setting considered, we provide a result reporting the price of anarchy bound under an optimal toll, as well as a subsidiary result reporting the optimal scaled marginal-cost toll when applicable. The main conclusions can be observed numerically in Table I, while the analytic expressions are given in Sections III-A–III-G. In this work, we limit the search for optimal tolls to a search over scaled marginal-cost tolling mechanisms. Taxation mechanisms of this form can be parameterized by a single scaling factor k and will be denoted $T(k)$ which assigns to an edge e a toll $\tau_e(f_e) = k a_e f_e$. As discussed in Section II-B, these tolls possess desirable properties in that they are naturally bounded when the latency is finite, and they can be reasonably implemented in network-aware and network-agnostic settings; furthermore, Brown and Marden [27, Lemma 2.2] show that the optimal *bounded* toll can be found by searching for the optimal scaled marginal-cost toll in the network-agnostic case. In the network-aware case, it is an open question as to what form an optimal, bounded taxation mechanism will take.

A. Network-Agnostic, Bound-Aware

The first scenario we consider is when the system designer is agnostic of the exact network characteristics and knows only the lower and upper bound on user sensitivities; we provide a bound on the price of anarchy under an optimal scaled marginal-cost taxation mechanism in this setting.

Theorem 1: When only S_L and S_U are known, the price of anarchy under an optimal scaled marginal-cost tolling mechanism is

$$\mathrm{PoA}^*(\mathcal{G}, \mathcal{S}) = \frac{\left(q - 1 + \sqrt{q^2 + 14q + 1}\right)^2}{8q\left(-q - 1 + \sqrt{q^2 + 14q + 1}\right)} \tag{8}$$

where $q := S_L/S_U$.

This result gives a performance guarantee for the setting where the system designer has minimal information about the population and network characteristics. This result is a generalization of [30, Theorem 1], where we now consider networks that need not have traffic on every edge in a Nash flow. As a means to find this upper bound, we first derive the optimal scaled marginal-cost toll. In the proof of Theorem 1, at the end of this section, we will use this toll to show the associated price of anarchy bound.

Proposition 1: When only S_L and S_U are known, the optimal network-agnostic marginal-cost toll scaling factor is

$$k_{\text{agn}} = \frac{-S_L - S_U + \sqrt{S_L^2 + 14 S_L S_U + S_U^2}}{2 S_L S_U}. \tag{9}$$

Proof: We start by finding the scaling factor k_{agn} for the optimal scaled marginal-cost toll. In this information setting, the optimal scaling factor was found in [32] to be the solution to the equation

$$\frac{4}{4\left(1 + k_{\text{agn}} S_L\right) - \left(1 + k_{\text{agn}} S_L\right)^2} = \frac{\left(1 + k_{\text{agn}} S_U\right)^2}{4 k_{\text{agn}} S_U}. \tag{10}$$

It is shown in [32] that when $S_L < S_U$, (10) always has exactly one solution on the interval $(1/S_U, 1/S_L)$ and that solution is the desired optimal scale factor. Equation (9) is a solution to (10), so we show here that (9) describes this desired solution by showing that k_{agn} is in the interval $(1/S_U, 1/S_L)$. Define the term $p = S_U/S_L$. Because $p > 1$, we have

$$1 + 14p + p^2 > 1 + 14p + p^2 + 8(1 - p) = (p + 3)^2.$$

Thus, (9) can be lower bounded by

$$k_{\text{agn}} > \frac{-1 - p + \sqrt{(p+3)^2}}{2 S_U} = \frac{1}{S_U}. \tag{11}$$

Likewise, define $q = S_L/S_U$ (so that $q < 1$), we have

$$1 + 14q + q^2 < 1 + 14q + q^2 + 8(1 - q) = (q + 3)^2$$

yielding a lower bound on k_{agn} of

$$k_{\text{agn}} < \frac{-1 - q + \sqrt{(q+3)^2}}{2 S_U} = \frac{1}{S_L}. \tag{12}$$

Thus, the scaling factor k_{agn} defined in (9) exists in the interval $(1/S_U, 1/S_L)$. It can be shown by substitution that (9) satisfies (10). □

Proof of Theorem 1: Using the scaling factor from Proposition 1, (8) can be found by substituting (9) into (10). □

B. Network-Agnostic, Mean-Aware

We consider the mean sensitivity as additional information to just the bounds S_L, S_U. If the system designer is aware that the mean user sensitivity is $\bar{s}$, the set of possible sensitivity distributions is reduced to the set $\mathcal{S}(\bar{s}) \subset \mathcal{S}$. Using this information, the toll designer is able to refine the optimal mechanism and improve the performance guarantees. The price of anarchy of an optimal toll in this setting is shown in Table I, where value (B) is noticeably lower than (A) demonstrating the value of information to the system designer.

In deriving this bound, we perform a series of reductions in the set of feasible instances to one which realizes the worst-case price of anarchy. In Lemma 1, we show that any Nash flow can emerge by a population with a bimodal sensitivity distribution, thus reducing our search for worst-case instances to those with bimodal sensitivity populations. In Lemma 2, we further identify two distributions, one of which will realize worst-case inefficiency. In Lemma 3, we reduce the search over networks to those with only one linear and one constant edge, and in Lemma 4, we identify two such networks that constitute worst-case instances. We will first prove these lemmas that will aid in proving several later theorems.

We say users x, y have the same *type* if $s(x) = s(y)$. Furthermore, let a *bimodal* distribution be one in which there exist exactly two user types; the set of such distributions is denoted $\mathcal{S}^{\text{bi}}(\bar{s}) \subset \mathcal{S}(\bar{s})$. We denote a bimodal distribution with types S_1 and S_2 by (S_1, S_2). Note that for a given $\bar{s}$, S_1 and S_2, the mass of users with each sensitivity is well defined. Additionally, we adopt the convention used elsewhere that the network links are indexed such that $b_1 \leq b_2$.

Lemma 1: A Nash flow f for a sensitivity distribution $s \in \mathcal{S}(\bar{s})$, under a linear tax T, is likewise a Nash flow for some distribution $s' \in \mathcal{S}^{\text{bi}}(\bar{s})$ in which one type of user is indifferent between the two edges and all users on each edge are of a single type. This implies the price of anarchy over sensitivities in $\mathcal{S}(\bar{s})$ is equal to the price of anarchy over bimodal distributions in $\mathcal{S}^{\text{bi}}(\bar{s})$, that is,

$$\text{PoA}(\mathcal{G}, \mathcal{S}(\bar{s}), T) = \text{PoA}\left(\mathcal{G}, \mathcal{S}^{\text{bi}}(\bar{s}), T\right). \tag{13}$$

Proof: Let $s_1 \in \mathcal{S}(\bar{s})$ be some distribution of users' sensitivities, and let S_{ind} be the sensitivity that has equal cost between the two links in the Nash flow f^{Nf}, that is, solution to

$$(1 + S_{\text{ind}} k) a_1 f_1^{\text{Nf}} + b_1 = (1 + S_{\text{ind}} k) a_2 f_2^{\text{Nf}} + b_2. \tag{14}$$

Note that in the case where $S_{\text{ind}} > S_U$ or $S_{\text{ind}} < S_L$, any distribution $s \in \mathcal{S}$ will have the same Nash flow with all users choosing the same edge. First, consider the case where $S_{\text{ind}} < \mu(s_1)$, where $\mu(\cdot)$ is the mean of the distribution. From Claim 1.1.2 in [30], if a user has a sensitivity $S < S_{\text{ind}}$, then they strictly prefer the first link; if they have a sensitivity $S > S_{\text{ind}}$, then they strictly prefer the second.

Now, let s_2 be a new distribution where each user who had chosen edge 1 now has sensitivity S_{ind}. The Nash flows from s_1 and s_2 are the same, as the same number of users have a sensitivity $S \leq S_{\text{ind}}$ and thus the same users choose the first edge. It is clear that $\mu(s_2) > \mu(s_1)$ as no user has a lower sensitivity and some have higher.

Now, consider a third distribution s_3, where users who chose edge 2 now have some sensitivity $S' \in (S_{\text{ind}}, S_U]$; these users will now strictly prefer the second edge of the network but the Nash flow will remain unchanged. If we pick $S' = S_U$, the mean has surely increased again; if we pick $S' = S_{\text{ind}}$, because we are in the case $S_{\text{ind}} < \bar{s}$, the mean is lower than $\mu(s_1)$. Because $\mu(s_3)$ is continuous with S', we can select S' so that $\mu(s_3) = \mu(s_1)$. The case of $S_{\text{ind}} > \mu(s_1)$ is similar.

The distribution $s_3 = (S_{\text{ind}}, S')$ induces the same Nash flow as s_1 and now, one set of users is indifferent and users of the same type exist on the same edge only. □

Having shown in Lemma 1 that any feasible Nash flow can be realized by a population with a bimodal sensitivity distribution, we note that the worst-case price of anarchy can be realized by a bimodal distribution. Our search further reduces as we characterize two specific distributions that give worst-case inefficiency.

Lemma 2: For a given network $G \in \mathcal{G}$ and scaled marginal-cost tax T with toll scaling factor k, two distributions $s_\ell^{(\bar{s},G,k)}$ and $s_u^{(\bar{s},G,k)}$, that maximize and minimize (respectively) the flow on the first edge of the network, realize the price of anarchy over those in $\mathcal{S}^{\text{bi}}(\bar{s})$

$$\text{PoA}\big(G, \mathcal{S}^{\text{bi}}(\bar{s}), T\big) = \text{PoA}\Big(G, \Big\{s_\ell^{(\bar{s},G,k)}, s_u^{(\bar{s},G,k)}\Big\}, T\Big). \quad (15)$$

Proof: The proof follows from the fact that total latency is quadratic in the flow, thus the largest price of anarchy will come from the flow that is furthest from optimal. From Lemma 1, we see that any flow induced by a distribution $s \in \mathcal{S}(\bar{s})$ can be realized by a bimodal distribution that has one set of users observing equal cost between the links and each edge containing only one sensitivity type. We, therefore, define $s_\ell^{(\bar{s},G,k)}$ as the distribution that maximizes f_1^{Nf} and $s_u^{(\bar{s},G,k)}$ as the distribution which maximizes f_1^{Nf}. □

Next, we focus on which networks exhibit worst-case inefficiency and reduce our search to the set of instances with have one linear latency function and one constant; the set of such networks is defined as

$$\mathcal{G}^{\text{lc}} = \{G | \ell_1(f_1) = a_1 f_1,\ \ \ell_2(f_2) = b_2,\ \ a_1, b_2 \geq 0\}.$$

Lemma 3: For any $G \in \mathcal{G}$, there exists a $\hat{G} \in \mathcal{G}^{\text{lc}} \subset \mathcal{G}$ that, under the same scaled marginal-cost tolling mechanism $T(k)$, has a higher price of anarchy, implying

$$\text{PoA}(\mathcal{G}, \mathcal{S}(\bar{s}), T(k)) = \text{PoA}\big(\mathcal{G}^{\text{lc}}, \mathcal{S}(\bar{s}), T(k)\big). \quad (16)$$

The proof of Lemma 3 appears in the Appendix.

Finally, we identify two specific networks that demonstrate worst-case inefficiency. For a given set of distributions $\mathcal{S}(\bar{s})$ and toll scaling factor k, we define two networks.

1) $G_\beta \in \mathcal{G}^{\text{lc}}$ with latency functions $\ell_1(f_1) = f_1$ and $\ell_2(f_2) = \beta$ and satisfies $s_\ell^{(\bar{s},G_\beta,k)} = (S_L, S_U)$, and
2) $G_\alpha \in \mathcal{G}^{\text{lc}}$ with latency functions $\ell_1(f_1) = f_1$ and $\ell_2(f_2) = \alpha$ and satisfies $s_u^{(\bar{s},G_\alpha,k)} = (S_L, S_U)$. Due to the discussion in the proof of Lemma 3, any network in $\mathcal{G}^{\text{lc}}$ with cost functions satisfying $b_2/a_1 = \beta$ will have the same price of anarchy as G_β, and the same is true for for G_α.

Lemma 4: For linear constant networks, under sensitivity distributions in $\mathcal{S}(\bar{s})$ with toll scaling factor k, the network G_α or G_β will realize the upper bound on the price of anarchy, that is,

$$\text{PoA}\big(\mathcal{G}^{\text{lc}}, \mathcal{S}(\bar{s}), T(k)\big) = \text{PoA}\big(\big\{G_\alpha, G_\beta\big\}, \mathcal{S}(\bar{s}), T(k)\big).$$

Proof: It can be seen by differentiation of (33), the price of anarchy increases with the value of the indifferent sensitivity when $f_1^{\text{Nf}} < f_1^{\text{opt}}$ and decreases when $f_1^{\text{Nf}} < f_1^{\text{opt}}$. Recall that $s_\ell^{(\bar{s},G,k)}$ has $f_{1\ell} > f_1^{\text{opt}}$ and indifferent sensitivity $S_{\ell 1}$; similarly, $s_u^{(\bar{s},G,k)}$ has $f_{1u} < f_1^{\text{opt}}$ and indifferent sensitivity S_{u2}. It is therefore true that having $S_{\ell 1} = S_L$ or $S_{u2} = S_U$ is a necessary condition for the network which maximizes the price of anarchy.

Furthermore, in bimodal distributions (S_1, S_2) where users are homogeneous on either link, $f_1^{\text{Nf}} = (S_2 - \bar{s})/(S_2 - S_1)$. For $s_\ell^{(\bar{s},G,k)}$ when users with sensitivity $S_{\ell 1} = S_L$ are indifferent, the largest flow that can occur on f_1 occurs when $s_\ell^{(\bar{s},G,k)} = (S_L, S_U)$. Similarly, for $s_u^{(\bar{s},G,k)}$, when users with sensitivity $S_{u2} = S_U$ are indifferent, the least flow in f_1 has $s_u^{(\bar{s},G,k)} = (S_L, S_U)$. One of these two conditions must be met by a network $G \in \mathcal{G}^{\text{lc}}$ to maximize the price of anarchy. Those networks are the defined G_α and G_β. □

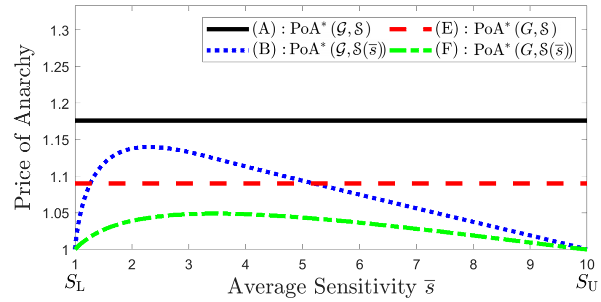

Fig. 2. Price of anarchy in each information setting over various mean sensitivities. Each plot represents a bound for one of the introduced tolling mechanisms: (A) network-agnostic, mean-agnostic toll, (B) network-agnostic, mean-aware toll, (E) network-aware, mean-agnostic toll, and (F) network-aware, mean-aware toll. Each toll gives superior performance guarantees than the untolled price of anarchy of 4/3. Price sensitivity bounds $S_L = 1$ and $S_U = 10$ are shown; changing these values has a minimal effect on the relation between the lines.

For ease of notation, we will define a change of variable $z(x) = (1/(1 + s(x)k))$ and $z_L = (1/(1 + S_L k))$ and $z_U = (1/(1 + S_U k))$.

Proposition 2: When S_L, S_U and the mean sensitivity $\bar{s}$ are known, the optimal network-agnostic marginal-cost toll scaling factor $k_{(\bar{s})}$ will be the solution on $(1/S_U, 1/S_L)$ to

$$\text{PoA}\big(G_\beta, (S_L, S_U), T(k)\big) = \text{PoA}(G_\alpha, (S_L, S_U), T(k)) \quad (17)$$

where

$$\beta = R/z_L, \quad \alpha = \begin{cases} \dfrac{1}{2\big(z_U - z_U^2\big)}, & \dfrac{1}{2(1 - z_U)} \geq R \\ R/z_U, & \text{otherwise} \end{cases} \quad (18)$$

where $R := (S_U - \bar{s})/(S_U - S_L)$.

The proof of Proposition 2 appears in the Appendix.

Finally, the price of anarchy under the optimal tolling mechanism can be expressed as in the following theorem.

Theorem 2: When S_L, S_U and the mean sensitivity $\bar{s}$ are known, the price of anarchy under an optimal, scaled marginal-cost toll is given by

$$\text{PoA}^*(\mathcal{G}, \mathcal{S}(\bar{s})) = \frac{R^2 - \beta R + \beta}{\beta - \beta^2/4} \quad (19)$$

where $R := (S_U - \bar{s})/(S_U - S_L)$, and $\beta = (1 + S_L k_{(\bar{s})})R$, with $k_{(\bar{s})}$ being the solution to (17).

In Fig. 2, we show the price of anarchy bound of the network-agnostic, mean-aware tolls alongside the price of anarchy bound in several other settings. As mentioned before, the value of knowing certain pieces of information (in this case the mean) is highly contextual: when the mean $\bar{s}$ is close to one of the lower or upper bound S_L and S_U, the mean is very informative as users' sensitivities must be more concentrated around the average. However, in worst case, the mean sensitivity does not offer as much value to the toll designer as the knowledge of the edge latency functions as in the network-aware, mean-agnostic case.

Proof of Theorem 2: From Lemma 4, a network G_β realizes the price of anarchy when the toll scaling factor is chosen optimally as in Proposition 2. The price of anarchy for this network is found by substituting β from (38) into the latency function ratio in (36). □

C. Network-Agnostic, Distribution-Aware

When the system designer is informed of the average user sensitivity, they are able to improve the price of anarchy ratio by utilizing the new information. It would seem that having precise knowledge would allow further reductions in the price of anarchy; however, in Theorem 3, it is shown that full information on the user sensitivities does not improve the price of anarchy.

Theorem 3: The worst-case performance guarantee for the network-agnostic taxation mechanism with knowledge of the full user sensitivity distribution is no better than that of the network-agnostic, mean-aware

$$\max_{s\in\mathcal{S}(\bar{s})} \mathrm{PoA}^*(\mathcal{G}, s) = \mathrm{PoA}^*(\mathcal{G}, \mathcal{S}(\bar{s})). \tag{20}$$

When the system designer is uncertain of the network characteristics, the full sensitivity distribution information is no more valuable than the average of the users sensitivities, in worst case; this is highlighted in Table I where the price of anarchy in box (B) and (C) are equal.

Proof: The proof follows similarly from Section III-B, utilizing Lemmas 1, 2, 3, and 4. Observe the two worst-case problem instances: G_α with bimodal distribution (S_L, S_U) with mean $\bar{s}$, and G_β with bimodal distribution (S_L, S_U) with mean $\bar{s}$. Because the user sensitivity distribution is the same in both instances, if this distribution was known a priori, the networks G_α and G_β would still constitute worst-case instances and the optimal tolling mechanism must be selected as in Proposition 2 and give the same performance guarantee as if only the mean was known. □

D. Network-Aware, Sensitivity-Agnostic

In Sections III-A–III-C, it is shown that additional information about the population of users may help improve the performance guarantees of an optimal taxation mechanism. In Sections III-E–III-G, we will also see how full knowledge of the network characteristics can improve the efficacy of tolls. Specifically, in this section, we consider the case where the system designer has full information on the network characteristics but knows nothing about the users' sensitivities except that they are bounded away from zero; in this setting, the additional information will not help.

Proposition 3: When the exact network G is known, but users' sensitivities $s \in \mathcal{S}_{>0}$ are only known to be bounded away from zero, no taxation mechanism can improve the price of anarchy, that is,

$$\sup_{G\in\mathcal{G}} \mathrm{PoA}^*(G, \mathcal{S}_{>0}) = \mathrm{PoA}^*(\mathcal{G}, \mathcal{S}_{>0}) = 4/3. \tag{21}$$

Proposition 3 shows that even if the exact network characteristics are known, some information about the population's response is required to improve worst-case performance guarantees. This is consistent with box (D) of Table I. Furthermore, this implies the top left box as well: when no information on network or population is present, no toll can lower the price of anarchy below 4/3.

Proof: If each user has the same sensitivity S; further, consider the classic Pigou network with two parallel edges, one with latency $\ell_1(f_1) = f_1$ and $\ell_e(f_2) = 1$. In the absence of any tolling, the Nash flow is $f^{\mathrm{Nf}} = (1, 0)$ and the optimal is $f^{\mathrm{opt}} = (1/2, 1/2)$, giving a price of anarchy of $4/3$. When all users travel on the same link (either the first or the second) in the Nash flow, the price of anarchy is $4/3$, the same as the untolled case. Any tolling mechanism that incentives users to utilize the second link (i.e., $\tau_1 > \tau_2$) can be made arbitrarily ineffective by letting $S \to \infty$, causing the Nash flow to be $f^{\mathrm{Nf}} = (0, 1)$ and the price of anarchy to be $4/3$. □

E. Network-Aware, Mean-Agnostic

As seen in Section III-D, network information will not help a system designer that has no knowledge of the population. When the system designer at least has bounds on the possible sensitivities of users, the optimal toll will be able to improve performance.

Theorem 4: When only S_L and S_U are known, the price of anarchy under an optimal, network-aware, scaled marginal-cost toll is tightly upper bounded by

$$\mathrm{PoA}^*(G, \mathcal{S}) \le \frac{4}{3}\left(1 - \frac{\sqrt{q}}{\left(1+\sqrt{q}\right)^2}\right) \tag{22}$$

where $q := S_L/S_U$.

By comparing box (B), (C), and (E) in Table I, one can observe that network information is significantly more valuable than additional population information beyond the lower and upper bound.

To prove this bound, we assume the toll designer can determine the Nash flow of a possible homogeneous low-sensitivity population associated with each toll scaling factor; this assumption is reasonable as the Nash flow of a homogeneous population can be found by solving a convex optimization problem [16]. Let $f_i^{\mathrm{Nf}}(G, S, k)$ be the mass of traffic on edge i in network G in a Nash flow of a population of users with homogeneous price-sensitivity S and tolling factor k.

Proposition 4: For any network $G \in \mathcal{G}$ and any $S_U \ge S_L > 0$, let $k^{\mathrm{gm}} = (S_L S_U)^{-1/2}$. The following is an optimal network-aware marginal-cost toll scaling factor:

$$k_{(G)} = \begin{cases} 0, & \text{if } f_2^{\mathrm{Nf}}\left(G, S_L, k^{\mathrm{gm}}\right) = 0 \\ k^{\mathrm{gm}}, & \text{otherwise.} \end{cases} \tag{23}$$

Proof: Consider the following cases, differentiated by the structure of Nash flows resulting from $k = k^{\mathrm{gm}} := (S_L S_U)^{-1/2}$:

1) $f_2^{\mathrm{Nf}}(G, S_L, k^{\mathrm{gm}}) > 0$, and
2) $f_2^{\mathrm{Nf}}(G, S_L, k^{\mathrm{gm}}) = 0$.

It is shown in [30] that in Case (1), it must be true that $\mathcal{L}^{\mathrm{Nf}}(G, S_L, k) = \mathcal{L}^{\mathrm{Nf}}(G, S_U, k)$ and that this choice of k

is uniquely optimal, resulting in the price of anarchy given in (22).

Consider Case (2). Here, the extreme low-sensitivity population with $s = S_L$ strictly prefers link 1 when $k = k^{\text{gm}}$, effectively stripping the designer of their influence over the price of anarchy. It can easily be shown (using, e.g., tools from [30]) that

$$k' \leq k^{\text{gm}} \implies \mathcal{L}^{\text{Nf}}(G, S_L, k) = \mathcal{L}^{\text{Nf}}(G, \emptyset) \tag{24}$$

but that

$$k^{\dagger} > k^{\text{gm}} \implies \mathcal{L}^{\text{Nf}}\big(G, S_U, k^{\dagger}\big) > \mathcal{L}^{\text{Nf}}(G, \emptyset). \tag{25}$$

That is, in this regime, the designer cannot change the behavior of $s = S_L$ without increasing tolls, but cannot increase tolls because this would cause the high-sensitivity population with $s = S_U$ to route more inefficiently. That is, $k = 0$ is an optimal tolling coefficient in this case.[4] □

Proof of Theorem 4 It follows easily from the results in [30] that in Case (2) when $k \leq k^{\text{gm}}$, it is true for any s that $\mathcal{L}^{\text{Nf}}(G, s, k) \leq \mathcal{L}^{\text{Nf}}(G, S_U, k^{\text{gm}})$; the price of anarchy bound for this scenario is thus precisely that in [30], where now we include games which need not have flow on every edge in an untolled Nash flow. □

F. Network-Aware, Mean-Aware

Next, we consider when the system designer is network-aware and mean-aware to illustrate the gain in performance when the system designer has knowledge of the network and partial information of the population.

Theorem 5: When S_L, S_U and the mean sensitivity $\bar{s}$ are known, under an optimal, network-aware, scaled marginal-cost tolling mechanism, the price of anarchy is tightly upper bounded by

$$\text{PoA}^*(G, \mathcal{S}(\bar{s})) \leq \frac{R^2 - \beta R + \beta}{\beta - \beta^2/4} \tag{26}$$

where $R = (S_U - \bar{s})/(S_U - S_L)$ and β is the unique solution on the interval $[0, 2]$ to

$$\beta = R\left(1 + \sqrt{\frac{1 + R - \beta}{\bar{s}/S_L + R - \beta}}\right). \tag{27}$$

To prove this, we start by making several of the same reductions as in Section III-B. In this setting, the optimal network-aware toll is found and denoted by the scaling factor $k_{(\bar{s},G)}$.

Proposition 5: For a network $G \in \mathcal{G}$ with price sensitivity distributions $s \in \mathcal{S}(\bar{s})$ with extreme sensitivity distributions $s_\ell^{(\bar{s},G,k)} = (S_{\ell 1}, S_{\ell 2})$ and $s_u^{(\bar{s},G,k)} = (S_{u1}, S_{u2})$, the optimal toll scaling factor for a linear toll will take the form

$$k_{(\bar{s},G)} = \frac{1}{\sqrt{S_{\ell 1} S_{u2}}}. \tag{28}$$

[4] In this case, the set of price-of-anarchy-minimizing tolling coefficients is not a singleton in general: any coefficient satisfying $\mathcal{L}^{\text{Nf}}(G, S_L, k) \geq \mathcal{L}^{\text{Nf}}(G, S_U, k)$ is optimal. Implication (24) means that this set always contains $k = 0$.

Proof of Proposition 5: From Lemma 1, under the same tolling mechanism, the set of Nash flows caused by $\mathcal{S}(\bar{s})$ is equal to those caused by distributions with bounds $[S_{\ell 1}, S_{u2}]$ and no mean constraint. The optimal scaling factor will therefore minimize the price of anarchy over this set of distributions. From [30], the optimal scaling factor for a linear toll will take this form. □

Lemma 3 shows that a transformation from a network $G \in \mathcal{G}$ to a network $\hat{G} \in \mathcal{G}^{\text{lc}}$ will increase the price of anarchy; we also note that this transformation had no dependence on the toll scaling factor and we can thus choose a k that is optimal for the resulting network.

Corollary 1: When making a reduction from $G \in \mathcal{G}$ to $\hat{G} \in \mathcal{G}^{\text{lc}}$, the price of anarchy increases regardless of the toll scaling factor k, including when $k = k_{(\bar{s},G)}$ for each network before and after the reduction.

Proof: In the proof of Lemma 3, the relation between k and the price of anarchy was not used; instead, it was shown that the price of anarchy increases as the network is transformed from any two link networks, to one that was in $\mathcal{G}^{\text{lc}}$. Consider having network G with the nonoptimal toll scaling factor $k_{(\bar{s},\hat{G})}$. When the reduction from G to $\hat{G}$ is done, by Lemma 3, we have

$$\begin{aligned}\text{PoA}\big(G, \mathcal{S}(\bar{s}), T\left(k_{(\bar{s},G)}\right)\big) &\leq \text{PoA}\Big(G, \mathcal{S}(\bar{s}), T\Big(k_{(\bar{s},\hat{G})}\Big)\Big) \\ &\leq \text{PoA}\Big(\hat{G}, \mathcal{S}(\bar{s}), T\Big(k_{(\bar{s},\hat{G})}\Big)\Big).\end{aligned}$$

□

Proof of Theorem 5: It is shown in Lemma 4 that a set of two networks realizes the price of anarchy. The price of anarchy for the network G_β is found by (36). Now, let G', defined by β', be a network that has the same price of anarchy when the flow R is on the first link, One solution is clearly $\beta = \beta'$, the other is $\beta' = (((4-\beta)R)/(R^2 - \beta R + \beta))$. Using the cost function of network G, we have $\beta = (1 + S_L k_{(\bar{s},G)})R$. Thus, if β' satisfied $s_u^{(\bar{s},G,k)} = (S_L, S_U)$ for the same mean sensitivity, then

$$\beta' = \frac{(4-\beta)R}{R^2 - \beta R + \beta} = \big(1 + S_U k_{(\bar{s},G')}\big)R. \tag{29}$$

However, it can be shown that the right-hand side of (37) is strictly less than the left-hand side. This imposes that the flow $f_1 = R$ cannot be a Nash flow in G' under distributions in $\mathcal{S}(\bar{s})$ and therefore not achieve the same price of anarchy as G. This implies that the price of anarchy for G_β is greater than that of G_α when both are tolled optimally with respect to Proposition 5. As this network is optimally tolled, from Proposition 5, it will be the case that

$$S_{u2} = \frac{\bar{s} - S_L}{1 + R - \beta} + S_L. \tag{30}$$

Now, in sensitivity distribution $s_\ell^{(\bar{s},G,k)} = (S_L, S_U)$, users with sensitivity S_L are indifferent with optimally scaled toll $k_{(\bar{s},G)}$. Using β from (37) and substituting the optimal scaling factor with extreme sensitivity from (30) leads to the characterization of β in the theorem statement, and the price of anarchy is found by substituting this into (36). □

G. Network-Aware, Distribution-Aware

Fleischer et al. [25] show that in the fully informed setting, there exist fixed tolls that will incentivize the users to self-route optimally. We will extend this work to our framework to show that, when the system designer known the full user sensitivity distribution and the network characteristics, they can always design a toll that gives price of anarchy of one.

Let $F(S)$ be a cumulative distribution function of the users' sensitivities in population s. Furthermore, let $F^{-1}(f)$ be a preimage of $[S_L, S_U]$ under $F(S)$ where if the preimage is nonsingleton, the minimum sensitivity is used, that is, $F^{-1}(f)$ is the sensitivity at which f mass of users have a lower sensitivity.

Theorem 6: For the network G with population s, the linear toll

$$T(af+b) = \frac{1}{F^{-1}\left(f_1^{\text{opt}}\right)} af \tag{31}$$

where $f_1^{\text{opt}} = ((2a_2 + b_2 - b_1)/(2(a_1 + a_2)))$ will have price of anarchy one, that is, $\text{PoA}^*(G, s) = 1$ for any G and s.

Box (G) in Table I shows that, when sufficient information is available, the inefficiency can be entirely eliminated, regardless of the problem instance.

Proof: First, note that for a network G, the optimal flow will be $f_1^{\text{opt}} = ((2a_2 + b_2 - b_1)/(2(a_1 + a_2)))$. If S is the sensitivity at which a user is indifferent between the two paths in the optimal flow, then any user with lower sensitivity will use the first edge. If the indifferent sensitivity is $S^\star = F^{-1}(f_1^{\text{opt}})$ then picking k such that

$$\left(1 + S^\star k\right) a_1 f_1^{\text{opt}} + b_1 = \left(1 + S^\star k\right) a_2 \left(1 - f_1^{\text{opt}}\right) + b_2 \tag{32}$$

is satisfied, the equilibrium flow will be f^{opt}. Substituting $S^\star$ and solving for k gives $k = (1/(F^{-1}(f_1^{\text{opt}})))$. □

IV. Empirical Study

The theoretical claims of this work are presented in the setting of two-link, affine-latency congestion games with scaled marginal cost tolls. As mentioned in Section II, some of these results generalize beyond this simple class of networks; we presented each result in this reduced setting to improve uniformity in presentation. Specifically, by matching the upper bound in [16], Proposition 3 holds for all networks, even those with multiple source-destination pairs. Theorem 6 easily generalizes to multilink parallel networks following the same steps as presented in this work. Theorem 1 and Theorem 4 can be shown to hold after a slight transformation for multilink parallel networks by following steps from [27]. This leaves Theorems 2, 3, and 5 as results where it is not known if our findings generalize to other classes of networks. In this section, we will motivate why we believe these results offer insights more broadly and the relationships likely hold more generally.

To understand how the results of this work extend to more general networks, we focus on understanding how the derived, optimal, scaled marginal-cost tolls perform in networks of greater than two links. We do this empirically by randomly generating a large number of five-link parallel networks with different population sensitivity distributions and recording the price of anarchy. Fig. 3 shows the empirical price of anarchy values and demonstrates that every found parallel network has a strictly better price of anarchy than the two-link bound.

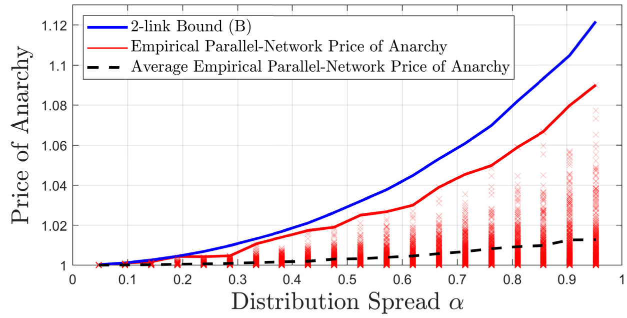

Fig. 3. Empirical price of anarchy in five-link parallel congestion networks compared to two-link bound. Specifically, the study is in the mean-aware, network-agnostic information setting, where in each game the optimal scaled marginal cost toll $k_{(\bar{s})}$ from Proposition 1 is used. With average sensitivity $\bar{s} = 1$, the population sensitivity distributions were chosen with lower-bound $S_L = \bar{s} - \alpha$ and upper-bound $S_U = \bar{s} + \alpha$. For varying values of $\alpha \in (0, 1)$, 500 parallel networks with five links and 100 population sensitivity distributions (including demonstrably worst case distributions) were randomly generated and the price of anarchy while using $k_{(\bar{s})}$ was recorded for each realization.

We focus our simulation on the mean aware setting to understand how these results generalize. To garner these empirical results, we set $\bar{s} = 1$ and vary the lower and upper bound over several values. For many values of $\alpha \in (0, 1)$, such that $S_L = \bar{s} - \alpha$ and $S_U = \bar{s} + \alpha$, we randomly generate 500 networks with five parallel edges, where the coefficients a_e and b_e in the latency function $\ell_e(f_e) = a_e f_e + b_e$ are independently drawn uniformly at random from $[0, 1]$. For each realized network, a set of sensitivity distribution were generated; the generated distributions included ones randomly created with 2, 3, 4, and 5 sensitivity values with positive weight. Fig. 3 shows a scatter plot of each recorded price of anarchy along with the maximum over these empirical samples.

It is not surprising that the two-link bound appears to hold over the class of parallel networks; in [31], it is shown that the price of anarchy in nonatomic congestion games is independent of the network structure, and worst-case examples are realized by two-link networks. Though it is not obvious this relationship holds with the introduction of tolling, in Brown and Marden [30] show that scaled-marginal cost tolls similarly experience worst-case performance over parallel networks in two-link networks.

V. Conclusion

This work studies the value of different types of information to a toll designer. When comparing the performance guarantees awarded to toll designers with differing available information, we observe that, though possessing additional information can give a system designer greater capabilities, it is not trivial which specific pieces of information are most helpful in influencing user behavior. The results of this work offer comparisons between the value of different types of information, including when additional information is helpful and when it is not.

APPENDIX

Proof of Lemma 3: Consider a network $G \in \mathcal{G}$ with affine latency functions on each link $\ell_i(f) = a_i f + b_i$. Let $\hat{G}$ have cost functions $\hat{\ell}_i(f) = \hat{a}_i f + \hat{b}_i$ with $\hat{a}_i \geq 0$ and $\hat{b}_i \geq 0$. We first show that simply removing the constant latency term on the first edge b_1 strictly increases the price of anarchy under any scaled marginal-cost toll. Using the optimal and Nash flow in (33), if $\hat{b}_2 = b_2 - b_1$ and $\hat{b}_1 = 0$ then G and $\hat{G}$ will have the same optimal flow and Nash flow for a distribution s. From (2), we observe that $\mathcal{L}^{\text{opt}}(G) = \mathcal{L}^{\text{opt}}(\hat{G}) + b_1$ as well as $\mathcal{L}^{\text{Nf}}(G, s, k) = \mathcal{L}^{\text{Nf}}(\hat{G}, s, k) + b_1$; therefore,

$$\begin{aligned}\text{PoA}(G, \mathcal{S}(\bar{s}), T(k)) &= \frac{\mathcal{L}^{\text{Nf}}\big(\hat{G}, s, T(k)\big) + b_1}{\mathcal{L}^{\text{opt}}\big(\hat{G}\big) + b_1} \\ &\leq \frac{\mathcal{L}^{\text{Nf}}\big(\hat{G}, s, T(k)\big)}{\mathcal{L}^{\text{opt}}\big(\hat{G}\big)} \\ &= \text{PoA}\big(\hat{G}, \mathcal{S}(\bar{s}), T(k)\big).\end{aligned}$$

Thus, for any network $G \in \mathcal{G}$, there exists a network $\hat{G}$ with a linear latency function on an edge with higher price of anarchy.

Next, we show a network $G \in \mathcal{G}$ will have the same price of anarchy as a network $\hat{G} \in \mathcal{G}$ under the same linear toll if the latency functions of $\hat{G}$ equal the latency functions of G times a scaling factor c. Under a distribution $s \in \mathcal{S}$, G and $\hat{G}$ will have the same Nash flow. Using the indifferent sensitivity S_{ind} that is the solution to (14), the Nash flow and optimal flow on the first edge are

$$f_1^{\text{opt}} = \frac{2a_2 + b_2 - b_1}{2(a_1 + a_2)}, \quad f_1^{\text{Nf}} = \frac{(1 + S_{\text{ind}}k)a_2 + b_2 - b_1}{(1 + S_{\text{ind}}k)(a_1 + a_2)}. \tag{33}$$

Under the same distribution s, S_{ind} will satisfy

$$(1 + S_{\text{ind}}k)ca_1 f_1^{\text{Nf}} + cb_1 = (1 + S_{\text{ind}}k)ca_2 f_2^{\text{Nf}} + cb_2 \tag{34}$$

which are the latency functions for the network $\hat{G}$. It is now clear that G and $\hat{G}$ will have the same Nash and optimal flows. From the definition of total latency in (2), the latency in $\hat{G}$ will be c times the latency in G under the same flow. The price of anarchy, which is the ratio of two total latencies, will be identical in G and G'.

Lastly, we show that by decreasing a_2 in a network, the price of anarchy will increase. In Lemma 1, it was shown that any feasible Nash flow can be induced by a bimodal sensitivity distribution in which users are segregated on either link by their sensitivity. The price of anarchy for the network G with a Nash flow caused by s will therefore be,

$$\text{PoA}(G, s, T(k)) = \frac{\ell_1\big(f_1^{\text{Nf}}\big) f_1^{\text{Nf}} + \ell_2\big(f_2^{\text{Nf}}\big) f_2^{\text{Nf}}}{\ell_1\big(f_1^{\text{opt}}\big) f_1^{\text{opt}} + \ell_2\big(f_2^{\text{opt}}\big) f_2^{\text{opt}}}. \tag{35}$$

Let us consider the case where $f_2^{\text{Nf}} > f_2^{\text{opt}}$. Now, consider a new network, $\hat{G}$ which replaces latency function $\ell_2(f) = a_2 f + b_2$ in G with $\hat{\ell}_2(f) = a_2 f + \hat{b}_2$ where $\hat{b}_2 = b_2 + \delta$ such that $\delta > 0$. Because the users are segregated on the links, the Nash flow will not change. Note that because $f_2^{\text{Nf}} > f_2^{\text{opt}}$

$$\frac{\ell_2\big(f_2^{\text{Nf}}\big)}{\ell_2\big(f_2^{\text{opt}}\big)} = \frac{a_2 f_2^{\text{Nf}} + b_2}{a_2 f_2^{\text{opt}} + b_2} < \frac{f_2^{\text{Nf}}}{f_2^{\text{opt}}}.$$

It can now be shown that

$$\begin{aligned}\frac{\mathcal{L}^{\text{Nf}}(G, s, T(k))}{\mathcal{L}^{\text{opt}}(G)} &< \frac{\mathcal{L}^{\text{Nf}}(G, s, T(k)) + \delta f_2^{\text{Nf}}}{\mathcal{L}^{\text{opt}}(G) + \delta f_2^{\text{opt}}} \\ &= \frac{\mathcal{L}^{\text{Nf}}\big(\hat{G}, s, T(k)\big)}{\mathcal{L}^{\text{opt}}\big(\hat{G}\big)}.\end{aligned}$$

Thus, the price of anarchy has increased in the new network $\hat{G}$, under the same sensitivity distribution and toll, when b_2 was increased, which has the same effect as decreasing the other terms and holding b_2 constant. A very similar argument can be followed for when $f_2^{\text{Nf}} < f_2^{\text{opt}}$ by picking $\hat{a}_2 = a_2 - \delta$, and the price of anarchy then again increases. □

Proof of Proposition 2: The k that solves (17) equates the price of anarchy for G_β and G_α. It is shown in Lemma 4 that these networks realize the worst-case inefficiency and in Lemma 2 it is shown that the worst-case distribution will be $s_\ell^{(\bar{s},G,k)}$ and $s_u^{(\bar{s},G,k)}$, respectively, both defined as the bimodal distribution (S_L, S_U) with mean $\bar{s}$ and mass R of the lower sensitivity type. To show this is optimal, it is sufficient to show that the price of anarchy for the network G_β is decreasing with k while the price of anarchy for the network G_α is increasing with k. If the networks have this relation with k, then the k that minimizes the price of anarchy must equalize them.

Consider a network $G \in \mathcal{G}^{\text{lc}}$ characterized by $\gamma = b_2/a_1$. If this satisfies that $s_\ell^{(\bar{s},G,k)} = (S_L, S_U)$ or $s_u^{(\bar{s},G,k)} = (S_L, S_U)$, then the price of anarchy for this network will be

$$\text{PoA}(G, \mathcal{S}(\bar{s}), T) = \begin{cases} \dfrac{R^2 - \gamma R + \gamma}{\gamma - \gamma^2/4}, & \gamma < 2 \\ R^2 - \gamma R + \gamma, & \gamma \geq 2. \end{cases} \tag{36}$$

This piecewise-continuous expression is locally minimized by $\gamma = 2R$, furthermore, by differentiation, it can be observed that it is monotone decreasing for $0 < \gamma < 2R$ and monotone increasing for $\gamma > 2R$.

For the previously defined network G_β, under the bimodal distribution (S_L, S_U)

$$\beta = (1 + S_L k)R = R/z_L \tag{37}$$

where β is dependent on the scaling factor k. From [32], the optimal scaling factor k will be in $(1/S_U, 1/S_L)$. Therefore, for any k, $\beta < 2R$. The price of anarchy for this network is therefore monotone decreasing with β, and from (37), β is clearly increasing with k. The price of anarchy of the network is therefore decreasing with k. Similarly, for G_α, under the distribution (S_L, S_U), the worst network is found by maximizing (36) over $\gamma = \alpha$, giving

$$\frac{1}{2\big(z_U - z_U^2\big)} = \alpha > 2R \tag{38}$$

when $(1/(2(z_U - z_U^2))) \geq R$, and $R/z_U = \alpha > 2R$, and the price of anarchy will be increasing with k. □

REFERENCES

[1] B. L. Ferguson, P. N. Brown, and J. R. Marden, “Utilizing information optimally to influence distributed network routing,” in *Proc. IEEE 58th Conf. Decis. Control (CDC)*, Dec. 2019, pp. 5008–5013.

[2] A. C. Pigou, *The Economics of Welfare*. New York, NY, USA: Macmillan, 1920.

[3] C. Papadimitriou, "Algorithms, games, and the internet," in *Proc. 33rd Annu. ACM Symp. Theory Comput.*, Jul. 2001, pp. 749–753.

[4] R. Johari and J. N. Tsitsiklis, "Efficiency loss in a network resource allocation game," *Math. Oper. Res.*, vol. 29, no. 3, pp. 407–435, Aug. 2004.

[5] J. R. Marden and J. S. Shamma, "Game theory and distributed control," in *Handbook of Game Theory*, vol. 4, H. P. Young and S. Zamir, Eds. Amsterdam, The Netherlands: Elsevier, 2014.

[6] G. Piliouras, E. Nikolova, and J. S. Shamma, "Risk sensitivity of price of anarchy under uncertainty," in *Proc. 14th ACM Conf. Electron. Commerce*, New York, NY, USA, Jun. 2013, pp. 715–732.

[7] L. J. Ratliff, R. Dong, S. Sekar, and T. Fiez, "A perspective on incentive design: Challenges and opportunities," *Annu. Rev. Control, Robot., Auto. Syst.*, vol. 2, pp. 305–338, Dec. 2018.

[8] J. M. Hoque, G. D. Erhardt, D. Schmitt, M. Chen, and M. Wachs, "Estimating the uncertainty of traffic forecasts from their historical accuracy," *Transp. Res. A, Policy Pract.*, vol. 147, pp. 339–349, May 2021.

[9] R. Tang, S. Wang, and H. Li, "Game theory based interactive demand side management responding to dynamic pricing in price-based demand response of smart grids," *Appl. Energy*, vol. 250, pp. 118–130, Sep. 2019.

[10] V. Ranganathan, P. Kumar, U. Kaur, S. H. Q. Li, T. Chakraborty, and R. Chandra, "Re-inventing the food supply chain with IoT: A data-driven solution to reduce food loss," *IEEE Internet Things Mag.*, vol. 5, no. 1, pp. 41–47, Mar. 2022.

[11] R. Schroll and R. Grohs, "Uncertainty in prerelease advertising," *J. Advertising*, vol. 48, no. 2, pp. 167–180, Mar. 2019.

[12] J. Verbree and A. Cherukuri, "Inferring the prior in routing games using public signalling," 2021, *arXiv:2109.05895*.

[13] G. Kreindler, "Peak-hour road congestion pricing: Experimental evidence and equilibrium implications," Dept. Econ., Harvard Univ., Cambridge, MA, USA, Tech. Rep., 2020, doi: 10.3386/w30903.

[14] P. A. Akbar, V. Couture, G. Duranton, and A. Storeygard, "Mobility and congestion in urban India," *Amer. Econ. Rev.*, vol. 113, no. 4, pp. 1083–1111, Apr. 2023, doi: 10.1257/aer.20181662.

[15] N. Buchholz, L. Doval, J. Kastl, F. Matejka, and T. Salz, "The value of time: Evidence from auctioned cab rides," Nat. Bureau Economic Res., Princeton, NJ, USA, Tech. Rep., May 2020, doi: 10.3386/w27087.

[16] T. Roughgarden, "How bad is selfish routing?" *J. ACM*, vol. 49, no. 2, pp. 236–259, 2002.

[17] D. Paccagnan, R. Chandan, B. L. Ferguson, and J. R. Marden, "Optimal taxes in atomic congestion games," *ACM Trans. Econ. Comput.*, vol. 9, no. 3, pp. 1–33, Aug. 2021.

[18] V. Bilò and C. Vinci, "Dynamic taxes for polynomial congestion games," in *Proc. ACM Conf. Econ. Comput.*, New York, NY, USA, Jul. 2016, pp. 839–856.

[19] F. Farokhi and K. H. Johansson, "A piecewise-constant congestion taxing policy for repeated routing games," *Transp. Res. B, Methodol.*, vol. 78, pp. 123–143, Aug. 2015.

[20] P.-A. Chen, B. D. Keijzer, D. Kempe, and G. Schäfer, "Altruism and its impact on the price of anarchy," *ACM Trans. Econ. Comput.*, vol. 2, no. 4, pp. 1–45, Oct. 2014.

[21] P. Kleer and G. Schäfer, "Path deviations outperform approximate stability in heterogeneous congestion games," in *Proc. Int. Symp. Algorithmic Game Theory*, 2017, pp. 212–224.

[22] B. L. Ferguson, P. N. Brown, and J. R. Marden, "The effectiveness of subsidies and tolls in congestion games," *IEEE Trans. Autom. Control*, vol. 67, no. 6, pp. 2729–2742, Jun. 2022.

[23] P. N. Brown and J. R. Marden, "Can taxes improve congestion on all networks?" *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 4, pp. 1643–1653, Dec. 2020.

[24] R. Cole, Y. Dodis, and T. Roughgarden, "Pricing network edges for heterogeneous selfish users," in *Proc. 34th Annu. ACM Symp. Theory Comput.*, New York, NY, USA, Jun. 2003, pp. 521–530.

[25] L. Fleischer, K. Jain, and M. Mahdian, "Tolls for heterogeneous selfish users in multicommodity networks and generalized congestion games," in *Proc. 45th Annu. IEEE Symp. Found. Comput. Sci.*, Rome, Italy, 2004, pp. 277–285.

[26] D. Fotakis, G. Karakostas, and S. G. Kolliopoulos, "On the existence of optimal taxes for network congestion games with heterogeneous users," in *Proc. Int. Symp. Algorithmic Game Theory* (Lecture Notes in Computer Science), vol. 6386, 2010, pp. 162–173.

[27] P. N. Brown and J. R. Marden, "Optimal mechanisms for robust coordination in congestion games," *IEEE Trans. Autom. Control*, vol. 63, no. 8, pp. 2437–2448, Aug. 2018.

[28] J. Wardrop, "Some theoretical aspects of road traffic research," *Proc. Inst. Civil Eng., II*, vol. 1, pp. 352–362, Jun. 1952.

[29] A. Mas-Colell, "On a theorem of schmeidler," *J. Math. Econ.*, vol. 13, no. 3, pp. 201–206, Dec. 1984.

[30] P. N. Brown and J. R. Marden, "The robustness of marginal-cost taxes in affine congestion games," *IEEE Trans. Autom. Control*, vol. 62, no. 8, pp. 3999–4004, Aug. 2017.

[31] T. Roughgarden, "The price of anarchy is independent of the network topology," *J. Comput. Syst. Sci.*, vol. 67, no. 2, pp. 341–364, Sep. 2003.

[32] P. N. Brown and J. R. Marden, "The benefit of perversity in distributed network routing," in *Proc. 56th IEEE Conf. Decis. Control*, Melbourne, VIC, Australia, Dec. 2017, pp. 6229–6234.

Bryce L. Ferguson (Graduate Student Member, IEEE) received the A.A. degree in mathematics from Santa Rosa Junior College, Santa Rosa, CA, USA, in 2016, and the B.S. and M.S. degrees in electrical engineering from the University of California, Santa Barbara, CA, in 2018 and 2020, respectively. He is currently pursuing the Ph.D. degree with the Electrical and Computer Engineering Department, University of California, Santa Barbara.

His research interests focus on using game theoretic methods for describing and controlling both societal and engineered multiagent systems.

Mr. Ferguson was named a 2022 CPS Rising Star and was a finalist for the Best Student Paper Award at the 2020 American Controls Conference.

Philip N. Brown (Member, IEEE) received the B.Sc. degree in electrical engineering from Georgia Tech, Atlanta, GA, USA, in 2007, the M.Sc. degree in electrical engineering from the University of Colorado at Boulder, Boulder, CO, USA, in 2015, under the supervision of Jason R. Marden, and the Ph.D. degree in electrical and computer engineering from the University of California, Santa Barbara, CA, USA, under the supervision of Jason R. Marden.

After 2007, he spent several years designing control systems and process technology for the biodiesel industry. He is currently an Assistant Professor at the Department of Computer Science, University of Colorado, Colorado Springs, CO, USA. He is interested in the interactions between engineered and social systems.

Dr. Brown was a recipient of the University of Colorado Chancellor's Fellowship from the University of Colorado at Boulder. He was a finalist for the Best Student Paper Award from the 2016 and 2017 IEEE Conferences on Decision and Control, the 2018 CCDC Best Ph.D. Thesis Award from UCSB, and the AFOSR Young Investigator Award.

Jason R. Marden (Senior Member, IEEE) received the B.Sc. and Ph.D. degrees in mechanical engineering from University of California, Los Angeles (UCLA), Los Angeles, CA, USA, in 2001 and 2007, respectively, under the supervision of Jeff S. Shamma.

After graduating from UCLA, he served as a Junior Fellow at the Social and Information Sciences Laboratory, California Institute of Technology, Pasadena, CA, USA, until 2010, and then as an Assistant Professor at the University of Colorado until 2015. He is currently a Professor at the Department of Electrical and Computer Engineering, University of California, Santa Barbara, Santa Barbara, CA, USA. His research interests focus on game theoretic methods for the control of distributed multiagent systems.

Dr. Marden was awarded the Outstanding Graduating Ph.D. Student in mechanical engineering from UCLA. He is a recipient of an ONR Young Investigator Award in 2015, NSF Career Award in 2014, AFOSR Young Investigator Award in 2012, SIAM CST Best Sicon Paper Award in 2015, and the American Automatic Control Council Donald P. Eckman Award in 2012.