Players A, B, C, and D are playing a dice game in which each player rolls a die until they roll a 5. What is the probability that at least 2 players tie for the most rolls attempted before rolling a 5?

Allow "$\rightarrow$" to denote a difference in attempted rolls between players. Allow groupings of letters to denote multiple players tying in attempted rolls (e.g. ABC indicates that it took players A, B, and C the same number of rolls to roll a 5).

Possible Outcomes:

Outcome 1: All players tie (1)

1. ABCD

Outcome 2: 1 succeeds, 3 tie (4)

1. A $\rightarrow$ BCD
2. B $\rightarrow$ ACD
3. C $\rightarrow$ ABD
4. D $\rightarrow$ ABC

Outcome 3: 2 tie, then 2 tie (6)

1. AB $\rightarrow$ CD
2. AC $\rightarrow$ BD
3. AD $\rightarrow$ BC
4. BC $\rightarrow$ AD
5. BD $\rightarrow$ AC
6. CD $\rightarrow$ AB

Outcome 4: 3 tie, then 1 succeeds (4)

1. ABC $\rightarrow$ D
2. ABD $\rightarrow$ C
3. ACD $\rightarrow$ B
4. BCD $\rightarrow$ A

Outcome 5: 1, then 1, then 2 tie (12)

1. A $\rightarrow$ B $\rightarrow$ CD
2. A $\rightarrow$ C $\rightarrow$ BD
3. A $\rightarrow$ D $\rightarrow$ BC
4. B $\rightarrow$ A $\rightarrow$ CD
5. B $\rightarrow$ C $\rightarrow$ AD
6. B $\rightarrow$ D $\rightarrow$ AC
7. C $\rightarrow$ A $\rightarrow$ BD
8. C $\rightarrow$ B $\rightarrow$ AD
9. C $\rightarrow$ D $\rightarrow$ AB
10. D $\rightarrow$ A $\rightarrow$ BC
11. D $\rightarrow$ B $\rightarrow$ AC
12. D $\rightarrow$ C $\rightarrow$ AB

Outcome 6: 1, then 1, then 1, then 1 (24)

1. A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ D
2. A $\rightarrow$ B $\rightarrow$ D $\rightarrow$ C
3. A $\rightarrow$ C $\rightarrow$ B $\rightarrow$ D
4. A $\rightarrow$ C $\rightarrow$ D $\rightarrow$ B
5. A $\rightarrow$ D $\rightarrow$ B $\rightarrow$ C
6. A $\rightarrow$ D $\rightarrow$ C $\rightarrow$ B
7. B $\rightarrow$ A $\rightarrow$ C $\rightarrow$ D
8. B $\rightarrow$ A $\rightarrow$ D $\rightarrow$ C
9. B $\rightarrow$ C $\rightarrow$ A $\rightarrow$ D
10. B $\rightarrow$ C $\rightarrow$ D $\rightarrow$ A
11. B $\rightarrow$ D $\rightarrow$ A $\rightarrow$ C
12. B $\rightarrow$ D $\rightarrow$ C $\rightarrow$ A
13. C $\rightarrow$ A $\rightarrow$ B $\rightarrow$ D
14. C $\rightarrow$ A $\rightarrow$ D $\rightarrow$ B
15. C $\rightarrow$ B $\rightarrow$ A $\rightarrow$ D
16. C $\rightarrow$ B $\rightarrow$ D $\rightarrow$ A
17. C $\rightarrow$ D $\rightarrow$ A $\rightarrow$ B
18. C $\rightarrow$ D $\rightarrow$ B $\rightarrow$ A
19. D $\rightarrow$ A $\rightarrow$ B $\rightarrow$ C
20. D $\rightarrow$ A $\rightarrow$ C $\rightarrow$ B
21. D $\rightarrow$ B $\rightarrow$ A $\rightarrow$ C
22. D $\rightarrow$ B $\rightarrow$ C $\rightarrow$ A
23. D $\rightarrow$ C $\rightarrow$ A $\rightarrow$ B
24. D $\rightarrow$ C $\rightarrow$ B $\rightarrow$ A

Outcome 7: 1, then 2, then 1 (12)

1. A $\rightarrow$ BC $\rightarrow$ D
2. A $\rightarrow$ BD $\rightarrow$ C
3. A $\rightarrow$ CD $\rightarrow$ B
4. B $\rightarrow$ AC $\rightarrow$ D
5. B $\rightarrow$ AD $\rightarrow$ C
6. B $\rightarrow$ CD $\rightarrow$ A
7. C $\rightarrow$ AB $\rightarrow$ D
8. C $\rightarrow$ AD $\rightarrow$ B
9. C $\rightarrow$ BD $\rightarrow$ A
10. D $\rightarrow$ AB $\rightarrow$ C
11. D $\rightarrow$ AC $\rightarrow$ B
12. D $\rightarrow$ BC $\rightarrow$ A

Outcome 8: 2, then 1, then 1 (12)

1. AB $\rightarrow$ C $\rightarrow$ D
2. AB $\rightarrow$ D $\rightarrow$ C
3. AC $\rightarrow$ B $\rightarrow$ D
4. AC $\rightarrow$ D $\rightarrow$ B
5. AD $\rightarrow$ B $\rightarrow$ C
6. AD $\rightarrow$ D $\rightarrow$ B
7. BC $\rightarrow$ A $\rightarrow$ D
8. BC $\rightarrow$ D $\rightarrow$ A
9. BD $\rightarrow$ A $\rightarrow$ C
10. BD $\rightarrow$ C $\rightarrow$ A
11. CD $\rightarrow$ A $\rightarrow$ B
12. CD $\rightarrow$ B $\rightarrow$ A

As shown, each grouping of scenarios (1, 2-5, 6-11, 12-15, 16-27, 28-51, 52-63, 64-75) represents each of the 8 possible outcomes of the game. There are 78 distinct scenarios within this game. The probabilities of each outcome will be calculated below.

Outcome 1: 1234

$$P = \sum_{n=1}^{\infty} \left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \right]^4$$

Outcome 2: $1 \rightarrow 234$

$$P = 4 * \sum_{n=1}^{\infty} \left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \left[\left(\frac{5}{6}\right)^{n} \right]^3 * \sum_{m=n+1}^{\infty} \left[\left(\frac{5}{6}\right)^{m-n-1} \left(\frac{1}{6}\right) \right]^3 \right]$$

Outcome 3: $12 \rightarrow 34$

$$P = 6 * \sum_{n=1}^{\infty} \left[\left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \right]^2 \left[\left(\frac{5}{6}\right)^{n} \right]^2 * \sum_{m=n+1}^{\infty} \left[\left(\frac{5}{6}\right)^{m-n-1} \left(\frac{1}{6}\right) \right]^2 \right]$$

Outcome 4: $123 \rightarrow 4$

$$P = 4 * \sum_{n=1}^{\infty} \left[\left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \right]^3 \left(\frac{5}{6}\right)^{n} * \sum_{m=n+1}^{\infty} \left[\left(\frac{5}{6}\right)^{m-n-1} \left(\frac{1}{6}\right) \right] \right]$$

Outcome 5: $1 \rightarrow 2 \rightarrow 34$

$$P = 12 * \sum_{n=1}^{\infty} \left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \left[\left(\frac{5}{6}\right)^{n} \right]^3 * \sum_{m=n+1}^{\infty} \left[\left(\frac{5}{6}\right)^{m-n-1} \left(\frac{1}{6}\right) \left[\left(\frac{5}{6}\right)^{m-n} \right]^2 \right.\right.$$
$$\left.\left. * \sum_{p=m+1}^{\infty} \left[\left(\frac{5}{6}\right)^{p-m-1} \left(\frac{1}{6}\right) \right]^2 \right] \right]$$

Outcome 6: $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$

$$P = 24 * \sum_{n=1}^{\infty} \left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \left[\left(\frac{5}{6}\right)^{n} \right]^3 * \sum_{m=n+1}^{\infty} \left[\left(\frac{5}{6}\right)^{m-n-1} \left(\frac{1}{6}\right) \left[\left(\frac{5}{6}\right)^{m-n} \right]^2 \right.\right.$$
$$\left.\left. * \sum_{p=m+1}^{\infty} \left[\left(\frac{5}{6}\right)^{p-m-1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^{p-m} * \sum_{q=p+1}^{\infty} \left(\frac{5}{6}\right)^{q-p-1} \left(\frac{1}{6}\right) \right] \right] \right]$$

Outcome 7: $1 \rightarrow 23 \rightarrow 4$

$$P = 12 * \sum_{n=1}^{\infty} \left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \left[\left(\frac{5}{6}\right)^{n} \right]^3 * \sum_{m=n+1}^{\infty} \left[\left(\left(\frac{5}{6}\right)^{m-n-1} \left(\frac{1}{6}\right) \right)^2 \left(\frac{5}{6}\right)^{m-n} \right.\right.$$
$$\left.\left. * \sum_{p=m+1}^{\infty} \left(\frac{5}{6}\right)^{p-m-1} \left(\frac{1}{6}\right) \right] \right]$$

Outcome 8: $12 \rightarrow 3 \rightarrow 4$

$$P = 12 * \sum_{n=1}^{\infty} \left[\left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \right]^2 \left[\left(\frac{5}{6}\right)^{n} \right]^2 * \sum_{m=n+1}^{\infty} \left[\left(\frac{5}{6}\right)^{m-n-1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^{m-n} \right.\right.$$
$$\left.\left. * \sum_{p=m+1}^{\infty} \left(\frac{5}{6}\right)^{p-m-1} \left(\frac{1}{6}\right) \right] \right]$$

Thus, the sum of outcomes 1-8 is equal to 1:

$$
\begin{gathered}
\sum_{n=1}^{\infty}\left[\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)\right]^{4} \\
+4*\sum_{n=1}^{\infty}\left[\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)\left[\left(\frac{5}{6}\right)^{n}\right]^{3}*\sum_{m=n+1}^{\infty}\left[\left(\frac{5}{6}\right)^{m-n-1}\left(\frac{1}{6}\right)\right]^{3}\right] \\
+6*\sum_{n=1}^{\infty}\left[\left[\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)\right]^{2}\left[\left(\frac{5}{6}\right)^{n}\right]^{2}*\sum_{m=n+1}^{\infty}\left[\left(\frac{5}{6}\right)^{m-n-1}\left(\frac{1}{6}\right)\right]^{2}\right] \\
+4*\sum_{n=1}^{\infty}\left[\left[\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)\right]^{3}\left(\frac{5}{6}\right)^{n}*\sum_{m=n+1}^{\infty}\left[\left(\frac{5}{6}\right)^{m-n-1}\left(\frac{1}{6}\right)\right]\right] \\
+12*\sum_{n=1}^{\infty}\Biggl[\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)\left[\left(\frac{5}{6}\right)^{n}\right]^{3}*\sum_{m=n+1}^{\infty}\Biggl[\left(\frac{5}{6}\right)^{m-n-1}\left(\frac{1}{6}\right)\left[\left(\frac{5}{6}\right)^{m-n}\right]^{2} \\
*\sum_{p=m+1}^{\infty}\left[\left(\frac{5}{6}\right)^{p-m-1}\left(\frac{1}{6}\right)\right]^{2}\Biggr]\Biggr] \\
+24*\sum_{n=1}^{\infty}\Biggl[\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)\left[\left(\frac{5}{6}\right)^{n}\right]^{3}*\sum_{m=n+1}^{\infty}\Biggl[\left(\frac{5}{6}\right)^{m-n-1}\left(\frac{1}{6}\right)\left[\left(\frac{5}{6}\right)^{m-n}\right]^{2} \\
\sum_{p=m+1}^{\infty}\left[\left(\frac{5}{6}\right)^{p-m-1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{p-m}\sum_{q=p+1}^{\infty}\left(\frac{5}{6}\right)^{q-p-1}\left(\frac{1}{6}\right)\right]\Biggr]\Biggr] \\
+12*\sum_{n=1}^{\infty}\Biggl[\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)\left[\left(\frac{5}{6}\right)^{n}\right]^{3}*\sum_{m=n+1}^{\infty}\Biggl[\left(\left(\frac{5}{6}\right)^{m-n-1}\left(\frac{1}{6}\right)\right)^{2}\left(\frac{5}{6}\right)^{m-n} \\
*\sum_{p=m+1}^{\infty}\left(\frac{5}{6}\right)^{p-m-1}\left(\frac{1}{6}\right)\Biggr]\Biggr] \\
+12*\sum_{n=1}^{\infty}\Biggl[\left[\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)\right]^{2}\left[\left(\frac{5}{6}\right)^{n}\right]^{2}*\sum_{m=n+1}^{\infty}\Biggl[\left(\frac{5}{6}\right)^{m-n-1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{m-n} \\
*\sum_{p=m+1}^{\infty}\left(\frac{5}{6}\right)^{p-m-1}\left(\frac{1}{6}\right)\Biggr]\Biggr]=1
\end{gathered}
$$

Therefore, the methods are proved and the probability of a tie can be calculated by summing the probabilities of the scenarios that end in more than one player (scenarios 1, 2, 3, and 5):

$$P = \sum_{n=1}^{\infty} \left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \right]^4$$

$$+4 * \sum_{n=1}^{\infty} \left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \left[\left(\frac{5}{6}\right)^{n} \right]^3 * \sum_{m=n+1}^{\infty} \left[\left(\frac{5}{6}\right)^{m-n-1} \left(\frac{1}{6}\right) \right]^3 \right]$$

$$+6 * \sum_{n=1}^{\infty} \left[\left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \right]^2 \left[\left(\frac{5}{6}\right)^{n} \right]^2 * \sum_{m=n+1}^{\infty} \left[\left(\frac{5}{6}\right)^{m-n-1} \left(\frac{1}{6}\right) \right]^2 \right]$$

$$+12 * \sum_{n=1}^{\infty} \left[\left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) \left[\left(\frac{5}{6}\right)^{n} \right]^3 * \sum_{m=n+1}^{\infty} \left[\left(\frac{5}{6}\right)^{m-n-1} \left(\frac{1}{6}\right) \left[\left(\frac{5}{6}\right)^{m-n} \right]^2 \right.\right.$$

$$\left.\left. * \sum_{p=m+1}^{\infty} \left[\left(\frac{5}{6}\right)^{p-m-1} \left(\frac{1}{6}\right) \right]^2 \right] \right] = \frac{114033471}{1894783891} \approx 6.0183\%$$