Notes on Finite State Machines and Regular Languages

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Abstract

As the major concerns of computer science are *computation* and the nature and variety of the *machines* that support it, in all their glorious but unwieldy variety, a particularly productive strategy in the study of computer science is to focus on special cases of computation (and machine) that exhibit two opposing qualities: they are special enough to make a detailed study feasible, yet they are general enough to provide useful insight into many broader questions. In the whole of computer science, there is no better example of a subject that meets these two requirements than the one involving Finite State Machines and Regular Languages.

These notes present the standard definitions, techniques, and results of the subject, occasionally in new ways, with a strong emphasis on inductive definitions and proofs. They are currently in draft form and contain parts in various stages of completion, from bare outlines to polished text, so please do not distribute them without permission of the author.

1 Introduction

Suppose we are given a string of characters, say `twilson@csufresno.edu`, and need to determine whether this string represents a valid email address. Or we are given a string and need to determine whether it represents a valid date, or zip code, or system log entry, or programming-language statement. These are all examples of *string recognition problems*, and these problems form the context of our study.

We will be studying simple machines that do nothing but read strings, a character at a time, from left to right, indicating after reading each character whether or not they *accept* (or *recognize*) the string read so far. Each such machine will have a fixed, finite amount of "memory" it can use to remember information about the characters it has read, or "reference" information it needs to use, to help it decide whether or not to accept the string it has read up to that point. It is not allowed to look ahead at the characters it has yet to read, nor look back at or re-read the characters it has already read. And it is completely *deterministic*, in that it always produces the same result when given the same string as input. Such machines will be called *Finite State Machines*, or FSMs for short.

If you'd like to have a picture in mind, consider an FSM to be a black box that has access to a *tape*, onto which is written, in distinct cells from left to right, the characters of an *input string.* The machine accesses this tape by means of a *read head*, which is always positioned either just to the left of one of the characters on the tape or at the end. The box also has a *light*, its only means of output, which is either on or off at any given time. Finally, the box has two buttons: a "reset" button that moves the read head back to the beginning of the tape and puts the machine in its *initial state*, which includes turning its light on or off; and a "read" button that causes the machine's read head to pass over and read the next character to its right (the only way the machine gets information about this character), and adjust the status of its light. The machine *accepts* a particular string when, after we write the string on the tape, push the "reset" button, and then push the "read" button for each character in the string (if any), the machine has its light on at the end of the process.

Note that, as I've described it, it makes sense to consider whether or not our machine accepts the *empty string*, the string with no characters and thus length 0: it does so if its light goes on when the machine is reset. It also makes sense to consider whether our machine accepts the various *prefixes* of the input string, since its light is on or off at the end of reading each prefix, and the rest of the characters have no influence on what the machine has done up to that point.

Although I've described the acceptance process of an FSM in physical terms, its input/output behavior, being deterministic, is completely characterized by the set of strings it accepts. Taking advantage of a linguistic metaphor, I will use the term *language* to mean any sets of strings; thus, every machine accepts a particular language. One important component of our study will be to understand the class of languages accepted by FSMs. We will call these languages *regular.*

Examples of the kinds of questions we would like to answer about FSMs and regular languages are:

- Given a set of strings, is it regular? In other words, is there an FSM that accepts exactly these strings?

- Are there other ways of representing regular languages besides using FSMs that might be more convenient for other purposes?

- What are the closure properties of regular languages? In other words, what operations, when performed on regular languages, always result in regular languages? (I'm thinking of such operations as union, intersection, and difference, as well as many more complicated ones we'll define later.) How can the machines associated with these languages be constructed?

- Are there extensions to the notion of FSM we can make that do not change the class of languages they accept? This would allow us to investigate "higher-level machines" that might be easier to construct or work with than FSMs but still accept the same languages, the same way that higher-level programming languages make it easier to write programs than assembly languages do, even though they don't allow us to write any new programs.
- How can we show that a particular language is *not* regular? It's one thing to show that a language is regular—you just need to construct a FSM that accepts it—but it's something altogether different to show that a language is not regular: you have to show somehow that, no matter how clever you are, you will never be able to construct an accepting FSM.
- How much of the theory of FSMs and regular languages can be implemented on a computer (for example in Haskell)?
- What are the applications of FSMs to other areas of computer science? In particular, how can we use FSMs and regular languages to solve string-recognition problems, which come up all over the place?
- What other directions are there that can be explored further?

2 Basic Definitions

We now begin the formal treatment of our subject. The initial goal is to capture the informal notions introduced in the previous section as precise, mathematical defintions, most of which are inductive, and to prove our results precisely, mostly by induction.

2.1 Characters and Strings

We start with a fixed, finite set Σ of *characters*, which we will use to build our strings. We call Σ the *alphabet*.

Definition 2.1 (Strings) We define the set Σ^* of *strings* over Σ by the following rules:

$$\frac{}{\varepsilon \in \Sigma^*}\ \textsc{Empty} \qquad \frac{a \in \Sigma \qquad w \in \Sigma^*}{aw \in \Sigma^*}\ \textsc{Cons}$$

Here, ε stands for the empty string, and aw is the result of adding the character a to the beginning of the string w to get a string whose length is one greater.

So, for example, if $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, then

$$\Sigma^* = \{\varepsilon, \mathtt{a}\varepsilon, \mathtt{b}\varepsilon, \mathtt{aa}\varepsilon, \mathtt{ab}\varepsilon, \mathtt{ba}\varepsilon, \mathtt{bb}\varepsilon, \mathtt{aaa}\varepsilon, \mathtt{aab}\varepsilon, \dots\}.$$

For economy of notation, I will often omit the final ε when writing a concrete string with particular characters, since we know that it is always there; thus the set of strings above becomes $\{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{aa}, \mathtt{ab}, \mathtt{ba}, \mathtt{bb}, \mathtt{aaa}, \mathtt{aab}, \dots\}$.

I will now define a few operations on strings by recursion and prove some properties of these operations by induction, both because we will need these later and to illustrate basic recursive definitions and inductive proofs.

DEFINITION 2.2 (Operations on strings) The length, $|w|$, of a string w is defined by recursion on w:

$$|\varepsilon| = 0, \qquad \text{(LenEmpty)}$$
$$|aw| = 1 + |w|; \qquad \text{(LenCons)}$$

The concatenation, $w \cdot u$, of two strings w and u is defined by recursion on w:

$$\varepsilon \cdot u = u, \qquad \text{(CatEmpty)}$$
$$aw \cdot u = a(w \cdot u). \qquad \text{(CatCons)}$$

The reversal, $\mathsf{rev}(w)$, of a string w is defined by recursion on w:

$$\mathsf{rev}(\varepsilon) = \varepsilon, \qquad \text{(RevEmpty)}$$
$$\mathsf{rev}(aw) = \mathsf{rev}(w) \cdot a\varepsilon. \qquad \text{(RevCons)}$$

So, for example, using my convention of omitting the final ε,

- $|\mathtt{abc}| = 3$
- $\mathtt{abc} \cdot \mathtt{def} = \mathtt{abcdef}$
- $\mathsf{rev}(\mathtt{abc}) = \mathtt{cba}$.

THEOREM 2.3 *(Length of concatenation) For all $w, u \in \Sigma^*$, $|w \cdot u| = |w| + |u|$.*

PROOF. By induction on w.

Case EMPTY: $w = \varepsilon$. For all $u \in \Sigma^*$, we have

$$\begin{aligned} |\varepsilon \cdot u| &= |u| && \text{(by CatEmpty)} \\ &= 0 + |u| \\ &= |\varepsilon| + |u| && \text{(by LenEmpty)} \end{aligned}$$

Case CONS: $w = aw'$, where $a \in \Sigma$ and $w' \in \Sigma^*$. For all $u \in \Sigma^*$, we have

$$\begin{aligned} |aw' \cdot u| &= |a(w' \cdot u)| && \text{(by CatCons)} \\ &= 1 + |w' \cdot u| && \text{(by LenCons)} \\ &= 1 + |w'| + |u| && \text{(by the IH)} \\ &= |aw'| + |u| && \text{(by LenCons)} \end{aligned}$$

⊣

THEOREM 2.4 *(Length of reversal) For all $w \in \Sigma^*$, $|\mathsf{rev}(w)| = |w|$.*

PROOF. By induction on w.

Case EMPTY: $w = \varepsilon$. Then, $|\mathsf{rev}(\varepsilon)| = |\varepsilon|$ by (REVEMPTY).

Case CONS: $w = aw'$, where $a \in \Sigma$ and $w' \in \Sigma^*$. Then,

$$\begin{aligned}
|\mathsf{rev}(aw')| &= |\mathsf{rev}(w') \cdot a\varepsilon| && \text{(by RevCons)}\\
&= |\mathsf{rev}(w')| + |a\varepsilon| && \text{(by Theorem 2.3)}\\
&= |w'| + 1 && \text{(by the IH and } |a\varepsilon| = 1)\\
&= |aw'| && \text{(by LenCons)}
\end{aligned}$$

⊣

THEOREM 2.5 *(Right identity and associativity of concatenation)*
For all $w, u, v \in \Sigma^*$,

1. $w \cdot \varepsilon = w$
2. $w \cdot (u \cdot v) = (w \cdot u) \cdot v$

PROOF. I'll prove these together by induction on w, although they could just as easily be proven individually the same way.

Case EMPTY: $w = \varepsilon$. Then $\varepsilon \cdot \varepsilon = \varepsilon$ by (CATEMPTY), and for every $u, v \in \Sigma^*$, we have $\varepsilon \cdot (u \cdot v) = u \cdot v = (\varepsilon \cdot u) \cdot v$ by two instances of (CATCONS).

Case CONS: $w = aw'$, where $a \in \Sigma$ and $w' \in \Sigma^*$. Then $(aw') \cdot \varepsilon = a(w' \cdot \varepsilon) = aw'$ by (CATCONS) and the IH, and for all $u, v \in \Sigma^*$, we have

$$\begin{aligned}
aw' \cdot (u \cdot v) &= a(w' \cdot (u \cdot v)) && \text{(by CatCons)}\\
&= a((w' \cdot u) \cdot v) && \text{(by the IH)}\\
&= a(w' \cdot u) \cdot v && \text{(by CatCons)}\\
&= (aw' \cdot u) \cdot v && \text{(by CatCons)}
\end{aligned}$$

⊣

THEOREM 2.6 *(Reverse of concatenation) For all $w, u \in \Sigma^*$,*

$$\mathsf{rev}(w \cdot u) = \mathsf{rev}(u) \cdot \mathsf{rev}(w).$$

PROOF. By induction on w.

Case EMPTY: $w = \varepsilon$. For all $u \in \Sigma^*$, we have

$$\begin{aligned}
\mathsf{rev}(\varepsilon \cdot u) &= \mathsf{rev}(u) && \text{(by CatEmpty)}\\
&= \mathsf{rev}(u) \cdot \varepsilon && \text{(by Theorem 2.5.1)}\\
&= \mathsf{rev}(u) \cdot \mathsf{rev}(\varepsilon) && \text{(by RevEmpty)}
\end{aligned}$$

Case Cons: $w = aw'$, where $a \in \Sigma$ and $w' \in \Sigma^*$. For all $u \in \Sigma^*$, we have

$$\begin{aligned}
\mathsf{rev}(aw' \cdot u) &= \mathsf{rev}(a(w' \cdot u)) && \text{(by CatCons)}\\
&= \mathsf{rev}(w' \cdot u) \cdot a\varepsilon && \text{(by RevCons)}\\
&= (\mathsf{rev}(u) \cdot \mathsf{rev}(w')) \cdot a\varepsilon && \text{(by the IH)}\\
&= \mathsf{rev}(u) \cdot (\mathsf{rev}(w') \cdot a\varepsilon) && \text{(by Theorem 2.5.2)}\\
&= \mathsf{rev}(u) \cdot \mathsf{rev}(aw') && \text{(by RevCons)}
\end{aligned}$$

$\dashv$

Exercise: Prove that $\mathsf{rev}(\mathsf{rev}(w)) = w$ for all $w \in \Sigma^*$.

2.2 Languages and Language Operations

As I mentioned in the introduction, I'll use the term *language* to mean any set of strings. Since Σ^* is the set of all strings (over our fixed alphabet Σ), saying that L is a language is the same as saying $L \subseteq \Sigma^*$, or equivalently $L \in \mathsf{Pow}(\Sigma^*)$. This makes Σ^* the maximum, or largest, language. On the other end of the spectrum, the empty set, $\emptyset$, is the minimum, or smallest, language. Next up in size are the languages that contain one string—that is, $\{w\}$, where $w \in \Sigma^*$—which also includes the language $\{\varepsilon\}$. Another example is the *language of letters*, $\Sigma\varepsilon = \{a\varepsilon \mid a \in \Sigma\}$, which are all the strings of length 1. These examples show that the size of the strings in a language (their lengths) has nothing to do with the size of the language itself (its cardinality as a set).

Basic set operations. Because languages are sets, all of the usual set operations can be performed on languages and result in languages:

- $L_1 \cup L_2 = \{w \mid w \in L_1 \vee w \in L_2\}$ (union),
- $L_1 \cap L_2 = \{w \mid w \in L_1 \wedge w \in L_2\}$ (intersection),
- $-L = \{w \mid w \in \Sigma^* \wedge w \notin L\}$ (complement),[1]
- $L_1 - L_2 = \{w \mid w \in L_1 \wedge w \notin L_2\}$ (set difference, or relative complement).

Given a *set* of languages, $\mathcal{L}$, we can also take the union and intersection of *all* the languages in $\mathcal{L}$:

- $\bigcup \mathcal{L} = \{w \mid \exists L\ L \in \mathcal{L} \wedge w \in L\}$,
- $\bigcap \mathcal{L} = \{w \mid \forall L\ L \in \mathcal{L} \wedge w \in L\}$.

These operations are functions $\bigcup, \bigcap : \mathsf{Pow}(\mathsf{Pow}(\Sigma^*)) \to \mathsf{Pow}(\Sigma^*)$. And note that binary union and intersection and the constants $\emptyset$ and Σ^* are just special cases: $L_1 \cup L_2 = \bigcup\{L_1, L_2\}$, $\emptyset = \bigcup\{\}$, $L_1 \cap L_2 = \bigcap\{L_1, L_2\}$, and $\Sigma^* = \bigcap\{\}$.

Finally, if $\phi(w)$ is any property of strings, then $\{w \mid \phi(w)\}$ is a language, so we can define languages such as $\{w \mid w \text{ is the password of an NSA executive}\}$ without even knowing what strings are included!

[1]Note: many authors write $\overline{L}$ instead of $-L$.

Concatenation. Because strings are not just atomic elements, but have an internal structure involving characters and sequencing, we can also define language operations in terms of this structure, starting with concatenation.

Definition 2.7 (Concatenation of languages) If $L_1, L_2 \subseteq \Sigma^*$, then we define the *concatenation* of L_1 and L_2 to be the language

$$L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}.$$

That is,

$$w \in L_1 \cdot L_2 \quad \text{iff} \quad \exists w_1\, \exists w_2\; w = w_1 \cdot w_2 \wedge w_1 \in L_1 \wedge w_2 \in L_2,$$

or, in other words, $w \in L_1 \cdot L_2$ precisely when we can split w up into two parts (in at least one but possibly more than one way), so that the first part is in L_1 and the second part is in L_2. And note that either part might be empty: the string `abc` can be split in four ways: (ε,`abc`), (`a`,`bc`), (`ab`,`c`), and (`abc`,ε).

The following theorem lists some properties of this operation:

Theorem 2.8 *(Properties of language concatenation)*

1. $\emptyset \cdot L = \emptyset$ *and* $L \cdot \emptyset = \emptyset$
2. $\{\varepsilon\} \cdot L = L$ *and* $L \cdot \{\varepsilon\} = L$
3. $L_1 \cdot (L_2 \cdot L_3) = (L_1 \cdot L_2) \cdot L_3$
4. $L_1 \subseteq L_2$ *implies* $L \cdot L_1 \subseteq L \cdot L_2$ *and* $L_1 \cdot L \subseteq L_2 \cdot L$
5. $L \cdot (L_1 \cup L_2) = (L \cdot L_1) \cup (L \cdot L_2)$ *and* $(L_1 \cup L_2) \cdot L = (L_1 \cdot L) \cup (L_2 \cdot L)$
6. $L \cdot (L_1 \cap L_2) \subseteq (L \cdot L_1) \cap (L \cdot L_2)$ *and* $(L_1 \cap L_2) \cdot L \subseteq (L_1 \cdot L) \cap (L_2 \cdot L)$, *but not necessarily the reverse.*

Proof. I'll prove the first part of (6); the proofs of the rest are left as exercises for the reader.

Let $L, L_1, L_2 \subseteq \Sigma^*$ be arbitrary, and assume $w \in L \cdot (L_1 \cap L_2)$. Then by Definition 2.7 and the comment afterward, $w = w_1 \cdot w_2$ for some $w_1, w_2 \in \Sigma^*$ with $w_1 \in L$ and $w_2 \in L_1 \cap L_2$. By the definition of intersection, this means $w_2 \in L_1$ and $w_2 \in L_2$. Therefore $w_1 \cdot w_2 \in L \cdot L_1$ and $w_1 \cdot w_2 \in L \cdot L_2$, showing that $w = w_1 \cdot w_2 \in (L \cdot L_1) \cap (L \cdot L_2)$.

Why isn't the reverse necessarily true? Intuitively, if $w \in (L \cdot L_1) \cap (L \cdot L_2)$, then $w \in L \cdot L_1$ and $w \in L \cdot L_2$, and so we can write $w = w' \cdot w_1$ with $w' \in L$ and $w_1 \in L_1$, and write $w = w'' \cdot w_2$ with $w'' \in L$ and $w_2 \in L_2$, but there is no guarantee that $w' = w''$ or $w_1 = w_2$, which we would need in order to claim $w \in L \cdot (L_1 \cap L_2)$. However, to really disprove this, we would need to find a *counterexample*, so here is one that realizes this intuitive idea: let $L = \{\texttt{a}, \texttt{aa}\}$,

$L_1 = \{\texttt{ab}\}$, and $L_2 = \{\texttt{b}\}$; then the string aab is in both $L \cdot L_1$ and $L \cdot L_2$ (as $\texttt{a} \cdot \texttt{ab}$ and $\texttt{aa} \cdot \texttt{b}$, respectively) but is not in $L \cdot (L_1 \cap L_2)$, since $L_1 \cap L_2 = \emptyset$. ⊣

Note that, according to parts 1 and 2 of this Theorem, $\emptyset$ is a *zero element* for concatenation, and $\{\varepsilon\}$ is a *unit element* for concatenation. That is, in the numerical analogy where concatenation is multiplication, $\emptyset$ acts like 0 and $\{\varepsilon\}$ acts like 1. For this reason, we will sometimes denote these languages **0** and **1**, respectively. Also, again like multiplication, the associativity of concatenation means that there is no ambiguity in writing $L_1 \cdot L_2 \cdot L_3$ for the concatenation of three languages, since either way of grouping the concatenations produces the same result. However, be careful with the numerical analogy: unlike multilplication, concatenation is not commutative: $L_1 \cdot L_2$ is not in general equal to $L_2 \cdot L_1$.

Kleene star. We now consider an operation of *iterated* concatenation, which, because of its importance for our subject, we investigate in detail.

Definition 2.9 (Kleene star) If $L \subseteq \Sigma^*$, then the *Kleene star* of L is the language L^* defined inductively by these rules:

$$\frac{}{\varepsilon \in L^*}\ {}^*\textsc{Empty} \qquad \frac{w \in L \qquad u \in L^*}{w \cdot u \in L^*}\ {}^*\textsc{Cat}$$

The essence of this inductive defintion is captured by the following theorem:

Theorem 2.10 *For any language L, we have $L^* = \{\varepsilon\} \cup L \cdot L^*$. Furthermore, if X is any language such that $X = \{\varepsilon\} \cup L \cdot X$, then $L^* \subseteq X$.*

Proof. For the first part, we show both $L^* \subseteq \{\varepsilon\} \cup L \cdot L^*$ and $\{\varepsilon\} \cup L \cdot L^* \subseteq L^*$.

First, suppose that $v \in L^*$. By inversion on the derivation of $v \in L^*$, we have two cases:

Case *Empty: $v = \varepsilon$. Then $v \in \{\varepsilon\}$ and thus $v \in \{\varepsilon\} \cup L \cdot L^*$.

Case *Cat: $v = w \cdot u$ where $w \in L$ and $u \in L^*$. Then $v \in L \cdot L^*$ by the definition of concatenation and thus $v \in \{\varepsilon\} \cup L \cdot L^*$.

In either case, $v \in \{\varepsilon\} \cup L \cdot L^*$, and so we've shown $L^* \subseteq \{\varepsilon\} \cup L \cdot L^*$.

In the other direction, suppose $v \in \{\varepsilon\} \cup L \cdot L^*$. Then either $v = \varepsilon$, in which case $v \in L^*$ by (*Empty) or $v = w \cdot u$ where $w \in L$ and $u \in L^*$, in which case $v \in L^*$ by (*Cat). It follows that $\{\varepsilon\} \cup L \cdot L^* \subseteq L^*$.

For the second part of the theorem, let X be arbitrary and assume that $X = \{\varepsilon\} \cup L \cdot X$. We prove by induction on $w \in L^*$ that $w \in X$.

Case *Empty: $w = \varepsilon$. Then $w \in \{\varepsilon\} \subseteq \{\varepsilon\} \cup L \cdot X = X$, and so $w \in X$.

Case *Cat: $w = w_1 \cdot w_2$ where $w_1 \in L$ and $w_2 \in L^*$. By the IH, $w_2 \in X$, and so $w = w_1 \cdot w_2 \in L \cdot X \subseteq \{\varepsilon\} \cup L \cdot X = X$, and so $w \in X$. ⊣

We can summarize this theorem by saying that L^* is the *least* solution to the language equation $X = \{\varepsilon\} \cup L \cdot X$.

Once we know that $L^* = \{\varepsilon\} \cup L \cdot L^*$, we can "unfold" this equation one time and use 2.8.5 and 2.8.2 to get

$$L^* = \{\varepsilon\} \cup L \cdot L^* = \{\varepsilon\} \cup L \cdot (\{\varepsilon\} \cup L \cdot L^*) = \{\varepsilon\} \cup L \cup L \cdot L \cdot L^*.$$

Continuing in this way, if we write $L^n = L \cdot L \cdots L$ (n times), i.e., if we define

$$\begin{aligned} L^0 &= \mathbf{1} \quad \text{(i.e., } \{\varepsilon\}\text{)} \\ L^{n+1} &= L \cdot L^n, \end{aligned}$$

then, for any $n \geq 0$, we have

$$L^* = L^0 \cup L^1 \cup L^2 \cup \cdots \cup L^n \cdot L^*,$$

as well as this infinite version:

$$L^* = L^0 \cup L^1 \cup L^2 \cup L^3 \cup \cdots = \bigcup_{i=0}^{\infty} L^i.$$

We won't use these characterizations of L^* in what follows, but you can try proving them as exercises.[2]

The following theorem lists some other properties of the Kleene star:

THEOREM 2.11 *(Properties of the Kleene star)*

1. $\emptyset^* = \{\varepsilon\}$ *(i.e.,* $\mathbf{0}^* = \mathbf{1}$*)*
2. $\{\varepsilon\}^* = \{\varepsilon\}$ *(i.e.,* $\mathbf{1}^* = \mathbf{1}$*)*
3. $(\Sigma\varepsilon)^* = \Sigma^*$, *justifying our use of the notation* Σ^* *for all strings on the alphabet* Σ
4. $L \subseteq L^*$
5. $L_1 \subseteq L_2$ *implies* $L_1^* \subseteq L_2^*$
6. $L^* \cdot L^* \subseteq L^*$, *i.e.,* L^* *is closed under concatenation*
7. $(L^*)^* \subseteq L^*$, *i.e.,* L^* *is closed under Kleene star*

Moreover, the subset relationships in (6) and (7) can be replaced by equalities.

PROOF. I'll prove (6) and leave the proofs of the rest as exercises.

By the observation following Definition 2.7 above, it will suffice to show that $w_1 \in L^*$ and $w_2 \in L^*$ imply $w_1 \cdot w_2 \in L^*$, which I'll do for an arbitrary $w_2 \in L^*$ by induction on the derivation of $w_1 \in L^*$.

[2] Many authors *define* the Kleene star using the last of these equations, but this is a logically much more complicated approach than ours, since it involves natural numbers, arbitrarly powers of a language, and an infinite union.

Case *Empty: $w_1 = \varepsilon$. Then $w_1 \cdot w_2 = w_2$ by (CatEmpty), which is in L^* by assumption.

Case *Cat: $w_1 = w_1' \cdot u$ where $w_1' \in L$ and $u \in L^*$. Then

$$w_1 \cdot w_2 = (w_1' \cdot u) \cdot w_2 = w_1' \cdot (u \cdot w_2)$$

by Theorem 2.5.2. But by the IH on $u \in L^*$, we have $u \cdot w_2 \in L^*$, and so $w_1' \cdot (u \cdot w_2) \in L^*$ by *Cat, and therefore $w_1' \cdot (u \cdot w_2) = (w_1' \cdot u) \cdot w_2 = w_1 \cdot w_2$ again by Theorem 2.5.2.

To justify the statement that the subset relationship in (6) can be replaced by an equality, I need to prove the converse inclusion: $L^* \subseteq L^* \cdot L^*$. But if $w \in L^*$, then since $\varepsilon \in L^*$ by *Empty, we have $w = \varepsilon \cdot w \in L^* \cdot L^*$. ⊣

The conditions $\varepsilon \in L^*$, $L \subseteq L^*$, and $L^* \cdot L^* \subseteq L^*$ from *Empty and Theorem 2.11.6–7 can be used to give an alternative inductive definition of L^*:

Theorem 2.12 *(Alternative definition of Kleene star) L^* can be equivalently defined by the rules*

$$\frac{}{\varepsilon \in L^*} \qquad \frac{w \in L}{w \in L^*} \qquad \frac{w \in L^* \qquad u \in L^*}{w \cdot u \in L^*}$$

i.e., it is the smallest set containing ε (as an element) and L (as a subset) and closed under concatenation.

Proof. Let's number these rules 1, 2, and 3, and for the purposes of this proof use L^{*1} and L^{*2} to stand for the sets defined by the rules in Definition 2.9 and the present theorem, respectively. I'll first show that $s \in L^{*1}$ implies $s \in L^{*2}$ for every $s \in \Sigma^*$ by induction on the derivation of $s \in L^{*1}$:

Case *Empty: $s = \varepsilon$. Then $s \in L^{*2}$ by rule 1.

Case *Cat: $s = w \cdot u$ where $w \in L$ and $u \in L^{*1}$. By the IH, $u \in L^{*2}$, and since $w \in L^{*2}$ by rule 2, we have $w \cdot u \in L^{*2}$ by rule 3.

Second, I'll show the converse, that $t \in L^{*2}$ implies $t \in L^{*1}$ for every $t \in \Sigma^*$, by induction on the derivation of $t \in L^{*2}$:

Case 1: $t = \varepsilon$. Then $t \in L^{*1}$ by *Empty.

Case 2: $t \in L$. Since $\varepsilon \in L^{*1}$ by *Empty and $t = t \cdot \varepsilon$ by Theorem 2.5, we have $t \in L^{*1}$ by *Cat.

Case 3: $t = w \cdot u$ where $w \in L^{*2}$ and $u \in L^{*2}$. By the IH, both $w \in L^{*1}$ and $u \in L^{*1}$, and so $t \in L^{*1}$ by Theorem 2.11.6. ⊣

Let me record one final way of defining the Kleene star operation, which makes it clear that L^* is the collection of all possible concatenations of elements of L. My notation is inspired by Haskell.

Given a set L, the set $[L]$ of *lists over* L can be defined inductively by the rules

$$\frac{}{[\,] \in [L]} \qquad \frac{w \in L \qquad \overline{w} \in [L]}{w : \overline{w} \in [L]},$$

where $\overline{w}$ is a variable standing for an arbitrary list. If L is a language, we can define an operation $\mathsf{concat} : [L] \to \Sigma^*$ by

$$\mathsf{concat}\ [\,] = \varepsilon \qquad (\textsc{ConcatEmpty})$$
$$\mathsf{concat}\ (w : \overline{w}) = w \cdot \mathsf{concat}\ \overline{w} \qquad (\textsc{ConcatCons})$$

Theorem 2.13 *(Yet another definition of Kleene star) L^* can be equivalently defined by the equation*

$$L^* = \{\mathsf{concat}\ \overline{w} \mid \overline{w} \in [L]\}.$$

Proof. Let's number the rules defining $[L]$ as 1 and 2. I'll first show by induction on the derivation of $s \in L^*$ that $s = \mathsf{concat}\ \overline{w}$ for some $\overline{w} \in [L]$. So, if $s = \varepsilon$, then we can take $\overline{w} = [\,]$, since $\overline{w} \in [L]$ by 1 and $\mathsf{concat}\ \overline{w} = \varepsilon$ by (ConcatEmpty). And if $s = w \cdot u$ where $w \in L$ and $u \in L^*$, then by the IH, $u = \mathsf{concat}\ \overline{w}'$ for some $\overline{w}' \in [L]$, so we can take $\overline{w} = w : \overline{w}'$, and get $\overline{w} \in [L]$ by 2 and $\mathsf{concat}\ \overline{w} = s$ by (ConcatCons).

In the reverse direction, I'll show by induction on the derivation of $\overline{w} \in [L]$ that $\mathsf{concat}\ \overline{w} \in L^*$. So, if $\overline{w} = [\,]$, then $\mathsf{concat}\ \overline{w} = \varepsilon$ by (ConcatEmpty), which is in L^* by (*Empty). And if $l = w : \overline{w}'$ where $w \in L$ and $\overline{w}' \in [L]$, then $\mathsf{concat}\ \overline{w} = w \cdot \mathsf{concat}\ \overline{w}'$ by (ConcatCons), and since $\mathsf{concat}\ \overline{w}' \in L^*$ by the IH, we have $\mathsf{concat}\ \overline{w} \in L^*$ by (*Cat). ⊣

We end our discussion of the Kleene star by listing several more of its properties, the proofs of which we leave as exercises:

- $L^* = \{\varepsilon\} \cup L^* \cdot L$
- $(L_1 \cup L_2)^* = (L_1^* \cdot L_2^*)^*$
- $(L_1 \cup L_2)^* = (L_1^* \cdot L_2)^* \cdot L_1^*$
- $L_1 \cdot (L_2 \cdot L_1)^* = (L_1 \cdot L_2)^* \cdot L_1$
- If $L_1 \cdot L_3 \subseteq L_3$ and $L_2 \subseteq L_3$, then $L_1^* \cdot L_2 \subseteq L_3$
- If $L_2 \subseteq L_3$ and $L_3 \cdot L_1 \subseteq L_3$, then $L_2 \cdot L_1^* \subseteq L_3$.

Reversal. In Section 2.1, I defined the reversal operation on strings. We can extend this to languages by applying it to each string in the language.

Definition 2.14 (Language Reversal) If $L \subseteq \Sigma^*$, then we define the *reversal* of L by

$$\mathsf{rev}(L) = \{\mathsf{rev}(w) \mid w \in L\}.$$

In other words, a string is in $\mathsf{rev}(L)$ iff it is the reverse of a string in L.

The following theorem lists some properties of this operation:

THEOREM 2.15 *(Properties of language reversal)*

1. *Reversal preserves all of the basic set operations (and constants):*
   (a) $\mathsf{rev}(-L) = -\mathsf{rev}(L)$,
   (b) $\mathsf{rev}(\bigcup\{L_i \mid i \in I\}) = \bigcup\{\mathsf{rev}(L_i) \mid i \in I\}$,
   (c) $\mathsf{rev}(\bigcap\{L_i \mid i \in I\}) = \bigcap\{\mathsf{rev}(L_i) \mid i \in I\}$.
2. $L_1 \subseteq L_2$ *implies* $\mathsf{rev}(L_1) \subseteq \mathsf{rev}(L_2)$
3. $\mathsf{rev}(L_1 \cdot L_2) = \mathsf{rev}(L_2) \cdot \mathsf{rev}(L_1)$
4. $\mathsf{rev}(L^*) = \mathsf{rev}(L)^*$
5. $\mathsf{rev}(\mathsf{rev}(L)) = L$.

PROOF. I'll prove 1(b) and leave the rest as exercises. Note that 1(b) includes, as special cases, $\mathsf{rev}(L_1 \cup L_2) = \mathsf{rev}(L_1) \cup \mathsf{rev}(L_2)$ and $\mathsf{rev}(\emptyset) = \emptyset$.

From left to right, suppose $w \in \mathsf{rev}(\bigcup\{L_i \mid i \in I\})$. Then by the definition of reversal, $w = \mathsf{rev}(w_r)$ where $w_r \in \bigcup\{L_i \mid i \in I\})$. By the definition of indexed union, that means that there exists some $i \in I$ such that $w_r \in L_i$, and so $w = \mathsf{rev}(w_r) \in \mathsf{rev}(L_i)$, again by the definition of reversal. Thus we have $w \in \bigcup\{\mathsf{rev}(L_i) \mid i \in I\}$ by the definition of indexed union, as required.

From right to left, suppose $w \in \bigcup\{\mathsf{rev}(L_i) \mid i \in I\}$. Then $w \in \mathsf{rev}(L_i)$ for some $i \in I$, and so $w = \mathsf{rev}(w_r)$ for some $w_r \in L_i$. Therefore, $w_r \in \bigcup\{L_i \mid i \in I\}$, and so $w = \mathsf{rev}(w_r) \in \mathsf{rev}(\bigcup\{L_i \mid i \in I\})$, as required. ⊣

Left and right quotient. The last two operations I want to consider are each a kind of inverse to concatenation. If we think of concatenation as analogous to multiplication, that would make these operations analogous to division.

DEFINITION 2.16 (Left and right quotient) If $L \subseteq \Sigma^*$ and $s \in \Sigma^*$, then we define the *left and right quotients* of L by s as follows:

$$\begin{aligned} s\backslash L &= \{w \mid s \cdot w \in L\} && \text{(left quotient)} \\ L/s &= \{w \mid w \cdot s \in L\} && \text{(right quotient)} \end{aligned}$$

So, $s\backslash L$ is the set of *suffixes* of strings in L that start with s, and L/s is the set of *prefixes* of strings in L that end in s. Or, put another way, $s\backslash L$ is the result of "cancelling" s from the beginning of all strings in L (and throwing away the ones that don't start with s), and L/s similarly cancels s from the end.

These operations can be extended to languages using indexed union. Let $S \subseteq \Sigma^*$. Then we define[3]

$$\begin{aligned} S\backslash L &= \bigcup\{s\backslash L \mid s \in S\} = \{w \mid \exists s \in S\ s \cdot w \in L\} \\ L/S &= \bigcup\{L/s \mid s \in S\} = \{w \mid \exists s \in S\ w \cdot s \in L\}. \end{aligned}$$

[3]Unfortunately, if you search around the Internet, you will find several authors who write $L\backslash S$ for what I (and Wikipedia) call $S\backslash L$. Caveat emptor!

The following theorem lists some properites of the left quotient operation (analogous properties hold for the right quotient):

Theorem 2.17 *(Properties of left quotient)*

1. $L_1 \subseteq s\backslash L_2$ *iff* $\{s\} \cdot L_1 \subseteq L_2$
2. $S\backslash \bigcup\{L_i \mid i \in I\} = \bigcup\{S\backslash L_i \mid i \in I\}$
3. $S\backslash(L_1 \cdot L_2) = \begin{cases} (S\backslash L_1) \cdot L_2 \cup (S\backslash L_2), & \textit{if } \varepsilon \in L_1, \textit{ and} \\ (S\backslash L_1) \cdot L_2, & \textit{otherwise.} \end{cases}$
4. $S\backslash L^* = (S\backslash L) \cdot L^*$
5. $\mathsf{rev}(S\backslash L) = \mathsf{rev}(L)/\mathsf{rev}(S)$

The proofs are left as exercises. Note again that Theorem 2.17.2 includes, as special cases, $S\backslash(L_1 \cup L_2) = (S\backslash L_1) \cup (S\backslash L_2)$ and $S\backslash\emptyset = \emptyset$. Also note that, if we apply the rev operation to both sides of Theorem 2.17.5 and then use Theorem 2.15.5, we get the identity $S\backslash L = \mathsf{rev}(\mathsf{rev}(L)/\mathsf{rev}(S))$, showing that we can define left quotient in terms of reversal and right quotient (and vice-versa). This identity can also be used to prove the analogous statements about right quotient.

The case-split in Theorem 2.17.3, and similar ones that we'll see later, are sometimes inconvenient, but we can get around them with the following device: if we use a logical statemet, ϕ, as if it were a language, then we will take it to refer to the language $\{\varepsilon \mid \phi\}$, which is $\mathbf{1}$ is ϕ is true and $\mathbf{0}$ if ϕ is false.[4] Using this convention, we can rewrite Theorem 2.15.3 as

$$S\backslash(L_1 \cdot L_2) = (S\backslash L_1) \cdot L_2 \cup (\varepsilon \in L_1) \cdot (S\backslash L_2).$$

Nonempty part. Finally, we will occasionally have to consider the result of removing the empty string from a language, which we formalize as an operation.

Definition 2.18 (Nonempty part) If $L \subseteq \Sigma^*$, then we define the *nonempty part* of L by

$$L^- = L - \{\varepsilon\}.$$

The following theorem lists some properties of this operation, which, interestingly, are very similar to the properties of the left quotient:

Theorem 2.19 *(Properties of nonempty part)*

1. $(L_1 \cup L_2)^- = L_1^- \cup L_2^-$
2. $(L_1 \cdot L_2)^- = L_1^- \cdot L_2 \cup (\varepsilon \in L_1) \cdot L_2^-$
3. $(L^*)^- = L^- \cdot L^*$

The proofs are left as exercises.

[4]This is not unlike the practice in programming languages like C where boolean values are taken to be the integers 0 and 1.

3 Regular languages and regular expressions

In the introduction, I defined regular languages as those accepted by FSMs. Here, I am going to take an indirect approach: I am going to give an inductive definition of what it means to be a "regular language" and then *prove*, in later sections, that these so-called regular languages are precisely the languages accepted by FSMs and thus are deserving of the name. The end point is the same, but this will allow me to do some preliminary study of regular languages before formally introducing FSMs and developing the tools necessary for the proof.

3.1 Regular languages

DEFINITION 3.1 (REG) We define the class REG of *regular languages* by the following rules:

$$\frac{}{\emptyset \in \mathsf{REG}}\ \textsc{RegEmpty} \qquad \frac{a \in \Sigma}{\{a\varepsilon\} \in \mathsf{REG}}\ \textsc{RegLetter}$$

$$\frac{L_1 \in \mathsf{REG} \qquad L_2 \in \mathsf{REG}}{L_1 \cup L_2 \in \mathsf{REG}}\ \textsc{RegUnion}$$

$$\frac{L_1 \in \mathsf{REG} \qquad L_2 \in \mathsf{REG}}{L_1 \cdot L_2 \in \mathsf{REG}}\ \textsc{RegCat} \qquad \frac{L \in \mathsf{REG}}{L^* \in \mathsf{REG}}\ \textsc{RegStar}$$

Note that these rules define a *set* of languages, i.e., $\mathsf{REG} \subseteq \mathsf{Pow}(\Sigma^*)$ or, equivalently, $\mathsf{REG} \in \mathsf{Pow}(\mathsf{Pow}(\Sigma^*))$. It is the smallest set of languages that contains the empty and single-letter languages and is closed under union, concatenation, and Kleene star.

Let's play around with this definition a bit to get a sense of what's here. As a start, we can prove that every finite language is regular:

THEOREM 3.2 *(Finite languages are regular) If $L \subseteq \Sigma^*$ is finite, then $L \in \mathsf{REG}$.*

The proof is to observe that, since the one-letter languages $\{a\varepsilon\}$ ($a \in \Sigma$) are all regular by (RegLetter) and regular languages are closed under concatenation by (RegCat), it follows that the *one-string* languages $\{w\}$ ($w \in \Sigma^*$) are also regular. And since every finite set is a union of one-element sets, we get that every finite language $F \subseteq \Sigma^*$ is regular.

Of course, regular languages can be infinite, too: if $\mathtt{a} \in \Sigma$, then by using (RegLetter) and (RegStar), we see that

$$\{\mathtt{a}\}^* = \{\varepsilon, \mathtt{a}, \mathtt{aa}, \mathtt{aaa}, \ldots\}$$

is regular. Finally, let's observe that, because of Theorem 3.2 and (RegUnion), any *finite modification* of a regular language is regular, where by finite modification I mean adding an arbitrary finite set of strings. (We'll see later that we can also *subtract* a finite set of strings and still have a regular language.)

3.2 Regular expressions

Regular expressions are just *names* of regular sets. Since, by Definition 3.1, regular sets are constructed using certain fixed operations, we can create a term language with constructors for each of these operations, and the terms will then correspond to the regular sets. Let's formalize this idea.

Let RE be the set of terms generated by this syntactic specification:

$$r \in \mathsf{RE} \quad ::= \quad 0 \mid \mathtt{a} \mid r_1 + r_2 \mid r_1 r_2 \mid r^* \qquad (\mathtt{a} \in \Sigma)$$

In writing such terms, it will be convenient to omit parentheses by declaring that the star operation has the highest precedence, followed by concatenation, and finally plus, and that both plus and concatenation associate to the right. Thus, the regular expression

$$\mathtt{b} + \mathtt{abc} + \mathtt{bc}^* \quad \text{is really} \quad (\mathtt{b} + (((\mathtt{a})((\mathtt{b})(\mathtt{c}))) + ((\mathtt{b})((\mathtt{c})^*)))),$$

when a pair of parentheses is added around the result of every constructor.

If r is a regular expression, let us write $[\![r]\!]$ for the regular set named by r, often called its *denotation*. This can be defined recursively as follows:

$$\begin{aligned} [\![0]\!] &= \emptyset \\ [\![\mathtt{a}]\!] &= \{\mathtt{a}\} \\ [\![r_1 + r_2]\!] &= [\![r_1]\!] \cup [\![r_2]\!] \\ [\![r_1 r_2]\!] &= [\![r_1]\!] \cdot [\![r_2]\!] \\ [\![r^*]\!] &= [\![r]\!]^*. \end{aligned}$$

Of course, because of the nature of the regular operators, many different regular expressions can denote the same regular set. For example, both $\mathtt{a}+\mathtt{b}$ and $\mathtt{b}+\mathtt{a}$ denote the regular set $\{\mathtt{a}, \mathtt{b}\}$; both $\mathtt{a}+(\mathtt{b}+\mathtt{c})$ and $(\mathtt{a}+\mathtt{b})+\mathtt{c}$ denote the regular set $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$; and both $\mathtt{a}+\mathtt{a}$ and $\mathtt{a}$ denote the regular set $\{\mathtt{a}\}$. Moreover, for any regular expression r, the regular expressions r^*, $(r^*)^*$, and $r^* r^*$, all denote the regular set $[\![r]\!]^*$, as a consequence of Theorem 2.11.

If w is a string and r is a regular expression, then we say that w *matches* r, and write $r \mathrel{?} w$, if $w \in [\![r]\!]$.

Examples. I'm going to give a few easier examples of regular expressions over the two-letter alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and then leave the harder ones as exercises for you!

1. The regular expression $r = (\mathtt{a}+\mathtt{b})^*$ matches all strings on Σ, i.e., every string of a's and b's, of any length, including the empty string.

2. Both regular expressions $r_1 = (\mathtt{a}+\mathtt{b})(\mathtt{a}+\mathtt{b})^*$ and $r_2 = (\mathtt{a}+\mathtt{b})^*(\mathtt{a}+\mathtt{b})$ match all non-empty strings, since the $(\mathtt{a}+\mathtt{b})$ term matches exactly one character.

3. All three regular expressions $r_1 = \mathtt{b}^*\mathtt{a}(\mathtt{a}+\mathtt{b})^*$, $r_2 = (\mathtt{a}+\mathtt{b})^*\mathtt{a}(\mathtt{a}+\mathtt{b})^*$ and $r_3 = (\mathtt{a}+\mathtt{b})^*\mathtt{ab}^*$ match all strings containing at least one `a`: r_1 singles out the first `a`, r_2 singles out any `a`, and r_3 singles out the last `a`.

4. The regular expression $r = \mathtt{a}^* + \mathtt{a}^*\mathtt{ba}^*$ matches all strings with at most one `b`: $\mathtt{a}^*$ matches strings with no `b`'s, and $\mathtt{a}^*\mathtt{ba}^*$ matches strings with exactly one `b`.

Exercises. Now, try your hand at constructing regular expressions for each of the following, progressively more difficult languages, still over the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$:

1. all strings in which every `a` is immediately followed by `bb`
2. all strings with an even number of `a`'s
3. all strings with at least one `a` and at least one `b`
4. all strings with no two adjacent letters the same
5. all strings with no instance of `bbb` (as a substring)
6. all strings with no instance of `aba`
7. all strings with every instance of `aa` coming before every instance of `bb`
8. all strings with an even number of `a`'s and an even number of `b`'s.

3.3 Recursive predicates and operations on REs

Many properties of, and operations on, the languages denoted by regular expressions can be computed recursively from the regular expressions themselves.

Emptiness. A regular expression r is *empty* if $[\![r]\!] = \mathbf{0}$. Let's denote this condition by $0(r)$; it can be defined as follows:

- $0(0)$ is true;
- $0(\mathtt{a})$ is false;
- $0(r_1 + r_2)$ iff $0(r_1)$ and $0(r_2)$;
- $0(r_1 r_2)$ iff $0(r_1)$ or $0(r_2)$;
- $0(r_1^*)$ is false.

Unitarity. A regular expression r is *unitary* if $[\![r]\!] = \mathbf{1}$. Let's denote this condition by $1(r)$; it can be defined as follows:

- $1(0)$ and $1(\mathsf{a})$ are false;
- $1(r_1 + r_2)$ iff either $1(r_1)$ and $0(r_2)$, or $0(r_1)$ and $1(r_2)$, or $1(r_1)$ and $1(r_2)$;
- $1(r_1 r_2)$ iff $1(r_1)$ and $1(r_2)$;
- $1(r_1^*)$ iff $0(r_1)$ or $1(r_1)$.

Bypassability. A regular expression r is *bypassable* if $\varepsilon \in [\![r]\!]$ (or, equivalently, $\mathbf{1} \subseteq [\![r]\!]$). This condition, denoted $b(r)$, can be defined as follows:

- $b(0)$ and $b(\mathsf{a})$ are false;
- $b(r_1 + r_2)$ iff $b(r_1)$ or $b(r_2)$;
- $b(r_1 r_2)$ iff $b(r_1)$ and $b(r_2)$;
- $b(r_1^*)$ is true.

Infiniteness. A regular expression r is *infinite* if $[\![r]\!]$ is an infinite set. This condition, denoted $\infty(r)$, can be defined as follows:

- $\infty(0)$ and $\infty(\mathsf{a})$ are false;
- $\infty(r_1 + r_2)$ iff $\infty(r_1)$ or $\infty(r_2)$;
- $\infty(r_1 r_2)$ iff $\infty(r_1)$ and not $0(r_2)$, or $\infty(r_2)$ and not $0(r_1)$;
- $\infty(r_1^*)$ iff not $0(r_1)$ and not $1(r_1)$.

Here are some examples of operations.

Reversal. By the results of Theorem 2.15, we can define the reversal of a regular expression by

- $\mathsf{rev}(0) = 0$;
- $\mathsf{rev}(\mathsf{a}) = \mathsf{a}$;
- $\mathsf{rev}(r_1 + r_2) = \mathsf{rev}(r_1) + \mathsf{rev}(r_2)$;
- $\mathsf{rev}(r_1 r_2) = \mathsf{rev}(r_2)\mathsf{rev}(r_1)$;
- $\mathsf{rev}(r_1^*) = \mathsf{rev}(r_1)^*$.

Left quotient. By the results of Theorem 2.17, we can define the left quotient of a regular expression by a letter $s \in \Sigma$ by

- $s\backslash 0 = 0$;
- $s\backslash \mathsf{a} = 1$ if $s = \mathsf{a}$, and 0 otherwise (here, $1 = 0^*$);
- $s\backslash(r_1 + r_2) = s\backslash r_1 + s\backslash r_2$;
- $s\backslash(r_1 r_2) = (s\backslash r_1)r_2 + s\backslash r_2$ if $b(r_1)$, and $(s\backslash r_1)r_2$ otherwise;
- $s\backslash(r_1^*) = (s\backslash r_1)r_1^*$.

Nonempty part. By the results of Theorem 2.19, we can define the nonempty part of a regular expression by

- $\mathsf{nep}(0) = 0$;
- $\mathsf{nep}(\mathsf{a}) = \mathsf{a}$;
- $\mathsf{nep}(r_1 + r_2) = \mathsf{nep}(r_1) + \mathsf{nep}(r_2)$;
- $\mathsf{nep}(r_1 r_2) = \mathsf{nep}(r_1)r_2 + \mathsf{nep}(r_2)$ if $b(r_1)$, and $\mathsf{nep}(r_1)r_2$ otherwise;
- $\mathsf{nep}(r_1^*) = \mathsf{nep}(r_1)r_1^*$.

I will again leave it to you to ponder the interesting similarity pointed out at the end of Section 2.2 between the definitions of left quotient and nonempty part, which is even more clearly visible here.

3.4 Recursive RE matching

The relation $r\ ?\ w$, defined in Section 3.2 as $w \in [\![r]\!]$, can also be defined by recursion on r (with a twist), as we show in this section in two different ways. But first, let me prove a sharper version of Theorem 2.10, which captured the inversion principle for L^*, namely that any $w \in L^*$ is either ε or splits as $w = u{\cdot}v$ with $u \in L$ and $v \in L^*$. The sharper version shows that we can insist that u be nonempty:

THEOREM 3.3 *For any language L, we have $L^* = \{\varepsilon\} \cup L^- \cdot L^*$.*

PROOF. Let $L^\# = \{\varepsilon\} \cup L^- \cdot L^*$. If $w \in L^\#$, then either $w = \varepsilon$, in which case $w \in L^*$ by (*EMPTY), or $w = u \cdot v$, where $u \in L^-$ and $v \in L^*$, in which case $u \in L$ and thus $w \in L^*$ by (*CAT), showing that $L^\# \subseteq L^*$.

Conversely, we prove by induction on $w \in L^*$ that $w \in L^\#$. Clearly $\varepsilon \in L^\#$, so suppose $w = u \cdot v$ with $u \in L$ and $v \in L^*$. By the IH, $v \in L^\#$. Now, if $u = \varepsilon$, then $w = v$ and so $w \in L^\#$. If $u \neq \varepsilon$, then $w = u \cdot v \in L^- \cdot L^*$, and so again $w \in L^\#$. $\dashv$

We can now give our first regular-expression matching "algorithm":

Definition 3.4 (Matching Algorithm 1) We define the relation $r \mathrel{?} w$ between a regular expression $r \in \mathsf{RE}$ and a string $w \in \Sigma^*$ inductively by these clauses:

- $0 \mathrel{?} w$ is false;
- $\mathsf{a} \mathrel{?} w$ iff $w = \mathsf{a}$;
- $r_1 + r_2 \mathrel{?} w$ iff $r_1 \mathrel{?} w$ or $r_2 \mathrel{?} w$;
- $r_1 r_2 \mathrel{?} w$ iff we can write $w = w_1 \cdot w_2$ such that $r_1 \mathrel{?} w_1$ and $r_2 \mathrel{?} w_2$;
- $r_1^* \mathrel{?} w$ iff either $w = \varepsilon$ or we can write $w = w_1 \cdot w_2$, where $w_1 \neq \varepsilon$, $r_1 \mathrel{?} w_1$, and $r_1^* \mathrel{?} w_2$.

The reason I put "algorithm" in scare quotes above is that it is not immediately clear that this inductively defined *relation* actually constitutes a recursive *algorithm*. The worry is in the last clause, where we are trying to define what it means to match against r_1^* by appealing recursively to a match against the same r_1^*, which appears circular: aren't we introducing a potential infinite loop if we turn this definition into a recursive function? Of course, all recursions are circular in some sense, but what makes a recursion successful is that we're only allowed to call the same function on *smaller* values of the recursive argument, and what saves the day in this case is that, although the *regular expression* doesn't get any smaller here, the *string* we are matching against does! This is the whole point of insisting that $w_1 \neq \varepsilon$, because then w_2 is a smaller string than w. And this insistence, in turn, is made possible by Theorem 3.3.

The upshot is that this algorithm is actually recursive on *both* input arguments, where we do the recursion first on the string argument and then on the regular expression argument: either the string gets smaller on a recursive call (as in the second call in the star case), or the string stays the same but the regular expression gets smaller (as in all other recursive calls).

Here is how we can organize these observations into a theorem and proof:

Theorem 3.5 *Let $w \in \Sigma^*$ and $r \in \mathsf{RE}$. Then $w \in [\![r]\!]$ iff $r \mathrel{?} w$.*

Proof. I will prove by strong induction that, for every natural number n,

$$\forall w \in \Sigma^* \; |w| = n \to \forall r \in \mathsf{RE} \; w \in [\![r]\!] \leftrightarrow r \mathrel{?} w. \qquad (*)$$

Since every $w \in \Sigma^*$ satisfies $|w| = n$ for some n, this will suffice to prove the theorem.

So, let k be a natural number and suppose (*) is true for all $n < k$. I have to show that (*) is also true for $n = k$. To that end, let $w \in \Sigma^*$ be such that $|w| = k$, and let me show for this w that $\forall r \in \mathsf{RE} \; w \in [\![r]\!] \leftrightarrow r \mathrel{?} w$. I will do this by a second, "inner" induction on r. We have one case for each kind of regular expression.

Case $r = 0$: in this case $[\![r]\!] = \emptyset$, and so $w \in [\![r]\!]$ is false, which is equivalent by definition to $0 \mathrel{?} w$.

Case $r = \mathtt{a}$: Then $[\![r]\!] = \{\mathtt{a}\}$, and so $w \in [\![r]\!]$ iff $w = \mathtt{a}$, which is equivalent by definition to $\mathtt{a} \mathbin{?} w$.

Case $r = r_1 + r_2$: Then $[\![r]\!] = [\![r_1]\!] \cup [\![r_2]\!]$, and so we have $w \in [\![r]\!]$ iff $w \in [\![r_1]\!] \vee w \in [\![r_2]\!]$, which, by the (inner) IH on r_1 and r_2, is equivalent to $r_1 \mathbin{?} w$ or $r_2 \mathbin{?} w$, which is further equivalent by definition to $r_1 + r_2 \mathbin{?} w$.

Case $r = r_1 r_2$: Then $[\![r]\!] = [\![r_1]\!] \cdot [\![r_2]\!]$, and so we have $w \in [\![r]\!]$ iff

$$\exists w_1\, \exists w_2\ \ w = w_1 \cdot w_2 \wedge w_1 \in [\![r_1]\!] \wedge w_2 \in [\![r_2]\!],$$

which, by the (inner) IH on r_1 and r_2, is equivalent to being able to write $w = w_1 \cdot w_2$ where $r_1 \mathbin{?} w_1$ and $r_2 \mathbin{?} w_2$, which is further equivalent by definition to $r_1 r_2 \mathbin{?} w$.

Case $r = r_1^*$: Then $[\![r]\!] = [\![r_1]\!]^*$, and so by Theorem 3.3, we have $w \in [\![r]\!]$ iff

$$w = \varepsilon\ \vee\ \exists u\, \exists v\ \ w = u \cdot v\ \wedge\ u \in [\![r_1]\!]^-\ \wedge\ v \in [\![r_1]\!]^*.$$

By the (inner) IH on r_1, the condition $u \in [\![r_1]\!]^-$ is equivalent to $r_1 \mathbin{?} u$ and $u \neq \varepsilon$, and if $u \neq \varepsilon$, then $|v| < k$, so by the (outer) IH on v, the condition $v \in [\![r_1]\!]^*$ is equivalent to $r_1^* \mathbin{?} v$, showing that the whole condition is equivalent by definition to $r_1^* \mathbin{?} w$. $\dashv$

The inductive definition of $r \mathbin{?} w$ above can be turned into a recursive algorithm in a straightforward way, where we implement the existential quantifiers over w_1 and w_2 by generating a list of the possible splits of w into w_1 and w_2 (with $w_1 \neq \varepsilon$ in the star case) and recursively testing the resulting matches. There is an inefficiency here, however, even if the splits and recursive matches are computed lazily. Suppose, for example, we are trying to check $\mathtt{a}r \mathbin{?} \mathtt{b}w$ for some $r \in \mathsf{RE}$ and $w \in \Sigma^*$. Then we will try all $|w| + 2$ splits of $\mathtt{b}w$ into w_1 and w_2, but in each case w_1 will fail to match $\mathtt{a}r$, since w_1 is either empty or begins with $\mathtt{b}$. We discover these failures fairly quickly, but we could replace $\mathtt{a}$ by something more complicated and for which failure would take longer, and then much work would be wasted.

This leads us to our second recursive definition of $r \mathbin{?} w$. The idea will be that, instead of checking $r_1 r_2 \mathbin{?} w$ by trying to match r_1 and r_2 against all splits of w, we will break this task up into a list of two tasks, written $[r_1, r_2] \mathbin{?} w$ and meaning "match r_1 against some initial part of w (determined by r_1) and then match r_2 against the rest." In this way, we let the regular expressions themselves determine which splits of w we check, and there is much less failing.

To turn this into an inductive definition (and ultimately a recursive algorithm), we have to make two adjustments to our previous definition. First, we will in general be matching a *list* of regular expressions, $\overline{r} \in [\mathsf{RE}]$, against a string, w (recall that lists and the associated operation concat we defined in Section 2.2). Second, we will need a way, when matching r_1^* by splitting the match into r_1 followed by r_1^*, to insure that r_1 does not match the empty string, and thus insure in turn that r_1^* is matched against a smaller string, preventing an infinite loop.

We will achieve both of these by defining relations $\overline{r} \mathbin{?_c} w$ $(c = 0, 1)$, where $c = 0$ means that if $\overline{r} = r : \overline{r}'$ then r is allowed to match ε, whereas $c = 1$ means that it is not—in other words, c is the minimal number of characters that r is allowed to match.

Definition 3.6 (Matching Algorithm 2) We define the relation $\overline{r} \mathrel{?_c} w$ between a list of regular expressions $\overline{r} \in [\mathsf{RE}]$, a string $w \in \Sigma^*$, and $c \in \{0,1\}$ inductively by these clauses:

- $[\,] \mathrel{?_c} w$ iff $w = \varepsilon$
- $0 : \overline{r} \mathrel{?_c} w$ is false
- $\mathtt{a} : \overline{r} \mathrel{?_c} w$ iff $w = \mathtt{a}w'$ and $\overline{r} \mathrel{?_0} w'$
- $r_1 + r_2 : \overline{r} \mathrel{?_c} w$ iff $r_1 : \overline{r} \mathrel{?_c} w$ or $r_2 : \overline{r} \mathrel{?_c} w$
- $r_1 r_2 : \overline{r} \mathrel{?_c} w$ iff $r_1 : r_2 : \overline{r} \mathrel{?_c} w$ or $c = 1$ and $b(r_1)$ and $r_2 : \overline{r} \mathrel{?_1} w$
- $r_1^* : \overline{r} \mathrel{?_c} w$ iff $c = 0$ and $\overline{r} \mathrel{?_0} w$ or $r_1 : r_1^* : \overline{r} \mathrel{?_1} w$.

We then define $r \mathrel{?} w$ as $[r : [\,]] \mathrel{?_0} w$.

Here, $b(r_1)$ means that r_1 is bypassable, as defined in Section 3.3, and, as usual, "and" has higher precedence than "or". Note that the definition of $?_0$ uses $?_1$ in the star case, and the definition of $?_1$ uses $?_0$ in the letter case, so the individual relations $?_0$ and $?_1$ are defined in terms of each other ("mutually").

As with the first definition of $r \mathrel{?} w$, this definition can also be turned into a recursive function in a straightforward way. Verifying the correctness and termination of this function, however, is a bit more involved. Let me first introduce some auxiliary notions. For any language L and $c \in \{0,1\}$, let me define $L^{[c]}$ by $L^{[0]} = L$ and $L^{[1]} = L^- = L - \{\varepsilon\}$. Using this notation, I will extend the notion of *denotation* to lists of regular expression by recursion as:

- $[\![[\,]]\!]_c = \{\varepsilon\} \quad (c = 0, 1)$;
- $[\![r : \overline{r}]\!]_c = [\![r]\!]^{[c]} \cdot [\![\overline{r}]\!]_0$.

Thus, for example,

$$[\![r_1 : r_2 : r_3 : [\,]]\!]_0 = [\![r_1]\!] \cdot [\![r_2]\!] \cdot [\![r_3]\!] \quad \text{and} \quad [\![r_1 : r_2 : r_3 : [\,]]\!]_1 = [\![r_1]\!]^- \cdot [\![r_2]\!] \cdot [\![r_3]\!],$$

where you will note that the subscript only affects the interpretation of the first regular expression in the list.

The key to showing that a recursive function terminates is to find a measure of the inputs that gets smaller on each recursive call. Often it's just a single argument that gets smaller in an obvious way, but in some cases, like this one, it can involve multiple arguments and be somewhat tricky to identify.

First, given a regular expression $r \in \mathsf{RE}$, I define its *size*, denoted $|r|$, by this recursion:

- $|0| = |\mathtt{a}| = 1$

- $|r_1 + r_2| = |r_1| + |r_2|$
- $|r_1 r_2| = |r_1| + |r_2| + 1$
- $|r_1^*| = |r_1| + 1$.

The key properties of size, which you can easily verify, are

- $|r| > 0$
- $|r_1 + r_2| > |r_1|$ and $|r_1 + r_2| > |r_2|$
- $|r_1 r_2| > |r_1| + |r_2|$
- $|r_1^*| > |r_1|$.

DEFINITION 3.7 The *size* of a call $\overline{r}\,?_c\,w$ in Matching Algorithm 2 is the number $|\overline{r}|_c$ defined by recursion as follows:

- $|[\,]|_c = 0 \quad (c \in \{0, 1\})$
- $|r : \overline{r}|_0 = |r| + |\overline{r}|_0$
- $|r : \overline{r}|_1 = |r|$

LEMMA 3.8 *For every recursive call in Matching Algorithm 2 where the string stays the same, the size of the call gets smaller in the sense of Definition 3.7.*

PROOF. I look individually at each recursive call that is made in Matching Algorithm 2. Since $|r| \geq 1$ for each regular expression r, I can, by Definition 3.7, reduce each case to checking an inequality on positive natural numbers $n_1 = |r_1|$, $n_2 = |r_2|$, and $m = |\overline{r}|_0$, the details of which I leave to you.

Case Letter. The recursive call in this case is made on a different (in fact smaller) string, so there is nothing to prove.

Case Union: $|r_1 + r_2 : \overline{r}|_c > |r_1 : \overline{r}|_c$ and $|r_1 + r_2 : \overline{r}|_c > |r_2 : \overline{r}|_c$. For the first, you need to check (when $c = 0$) that $n_1 + n_2 + m > n_1 + m$ and (when $c = 1$) that $n_1 + n_2 > n_1$. The second is similar.

Case Cat: $|r_1 r_2 : \overline{r}|_c > |r_1 : r_2 : \overline{r}|_c$ and $|r_1 r_2 : \overline{r}|_1 > |r_2 : \overline{r}|_1$. For the first, you need to check $n_1 + n_2 + 1 + m > n_1 + n_2 + m$ and $n_1 + n_2 + 1 > n_1$. For the second, you need to check $n_1 + n_2 + 1 > n_2$.

Case Star: $|r_1^* : \overline{r}|_0 > |\overline{r}|_0$ and $|r_1^* : \overline{r}|_c > |r_1 : r_1^* : \overline{r}|_1$. For the first, you need to check $n_1 + 1 + m > m$. For the second, you need to check $n_1 + 1 + m > n_1$ and $n_1 + 1 > n_1$. $\dashv$

I can now state and prove our correctness theorem:

THEOREM 3.9 *Let $w \in \Sigma^*$, $\overline{r} \in [\mathsf{RE}]$, and $c \in \{0, 1\}$. Then $w \in [\![\overline{r}]\!]_c$ iff $\overline{r}\,?_c\,w$.*

Proof. I will prove the statement

$$w \in [\![\overline{r}]\!]_c \leftrightarrow \overline{r} \mathrel{?_c} w \tag{*}$$

by an outer strong induction on $|w|$ and an inner strong induction on $|\overline{r}|_c$. So assume (*) is true for all strings $w \in \Sigma^*$ of length $n < k$ and all lists of regular expressions $\overline{r} \in [\mathsf{RE}]$ and $c \in \{0,1\}$, and consider a string w of length k. For this particular w, further assume that (*) is true for all lists of regular expressions $\overline{r} \in [\mathsf{RE}]$ and $c \in \{0,1\}$ such that $|\overline{r}|_c < s$ and consider $\overline{r} \in [\mathsf{RE}]$ and $c \in \{0,1\}$ such that $|\overline{r}|_c = s$. The rest of the proof will use the various parts of Theorem 2.8 without mention.

If $\overline{r} = [\,]$, then by the definitions, $w \in [\![\overline{r}]\!]_c$ iff $w = \varepsilon$ iff $\overline{r} \mathrel{?_c} w$, as required. So suppose $\overline{r} = r : \overline{r}'$ and proceed by cases on r.

Case $r = 0$. Then $[\![r]\!] = \emptyset$, and since $\emptyset^{[c]} = \emptyset$ for each $c \in \{0,1\}$, it follows that $[\![r : \overline{r}']\!]_c = \emptyset \cdot [\![\overline{r}']\!]_c = \emptyset$, so that $w \in [\![0 : \overline{r}']\!]_c$ and $0 : \overline{r}' \mathrel{?_c} w$ are both false and thus equivalent.

Case $r = \mathtt{a}$. Then $[\![r]\!] = \{\mathtt{a}\}$, and since $\{\mathtt{a}\}^{[c]} = \{\mathtt{a}\}$ for each $c \in \{0,1\}$, it follows that $[\![r : \overline{r}']\!]_c = \{\mathtt{a}\} \cdot [\![\overline{r}']\!]_0$, and so $w \in [\![\mathtt{a} : \overline{r}']\!]_c$ iff $w = \mathtt{a}w'$ for some $w' \in [\![\overline{r}']\!]_0$. But w' is a smaller string than w, so by the (outer) IH, $w' \in [\![\overline{r}']\!]_0$ is equivalent to $\overline{r}' \mathrel{?_0} w'$, and so the whole statement is equivalent to $\mathtt{a} : \overline{r}' \mathrel{?_c} w$.

Case $r = r_1 + r_2$. Then $[\![r]\!] = [\![r_1]\!] \cup [\![r_2]\!]$, and since $([\![r_1]\!] \cup [\![r_2]\!])^{[c]} = [\![r_1]\!]^{[c]} \cup [\![r_2]\!]^{[c]}$ for each $c \in \{0,1\}$ (the case $c = 0$ is trivial and the case $c = 1$ is Theorem 2.19.1), it follows that

$$\begin{aligned}
[\![r_1 + r_2 : \overline{r}']\!]_c &= [\![r_1 + r_2]\!]^{[c]} \cdot [\![\overline{r}']\!]_0 \\
&= ([\![r_1]\!] \cup [\![r_2]\!])^{[c]} \cdot [\![\overline{r}']\!]_0 \\
&= ([\![r_1]\!]^{[c]} \cup [\![r_2]\!]^{[c]}) \cdot [\![\overline{r}']\!]_0 \\
&= ([\![r_1]\!]^{[c]} \cdot [\![\overline{r}']\!]_0) \cup ([\![r_2]\!]^{[c]} \cdot [\![\overline{r}']\!]_0) \\
&= [\![r_1 : \overline{r}']\!]_c \cup [\![r_2 : \overline{r}']\!]_c,
\end{aligned}$$

and so we have $w \in [\![r_1 + r_2 : \overline{r}']\!]_c$ iff $w \in [\![r_1 : \overline{r}']\!]_c$ or $w \in [\![r_2 : \overline{r}']\!]_c$. By the Lemma, both $|r_1 : \overline{r}'|_c$ and $|r_2 : \overline{r}'|_c$ are smaller than $|r_1 + r_2 : \overline{r}'|_c$, and so, by the (inner) IH, this is further equivalent to $r_1 : \overline{r}' \mathrel{?_c} w$ or $r_2 : \overline{r}' \mathrel{?_c} w$, which is equivalent to $r_1 + r_2 : \overline{r}' \mathrel{?_c} w$.

Case $r = r_1 r_2$. Then $[\![r]\!] = [\![r_1]\!] \cdot [\![r_2]\!]$, and since

$$([\![r_1]\!] \cdot [\![r_2]\!])^{[c]} = [\![r_1]\!]^{[c]} \cdot [\![r_2]\!] \cup (c = 1) \cdot b(r_1) \cdot [\![r_2]\!]^{[c]}$$

for each $c \in \{0,1\}$ (in the case $c = 0$, the second term disappears and the result is trivial, and the case $c = 1$ is Theorem 2.19.2), it follows that

$$\begin{aligned}
[\![r_1 r_2 : \overline{r}']\!]_c &= [\![r_1 r_2]\!]^{[c]} \cdot [\![\overline{r}']\!]_0 \\
&= ([\![r_1]\!] \cdot [\![r_2]\!])^{[c]} \cdot [\![\overline{r}']\!]_0 \\
&= ([\![r_1]\!]^{[c]} \cdot [\![r_2]\!] \cup (c = 1) \cdot b(r_1) \cdot [\![r_2]\!]^{[c]}) \cdot [\![\overline{r}']\!]_0 \\
&= [\![r_1]\!]^{[c]} \cdot [\![r_2]\!] \cdot [\![\overline{r}']\!]_0 \cup (c = 1) \cdot b(r_1) \cdot [\![r_2]\!]^{[c]} \cdot [\![\overline{r}']\!]_0 \\
&= [\![r_1 : r_2 : \overline{r}']\!]_c \cup (c = 1) \cdot b(r_1) \cdot [\![r_2 : \overline{r}']\!]_1,
\end{aligned}$$

and so we have $w \in [\![r_1 r_2 : \overline{r}']\!]_c$ iff $w \in [\![r_1 : r_2 : \overline{r}']\!]_c$ or $c = 1$ and $b(r_1)$ and $w \in [\![r_2 : \overline{r}']\!]_1$. By the Lemma, $|r_1 : r_2 : \overline{r}'|_c$ is smaller than $|r_1 r_2 : \overline{r}'|_c$, and $|r_2 : \overline{r}'|_1$ is smaller than $|r_1 r_2 : \overline{r}'|_1$, and so, by the (inner) IH, this is further equivalent to $r_1 : r_2 : \overline{r}' \mathbin{?_c} w$ or $c = 1$ and $b(r_1)$ and $r_2 : \overline{r}' \mathbin{?_1} w$, which is equivalent to $r_1 r_2 : \overline{r}' \mathbin{?_c} w$.

Case $r = r_1^*$: Then $[\![r]\!] = [\![r_1]\!]^*$, and since $([\![r_1]\!]^*)^{[c]} = (c = 0) \cup [\![r_1]\!]^{[1]} \cdot [\![r_1]\!]^*$ for each $c \in \{0, 1\}$ (the case $c = 0$ is Theorem 3.3 and the case $c = 1$ is Theorem 2.19.3), it follows that

$$\begin{aligned}
[\![r_1^* : \overline{r}']\!]_c &= ([\![r_1]\!]^*)^{[c]} \cdot [\![\overline{r}']\!]_0 \\
&= ((c = 0) \cup [\![r_1]\!]^{[1]} \cdot [\![r_1]\!]^*) \cdot [\![\overline{r}']\!]_0 \\
&= (c = 0) \cdot [\![\overline{r}']\!]_0 \cup [\![r_1]\!]^{[1]} \cdot [\![r_1]\!]^* \cdot [\![\overline{r}']\!]_0 \\
&= (c = 0) \cdot [\![\overline{r}']\!]_0 \cup [\![r_1 : r_1^* : \overline{r}']\!]_1,
\end{aligned}$$

and so we have $w \in [\![r_1^* : \overline{r}']\!]_c$ iff $c = 0$ and $w \in [\![\overline{r}']\!]_0$ or $w \in [\![r_1 : r_1^* : \overline{r}']\!]_1$. By the Lemma, $|\overline{r}'|_0$ is smaller than $|r_1^* : \overline{r}'|_0$, and $|r_1 : r_1^* : \overline{r}'|_1$ is smaller than $|r_1^* : \overline{r}'|_c$, and so, by the (inner) IH, this is further equivalent to $c = 0$ and $\overline{r}' \mathbin{?_0} w$ or $r_1 : r_1^* : \overline{r}' \mathbin{?_1} w$, which is equivalent to $r_1^* : \overline{r}' : ?_c w$. ⊣

4 Finite State Machines

We now come to the problem of turning our previous informal description of an FSM into a formal definition. Recall from the Introduction that an FSM

- reads characters one at a time from left to right, indicating after each character whether it has accepted the string read so far,
- can only be in one of a fixed, finite number of states, and
- is deterministic (i.e., always starts in the same state and behaves the same way given the same input).

We can represent the states of an FSM as a finite set Q, and let $s \in Q$ be the state in which the machine starts (its *start state*). Since the machine is deterministic, and the only "memory" available to it is its set of states, all that matters in determining the machine's next state is the state that it is currently in and the character it just read. It follows that the machine's transition behavior is completely described by a function $\delta : Q \times \Sigma \to Q$, so that if the machine is in state $q \in Q$ and reads the character $x \in \Sigma$, it moves to the state $\delta(q, x)$.

That leaves the question of acceptance: how does the machine indicate it has accepted the string read so far? Its decision to accept a string or not must be completely determined by the state it's in after reading the last character, so that leaves us really only one choice: we designate some subset $F \subseteq Q$ of the states as *accepting (or final) states* and say that if the machine enters an accepting state then it has accepted the string it has read so far. With these preliminaries, we can now give the formal definition.

4.1 Definition of an FSM

Definition 4.1 (Finite State Machine) A *Finite State Machine* is a tuple (Q, s, F, δ), where

- Q is a finite set (the *states*)
- $s \in Q$ (the *start state*)
- $F \subseteq Q$ (the *final states*)
- $\delta : Q \times \Sigma \to Q$ (the *transition function*)

We denote by FSM the set of all finite state machines.

We want to have no limitation on what Q can be, other than that it is finite, so that we have maximum flexibility in constructing machines. A much more restrictive alternative would have been to replace Q with a single number N and call the states $\{0, 1, 2, \ldots, N-1\}$. Although this would work, and would standardize machines more, it would make it much harder to construct new machines from old ones.

4.2 Language Accepted by an FSM

Of course, the main purpose of an FSM is to accept a particular language, so we have to give a formal definition of the language accepted by an FSM $M = (Q, s, F, \delta)$ in terms of its components Q, s, F, and δ. We will denote this language by $L(M)$, which you can read as "the language of M." I'm going to give two such definitions, each of which proceeds by recursion on the input string, and then prove (by induction on the input string) that they are equivalent.

Definition 1. For this definition we extend the function $\delta : Q \times \Sigma \to Q$, which tells us what M does from each state on each input symbol, to a function $\delta^* : Q \times \Sigma^* \to Q$, which tells us what M does from each state on each input *string*. Since a string is just a sequence of input symbols, this can be achieved, simultaneously for each $q \in Q$, by recursion on the input string:

$$\begin{aligned} \delta^*(q, \varepsilon) &= q \\ \delta^*(q, aw) &= \delta^*(\delta(q, a), w). \end{aligned}$$

That is, after "reading" the empty string, we are still in the same state, and after reading aw, we are in the state we get to by first making the transition from q to $\delta(q, a)$ on a, and then reading the rest of the string w from there.

Now, using δ^* we can define $L(M)$:

$$w \in L(M) \quad \text{iff} \quad \delta^*(s, w) \in F.$$

In English: a string w is accepted by the FSM M if, when we start the machine in state s and read the string w, we end up in a final state.

Definition 2. For this defintion, we generalize $L(M)$ to $L_q(M)$, which is the language accepted by M when we start the machine in state q rather than s; we call this the *state language* of q. The judgment $w \in L_q(M)$ can be defined, simultaneously for every $q \in Q$, by these rules:

$$\frac{q \in F}{\varepsilon \in L_q(M)}\ \textsc{AcceptEmpty} \qquad \frac{w \in L_{\delta(q,a)}(M)}{aw \in L_q(M)}\ \textsc{AcceptCons}$$

In English: we accept the empty string starting in state q if q is already a final state, and we accept the string aw starting in state q if we accept the string w starting in state $\delta(q, a)$, which is where we are after making the transition from q on symbol a.

Using this judgment, we can give our second definition of $L(M)$:

$$w \in L(M) \quad \text{iff} \quad w \in L_s(M).$$

Before considering the equivalence of these two definitions, let me pause briefly to note a special feature of the inductive definition of $L_q(M)$ that is not shared by inductive definitions in general (for example, it is not true of the definition of L^*). Namely, for every $w \in \Sigma^*$, there is *at most one* derivation that $w \in L_q(M)$. That's because each string w satisfies either $w = \varepsilon$ or $w = aw'$, but not both, and in the latter case, $a \in \Sigma$ and $w' \in \Sigma^*$ are uniquely determined by w. Thus, the rule used to derive $w \in L_q(M)$ is also uniquely determined, and that continues to be true for the rest of the derivation tree. It follows that these rules are not just implications but *equivalences*:

Theorem 4.2 *Suppose $M = (Q, s, F, \delta)$ is a FSM and $L_q(M)$ is defined as above. Then,*

1. *$\varepsilon \in L_q(M)$ iff $q \in F$, for all $q \in Q$*
2. *$aw \in L_q(M)$ iff $w \in L_{\delta(q,a)}(M)$, for all $q \in Q$, $a \in \Sigma$, and $w \in \Sigma^*$.*

And note that the equivalence in condition 2 can be reformulated more simply using the left-quotient operation from the end of Section 3:

$$a \backslash L_q(M) = L_{\delta(q,a)}(M).$$

The equivalence of Definitions 1 and 2. Now, how do we prove the equivalence of our two definitions of language acceptance? Clearly, given the two ways we've defined $L(M)$, this will amount to showing

$$\delta^*(s, w) \in F \quad \text{iff} \quad w \in L_s(M),$$

for all w, and we would be tempted to prove this by induction on w, since both of the definitions proceeded by recursion or induction on w. However, this wouldn't work, because $\delta^*(s, aw)$ is not defined in terms of $\delta^*(s, w)$, but rather in terms of $\delta^*(\delta(s, a), w)$, and similarly, $aw \in L_s(M)$ is not defined in terms of

$w \in L_s(M)$, but rather in terms of $w \in L_{\delta(q,a)}(M)$. In short, the problem is that Definitions 1 and 2, although both involving a recursion on $w \in \Sigma^*$, are given *simultaneously* for each state $q \in Q$, since the definitions of $\delta^*(q, aw)$ and $aw \in L_q(M)$ potentially depend on $\delta^*(q', w)$ and $w \in L_{q'}(M)$ being defined for *all* $q' \in Q$, not just for q. It follows that we also have to prove the equivalence (and anything else about these definitions) simultaneously for all $q \in Q$; that is, we need is a stronger IH that quantifies over all states:

THEOREM 4.3 *Suppose $M = (Q, s, F, \delta)$ is a FSM, and δ^* and $L_q(M)$ are defined as above. Then for any string $w \in \Sigma^*$, we have*

$$\text{for all } q \in Q,\ \ \delta^*(q, w) \in F \text{ iff } w \in L_q(M).$$

PROOF. By induction on w (where we note that the quantification over states is now included in the IH).

Case EMPTY: $w = \varepsilon$. Let q be arbitrary. Then, by the definitions of δ^* and $L_q(M)$, we have $\delta^*(q, w) \in F$ iff $q \in F$ iff $w \in L_q(M)$, as required.

Case CONS: $w = aw'$, where $a \in \Sigma$ and $w' \in \Sigma^*$. Let q be arbitrary. Then,

$$\begin{aligned} \delta^*(q, aw') \in F \quad &\text{iff} \quad \delta^*(\delta(q, a), w') \in F && \text{(by def of } \delta^*) \\ &\text{iff} \quad w' \in L_{\delta(q,a)}(M) && \text{(by the IH on state } \delta(q, a)) \\ &\text{iff} \quad aw' \in L_q(M), && \text{(by def of } L_q(M)) \end{aligned}$$

as required (and note how the stronger IH lets us use the equivalence at any state, in this case $\delta(q, a)$, even though we have to prove it at q). ⊣

The equivalence of the two definitions of $L(M)$ now follows easily from this Theorem by choosing q to be s. The practical implication of this equivalence is that we are free to use either definition where it is convenient.

The construction of the language $L(M)$ from an FSM M determines a function $L : \mathsf{FSM} \to \mathsf{Pow}(\Sigma^*)$, but will see in Section 5 below that $L(M)$ is, in fact, regular for every $M \in \mathsf{FSM}$, meaning that our function is actually $L : \mathsf{FSM} \to \mathsf{REG}$.

4.3 Representing FSMs Visually

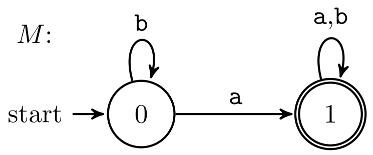
In Section 3, we represented (or named) regular languages using regular expressions, and we would like to do something similar with FSMs, but we quickly realize that the amount of information that goes into a description of a FSM is too much for a useful textual description. Fortunately, there is a nice graphical description that will do the job.

We will represent an FSM $M = (Q, s, F, \delta)$ graphically by drawing a circle for each state; drawing an arrow connecting two circles, labeled with a list (or set) of input symbols, if there are transitions from the source state to the destination state for each of input symbols in the label; drawing an arrow from the word "start" into the initial state; and making the final state(s) double circles.

For example, consider the FSM over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ with

- $Q = \{0, 1\}$
- $s = 0$
- $F = \{1\}$
- $\delta = \{(0, \mathtt{a}, 1), (0, \mathtt{b}, 0), (1, \mathtt{a}, 1), (1, \mathtt{b}, 1)\}$,

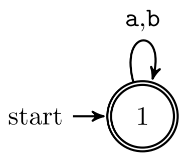
where I've given δ by its "graph;" thus, $\delta(0, \mathtt{a}) = 1$, $\delta(0, \mathtt{b}) = 0$, $\delta(1, \mathtt{a}) = 1$, and $\delta(1, \mathtt{b}) = 1$. This FSM can be represented graphically as follows:



4.4 Examples and Exercises

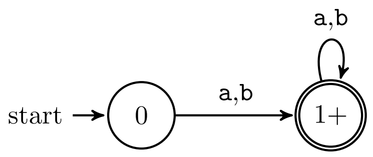
We will now revisit the examples and exercises from Section 3, this time constructing FSMs that accept the given languages. As before, I will do the easier ones and leave the more difficult ones for you! (Part of the fun of learning how to design FSMs is figuring out the tricks and techniques yourself, but I'll give you a few hints to get you started.) Again, we are keeping to a two-letter alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

1. The language consisting of all strings can be recognized by a one-state FSM:

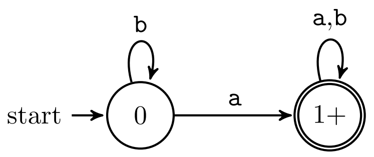
   Note, by the way, that if the single state in this machine were not a final state, then the machine would accept the empty language—consider that a bonus example!

2. The language consisting of all nonempty strings can be recognized by a machine that adds a new initial state to the previous machine:

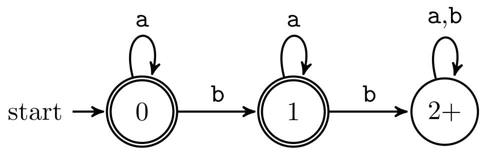
   Here, the names I have chosen for the states are suggestive: 0 is the state where we have (so far) read 0 characters, and 1+ is the state where we have read 1 or more characters.

3. The language consisting of all strings with at least one `a` is recognized by the example machine from the previous subsection, which I'll repeat here for convenience (with a new name for state 1):



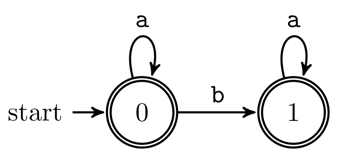
State 0 of this machine is the state where we have read 0 `a`'s, and state 1+ is the state where we have read one or more `a`'s; note that reading a `b` doesn't change the number of `a`'s we have read, so these transitions just loop back to the same state.

4. Finally, the language consisting of all strings with at most one `b` can be recognized by this three-state FSM:



Here, state 0 is the state where we have read 0 `b`'s, state 1 is the state where we have read 1 `b`, and state 2+ is the state where we have read at least 2 `b`'s. Both 0 and 1 are final states, because we want to accept strings that have zero `b`'s or exactly one `b`. However, once we read a second `b`, we will no longer accept the string, whatever the remaining characters may be, and so 2+ is a state that is not final but to which all transitions lead back. Such a state is called a "trap" state, because once you "fall" into it, you can't get back out.

Trap states are both useful and common, and so we will adopt a convention when drawing machines with trap states that will keep our diagrams simpler: if, in an FSM diagram, there is at least one state and input symbol for which no transition is indicated, then the machine is assumed to have a single trap state, and every missing transition is taken to be a transition to that state. Thus, we could simplify the previous machine as follows:



Now, with these examples in hand, go back to the exercises in Section 3 and construct FSMs for each of those languages. In some cases, you will be able to use the trap-state convention to keep your machines simpler.

4.5 Proving FSM Language Acceptance

If you tried the exercises from the previous section, you may have found yourself wondering: how do I know whether my machine constructions are correct? (And if you haven't yet tried the exercises, then what are you doing reading this? Get back and try those exercises!) Or maybe you have a machine M and have made a guess at what $L(M)$ is, but need a way to verify your guess.

The solution lies is Theorem 4.2, which showed that the languages $L_q(M)$ satisfy two conditions involving the set of final states of M and the various left quotients with respect to the letters $a \in \Sigma$. For it turns out that the languages $L_q(M)$ are the *only* languages satisyfing these conditions:

THEOREM 4.4 *Suppose $M = (Q, s, F, \delta)$ is a FSM. If $\{L_q\}_{q\in Q}$ is a collection of languages, indexed by Q, that satisfies the two conditions*

1. *$\varepsilon \in L_q$ iff $q \in F$, for all $q \in Q$*
2. *$a\backslash L_q = L_{\delta(q,a)}$, for all $q \in Q$ and $a \in \Sigma$,*

then $L_q = L_q(M)$ for all $q \in Q$.

PROOF. Assume that the two conditions above hold. We will prove the following statement for all $w \in \Sigma^*$, by induction on w:

$$\text{for all } q \in Q,\ w \in L_q \text{ iff } w \in L_q(M)$$

Case EMPTY: $w = \varepsilon$. Let q be arbitrary. Then, by condition 1 and 4.2.1,

$$\varepsilon \in L_q \text{ iff } q \in F \text{ iff } q \in L_q(M).$$

Case CONS: $w = aw'$, where $a \in \Sigma$ and $w' \in \Sigma^*$. Let q be arbitrary. Then, for any $a \in \Sigma$, we have $a\backslash L_q = L_{\delta(q,a)} = L_{\delta(q,a)}(M) = a\backslash L_q(M)$ by condition 2, the IH, and 4.2.2, respectively. Therefore, by the definition of left quotient,

$$aw' \in L_q \text{ iff } w' \in a\backslash L_q \text{ iff } w' \in a\backslash L_q(M) \text{ iff } aw' \in L_q(M),$$

as required. $\dashv$

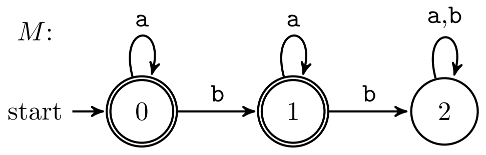
This theorem allows us to formulate a method (or "proof technique") for showing that $L(M) = L$ for an FSM $M = (Q, s, F, \delta)$ and language L:

- First, define a collection of languages, $\{L_q\}_{q\in Q}$, one for each state of M, where L_q is the proposed value of $L_q(M)$. It should be that $L_s = L$.
- Show that $\varepsilon \in L_q$ iff $q \in F$.
- Show that $\delta(q, a) = q'$ implies $a\backslash L_q = L_{q'}$.

In the case where the languages $\{L_q\}_{q\in Q}$ are given as regular expressions $\{r_q\}_{q\in Q}$ where $[\![r_q]\!] = L_q$ for each $q \in Q$, the method reduces to checking bypassability and calculating left quotients of regular expressions (see Section 3.3):

- Show that $b(r_q)$ iff $q \in F$
- Show that $a \backslash r_q = r_{q'}$ for all $a \in \Sigma$, where the equality here means that both regular expressions denote the same language.

Let me illustrate this method by applying it to the machine from the previous section that accepted all strings on $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ with at most one $\mathtt{b}$, which we had identified in Example 4 of Section 3.2 as being the language denoted by the regular expression $\mathtt{a}^* + \mathtt{a}^*\mathtt{b}\mathtt{a}^*$:



For the first step of the method, we define $r_0 = \mathtt{a}^* + \mathtt{a}^*\mathtt{b}\mathtt{a}^*$, $r_1 = \mathtt{a}^*$, and $r_2 = 0$; these are our "guesses" at what the languages $L_0(M)$, $L_1(M)$, and $L_2(M)$ will be, where r_0 is the language we want to show the machine accepts, since 0 is the start state.

For the second step, we can use the definition of bypassability to check that $b(r_0)$ and $b(r_1)$ are true but $b(r_2)$ is false, showing that $b(r_q)$ iff $q \in F$.

For the third step, we have to check the transitions of the machine to make sure that we get the appropriate left-quotient equations, namely,

- $\mathtt{a} \backslash r_0 = r_0$ and $\mathtt{b} \backslash r_0 = r_1$
- $\mathtt{a} \backslash r_1 = r_1$ and $\mathtt{b} \backslash r_1 = r_2$
- $\mathtt{a} \backslash r_2 = r_2$ and $\mathtt{b} \backslash r_2 = r_2$.

These are straightforward to check using the definition of left quotient of regular expressions.

It follows that our guesses were correct.

5 Regular Languages are Accepted by FSMs

The goal of this section is to prove one half of the equivalence of FSMs and regular languages:

Theorem 5.1 *For every regular language R, there exists a finite state machine M such that $R = L(M)$.*

Our strategy is simple, even if the individual steps are more complicated. The class of regular languages has an inductive definition (3.1), according to which the empty language and, for every $a \in \Sigma$, the single-letter language, $\{a\}$, are regular; and if L_1 and L_2 are regular, then $L_1 \cup L_2$, $L_1 \cdot L_2$, and L_1^* are regular. Moreover, we have a system of names for these regular languages, which we called regular expressions (the set of which was denoted RE), with a constructor for each of these base languages and regular operators; if $r \in \mathsf{RE}$, then $[\![r]\!] \in \mathsf{REG}$ was the regular language named by r. Therefore, to show that every regular language is accepted by some FSM, it will be enough to

- construct a machine $M(0)$ that accepts the (empty) language, $\emptyset$;
- construct, for every $a \in \Sigma$, a machine $M(a)$ that accepts the (single-letter) language, $\{a\}$;
- given machines M_1 and M_2 that accept the languages L_1 and L_2, construct a machine $M_1 \cup M_2$ that accepts the language $L_1 \cup L_2$;
- given machines M_1 and M_2 that accept the languages L_1 and L_2, construct a machine $M_1 \cdot M_2$ that accepts the language $L_1 \cdot L_2$; and
- given a machine M that accepts the language L, construct a machine M^* that accepts the language L^*.

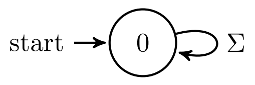
This collection of constructions can then be used to give a recursive definition of a function $M : \mathsf{RE} \to \mathsf{FSM}$ that satisfies the condition $\forall r \in \mathsf{RE}\ L(M(r))) = [\![r]\!]$. Since every regular language, R, satisfies $[\![r]\!] = R$ for its name, r, the machine $M(r)$ will be therefore be the required machine recognizing R.

Each one of the above bulleted constructions is the subject of its own subsection below.

5.1 Empty

I've already given a machine accepting the empty language—it was the "bonus" machine from Example 4.4.1: let $M(0)$ be the machine, pictured below, with

- $Q = \{0\}$
- $s = 0$
- $F = \emptyset$
- $\delta(0, a) = 0$ for all $a \in \Sigma$.



Let me record the obvious:

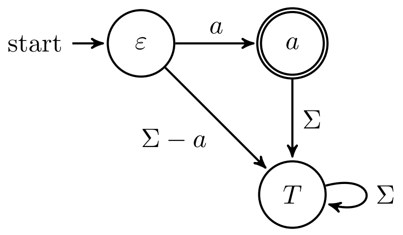
LEMMA 5.2 $L(M(0)) = \emptyset$.

PROOF. By the first definition of $L(M)$, $w \in L(M(0))$ iff $\delta^*(0, w) \in \emptyset$. But the latter is false for every w, and so $L(M(0))$ is empty. ⊣

5.2 Letter

For any $a \in \Sigma$, let $M(a)$ be the machine, pictured below, with

- $Q = \{\varepsilon, a, T\}$
- $s = \varepsilon$
- $F = \{a\}$
- $\delta(q, x) = \begin{cases} a, & \text{if } q = \varepsilon \text{ and } x = a; \\ T, & \text{otherwise.} \end{cases}$



.

I've chosen the names of the (non-trap) states to indicate the strings that must have already been read by the machine if it ends up in that particular state.

Now, it's probably obvious to you that this machine accepts exactly the language $\{a\}$. And it is, using Theorem 4.4:

LEMMA 5.3 *For $q \in \{\varepsilon, a, T\}$, we have*

$$L_q(M(a)) = \begin{cases} \{a\}, & \textit{if } q = \varepsilon; \\ \mathbf{1}, & \textit{if } q = a; \\ \mathbf{0}, & \textit{if } q = T. \end{cases}$$

In particular, $L(M(a)) = \{a\}$.

PROOF. The only language listed that contains ε is $\mathbf{1}$, which agrees with the final states of the machine. By Theorem 4.4, the rest of the proof is a few simple calculations involving left quotient: $a\backslash\{a\} = \mathbf{1}$, $(\Sigma - \{a\})\backslash\{a\} = \mathbf{0}$, $\Sigma\backslash\mathbf{1} = \mathbf{0}$, and $\Sigma\backslash\mathbf{0} = \mathbf{0}$. $\dashv$

5.3 Union

In this case, we are given two machines

$$M_1 = (Q_1, s_1, F_1, \delta_1) \qquad \text{and} \qquad M_2 = (Q_2, s_2, F_2, \delta_2),$$

and need to construct a machine $M_1 \cup M_2$ with $L(M_1 \cup M_2) = L(M_1) \cup L(M_2)$. Here is the idea. Given an input string w, we are going to start both machines M_1 and M_2 in their respective start states simultaneously reading w. As they each make their respective transitions, we will record the simultaneous state of both machines as a pair (q_1, q_2), where q_1 is the state of machine M_1 and q_2 is the state of machine M_2. Since we want $M_1 \cup M_2$ to accept a string if it is accepted by *either* M_1 or M_2 (or both), we can simply declare the final states of $M_1 \cup M_2$ to be the states (q_1, q_2) where $q_1 \in F_1$ or $q_2 \in F_2$.

Here, then, is the formal construction. Let $M_1 \cup M_2$ be the machine with

- $Q = Q_1 \times Q_2$

- $s = (s_1, s_2)$
- $F = \{(q_1, q_2) \mid q_1 \in F_1 \vee q_2 \in F_2\} = (F_1 \times Q_2) \cup (Q_1 \times F_2)$
- $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$.

To prove that this construction works, we give the appropriate characterization of the state languages for this machine:

LEMMA 5.4 *For all $(q_1, q_2) \in Q$, we have*

$$L_{(q_1,q_2)}(M_1 \times M_2) = L_{q_1}(M_1) \cup L_{q_2}(M_2).$$

In particular $L(M_1 \times M_2) = L(M_1) \cup L(M_2)$.

PROOF. For $(q_1, q_2) \in Q$, define $L_{(q_1,q_2)} = L_{q_1}(M_1) \cup L_{q_2}(M_2)$. We will use Theorem 4.4 to show that $L_{(q_1,q_2)} = L_{(q_1,q_2)}(M_1 \times M_2)$ for all $(q_1, q_2) \in Q$.

To check the first condition, suppose $(q_1, q_2) \in Q$. Then,

$$\begin{aligned}
\varepsilon \in L_{(q_1,q_2)} &\text{ iff } \varepsilon \in L_{q_1}(M_1) \cup L_{q_2}(M_2) && \text{(def of } L_{(-)}) \\
&\text{ iff } \varepsilon \in L_{q_1}(M_1) \vee \varepsilon \in L_{q_2}(M_2) && \text{(def of } \cup) \\
&\text{ iff } q_1 \in F_1 \vee q_2 \in F_2 && \text{(Thm 4.2)} \\
&\text{ iff } (q_1, q_2) \in F. && \text{(def of } F)
\end{aligned}$$

To check the second condition, suppose $(q_1, q_2) \in Q$ and $a \in \Sigma$. Then,

$$\begin{aligned}
a\backslash L_{(q_1,q_2)} &= a\backslash\big(L_{q_1}(M_1) \cup L_{q_2}(M_2)\big) && \text{(def of } L_{(-)}) \\
&= a\backslash L_{q_1}(M_1) \cup a\backslash L_{q_2}(M_2) && \text{(Theorem 2.17.2)} \\
&= L_{\delta_1(q_1,a)}(M_1) \cup L_{\delta_2(q_2,a)}(M_2) && \text{(Thm 4.2)} \\
&= L_{(\delta_1(q_1,a),\delta_2(q_2,a))} && \text{(def of } L_{(-)}) \\
&= L_{\delta((q1,q2),a)}. && \text{(def of } \delta)
\end{aligned}$$

The final statement follows because $s = (s_1, s_2)$. ⊣

Exercise. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Draw the machine $M(\mathtt{a}) \cup M(\mathtt{b})$, first without using the trap-state convention, and then again using it. (Hint: the first machine will have 9 states, 5 of which are final, and 14 arrows; the second machine will have 8 states and 6 arrows.) What's the smallest machine you can draw that accepts the same language, i.e., $\{\mathtt{a}, \mathtt{b}\}$? What's going on with the extra states in $M(\mathtt{a}) \cup M(\mathtt{b})$? (The answer will be given in Section 5.6.)

5.4 Concatenation

Similarly to the union, we are given two machines M_1 and M_2 and need to construct a machine $M_1 \cdot M_2$ such that $L(M_1 \cdot M_2) = L(M_1) \cdot L(M_2)$. This construction is a bit more complicated, however, so let's first get a better idea of what's required. By definition of concatenation, $w \in L(M_1) \cdot L(M_2)$ iff we can write $w = w_1 \cdot w_2$, where $w_1 \in L(M_1)$ and $w_2 \in L(M_2)$. Now if there were just one possible way to divide w into w_1 and w_2 such that $w_1 \in L(M_1)$, then we could proceed as follows:

- Start M_1 reading w.
- If M_1 enters a final state after reading w_1 (with w_2 still to read), then abandon machine M_1 and start M_2 in its start state, s_2, reading w_2.
- If M_2 is in a final state after reading w_2, then accept, otherwise reject.

The problem is that there might be several ways to divide w as $w_1 \cdot w_2$ with $w_1 \in L(M_1)$, only one of which has $w_2 \in L(M_2)$. If we commit to running M_2 on w_2 as soon as we find the *first* w_1 such that $w = w_1 \cdot w_2$ and $w_1 \in L(M_1)$, it may be that M_2 rejects *that* w_2, whereas a different division of w into $w_1 \cdot w_2$ would have succeeded.

What to do? The solution is a combination of two insights. First, in order to keep all of our options open, we won't abandon machine M_1 when it gets into a final state, but keep it running simultaneously with M_2, which we have just started in its start state. Then, if M_1 gets into a final state again later on, we'll start *another copy* of M_2 in its start state, so that we'll have three machines running simultaneously. And, as M_1 might continue returning to a final state many more times while reading w, we might end up having many copies of M_2 running simultaneously, each in its own state. Finally, when we get to the end of the string w, we will say that this system of machines accepts w if *at least one* of the copies of M_2 is in a final state.

The problem with this idea is that there would be no limit to the number of copies of M_2 we'd have to keep track of simultaneously, and we only have available a fixed, finite amount of state that is independent of the length of the input string. This is where the second insight comes in. Since all we care about this set of copies of M_2 is that at least one of them gets into a final state after finishing w, all we really need to keep track of is the particular *subset* of states of M_2 that the system of machines is in, as an aggregate, at any one time—having more than one copy of M_2 in the same state doesn't make it any easier or harder to accept the string w, since they will all accept or reject together. And, crucially, if Q_2 is a finite set, then $\mathsf{Pow}(Q_2)$ is also finite.

Thus, the states of the machine $M_1 \cdot M_2$ will be pairs (q, X), where $q \in Q_1$ and $X \subseteq Q_2$, with X representing the aggregate of the states the copies of the machine M_2 are in at the moment. There is one catch, however: if $q \in F_1$ then we should have $s_2 \in X$ as well, since we will be starting another copy of M_2 in its start state at the same time. This condition on states can insured by the following device: if $q \in Q_1$ and $X \subseteq Q_2$, then we define

$$X^q = X \cup \{s_2 \mid q \in F_1\},$$

which "corrects" X by adding s_2 to X if q is final in M_1 and leaves it unchanged otherwise. Note that the correction is *idempotent*, in the sense that correcting a pair that's already corrected doesn't result in any further changes: $(X^q)^q = X^q$.

In defining the final states, I will use the notation $X \sqcap F_2$ to mean that the sets X and F_2 *overlap*, i.e., that there exists $q \in X \cap F_2$ (or, written another way, $X \cap F_2 \neq \emptyset$), meaning that at least one of the copies of M_2 is in a final state.

Finally, in defining the transition function, I will make use of the function $\hat{\delta}_2 : \mathsf{Pow}(Q_2) \times \Sigma \to \mathsf{Pow}(Q_2)$ defined by

$$\hat{\delta}_2(X_2, a) = \{\delta_2(q_2, a) \mid q_2 \in X_2\},$$

which are the states to which the various copies of M_2 transition on symbol a.

With these preliminaries out of the way, I can now give the formal construction: let $M_1 \cdot M_2$ be the machine with

- $Q = \{(q, X) \mid q \in Q_1, X \subseteq Q_2, X^q = X\}$
- $s = (s_1, \emptyset^{s_1})$
- $F = \{(q, X) \in Q \mid X \sqcap F_2\}$
- $\delta\big((q, X), a\big) = \big(q', \hat{\delta}_2(X, a)^{q'}\big)$, where $q' = \delta_1(q, a)$.

Note that we've restricted the states of $M_1 \cdot M_2$ to just the corrected ones, but this doesn't cause a problem for the validity of our machine, since s is corrected, every state in F is corrected, and the result of every transition is corrected.

To prove that this construction works, we again give the appropriate characterization of the state languages for this machine, which involves this definition of set-of-states languages:

$$\hat{L}_X(M) = \bigcup\{L_q(M) \mid q \in X\}.$$

Lemma 5.5 *For all $(q, X) \in Q$, we have*

$$L_{(q,X)}(M_1 \cdot M_2) = L_q(M_1) \cdot L(M_2) \cup \hat{L}_X(M_2).$$

In particular, $L(M_1 \cdot M_2) = L(M_1) \cdot L(M_2)$.

Proof. For $q \in Q_1$ and $X \subseteq Q_2$, define $L_{(q,X)} = L_q(M_1) \cdot L(M_2) \cup \hat{L}_X(M_2)$. I will use Theorem 4.4 to show that $L_{(q,X)} = L_{(q,X)}(M_1 \cdot M_2)$ for all $(q, X) \in Q$. To save on symbols, let me write L^1_q for $L_q(M_1)$, L^2 for $L(M_2)$, and $\hat{L}^2_X$ for $\hat{L}_X(M_2)$. To check the first condition, suppose $(q, X) \in Q$. Then,

$$\begin{aligned}
\varepsilon \in L_{(q,X)} &\text{ iff } \varepsilon \in L^1_q \cdot L^2 \cup \hat{L}^2_X && \text{(def of } L_{(-)}\text{)}\\
&\text{ iff } \varepsilon \in L^1_q \cdot L^2 \vee \varepsilon \in \hat{L}^2_X && \text{(def of } \cup\text{)}\\
&\text{ iff } (\varepsilon \in L^1_q \wedge \varepsilon \in L^2) \vee \varepsilon \in \hat{L}^2_X && \text{(def of } \cdot\text{)}\\
&\text{ iff } (q \in F_1 \wedge s_2 \in F_2) \vee X \sqcap F_2 && \text{(Thm 4.2)}\\
&\text{ iff } X \sqcap F_2,
\end{aligned}$$

where this last step follows because X is corrected: $q \in F_1$ implies $s_2 \in X$, and so if $s_2 \in F_2$ as well, then $X \sqcap F_2$. Finally, $X \sqcap F_2$ is equivalent to $(q, X) \in F$.

Before checking the second condition, let me show first that our languages $L_{(q,X)}$ are unaffected by correction, i.e., $L_{(q,X)} = L_{(q,X^q)}$ for all $q \in Q_1$ and $X \subseteq Q_2$. That's because $\hat{L}_{X^q}(M_2) = \hat{L}_X(M_2) \cup (q \in F_1) \cdot L(M_2)$ and, by Theorem 4.2, $(q \in F_1) \subseteq L_q(M_1)$, so $(q \in F_1) \cdot L(M_2) \subseteq L_q(M_1) \cdot L(M_2)$ by Theorem 2.8.4.

Now, turning to the second condition, suppose $(q, X) \in Q$ and $a \in \Sigma$, and let $q' = \delta_1(q, a)$. Then,

$$\begin{aligned}
a\backslash L_{(q,X)} &= a\backslash\big(L^1_q \cdot L^2 \cup \hat{L}^2_X\big) && \text{(def of } L_{(-)}\text{)}\\
&= (a\backslash L^1_q) \cdot L^2 \cup (\varepsilon \in L^1_q) \cdot (a\backslash L^2) \cup a\backslash \hat{L}^2_X && \text{(Thm 2.17)}\\
&= L^1_{q'} \cdot L^2 \cup (q \in F_1) \cdot L^2_{\delta_2(s_2,a)} \cup \hat{L}^2_{\hat{\delta}_2(X,a)} && \text{(Thm 4.2)}\\
&= L^1_{q'} \cdot L^2 \cup \hat{L}^2_{\hat{\delta}_2(X^q,a)} && \text{(defs of } \hat{L}^2,\ \hat{\delta}_2,\ X^q\text{)}\\
&= L_{(q',\hat{\delta}_2(X,a))}, && (X^q = X, \text{ def of } L_{(-)})\\
&= L_{\delta((q,X),a)},
\end{aligned}$$

where this last step follows by the definition of δ and the observation above about our languages being unaffected by correction.

As for the final statement, the results above give us that $L_{(s_1,\emptyset)^c}(M_1 \cdot M_2)$ is equal to $L_{(s_1,\emptyset)^c}$ and thus $L_{(s_1,\emptyset)}$, but since $\hat{L}_\emptyset(M_2) = \emptyset$, this is equal to $L_{s_1}(M_1) \cdot L(M_2)$ and thus $L(M_1) \cdot L(M_2)$, as required. $\dashv$

Exercise. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Draw the machine $M(\mathtt{a}) \cdot M(\mathtt{b})$. (Hint: it will have 20 states, 10 of which are final.) What's the smallest machine you can draw that accepts the same language, i.e., $\{\mathtt{ab}\}$? Again, what's going on with all these extra states? (Again, the answer will be given in Section 5.6.)

5.5 Star

For the final construction, we are given a machine $M_1 = (Q_1, s_1, F_1, \delta_1)$ and we need to construct a machine M_1^* such that $L(M_1^*) = L(M_1)^*$. The idea is similar to the concatenation machine, in that we will need to run multiple copies of M_1 simultaneously, so we will use subsets $X \subseteq Q_1$ to keep track of the aggregate set of states a collection of copies of M_1 are in at any given time, with the catch that, if $X \sqcap F_1$, then we require $s_1 \in X$; that is, when at least one copy of M_1 is in a final state, then a new copy of M_1 will be started simultaneously in its initial state, which we will enforce by defining, for $X \subseteq Q_1$,

$$X^c = X \cup \{s_1 \mid X \sqcap F_1\}.$$

A set of states is final if it includes at least one final state. One final twist: since we always have $\varepsilon \in M_1^*$, the start state of M_1^* must also be final, even if the start state of M_1 is not. We will achieve this by using $\emptyset$ as the start state of M_1^*, which is not otherwise used in the construction, and set the transitions out of this state to mirror the transitions of M_1 out of s_1.

Here, then is the formal construction. Let M_1^* be the machine with

- $Q = \{X \mid X^c = X\} \subseteq \mathsf{Pow}(Q_1)$
- $s = \emptyset$
- $F = \{\emptyset\} \cup \{X \in Q \mid X \sqcap F_1\}$
- $\delta(X, a) = \begin{cases} \{\delta_1(s_1, a)\}^c, & \text{if } X = \emptyset; \\ \hat{\delta}_1(X, a)^c, & \text{otherwise.} \end{cases}$

Note that, since $\emptyset^c = \emptyset$ and the results of each transition are corrected, we have $s \in Q$ and $\delta : Q \times \Sigma \to Q$, as required in the definition of an FSM.

To prove that this construction works, we again give the appropriate characterization of the state languages for this machine.

Lemma 5.6 *For all $X \in Q$, we have*

$$L_X(M_1^*) = \begin{cases} L(M_1)^*, & \textit{if } X = \emptyset; \\ \hat{L}_X(M_1) \cdot L(M_1)^*, & \textit{otherwise.} \end{cases}$$

In particular, $L(M_1^*) = L(M_1)^*$.

Proof. Define $L_\emptyset = L(M_1)^*$ and $L_X = \hat{L}_X(M_1) \cdot L(M_1)^*$ for $X \in Q - \{\emptyset\}$. We will use Theorem 4.4 to show that $L_X = L_X(M_1^*)$ for all $X \in Q$.

For the first condition, note that $\varepsilon \in L_\emptyset$ is always true by the definition of $L_\emptyset$ and by (*Empty), and for $X \in Q - \{\emptyset\}$, we have

$$\begin{aligned} \varepsilon \in L_X \text{ iff } & \varepsilon \in \hat{L}_X(M_1) \cdot L(M_1)^* && \text{(def of } L_X\text{)} \\ \text{iff } & \varepsilon \in \hat{L}_X(M_1) && \text{(def of } \cdot\text{)} \\ \text{iff } & X \sqcap F_1. && \text{(Thm 4.2)} \end{aligned}$$

So, for all $X \in Q$, we have $\varepsilon \in L_X$ iff $X \in \{\emptyset\} \cup \{X \mid X \sqcap F_1\} = F$, as required.

For the second condition, note first that, for any $a \in \Sigma$, we have

$$\begin{aligned} a \backslash L_\emptyset &= a \backslash L(M_1)^* && \text{(def of } L_\emptyset\text{)} \\ &= (a \backslash L(M_1)) \cdot L(M_1)^* && \text{(Thm 2.17.4)} \\ &= L_{\delta_1(s_1,a)}(M_1) \cdot L(M_1)^* && \text{(Thm 4.2)} \\ &= \hat{L}_{\{\delta_1(s_1,a)\}}(M_1) \cdot L(M_1)^* && \text{(def of } \hat{L}\text{)} \\ &= L_{\delta((q,X),a)}. && \text{(defs of } \delta \text{ and } (-)^c\text{)} \end{aligned}$$

$a \backslash L_= a$Ł$(M_1)^* = L_{\delta_1(s_1,a)}(M_1)$ by definition of $L_\emptyset$ and Theorem 4.2.

suppose $(q, X) \in Q$ and $a \in \Sigma$. Then,

$$\begin{aligned} a\backslash L_{(q,X)} &= a\backslash\big(L^1_q \cdot L^2 \cup L^2_X\big) && \text{(def of } L_{(-)}\text{)} \\ &= (a\backslash L^1_q) \cdot L^2 \cup (\varepsilon \in L^1_q) \cdot (a\backslash L^2) \cup a\backslash L^2_X && \text{(Thm 2.17)} \\ &= L^1_{\delta_1(q,a)} \cdot L^2 \cup (q \in F_1) \cdot L^2_{\delta_2(s_2,a)} \cup L^2_{\hat\delta_2(X,a)} && \text{(Thm 4.2)} \\ &= L^1_{\delta_1(q,a)} \cdot L^2 \cup L^2_{\hat\delta_2(X \cup \{s_2 \mid q \in F_1\},a)} && \text{(def of } \hat\delta_2\text{)} \\ &= L_{\delta((q,X),a)}. && \text{(defs of } \delta \text{ and } (-)^c\text{)} \end{aligned}$$

As for the final statement, we check

$$\begin{aligned} L_{(s_1,\emptyset)^c}(M_1 \cdot M_2) &= L_{s_1}(M_1) \cdot L(M_2) \cup L_{\{s_2 \mid s_1 \in F_1\}}(M_2) && \text{(result above)} \\ &= L(M_1) \cdot L(M_2) \cup (s_1 \in F_1) \cdot L(M_2) \\ &= (L(M_1) \cup (s_1 \in F_1)) \cdot L(M_2) && \text{(Theorem 2.8.5)} \\ &= L(M_1) \cdot L(M_2). && (s_1 \in F_1 \text{ implies } \varepsilon \in L(M_1)) \end{aligned}$$

$\dashv$

Exercise. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Draw the machine $M(\mathtt{a})^*$ (Hint: it will have 6 states, 3 of which are final.) What's the smallest machine you can draw that accepts the same language, i.e., $\{\mathtt{a}\}^*$. For the last time, what's going on with these extra states? (For the answer, read on!)

5.6 An aside: Reachability

If you tried the exercises in the earlier part of this section, you realized that the constructions for union, concatenation, and star tend to produce machines that are much (and often staggeringly) bigger than they have to be. For example, if the machine M_1 has n_1 states and the machine M_2 has n_2 states, then machine $M_1 \cdot M_2$ can have up to $n_1 \cdot 2^{n_2}$ states, meaning that a machine as "simple" as $M(\mathtt{a}) \cdot (M(\mathtt{b}) \cdot M(\mathtt{c}))$, which accepts the singleton language $\{\mathtt{abc}\}$, could have up to $3 \cdot 2^{3 \cdot 2^3}$ or 50,331,648 states, even though it is easy to construct a machine that accepts this language with just 5 states—talk about overkill!

The reason that our constructions are this "wasteful" is that they are completely *general*: since they don't know anything about the constituent machines, they have to include every possibility of interaction between them. In any particular case, however, the constructions will include many states that never get used, in the sense that the machine, staring in its start state and reading an arbitrary string, can never reach those states. For example, the colossal machine mentioned above has only 6 states that can be reached from its start state, and thus 50,331,642 states that might as well not even have been included in the construction!

Let's now formalize the idea of the "reachable" portion of a machine and show that we can simultaneously throw away all of the states that are not reachable and still have a machine that accepts the same language.

Let $M = (Q, s, F, \delta)$ be a FSM. Then we can define a subset $Q_r \subseteq Q$, called the *reachable* states of M, by the following rules:

$$\frac{}{s \in Q_r} \qquad \frac{q \in Q_r \qquad a \in \Sigma}{\delta(q, a) \in Q_r} .$$

Why is Q_r a subset of Q? It follows by a straightforward induction on these rules, since $s \in Q$ and $\delta(q, a) \in Q$ for every $q \in Q$ and $a \in \Sigma$. Now let M_r be the machine $(Q_r, s_r, F_r, \delta_r)$, where

- $s_r = s$
- $F_r = F \cap Q_r$
- $\delta_r(q, a) = \delta(q, a)$, for $q \in Q_r$ and $a \in \Sigma$.

Clearly, $s_r \in Q_r$ and $F_r \subseteq Q_r$, and the proposition that $q \in Q_r$ and $a \in \Sigma$ imply $\delta_r(q, a) \in Q_r$ is just the second rule above, showing that $\delta_r : Q_r \times \Sigma \to Q_r$ and thus that M_r is a FSM.

LEMMA 5.7 *For all $q \in Q_r$ and $w \in \Sigma^*$,*

$$\delta_r^*(q, w) = \delta^*(q, w) \in Q_r$$

PROOF. A straightforward induction on w, which we omit. $\dashv$

THEOREM 5.8 *$L(M_r) = L(M)$.*

PROOF. Let $w \in L(M_r)$ be arbitrary. Then, by the definitions of the machine and acceptance, $\delta_r^*(s, w) \in F \cap Q_r$, so by the Lemma, $\delta^*(s, w) = \delta_r^*(s, w) \in F$ and thus $w \in L(M)$.

Conversely, let $w \in L(M)$ be arbitrary. The $\delta^*(s, w) \in F$, and so by the Lemma and the definitions, $\delta_r^*(s, w) = \delta^*(s, w) \in F \cap Q_r$, and so $w \in L(M_r)$, as required. $\dashv$

Exercise. Go back over the exercises in this section and determine which of the states in the machines you constructed were reachable. How does that number compare with the minimal machines you were able to find? Is there still some room for improvement? (This question will be settled completely in Section 10.2 below.)

6 FSMs Accept Regular Languages

We established in the previous section (with a brief detour into reachability) that every regular language is accepted by some FSM, so now let's move on in this section to the problem of proving the converse:

Theorem 6.1 *For every finite state machine, M, the language $L(M)$ is regular.*

To do this, I'll first introduce the notion of a *system of proper linear equations* (SPLE) in a finite list of variables and define what it means for a list of languages to satisfy such a system. I'll then show that every SPLE has a unique solution and that every component of that solution is regular. Finally I'll show how we can construct, given an FSM M, a SPLE using the states of M as the variables so that the components of the solution are exactly the $L_q(M)$.

6.1 Systems of Proper Linear Equations

A *linear equation* between languages is an equation of the form

$$X = L_1 \cdot X \cup L_2, \tag{1}$$

where L_1 and L_2 are languages and, by convention, the right-hand side is grouped $(L_1 \cdot X) \cup L_2$, as $\cdot$ has higher precedence than $\cup$. A language L is a *solution* to (1) if, naturally, $L = L_1 \cdot L \cup L_2$.

If $\varepsilon \in L_1$, then the solutions to (1) are numerous and not very interesting: for example, $L = \Sigma^*$ is always a solution, and it is just the largest of a generally infinite (even *uncountable*, if that means anything to you) number of solutions. For example, at the extreme, when $L_1 = \{\varepsilon\}$ and $L_2 = \emptyset$ (i.e., when $L_1 = \mathbf{1}$ and $L_2 = \mathbf{0}$), *every* language L is a solution to (1). On the other hand, if we call a language L_1 *proper* when $|w| \geq 1$ for all $w \in L_1$ (which is just another way of saying $\varepsilon \notin L_1$, since ε is the only string of length 0), then we have the following remarkable result:

Lemma 6.2 *(Arden's Lemma) For any two languages L_1 and L_2, the language $L = L_1^* \cdot L_2$ is the smallest solution to* (1)*; i.e., L is a solution, and if L' is also a solution then $L \subseteq L'$. Moreover, if L_1 is proper, then L is the unique solution.*

Proof. Let L_1 and L_2 be arbitrary languages, and let $L = L_1^* \cdot L_2$. Then we can use the theorems of Section 2.2 to show that L is a solution to (1):

$$\begin{aligned} L_1 \cdot (L_1^* \cdot L_2) \cup L_2 &= (L_1 \cdot L_1^*) \cdot L_2 \cup L_2 && \text{(by 2.8.3)} \\ &= (L_1 \cdot L_1^*) \cdot L_2 \cup \{\varepsilon\} \cdot L_2 && \text{(by 2.8.2)} \\ &= (L_1 \cdot L_1^* \cup \{\varepsilon\}) \cdot L_2 && \text{(by 2.8.5)} \\ &= L_1^* \cdot L_2. && \text{(by 2.10)} \end{aligned}$$

To show that it's the smallest solution, suppose $L' = L_1 \cdot L' \cup L_2$, and let's show by induction on $w_1 \in L_1^*$ that, for every $w_2 \in L_2$, we have $w_1 \cdot w_2 \in L'$. If $w_1 = \varepsilon$ and $w_2 \in L_2$, then

$$w_1 \cdot w_2 = w_2 \in L_2 \subseteq L_1 \cdot L' \cup L_2 = L'.$$

And if $w_1 = w \cdot w_1'$, with $w \in L_1$ and $w_1' \in L_1^*$ and $w_2 \in L_2$, then we have, using associativity of $\cdot$ and the IH,

$$w_1 \cdot w_2 = (w \cdot w_1') \cdot w_2 = w \cdot (w_1' \cdot w_2) \in L_1 \cdot L' \subseteq L_1 \cdot L' \cup L_2 = L'.$$

Finally, let's show that if L_1 is proper, then L is also the largest—and thus *only*—solution to (1), in that if L' is also a solution, then $L' \subseteq L$. So, assume L_1 is proper and $L' = L_1 \cdot L' \cup L_2$. By Theorem 2.8, we have

$$L_1 \cdot L = L_1 \cdot (L_1^* \cdot L_2) = (L_1 \cdot L_1^*) \cdot L_2 \subseteq L_1^* \cdot L_2 = L,$$

and so $L_1 \cdot L \subseteq L$. I will now prove by strong induction on $|w|$ that

$$w \in L' \text{ implies } w \in L. \tag{2}$$

So let n be a natural number, assume (2) holds for all strings w' with $|w'| < n$, let w be such that $|w| = n$, and assume $w \in L'$. Since $L' = L_1 \cdot L' \cup L_2$, either $w \in L_1 \cdot L'$ or $w \in L_2$. In the first case, there exist $w_1 \in L_1$ and $w' \in L'$ such that $w = w_1 \cdot w'$. Since L_1 is proper, we have $|w_1| \geq 1$, and so $|w'| < n$. Therefore, by the IH, $w' \in L$, and so $w = w_1 \cdot w' \in L_1 \cdot L \subseteq L$, showing $w \in L$. In the second case, we use $w = \varepsilon \cdot w$ and $\varepsilon \in L_1^*$ to conclude $w \in L_1^* \cdot L_2 = L$. $\dashv$

We will say that the equation (1) is *proper* if L_1 is proper. Thus, Arden's Lemma gives us a unique solution to any proper linear equation. Notice, moreover, the crucial point that if L_1 and L_2 are regular, then the unique solution to (1) is also regular, since it is constructed from L_1 and L_2 using Kleene star and concatenation.

Now that we have a way to solve proper linear equations in one variable, we will extend the method to systems of proper linear equations of the form

$$\begin{array}{ccccccccccc}
X_1 & = & L_{1,1} \cdot X_1 & \cup & L_{1,2} \cdot X_2 & \cup & \cdots & \cup & L_{1,n} \cdot X_n & \cup & L'_1 \\
X_2 & = & L_{2,1} \cdot X_1 & \cup & L_{2,2} \cdot X_2 & \cup & \cdots & \cup & L_{2,n} \cdot X_n & \cup & L'_2 \\
 & & \multicolumn{9}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\
X_n & = & L_{n,1} \cdot X_1 & \cup & L_{n,2} \cdot X_2 & \cup & \cdots & \cup & L_{n,n} \cdot X_n & \cup & L'_n,
\end{array} \tag{3}$$

with n equations in n variables $X_1, X_2, \ldots, X_n$, where every $L_{i,j}$ is proper. Here is the formal definition:

Definition 6.3 A *system of n proper linear equations* (or n-SPLE), where $n \geq 0$, is a pair $(\{L_{i,j}\}_{ij}, \{L'_i\}_i)$, where

- $\{L_{i,j}\}_{ij}$ is an n-by-n matrix of proper languages, called the *coefficients* of the system, and
- $\{L'_i\}_i$ is an n-vector of languages, called the *constants* of the system.

This system of equations is called *regular* if every coefficient $L_{i,j}$ and every constant L'_i is a regular language. An n-vector of languages, $\{L_i\}_i$, is a *solution* to this system if, for every i,

$$L_i = \big(\bigcup_{j=1}^{n} L_{i,j} \cdot L_j\big) \cup L'_i. \tag{4}$$

Now, Arden's Lemma gives us a recursive method to solve any n-SPLE: if $n = 0$, then the system is empty, and has an empty solution; if $n > 1$, then we

- rewrite the first equation in (3) to the form

$$X_1 = L_{1,1} \cdot X_1 \cup (L_{1,2} \cdot X_2 \cup \cdots \cup L_{1,n} \cdot X_n \cup L'_1),$$

  and use Arden's Lemma to get a solution for X_1 in terms of the other variables:

$$X_1 = L^*_{1,1} \cdot (L_{1,2} \cdot X_2 \cup \cdots \cup L_{1,n} \cdot X_n \cup L'_1); \qquad (5)$$

- substitute this solution in for the other instances of X_1 in (3), and then use the properties of $\cdot$ and $\cup$ to rearrange the result to get an $(n-1)$-SPLE, $(\{\bar{L}_{i,j}\}_{ij}, \{\bar{L}'_i\}_i)$, where we find that

$$\begin{aligned} \bar{L}_{i,j} &= L_{i,1} \cdot L^*_{1,1} \cdot L_{1,j} \cup L_{i,j} && (i, j \in \{2, \ldots, n\}), && (6) \\ \bar{L}'_i &= L_{i,1} \cdot L^*_{1,1} \cdot L'_1 \cup L'_i && (i \in \{2, \ldots, n\}); && (7) \end{aligned}$$

- recursively solve this $(n-1)$-SPLE to get solutions $L_2, \ldots, L_n$ for the variables $X_2, \ldots, X_n$; and finally
- extend this solution to include L_1, which we get as the value of X_1 in (5) after substituting $L_2, \ldots, L_n$ in for the variables $X_2, \ldots, X_n$.

The following theorem establishes that this method produces unique regular solutions to regular systems of equations:

THEOREM 6.4 *Every regular n-SPLE* $(\{L_{i,j}\}_{ij}, \{L'_i\}_i)$, *for* $n \geq 0$, *has a unique solution, all of the components of which are regular.*

PROOF. By induction on n. The case $n = 0$ is trivial, so suppose $n > 0$ and $(\{L_{i,j}\}_{ij}, \{L'_i\}_i)$ is regular. Let $\{\bar{L}_{i,j}\}_{ij}$ and $\{\bar{L}'_i\}_i$ be given by (6) and (7). For every $i, j \geq 2$, the language $\bar{L}_{i,j}$ is proper, since $L_{i,1}$ and $L_{i,j}$ are proper, and every $\bar{L}_{i,j}$ and $\bar{L}'_i$ is regular, since these languages are constructed from regular languages using the regular operations of union, concatenation, and Kleene star. By the IH, this $(n-1)$-SPLE has a unique solution $\{L_i\}_i$, where for every $i \geq 2$, L_i is regular and satisfies

$$L_i = \big(\bigcup_{j=2}^{n} \bar{L}_{i,j} \cdot L_j\big) \cup \bar{L}'_i \qquad (i \in \{2, \ldots, n\}), \qquad (8)$$

Define

$$L_1 = L^*_{1,1} \cdot \Big(\big(\bigcup_{j=2}^{n} L_{1,j} \cdot L_j\big) \cup L'_1\Big),$$

which we again note is regular for the same reason noted above. We need to show that $\{L_i\}_i$ $(1 \leq i \leq n)$ is the unique solution to the original system, i.e., satisfies (4) for every $i \geq 1$. The case $i = 1$ (including the uniqueness of L_1) follows directly from Arden's Lemma and the definition of L_1. The cases where $i > 1$ follow from the definition of L_1, the properties of $\cdot$ and $\cup$, and (6), (7), and (8), as follows:

$$
\begin{aligned}
&(\bigcup_{j=1}^{n} L_{i,j} \cdot L_j) \cup L'_i \\
&\qquad = L_{i,1} \cdot L_1 \cup (\bigcup_{j=2}^{n} L_{i,j} \cdot L_j) \cup L'_i \\
&\qquad = L_{i,1} \cdot \Big(L^*_{1,1} \cdot \Big((\bigcup_{j=2}^{n} L_{1,j} \cdot L_j) \cup L'_1\Big)\Big) \cup (\bigcup_{j=2}^{n} L_{i,j} \cdot L_j) \cup L'_i \\
&\qquad = (\bigcup_{j=2}^{n} L_{i,1} \cdot L^*_{1,1} \cdot L_{1,j} \cdot L_j) \cup L_{i,1} \cdot L^*_{1,1} \cdot L'_1 \cup (\bigcup_{j=2}^{n} L_{i,j} \cdot L_j) \cup L'_i \\
&\qquad = (\bigcup_{j=2}^{n} L_{i,1} \cdot L^*_{1,1} \cdot L_{1,j} \cdot L_j) \cup (\bigcup_{j=2}^{n} L_{i,j} \cdot L_j) \cup \bar{L}'_i \\
&\qquad = (\bigcup_{j=2}^{n} L_{i,1} \cdot L^*_{1,1} \cdot L_{1,j} \cdot L_j \cup L_{i,j} \cdot L_j) \cup \bar{L}'_i \\
&\qquad = (\bigcup_{j=2}^{n} (L_{i,1} \cdot L^*_{1,1} \cdot L_{1,j} \cup L_{i,j}) \cdot L_j) \cup \bar{L}'_i \\
&\qquad = (\bigcup_{j=2}^{n} \bar{L}_{i,j} \cdot L_j) \cup \bar{L}'_i, \\
&\qquad = L_i \qquad\qquad \dashv
\end{aligned}
$$

In what follows, all of our SPLEs will be regular, so it will often be convenient to use regular expressions, rather than the regular languages they name, to specify these systems and their solutions.

Exercise. Use the algorithm above to find the unique solution to this regular 3-SPLE in the variables X_0, X_1, X_2, which we write using regular expressions:

$$
\begin{aligned}
X_0 &= b \cdot X_0 + a \cdot X_1 \\
X_1 &= b \cdot X_1 + a \cdot X_2 + 1 \\
X_2 &= a \cdot X_0 + b \cdot X_2
\end{aligned}
$$

Note that several of the coefficients and constants in this SPLE are 0.

6.2 From FSMs to SPLEs

Suppose we are given an FSM $M = (Q, s, F, \delta)$, where $Q = \{q_1, q_2, \ldots, q_n\}$, i.e., where we have numbered the states from 1 to n. We will now construct a regular n-SPLE $(\{L_{i,j}\}_{ij}, \{L'_i\}_i)$ whose unique solution is $\{L_{q_i}(M)\}_i$, i.e., where the ith component of the solution is the language accepted by M if the machine is started in state q_i. Since every component of this unique solution is regular by Theorem 6.4, it will follow that $L(M)$, which is just $L_{q_i}(M)$ for the i such that $s = q_i$, is regular, achieving our goal for this section.

THEOREM 6.5 *Let $M = (Q, s, F, \delta)$ be an FSM, where $Q = \{q_1, q_2, \ldots, q_n\}$, and define, for all $i, j \in \{1, \ldots, n\}$,*

$$\begin{aligned} L_{i,j} &= \{a\varepsilon \mid a \in \Sigma \text{ and } \delta(q_i, a) = q_j\} \\ L'_i &= \{\varepsilon \mid q_i \in F\}. \end{aligned}$$

Then $(\{L_{i,j}\}_{ij}, \{L'_i\}_i)$ is a regular n-SPLE whose unique solution is $\{L_{q_i}(M)\}_i$.

PROOF. Note first that each $L_{i,j}$ is a finite set of strings of length 1, and thus regular and proper, and each L'_i is either $\mathbf{1}$ or $\mathbf{0}$, and thus regular, so that $(\{L_{i,j}\}_{ij}, \{L'_i\}_i)$ is indeed a regular n-SPLE. By Theorem 6.4, there is a unique vector of languages, $\{L_i\}_i$, satisfying (4) for every $i \in \{1, \ldots, n\}$. Since each $L_{i,j}$ is proper, it follows from (4) that, for every i,

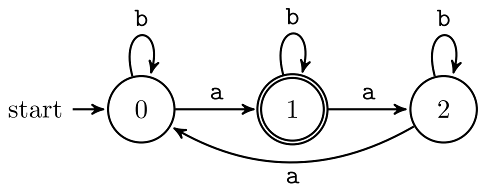
$$\varepsilon \in L_i \text{ iff } \varepsilon \in L'_i \text{ iff } q_i \in F.$$

Furthermore, if $aw \in \Sigma^*$, then it also follows from (4) that, for every i,

$$aw \in L_i \text{ iff } w \in L_{\delta(q_i,a)}.$$

Thus, the vector of languages $\{L_i\}_i$ satisfies the same recursion equations as the vector of languages $\{L_{q_i}(M)\}_i$, proving that they are equal. $\dashv$

As an example, consider this FSM over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, which accepts exactly the strings with n $\mathtt{a}$'s, where $n \equiv 1 \pmod 3$:



You can check that the 3-SPLE corresponding to this machine is exactly the 3-SPLE from the Exercise in the previous subsection, and that a solution $\{r_0, r_1, r_2\}$ to that system therefore provides a regular expression r_0 matching exactly the strings singled out above.

6.3 An aside: Simpflication of REs

Just as in Section 5, where the construction of FSMs from regular expressions produced very large machines that can be simplified by considering only reachable states, so too here the construction of regular expressions from FSMs can produce large expressions that can be simplified.

7 More closure properties of regular languages

In this section, we show, as a consequence of the equivalence proved in Sections 5 and 6, that regular languages, besides being closed under union, concatenation, and Kleene star, are also closed under intersection, complement, reversal, homomorphism, reverse homorphism, and left and right quotients by an arbitrary language.

7.1 Intersection and Complement

Given

$$M_1 = (Q_1, s_1, F_1, \delta_1) \qquad \text{and} \qquad M_2 = (Q_2, s_2, F_2, \delta_2),$$

we define a machine $M_1 \cap M_2$ as follows:

- $Q = Q_1 \times Q_2$
- $s = (s_1, s_2)$
- $F = \{(q_1, q_2) \mid q_1 \in F_1 \wedge q_2 \in F_2\} = F_1 \times F_2$
- $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$.

It follows that $L(M_1 \cap M_2) = L(M_1) \cap L(M_2)$. We can also define a machine $\overline{M_1}$ as follows:

- $Q = Q_1$
- $s = s_1$
- $F = Q_1 - F_1$
- $\delta = \delta_1$

It follows that $L(\overline{M_1}) = \Sigma^* - L(M_1)$.

7.2 Reversal

We define the reversal of a string w as follows:

$$\mathsf{rev}(\varepsilon) = \varepsilon$$
$$\mathsf{rev}(aw) = \mathsf{rev}(w) \cdot (a\varepsilon)$$

and the reversal of a language L by

$$\mathsf{rev}(L) = \{\mathsf{rev}(w) \mid w \in L\}.$$

It follows by induction on w_1 that

- $\mathsf{rev}(w_1 \cdot w_2) = \mathsf{rev}(w_2) \cdot \mathsf{rev}(w_1)$ for all w_2, and
- $\mathsf{rev}(\mathsf{rev}(w_1)) = w_1$.

It also follows for all languages L_1 and L_2 that

- $\mathsf{rev}(L_1 \cdot L_2) = \mathsf{rev}(L_2) \cdot \mathsf{rev}(L_1)$, and
- $\mathsf{rev}(\mathsf{rev}(L_1)) = L_1$.

Given a regular expression r, we define a regular expression $\mathsf{rev}(r)$ by induction on r:

$$\begin{aligned}
\mathsf{rev}(0) &= 0 \\
\mathsf{rev}(\mathsf{a}) &= \mathsf{a} \\
\mathsf{rev}(r_1 + r_2) &= \mathsf{rev}(r_1) + \mathsf{rev}(r_2) \\
\mathsf{rev}(r_1 r_2) &= \mathsf{rev}(r_2)\mathsf{rev}(r_1) \\
\mathsf{rev}(r^*) &= \mathsf{rev}(r)^*
\end{aligned}$$

The result is:

$$[\![\mathsf{rev}(r)]\!] = \mathsf{rev}([\![r]\!]).$$

It follows that, if R is regular, then $\mathsf{rev}(R)$ is also regular.

7.3 Homomorphisms

A homomorphism is a function $h : \Sigma \to \Sigma^*$. Given such a homomorphism, we can define a function $h^* : \Sigma^* \to \Sigma^*$ by

$$\begin{aligned}
h^*(\varepsilon) &= \varepsilon \\
h^*(aw) &= h(a) \cdot h^*(w)
\end{aligned}$$

The first equation says that h^* preserves ε, and it easily follows by induction that h^* preserves concatenation, in the sense that

$$h^*(w \cdot u) = h^*(w) \cdot h^*(u).$$

Conversely, if $k : \Sigma^* \to \Sigma^*$ is a function that preserves both ε and concatenation, then $k = h^*$ for the homomorphism h defined by

$$h(a) = k(a\varepsilon).$$

Given a homomorphism h and language L, both the image of L under h^*,

$$h^*(L) = \{h^*(w) \mid w \in L\},$$

and the inverse image of L under h^*,

$$(h^*)^{-1}(L) = \{w \mid h^*(w) \in L\},$$

are regular.

First, given a string $w \in \Sigma^*$, we can define, by recursion on w, a regular expression $\overline{w}$ that matches only w:

$$\begin{aligned}
\overline{\varepsilon} &= 1 \\
\overline{a\varepsilon} &= a \\
\overline{aw} &= a\overline{w} \quad (w \neq \varepsilon).
\end{aligned}$$

Then, for an arbitrary regular expression r, we define a regular expression, $h(r)$, by recursion as follows:

$$\begin{aligned}
h(0) &= 0 \\
h(\mathtt{a}) &= \overline{h(\mathtt{a})} \\
h(r_1 + r_2) &= h(r_1) + h(r_2) \\
h(r_1 r_2) &= h(r_1)h(r_2) \\
h(r_1^*) &= h(r_1)^*.
\end{aligned}$$

The result is

$$[\![h(r)]\!] = h([\![r]\!]),$$

showing that the homomorphic image of a regular set is regular.

Now let $M_1 = (Q_1, s_1, F_1, \delta_1)$ be an FSM and define a machine $h^{-1}(M_1)$ as follows:

- $Q = Q_1$
- $s = s_1$
- $F = F_1$
- $\delta(q, a) = \delta_1^*(q, h(a))$.

The result is

$$L(h^{-1}(M_1)) = (h^*)^{-1}(L(M_1)),$$

showing that the inverse homomorphic image of a regular set is also regular.

7.4 Quotients

Given two languages L_1 and L_2, we define the right and left quotients of L_1 by L_2, respectively, as follows:

$$\begin{aligned}
L_1/L_2 &= \{w \mid \exists w_2(w_2 \in L_2 \wedge w \cdot w_2 \in L_1)\} \\
L_2\backslash L_1 &= \{w \mid \exists w_2(w_2 \in L_2 \wedge w_2 \cdot w \in L_1)\}.
\end{aligned}$$

We show that if L_1 is regular, then both of these languages are regular for an arbitrary language L_2.

First, let $M_1 = (Q_1, s_1, F_1, \delta_1)$ be an FSM accepting the language L_1. We define a machine M_1/L_2 as follows:

- $Q = Q_1$
- $s = s_1$
- $F = \{q \mid \exists w(w \in L_2 \land \delta_1^*(q, w) \in F_1)\}$
- $\delta = \delta_1$

It follows that $L(M_1/L_2) = L(M_1)/L_2$, and thus that L_1/L_2 is regular.

Next, we prove the following relationship between right and left quotients:

$$\mathsf{rev}(L_2 \backslash L_1) = \mathsf{rev}(L_1)/\mathsf{rev}(L_2).$$

It follows, by applying rev to both sides and using the fact that $\mathsf{rev}(\mathsf{rev}(L)) = L$, that

$$L_2 \backslash L_1 = \mathsf{rev}(\mathsf{rev}(L_1)/\mathsf{rev}(L_2)),$$

and therefore that $L_2 \backslash L_1$ is regular if L_1 is.

8 Variants of FSMs

In this section, we look at two variants of the notion of FSM that seem to add additional power to the model, thereby making it easier to construct machines accepting particular languages, and possibly also increasing the set of languages accepted. In each case, however, we see by means of a reduction that the FSM variant defines the same set of languages as the basic FSMs, i.e., the regular langauges.

8.1 Nondeterministic FSMs

In an ordinary—or, as we say, *deterministic*—FSM, we have a single start state, and every transition from a state of the machine on a letter of the alphabet leads to a unique state. In a *nondeterministic* FSM, or NFSM, we allow ourselves a *set* of start states and a *set* of transitions from a state of the machine on a letter of the alphabet.

Definition 8.1 (Nondeterministic Finite State Machine) A *Nondeterministic Finite State Machine* is a tuple (Q, S, F, δ), where

- Q is a finite set (the *states*)
- $S \subseteq Q$ (the *start states*)
- $F \subseteq Q$ (the *final states*)
- $\delta : Q \times \Sigma \to \mathsf{Pow}(Q)$ (the *transition function*)

Given a state $q \in Q$ and a letter $a \in \Sigma$, we have $\delta(q,a) \subseteq Q$. If $q' \in \delta(q,a)$ we say that there is a transition from q to q' on a; there may be many such transitions from q on a, or there may be none. We can draw an NFSM the same way that we draw an FSM, except that multiple states can be labeled "start" and we can have many (or no) transitions with the same label leaving a particular state.

To define acceptance of a string by an NFSM, we define $\delta^* : Q \times \Sigma \to \mathsf{Pow}(Q)$ by recursion as follows:

$$\begin{aligned}
\delta^*(q,\varepsilon) &= \{q\} \\
\delta^*(q,aw) &= \bigcup \{\delta^*(q',w) \mid q' \in \delta(q,a)\} \\
&= \{q'' \mid \exists q'\ q' \in \delta(q,a) \wedge q'' \in \delta^*(q',w)\}.
\end{aligned}$$

We then define

$$L(M) = \{w \mid \exists s \in S\ \delta^*(s,w) \sqcap F\};$$

that is, a string w is accepted by M if there is a start state $s \in S$ such that $\delta^*(s,w)$ contains a final state.

How should we think of NFSMs as machines? We already have some experience with this in connection with our constructions of the concatenation and star machines from Section 5. We can think of the machine as forking copies of itself whenever there are multiple transition on an input letter, with one copy for each possible transition, and destroying such copies if there are no transitions; a string will be accepted by the system if it is accepted by one of the copies. Alternately, we can think of the machine as choosing, in some unobservable manner from the transitions that are available, the next state that it goes to when reading an input letter. In this way, we can only say what the *possible* states of the machine will be after reading a string, and the machine accepts a string if it is *possible* for it to be in a final state after reading the entire string.

NFSMs, despite their extra flexibility, are equivalent to FSMs as language acceptors. In one direction, clearly every FSM can be considered an NFSM for which every $\delta(q,a)$ is a singleton. In the other direction, we can simulate an NFSM, $M_1 = (Q_1, S_1, F_1, \delta_1)$, using an FSM, $\mathsf{Pow}(M_1)$, whose states are subsets of Q_1; we define the machine $\mathsf{Pow}(M_1)$ as follows:

- $Q = \mathsf{Pow}(Q_1)$
- $s = S_1$
- $F = \{X \in Q \mid X \sqcap F_2\}$
- $\delta(X,a) = \bigcup\{\delta_1(q,a) \mid q \in X\} = \{q' \mid \exists q \in X\ q' \in \delta_1(q,a)\}$.

The result is that $L(M_1) = L(\mathsf{Pow}(M_1))$, completing the equivalence and showing that NFSMs accept only regular sets.

As an example of the usefulness of NFSMs, consider this alternate proof that regular sets are closed under reversal. Let $M_1 = (Q_1, s_1, F_1, \delta_1)$ be a FSM; we will construct an NFSM, $\mathsf{rev}(M_1)$, that accepts $\mathsf{rev}(L(M_1))$. This machine is defined as follows:

- $Q = Q_1$
- $S = F_1$
- $F = \{s_1\}$
- $\delta(q, a) = \delta_1^{-1}(q, a) = \{q' \mid \delta_1(q', a) = q\}$

That is, we just take the machine M_1 and run it nondeterministically in reverse! We start the machine in the final states of M_1, reverse the direction of all of the transitions in δ_1, and accept the string if it is possible to end up in M_1's start state, s_1.

8.2 Adding ε-moves

We can gain even more flexibility in defining machines if we can allow our machines the ability to switch states spontaneously, without reading any input symbols. We call such transitions ε-moves. Here is the definition.

Definition 8.2 (EFSM) A *Nondeterministic Finite State Machine with ε-moves* is a tuple $(Q, S, F, \delta, \varepsilon)$, where

- Q is a finite set (the *states*)
- $S \subseteq Q$ (the *start states*)
- $F \subseteq Q$ (the *final states*)
- $\delta : Q \times \Sigma \to \mathsf{Pow}(Q)$ (the *transition function*)
- $\varepsilon : Q \to \mathsf{Pow}(Q)$ (the *ε-transition function*)

If $q' \in \varepsilon(q)$, we say that the machine can make an ε-transition from q to q'.

- Since the machine is nondeterministic, it doesn't *have* to make an available ε-transition; it can stay where it is.
- The machine can make any number of ε-transitions in a row, all without consuming any characters from the input, if such transitions are possible.

Given a set of states $X \subseteq Q$, we define the ε-closure of X, written X^ε, by the following rules:

$$\frac{q \in X}{q \in X^\varepsilon} \qquad \frac{q \in X^\varepsilon \qquad q' \in \varepsilon(q)}{q' \in X^\varepsilon}\ .$$

A set of states $X \subseteq Q$ is called *ε-closed* if $X^\varepsilon = X$.

We extend the orignal function $\delta : Q \times \Sigma \to \mathsf{Pow}(Q)$ on states to a function $\hat{\delta} : \mathsf{Pow}(Q) \times \Sigma \to \mathsf{Pow}(Q)$ on sets of states by

$$\hat{\delta}(X, a) = \bigcup\{\delta(q, a) \mid q \in X\},$$

and then extend this further to a function $\hat{\delta}^* : \mathsf{Pow}(Q) \times \Sigma^* \to \mathsf{Pow}(Q)$ on strings by induction as follows:

$$\hat{\delta}^*(X, \varepsilon) = X$$
$$\hat{\delta}^*(X, aw) = \hat{\delta}^*(\hat{\delta}(X, a)^\varepsilon, w)$$

It is easy to see by induction that if X is ε-closed, then so is $\hat{\delta}^*(X, w)$ for all strings w. Finally, we define the language accepted by an EFSM:

$$L(M) = \{w \mid \hat{\delta}^*(S^\varepsilon, w) \sqcap F\}.$$

As with NFSMs, we can show the equivalence of EFSMs and FSMs by simulating any EFSM, $M_1 = (Q_1, S_1, F_1, \delta_1, \varepsilon_1)$, using an FSM, $\mathsf{Pow}^\varepsilon(M_1)$ as follows:

- $Q = \{X \mid X \subseteq Q_1 \wedge X^\varepsilon = X\}$
- $S = S_1^\varepsilon$
- $F = \{X \in Q \mid X \sqcap F_1\}$
- $\delta(X, a) = \hat{\delta}(X, a)^\varepsilon$

The states of this machine are thus the ε-closed sets of states of M_1. The start state is ε-closed, as is the result of each transition, so this is indeed an FSM. The result is that $L(M_1) = L(\mathsf{Pow}^\varepsilon(M_1))$.

9 From RE to FSM, Redux

In this section, I will describe three additional constructions that produce machines from regular expressions; the first produces an EFSM, the second produces an NFSM, and the third produces an FSM.

9.1 A standard construction

Our first construction produces an EFSM from any regular expression r, and parallels the recursive construction we gave in Section 5. The machines $M(0)$ and $M(a)$ there can be viewed as EFSMs, so we will just have to show how to form the union, concatenation, and star of EFSMs.

Given disjoint machines $M_1 = (Q_1, S_1, F_1, \delta_1, \varepsilon_1)$ and $M_2 = (Q_2, S_2, F_2, \delta_2, \varepsilon_2)$, we define the machine $M_1 \cup M_2$ as follows:

- $Q = Q_1 \cup Q_2$
- $S = S_1 \cup S_2$
- $F = F_1 \cup F_2$
- $\delta = \delta_1 \cup \delta_2$

- $\varepsilon = \varepsilon_1 \cup \varepsilon_2$.

This is an EFSM and satisfies $L(M_1 \cup M_2) = L(M_1) \cup L(M_2)$.

Given the same M_1 and M_2, we define the machine $M_1 \cdot M_2$ as follows:

- $Q = Q_1 \cup Q_2$
- $S = S_1$
- $F = F_2$
- $\delta = \delta_1 \cup \delta_2$
- $\varepsilon = \varepsilon_1 \cup \varepsilon_2 \cup \{(q_1, q_2) \mid q_1 \in F_1 \wedge q_2 \in S_2\}$.

This is an EFSM and satisfies $L(M_1 \cdot M_2) = L(M_1) \cdot L(M_2)$.

Given M_1 as above, we let q_0 be a new state not already in Q_1 and define the machine M_1^* as follows:

- $Q = Q_1 \cup \{q_0\}$
- $S = \{q_0\}$
- $F = F_1 \cup \{q_0\}$
- $\delta = \delta_1 \cup \{(q_0, a, q)) \mid a \in \Sigma \wedge \exists s(s \in S_1 \wedge q \in \delta(s, a))\}$
- $\varepsilon = \varepsilon_1 \cup \{(q, s) \mid q \in F_1 \wedge s \in S_1\}$.

This is an EFSM and satisfies $L(M_1^*) = L(M_1)^*$.

9.2 A construction of Glushkov

Let's call an NFSM $M = (Q, S, F, \delta)$ *standard* if Q is a subset of the natural numbers, $F = \{0\}$, and all states in Q are reachable from S using δ. A standard machine can be specified using only the pair (S, δ), since F is determined and Q can be recovered from δ. Let's write $\#M = 1 + \max(S)$, which is the smallest natural number that is greater than all states of M.

Our second construction, due to Victor Glushkov—and, independently, to Robert McNaughton and Hisao Yamada—produces, for any standard NFSM $M = (S, \delta)$, regular expression r, and natural number $n \geq \#M$, a standard machine $r \cdot M@n$ such that $L(r \cdot M@n) = [\![r]\!] \cdot L(M)$ and any state q of $r \cdot M@n$ that is not a state of M satisfies $q \geq n$. Here are the constructions.

Empty. The machine $0 \cdot M@n$ is $(\emptyset, \emptyset)$.

Letter. The machine $\mathtt{a} \cdot M@n$ is $(\{n\}, \delta \cup \{(n, \mathtt{a}, S)\})$.

Union. Let $M_2 = (S_2, \delta_2)$ be the machine $r_2 \cdot M@n$, and let $(S_1, \delta_1) = r_1 \cdot M@\#M_2$. Then the machine $(r_1 + r_2) \cdot M@n$ is $(S_1 \cup S_2, \delta_1 \cup \delta_2)$.

Concatenation. Let $M_2 = (S_2, \delta_2)$ be the machine $r_2 \cdot M@n$. Then the machine $(r_1 \cdot r_2) \cdot M@n$ is $r_1 \cdot M_2@\#M_2$.

Star. Let S_1 be the smallest set such that $(S_1, \delta_1) = r_1 \cdot (S_1 \cup S, \delta)@n$. Then the machine $r_1^* \cdot M@n$ is (S_1, δ_1).

9.3 A construction of Brzozowski

10 Beyond Regularity

10.1 The Pumping Lemma

In this section, we prove a basic result that allows us to show that certain languages are *not* regular.

10.2 The Myhill-Nerode Theorem

In this section, we prove a characterization of regular languages that gives us an even more powerful—and theoretically completely general—method for proving that languages are not regular, which moreover has the very useful side-effect of giving us a method to construct the *minimal* FSM accepting a given regular language. The method defines, for any language L, an equivalence relation $\equiv_L$ on Σ^*, and then constructs a "machine" M whose states, Q, are the equivalence classes of $\equiv_L$. The result is that L is regular iff Q is *finite*, in which case M is an FSM, $L(M) = L$, and M the minimum number of states of any machine accepting L.

10.3 Context-Free Grammars

10.4 Push-Down Automata

10.5 Other topics