How to Train Your Dragon (Model)

Introduction

In the fictional universe of *Game of Thrones*, Daenerys Targaryen, the "Mother of Dragons," raises three dragons that grow and thrive depending on their environment and food availability. For this project, we analyze the weight dynamics of these dragons over time, focusing on their interactions with prey populations and environmental conditions. Dragon weight, a critical indicator of health, evolves with prey availability, hunting success, and environmental stressors such as droughts or injuries from hunters. Using a mathematical model, we simulate the growth and survival of the dragons while accounting for prey population dynamics, hunting behaviors, and external events.

Background

The problem centers on simulating and understanding how dragon weights vary over time under stochastic and deterministic influences. Dragons grow in weight by successfully hunting prey, including cows, sheep, and chickens, whose populations fluctuate due to natural growth, carrying capacities, and external events like droughts. Additionally, the model incorporates dragon hunting success rates, prey preferences, and health impacts from failed hunts. Environmental disruptions, such as periodic hunter attacks and droughts, add complexity to the system, making this model an example of predator-prey dynamics under uncertainty.

The primary goal is to determine how dragon weights differ over time under these varying conditions and evaluate factors influencing their growth, health, and survival.

Problem Statement

When hatched, the dragons are small, roughly 10 kg. They continue to grow throughout their life depending on the conditions and amount of food available to them. For the purposes of this problem, we will consider these three fictional dragons that are living today. We will analyze dragon weight based on the following question: How do the dragons' weights differ over time?

The maximum weight of a dragon is 2,000 kg. If the weight of a dragon drops below 10 kg, it shall die.

The dragons' success, or weight gain, is defined by three distinct environmental conditions: availability of food sources, in the form of potential prey, unique characteristics of each dragon, and environmental catastrophes. These factors will contribute towards the variability in each dragons' weight.

In the dragons' environment, cows, sheep, and chickens are the only sources of food. Each of the prey has unique characteristics. For instance, we assume that the cows' initial population is 100. To simulate realistic environmental conditions, we want to observe fluctuations in the cows' growth rate. Therefore, we assume that cows have an average growth rate of 10% population/month, with an SD of 0.01. If a dragon is able to successfully hunt cows during the month, the dragon gains 150 kg from 10 cows and the cow population also loses 10 cows. To prevent overpopulation, we assume cows have a carrying capacity of 350. To prevent extinction, dragons cannot hunt cows if the cow population drops below 15.

Sheep and chicken follow similar assumptions. We assume that sheep have an initial population of 200, and have an average growth rate of 15% population/month with a SD of 0.05. If a dragon is able to successfully hunt sheep during the month, the dragon will gain 90 kg from 15 sheep. The sheep population also loses 15 sheep respectively. Sheep have a carrying capacity of 700, and dragons cannot hunt sheep if the sheep population drops below 50. The initial chicken population is 500. Chickens have an average growth rate of 30% population/month, with a SD of 0.10. The dragons will gain 30 kg from 75 chickens, if dragons successfully hunt chickens during the month. The chicken population will also lose 75 chickens respectively. We cap the chicken population at 1000 chickens, and the dragons will be unable to hunt chickens if the chicken population drops below 100.

If the dragons successfully hunt, but the respective prey population is too low, the dragon will experience a penalty, losing 10% kg of their current weight. If the dragons fail to hunt during any month, they experience another penalty and lose a percentage of their weight depending on the prey they tried to hunt. During any month, the dragons will lose 30%, 20%, 10% of their weight if they fail to hunt cows, sheep, or chickens respectively.

Each dragon has unique characteristics (aggressive, balanced, opportunistic) that distinguishes it from others. Dragon 1 is an aggressive hunter, with a hunting success rate of 60%. Dragon 1 is frequently successful due to experience but expends more energy hunting. Upon success, Dragon 1 has a 30% chance of hunting cows (prefers larger prey for more food but it's riskier), 20% chance of hunting sheep (moderate preference, easier than cows), and 50% chance of hunting chickens (relatively easier prey with negligible energy cost) as its food source for that month.

Dragon 2 is a balanced hunter, with a hunting success rate of 70%. Dragon 2 is adaptable and balances energy cost and prey. Upon hunting success, Dragon 2 has a 25% chance of hunting cows (targets cows less frequently, balancing energy cost), 50% chance of hunting sheep (optimal target for energy expenditure and reward), and 25% chance of hunting chickens (reserves chicken hunts for when prey availability is low).

Dragon 3 is an opportunistic hunter, with a hunting success rate of 50%. Dragon 3 is less skilled and easily affected by lower weight or health. Upon hunting success, Dragon 3 has a 20% chance of hunting cows (low energy availability reduces hunting effort for large prey), 30% chance of hunting sheep (moderate effort and reward balance), and 50% chance of hunting chickens (relies heavily on chickens for sustenance due to ease).

The dragons' environment is also marked by two distinct, environmental catastrophes. Every 30 months, a hunter comes to hunt the dragons. The hunter has a 1% chance to severely wound the dragons, reducing their weight by 40%; a 30% chance to mildly wound the dragons, reducing their weight by 20%; and a 69% chance to lightly wound the dragons, reducing their weight by 10%.

Additionally, every 30 months, there is a 5% chance of a drought occurring that reduces the chicken population by 40%.

To also simulate our dragons' environment under ideal conditions, we also incorporated a simplified version of our stochastic model, the deterministic model. In this model, the growth rates for the prey do not vary by month. Dragons are always successful at hunting unless the prey population is below their respective threshold. Since Dragon 1 is an aggressive hunter, it only eats cows. Since Dragon 2 is a balanced hunter, it only eats sheep. Finally, since Dragon 3 is an opportunistic hunter, it only eats chickens. There is no longer a hunter or drought.

In order to define our many parameters and assumptions, we consulted ChatGPT to brainstorm numbers and then scaled them appropriately for our model.

Results and Analysis

Deterministic Model:

Dragons:

All three dragons gain weight linearly and remain at constant weight once they reach the maximum (2000 kilograms) or, in the case of Dragon 1, begin losing weight due to insufficient prey population.

In the deterministic model, Dragon 1 dies at month 38 because it hunts the cow population, which declines to 15 cows at month 18, to near-extinction. From month 19 to 38—the period it takes for the cow population to return to at least 15, the minimum number required for the dragon to successfully hunt—the dragon loses 30% of its body weight each month, eventually dying at month 38.

While Dragon 1's choice of prey allows it to reach maximum weight faster than the other two dragons, the cows' low initial population and slow growth rate cause it to die less than halfway through the runtime of the simulation.

Dragon 2 and Dragon 3 follow a linearly positive pattern of growth until they achieve 2000 kilograms, at which point their weights remain constant. In ideal conditions, Dragon 2 exclusively hunts sheep, gaining a total of 90 kilograms each iteration, while Dragon 3 hunts chickens, gaining 30 kilograms; this allows Dragon 2 to grow at thrice the rate of Dragon 3, while both avoid competing with each other.

Prey:

The cow population is initially described by the equation:

$$x(n + 1) = x(n) + (1 - x(n) / 350) \cdot 0.10 \cdot x(n) - 10$$

The population declines to 7 at n = 20, at which point Dragon 1 stops hunting the cows and the equation becomes:

$$x(n + 1) = x(n) + (1 - x(n)/ 350) \cdot 0.10 \cdot x(n)$$

The population growth rate remains mostly positive, with the exception of month 27, during which the dragon hunts a final time before beginning to die. From month 28 onward, the population follows the equation:

$$x(n + 1) = x(n) + (1 - x(n) / 350) \cdot 0.10 \cdot x(n)$$

reaching a stable fixed point at 350.0.

The sheep population is described by the equation:

$$y(n + 1) = y(n) + (1 - y(n) / 700) * 0.15 * y(n) - 15$$

which has fixed points at y(n) = 120.87 and 579.13. Because the initial population is above 120.87 and the growth rate remains positive, the sheep population stabilizes at 579.

The chicken population is described by the equation:

$$z(n + 1) = z(n) + (1 - z(n) / 1000) * 0.30 * z(n) - 75$$

which reaches a fixed point at z(n) = 500. Because the initial population is 500, the growth rate equals zero in each time step, resulting in a constant population under ideal conditions.

Stochastic Model:

Dragons:

In general, dragon weight growth seems to be logistic, increasing quickly initially but gradually flattening out around the 5-year mark. While each dragon's weight has minor fluctuations due to catastrophic events (such as being hit by a hunter), the overall direction of growth is positive.

Due to elements of demographic stochasticity (the dragon's hunting success rate) and environmental stochasticity (the hunter's success rate), there is wide variation in the final distribution of dragon weights between trials. Dragon 1's final weight distribution across all trials had an IQR of about 250 kilograms, for instance, with Dragon 2 showing more variation (at an IQR of 750 kilograms) and Dragon 3 showing less variation (with an IQR of about 250 kilograms). This may be linked to the dragons' initial weights and growth rates—because Dragon 2 generally has the highest weight, any weight loss calculated as a percentage of its total weight (such as a 20% decrease in the case of a "mild wound" from the hunter) is also greater in absolute value. This leads to a wider range of possible weights for Dragon 2.

The different success rates also caused Dragon 2 to achieve the highest median weight, followed by Dragon 1 and Dragon 3.

In all three dragons' models, we see dips in the average weight across trials every 30 months, when the hunter arrives.

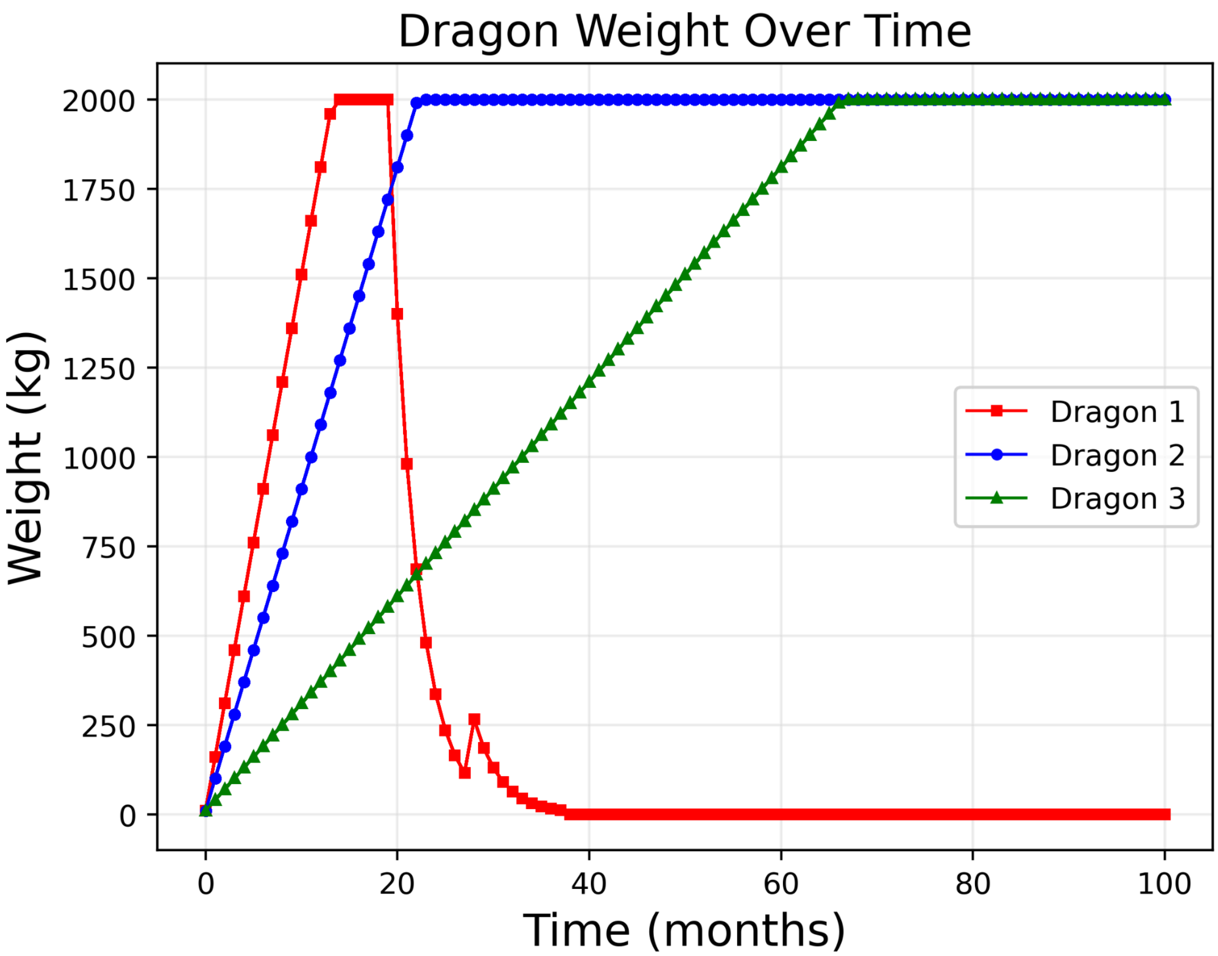
Prey:

In the stochastic model, the sheep and cows generally follow logistic patterns of growth. Individual chicken populations follow highly variable patterns of growth, though the average across all trials decreases over time as more trials are hit by catastrophe. Because only the chickens are affected by the drought, their population across trials has significantly more variability than those of the sheep and cows. The sheep and cows achieve fixed points at roughly 620 and 300, respectively, while the average chicken population declines at an average rate of 2 chickens a month.

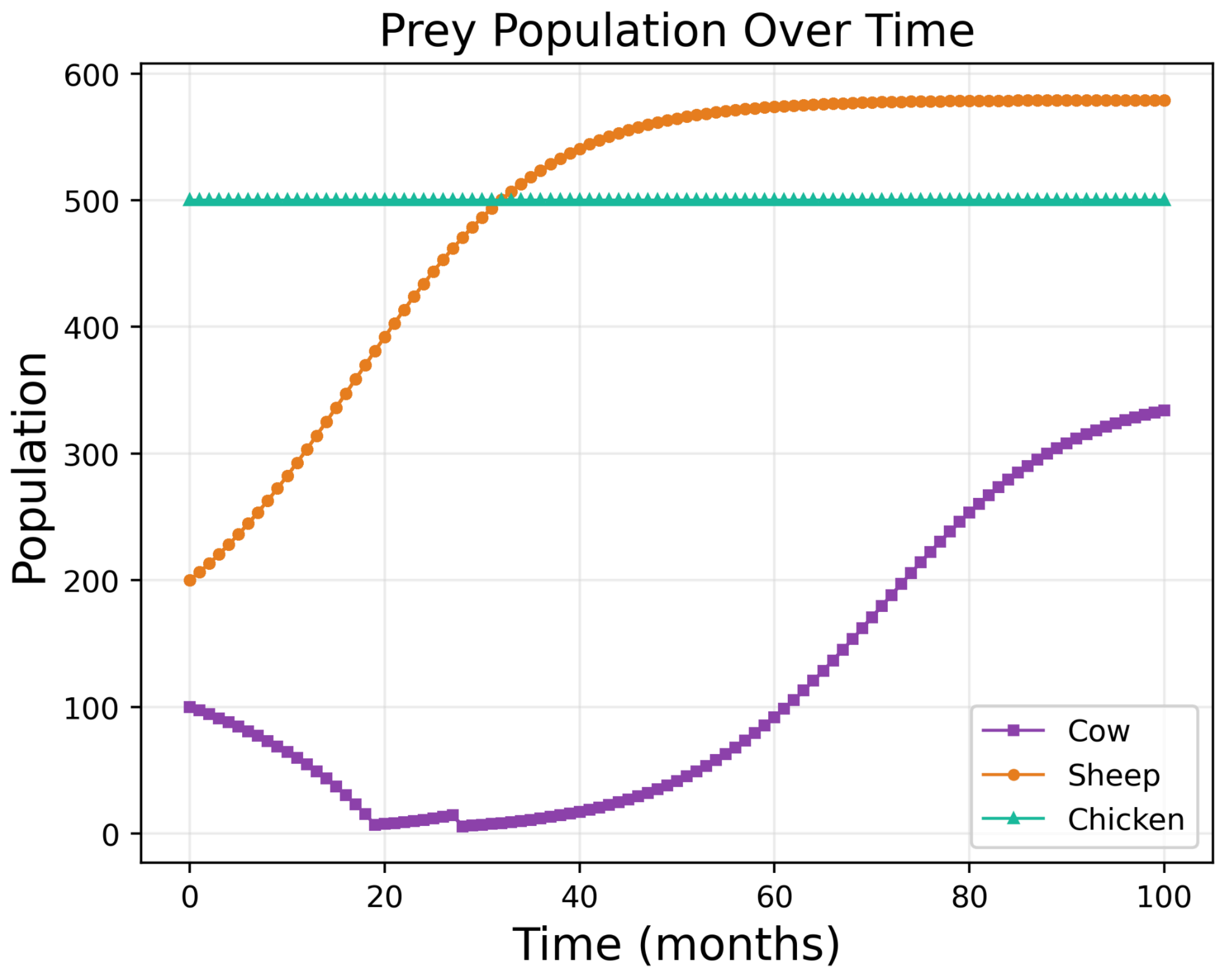
The final distribution of chicken populations is also skewed right, likely due to a cluster of trials in which the chicken population was hit early by one or more catastrophes and remained low for the remainder of the trial.

Figures and plots of results

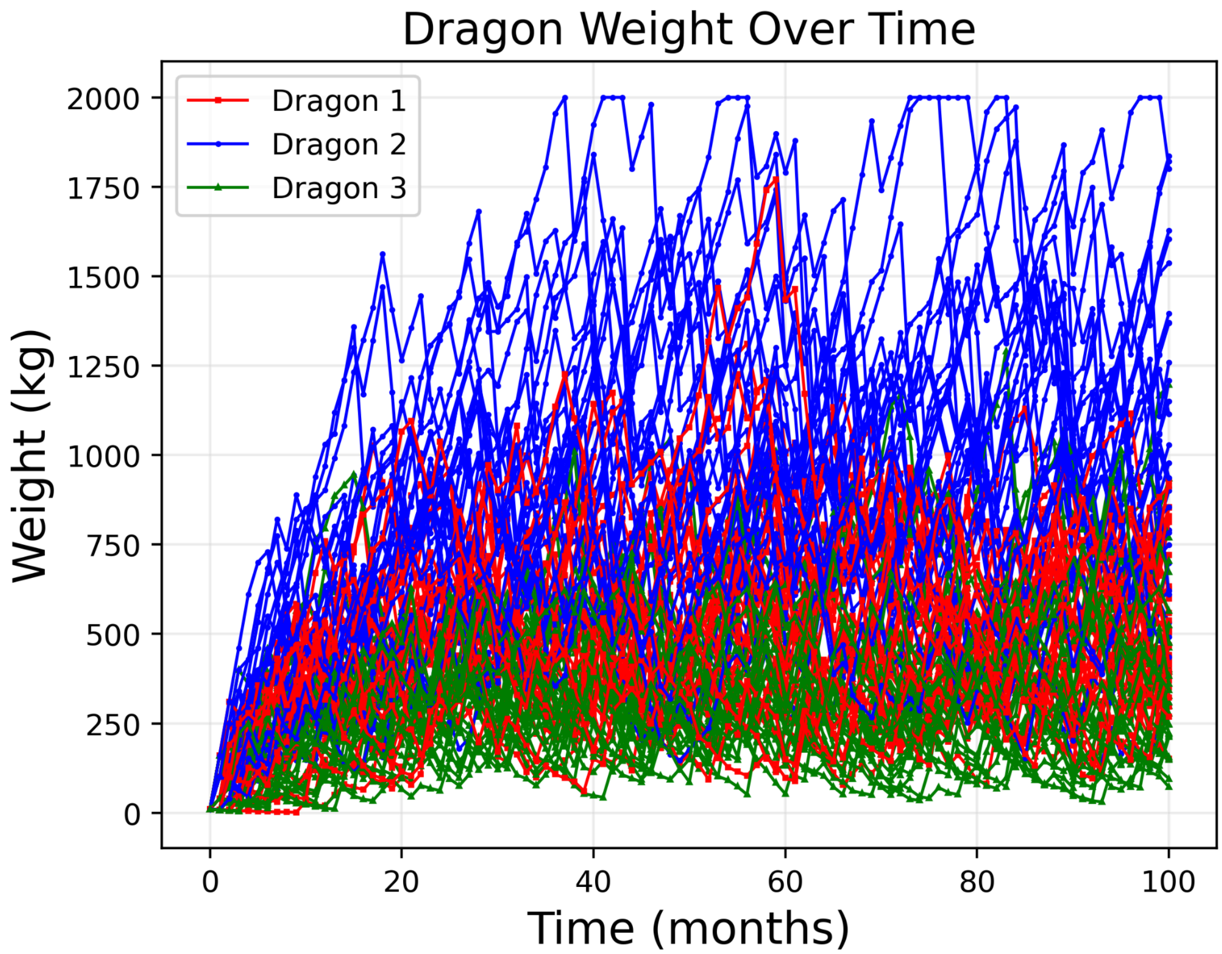
Deterministic Model:

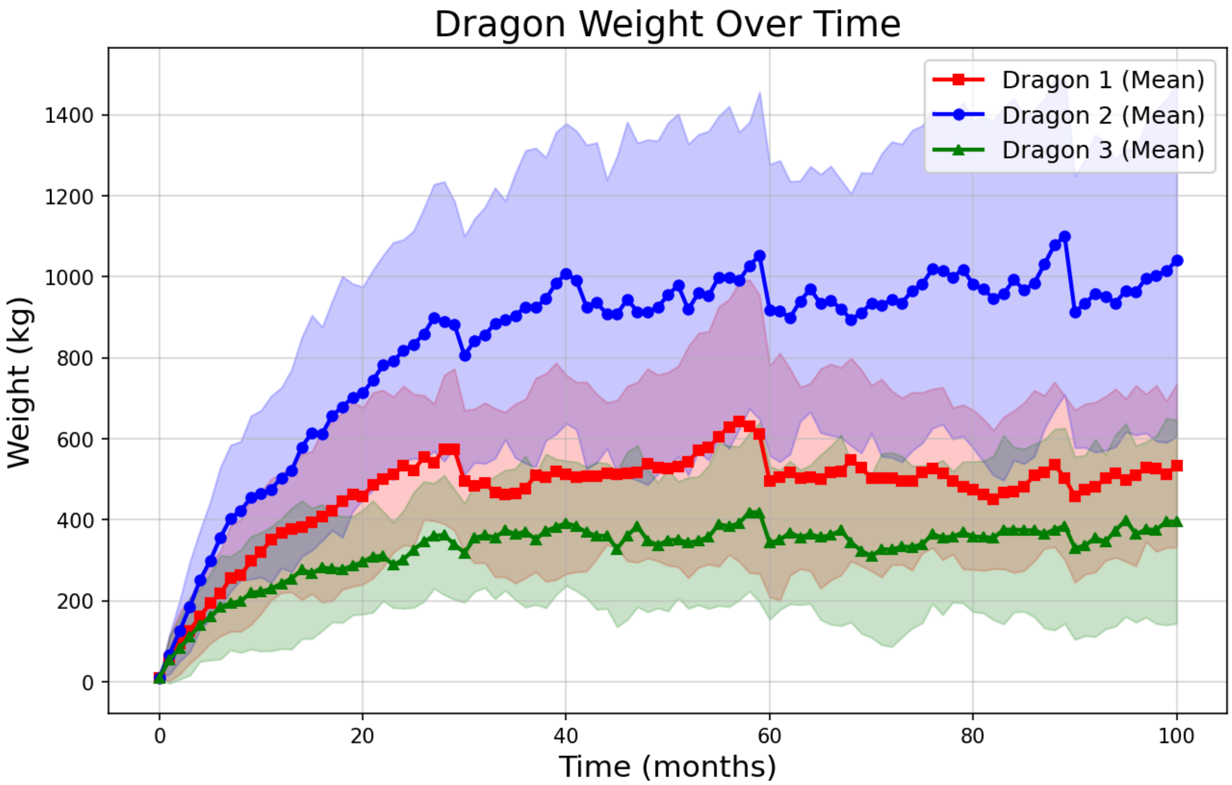


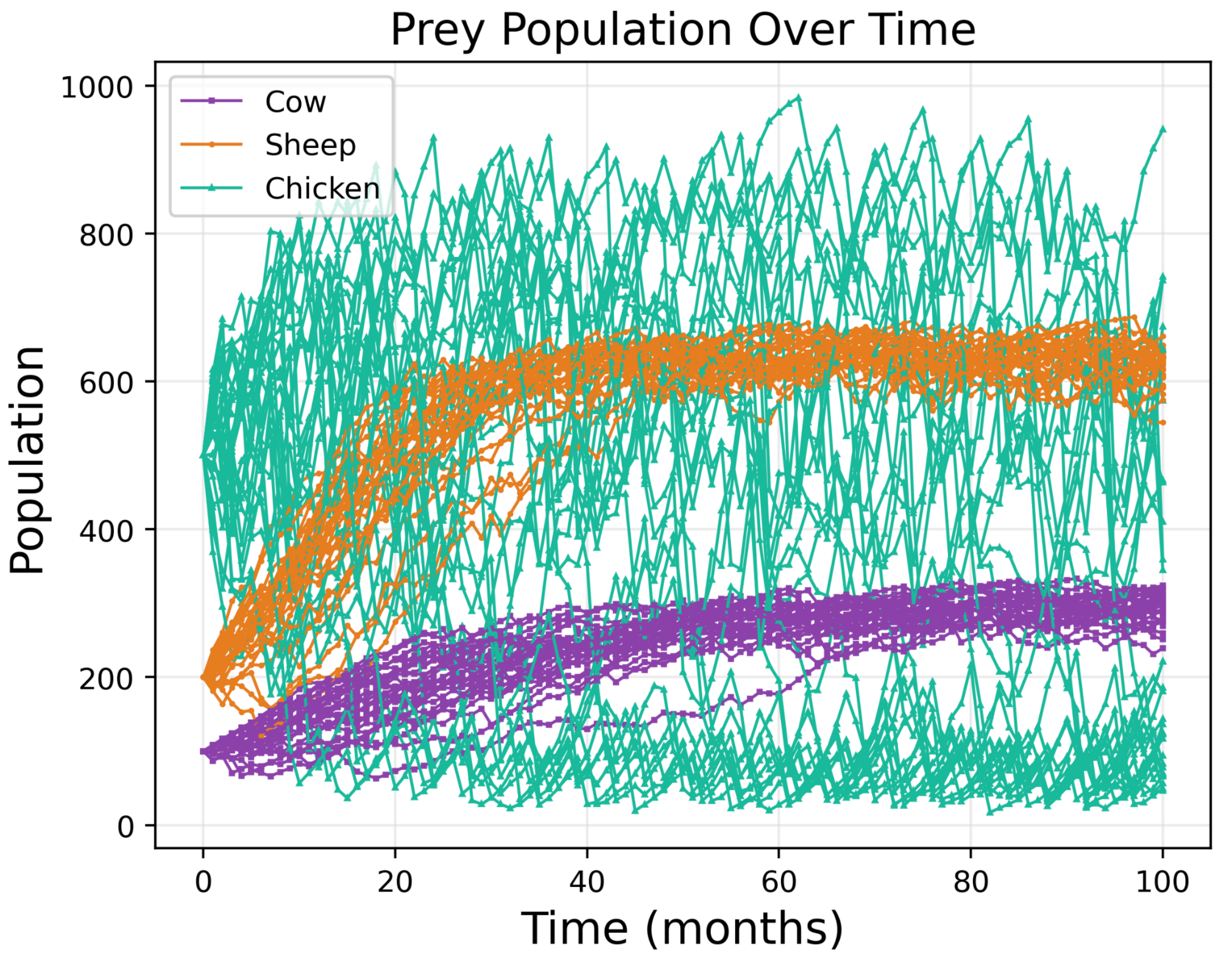
Dragon 1 dies at time step 38


Prey Population Over Time
Population
Time (months)
600
500
400
300
200
100
0
0
20
40
60
80
100
Cow
Sheep
Chicken

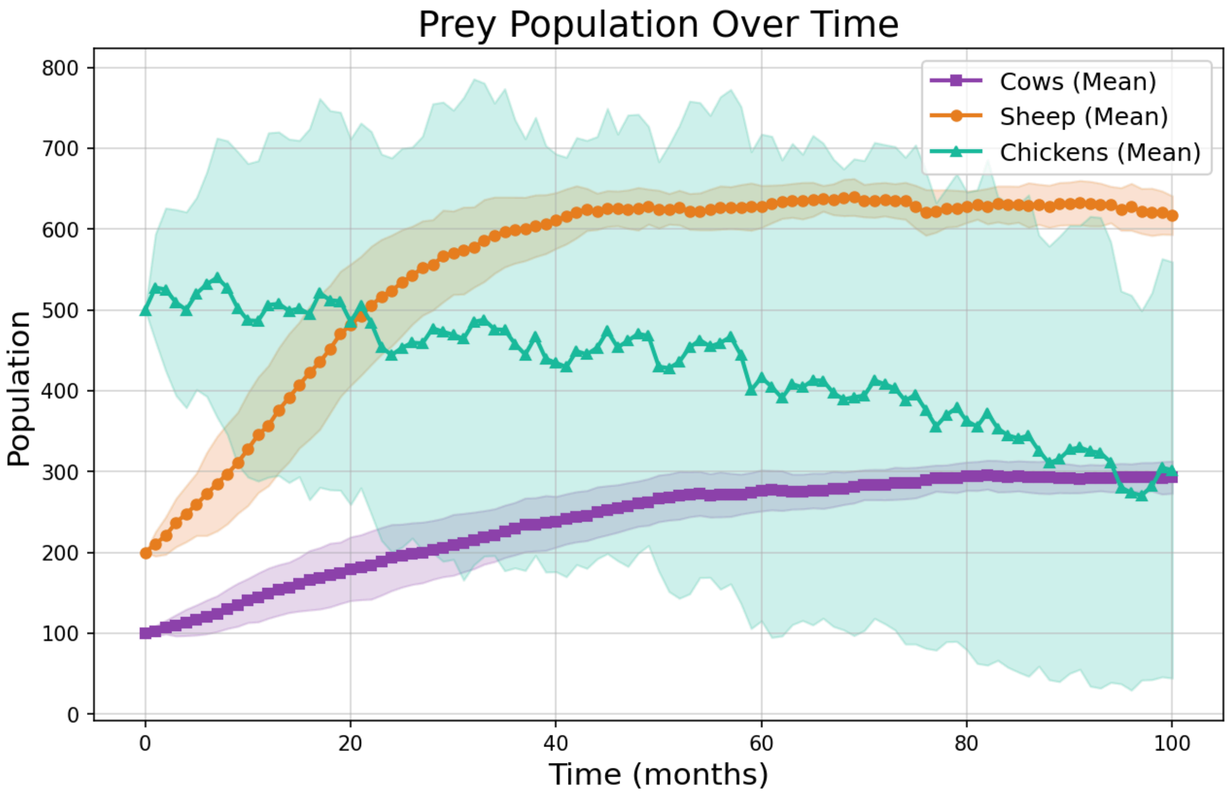
Stochastic Model

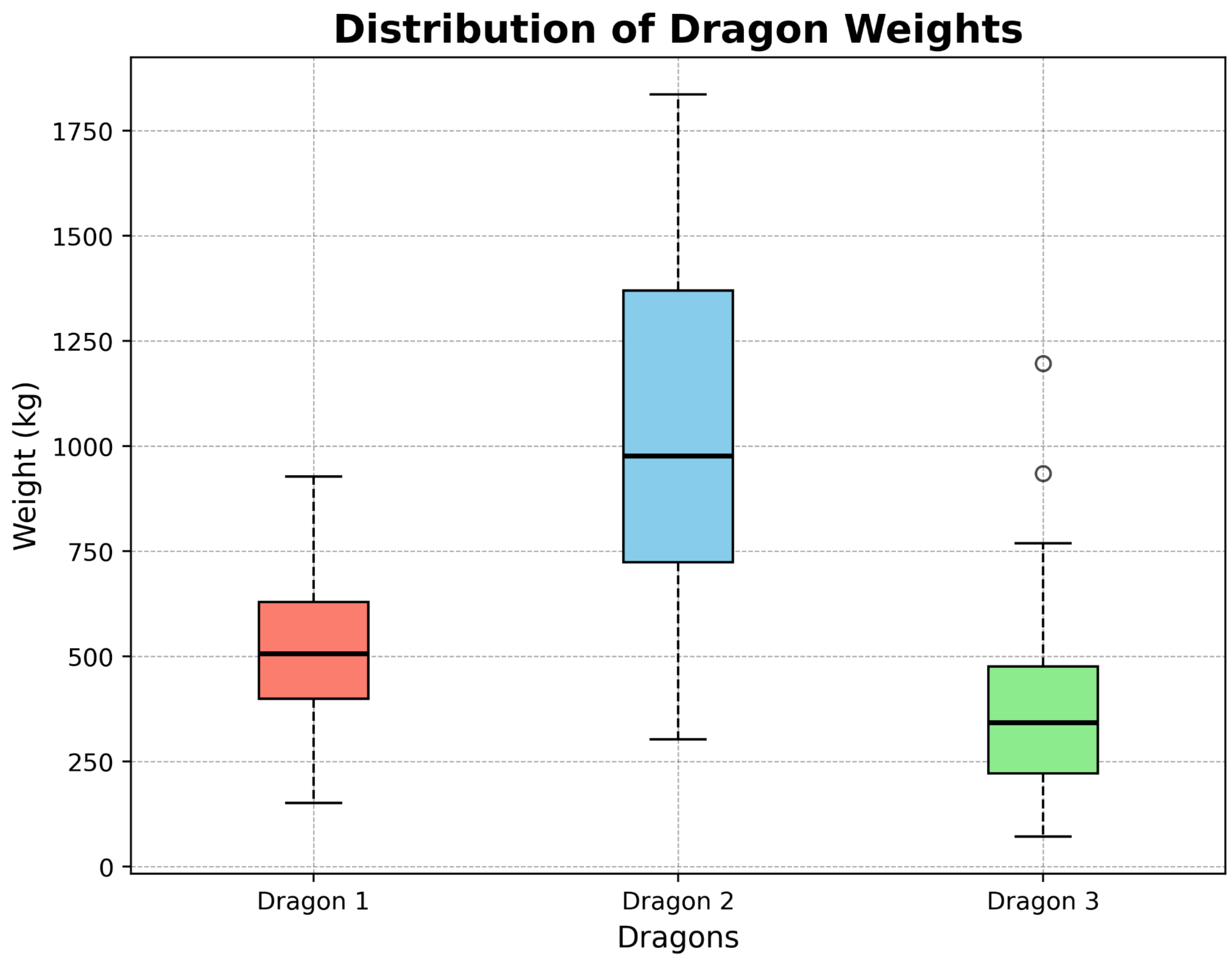


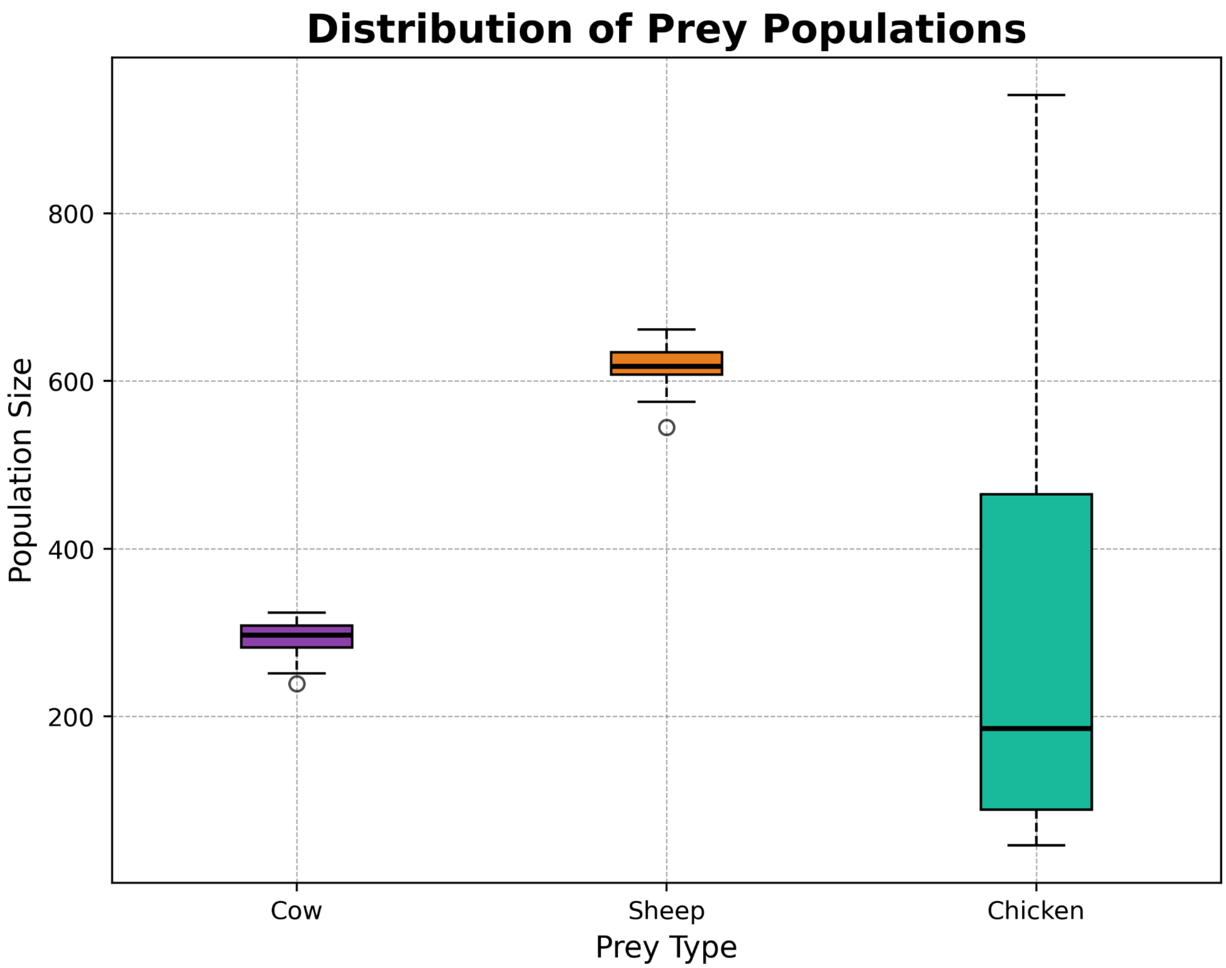

Dragon Weight Over Time
Dragon 1 (Mean)
Dragon 2 (Mean)
Dragon 3 (Mean)
Weight (kg)
Time (months)
0
200
400
600
800
1000
1200
1400
0
20
40
60
80
100



Prey Population Over Time
Cow
Sheep
Chicken
Population
Time (months)
0
200
400
600
800
1000
0
20
40
60
80
100


Prey Population Over Time
Cows (Mean)
Sheep (Mean)
Chickens (Mean)
Population
Time (months)
0
100
200
300
400
500
600
700
800
0
20
40
60
80
100


Distribution of Dragon Weights
Weight (kg)
1750
1500
1250
1000
750
500
250
0
Dragon 1
Dragon 2
Dragon 3
Dragons


Distribution of Prey Populations
Population Size
800
600
400
200
Cow
Sheep
Chicken
Prey Type

Conclusion

Three dragons and their prey populations—cows, sheep, and chickens—interacted in these simulations under ambiguous circumstances that were impacted by several environmental factors. Stochastic growth rates for the prey populations, surplus prey availability, thresholds that make hunting inefficient, and two catastrophic events—hunters targeting dragons and drought affecting chicken populations—were all included in this model as factors that affect the dragons' capacity to gain weight. These factors simulated a dynamic model in which populations of both predators and prey were subject to instability and fluctuation.

As for the deterministic model, the dragons reveal certain setbacks due to their hunting strategies, where each dragon targets a certain prey. This model highlights the dynamics of competition, the interplay between prey availability, and their carrying capacity. As Dragon 1 hunts cows only, they tend to face the challenge of the cows having the smallest carrying capacity ($K = 350$), ultimately allowing their population to stabilize at a lower level compared to the other prey species. Hunting cows frequently depletes the cow population quickly, driving it below the critical threshold ($T = 15$). From this, Dragon 1's weight begins to fall and without an abundance of cows, the dragon eventually dies. In contrast, Dragons 2 and 3, that target sheep and chickens, reach stable outcomes. Since sheep and chickens have higher carrying capacities ($K = 700$ for sheep and $K = 1000$ for chickens), their populations can recover more efficiently from hunting. With consistent hunting, Dragons 2 and 3 eventually reach their weight capacity.

The simulations reveal key observations about dragon survival and weight dynamics. Initially, the dragons all start at a low weight and attempt to rely on the prey populations to sustain themselves. The dragons' survival is closely tied to prey availability, and when prey populations exceed a certain threshold, dragons can hunt efficiently, leading to weight gain. However, as prey populations decline below this threshold, hunting efficiency diminishes, causing the dragons to lose weight and increasing their risk of starvation. This relationship underscores the delicate balance between predator and prey populations.

For prey populations, the three species exhibit stochastic logistic growth patterns influenced by environmental and catastrophic factors. The cows and sheep generally reach their carrying capacities in most simulations, indicating their populations can stabilize under typical conditions. In contrast, the chickens experience significant fluctuations due to the drought, which reduces their population and adds an element of unpredictability to the ecosystem. These catastrophic events highlight how random disturbances destabilize the environment, leading to cascading effects on the dragon populations. For example, when hunters enter the environment, the dragons are unable to maintain their hunting efficiency, losing weight and weakening their ability to sustain themselves in some trials.

Limitations of the Current Model

The current model simplifies the intricate dynamics between dragons and their prey by utilizing a logistic growth framework and focusing on two types of catastrophic events: drought and hunters. While this approach provides a foundation for understanding the interactions between predators and prey, it excludes other ecological factors that might influence population dynamics. For example, the absence of competitors that might also hunt the prey species limits the complexity of the ecosystem. Competitors would reduce prey abundance further, affecting the dragons' access to resources and causing shifts in their weight and survival rates. Additionally, the model does not account for interspecies interactions among prey, such as competition for resources, which could introduce further variability into the system.

Potential Improvements for Future Models

During the presentation, an additional suggestion was to increase the death threshold at which dragons die. To better understand the complexity of our model, we ran the simulation two more times after, increasing the death threshold to 50 and 100 kg respectively. Notably, increasing this parameter caused more dragons to die throughout the simulation. Furthermore, when we analyzed the trends of the average weights of each dragon, there was a noticeable decline in mortality for both Dragons 1 and 3 under these new conditions. In contrast, Dragon 2's mortality remained largely unchanged. The most noticeable difference in Dragon 2 was a reduction in weight variability, a smaller IQR spread, and a lower median weight at the end of the simulation. These changes all suggest that under our given assumptions and conditions, Dragon 2 is the best suited for our environment.

Future improvements of this model could enhance its real complexity by incorporating several additional features. First, emphasizing the relationship between prey and predators by adjusting prey growth rates dynamically based on hunting would create a more intertwined system. Seasonal variations could also be added to simulate changes in prey availability and behavior over a given period, showing real-world cycles that implement predator-prey dynamics.

Additionally, refining the parameters in the model to capture more detailed interactions would provide deeper impacts into the dynamics of the environment. For example, introducing specific metabolic rates for dragons or including energetic costs of hunting could reveal how different strategies affect survival.

Finally, incorporating broader stochastic events, such as disease outbreaks, would add complexity and reflect fluctuation among environmental strains. These events could influence not only prey populations but also the dragons' health and performance in hunting. After many trials and adjustments, the model has become better understood compared to the first simulation, but there is potential for further refinement to explore these complex interactions.

Bibliography

- Past problem from the Mathematical Contest in Modeling: https://www.mathmodels.org/Problems/2019/MCM-A/index.html
- HW3, HW5
- ChatGPT