Image Compression and the Singular Value Decomposition

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MAT-436 Linear Algebra

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http://linalg.beastman.ca

http://cs.redeemer.ca/∼beastman/linalg

Intro

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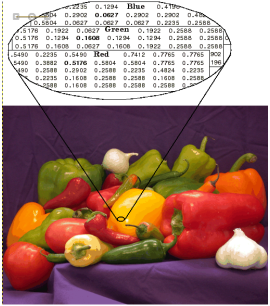

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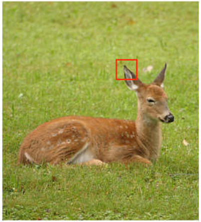

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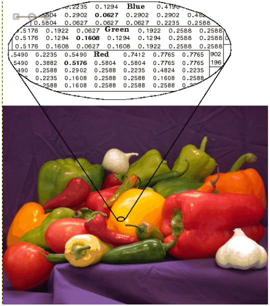

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Why We Need Image Compression

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This is a picture I uploaded to Facebook last summer.
The camera I used stored the $35,426,304$ numbers ($\approx$ 65MB)
On Facebook it was only 12 MB. What happened?

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A matrix A is diagonalisable if we can find an invertible matrix P such that $A = PDP^{-1}$ where D is a diagonal matrix.

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Singular Values

If A is an $m \times n$ matrix, the singular values of A are the square roots of the eigenvalues of $A^\intercal A$ and are denoted $\sigma_1, \sigma_2, \ldots, \sigma_n$. By convention $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.

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Where $\Sigma = \left[\begin{array}{c|c} D & O \\ \hline O & O \end{array}\right]$ and $D = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}$

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Examples of Σ

Some examples of Σ that may show up if $r = 2$:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 9 & 0 \\ 0 & 6 \end{bmatrix}$$

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Now recall that $\sigma_i = ||A\vec{v}_i||$ and r of these are nonzero. Thus

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Recall $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

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So for A_k we're really only storing $k(1 + n + m)$ numbers.

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