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# 1. Laboratory Objectives

The inverted pendulum is a classic experiment used to teach dynamics and control systems. In this laboratory, the pendulum dynamics are derived using Lagrangian equations and an introduction to nonlinear control is made.

There are two control challenges: designing a balance controller and designing a swing-up control. After manually initializing the pendulum in the upright vertical position, the balance controller moves the rotary arm to keep the pendulum in this upright position. It is designed using the Linear-Quadratic Regulator technique on a linearized model of the rotary pendulum system.

The swing-up controller drives the pendulum from its suspended downward position to the vertical upright position, where the balance controller can then be used to balance the link. The pendulum equation of motion is derived using Lagrangian principles and the pendulum moment of inertia is identified experimentally to obtain a model that represents the pendulum more accurately. The swing-up controller is designed using the pendulum model and a Lyapunov function. Lyapunov functions are commonly used in control theory and design and it will be introduced to design the nonlinear swing-up control.

***Regarding Gray Boxes:***

Gray boxes present in the **instructor manual** are not intended for the students as they provide solutions to the pre-lab assignments and contain typical experimental results from the laboratory procedure.

# 2. References

[1] *QNET-ROTPEN User Manual.*
[2] *NI-ELVIS User Manual.*
[3] *QNET Experiment #03: ROTPEN Gantry Control*

# 3. ROTPEN Plant Presentation

## 3.1. Component Nomenclature

As a quick nomenclature, Table 1, below, provides a list of the principal elements composing the Rotary Pendulum (ROTPEN) Trainer system. Every element is located and identified, through a unique identification (ID) number, on the ROTPEN plant represented in Figure 1, below.

| ID # | Description | ID # | Description |
|---|---|---|---|
| **1** | DC Motor | **3** | Arm |
| **2** | Motor/Arm Encoder | **4** | Pendulum |

Table 1 ROTPEN Component Nomenclature


Figure 1 ROTPEN System

## 3.2. ROTPEN Plant Description

The QNET-ROTPEN Trainer system consists of a 24-Volt DC motor that is coupled with an encoder and is mounted vertically in the metal chamber. The L-shaped arm, or hub, is connected to the motor shaft and pivots between ±180 degrees. At the end of the arm, there is a suspended pendulum attached. The pendulum angle is measured by an encoder.

# 4. Pre-Lab Assignments

**This section must be read, understood, and performed before you go to the laboratory session.**

The first section, Section 4.1, summarizes the control design method using the linear-quadratic regulator technique to construct the balance control. Section 4.2 is the first pre-lab exercise and involves modeling the open-loop pendulum using Lagrangian. The second pre-lab assignment, in Section 4.3, develops the equations needed to experimentally identify the pendulum inertia. Lastly, the last pre-lab exercise in Section 4.4 is designing the swing-up control.

## 4.1. Balance Control Design

Section 4.1.1 discusses the model of the inverted pendulum and the resulting linear state-space representation of the device. The design of a controller that balances an inverted pendulum is summarized in Section 4.1.2.

### 4.1.1. Open-Loop Modeling

As already discussed in the gantry experiment, ROTPEN Laboratory #3, the ROTPEN plant is free to move in two rotary directions. Thus it is a two degree of freedom, or 2 DOF, system. As described in Figure 2, the arm rotates about the $Y_0$ axis and its angle is denoted by the symbol $\theta$ while the pendulum attached to the arm rotates about its pivot and its angle is called $\alpha$. The shaft of the DC motor is connected to the arm pivot and the input voltage of the motor is the control variable.

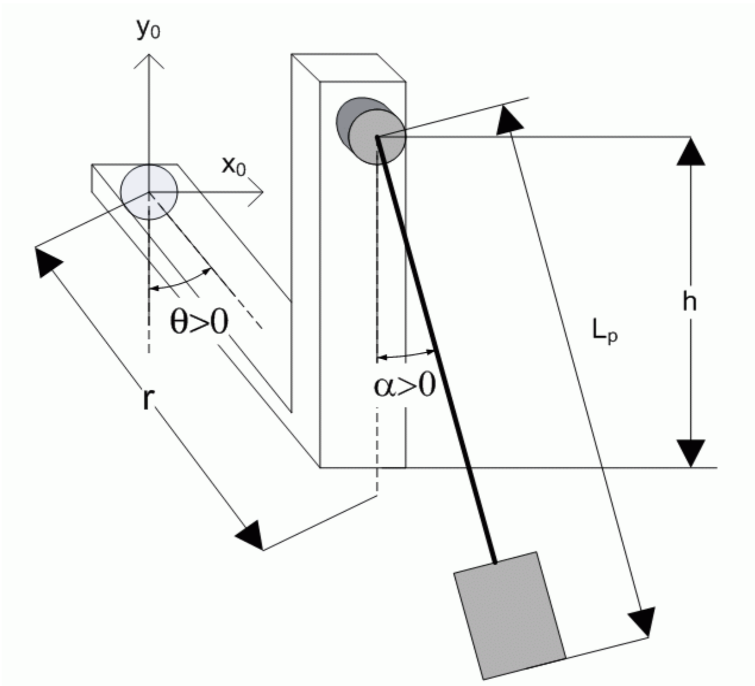


Figure 2 Rotary Pendulum System

In the inverted pendulum experiment, the pendulum angle, $\alpha$, is defined to be positive when the it rotates counter-clockwise. That is, as the arm moves in the positive clockwise direction, the *inverted* pendulum moves clockwise (i.e. the *suspended* pendulum moves counter-clockwise) and that is defined as $\alpha>0$. Recall that in the gantry device, when the arm rotates in the positive clockwise direction the pendulum moves clockwise, which in turn is defined as being positive.

The nonlinear dynamics between the angle of the arm, $\theta$, the angle of the pendulum, $\alpha$, and the torque applied at the arm pivot, $\tau_{output}$, are

$$\frac{d^2}{dt^2}\theta(t) = -\frac{M_p{}^2\, g\, l_p{}^2\, r\cos(\theta(t))\,\alpha(t)}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p{}^2\, J_{eq}}$$
$$- \frac{J_p\, M_p\, r^2 \cos(\theta(t))\sin(\theta(t))\left(\frac{d}{dt}\theta(t)\right)^2}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p{}^2\, J_{eq}}$$
$$- \frac{J_p\, \tau_{output} + M_p\, l_p{}^2\, \tau_{output}}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p{}^2\, J_{eq}} \qquad [1]$$

$$\frac{d^2}{dt^2}\alpha(t) = -\frac{l_p\, M_p\, (-J_{eq}\, g + M_p\, r^2 \sin(\theta(t))^2\, g - M_p\, r^2\, g)\,\alpha(t)}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p{}^2\, J_{eq}}$$
$$- \frac{l_p\, M_p\, r \sin(\theta(t))\, J_{eq}\left(\frac{d}{dt}\theta(t)\right)^2}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p{}^2\, J_{eq}}$$
$$+ \frac{l_p\, M_p\, r\, \tau_{output} \cos(\theta(t))}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p{}^2\, J_{eq}}$$

where the torque generated at the arm pivot by the motor voltage $V_m$ is

$$\tau_{output} = \frac{K_t\left(V_m - K_m\left(\frac{d}{dt}\theta(t)\right)\right)}{R_m}. \qquad [2]$$

The ROTPEN model parameters used in [1] and [2] are defined in Table 2.

| *Symbol* | *Description* | *Value* | *Unit* |
|---|---|---|---|
| $M_p$ | Mass of the pendulum assembly (weight and link combined). | 0.027 | kg |
| $l_p$ | Length of pendulum center of mass from pivot. | 0.153 | m |
| $L_p$ | Total length of pendulum. | 0.191 | m |
| r | Length of arm pivot to pendulum pivot. | 0.08260 | m |

| *Symbol* | *Description* | *Value* | *Unit* |
|---|---|---|---|
| $J_m$ | Motor shaft moment of inertia. | 3.00E-005 | $kg \cdot m^2$ |
| $M_{arm}$ | Mass of arm. | 0.028 | kg |
| g | Gravitational acceleration constant. | 9.810 | $m/s^2$ |
| $J_{eq}$ | Equivalent moment of inertia about motor shaft pivot axis. | 1.23E-004 | $kg \cdot m^2$ |
| $J_p$ | Pendulum moment of inertia about its pivot axis. | 1.10E-004 | $kg \cdot m^2$ |
| $B_{eq}$ | Arm viscous damping. | 0.000 | N·m/(rad/s) |
| $B_p$ | Pendulum viscous damping. | 0.000 | N·m/(rad/s) |
| $R_m$ | Motor armature resistance. | 3.30 | Ω |
| $K_t$ | Motor torque constant. | 0.02797 | N·m |
| $K_m$ | Motor back-electromotive force constant. | 0.02797 | V/(rad/s) |

Table 2 ROTPEN Model Nomenclature

The pendulum center of mass, $l_p$, is not given in Table 2 since it was calculated in the previous experiment, *ROTPEN Laboratory #3 – Gantry*. The moment of inertia parameter, $J_p$, is not given because it will be determined experimentally in this laboratory. However, the $J_p$ that was calculated in *ROTPEN Laboratory #3 – Gantry* is still used in this experiment for comparison purposes. The viscous damping parameters of the pendulum, $B_p$, and of the arm, $B_{eq}$, are regarded as being negligible in this laboratory.

Similarly in *ROTPEN Laboratory #3*, the linear equations of motion of the system are found by linearizing the nonlinear equations of motions, or EOMs, presented in [1] about the operation point $\alpha = \pi$ and solving for the acceleration of the terms θ and α. For the state

$$x = [x_1, x_2, x_3, x_4]^T \qquad [3]$$

where

$$x_1 = \theta \text{ , } x_2 = \alpha \text{ , } x_3 = \frac{\partial}{\partial t}\theta \text{ , and } x_4 = \frac{\partial}{\partial t}\alpha \qquad [4]$$

the linear state-space representation of the *ROTPEN Inverted Pendulum* is

$$\frac{d}{dt}\mathrm{x}(t) = A\,\mathrm{x}(t) + B\,\mathrm{u}(x)$$
$$\mathrm{y}(t) = C\,\mathrm{x}(t) + \mathrm{D}\,\mathrm{u}(x) \qquad [5]$$

where $u(x) = V_m$ and the $A$, $B$, $C$, and $D$ matrices are

| ***State-Space Matrix*** | ***Expression*** |
| --- | --- |
| A | $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{r M_p^2 l_p^2 g}{J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2} & -\dfrac{K_t K_m (J_p + M_p l_p^2)}{(J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2) R_m} & 0 \\ 0 & \dfrac{M_p l_p g (J_{eq} + M_p r^2)}{J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2} & \dfrac{-M_p l_p K_t r K_m}{(J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2) R_m} & 0 \end{bmatrix}$ |
| B | $\begin{bmatrix} 0 \\ 0 \\ \dfrac{K_t (J_p + M_p l_p^2)}{(J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2) R_m} \\ \dfrac{M_p l_p K_t r}{(J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2) R_m} \end{bmatrix}$ |
| C | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| D | $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ |

Table 3 Linear State-Space Matrices

### 4.1.2. LQR Control Design

The problem of balancing an inverted pendulum is like balancing a vertical stick with your hand by moving it back and forth. Thus by supplying the appropriate linear force, the stick can be kept more-or-less vertical. In this case, the pendulum is being balanced by applying torque to the arm. The balance controller supplies a motor voltage that applies a torque to the pendulum pivot and the amount of voltage supplied depends on the angular position and speed of both the arm and the pendulum.

Recall that the *linear quadratic regulator* problem is: given a plant model

$$\frac{d}{dt}\mathrm{x}(t) = A\,\mathrm{x}(t) + B\,\mathrm{u}(t) \qquad [6]$$

find a control input $u$ that minimizes the cost function

$$J = \int_0^\infty \mathrm{x}(t)^T Q\,\mathrm{x}(t) + \mathrm{u}(t)^T R\,\mathrm{u}(t)\,dt \qquad [7]$$

where Q is an $n \times n$ positive semidefinite weighing matrix and R is an $r \times r$ positive definite symmetric matrix. That is, find a control gain $K$ in the state feedback control law

$$u = K\,x \qquad [8]$$

such that the quadratic cost function $J$ is minimized.

The $Q$ and $R$ matrices set by the user affects the optimal control gain that is generated to minimize $J$. The closed-loop control performance is affected by changing the $Q$ and $R$ weighing matrices. Generally, the control input $u$ will work harder and therefore a larger gain, $K$, will be generated if $Q$ is made larger. Likewise, a larger gain will be computed by the LQR algorithm if the $R$ weighing matrix is made smaller.

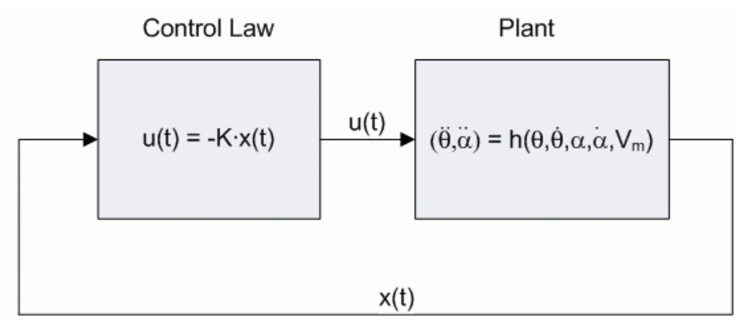


Figure 3 Closed-Loop Control System

The closed-loop system that balances the pendulum is shown in Figure 3. The controller computes a voltage $V_m$ depending on the position and velocity of the arm and pendulum angles. The box labeled *Plant* shown in Figure 3 represents the nonlinear dynamics given in [1] and [2]. Similarly to the gantry experiment, the LQR gain $K$ is automatically generated in the LabView Virtual Instrument by tuning the $Q$ and $R$ matrix.

### 4.1.3. Inverted Pendulum Control Specifications

Design an LQR control, that is tune the *Q* weighing matrix, such that the closed-loop response meets the following specifications:

(1) **Arm Regulation**: $|\theta(t)| < 30°$

(2) **Pendulum Regulation**: $|\alpha(t)| < 3°$

(3) **Control input limit**: $V_m < 12$ V

Thus the control should regulate the arm about zero degrees within 30° as it balances the pendulum without angle $|\alpha|$ going beyond 3°. The arm angle is re-defined to zero degrees, $\theta = 0°$, when the balance controller is activated. Additionally, the control input must be kept under the voltage range of the motor, 12 Volts.

## 4.2. Pre-Lab Assignment #1: Open-loop Modeling of the Pendulum

In Reference [3], the full model representing the two degrees-of-freedom motion of the gantry was developed using Lagrange. The following exercises deals instead with modeling only the pendulum shown in Figure 4 and assuming that the torque at the pendulum pivot, which is not directly actuated, is a control variable. Later, the dynamics between the input voltage of the DC motor and the torque applied to the pendulum pivot will be expressed. The Lagrange method will be used to find the *nonlinear* equations of motion of the pendulum. Thus the kinematics, potential energy, and kinetic energy are first calculated and the equations of motion are found using Euler-Lagrange.

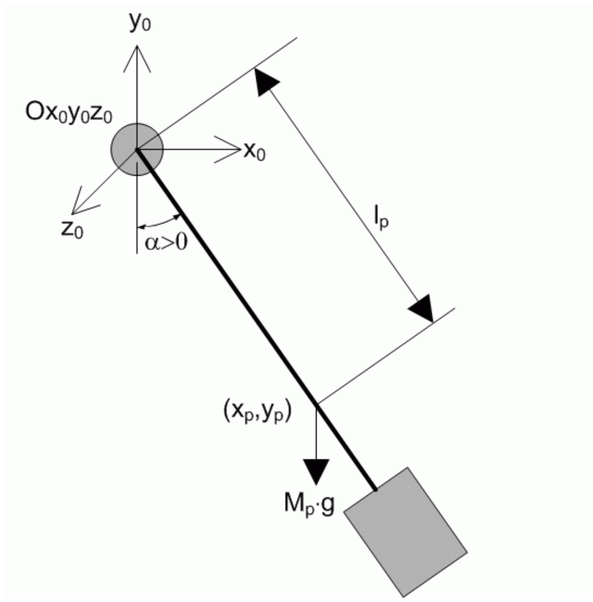


Figure 4 Free body diagram of pendulum considered a single rigid body.

### 4.2.1. Exercise: Kinematics

Figure 4 is the pendulum of the ROTPEN system when being considered as a single rigid object. It rotates about the axis $z_0$, at an angle $\alpha$ that is positive, by convention, when the pendulum moves in the counter-clockwise fashion. Further, $\alpha = 0$ when the pendulum is in the vertical downward position. Find the forward kinematics of the center of mass, or CM, of the pendulum with respect to the base frame $o_0x_0y_0z_0$, as shown in Figure 4 More specifically, express the position, $x_p$ and $y_p$, of the CM and the velocity, $xd_p$ and $yd_p$, of the pendulum CM in terms of the angle $\alpha$.

***Solution:***
The kinematics of the pendulum CM relative to the base coordinate system $o_0x_0y_0z_0$ is

$$x_p = l_p \sin(\alpha(t))$$
$$y_p = -l_p \cos(\alpha(t)) \qquad [s1]$$

and the velocity components are found by taking the derivative of [s1] with respect to time

$$xd_p = l_p \cos(\alpha(t)) \left(\frac{d}{dt}\alpha(t)\right)$$
$$yd_p = l_p \sin(\alpha(t)) \left(\frac{d}{dt}\alpha(t)\right). \qquad [s2]$$

### 4.2.2. Exercise: Potential Energy

Express the total potential energy, to be denoted as $U_T(\alpha)$, of the rotary pendulum system. The gravitational potential energy depends on the vertical position of the pendulum center-of-mass. The potential energy expression should be 0 Joules when the pendulum is at $\alpha = 0$, the downward position, and is positive when the $\alpha > 0$. It should reach its maximum value when the pendulum is upright and perfectly vertical.

***Solution:***
The gravitational potential energy is dependent on the vertical position of the pendulum CM. Thus using expression $y_p$ from the above exercise, the total potential energy of the system is

$$U_t(\alpha) = M_p\, g\, l_p\, (1 - \cos(\alpha(t))). \qquad [s3]$$

The potential energy is zero when the pendulum is at rest suspending, $U_t(0) = 0\ J$, and is at its maximum value when the pendulum is brought in upright vertical position, $U_t(\pi) = 2M_p g l_p$.

### 4.2.3. Exercise: Kinetic Energy

Find the total kinetic energy, $T_t$, of the pendulum. In this case, the system being considered is a pendulum that rotates about a fixed pivot, therefore the entire kinetic energy is rotational kinetic energy.

***Solution:***
The total kinetic energy of the system can be described is

$$T_t = \frac{1}{2} J_p \left( \frac{d}{dt} \alpha(t) \right)^2 . \qquad [s4]$$

The pendulum inertia parameter, $J_p$, is found experimentally in a later exercise.

### 4.2.4. Exercise: Lagrange of System

Calculate the *Lagrangian* of the pendulum

$$L\left( \alpha, \frac{d}{dt} \alpha(t) \right) = T_t - U_t . \qquad [9]$$

where $T_t$ is the total kinetic energy calculated in Exercise 4.2.3, and $U_t$ is the total potential energy of the system calculated in Exercise 4.2.2.

***Solution:***
The Lagrange of the system for the position and velocity of α is

$$L\left( \alpha, \frac{d}{dt} \alpha(t) \right) = \frac{1}{2} J_p \left( \frac{d}{dt} \alpha(t) \right)^2 - M_p \, g \, l_p \, (1 - \cos(\alpha(t))) . \qquad [s5]$$

### 4.2.5. Exercise: Euler-Lagrange Equations of Motions

The Euler-Lagrange equations of motion are calculated from the Lagrangian of a system using

$$\left( \frac{\partial^2}{\partial t \, \partial qdot_i} L \right) - \left( \frac{\partial}{\partial q_i} L \right) = Q_i \qquad [10]$$

where for an $n$ degree-of-freedom, or $n$ DOF, structure $i$ = {1,..,$n$}, $q_i$ is a generalized coordinate, and $Q_i$ is a generalized force.

For the 1 DOF pendulum being considered, $q_1(t) = \alpha(t)$ and the generalized force is,

$$Q_1 = \tau_{pend} - B_p \left( \frac{d}{dt} \alpha(t) \right) \qquad [11]$$

where $\tau_{pend}$ is the torque applied to the pendulum pivot. The generalized force, expression

[11] above, becomes $Q_1 = \tau_{pend}$ since the viscous damping of the pendulum, $B_p$, is regarded as being negligible.

Calculate the nonlinear equation of motion of the pendulum using [10] on the Lagrange calculated in Exercise 4.2.4. The answer should be in the form

$$\mathrm{f}\left(\alpha, \frac{\partial^2}{\partial t^2}\alpha\right) = \tau_{pend}, \qquad [12]$$

where the function *f* represents the differential equation in terms of the position and acceleration of the pendulum angle α. Do *not* express in terms of generalized coordinates.

***Solution:***

The two differentiations of [s5] for solving the Euler-Lagrange given in [10] are

$$\frac{\partial}{\partial\alpha} L = -M_p\, g\, l_p \sin(\alpha) \qquad [s6]$$

and

$$\frac{\partial^2}{\partial t\, \partial alpha\_dot} L = J_p \left(\frac{d^2}{dt^2}\alpha(t)\right). \qquad [s7]$$

The nonlinear equation of motion of the pendulum using Euler-Lagrange is

$$J_p \left(\frac{d^2}{dt^2}\alpha(t)\right) + M_p\, g\, l_p \sin(\alpha(t)) = \tau_{pend}. \qquad [s8]$$

## 4.3. Pre-Lab Assignment #2: Finding Inertia of the Pendulum Experimentally

The inertia of the pendulum about its pivot point was calculated analytically using integrals in the previous gantry experiment, *ROTPEN Laboratory #3*. In this laboratory, the inertia of the pendulum is found experimentally by measuring the frequency at which the pendulum freely oscillates. The nonlinear equation of motion derived in the previous exercise is used to find a formula that relates frequency and inertia. The nonlinear equation of motion must first be linearized about a point and then solved for angle α.

### 4.3.1. Exercise: Linearize Nonlinear EOMs of Pendulum

The inertia is found by measuring the frequency of the pendulum when it is allowed to swing freely, or without actuation. Thus the torque at the pivot is zero, $\tau_{pend} = 0$, and the nonlinear EOM found in [12] becomes

$$f\left(\alpha, \frac{\partial^2}{\partial t^2}\alpha\right) = 0 \qquad [13]$$

where *f* is the differential expression in [12] that represents the motions of the pendulum.

Linearize function [13] about the operating point $\alpha = 0°$, which is the angle the pendulum will be swinging about in order to measure its frequency.

***Solution:***

The only nonlinear component in [s8] is the trigonometric term *sin*(α). For small angles about α= 0°, it can be approximated to

$$\sin(\alpha) = \alpha \,. \qquad [s9]$$

The linearization of the nonlinear equations of motions in [s8] is therefore

$$J_p\left(\frac{d^2}{dt^2}\alpha(t)\right) + M_p\, g\, l_p\, \alpha(t) = 0 \,. \qquad [s10]$$

### 4.3.2. Exercise: Differential Equation Solution

Solve the linear differential equation found in [13] for α(t) given that its initial conditions are

$$\alpha(0) = \alpha_0 \text{ and } \frac{d}{dt}\alpha(0) = 0 \,. \qquad [14]$$

The solution should be in the form

$$\alpha(t) = \alpha_0 \cos(2\,\pi\, f\, t) \,, \qquad [15]$$

where *f* is the frequency of the pendulum in *Hertz*.

***Solution:***
The Laplace transform of the linear equation of motion of the pendulum is

$$J_p\,(s^2\,\alpha(s) - \alpha_o) + M_p\,g\,l_p\,\alpha(s) = 0 \quad . \qquad [s11]$$

Solving for α(s), [s11] becomes

$$\alpha(s) = \frac{\alpha_0}{s^2 + \frac{M_p\,g\,l_p}{J_p}} \quad . \qquad [s12]$$

The inverse Laplace of [s12] equals

$$\alpha(t) = \alpha_0 \cos\left(\sqrt{\frac{M_p\,g\,l_p}{J_p}}\;t\right) \quad . \qquad [s13]$$

The frequency of the pendulum is therefore

$$f = \frac{1}{2}\,\frac{\sqrt{\frac{M_p\,g\,l_p}{J_p}}}{\pi} \quad . \qquad [s14]$$

### 4.3.3. Relating Pendulum Inertia and Frequency

Solving the frequency expression in [15] for the moment of inertia of the pendulum, $J_p$, should yield the equation

$$J_p = \frac{1}{4}\,\frac{M_p\,g\,l_p}{\pi^2\,f^2} \quad , \qquad [16]$$

where $M_p$ is the mass of the pendulum assembly, $l_p$ is the center of mass of the pendulum system, $g$ is the gravitational acceleration, and $f$ is the frequency of the pendulum. Expression [16] will be used in the in-lab session to find the pendulum moment of inertia in terms of the frequency measured when the pendulum is allowed to swing freely after a perturbation. The frequency can be measured using

$$f = \frac{n_{cyc}}{t_1 - t_0}, \qquad [17]$$

where $n_{cyc}$ is the number of cycles within the time duration $t_1$-$t_0$, $t_0$ is the time when the first cycle begins, and $t_1$ is the time of the last cycle.

## 4.4. Pre-Lab Assignment #3: Swing-Up Control Design

The controller using the Linear-Quadratic Regulator technique in Section 4.1 balances the pendulum in the upright vertical position after it is *manually* rotated within a certain range about its upright vertical angle. In this section, a controller is designed to automatically swing the pendulum in the upwards vertical position. Once the pendulum is within the range of the balance controller, it kicks-in and balances the pendulum. The closed-loop system that uses the swing-up controller and the balance controller is depicted in Figure 5.

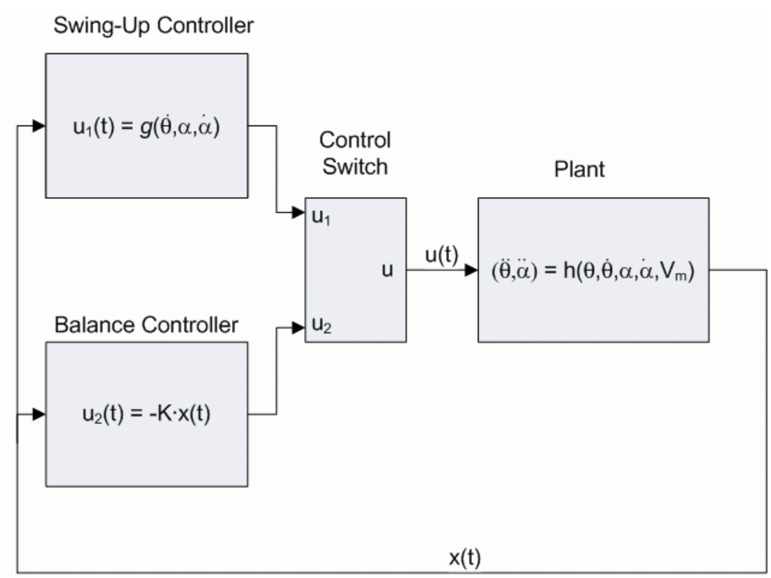


Figure 5 Swing-Up/Balance Closed-Loop System

The swing-up controller computes the torque that needs to be applied at the base of arm such that the pendulum can be rotated upwards. It is a nonlinear control that uses the pendulum energy to self-erect the pendulum. The swing-up controller will be designed using a Lyapunov function. Lyapunov functions are often used to study the stability properties of systems and can be used to design controllers.

### 4.4.1. Exercise: Re-defining System Dynamics

The controller that will be designed attempts to minimize an expression that is a function of the system's total energy. In order to rotate the pendulum into its upwards vertical position, the total energy of the pendulum and its dynamics must be redefined in terms of the angle

$$\alpha_{up} = \alpha - \pi \quad , \qquad [18]$$

resulting in the system shown in Figure 6. Thus angle zero is defined to be when the pendulum is vertically upright. The translational acceleration of the pendulum pivot is denoted by the variable *u* and is $m/s^2$.

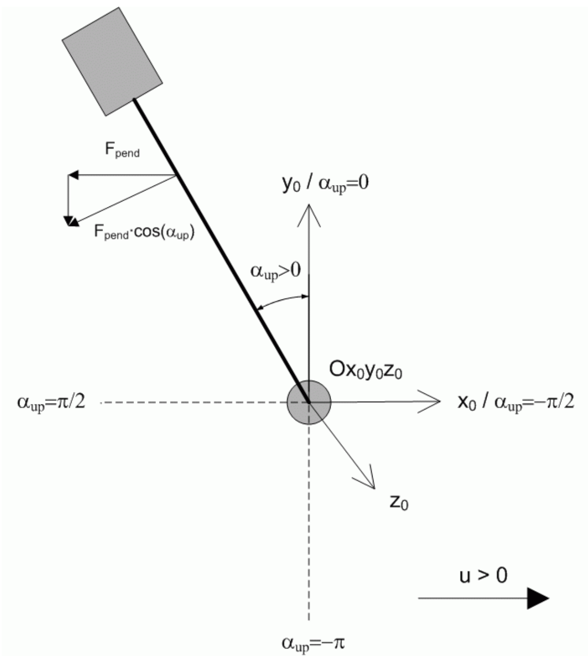

Figure 6 New Angle Definition

Re-define the nonlinear pendulum equations of motion found in Exercise 4.2.5 in terms of $\alpha_{up}$,

$$f\left(\alpha_{up}, \frac{\partial^2}{\partial t^2}\alpha_{up}\right) = \tau_{pend}(\alpha_{up}, u) \qquad [19]$$

and for the Lagrange calculated in Exercise 4.2.4, express energy *E* with respect to the upright angle

$$E(\alpha_{up}) = L(\alpha_{up} + \pi) \quad . \qquad [20]$$

Given the pendulum is not moving, the pendulum energy should be zero when it is vertically upright, thus $E(0) = 0\ J$, and should be negative when in the vertically down position, more specifically

$$\mathrm{E}(-\pi) = -2\,M_p\,g\,l_p \tag{21}$$

**_Solution:_**
Using trigonometric identities, the re-defined nonlinear EOM of the pendulum is

$$J_p\left(\frac{d^2}{dt^2}\alpha_{up}(t)\right) - M_p\,g\,l_p\sin(\alpha_{up}(t)) = \tau_{pend}(\alpha_{up}, u) \tag{s15}$$

and the total energy of the pendulum defined in terms of $\alpha_{up}$ is

$$\mathrm{E}(\alpha_{up}) = \frac{1}{2}J_p\left(\frac{d}{dt}\alpha_{up}(t)\right)^2 + M_p\,g\,l_p\,(1-\cos(\alpha_{up}(t))) \tag{s16}$$

Given that the pendulum is motionless, the energy expression above reads $0J$ when the pendulum is upright and $-2M_pgl_p$ when it is vertically downward.

### 4.4.2. Exercise: Actuator Dynamics

The swing-up controller that will be designed generates an acceleration at which the pendulum pivot should be moving at, denoted as *u* in Figure 6. The pendulum pivot acceleration however is *not* directly controllable, the input voltage of the DC motor voltage of the ROTPEN system is the input that is controlled by the computer. The dynamics between the acceleration of the pendulum pivot, *u*, and the input motor voltage, $V_m$, is required to supply the acceleration that is commanded by the swing-up control.

The dynamics between the torque applied at the arm by the motor, which is already given in [2], is

$$\tau_{output} = \frac{K_t\left(V_m - K_m\left(\frac{d}{dt}\theta(t)\right)\right)}{R_m} \tag{22}$$

The torque applied to the arm moves the pendulum pivot, situated at the end of arm, at an acceleration *u*, thus

$$\tau_{ouput} = M_{arm}\, u\, r \qquad [23]$$

where $M_{arm}$ is the mass of the arm and *r* is the length between the arm pivot and pendulum pivot. These parameters are both defined in Table 2. As shown above in Figure 6, the resulting torque applied on the pivot of the pendulum from acceleration *u* is

$$\tau_{pend}(\alpha_{up}, u) = l_p\, F_{pend} \cos(\alpha_{up}), \qquad [24]$$

where $l_p$ is the length between the pendulum CM and its axis of rotation and

$$F_{pend} = -M_p\, u. \qquad [25]$$

As depicted in Figure 6, the force acting on the pendulum due to the pivot acceleration is defined as being negative in the $x_0$ direction for a positive *u* going along the $x_0$ axis.

The swing-up controller computes a desired acceleration, *u*, and a voltage must be given that can achieve that acceleration. Express the input DC motor voltage in terms of *u* using the above equations.

***Solution:***
By substituting expression [23] into equation [22] and solving for $V_m$, the input voltage of the DC motor as a function of the pendulum pivot acceleration is

$$V_m = \frac{R_m\, M_{arm}\, u\, r}{K_t} + K_m \left( \frac{d}{dt} \theta(t) \right). \qquad [s17]$$

### 4.4.3. Exercise: Lyapunov Function

The goal of the self-erecting control is for $\alpha_{up}(t)$ to converge to zero, or $\alpha_{up}(t) \rightarrow 0$ in a finite time *t*. Instead of dealing with the angle directly, the controller will be designed to stabilize the energy of the pendulum using expression [20]. The idea is that if $E \rightarrow 0$ *J* then $\alpha_{up}(t) \rightarrow 0$. Thus the controller will be designed to regulate the energy such that $E \rightarrow 0$ *J*.

The swing-up control computes a pivot acceleration that is required to bring *E* down to zero and self-erect the pendulum. The control will be designed using the following Lyapunov function

$$\mathrm{V}(E) = \frac{1}{2}\,\mathrm{E}(\alpha_{up})^2 , \qquad [26]$$

where $E(\alpha_{up})$ was found in Exercise 4.4.1. By *Lyapunov's stability theorem*, the equilibrium point $E(\alpha_{up})=0$ is stable if the following properties hold

(1) $\mathrm{V}(0) = 0$

(2) $0 < \mathrm{V}(E)$ for all values of $E(\alpha_{up}) \neq 0$ [27]

(3) $\frac{\partial}{\partial t}\mathrm{V}(E) \leq 0$ for all values of $E(\alpha_{up})$.

The equilibrium point $E(\alpha_{up}) = 0$ is stable if the time derivative of function *V(E)* is negative or zero for all values of $E(\alpha_{up})$. The function *V(E)* approaches zero when its time derivative is negative (i.e. its a decreasing function) and that implies its variable, $E(\alpha_{up})$, converges to zero as well. According to the energy expression in [20], this means the upright angle converges to zero as well.

The derivative of *V(E)* is given by

$$\frac{\partial}{\partial t}\mathrm{V}(E) = \mathrm{E}(\alpha_{up})\left(\frac{\partial}{\partial t}\mathrm{E}(\alpha_{up})\right) . \qquad [28]$$

Compute the Lyapunov derivative in [28]. First, calculate the time derivative of $E(\alpha_{up})$

$$\frac{\partial}{\partial t}\mathrm{E}(\alpha_{up}) , \qquad [29]$$

and make the corresponding substitutions using the re-defined dynamics in [19] that introduces the pivot acceleration control variable $u$. When expressing the Lyapunov derivative leave $E(\alpha_{up})$ as a variable in [28] and do *not* substitute the expression of $E(\alpha_{up})$ in [20].

***Solution:***
The time derivative of the energy expression is

$$\frac{\partial}{\partial t}\,\mathrm{E}(\alpha_{up}) = J_p\left(\frac{d^2}{dt^2}\,\alpha_{up}(t)\right) - M_p\,g\,l_p\,\sin(\alpha_{up}) \qquad [s18]$$

The above result equals the pendulum nonlinear dynamics in [s15], therefore

$$\frac{\partial}{\partial t}\,\mathrm{E}(\alpha_{up}) = -M_p\,u\,l_p\,\cos(\alpha_{up}) \qquad [s19]$$

This substitution is key as it introduces the control variable $u$ in the Lyapunov derivative. The Lyapunov derivative is

$$\frac{\partial}{\partial t}\,\mathrm{V}(E) = -\mathrm{E}(\alpha_{up})\,M_p\,u\,l_p\left(\frac{d}{dt}\,\alpha_{up}(t)\right)\cos(\alpha_{up}) \qquad [s20]$$

### 4.4.4. Exercise: Swing-Up Control Design

The preceding calculations should yield a Lyapunov derivative in the form

$$\frac{\partial}{\partial t}\,\mathrm{V}(E) = -\mathrm{E}(\alpha_{up})\,\mathrm{g}(\alpha_{up})\,u \qquad [30]$$

where $g(\alpha_{up})$ is real valued function that can be either negative or positive depending on the pendulum angle. The swing-up controller is an expression $u$ that guarantees [30] will be negative or zero, $V\_dot(E) \le 0$, for all values of $E(\alpha_{up})$.

For example, determine if $V\_dot(E) \le 0$ for the simple proportional controller $u = \mu$ where $\mu \ge 0$ is a user-defined control gain. Substituting the control $u$ inside [30] gives

$$\frac{\partial}{\partial t}\,\mathrm{V}(E) = -\mu\,\mathrm{E}(\alpha_{up})\,\mathrm{g}(\alpha_{up}) \qquad [31]$$

The equilibrium point $E(\alpha_{up}) = 0$ is shown as being *unstable* using the control $u = \mu$ because $V\_dot(E)$ is *not* negative for all values of $E(\alpha_{up})$. Since either the function $g(\alpha_{up})$ or $E(\alpha_{up})$ can be negative, [31] can become positive which means $V(E)$ would not be a decreasing function and, as a result, $E(\alpha_{up})$ is not guaranteed to approach zero. In conclusion, this control design is not suitable for swinging up the pendulum because there is *no* guarantee the proper acceleration $u$ will be generated such that $\alpha_{up}$ will converge towards zero.

Determine and explain if $E(\alpha_{up}) = 0$ is stable using the following controllers and the *V_dot*(*E*) calculated in Exercise 4.4.3

1) $u = \mu \cdot E(\alpha_{up})$, where $\mu \geq 0$ is a user-defined control gain.

***Solution:***
Substituting the control *u* above into function *V*(*E*) in [s20] gives

$$\frac{\partial}{\partial t} \mathrm{V}(E) = -\mu\, \mathrm{E}(\alpha_{up})^2 M_p\, l_p \left(\frac{d}{dt} \alpha_{up}(t)\right) \cos(\alpha_{up}) \quad \text{[s21]}$$

If either the angular velocity of the pendulum angle or $\cos(\alpha_{up})$ goes negative, when the pendulum is below the horizontal, expression [s21] becomes positive.
Therefore $E(\alpha_{up}) = 0$ is *unstable* using $u = \mu \cdot E(\alpha_{up})$.

2) $u = \mu \cdot E(\alpha_{up}) \cdot \cos(\alpha_{up})$

***Solution:***
Substituting the control *u* above into function *V*(*E*) in [s20] gives

$$\frac{\partial}{\partial t} \mathrm{V}(E) = -\mu\, \mathrm{E}(\alpha_{up})^2 M_p\, l_p \left(\frac{d}{dt} \alpha_{up}(t)\right) \cos(\alpha_{up})^2 \quad \text{[s22]}$$

If the angular velocity of the pendulum angle goes negative expression [s22] becomes positive. Therefore $E(\alpha_{up}) = 0$ is *unstable* using $u = \mu \cdot E(\alpha_{up}) \cdot \cos(\alpha_{up})$.

3) $u = \mu \cdot E(\alpha_{up}) \cdot (d\alpha_{up}(t)/dt) \cdot \cos(\alpha_{up})$

***Solution:***
Substituting the control *u* above into function *V*(*E*) in [s20] gives

$$\frac{\partial}{\partial t}\mathrm{V}(E) = -\mu\,\mathrm{E}(\alpha_{up})^2\,M_p\,l_p\left(\frac{d}{dt}\alpha_{up}(t)\right)^2\cos(\alpha_{up})^2 \qquad \text{[s23]}$$

The time derivative of the Lyapunov in [s23] is negative for all values of $E(\alpha_{up})$. Therefore $E(\alpha_{up}) = 0$ is *stable* using $u = \mu \cdot E(\alpha_{up}) \cdot (d\alpha_{up}(t)/dt) \cdot \cos(\alpha_{up})$.

4) $u = \mu \cdot sgn( E(\alpha_{up}) \cdot (d\alpha_{up}(t)/dt) \cdot \cos(\alpha_{up}) )$, where *sgn()* represents the *signum* function.

***Solution:***
Substituting the control *u* above into function *V*(*E*) in [s20] gives

$$\frac{\partial}{\partial t}\mathrm{V}(E) = -\mathrm{E}(\alpha_{up})\,M_p\,l_p\left(\frac{d}{dt}\alpha_{up}(t)\right)\cos(\alpha_{up})\,\mu\,\mathrm{sgn}\left(\mathrm{E}(\alpha_{up})\left(\frac{d}{dt}\alpha_{up}(t)\right)\cos(\alpha_{up})\right) \qquad \text{[s24]}$$

The time derivative of the Lyapunov in [s24] is negative for all values of $E(\alpha_{up})$. Therefore $E(\alpha_{up}) = 0$ is *stable* using $u = \mu \cdot sgn( E(\alpha_{up}) \cdot (d\alpha_{up}(t)/dt) \cdot \cos(\alpha_{up}) )$.

### 4.4.5. Controller Implementation

The controller that is implemented in LabView is

$$u = sat_{u_{max}}\left(\mu\,\mathrm{sgn}\left(\mathrm{E}(\alpha_{up})\left(\frac{d}{dt}\alpha_{up}(t)\right)\cos(\alpha_{up})\right)\right) \qquad [32]$$

where *sat()* is the *saturation* function and $u_{max}$ represents the maximum acceleration of the pendulum pivot. The *signum* function makes for a control with the largest variance and overall tends to perform very well. However, the problem with using a *signum* function is the switching is high-frequency and can cause the voltage of the motor to chatter. In

LabView, a smooth approximation of the *signum* function to help prevent motor damage.

Given that the maximum motor input voltage is $V_m$ = 10V and neglecting the motor back-electromotive force constant, $K_m$ = 0, calculate the maximum acceleration of the pendulum pivot $u_{max}$ using the equations supplied in Section 4.4.2.

***Solution:***
Solving for the pivot acceleration *u* in expression [s17] found earlier gives

$$u = \frac{K_t V_m}{M_{arm}\, r\, R_m} \qquad \text{[s25]}$$

Substituting the appropriate parameters from Table 2 and the maximum input voltage $V_m$ = 10V into [s25] results in

$$u_{max} = 36.7 \left[ \frac{m}{s^2} \right]. \qquad \text{[s26]}$$

This is the maximum acceleration at which the pivot point of the pendulum can move and the control gain, μ, should not be set beyond this.

The control gain, μ, is an acceleration and it basically changes the amount of torque the motor outputs. The maximum acceleration, $u_{max}$, is the maximum value that the control gain can be set.

# 5. In-Lab Session

## 5.1. System Hardware Configuration

This in-lab session is performed using the NI-ELVIS system equipped with a QNET-ROTPEN board and the Quanser Virtual Instrument (VI) controller file *QNET_ROTPEN_Lab_04_Inv_Pend_Control.vi*. See *QNET_ROTPEN_Lab_04_Inv_Pend_Control_Demo.vi* for the already tuned-and-ready version of the student VI. Please refer to Reference [2] for the setup and wiring information required to carry out the present control laboratory. Reference [2] also provides the specifications and a description of the main components composing your system.

Before beginning the lab session, ensure the system is configured as follows:

QNET Rotary Pendulum Control Trainer module is connected to the ELVIS.

ELVIS Communication Switch is set to BYPASS.

DC power supply is connected to the QNET Rotary Pendulum Control Trainer module.

The 4 LEDs +B, +15V, -15V, +5V on the QNET module should be ON.

## 5.2. Software User-Interface

Please follow the steps described below:

Step 1. Read through Section 5.1 and go through the setup guide in Reference [2].

Step 2. Run the VI controller *QNET_ROTPEN_Lab_04_Inv_Pend_Control.vi* shown in Figure 7.

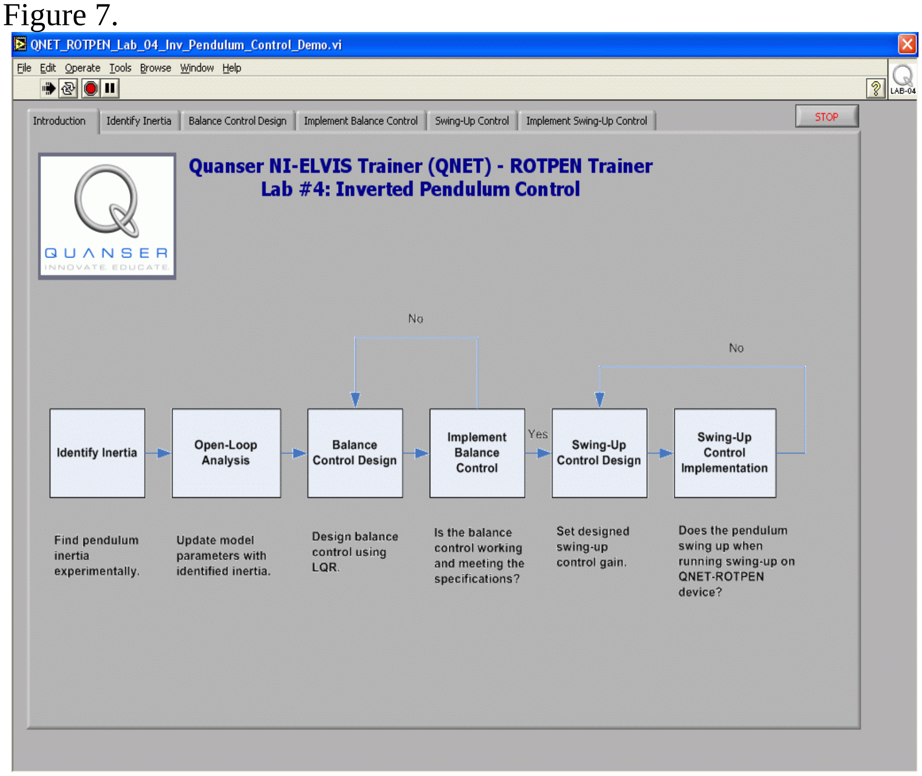


Figure 7 QNET-ROTPEN VI

Step 3. Select the *Identify Inertia* tab and the front panel shown in Figure 8 should load.

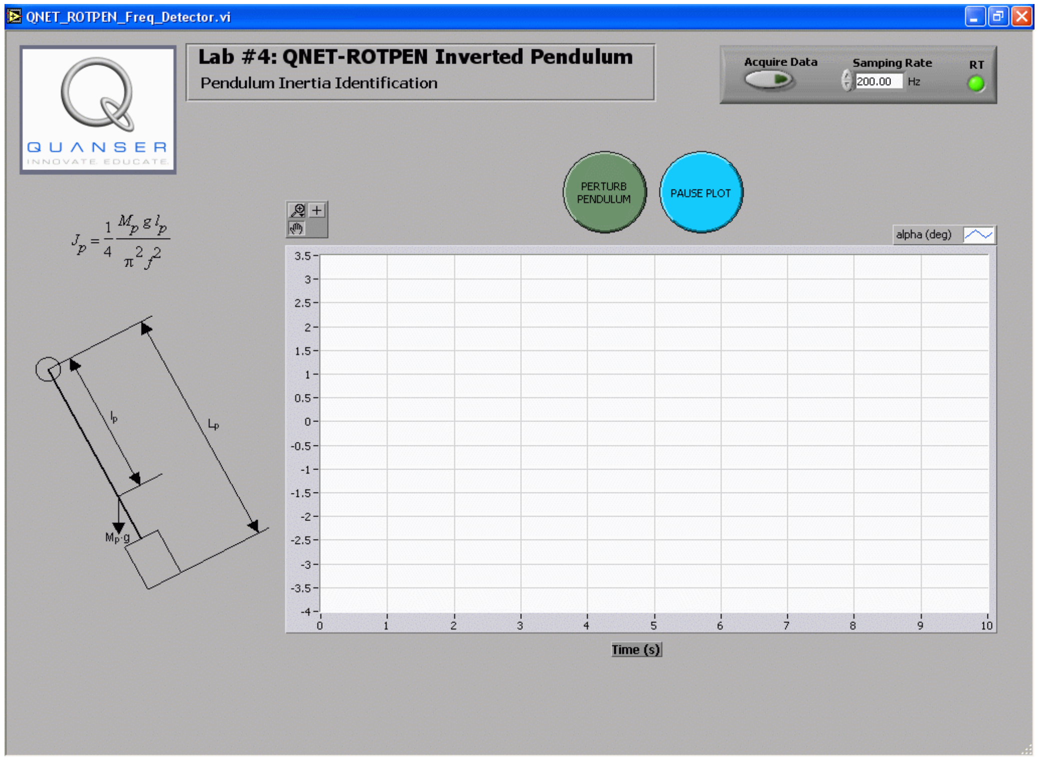


Figure 8 Identifying Inertia of Pendulum VI

The top-right corner has a panel with an *Acquire Data* button that stops this VI and goes to the next stage of the inverted pendulum laboratory. Also in the panel is the sampling rate for the implemented digital controller, which is by default set to 200 Hz. Adjust the rate according to the system's computing power. The RT LED indicates whether real-time is being sustained.

**If the RT light goes RED or flickers then the sampling rate needs to be decreased and the VI restarted.** The VI can be restarted by clicking on the *Acquire Data* button and selecting the *Identify Inertia* tab again.

The scope plots the angle of the pendulum, which is denoted by the variable $\alpha$, with respect to time. The scope can be frozen, for measuring purposes, by clicking on the PAUSE PLOT button and a small voltage can be applied to the DC motor by clicking on PERTURB PENDULUM.

Step 4. The moment of inertia of a pendulum that oscillates freely after being perturbed is given in [16]. The frequency can be measured accurately by taking into account many samples over a large span of time. The plot can be cleared by selecting the PAUSE PLOT button in the top-right corner to freeze the scope, clicking right on the scope, and selecting *Clear Plot* in the drop-down menu. The plot should now be initialized to $t = 0$ and paused. Un-pause the scope by clicking on the RESUME PLOT button and click on the PERTURB PENDULUM button to apply a slight impulse on the pendulum. Avoid perturbing the pendulum such that its oscillations exceed $\pm 10°$. The frequency of the pendulum can be found using [17], re-stated here

$$f = \frac{n_{cyc}}{t_1 - t_0}$$

where $n_{cyc}$ is the number of cycles within the time duration $t_1$-$t_0$, $t_0$ is the time when the first cycle begins, and $t_1$ is the time of the last cycle. Enter the measured number of cycles and the time duration as well as the calculated frequency and inertia, using expression [24], in Table 4.

| ***Parameter*** | ***Value*** | ***Unit*** |
|---|---|---|
| $t_0$ | 1.10 | s |
| $t_1$ | 7.90 | s |
| f | 3.10 | Hz |
| $J_p$ | 1.10E-004 | $kg \cdot m^2$ |

Table 4 Experimental Inertia Parameters

Step 5. Calculate the discrepancy between the experimentally derived inertia, $J_{p,e}$, in Table 4 and the inertia calculated analytically, $J_{p,a}$, in *Pre-Lab Exercise 4.3* from *ROTPEN Gantry Laboratory #3*. Enter the result below.

| ***Discrepancy*** | ***Value*** | ***Unit*** |
|---|---|---|
| $\frac{100\,(\lvert J_{p,a}\rvert - \lvert J_{p,e}\rvert)}{\lvert J_{p,a}\rvert + \lvert J_{p,e}\rvert}$ | 73.4 | % |

List one reason why the measured inertia is not the same as the calculated result.

***Solution:***

Here are possible sources of the discrepancy:

The analytical method does not take into account the friction at the pendulum pivot. The inertia measured experimental will tend therefore to be less than the theoretical result.

The moment of inertia, equation [14], is calculated using the center of mass of the pendulum. However, the pendulum center of mass is calculated analytically in *Pre-Lab Exercise 4.3* and may not represent the exact center of mass of the pendulum. This may contribute to the difference between the experimental and analytical results.

Step 6. Click on the *Acquire Data* button when the inertia has been identified and this will bring you to the *Control Design* tab shown in Figure 9.

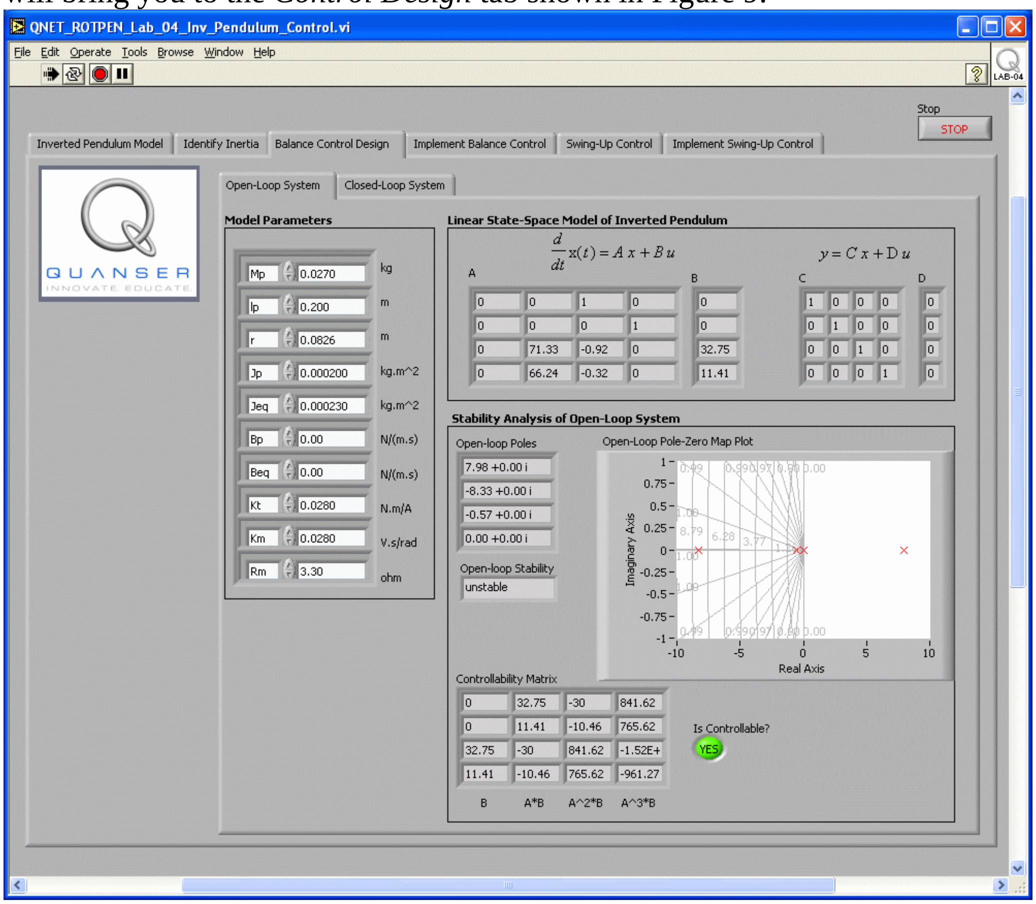


Figure 9 Open-Loop Stability Analysis

Step 7. Update the model parameter values in the top-right corner with the pendulum center of mass, $l_p$, calculated in *ROTPEN Gantry Laboratory #3* as well as the pendulum's moment of inertia just identified. The linear-state space model matrices *A* and *B*, on the top-right corner of the front panel, as well as the open-loop poles, situated directly below the state matrices, are automatically updated as the parameters are changed.

Step 8. Directly beneath the open-loop poles in this VI it indicates the stability of the inverted pendulum system as being *unstable*, as shown in Figure 9. According to the poles, why is the open-loop inverted pendulum considered to be *unstable*?

***Solution:***
There is an open-loop pole located in the right-hand plane. The open-loop gantry is therefore not considered to be *stable*.

Step 9. As depicted in Figure 9, the controllability matrix is shown in the bottom-right area of the front panel along with an LED indicating whether the system is controllable or not. The rank test of the controllability matrix gives

$$rank[B\ AB\ A^2B\ A3B] = 4$$

and is equal to the number of states in the system. This verifies that the inverted pendulum is controllable and, as a result, a controller can be constructed. Click on the *Closed-Loop System* tab shown in Figure 10 to begin the LQR control design.

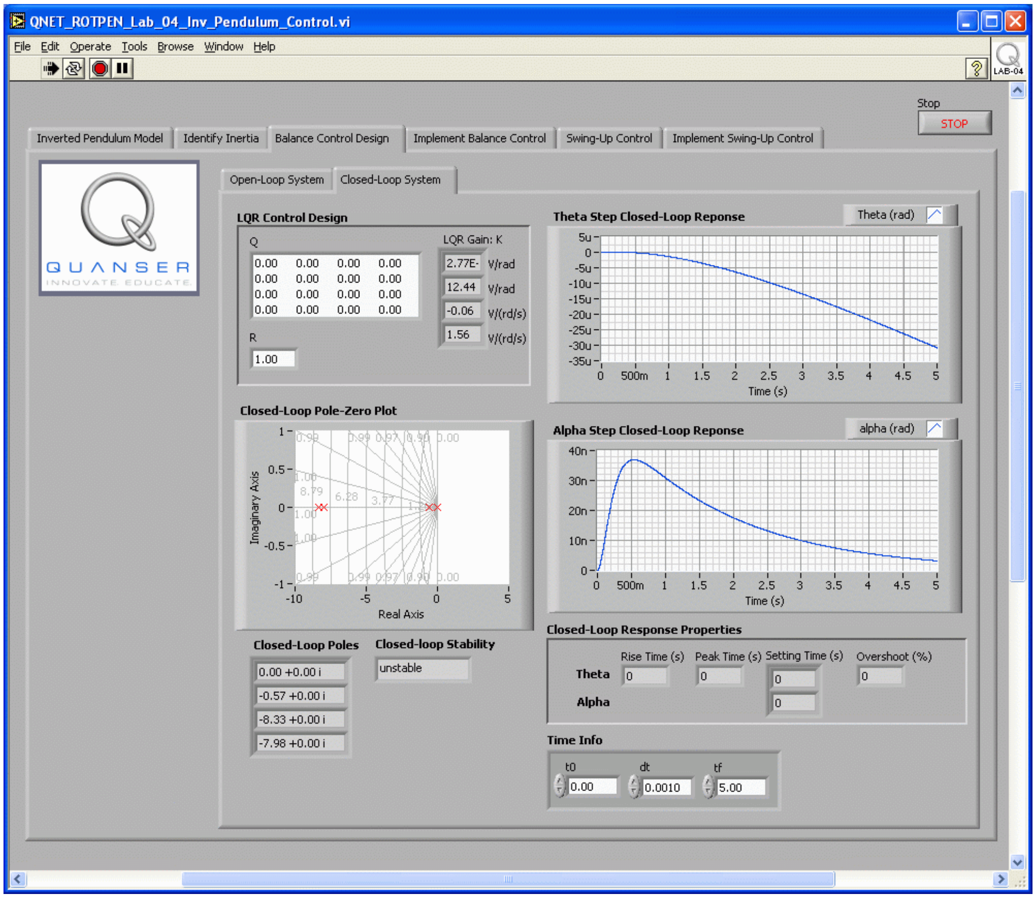

Figure 10 LQR Control Design Front Panel

Step 10. The *Q* and *R* weighing matrices and the resulting control gain *K* is in the top-left corner of the panel. Directly below the *LQR Control Design* section is a pole-zero plot that shows the locations of the closed-loop poles. The numerical value of the poles are given below the plot along with the resulting stability of the closed-loop system. The step response of the arm angle, $\theta(t)$, and the pendulum angle, $\alpha(t)$, are plotted in the two graphs on the right side of the VI, as shown in Figure 10. The rise time, peak time, settling time, and overshoot of the arm response and the settling time of the pendulum angle response is given. Further, the start time, duration, and final time of these responses can be changed in the *Time Info* section located at the bottom-right corner of the VI.

Step 11. The balance controller will be designed by tuning the weighing matrix *Q* and implementing the resulting control gain *K* on the ROTPEN system. The VI that runs the balance control is shown in Figure 11.

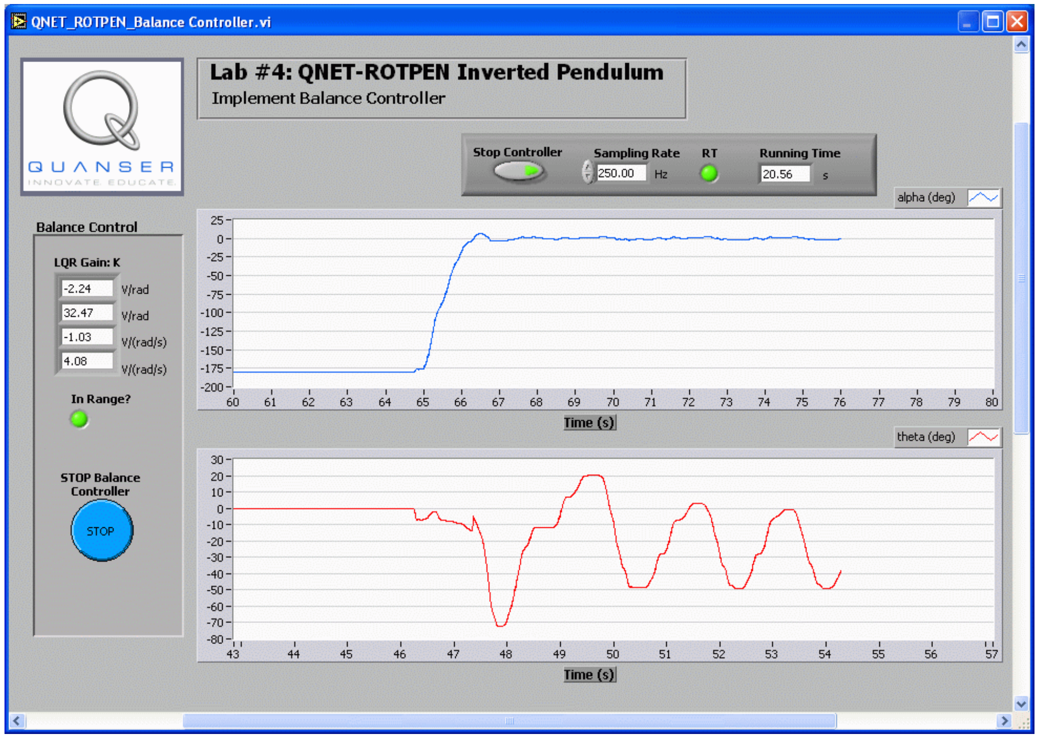

Figure 11 Implement Balance Controller VI

The pendulum angle measured by the encoder is shown in the above scope and the arm angle is plotted in the bottom scope. The VI, by default, begins with a sampling rate of 400 Hz. Adjust the rate according to the system's computing power. The RT LED indicates whether real-time is being sustained. **If the RT light goes RED or flickers then the sampling rate needs to be decreased and the VI restarted.** On the other hand, make sure the sampling rate is not set too low. Not attaining sufficient readings from the encoders can cause the digitally implemented controller to become unstable.

The *Stop Controller* button stops the control and returns the user to the control design tab where adjustments to the control can be made or the session can be ended. The balance controller gain generated by LQR in the *Control Design* tab is displayed in the panel along the left margin of the VI. The *In Range?* LED indicates whether the pendulum is placed within the angular range that activates the balance control. The *STOP Balance Controller* button disables the balance controller when it is pressed but does *not* stop the VI.

Step 12. For the *Q* and *R* weighing matrices

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix} \qquad [33]$$

$$R = 1$$

vary the $q_1$, $q_2$, $q_3$, and $q_4$ elements as specified in Table 5 and record the maximum amplitude range of the pendulum angle, α, and the arm angle, θ, in the same table. Thus in the *Control Design* tab, adjust the *Q* matrix accordingly and then click on the *Implement Balance Control* tab to run the controller on the ROTPEN system.

**Ensure the pendulum is motionless before clicking on the *Implement Balance Control* tab and running the controller.** Also, verify that the pendulum encoder cable is not positioned such that it will get entangled in the motor shaft.

However before implementing the updated LQR balance control with the newly changed *Q* matrix, observe its effects on the α(t) and θ(t) step responses. Referring to the feedback loop in Figure 3, for the LQR gain

$$K = [k_{p,\theta}, k_{p,\alpha}, k_{v,\theta}, k_{v,\alpha}]^T \qquad [34]$$

the control input *u(t)* that enters the DC motor input voltage is

$$V_m = k_{p,\theta}\, x_1 + k_{p,\alpha}\, x_2 + k_{v,\theta}\, x_3 + k_{v,\alpha}\, x_4 \qquad [35]$$

where $k_{p,\theta}$ is the proportional gain acting on the arm, $k_{p,\alpha}$ is the proportional gain of the pendulum angle, $k_{v,\theta}$ is the velocity gain of the arm, and $k_{v,\alpha}$ is the velocity gain of the pendulum. Observe the effects that changing the weighing matrix *Q* has on the gain *K* generated and, hence, how that effects the properties of the both step responses.

| | ***Q*** | | | | |
|---|---|---|---|---|---|
| ***q1*** | ***q2*** | ***q3*** | ***q4*** | ***Max \|α\| (deg)*** | ***Max \|θ\| (deg)*** |
| 5 | 0 | 0 | 0 | 2.5 | 30.0 |
| 1 | 0 | 0 | 0 | 4.5 | 63.0 |
| 10 | 0 | 0 | 0 | 2.5 | 32.0 |
| 8 | 0 | 0.5 | 0 | 1.3 | 26.0 |

Table 5 LQR Control Design

Step 13. Re-stating the balance control specifications given in Section 4.1.4:

(1) **Arm Regulation**: $|\theta(t)| < 30.0°$

(2) **Pendulum Regulation**: $|\alpha(t)| < 1.5°$

(3) **Control input limit**: $V_m < 10$ V

Find the $q_1$, $q_2$, $q_3$, and $q_4$ elements that results in specifications (1), (2), and (3) being satisfied. Record the *Q* matrix elements used and the resulting angle limits in Table 5.

Step 14. Click on the *Stop Controller* button to stop the balance control which returns to the *Control Design* tab. Select the tab labeled *Swing-Up Control* shown in Figure 12. The swing-up control law being implemented is shown in the VI. Set the maximum acceleration $u_{max}$ that was calculated in Exercise 4.4.5 and, can be set as well as the control gain for the swing-up controller, $\mu$. Set the maximum acceleration of the pendulum pivot, $u_{max}$, to the value that was found in *Exercise 4.4.5* and initially set the control gain to 50% of the maximum acceleration, thus set $\mu = 0.5 \cdot u_{max}$.

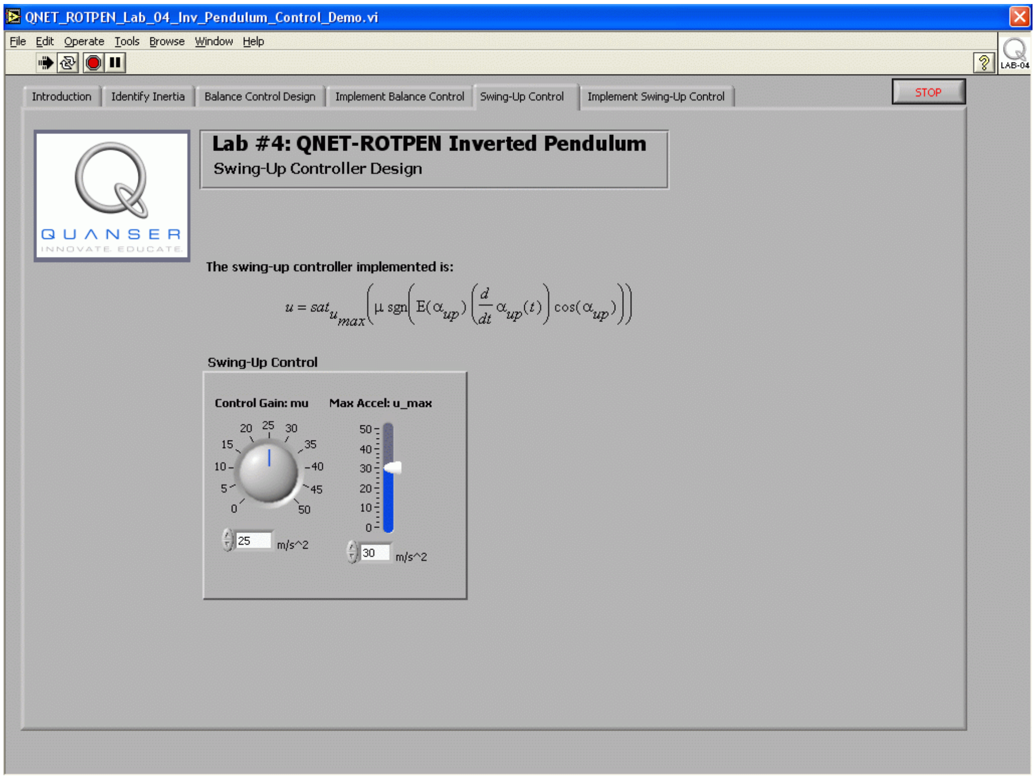


Figure 12 Swing-Up Controller Design VI

Step 15. Click on the *Implement Swing-Up Control* tab to run the swing-up controller and the VI shown in Figure 13 should load.

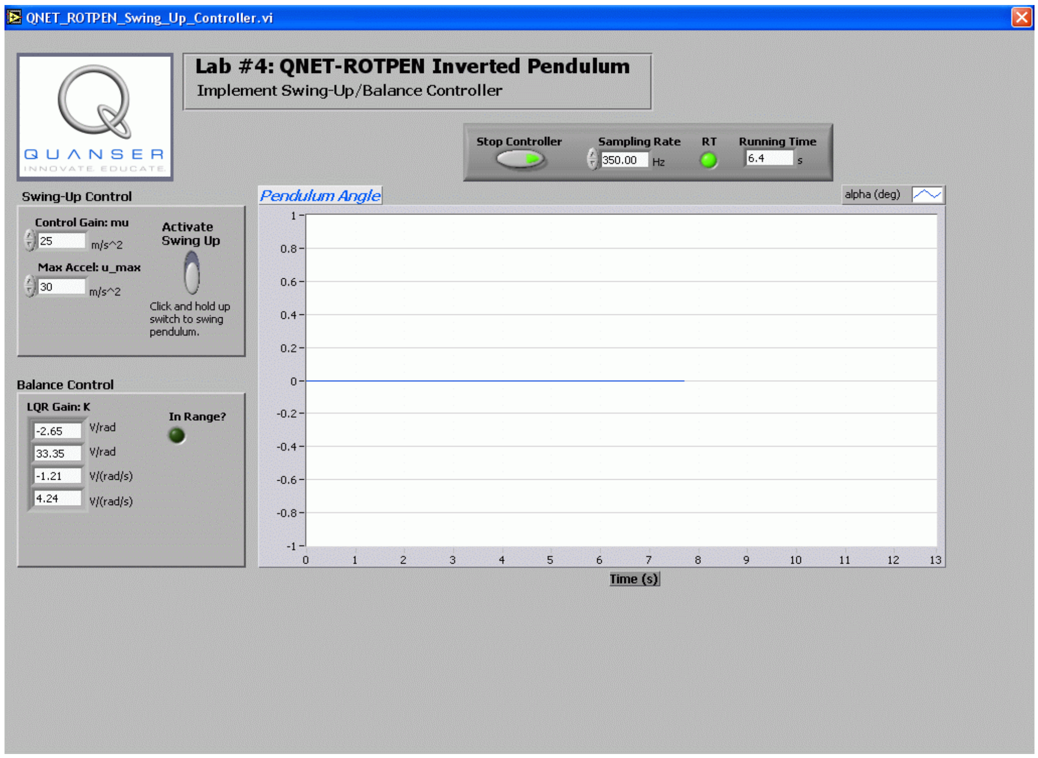


Figure 13 Implement Swing-Up Control VI

Step 16. The top-panel has the *Stop Controller* button that stops the VI and goes back to the *Swing-Up Control* tab where the swing-up gain can be tuned. Similarly to the balance controller implementation VI, the *Sampling Rate*, *RT*, and *Simulation Time* is given. By default, the sampling rate is set to 350 Hz. When the pendulum enters the range of the balance controller, the *In Range?* LED will become lit. The scope displays the measured pendulum angle.

⚠ **If the RT light goes RED or flickers then the sampling rate needs to be decreased and the VI restarted.** However it is very important to make sure that the sampling rate is not set too low. Not attaining sufficient readings from the encoders when the pendulum is swinging-up can result in the balance controller having difficulty "catching" the pendulum.

Step 17. **Before running the swing-up control, ensure the pendulum is motionless and the pendulum encoder cable is not entangled in any way.** When ready, click and hold the *Activate Swing-Up* switch to enable the motor and run the swing-up controller. On the other hand, ensure the button is released immediately when the pendulum goes unstable.

Step 18. Tuning the swing-up control gain is an iterative process. Thus based on the behaviour and performance of the controller, click on the *Stop Controller* button to return to the *Swing-Up Control* tab and adjust the $\mu$ accordingly. Avoid setting setting the gain too high, i.e. closer to $u_{max}$, since it can make the pendulum swing up too rapidly and cause the balance controller to have difficulty "catching" the pendulum in order to balance it. On the other hand, the gain has to be sufficient to drive the pendulum within ±30 degrees of its upright position. Finally, the pendulum can self-erect in one swing with a properly tuned controller.

Step 19. Once the pendulum can be swung-up and balanced, show the run to the teaching assistant and enter the resulting swing-up control gain $\mu$ and the balance control vector gain $K$ in Table 6. The balance control gain is formatted in Table 6 as in [34].

| *Gain* | *Value* | *Unit* |
|---|---|---|
| $\mu$ | 32.00 | $m/s^2$ |
| $k_{p,\theta}$ | -2.83 | V/rad |
| $k_{p,\alpha}$ | 44.16 | V/rad |
| $k_{v,\theta}$ | -1.58 | V/(rad/s) |
| $k_{v,\alpha}$ | 5.67 | V/(rad/s) |

Table 6 Final Swing-Up and Balance Control Gain Implemented

Step 20. Click on *Stop Controller* and the *Swing-Up Control* tab should become selected. If all the data necessary to fill the shaded regions of the tables is collected, end the *QNET-ROTPEN Inverted Pendulum* laboratory by turning off the *PROTOTYPING POWER BOARD* switch and the *SYSTEM POWER* switch at the back of the ELVIS unit. Unplug the module AC cord. Finally, end the laboratory session by selecting the *Stop* button on the VI.