The course-design report of signals and systems

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Abstract： This report talks about two questions and their expansions discussed in the signals and systems course. One (question one) is about radar systems, including techniques like pulse compression and matched filtering. The other (question five) is about amplitude modulation, including techniques like frequency modulation in non-linear signals and pre-distortion. All covered in this paper involves ideas discussed in the team or class, materials on the Internet and thoughts of the writer.

Key words： Pulse compression, Matched filtering, Frequency modulation, Pre-distortion

1. Question One

1.1 Linear Frequency Modulated Signal

1.1.1 Background

The radar often uses wide pulse transmission. It increases the average power so that a greater transmission distance can be achieved.

Frequency or phase modulated waveforms can be applied to achieve wider operating bandwidths mentioned above. As a matter of fact, Linear Frequency Modulation (LFM) is commonly used in the process.

1.1.2 The LFM Waveforms

As its name tells us, in linear frequency modulation, the frequency is swept linearly across the pulse-width, either upward (up-chirp) or downward (down-chirp). Figure 1 shows a typical example of an LFM waveform. The pulse width is τ, and the bandwidth is B.

The LFM up-chirp instantaneous phase can be expressed by

$$\psi(t)=2\pi(f_c t+\frac{k}{2}t^2) \qquad -\frac{\tau}{2}\le t\le\frac{\tau}{2}$$

where f_c is the radar center frequency and k is the slope of frequency modulation. Thus, the instantaneous frequency is

$$f(t)=\frac{1}{2\pi}\frac{d}{dt}\psi(t)=f_c+kt \qquad -\frac{\tau}{2}\le t\le\frac{\tau}{2}$$

Similarly, the down-chirp instantaneous phase and frequency are given, respectively, by

$$\psi(t)=2\pi(f_c t-\frac{k}{2}t^2) \qquad -\frac{\tau}{2}\le t\le\frac{\tau}{2}$$

$$f(t)=\frac{1}{2\pi}\frac{d}{dt}\psi(t)=f_c-kt \qquad -\frac{\tau}{2}\le t\le\frac{\tau}{2}$$

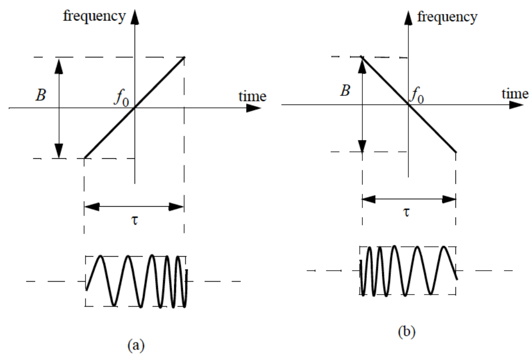

Figure 1. (a) up-chirp (b) down-chirp

A typical LFM waveform can be expressed by

$$s(t)=\mathrm{Re}ct(\frac{t}{\tau})e^{j2\pi(f_c t+\frac{k}{2}t^2)}$$

$Rect(\frac{t}{\tau})$ is the rectangular signal,

$$\mathrm{Re}ct(\frac{t}{\tau})=\begin{cases}1, |t|\le\frac{\tau}{2}\\ 0, |t|>\frac{\tau}{2}\end{cases}$$

Then,

we do Fourier Transform on $s(t)$,

$$S(\omega)=\tau\sqrt{\frac{1}{B\tau}}\,e^{-j\omega^2/(4\pi B)}\left\{\frac{[C(x_2)+C(x_1)]+j[S(x_2)+S(x_1)]}{\sqrt{2}}\right\}$$

where,

$$C(x)\approx\frac{1}{2}+\frac{1}{\pi x}\sin\left(\frac{\pi}{2}x^2\right) \qquad S(x)\approx\frac{1}{2}-\frac{1}{\pi x}\cos\left(\frac{\pi}{2}x^2\right)$$

The corresponding amplitude spectrum is,

$$|S(f)|=\sqrt{\frac{2}{k}}\mathrm{Re}ct(\frac{f-f_c}{B})$$

Particularly, in question one, part one, when

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$f_c = 0Hz, T = 10\mu s, B = 40MHz$, we can draw its waveform and the corresponding amplitude spectrum.

$$S(j\omega) = \frac{1}{2} e^{-j\frac{\pi}{4}(\omega^2 - 1)}$$

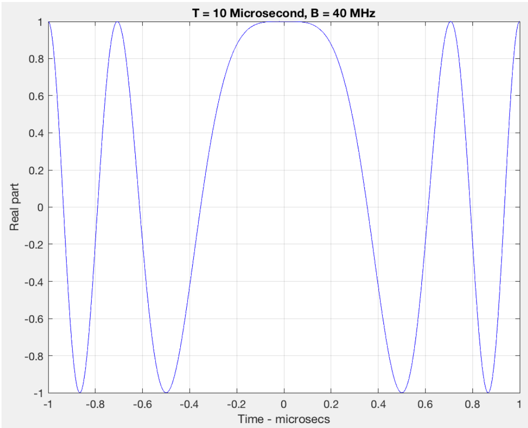


Figure 2. Real part of the signal

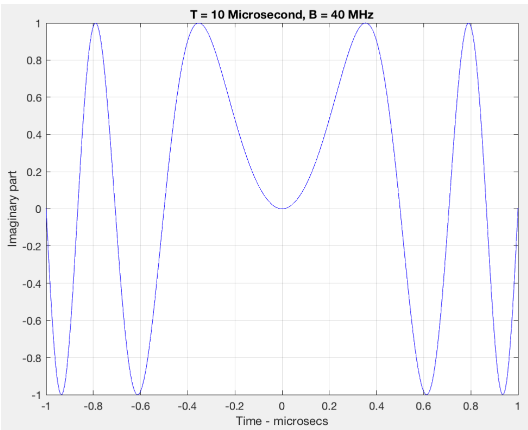


Figure 3. Imaginary part of the signal

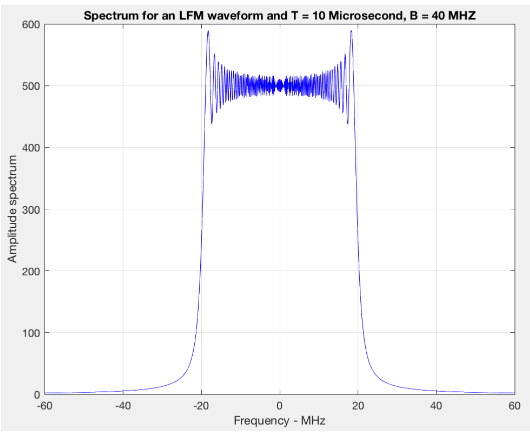


Figure 4. Amplitude spectrum of the signal

1.2 Pulse Compression

1.2.1 Background

As I mentioned above, in order to increase the transmission ability, we often use LFM waveforms which have a wide bandwidth. However, a wide bandwidth does not bring us a high range resolution in the receiving process.

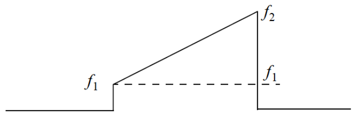
So, we often do pulse compression on the signal received in order to get a signal with a narrower bandwidth, which improves range resolution.

In summary, pulse compression allows us to achieve the average transmitted power of a relatively long pulse, while obtaining the range resolution corresponding to a short pulse.

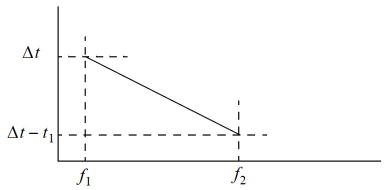
1.2.2 The LFM pulse compression

Linear Frequency Modulation pulse compression is accomplished by adding frequency modulation to a long pulse at transmission, and by using a matched filter receiver in order to compress the received signal. As a result, the matched filter output is compressed by a factor $D = B\tau'$, where B is the bandwidth and τ' is the pulse-width.

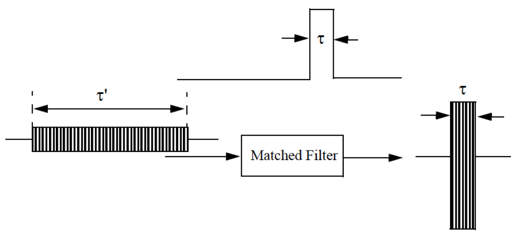
The following shows the ideal LFM pulse compression process.



(a) The frequency modulation



(b) Matched filter time-delay characteristic



(c) The Matched filter input/output waveforms

1.3 Noise Handling

1.3.1 Background

We know that noise can not be avoided in the signal transmission process. Actually, we often use Signal to Noise Ratio (SNR in short) to judge the quality of a receiver. The receiver with highest SNR is called matched filter.

1.3.2 Matched filter

Assume the input signal of an LIT system is $x(t) = s(t) + n(t)$, where $s(t)$ is the LFM signal, $n(t)$ is a stationary white noise with zero mean and its power spectral density is $N_0/2$. This is often the case discussed and let us talk about the matched filter of this system.

The energy of the input signal is,

$$E(s) = \int_{-\infty}^{\infty} s^2(t)\, dt < \infty$$

Consider the output signal $s_0(t) = s(t) * h(t)$, where $h(t)$ is the impulse response of the matched filter.

Then,

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-j\omega t}\, dt$$

$$S_0(\omega) = H(\omega) S(\omega)$$

$$s_0(t) = \frac{1}{2\pi} \int_{-\omega}^{\infty} H(\omega)\, S(\omega) e^{j\omega t} d\omega$$

The average power of the input noise is,

$$E[n_0^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_{n_0}(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} H^2(\omega) P_n(\omega)\, d\omega$$

So, the SNR is,

$$SNR = \frac{(\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) S(\omega) e^{j\omega t_0}\, d\omega)^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty} (H(\omega))^2 P_n(\omega) d\omega}$$

Using Schwarz inequality,

$$SNR \le \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(S(\omega))^2}{P_n(\omega)} d\omega$$

The condition of the equality is,

$$H(\omega) = \frac{\alpha S^*(\omega)}{P_n(\omega)} e^{-j\omega t_0} = k S^*(\omega) e^{-j\omega t_0}, k = \frac{2\alpha}{N_0}$$

So, when the impulse response of the matched filter is,

$$h(t) = k s^*(t_0 - t)$$

And we can achieve the max SNR,

$$SNR_{\max} = \frac{2E_s}{N_0}$$

1.4 The pulse compression of the Matched filter in question one

1.4.1 Pulse compression ability

In question one, part two, the impulse response of the matched filter is $h(t) = s^*(-t)$.

Let us see, after the matched filtering, what the signal is.

I firstly compute the general result. I omit some computing details in the process.

$$h(t) = s^*(-t) = \mathrm{Re}\,ct(\frac{t}{T}) e^{-j\pi k t^2} e^{j2\pi f_c t}$$

$$s_0(t) = s(t) * h(t) = \int_{-\infty}^{\infty} h(u) s(t-u) du$$

$$s_0(t) = \int_{-\infty}^{\infty} e^{j\pi k t^2} e^{-j2\pi k t u} \mathrm{Re}\,ct(\frac{u}{T}) \mathrm{Re}\,ct(\frac{t-u}{T}) du$$

When $0 < t \le T/2$,

$$s_0(t) = \int_{t-\frac{1}{2}T}^{\frac{1}{2}T} e^{j\pi k t^2} e^{-j2\pi k t u} du$$

$$s_0(t) = \frac{\sin[\pi k (T-t) t]}{\pi k t} e^{j2\pi f_c t}$$

When $-T/2 < t \le 0$,

$$s_0(t) = \int_{-\frac{1}{2}T}^{t+\frac{1}{2}T} e^{j\pi k t^2} e^{-j2\pi k t u} du$$

$$s_0(t) = \frac{\sin[\pi k (T+t) t]}{\pi k t} e^{j2\pi f_c t}$$

In summary, when $f_c = 0$,

$$s_0(t) = T \frac{\sin[\pi k T t (1 - \frac{|t|}{T})]}{\pi k T t} \mathrm{Re}\,ct(\frac{t}{T})$$

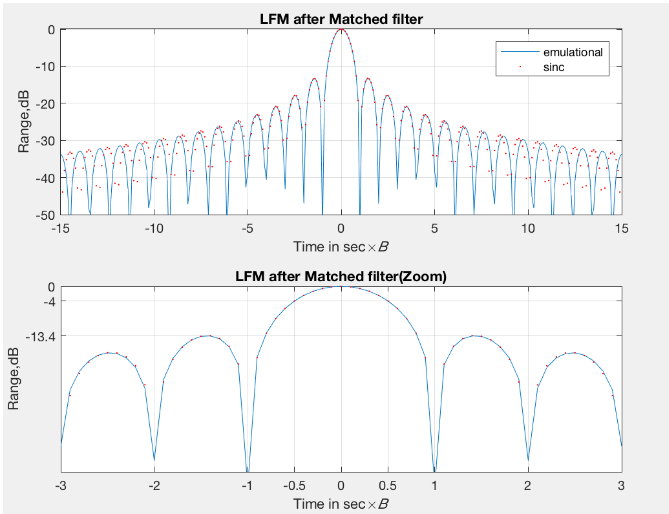
$$s_0(t) \approx T sa(\pi k T t) \mathrm{Re}\,ct(\frac{t}{T})$$

$$s_0(t) \approx T sa(\pi B t) \mathrm{Re}\,ct(\frac{t}{T})$$

We can see that the output of a LFM signal is like a sinc function in time domain.

We know that according the pulse compression, the bandwidth of the compressed signal is $1/B$.

We use Matlab to draw the figure and its zoom figure of the compressed signal, using $t * B$ as the x-axis.



It is clear seen that the first zero is at $t * B = 1$, which tells that the bandwidth of the compressed signal is $1/B$, verifying the statement above.

1.4.2 The discussions on the output signal

It is found that generally, the output of a LFM signal through the Matched filter above is approximately a sinc function.

However, $B * T$ in this case is 400, which is bigger.

The figure of the output signal in our question is shown in Figure 5.

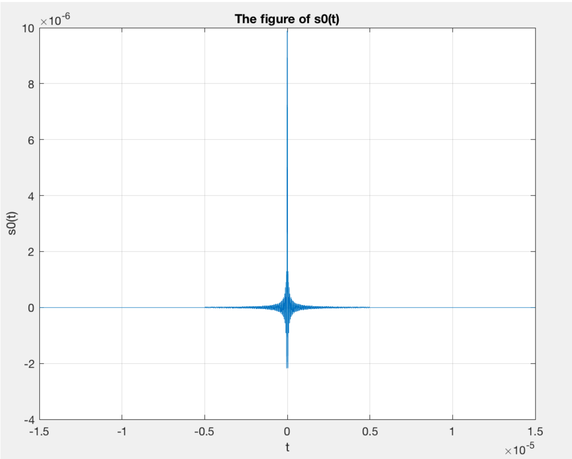


Figure 5. The figure of s0(t)

In such circumstances when $B * T$ is huge, the output signal is approximately an impulse response.

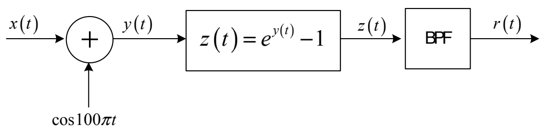
Actually, using precise calculation instead of approximate sinc function, we can get the precise result of the output signal,

$$s_0(t) = \frac{1}{4}\delta(t)$$

In our daily lives, I think, it is a good way for us to get an impulse response with less time and higher precise.

2. Question Two

2. 1 Brief analysis and problems we meet



Firstly, we know that $y(t) = x(t) + \cos(100\pi t)$,

Then, we get $z(t) = e^{y(t)} - 1 = e^{x(t)+\cos(100\pi t)} - 1$

Note that $e^x \approx 1 + x + \frac{1}{2}x^2$ (keep the first three expansions), we get,

$$z(t) = x(t) + \cos(100\pi t) + \frac{1}{2}x^2(t) + x(t)\cos((100\pi t) + \frac{1}{2}(\cos(100\pi t))^2$$

We do Fourier Transform on $z(t)$ and we get the signal in the frequency domain,

$$Z(j\omega) = X(j\omega) + \pi[\delta(\omega - 100\pi) + \delta(\omega + 100\pi) + \frac{1}{2}[X(j(\omega - 100\pi)) + X(j(\omega + 100\pi))] + \frac{x(t) * x(t)}{4\pi} + \frac{\pi[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)}{4} + \frac{\pi\delta(\omega)}{4}$$

In order to get $x(t)\cos(100\pi t)$, we ignore other irrelevant signals and focus on figures around $-99\pi \sim -101\pi$ and $99\pi \sim 101\pi$ where $\frac{1}{2}[X(j(\omega - 100\pi)) + X(j(\omega + 100\pi))]$ lies. (Figure 6.)

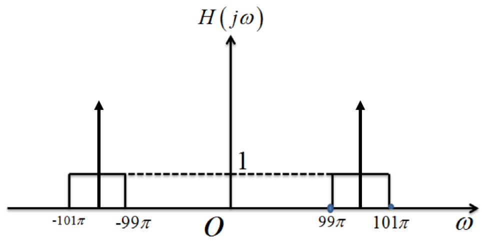


Figure 6. The figure at certain sections

However, we see that if we put two windows and extract two parts out, we get actually $(x(t) + 1)\cos(100\pi t)$, because two impulse responses interfere.

That is the main problem we meet.

2. 2 Possible solutions

One possible solution is adding a subtractor at the end, subtracting $\cos(100\pi t)$.

The other possible solution is subtracting $x(t)$ at the beginning by one, that is to say, $y(t) = [x(t) - 1] + \cos(100\pi t)$.

2.3 Discussion on the Fourier expansion

I think the question is not such easy and there exists much that deserves us to think about and discuss.

We observe that it keeps the first three expansions when doing Fourier expansion at figuring out $z(t)$. So, the question comes, why does it stop at the third expansion? Is it wrong or right?

We define $m(n) = \sum_{i=1}^{n} \frac{y^2(t)}{i!}$

$$m(n) = \sum_{i=1}^{n} \frac{(x(t) + \cos\omega_c t)^2}{i!}$$

$$m(n)=\frac{\cos^{n}(\omega_c t)}{n!}+[\frac{1+x(t)}{(n-1)!}]\cos^{n-1}(\omega_c t)+[\frac{1+x(t)+\frac{1}{2}x^2(t)}{(n-2)!}]\cos^{n-2}(\omega_c t)......$$

We know that,

$$\cos^{n}(\omega_c t)=(\frac{e^{j\omega_c t}+e^{-j\omega_c t}}{2})^{n}$$

$$\cos^{n}(\omega_c t)=\frac{1}{2^{n-1}}[C_n^0\cos(n\omega_c t)+C_n^1\cos((n-1)\omega_c t)+......]$$

So,

$$m(n)=[\frac{1}{n!}\frac{1}{2^{n-1}}]\cos(n\omega_c t)+[\frac{1+x(t)}{(n-1)!}\frac{1}{2^{n-2}}\cos((n-1)\omega_c t)$$
$$+[\frac{1+x(t)+\frac{1}{2}x^2(t)}{(n-2)!}\frac{1}{2^{n-3}}]\cos((n-2)\omega_c t)+......$$

We find that if n is greater than 3 (including 3), non-linear distortion like $x^2(t)$ will get introduced. In other words, non-linear distortion is obvious at lower frequency. So, if we pick n as 2, we will not involve higher-order $x(t)$, the non-linear distortion into our system. Therefore, it is wise that n is picked as 2 and Fourier expansion stops at the third one.

However, if we compare the exact figure of $z(t)=e^{x(t)+\cos(100\pi t)}-1$ and the figure of $z(t)$ after the Fourier expansion in the question,

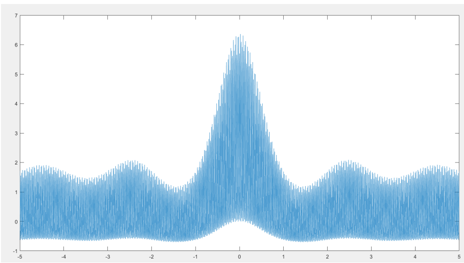


Figure 7. The exact figure of $z(t)$

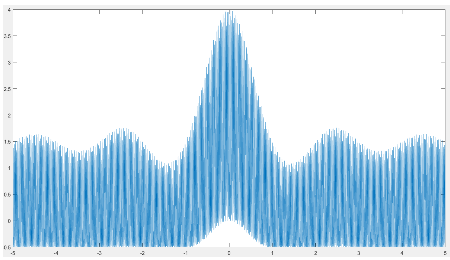


Figure 8. The figure of $z(t)$ after Fourier expansion

We can still see much distortion.

So, how on earth can we deal with distortion in the signal processing?

2.4 Pre-distortion

Mr. Shi introduces pre-distortion in class.

Pre-distortion deals with the non-linear distortion, for example, if $y(t)=x(t)+\frac{1}{2}x^2(t)$ and $x(t)$ is what we want to get, we can input $x(t)-\frac{1}{2}x^2(t)$ to replace $x(t)$.

Then,

$$y'(t)=[x(t)-\frac{1}{2}x^2(t)]+\frac{1}{2}[x(t)-\frac{1}{2}x^2(t)]^2$$
$$=x(t)-\frac{1}{2}x^3(t)+\frac{1}{8}x^4(t)$$

It can be seen that two-order distortion is transformed to higher-order distortion. So, if we apply this trick again and again, the distortion will get smaller.

Actually, pre-distortion is often used to improve the linearity of radio transmitter amplifiers. It is cost-saving and power efficiency compared to other techniques dealing with non-linear distortion.

3. Summary

This paper mainly discusses two course-design questions and their expansions talked about in the group and with other classmates in class.

In discussion courses, I am blessed with the opportunity to communicate with other mates about some problems related to signals and systems. By sharing ideas, I am impressed by some novel ideas from my classmates and I expand my horizons during the exploration in new areas.

I hope that I can keep my curiosity and the habit of thinking deeply in my future study.

Thanks for the devotion of Mr. Lv and Mr. Shi.

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