The Logic in Philosophy of Science by Hans Halvorson (Exercises and solutions)

Dustin Bryant

December 29, 2021

PART I

Invitation to Metatheory

1 Logical grammar

There are no exercises in this section.

2 Proof Theory

There are no exercises in this section.

3 Semantics

1.3.8 Show that if $\Delta, \phi \vDash \psi$ then $\Delta \vDash \phi \to \psi$.

Proof (Contrapositive): Suppose that $\Delta \nvDash \phi \to \psi$ then there exists a model ν of Δ such that $\nu(\phi \to \psi) = 0$. Therefore $\nu(\neg\phi \vee \psi) = 0$ and hence $\nu(\neg\phi) = 0$ and $\nu(\psi) = 0$ but then $\nu(\phi) = 1$ and hence $\Delta, \phi \nvDash \psi$.

Prood (Direct): Suppose $\Delta, \phi \vDash \psi$ then, for any v that is a model of Δ, ϕ, we have $v(\psi) = 1$, it thus follows that we have $v(\neg\phi \vee \psi) = 1$, and so $v(\phi \to \psi) = 1$. Since any model of Δ, ϕ is also a model of Δ, we then have that $\Delta \vDash \phi \to \psi$.

1.3.9 Show that $\Delta \vDash \phi$ if and only if $\Delta \cup \{\neg\phi\}$ is inconsistent.

Proof: $\Delta \cup \{\neg\phi\}$ is consistent iff there exists a model ν such that $\nu(\neg\phi) = 1$ and $\nu(\psi) = 1$ for all $\psi \in \Delta$ iff $\nu(\phi) = 0$ and $\nu(\psi) = 1$ for all $\psi \in \Delta$ iff $\Delta \nvDash \phi$.

1.3.16 Show that the theory T' from the previous example is not complete.

Proof: $T' = (\Delta', \Sigma') = (\{p\}, \{p, q\})$. Suppose that T' were complete then either $p \vDash q$ or $p \vDash \neg q$. By the Completeness Theorem if the former case is true then $p \vdash q$ however $p \nvdash q$; likewise, by the CT if the latter case is true then $p \vdash \neg q$ however $p \nvdash \neg q$. Therefore, T' is not complete.

1.3.17 Show that $\mathrm{Cn}(\mathrm{Cn}(T)) = \mathrm{Cn}(T)$.

Proof: If $p \in \mathrm{Cn}(\mathrm{Cn}(T))$ then there exists a proof of p from $\mathrm{Cn}(T)$. That is to say there exists a finite set $\Delta = \{q_1, \ldots, q_n\}$ of elements from $\mathrm{Cn}(T)$ such that $\Delta \vdash p$. Now consider one of these q_i, since $q_i \in \mathrm{Cn}(T)$ then by definition there exists a finite set $\Delta_i = \{q_{i_1}, \ldots, q_{i_m}\}$ of elememts from T such that $\Delta_i \vdash q_i$. Such a set Δ_i exists for every such q_i and $\Delta_1, \ldots \Delta_n \vdash \Delta$ and $\Delta \vdash p$ therefore $\Delta_1, \ldots \Delta_n \vdash p$. Hence it follows that $p \in \mathrm{Cn}(T)$.

Conversely, suppose that $p \in \mathrm{Cn}(T)$, by the Rule of Assumptions $p \vdash p$, therefore $p \in \mathrm{Cn}(\mathrm{Cn}(T))$.

1.3.18 Consider the signature $\Sigma = \{p\}$. How many complete theories are there in this signature? (We haven't been completely clear on the identity conditions of theories, and hence on how to count theories. For this exercise, assume that theories are deductively closed, and two theories are equal just in case they contain exactly the same sentences.)

Solution: There are precisely two!

Proof: $\Delta = \Sigma$ gives one complete theory since $\Delta \vdash p$ and Δ is consistent, simply define a model v such that $v(p) = 1$ and $\Delta' = \{\neg p\}$ gives another since $\Delta' \vdash \neg p$ and likewise it is consistent since we can define a model $w(\neg p) = 1$.

$\Delta'' = \{p \wedge \neg p\}$ is not a complete theory since it is obviously inconsistent: If u models Δ'' then $u(p \wedge \neg p) = 1$, hence $u(p) = 1$ and $u(\neg p) = 1$ and hence $u(p) = 0$ as well. This theory is equivalent to $\{p, \neg p\}$ since by $\wedge$ introduction and elimination they derive one another.

$\Delta''' = \{p \vee \neg p\}$ is not complete because on the one hand v defined above is a model for Δ''' however $v(\neg p) = 0$ therefore $\Delta''' \nvDash \neg p$ and so by the Soundness Theorem it follows that $\Delta''' \nvdash \neg p$. Likewise w defined above is a model for Δ''' however $w(p) = 0$ therefore $\Delta''' \nvDash p$ and therefore by the Soundness Theorem $\Delta''' \nvdash p$. It follows from these two cases that Δ''' is not complete.

Obviously the empty theory is not complete since p is not a tautology and the empty theory only proves tautologies. So there are only two complete theories on a singleton signature. There are no other possible theories since all other sentences defining our axioms coming from p can be reduced to the above examples. For example, $p \to \neg p = \neg p \vee \neg p = \neg p$ and $\neg p \to p = \neg\neg p \vee p = p$ and $p \to p = \neg p \vee p = p \vee \neg p = \neg\neg p \vee \neg p = \neg p \to \neg p$ and $p \wedge p = p$ and $\neg p \wedge \neg p = \neg p$ and $\neg p \wedge p = p \wedge \neg p$ and $p \vee p = p$ and $\neg p \vee p = p \vee \neg p$ and $\neg p \vee \neg p = \neg p$. And we already exhausted these options so there is nothing else to consider.

1.4.7 Prove that if v is a model of T', and $f : T \to T'$ is a translation, then $v \circ f$ is a model of T. Here $v \circ f$ is the interpretation of Σ obtained by applying f first, and then applying v.

Proof:

Since v is a model of T', we have that $v(\phi) = 1$ for all ϕ in T'. Let ψ be an arbitrary element of T, then $f(\psi)$ is in T', but by the assumption we have that $(v \circ f)(\psi) = v(f(\psi)) = 1$. So we have that $v \circ f$ is a model of T, since this holds for any ψ in T.

1.4.8 Prove that if $f : T \to T'$ is a translation, and T' is consistent, then T is consistent.

Proof: Suppose that $T = (\Delta, \Sigma)$ is inconsistent (i.e. Δ is inconsistent) then $\Delta \cup \{\top\}$ is inconsistent. By Exercise 1.3.9 we have that $\Delta \vDash \neg\top$ that is $\Delta \vDash \bot$ therefore $\Delta \vdash \bot$ by the Completeness theorem.

Since f is a translation we have $f(\Delta) \vdash f(\bot)$ but $f(\bot) = \bot$ as $f(\bot) = f(\phi \wedge \neg\phi) = f(\phi) \wedge f(\neg\phi) = f(\phi) \wedge \neg f(\phi)$; that is $f(\Delta) \vdash \bot$ but $f(\Delta) \subseteq \Delta'$ so that $\Delta' \vdash \bot$ by *monotonicity*.

Then, applying Soundness, we have $\Delta' \vDash \bot$. By Exercise 1.3.9, we then have that $\Delta' \cup \{\top\}$ is inconsistent from which is follows that Δ' is inconsistent (since the inconsistency does not arrive from $\top$ since $v(\top) = 1$ for any v).

PART II

The Category of Sets

1 Introduction

2.1.7 Show the following:

(a) If gf is monic, then f is monic.
Proof: Suppose $f : X \to Y$ and $g : Y \to Z$ and $s, t : A \to X$ such that $f \circ s = f \circ t$ then, composing with g on the left,

$$(g \circ f) \circ s = g \circ (f \circ s) = g \circ (f \circ t) = (g \circ f) \circ t$$

but as $g \circ f$ is monic it follows $s = t$ and hence f is monic.

(b) If fg is epi then f is epi.
Proof: If $hf = kf$ then $h(fg) = (hf)g = (kf)g = k(fg)$ hence $h = k$ since fg is epi.

(c) If f and g are monic, then gf is monic.
Proof: Suppose $(gf)h = (gf)k$ then $g(fh) = g(fk)$ and hence $fh = fk$ since g is monic, and then $h = k$ since f is monic as well.

(d) If f and g are epi, then gf is epi.
Proof: Suppose $h(gf) = k(gf)$ then $(hg)f = (kg)f$ so $hg = kg$ since f is epi, and likewise $h = k$ since g is epi.

(e) If f is an isomorphism, then f is epi and monic.
Proof: Let $f : X \to Y$ be an isomorphism then there exists $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$. So if $fh = fk$ then

$$h = 1_X h = (gf)h = g(fh) = g(fk) = (gf)k = 1_X k = k$$

and therefore f is monic. Now suppose $sf = tf$ then

$$s = s1_Y = s(fg) = (sf)g = (tf)g = t(fg) = t1_Y = t$$

and hence f is epi.

2.1.12 Show that $1_X \times 1_Y = 1_{X \times Y}$

Proof: From Definition 2.1.10, $1_X \times 1_Y = \langle 1_X \circ \pi_0, 1_Y \circ \pi_1 \rangle$ is defined as the *unique* morphism such that

$$\pi_0(1_X \times 1_Y) = 1_X \circ \pi_0 = \pi_0$$

$$\pi_1(1_X \times 1_Y) = 1_Y \circ \pi_1 = \pi_1$$

however $1_{X \times Y}$ does this as well..... so they must be equal by the uniqueness clause of Axiom 2.

2.1.14 Suppose that $f : X \to Y$ is a function. Show that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\delta_X"] & Y \arrow[d, "\delta_Y"] \\ X \times X \arrow[r, "f\times f"] & Y \times Y \end{tikzcd}$$

Proof: Let $q_0, q_1 : X \times X \to X$ be one set of canonical projections and $\pi_0, \pi_1 : Y \times Y \to Y$ be another set of canonical projections. Note that $f \times f = \langle f \circ q_0, f \circ q_1 \rangle$ and

$$\begin{aligned}
\pi_0 \circ (f \times f \circ \delta_X) &= (\pi_0 \circ f \times f) \circ \delta_X && \text{(associativity of } \circ) \\
&= (f \circ q_0) \circ \delta_X && \text{(Definition 2.1.10)} \\
&= f \circ (q_0 \circ \delta_X) && \text{(associativity of } \circ) \\
&= f \circ (q_0 \langle 1_X, 1_X \rangle) && \text{(Definition 2.1.9)} \\
&= f \circ 1_X && \text{(Axiom 2)} \\
&= f && \text{(Axiom 1.4: Definition of Identity)} \\
&= 1_Y \circ f && \text{(Axiom 1.4: Definition of Identity)} \\
&= (\pi_0 \circ \delta_Y) \circ f && \text{(Definition 2.1.9 and Axiom 2)} \\
&= \pi_0 \circ (\delta_Y \circ f) && \text{(associativity of } \circ)
\end{aligned}$$

Then we get that $\pi_0 \circ (f \times f \circ \delta_X) = \pi_0 \circ (\delta_Y \circ f)$ and similarly $\pi_1 \circ (f \times f \circ \delta_X) = \pi_1 \circ (\delta_Y \circ f)$.

Consider arbitrary $x : 1 \to X$ then $(f \times f \circ \delta_X)(x), (\delta_Y \circ f)(x) \in Y \times Y$ and by the above result if we apply π_0 to both of these elements we get $\pi_0[(f \times f \circ \delta_X)(x)] = [\pi_0(f \times f \circ \delta_X)](x) = [\pi_0(\delta_Y \circ f)](x) = \pi_0[(\delta_Y \circ f)(x)]$ and likewise we get the obvious analogous result when applying π_1. By Proposition 2.1.21 it follows that $(f \times f \circ \delta_X)(x) = (\delta_Y \circ f)(x)$ but $x \in X$ was arbitrary so by Axiom 3.2 if follows that $f \times f \circ \delta_X = \delta_Y \circ f$ and so indeed the diagram commutes.

Remark: I admit the proof feels a little gauche since I'm using Proposition 2.1.21 which comes two pages later and Axiom 3 which immediately follows this exercise.

2.1.15 If X and Y are terminal objects then $X \cong Y$.

Proof: Since Y is terminal there exists a unique morphism $\beta_X : X \to Y$ and since X is terminal there exists a unique morphism $\beta_Y : Y \to X$ then from composition of maps $\beta_X \circ \beta_Y : Y \to Y$ and $\beta_Y \circ \beta_X : X \to X$ however because X is terminal it has exactly one morphism from every object, including itself but necessarily X has an identity map, therefore $1_X = \beta_Y \circ \beta_X$ and $1_Y = \beta_X \circ \beta_Y$. It follows β_X is an isomorphism.

2.1.18 Show that any function $x : 1 \to X$ is monic.

Proof: Suppose $x \circ f = x \circ g$ for arbitrary $f, g : Z \rightrightarrows 1$. But then, from Axiom 3.1, $f = g = \beta_Z$, since β_Z is the unique function $Z \to 1$.

2.1.19 If $X \cong 1$ then X has exactly one element.

Proof: The proof for this in the textbook is unsound, so here we correct it.

There is at least one element $x : 1 \to X$ since if $X \cong 1$ then $1 \cong X$. Now let us suppose there is another element $y : 1 \to X$.

First observe that $\beta_1 : 1 \to 1$ is the unique map from 1 to 1 by Axiom 3. It follows that $\beta_1 = 1_1 = \beta_X \circ y = \beta_X \circ x$. But β_X is an isomorphism (since $X \cong 1$ we have *some* isomorphism $f : X \to 1$ but from Axiom 3 we have $f = \beta_X$) from which it follows that $y = x$ since β_X is an monomorphism (by Exercise 2.1.7(5)).

2.1.15b If $X \cong 1$ then X is terminal. Then use this fact to show that if $X \cong Y$ and at least one of them is terminal then they are both terminal.

Proof: Suppose $X \cong 1$ then by Exercise 2.1.19 X has exactly one element, say x. Let Y be an arbitrary set, then $\beta_Y : Y \to 1$ and $x \circ \beta_Y : Y \to X$. This shows that given any set Y we can always find at least one map from Y to X. Now suppose X was not terminal then there exists a set Y and distinct maps $f, g : Y \to X$. From Axiom 3.2, since $f \neq g$ there exists $y \in Y$ such that $f \circ y \neq g \circ y$. But $f \circ y : 1 \to X$ and $g \circ y : 1 \to X$; however, X only has a single element so $f \circ y = g \circ y$, a contradiction! We conclude that $x \circ \beta_Y$ is the unique mapping from Y to X. Therefore, X is terminal.

Now suppose that $X \cong Y$ and that Y is terminal then from Exercise 2.1.15 we have $Y \cong 1$ (since 1 is terminal). Since $X \cong Y$ and $Y \cong 1$ it follows that $X \cong 1$ and by the previous result we have that X is terminal.

2.1.26 Let $f : X \to Y$ be a function. Show that if f is monic, then f is injective.

Proof: Because $f : X \to Y$ is monic, then (in particular) for any pair of morphisms $s, t : 1 \to X$, $s = t$ whenever $f(s) = f(t)$.

2.1.30 Show that if $f : X \to Y$ is surjective then f is an epimorphism.

Proof: Suppose that Y is nonempty and let $f : X \to Y$ be surjective and suppose we have a pair of distinct morphisms $g, h : Y \rightrightarrows Z$ then by Axiom 3.2 there exists $y \in Y$ such that $g(y) \neq h(y)$. But as f is surjective there exists $x \in X$ such that $f(x) = y$. Then $g(f(x)) \neq h(f(x))$ but then $g \circ f \neq h \circ f$ again because 1 is a separator. (Proof by contraposition).

Now if Y is empty (not nonempty) then for any $g, h : Y \rightrightarrows Z$, it is vacuously true that $g(x) = h(x)$ for all $x \in Y$, so $g = h$ by Axiom 3.2. The condition of being an epimorphism is thus always fulfilled for f with such a codomain.

2.1.31 Suppose that (E, m) and (E', m') are both equalizers of f and g. Show that there is an isomorphism $k : E \to E'$.

Proof: Let $f, g : X \to Y$ be a pair of maps equalized by (E, m) and (E', m') then in particular $fm = gm$ and $fm' = gm'$. Moreover, since (E, m) equalizes (f, g) it follows that there exist a unique map $k : E' \to E$ that makes the top part of the diagram below commmute; likewise there exists a unique map $k' : E \to E'$ that makes the bottom part of the diagram commute since (E', m') is an equalizer

$$\begin{tikzcd}
E \arrow[r, "m"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
E' \arrow[u, dotted, "k"] \arrow[ur, "m'"'] \arrow[r, "m'"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
E \arrow[u, dotted, "k'"] \arrow[ur, "m"']
\end{tikzcd}$$

then in particular $fmkk' = gmkk'$. Now consider the following diagram

$$\begin{tikzcd}
E \arrow[r, "m"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
E \arrow[u, dotted, "v"] \arrow[ur, "m"']
\end{tikzcd}$$

since (E, m) equalizes (f, g) there exists a *unique* map $v : E \to E$ that satisfies this diagram but 1_E and kk' also satisfy this diagram, therefore $1_E = v = kk'$. Now repeat the entire argument switching the roles of (E', m') and (E, m) everywhere to see that $1_{E'} = k'k$. This proves that $E \cong E'$.

2.1.33 Let $f, g : X \rightrightarrows Y$, and let $m : E \to X$ be the equalizer of f and g. Let $x \in X$. Show that x factors through m if and only if $f(x) = g(x)$.

Proof: Suppose that $f(x) = g(x)$ then as (E, m) equalizes f and g the following diagram commmutes

$$\begin{tikzcd} E \arrow[r, "m"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ 1 \arrow[u, dotted, "k"] \arrow[ur, "x"'] & & \end{tikzcd}$$

from the diagram we see that $m \circ k = x$.

Conversely, now suppose that x factors through m then there exists k such that $m \circ k = x$. That is to say, the factorization induces the following diagram

$$\begin{tikzcd} E \arrow[r, "m"] & X \\ 1 \arrow[u, dotted, "k"] \arrow[ur, "x"'] & \end{tikzcd}$$

on the other hand as $m : E \to X$ is an equalizer of f and g we have that $fm = gm$ and therefore $f(x) = f(mk) = (fm)k = (gm)k = g(mk) = g(x)$

2.1.36 Show that if f is an epimorphism and a regular monomorphism, then f is an isomorphism.

Proof: Let $f : X \to Y$ be a regular monomophism, then f is an equalizer for some pair of arrows $g, h : Y \rightrightarrows Z$. Then we must have that $g \circ f = h \circ f$, however we also assume that f is epi so that $g = h$. We observe quite trivially, then, that $g \circ 1_Y = h \circ 1_Y$, but f is an equalizer so this induces a unique morphism $k : Y \to X$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & Z \\ Y \arrow[u, dotted, "k"] \arrow[ur, "1_Y"'] & & \end{tikzcd}$$

and so it follows that $f \circ k = 1_Y$. Notice then that

$$f \circ 1_X = f = 1_Y \circ f = (f \circ k) \circ f = f \circ (k \circ f)$$

but from Definition 2.1.35 regular monomorphisms are equalizers and by Proposition 2.1.34 equalizers are monomorphisms. Therefore, regular monomorphisms are monomorphisms and so it follows that $k \circ f = 1_X$ since f is monic. Therefore $X \cong Y$ via f.

2.1.44 Let $f : X \to Y$ be a function, and let $p_0, p_1 : X \times_Y X \rightrightarrows X$ be the kernel pair of f. Show that the following are equivalent:

(a) f is a monomorphism.

(b) p_0 and p_1 are isomorphisms.

(c) $p_0 = p_1$.

Proof:

(a) $(a) \to (c)$

Suppose that f is a monomorphism. Consider $p_0, p_1 : X \times_Y X \to X$ and pick an arbitrary element $\langle x, x' \rangle$ from $X \times_Y X$ then, since $X \times_Y X$ is a kernel pair,

$$f(p_0(\langle x, x' \rangle)) = f(x) = f(x') = f(p_1(\langle x, x' \rangle))$$

so that $f \circ p_0 = f \circ p_1$ but as f is a monomorphism it follows that $p_0 = p_1$.

(b) $(c) \to (a) \wedge (b)$

Suppose that $p_0 = p_1$. Let $x, x' \in X$ such that $f(x) = f(x')$ hence $\langle x, x' \rangle \in X \times_Y X$ then $x = p_0(\langle x, x' \rangle) = p_1(\langle x, x' \rangle) = x'$. Hence f is injective and hence monic by Proposition 2.1.27. Now since f is injective, it follows that $X \times_Y X = \{\langle x, x \rangle : x \in X\}$. Observe that

$$\pi_0(\delta \circ x) = \pi_0(\langle 1_X, 1_X \rangle \circ x) = (\pi_0 \langle 1_X, 1_X \rangle)x = 1_X x = x = \pi_0(\langle x, x \rangle)$$

(and a similar result holds with π_1) so that $\delta(x) = \langle x, x \rangle$ by Proposition 2.1.21 (as we should expect). Now in general $p_0 \circ \delta_X = 1_X$, and let $z \in X \times_Y X$ then $z = \langle x, x \rangle$ for some $x \in X$ and $(\delta \circ p_0)(z) = \delta(p_0(z)) = \delta(x) = z = 1_{X \times_Y X}(z)$ and hence $\delta \circ p_0 = 1_{X \times_Y X}$ by Axiom 3.2; therefore p_0 (and hence p_1) is/are isomorphisms.

(c) $(b) \to (a) \wedge (c)$

Suppose that p_0 and p_1 are isomorphisms then there exists q_0 and $q_1 : X \to X \times_Y X$ which satisfy the following equations

$$p_0 \circ q_0 = 1_X = p_1 \circ q_1$$

$$q_0 \circ p_0 = 1_{X \times_Y X} = q_1 \circ p_1$$

Now consider an arbitrary element $\langle x, x' \rangle \in X \times_Y X$ then

$$q_0(x) = (q_0 \circ p_0)(\langle x, x' \rangle) = \langle x, x' \rangle = (q_1 \circ p_1)(\langle x, x' \rangle) = q_1(x')$$

then it must follow that

$$x = 1_X(x) = (p_0 \circ q_0)(x) = (p_0 \circ q_1)(x')$$

However, if we apply the above equations to a diagonal element $\langle x', x' \rangle$ then we observe that $x' = (p_0 \circ q_1)(x')$. Combining these results together shows that $x = x'$ in the first set of equations, that is to say that $X \times_Y X = \{\langle x, x \rangle | x \in X\}$ and so f is injective and hence a monomorphism. *Also* since $(p_0 \circ q_1)(x) = x$ it follows that $q_1(x) = ((q_0 \circ p_0) \circ q_1)(x) = (q_0 \circ (p_0 \circ q_1))(x) = (q_0 \circ (p_0 \circ q_1)(x)) = q_0(x)$, and hence $q_0 = q_1$. Then $p_1 = (p_0 \circ q_0) \circ p_1 = p_0 \circ (q_1 \circ p_1) = p_0$. We note that strictly speaking we do not need to prove (c) here and so we do not need anything after "Also" in our proof.

2 Truth values and subsets

2.2.1 The terminal object 1 has exactly two subobjects (up to isomorphism).

Proof: . In the Axiom 5.2 let $B = X = 1$ and $m = \beta_1 = \mathrm{id}_1$ then it is clear that $(1, \mathrm{id}_1)$ is subobject of 1.

Now let $B = 1 \times_\Omega 1$ and $X = 1$ in Axiom 5.2. We will show that B is empty. Suppose $x \in B$ then let us consider the following pullback diagram

$$\begin{tikzcd}
1 \arrow[dr, dashed, "x"] \arrow[drr] \arrow[ddr] & & \\
& B \arrow[r] \arrow[d] & 1 \arrow[d, "\mathsf{t}"] \\
& 1 \arrow[r, "\mathsf{f}"'] & \Omega
\end{tikzcd}$$

this shows that "t" = "f" which is a contradiction; therefore, B is empty. Finally, if B is any set which causes the above diagram to commute we can likewise conclude that B is empty; however, all empty sets are isomorphic. Lasty, if B causes the following diagram to commute

$$\begin{tikzcd}
1 \arrow[dr, dashed, "x"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& B \arrow[r] \arrow[d] & 1 \arrow[d, "\mathsf{t}"] \\
& 1 \arrow[r, "\mathsf{t}"'] & \Omega
\end{tikzcd}$$

since we have that $x \in B$ is unique it must be that B has exactly one element so that $B \cong 1$. Since there are no commuting diagrams which associate subobjects and their morphisms with 1 we conclude that 1 contain exactly two subojects (up to isomorphism).

2.2.2 Show that $\Omega \times \Omega$ has exactly four elements.

Proof: *There are at most four elements!*

Suppose we have an element $g : 1 \to \Omega \times \Omega$. Then $\pi_0 \circ g : 1 \to \Omega$ is an element of Ω so, by Axiom 5, it is either t or f. Similarly, $\pi_1 \circ g : 1 \to \Omega$ is either t or f. There are four combinations:

- $\pi_0 g = \mathsf{t}$ and $\pi_1 g = \mathsf{t}$, so $g = \langle \mathsf{t}, \mathsf{t} \rangle$
- $\pi_0 g = \mathsf{t}$ and $\pi_1 g = \mathsf{f}$, so $g = \langle \mathsf{t}, \mathsf{f} \rangle$
- $\pi_0 g = \mathsf{f}$ and $\pi_1 g = \mathsf{t}$, so $g = \langle \mathsf{f}, \mathsf{t} \rangle$
- $\pi_0 g = \mathsf{f}$ and $\pi_1 g = \mathsf{f}$, so $g = \langle \mathsf{f}, \mathsf{f} \rangle$

The uniqueness of these follows from Axiom 2, so we have at most 4 possibilities for g.

There are at least four elements!

Note that $\pi_0 \langle \mathsf{t}, \mathsf{t} \rangle = \mathsf{t} \neq \mathsf{f} = \pi_0 \langle \mathsf{f}, \mathsf{t} \rangle$ by Proposition 2.1.21 this proves that $\langle \mathsf{t}, \mathsf{t} \rangle \neq \langle \mathsf{f}, \mathsf{t} \rangle$.

Likewise, $\pi_1 \langle \mathsf{t}, \mathsf{t} \rangle = \mathsf{t} \neq \mathsf{f} = \pi_1 \langle \mathsf{t}, \mathsf{f} \rangle$ by Proposition 2.1.21 this proves that $\langle \mathsf{t}, \mathsf{t} \rangle \neq \langle \mathsf{t}, \mathsf{f} \rangle$.

And, $\pi_1 \langle \mathsf{t}, \mathsf{t} \rangle = \mathsf{t} \neq \mathsf{f} = \pi_1 \langle \mathsf{f}, \mathsf{f} \rangle$ by Proposition 2.1.21 this proves that $\langle \mathsf{t}, \mathsf{t} \rangle \neq \langle \mathsf{f}, \mathsf{f} \rangle$.

And, $\pi_1 \langle \mathsf{f}, \mathsf{f} \rangle = \mathsf{f} \neq \mathsf{t} = \pi_1 \langle \mathsf{f}, \mathsf{t} \rangle$ by Proposition 2.1.21 this proves that $\langle \mathsf{f}, \mathsf{f} \rangle \neq \langle \mathsf{f}, \mathsf{t} \rangle$.

And, $\pi_0 \langle \mathsf{f}, \mathsf{f} \rangle = \mathsf{f} \neq \mathsf{t} = \pi_0 \langle \mathsf{t}, \mathsf{f} \rangle$ by Proposition 2.1.21 this proves that $\langle \mathsf{f}, \mathsf{f} \rangle \neq \langle \mathsf{t}, \mathsf{f} \rangle$.

And, $\pi_0 \langle \mathsf{f}, \mathsf{t} \rangle = \mathsf{f} \neq \mathsf{t} = \pi_0 \langle \mathsf{t}, \mathsf{f} \rangle$ by Proposition 2.1.21 this proves that $\langle \mathsf{f}, \mathsf{t} \rangle \neq \langle \mathsf{t}, \mathsf{f} \rangle$.

2.2.3 Show that in any category, if $f : X \to Y$ is a regular monomorphism, then f is monic.

Proof: It follows by definition that a regular monomorphism is the equalizer for some pair of maps; but by Proposition 21.34 an equalizer is a monomorphism.

3 Relations

2.3.1 Show that in any category, coequalizers are unique up to isomorphism.

Proof: I first review the definition of a coequalizer in a general category. A coequalizer is a object-morphism pair (C, c) such that for $f, g : X \rightrightarrows Y$ it is the case that $c : Y \to C$ and $cf = cg$ and for any other object-morphism pair $h : Y \to F$ if $hf = hg$ then there exists a unique map $k : C \to F$ such that $h = kc$.

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "c"] \arrow[dr, "h"'] & C \arrow[d, dotted, "k"] \\
& & F
\end{tikzcd}$$

Suppose that (C, n) and (C', n') both coequalize f and g then

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "n"] \arrow[dr, "n'"'] & C \arrow[d, dotted, "k"] \\ & & C' \end{tikzcd}$$

and as (C, n) coequalizes and $n'f = n'g$ it follows there exists a unique morphism $k : C \to C'$ such that $kn = n'$. On the other hand, since (C', n') coequalizes

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "n'"] \arrow[dr, "n"'] & C' \arrow[d, dotted, "k'"] \\ & & C \end{tikzcd}$$

and $nf = ng$ it follows that there exists a unique morphism $k' : C' \to C'$ such that $k'n' = n$.

Now consider the following diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "n'"] \arrow[dr, "n'"'] & C' \arrow[d, dotted, "k''"] \\ & & C' \end{tikzcd}$$

since (C', n') coequalizes it follows that there exists a unique morphism $k'' : C' \to C'$ such that $k''n' = n'$. But notice that $1_{C'}$ has this property as well, so that $k'' = 1_{C'}$.

Combining previous equations we see that $n' = kn = k(k'n') = (kk')n'$, but $kk' : C' \to C'$ and the uniqueness statement above proves that $kk' = 1_{C'}$.

Now consider the following diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "n"] \arrow[dr, "n"'] & C \arrow[d, dotted, "j"] \\ & & C \end{tikzcd}$$

since (C, n) coequalizes it follows that that there exists a unique morphism $j : C \to C$ such that $jn = n$. But notice that 1_C has this property as well, so that $j = 1_C$. Repeating as before, we see that $n = k'n' = k'(kn) = (k'k)n$. But $k'k : C \to C$ and the uniqueness statement above proves that $k'k = 1_C$.

2.3.2 Show that in any category, a coequalizer is an epimorphism.

Proof: Let $c : Y \to C$ coequalize $f, g : X \rightrightarrows Y$. Then suppose that $m, n : C \to A$ such that $mc = nc$. Notice $nc : Y \to A$ and that $ncf = ncg$ but as (C, c) coequalizes f and g it follows that there is a unique morphism $k : C \to A$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "c"] \arrow[dr, "nc"'] & C \arrow[d, dotted, "k"] \\ & & A \end{tikzcd}$$

Notice if we replace k with m or with n the diagram still commutes however we have that k is the *unique* morphism which makes the diagram commute, and therefore $m = k = n$. Hence c is an epimorphism.

2.3.3 For a function $f : X \to Y$, let $R = \{\langle x,y\rangle \in X \times X | f(x) = f(y)\}$. That is, R is the kernel pair of f. Show that R is an equivalence relation.

Proof: $(x,x) \in R$ since $f(x) = f(x)$. Now suppose that $(x,y) \in R$ then $f(x) = f(y)$ and hence $f(y) = f(x)$ therefore $(y,x) \in R$. Now suppose that $(x,y) \in R$ and $(y,z) \in R$ then $f(x) = f(y)$ and $f(y) = f(z)$ but then $f(x) = f(y) = f(z)$ so that $f(x) = f(z)$, in other words $(x,z) \in R$. Hence R is an equivalence relation.

2.3.5 Show that in any category, if $f : X \to Y$ is both a monomorphism and a regular epimorphism, then f is an isomorphism.

Proof: Since f is a regular epimorphism then it is the coequalizer for some pair of maps $g, h : W \to X$ and hence in particular $fg = fh$. But as f is also a monomorphism then it follows that $h = g$. It follows then, obviously, that $1_X \circ h = 1_X \circ g$. But then since f coequalizes g and h it must be that there exists a unique map $j : Y \to X$ which makes the following diagram commute:

$$\begin{tikzcd} W \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & X \arrow[r, "f"] \arrow[dr, "1_X"'] & Y \arrow[d, dashed, "j"] \\ & & X \end{tikzcd}$$

and in particular $jf = 1_X$. With a little more manipulation we get

$$1_Y \circ f = f = f \circ 1_X = f \circ (j \circ f) = (f \circ j) \circ f$$

but f is epi (by Exercise 2.3.2) so the above equation implies that $1_Y = f \circ j$. Therefore $X \cong Y$.

2.3.10 Use the previous result[1] to show that $A \subset f^{-1}(\exists_f(A))$, for any subset A of X.

Proof: Let $B = \exists_f(A)$. Very clearly $1_B : \exists_f(A) \to B$ is a monomorphism, so $\exists_f(A) \subseteq B$ and hence by Proposition 2.3.9 $A \subseteq f^{-1}(B)$. That is, $A \subseteq f^{-1}(\exists_f(A))$.

2.3.13 Show that graph(f) is a functional relation.

Proof: graph$(f) = \{\langle x,y\rangle | f(x) = y\}$. First the graph is certainly a relation since it is a subset of $X \times Y$. Uniqueness property: Suppose that $\langle x,y\rangle$ and $\langle x,y'\rangle$ are both elements of the graph, then $y = f(x) = y'$, hence $\langle x,y\rangle = \langle x,y'\rangle$. Existence property: Certainly given $x \in X$, we have that $\langle x, f(x)\rangle \in R$.

4 Colimits

2.4.6 Show that in any category with coproducts, if A is an initial object, then $X \amalg A \cong X$, for any object X.

Proof: Consider $1_X : X \to X$ and $\alpha_X : A \to X$ and let $i_0 : X \to X \amalg A$ and $i_1 : A \to X \amalg A$ be the coprojections of the coproduct then by Axiom 7 $1_X \amalg \alpha_X : X \amalg A \to X$ is induced and has the property that $(1_X \amalg \alpha_X) i_0 = 1_X$.

$$\begin{tikzcd} & X & \\ X \arrow[ur, "1_X"] \arrow[r, "i_0"'] & X \amalg A \arrow[u, dashed, "1_X \amalg \alpha_X"] & A \arrow[l, "i_1"] \arrow[ul, "\alpha_X"'] \end{tikzcd}$$

We want to show that $i_0(1_X \amalg \alpha_X) = 1_{X \amalg A}$. We first observe that the following properties hold

$$(i_0 \circ 1_X \amalg \alpha_X) \circ i_0 = i_0 \circ (1_X \amalg \alpha_X \circ i_0) = i_0 \circ 1_X = i_0$$

[1] *Proposition 2.3.9: For any $A \subseteq X$ and $B \subseteq Y$, we have $A \subseteq f^{-1}(B)$ if and only if $\exists_f(A) \subseteq B$.*

$$(i_0 \circ 1_X \amalg \alpha_X) \circ i_1 = i_0 \circ (1_X \amalg \alpha_X \circ i_1) = i_0 \circ \alpha_X = i_1$$

the last equality ($i_0 \circ \alpha_X = i_1$) is true since $i_0 \circ \alpha_X$ and i_1 are both maps from A to $X \amalg A$ but A is initial.

Now consider the following diagram:

$$\begin{tikzcd}
 & X \amalg A & \\
X \arrow[ur, "i_0"] \arrow[r, "i_0"'] & X \amalg A \arrow[u, dashed, "i_0 \amalg i_1"] & A \arrow[ul, "i_1"'] \arrow[l, "i_1"]
\end{tikzcd}$$

Then by Axiom 7 $i_0 \amalg i_1 : X \amalg A \to X \amalg A$ is the unique map such that $(i_0 \amalg i_1) \circ i_0 = i_0$ and $(i_0 \amalg i_1) \circ i_1 = i_1$. But the above centered equations show that $i_0 \circ 1_X \amalg \alpha_X : X \amalg A \to X \amalg A$ also satisfies these equations, as does $1_{X \amalg A}$. Therefore, $i_0 \circ 1_X \amalg \alpha_X = i_0 \amalg i_1 = 1_{X \amalg A}$. Therefore, $X \amalg A \cong A$.

5 Sets of functions and sets of subsets

2.5.7 Proposition: For any set, we have $X^1 \cong X$.

Note: The proof in the text is almost correct but is technically wrong in a few places.

Proof: Let $e_X : 1 \times X^1 \to X$ be the evaluation function from Axiom 9. We claim that $e = e_X$ is a bijection. Recall that there is a natural isomorphism $i : 1 \times 1 \to 1$. Consider the following diagram:

$$\begin{tikzcd}
1 \times X^1 \arrow[r, "e"] & X \\
1 \times 1 \arrow[u, "\mathrm{id}_1 \times (x \circ i)^{\#}"] \arrow[r, "i"] & 1 \arrow[u, "x"']
\end{tikzcd}$$

Epimorphism: Let $x \in X$, by Axiom 9 (see diagram above) we are given some map, namely $(x \circ i)^{\#} \in X^1$, such that the diagram commutes, equivalently $e(\mathrm{id}_1 \times (x \circ i)^{\#}) = x \circ i$. Let $i^{-1} : 1 \to 1 \times 1$ denote the inverse of i and let $y = \mathrm{id}_1 \times (x \circ i)^{\#} \circ i^{-1} \in 1 \times X^1$. Then we have

$$e(y) = e(\mathrm{id}_1 \times (x \circ i)^{\#} \circ i^{-1}) = e(\mathrm{id}_1 \times (x \circ i)^{\#}) \circ i^{-1} = (x \circ i) \circ i^{-1} = x$$

It follows e is a surjection and hence an epimorphism by Exercise 2.1.30.

Monomorphism: Suppose $e(x) = e(y)$ where $x, y \in 1 \times X^1$ then consider the following diagram

$$\begin{tikzcd}
1 \times X^1 \arrow[r, "e"] & X \\
1 \times 1 \arrow[u, "\mathrm{id}_1 \times (e(x) \circ i)^{\#}"] \arrow[r, "i"] & 1 \arrow[u, "e(x)"']
\end{tikzcd}$$

Note that $\pi_0(x), \pi_0(y), \mathrm{id}_1 : 1 \to 1$, it follows by Proposition 2.1.19 that $\pi_0(x) = \pi_0(y) = \mathrm{id}_1$. Since $\pi_0(x) = \mathrm{id}_1$ and $\pi_1(x) = \pi_1(x)$ it follows that $x = \langle \mathrm{id}_1, \pi_1(x) \rangle : 1 \to 1 \times X^1$ by Axiom 2 (Cartesian Products), and likewise $y = \langle \mathrm{id}_1, \pi_1(y) \rangle$ where $\pi_1(x), \pi_1(y) \in X^1$.

By Axiom 9 (Exponential Objects) we have that $(e(x) \circ i)^{\#} : 1 \to X^1$ is the unique function $F : 1 \to X^1$ such that $e(\mathrm{id}_1 \times F) = e(x) \circ i$. Likewise, $(e(y) \circ i)^{\#} : 1 \to X^1$ is the unique function $G : 1 \to X^1$ such that $e(\mathrm{id}_1 \times G) = e(y) \circ i$. To summarize, we have $e(x) \circ i = e(\mathrm{id}_1 \times (e(x) \circ i)^\#)$. Note that

$$\begin{aligned}
e(\mathrm{id}_1 \times \pi_1(x)) &= e((\mathrm{id}_1 \circ \mathrm{id}_1) \times (\pi_1(x) \circ \mathrm{id}_1)) \\
&= e((\mathrm{id}_1 \times \pi_1(x)) \circ (\mathrm{id}_1 \times \mathrm{id}_1)) \\
&= e((\mathrm{id}_1 \times \pi_1(x)) \circ (\delta_1 \circ i)) \\
&= e(((\mathrm{id}_1 \times \pi_1(x)) \circ \delta_1) \circ i) \\
&= e(((\mathrm{id}_1 \times \pi_1(x)) \circ \langle \mathrm{id}_1, \mathrm{id}_1 \rangle) \circ i) \\
&= e(\langle \mathrm{id}_1 \circ \mathrm{id}_1, \pi_1(x) \circ \mathrm{id}_1 \rangle \circ i) \\
&= e(\langle \mathrm{id}_1, \pi_1(x) \rangle \circ i) \\
&= e(x) \circ i
\end{aligned}$$

(Note that $\mathrm{id}_1 \times \mathrm{id}_1 = \delta_1 \circ i$, since $\delta_1 \in 1 \times 1$ and $i^{-1} \in 1 \times 1$, but 1×1 only has one element, so $\delta_1 \circ i = \mathrm{id}_{1\times 1}$, which is $\mathrm{id}_1 \times \mathrm{id}_1$ by Exercise 2.1.12) But, $(e(x) \circ i)^\# : 1 \to X^1$ is the *unique* function $F : 1 \to X^1$ such that $e(\mathrm{id}_1 \times F) = e(x) \circ i$, so it must be that $\pi_1(x) = (e(x) \circ i)^\#$. Likewise it must be that $\pi_1(y) = (e(y) \circ i)^\#$. It follows that

$$\pi_1(x) = (e(x) \circ i)^\# = (e(y) \circ i)^\# = \pi_1(y)$$

, and since $\pi_0(x) = \pi_0(y)$ and $\pi_1(x) = \pi_1(y)$, it follows that $x = y$ by Proposition 2.1.21. Therefore, e is injective and hence a monomorphism by Proposition 2.1.27.

Isomorphism: Since e_X is both a monomorphism and an epimorphism it is an isomorphism, by Proposition 2.2.5, so that $1 \times X^1 \cong X$. Finally, recall that $1 \times X^1 \cong X^1$, by Proposition 2.1.20. Lastly, by definition of "isomorphism", we have that the composition of isomorphisms is an isomorphism so that $X \cong X^1$.

2.5.1. Proposition: For every set X, $1^X \cong 1$.

Proof:

Note that for every $y \in 1^X$ we have $e_1 \circ (\mathrm{id}_X \times y) : X \times 1 \to 1$, yet 1 is terminal so it must be that $e_1 \circ (\mathrm{id}_X \times y) = \beta_{X\times 1}$ for *every* choice of $y \in 1^Y$. But from Axiom 9, if $f = \beta_{X\times 1}$ then $f^\# : 1 \to 1^X$ is the *unique* map which makes the following diagram commute:

$$\begin{tikzcd}
X \times 1^X \arrow[r, "e_1"] & 1 \\
X \times 1 \arrow[u, dotted, "\mathrm{id}_X \times f^\#"] \arrow[ur, "f=\beta_{X\times 1}"'] &
\end{tikzcd}$$

However, we argued that this is the case for *every* map $y : 1 \to 1^X$ so it must 1^X has only a single element hence $1 \cong 1^X$ by Proposition 2.1.19.

2.5.2. Proposition: For every set $X \times \varnothing \cong \varnothing$.

Proof: Observe that $\beta_X : X \to 1$ and $\mathrm{id}_\varnothing : \varnothing \to \varnothing$. Let $\pi_1 : 1 \times \varnothing \to \varnothing$ then $\pi_1 \circ (\beta_X \times \mathrm{id}_\varnothing) : X \times \varnothing \to \varnothing$ and this is an isomorphism by Proposition 2.4.7.

2.5.3. Proposition: For every *nonempty* set X, we have $\varnothing^X \cong \varnothing$.

Proof: Recall $A \cong 1 \times A$ via some isomorphism. Let $j : \varnothing^X \to 1 \times \varnothing^X$ be such an isomorphism. Since X is nonempty there is some $y : 1 \to X$ and $y \times \mathrm{id}_{\varnothing^X} : 1 \times \varnothing^X \to X \times \varnothing^X$.

By Axiom 9, there is a function $e_\varnothing : X \times \varnothing^X \to \varnothing$ ... which is enough. It follows $e_\varnothing(y \times \mathrm{id}_{\varnothing^X}) \circ j : \varnothing^X \to \varnothing$ and this is an isomorphism by Proposition 2.4.7.

2.5.4. Proposition: For every set X, $X^{\varnothing} \cong 1$.

Proof:

Let $f = \alpha_X \circ \pi_0$ then since $f : \varnothing \times 1 \to X$ we have there exists a unique map $f^\sharp : 1 \to X^{\varnothing}$ such that the following diagram commutes

$$\begin{tikzcd}
\varnothing \times X^{\varnothing} \arrow[r, "e_X"] & X \\
\varnothing \times 1 \arrow[u, dotted, "\mathrm{id}_{\varnothing} \times f^{\#}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

This proves $f^\sharp \in X^{\varnothing}$, now we prove this is the only element of $X^{\varnothing}$. Let $z \in X^{\varnothing}$ be arbitrary. Recall that $\varnothing \cong \varnothing \times 1$, so there exists an isomorphism $j : \varnothing \to \varnothing \times 1$ but $\varnothing$ is initial so $j = \alpha_{\varnothing \times 1}$ is an isomorphism. This means there is a map $\psi : \varnothing \times 1 \to \varnothing$ such that $\alpha_{\varnothing\times 1} \circ \psi = \mathrm{id}_{\varnothing \times 1}$ and $\psi \circ \alpha_{\varnothing \times 1} = \mathrm{id}_{\varnothing}$. We claim that ψ is the *only* map from $\varnothing \times 1$ to $\varnothing$. For suppose $b : \varnothing \times 1 \to \varnothing$ then $b \circ \alpha_{\varnothing\times 1} = \alpha_{\varnothing} = \mathrm{id}_{\varnothing}$ however $\alpha_{\varnothing \times 1}$ is (an isomorphism and hence) an epimorphism by Exercise 2.1.7(5) so that $b = \psi$. Now $\varnothing \times X^{\varnothing} \cong \varnothing$ so there is an isomorphism $\varphi : \varnothing \times X^{\varnothing} \to \varnothing$, and $\varphi \circ (\mathrm{id}_{\varnothing} \times z) : \varnothing \times 1 \to \varnothing$ so $\psi = \varphi \circ (\mathrm{id}_{\varnothing} \times z)$. However, by a similar argument by types we have that $\psi = \varphi \circ (\mathrm{id}_{\varnothing} \times f^\sharp)$, yet φ is an isomorphism so that $\mathrm{id}_{\varnothing} \times z = \mathrm{id}_{\varnothing} \times f^\sharp$ but this means that $\mathrm{id}_{\varnothing} \times z$ makes the first diagram commute as well hence $z = f^\sharp$. This proves that $f^\sharp$ is the unique element of $X^{\varnothing}$ hence we conclude that $X^{\varnothing} \cong 1$ by Proposition 2.1.19.

6 Cardinality

2.6.5 Let N' be a set, and let $z' : 1 \to N'$ and $s' : N' \to N'$ be functions that satisfy the conditions of the axiom above. Show that N' is isomorphic N. *Moreover, show that the converse is true.*

Proof: Since (N, z, s) is a natural number object the top part of the diagram below commutes, also the bottom part of the diagram since commutes since (N', z', s') also has the properties described in Axiom 10.

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[d, "1_1"'] \arrow[dr, "z'"] & N \arrow[r, "s"] \arrow[d, dotted, "u"] & N \arrow[d, dotted, "u"] \\
1 \arrow[r, "z'"] \arrow[dr, "z"'] & N' \arrow[r, "s'"] \arrow[d, dotted, "u'"] & N' \arrow[d, dotted, "u'"] \\
& N \arrow[r, "s"] & N
\end{tikzcd}$$

from the diagram we see that $u'uz = z$. Now consider the following diagram

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[dr, "z"'] & N \arrow[r, "s"] \arrow[d, dotted, "v"] & N \arrow[d, dotted, "v"] \\
& N \arrow[r, "s"] & N
\end{tikzcd}$$

since (N, z, s) is a NNO there exists a unique morphism $v : N \to N$ that makes the diagram above commute. However,both 1_N and $u'u$ make the diagram commute, therefore $1_N = v = u'u$. Repeating the entire argument switching the roles of (N, z, s) and (N', z', s') we see that $uu' = 1_{N'}$. Therefore $N \cong N'$.

Conversely, suppose that $N' \cong N$. Then there exists $i : N' \to N$ and $j : N \to N'$ such that $i \circ j = \mathrm{id}_N$ and $j \circ i = \mathrm{id}_{N'}$. Let $z' = j \circ z$ and let $s' = (j \circ s) \circ i$ then $z' : 1 \to N'$ and $s' : N' \to N'$. Now suppose that X is any other set with functions $q : 1 \to X$ and $f : X \to X$; we want to show that there exists a unique function $v : N' \to X$ such that the following diagram † commutes

$$\begin{tikzcd} 1 \arrow[r, "z'"] \arrow[dr, "q"'] & N' \arrow[r, "s'"] \arrow[d, dotted, "v"] & N' \arrow[d, dotted, "v"] \\ & X \arrow[r, "f"] & X \end{tikzcd}$$

Let $v = u \circ i$ where u is the unique map such that

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "q"'] & N \arrow[r, "s"] \arrow[d, dotted, "u"] & N \arrow[d, dotted, "u"] \\ & X \arrow[r, "f"] & X \end{tikzcd}$$

Then $v : N' \to X$ and $v \circ z' = (u \circ i) \circ (j \circ z) = u \circ z = q$, so that the left triangle of † commutes. Also $v \circ s' = (u \circ i) \circ (j \circ s \circ i) = (u \circ s) \circ i = (f \circ u) \circ i = f \circ v$. This shows that † above commutes. Now suppose $w : N' \to X$ made † commute; then $q = wz'$ and $ws' = fw$. We will use the uniqueness of u which makes the second diagram commute to establish that $w = v$. First observe $q = wz' = w(jz) = (w \circ j)z$. Secondly,

$$(w \circ j) \circ s \circ i = w \circ (j \circ s \circ i) = w \circ s' = f \circ w = ((f \circ w) \circ j) \circ i$$

But i is an isomorphism and hence an epimorphism so it must be that $(w \circ j) \circ s = (f \circ w) \circ j)$. To summarize, we have $w \circ j : N \to X$ where $w \circ j$ satisfies the second diagram; so it must be that $w \circ j = u$. It follows that $w = w \circ (j \circ i) = (w \circ j) \circ i = u \circ i = v$. Therefore, $v : N' \to X$ is the unique map which makes the first diagram commute, hence (N', z', s') is a NNO.

2.6.7 The function $s : N \to N$ is injective but not surjective. Thus, N is infinite. *Note:* The proof in the book is slightly off.

Proof: By Proposition 2.4.2, the function $i_1 : N \to 1 \coprod N$ is monic. Since $z \sqcup s$ is an isomorphism, $(z \sqcup s) \circ i_1 = s$ is monic. Now we show that s is not surjective. Suppose $s : N \to N$ was surjective, then there exists $n \in N$ such that $s(n) = z$. Since N is a NNO it follows there exists a unique $u : N \to N$ which makes the following diagram commute

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "\mathsf{t}"'] & N \arrow[r, "s"] \arrow[d, dotted, "u"] & N \arrow[d, dotted, "u"] \\ & \Omega \arrow[r, "\mathsf{f} \circ \beta_\Omega"'] & \Omega \end{tikzcd}$$

From the diagram we have

$$\mathsf{t} = u \circ z = u \circ s(n) = \mathsf{f}\beta_\Omega u(n) = \mathsf{f}(\beta_\Omega u(n)) = \mathsf{f}(\mathrm{id}_1) = \mathsf{f}$$

where we used the fact that $\beta_\Omega u(n) : 1 \to 1$ and 1 is terminal so $\beta_\Omega u(n) = \beta_1 = \mathrm{id}_1$. This show that $\mathsf{t} = \mathsf{f}$ but $\mathsf{t} \neq \mathsf{f}$, a contradiction! Therefore, s is not surjective.

2.6.11 Show that if A and B are countable then $A \cup B$ is countable.

Proof: Suppose that A and B are countably infinite then there exists $g : \mathbb{N} \to A$ and $f : \mathbb{N} \to B$ such that $g(n) = a_n$ and $f(n) = b_n$ are enumerations of their elements. Then $(a_0, b_0, a_1, b_1, a_2, \ldots) : \mathbb{N} \to A \cup B$ defines an enumeration of the elements and hence $A \cup B$ is countable.

Remark: This is not a valid proof within ETCS if $\cup$ stands for coproduct. Hint: First prove that $\mathbb{N} \cup \mathbb{N}$ is countable by defining a "parity" function that assigns to natural numbers the odds and the evens.

2.6.15 Show that there is an injective function $X \to \Omega^X$. (The proof is easy if you simply think of Ω^X as functions from X to $\{\mathtt{t}, \mathtt{f}\}$. For a bigger challenge, try to prove that it's true using the definition of the exponential set Ω^X.)

Easy Proof: Let $\triangle : X \to \Omega^X$ be defined by $x \mapsto \triangle_x$ where $\triangle_x : X \to \Omega$ is itself defined by $\triangle_x(y) = \mathtt{t}$ if $x = y$ and $\mathtt{f}$ otherwise. Suppose that $\triangle(p) = \triangle(q)$ then for all $x \in X$ we have that $\triangle_p(x) = \triangle_q(x)$ and in particular $\mathtt{t} = \triangle_p(p) = \triangle_q(p)$ and so it follows that $q = p$. Therefore $\triangle$ is injective.

Bigger Challenge Proof:

First, consider $\delta = \langle 1_X, 1_X \rangle : X \to X \times X$. This is monic (as noted in Definition 2.1.9), so it defines a subobject of $X \times X$. From Axiom 5, we obtain a function χ_δ such that the following is a pullback:

$$\begin{tikzcd}
X \arrow[r, "\beta_X"'] \arrow[d, "\delta"] & 1 \arrow[d, "\mathtt{t}"] \\
X \times X \arrow[r, "\chi_\delta"'] & \Omega
\end{tikzcd}$$

Note that $\chi_\delta\langle x, y\rangle = \mathtt{t}$ if and only if $x = y$. To see this, first assume that $x = y$, then we have $\chi_\delta\langle x, y\rangle = \chi_\delta\langle x, x\rangle = \chi_\delta\delta(x) = \mathtt{t}\beta_X(x) = \mathtt{t}$, since $\langle x, x\rangle = \delta(x)$ (see Exercise 2.1.44(b)'s proof or note they have the same components and Axiom 2 asserts uniqueness), the above diagram commutes and $\beta_X(x) = \beta_1 = 1_1$ (since β_1 is unique). Conversely, suppose $\chi_\delta\langle x, y\rangle = \mathtt{t}$ and consider the following diagram:

$$\begin{tikzcd}
1 \arrow[dr, dashed, "k"] \arrow[drr] \arrow[ddr, "\langle x,y\rangle"'] & & \\
& X \arrow[r, "\beta_X"'] \arrow[d, "\delta"] & 1 \arrow[d, "\mathtt{t}"] \\
& X \times X \arrow[r, "\chi_\delta"'] & \Omega
\end{tikzcd}$$

Since the assumption $\chi_\delta\langle x, y\rangle = \mathtt{t}$ means the outer quadrilateral with the upper-left 1 commutes, and the lower-right square is a pullback (by Axiom 5), we obtain $k : 1 \to X$ with $\delta(k) = \langle x, y\rangle$ and we have:

$$x = \pi_0\langle x, y\rangle = \pi_0(\delta(k)) = (\pi_0\langle 1_X, 1_X\rangle)(k) = 1_X(k) = (\pi_1\langle 1_X, 1_X\rangle)(k) = \pi_1(\delta(k)) = \pi_1\langle x, y\rangle = y$$

Now we take χ_δ and apply Axiom 9 to obtain $\chi_\delta^\sharp : X \to \Omega^X$ such that the following diagram commutes:

$$\begin{tikzcd}
X \times \Omega^X \arrow[r, "e_\Omega"] & \Omega \\
X \times X \arrow[u, "1_X \times \chi_\delta^\sharp"] \arrow[ur, "\chi_\delta"'] &
\end{tikzcd}$$

We claim that $\chi_\delta^\sharp$ is injective, that is, for any $x, y \in X$, if $\chi_\delta^\sharp(x) = \chi_\delta^\sharp(y)$ then $x = y$. Fix $x, y \in X$ and assume $\chi_\delta^\sharp(x) = \chi_\delta^\sharp(y)$. Consider $(1_X \times \chi_\delta^\sharp)\langle x, x\rangle$ and $(1_X \times \chi_\delta^\sharp)\langle x, y\rangle$ and note that

$$\chi_\delta\langle x, x\rangle = e_\Omega(1_X \times \chi_\delta^\sharp)\langle x, x\rangle = e_\Omega\langle x, \chi_\delta^\sharp(x)\rangle = e_\Omega\langle x, \chi_\delta^\sharp(y)\rangle = e_\Omega(1_X \times \chi_\delta^\sharp)\langle x, y\rangle = \chi_\delta\langle x, y\rangle$$

Note that we have $(f \times g)\langle a, b\rangle = \langle fa, gb\rangle$ for any f, g, a, b, since from Axiom 2 and Definition 2.10 we have:

$$\begin{tikzcd}
& A, \arrow[dl, "a"'] \arrow[d, "\langle a,b\rangle"] \arrow[dr, "b"] & \\
W \arrow[d, "f"'] & W \times X \arrow[l, "q_0"] \arrow[r, "q_1"'] \arrow[d, "f \times g"] & X \arrow[d, "g"] \\
Y & Y \times Z \arrow[l, "p_0"] \arrow[r, "p_1"'] & Z
\end{tikzcd}$$

We have that $\langle fa, gb\rangle : A \to Y \times Z$ is unique such that $p_0\langle fa, gb\rangle = fa$ and $p_1\langle fa, gb\rangle = gb$, but from the diagram we have $p_0(f \times g)\langle a, b\rangle = fq_0\langle a, b\rangle = fa$ and $p_1(f \times g)\langle a, b\rangle = gq_1\langle a, b\rangle = gb$, so $(f \times g)\langle a, b\rangle = \langle fa, gb\rangle$ by the *uniqueness* property of Axiom 2.

So, since we have that $\chi_\delta\langle x, x\rangle = \chi_\delta\langle x, y\rangle$, and since $\chi_\delta\langle x, x\rangle = \mathtt{t}$ (by the fact noted earlier), we have that $\chi_\delta\langle x, y\rangle = \mathtt{t}$. By the fact noted earlier, this implies that $x = y$, so we have that $\chi_\delta^\sharp(x) = \chi_\delta^\sharp(y)$ implies $x = y$. Thus we have obtained an injective function $\chi_\delta^\sharp : X \to \Omega^X$.

6.1 Finite implies countable

(Skip)

$$\mathrm{OR} : \Omega \times \Omega \to \Omega$$
$$\mathrm{EQ} = \chi_{\delta_X} : X \times X \to \Omega$$
$$\mathrm{distribute\_left} : X \times (Y \times Z) \to (X \times Y) \times (X \times Z)$$

Fix a finite non-empty X.

$$\begin{tikzcd}
X \times \Omega^X \arrow[r, "e_X"] & \Omega \\
X \times (\Omega^X \times X) \arrow[u, "\mathrm{id}\times(\mathrm{OR}\circ(e_X\times\mathrm{EQ})\circ\mathrm{distribute_left})^\sharp"] \arrow[ur, "\mathrm{OR}\circ(e_X\times\mathrm{EQ})\circ\mathrm{distribute_left}"'] &
\end{tikzcd}$$

Define $\mathrm{insert} = (\mathrm{OR} \circ (e_X \times \mathrm{EQ}) \circ \mathrm{distribute\_left})^\sharp : \Omega^X \times X \to \Omega^X$.
Then have $\mathrm{insert} \sqcup (\mathsf{f} \circ \beta_{X\times 1})^\sharp : (\Omega^X \times X) \coprod 1 \to \Omega^X$ is an epimorphism.
Then obtain from the axiom of choice a monomorphism $\mathrm{extract} : \Omega^X \to (\Omega^X \times X) \coprod 1$.
Then have $(\mathrm{extract} \circ \pi_0) \sqcup \mathrm{id} : (\Omega^X \times X) \coprod 1 \to (\Omega^X \times X) \coprod 1$ and $\mathrm{extract} \circ (\mathtt{t} \circ \beta_{X\times 1})^\sharp : 1 \to (\Omega^X \times X)$.

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[dr, "\mathrm{extract}\circ(\mathtt{t}\circ\beta_{X\times 1})^\sharp"'] & N \arrow[r, "s"] \arrow[d, "u"] & N \arrow[d, "u"] \\
& (\Omega^X \times X) \coprod 1 \arrow[r, "(\mathrm{extract}\circ\pi_0)\sqcup\mathrm{id}"'] & (\Omega^X \times X) \coprod 1
\end{tikzcd}$$

TO SHOW: $u : N \to (\Omega^X \times X) \coprod 1$ is an epimorphism.

Then pick an arbitrary element $x : 1 \to X$ (exists since X is finite, or apply the axiom of choice to β_X), and obtain $(\pi_1 \sqcup x) \circ u : N \to X$.

$(\pi_1 \sqcup x) : (\Omega^X \times X) \coprod 1 \to X$ should also be an epimorphism (any element of X can be output by providing it as part of the input, regardless of the Ω^X), so $(\pi_1 \sqcup x) \circ u$ is also an epimorphism, making X countable.

We can apply the axiom of choice to $(\pi_1 \sqcup x) \circ u$ to get the monomorphism definition of countable instead.

7 The Axiom of Choice

2.7.2 Prove that if f is a split epimorphism, then f is a regular epimorphism[2]. Prove that if s is a section, then s is a regular monomorphism[3].

(a) **Proof:** Suppose that $f : X \to Y$ is a split epi then there exists a morphism $s : Y \to X$ such that $f \circ s = 1_Y$. Then $f \circ (s \circ f) = (f \circ s) \circ f = 1_Y \circ f = f$ and $f \circ 1_X = f$. Now suppose that there exists a morphism $g : X \to Z$ such that $g \circ (s \circ f) = g = g \circ 1_X$. Then let $h : Y \to Z$ be defined by $h = g \circ s$, then we claim that this is the only morphism such that $h \circ f = g$. Indeed,

[2]Recall a function $f : X \to Y$ is said to be a regular epimorphism just in case f is a coequalizer.

[3]Recall we say that f is a regular monomorphism just in case f is the equalizer (up to isomorphis) of a pair of arrows $g, h : Y \leftrightarrow Z$.

$h \circ f = (g \circ s) \circ f = g \circ (s \circ f) = g$. Now suppose that there exists a morphism $k : Y \to Z$ such that $k \circ f = g$ then $k = k \circ 1_Y = k \circ (f \circ s) = (k \circ f) \circ s = g \circ s = h$, therefore $h = g \circ s$ uniquely solves $g = h \circ f$. Therefore (Y, f) coequalizes the pair of maps $(s \circ f, 1_X)$ and hence f is a regular epimorphism.

(b) **Proof:** Suppose that $s : Y \to X$ is a section then there exists a morphism $f : X \to Y$ such that $f \circ s = 1_Y$. It follows that $(s \circ f) \circ s = s \circ (f \circ s) = s \circ 1_Y = s$ and $1_X \circ s = s$. Now suppose that $t : Z \to X$ also has the property that $(s \circ f) \circ t = t = 1_X \circ t$. Then let $h : Z \to Y$ be defined by $h = f \circ t$. Then we claim that h is the unique map such that $s \circ h = t$. Indeed, $s \circ h = s \circ (f \circ t) = (s \circ f) \circ t = t$. Now suppose that $k : Z \to Y$ and $s \circ k = t$, then $k = 1_Y \circ k = (f \circ s) \circ k = f \circ (s \circ k) = f \circ t = h$. Therefore, $h := f \circ t$ is the unique map $h : Z \to Y$ such that $s \circ h = t$ and hence (Y, s) is an equalizer for the pair of maps $(s \circ f, 1_X)$ and hence s is a regular monomorphism.

We summarize the previous two exercises by stating that if $s : Y \to X$ is the section for a split epimorphism $f : X \to Y$ then the following diagram commutes and the dashed morphisms are uniquely defined

$$\begin{tikzcd}
Y \arrow[r, "s"] & X \arrow[r, shift left, "s\circ f"] \arrow[r, shift right, "1_X"'] & X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, dotted, "g\circ s"] \\
Z \arrow[u, dotted, "f\circ t"] \arrow[ur, "t"'] & & & Z
\end{tikzcd}$$

PART III

The Category of Propositional Theories

1 Basics

3.1.10 In Example 3.1.9[1], show that f and g are not essentially surjective.

Proof:

f is not essentially surjective:

$f : T \to T'$ is not essentially surjective if there exists $\phi \in Sent(\Sigma')$ such that for all $\psi \in Sent(\Sigma)$, $T' \nvdash \phi \leftrightarrow f(\psi)$. We will show this by demonstrating, in particular, that $T' \nvdash f(\psi) \to \phi$.

Let $\phi = q_0$ then clearly for all p_i we have $f(p_i) = q_{i+1}$ and $i + 1 > 0$.

Clearly $T' \nvdash f(\psi) \to q_0$ for any $\psi = p_i$ for some $i \geq 0$ as our only axioms are $q_0 \to q_i$.

A simple induction argument on the complexity of ψ shows that $T' \nvdash f(\psi) \to q_0$ for any ψ.

g is not essentially surjective:

$g : T' \to T$ is not essentially surjective if there exists $\phi \in Sent(\Sigma)$ such that for all $\psi \in Sent(\Sigma')$, $T \nvdash \phi \leftrightarrow g(\psi)$. We will show this by demonstrating, in particular, that $T \nvdash g(\psi) \to \phi$.

[1]**Example 3.1.9**: Let $\Sigma = \{p_0, p_1, \ldots\}$, and let T be the empty theory in Σ. Let $\Sigma' = \{q_0, q_1, \ldots\}$, and let T' be the theory with axioms $q_0 \to q_i$, for $i = 0, 1, \ldots$. We will show that there are conservative translations $f : T \to T'$ and $g : T' \to T$.

Define $f : \Sigma \to \mathrm{Sent}(\Sigma')$ be $f(p_i) = q_{i+1}$. Since T is the empty theory, f is a translation. Then for any valuation v of Σ', we have

$$f^* v(p_i) = v(f(p_i)) = v(q_{i+1})$$

Furthermore, for any sequence of zeros and ones, there is a valuation v of Σ' that assigns that sequence to $q_1, q_2, \ldots$. Thus, f^* is surjective, and f is conservative.

Now define $g : \Sigma' \to \mathrm{Sent}(\Sigma)$ by setting $g(q_i) = p_0 \vee p_i$. Since $T \vdash p_0 \vee p_0 \to p_o \vee p_i$, it follows that g is a translation. Furthermore, for any valuation v of Σ, we have

$$g^* v(q_i) = v(g(q_i)) = v(p_0 \vee p_i).$$

Recall that $M(T')$ splits into two parts: (1) a singleton set containing the valuation z where $z(q_i) = 1$ for all i, and (2) the infinitely many other valuations that assign 0 to q_0. Clearly, $z = g^* v$, where v is any valuation such that $v(p_0) = 1$. Furthermore, for any valuation w of Σ' such that $w(p_0) = 0$, we have $w = g^* v$, where $v(p_i) = w(q_i)$. Therefore, g^* is surjective, and g is conservative.

Let $\phi = p_1$. Then, for any q_i $(i \geq 0)$ we have $g(q_i) = p_0 \vee p_i$, so we want to show $T \nvdash p_0 \vee p_i \to p_1$:

$$\begin{aligned}
T \nvdash p_0 \vee p_i \to p_1 &\iff T \not\models p_0 \vee p_i \to p_1 && \text{(Theorem 1.3.10\&11)} \\
&\iff v(p_0 \vee p_i \to p_1) = 0 \text{ for some model } v \text{ of } T && \text{(Definition 1.3.7)} \\
&\iff v(\neg(p_0 \vee p_i) \vee p_1) = 0 \\
&\iff v(\neg(p_0 \vee p_i) \vee p_1) = 0 \\
&\iff v(\neg(p_0 \vee p_i)) = v(p_1) = 0 \\
&\iff v(p_0 \vee p_i) = 1 \text{ and } v(p_1) = 0 \\
&\iff \big(v(p_0) = 1 \text{ or } v(p_i) = 1\big) \text{ and } v(p_1) = 0
\end{aligned}$$

Let us define a model v such that $v(p_0) = 1$ and $v(p_i) = 0$ for $i > 0$. This is trivially a model of T, since T is empty. Since it satisfies the condition above, we thus have that $T \nvdash p_0 \vee p_i \to p_1$.

A simple induction argument on the complexity of ψ shows that $T \nvdash g(\psi) \to p_1$ for any ψ.

2 Boolean algebras

There were no exercises in this section.

3 Equivalent categories

There were no exercises in this section.

4 Propositional theories are Boolean algebras

There were no exercises in this section; but there are some unproven facts stated.

3.4.3 Show that $f(\phi) = f(\psi)$ if and only if $f(\phi \leftrightarrow \psi) = 1$ for interpretations f.

Proof: Note that $f(\phi) = f(\psi)$ is more appropriately written as $f(\phi) \simeq f(\psi)$.

$$\begin{aligned}
& f(\phi \leftrightarrow \psi) = 1 \\
&\iff f((\phi \to \psi) \wedge (\psi \to \phi)) = 1 \\
&\iff f((\neg\phi \vee \psi) \wedge (\neg\psi \vee \phi)) = 1 \\
&\iff (f(\neg\phi) \cup f(\psi)) \cap (f(\neg\psi) \cup f(\phi)) = 1 \\
&\iff (-f(\phi) \cup f(\psi)) \cap (-f(\psi) \cup f(\phi)) = 1 \\
&\iff -f(\phi) \cup f(\psi) \geq 1 \text{ and } -f(\psi) \cup f(\phi) \geq 1 && \text{(Proposition 3.2.6)} \\
&\iff -f(\phi) \cup f(\psi) = 1 \text{ and } -f(\psi) \cup f(\phi) = 1 \\
&\iff f(\phi) \cap f(\psi) = f(\phi) \text{ and } f(\psi) \cap f(\phi) = f(\psi) \\
&\iff f(\phi) \leq f(\psi) \text{ and } f(\psi) \leq f(\phi) = f(\psi) \\
&\iff f(\phi) = f(\psi)
\end{aligned}$$

We note that $\iff$ is justified by Exercise 3.5.10(1.a) below (this is not circular).

3.4.11 If f is an interpretation show that $\overline{f}(E_\phi) = f(\phi)$ is a Boolean homomorphism and that it is the unique homomorphism $\overline{f} : L(T) \to B$ such that $\overline{f} i_T = f$.

Proof:

$$\bar{f}(1) = \bar{f}(E_{\phi\vee\neg\phi}) = f(\phi \vee \neg\phi) = 1 \text{ since } T \vdash \phi \vee \neg\phi$$

$$\bar{f}(0) = \bar{f}(E_{\phi\wedge\neg\phi}) = f(\phi \wedge \neg\phi) = f(\phi) \cap f(\neg\phi) = f(\phi) \cap -f(\phi) = 0$$

$$\bar{f}(-E_\phi) = \bar{f}(E_{\neg\phi}) = f(\neg\phi) = -f(\phi) = -\bar{f}(E_\phi)$$

$$\bar{f}(E_\phi \cup E_\psi) = \bar{f}(E_{\phi\vee\psi}) = f(\phi \vee \psi) = f(\phi) \cup f(\psi) = \bar{f}(E_\phi) \cup \bar{f}(E_\psi)$$

$$\bar{f}(E_\phi \cap E_\psi) = \bar{f}(E_{\phi\wedge\psi}) = f(\phi \wedge \psi) = f(\phi) \cap f(\psi) = \bar{f}(E_\phi) \cap \bar{f}(E_\psi)$$

Suppose there is a homomorphism $h : L(T) \to B$ such that $h \circ i_T = f$, we want to show that, in fact, $h = \overline{f}$. From Proposition 3.4.10 we have i_T is surjective. We thus have that i_T is an epimorphism (proving by contraposition):

$$g \neq h \implies g(E_\phi) \neq h(E_\phi) \text{ for some } E_\phi \implies g(i_T(\phi)) \neq h(i_T(\phi)) \implies g \circ i_T \neq h \circ i_T$$

From which it follows that $h \circ i_T = \overline{f} \circ i_T$ implies $h = \overline{f}$, as desired!

5 Boolean algebras again

3.5.3 Let F be a filter. Show that F is proper if and only if $0 \notin F$.

Proof: We prove both directions by contraposition: Clearly if F is improper then $F = B$ and $0 \in B$. On the other hand, if $0 \in F$ then as $0 \leq b$ for all $b \in B$ it follows that $b \in F$ (by the definition of a filter).

3.5.10 (a) Let B be a Boolean algebra, and let $a, b, c \in B$. Show that the following hold:

i. $(a \to b) = 1$ iff $a \leq b$
Proof: Suppose that $(a \to b) = 1$ then

$$a = a \wedge 1 = a \wedge (\neg a \vee b) = (a \wedge \neg a) \vee (a \wedge b) = 0 \vee (a \wedge b) = a \wedge b$$

Conversely, suppose that $a \leq b$ then

$$\begin{aligned} a \to b = \neg a \vee b &= \neg(a \wedge b) \vee b \\ &= (\neg a \vee \neg b) \vee b \\ &= \neg a \vee (\neg b \vee b) \\ &= \neg a \vee 1 = 1 \end{aligned}$$

ii. $(a \wedge b) \leq c$ iff $a \leq (b \to c)$
Proof:

$$a \wedge b \leq c \quad \text{iff} \quad a \wedge b \to c = 1 \quad \text{by the above exercise}$$
$$\neg(a \wedge b) \vee c = 1 \quad \text{iff}$$
$$(\neg a \vee \neg b) \vee c = 1 \quad \text{iff}$$
$$\neg a \vee (\neg b \vee c) = 1 \quad \text{iff}$$
$$a \to (b \to c) = 1 \quad \text{iff } a \leq b \to c \quad \text{by the above exercise}$$

iii. $a \wedge (a \to b) \leq b$
Proof:

$$\begin{aligned} 1 &= \neg a \vee 1 = \neg a \vee (\neg b \vee b) \\ &= (\neg a \vee \neg b) \vee b = \neg(a \wedge b) \vee b \\ &= \neg(0 \vee (a \wedge b)) \vee b = \neg((a \wedge \neg a) \vee (a \wedge b)) \vee b \\ &= \neg(a \wedge (\neg a \vee b)) \vee b = \neg(a \wedge (a \to b)) \vee b \\ &= (a \wedge (a \to b)) \to b \end{aligned}$$

and hence by the first part of this exercise it follows that $a \wedge (a \to b) \leq b$

iv. $(a \leftrightarrow b) = (b \leftrightarrow a)$
Proof:

$$a \leftrightarrow b = (a \to b) \wedge (b \to a) = (b \to a) \wedge (a \to b) = b \leftrightarrow a$$

v. $(a \leftrightarrow a) = 1$
Proof:

$$(a \leftrightarrow a) = (a \to a) \wedge (a \to a) = (a \to a) = \neg a \vee a = 1$$

vi. $(a \leftrightarrow 1) = a$
Proof:

$$(a \leftrightarrow 1) = (a \to 1) \wedge (1 \to a) = (\neg a \vee 1) \wedge (\neg 1 \vee a) = 1 \wedge (0 \vee a) = 1 \wedge a = a$$

(b) Let $\mathcal{P}(N)$ be the powerset of the natural numbers, and let U be an ultrafilter on $\mathcal{P}(N)$. Show that if U contains a finite set F, then U contains a singleton set.

(Contrapositive) Proof: Suppose that $\{n\} \notin U$ for every $n \in N$, then $\neg\{n\} = \{n\}^c \in U$ for all n (by Proposition 3.5.6(2)) but each of these is an infinite set. Likewise, no finite subset of N could be an element of U, since if say $a = \{a_1, \ldots, a_n\} \in U$ (with $a_i \in N$) then as $a = \{a_1\} \vee (\{a_2\} \vee \ldots \vee \{a_n\})$ it follows that either $\{a_1\}$ is an element of U or $\{a_2\} \vee \ldots \vee \{a_n\}$ is, by Proposition 3.5.6(3); the former is impossible because of our opening assumption and the latter is impossible since we can repeat this argument until we consider only $\{a_{n-1}\} \vee \{a_n\}$. It follows that no finite set F can be an element of U.

3.5.15 Prove the remaining steps to Lemma 3.5.15[2]

Proof:

Suppose aRb, then $a \leftrightarrow b \in F$, but then also $b \leftrightarrow a \in F$ (by Exercise 3.5.10(d)) so bRa.

Suppose aRb and bRc then $a \leftrightarrow b, b \leftrightarrow c \in F$ then we need to simply show that $a \leftrightarrow c \in F$ from which it follows that aRc.

To this end, it will suffice to show that $(a \leftrightarrow b) \wedge (b \leftrightarrow c) \leq a \to c$ since if this is the case then $(a \leftrightarrow b) \wedge (b \leftrightarrow c) \leq c \to a$ as well. But since F is upward closed we will have $(a \to c), (c \to a) \in F$ and hence $(a \leftrightarrow c) = (a \to c) \wedge (c \to a) \in F$.

By Exercise 3.5.10(1.a) $x \leq y$ iff $(x \to y) = 1$. Now observe that

---

[2]**Lemma 3.5.15:**
Suppose that F is a filter on B. Let $R = \{\langle a, b\rangle \in B \times B | a \leftrightarrow b \in F\}$. Then R is an equivalence relation relation, and $F = \{a \in B | aR1\}$. *Proof*: Showing that R is an equivalence relation requires several straightforward verifications. For example, $a \leftrightarrow a = 1$, and $1 \in F$; therefore, aRa. **We leave the remaining verifications to the reader.**
Now suppose that $a \in F$. Since $a = (a \leftrightarrow 1)$, it follows that $a \leftrightarrow 1 \in F$, which means that $aR1$.

$$
\begin{aligned}
&((a \leftrightarrow b) \wedge (b \leftrightarrow c)) \to (a \to c) \\
&= (\neg(a \leftrightarrow b) \vee \neg(b \leftrightarrow c)) \vee (a \to c) && \text{(Definition of } \to \text{ and De Morgan)} \\
&= \neg(a \leftrightarrow b) \vee (\neg(b \leftrightarrow c) \vee (a \to c)) && \text{(Associativity of } \vee) \\
&= \neg(a \leftrightarrow b) \vee (\neg((\neg b \vee c) \wedge (\neg c \vee b)) \vee (a \to c)) && \text{(Definition of } \leftrightarrow) \\
&= \neg(a \leftrightarrow b) \vee ((\neg(\neg b \vee c) \vee \neg(\neg c \vee b)) \vee (a \to c)) && \text{(De Morgan)} \\
&\geq ((\neg(\neg b \vee c) \vee \neg(\neg c \vee b)) \vee (a \to c)) && (x \leq x \vee y \text{ by Absorpotion Law 1}) \\
&= (\neg(\neg b \vee c) \vee (\neg(\neg c \vee b) \vee (a \to c))) && \text{(Associativity of } \vee) \\
&= (\neg(\neg b \vee c) \vee ((c \wedge \neg b) \vee (a \to c))) && \text{(De Morgan and Proposition 3.2.14(Double Negation))} \\
&= (\neg(\neg b \vee c) \vee ((c \wedge \neg b) \vee (\neg a \vee c))) && \text{(Definition of } \to) \\
&= (\neg(\neg b \vee c) \vee ((c \wedge \neg b) \vee (c \vee \neg a))) && \text{(Commutativity of } \vee) \\
&= (\neg(\neg b \vee c) \vee (((c \wedge \neg b) \vee c) \vee \neg a)) && \text{(Associativity of } \vee) \\
&= [(b \wedge \neg c) \vee (((c \wedge \neg b) \vee c) \vee \neg a)] && \text{(De Morgan and Proposition 3.2.14(Double Negation))} \\
&= [(b \wedge \neg c) \vee (\neg a \vee ((c \wedge \neg b) \vee c))] && \text{(Commutativity } \vee) \\
&= [(b \wedge \neg c) \vee (\neg a \vee (c \vee (c \wedge \neg b)))] && \text{(Commutativity } \vee) \\
&= [(b \wedge \neg c) \vee (\neg a \vee ((c \vee c) \wedge (c \vee \neg b)))] && \text{(Distribution)} \\
&= [(b \wedge \neg c) \vee (\neg a \vee (1 \wedge (c \vee \neg b)))] && \text{(Idempotence)} \\
&= [(b \wedge \neg c) \vee (\neg a \vee (c \vee \neg b))] && \text{(Top)} \\
&= [(\neg a \vee (c \vee \neg b)) \vee (b \wedge \neg c)] && \text{(Commutativity } \vee) \\
&= [\neg a \vee (c \vee (\neg b \vee (b \wedge \neg c)))] && \text{(Associativity)} \\
&= [\neg a \vee (c \vee ((\neg b \vee b) \wedge (\neg b \vee \neg c)))] && \text{(Distribution)} \\
&= [\neg a \vee (c \vee (1 \wedge (\neg b \vee \neg c)))] && \text{(Excluded Middle)} \\
&= [\neg a \vee (c \vee (\neg b \vee \neg c))] && \text{(Top)} \\
&= [\neg a \vee ((c \vee \neg c) \vee \neg b)] && \text{(Commutativity and Associativity)} \\
&= [\neg a \vee (1 \vee \neg b)] && \text{(Excluded Middle)} \\
&= [\neg a \vee 1] && \text{(Absorption Law 3)} \\
&= 1 && \text{(Absorption Law 3)}
\end{aligned}
$$

It follows from Exercise 3.5.10(1.a) then that $((a \leftrightarrow b) \wedge (b \leftrightarrow c)) \leq (a \to c)$, as desired.

3.5.18 (This exercise presupposes knowledge of measure theory.) Let Σ be the Boolean algebra of Borel subsets of $[0,1]$, and let μ be the Lebesgue measure $[0,1]$. Let $\mathcal{F} = \{S \in \Sigma | \mu(S) = 1\}$. Show that $\mathcal{F}$ is a filter, and describe the equivalence relation on σ corresponding to $\mathcal{F}$.

(a) Show that F is a filter.
Clearly $1_\Sigma = [0,1]$ and $\mu(1_\Sigma) = 1$ so that $1_\Sigma \in F$. Suppose that $A, B \in F$. First we notice that

$$\mu([0,1]) = \mu(D \cup D^c) = \mu(D) + \mu(D^c)$$

hence it follows that $\mu(D^c) = 0$ if and only if $\mu(D) = 1$. Therefore,

$$0 \leq \mu[(A \cap B)^c] = \mu(A^c \cup B^c) \leq \mu(A^c) + \mu(B^c) = 0$$

so that $\mu[(A \cap B)^c] = 0$ and hence $A \wedge B \in F$. Finally, let $A \in F$ and $B \in \Sigma$ such that $A \leq B$ (that is, $A \subset B$). $B \subset [0,1]$ so that $\mu(B) \leq 1$ since μ is monotonic; on the other hand, $1 = \mu(A) \leq \mu(B)$. Hence, $\mu(B) = 1$ and $B \in F$. Therefore, F is a filter.

(b) Describe the equivalence relation on Σ corresponding to F.

Let $A \sim B$ if and only if $\mu(A \triangle B) = 0$. Then we will show $F = \{A | A \sim [0,1]\}$

Let's check that this is indeed an equivalence relation. We define $A \triangle B := (A \backslash B) \cup (B \backslash A)$ and note that in general, then, $A \triangle B \subset A \cup B$ and that $\mu(A \triangle A) = \mu(\varnothing) = 0$, hence $A \sim A$. From the definition of $\triangle$ it follows that $A \triangle B = B \triangle A$ so that if $A \sim B$ then $0 = \mu(A \triangle B) = \mu(B \triangle A)$ and hence $B \sim A$. Finally, suppose that $A \sim B$ and $B \sim C$. Since $A \sim B$, $\mu(A \backslash B) = \mu(B \backslash A) = 0$ as $\mu(A \triangle B) = 0$ and $A \triangle B$ is the union of disjoint sets. Likewise we see that $\mu(B \backslash C) = \mu(C \backslash B) = 0$. *Claim*: $A \triangle C = (A \triangle B) \triangle (B \triangle C)$. It follows from the claim that $A \triangle C \subset (A \triangle B) \cup (B \triangle C)$ and hence that $\mu(A \triangle C) \leq \mu(A \triangle B) + \mu(B \triangle C) = 0$. Therefore, $A \sim C$ provided we prove the Claim.

Proof of Claim:

We want first show that for any X, Y we have $X \triangle Y = (X \cup Y) \backslash (X \cap Y)$.

Let $x \in X \triangle Y$ then either $x \in X \backslash Y$ or $x \in Y \backslash X$ (these sets are clearly disjoint), suppose $x \in X \backslash Y$. As $X \backslash Y \subseteq (X \cup Y) \backslash Y$ it follows that $x \in (X \cup Y) \backslash Y$. We have that $x \notin Y$ so in particular we have that $x \notin X \cap Y$ from which it follows along with $x \in (X \cup Y) \backslash Y$ that $x \in (X \cup Y) \backslash (X \cap Y)$. The case where $x \in Y \smallsetminus X$ follows dually. This proves that $X \triangle Y \subseteq (X \cup Y) \backslash (X \cap Y)$. Conversely, suppose that $x \in (X \cup Y) \backslash (X \cap Y)$ then $x \in X \cup Y$ but $x \notin X \cap Y$. From the former fact, we have that $x \in X$ or $x \in Y$; let us suppose $x \in X$. Now $x \notin Y$ since if it were then $x \in X \cap Y$ from which it follows that $x \in (X \backslash Y)$ and the result follows. The case where $x \in Y$ follows dually.

$$X \triangle Y = (X \smallsetminus Y) \cup (Y \smallsetminus X)$$

Second, we show that $\triangle$ is associative for any sets X, Y and Z:

$$\begin{aligned}
X \triangle (Y \triangle Z) &= (X \cap (Y \triangle Z)^c) \cup ((Y \triangle Z) \cap X^c) \\
&= (X \cap ((Y \cap Z^c) \cup (Z \cap Y^c))^c) \cup (((Y \cap Z^c) \cup (Z \cap Y^c)) \cap X^c) \\
&= (X \cap ((Y \cap Z^c)^c \cap (Z \cap Y^c)^c)) \cup (((Y \cap Z^c) \cap X^c) \cup ((Z \cap Y^c) \cap X^c)) \\
&\overset{*}{=} (X \cap ((Y^c \cup Z) \cap (Z^c \cup Y))) \cup (((Y \cap Z^c) \cap X^c) \cup ((Z \cap Y^c) \cap X^c))
\end{aligned}$$

Now we use the fact that

$$\begin{aligned}
(Y^c \cup Z) \cap (Z^c \cup Y) &= ((Y^c \cup Z) \cap Z^c) \cup ((Y^c \cup Z) \cap Y) \\
&= (Y^c \cap Z^c) \cup (Z \cap Y)
\end{aligned}$$

and continuing our computation

$$\begin{aligned}
&\overset{*}{=} (X \cap ((Y^c \cap Z^c) \cup (Z \cap Y))) \cup (((Y \cap Z^c) \cap X^c) \cup ((Z \cap Y^c) \cap X^c)) \\
&= ((X \cap (Y^c \cap Z^c)) \cup (X \cap (Z \cap Y))) \cup (((Y \cap Z^c) \cap X^c) \cup ((Z \cap Y^c) \cap X^c))
\end{aligned}$$

By shuffling symbols we see that

$$Z \triangle (X \triangle Y) = ((Z \cap (X^c \cap Y^c)) \cup (Z \cap (Y \cap X))) \cup (((X \cap Y^c) \cap Z^c) \cup ((Y \cap X^c) \cap Z^c))$$

And by carefully comparing the terms we can see, in fact, $X \triangle (Y \triangle Z) = Z \triangle (X \triangle Y)$. But $\triangle$ is clearly commutative from which it follows that

$$X \triangle (Y \triangle Z) = Z \triangle (X \triangle Y) = (X \triangle Y) \triangle Z$$

Finally, let $D = A \triangle B$ then

$$(A\triangle B)\triangle(B\triangle C) = D\triangle(B\triangle C) = (D\triangle B)\triangle C = ((A\triangle B)\triangle B)\triangle C = (A\triangle(B\triangle B))\triangle C = (A\triangle\varnothing)\triangle C = A\triangle C$$

This concludes the proof of *Claim*. Now that we have shown that ~ is an equivalence relation, we will prove that this equivalence relation describes F, namely:

$$F = \{S \in \Sigma | \mu(S) = 1\} = \{A | A \sim [0,1]\}$$

Suppose that $A \in F$ then $\mu(A) = 1$ then $\mu(A \triangle [0,1]) = \mu((A\cup[0,1])\backslash(A\cap[0,1])) = \mu([0,1]\backslash A) = \mu(A^c) = 0$ and so it follows that $A \sim [0,1]$ so that A is an element of the latter set. Now suppose that $A \sim [0,1]$, then $0 = \mu(A \triangle [0,1]) = \mu(A\backslash[0,1]) + \mu([0,1]\backslash A) = \mu(\varnothing) + \mu(A^c)$. Therefore $\mu(A) = 1$ and hence $A \in F$.

It follows that ~ is the appropriate equivalence relation describing F.

3.5.20 Let $\mathcal{P}N$ be the powerset of the natural numbers. We say that a subset E of N is *cofinite* just in case $N\backslash E$ is finite. Let $\mathcal{F} \subseteq \mathcal{P}N$ be the set of cofinite subsets of N. Show that $\mathcal{F}$ is a filter, and show that there are infinitely many ultrafilters containing $\mathcal{F}$.

(a) Show that $\mathcal{F}$ is a filter.

Proof: $1_{\mathcal{P}(N)} = N$ and $|N^c| = 0$ so that $1_{\mathcal{P}(N)} \in \mathcal{F}$. If $A, B \in \mathcal{F}$ then $|(A\cap B)^c| = |A^c \cup B^c| \leq |A^c| + |B^c| < \infty$ and so $A \wedge B \in \mathcal{F}$. Now suppose that $A \in \mathcal{F}$ and $A \leq B$ then $B^c \subset A^c$ and hence $|B^c| \leq |A^c| < \infty$ therefore $B \in \mathcal{F}$. This proves that $\mathcal{F}$ is a filter.

(b) Show that there are infinitely many ultrafilters containing $\mathcal{F}$.

Proof:

We begin with a definition and two fact: A family $R \subseteq A$ is said to be *centered*, if for every finite set $\{a_1, \ldots, a_n\} \subseteq R, a_1 \cdots a_n > 0$. Moreover, every centered family can be extended to a filter. *Proof:* Exercise 3.9 of *Logic of Mathematics* by Adamowicz and Zbierski. Furthermore, every filter can be extended to an ultrafilter. *Proof:* Theorem 3.5 of Adamowicz and Zbierski – the proof is an obvious application of Zorn's lemma.

Now for the main proof. We construct an infinite tree. The first node is $\mathbb{N}$. It branches into evens, E, and odds, O. Now form the union of odds with $\mathcal{F}$. Clearly $\mathcal{F}$ is centered because if it were not then there exists a set $\{a_1, \ldots, a_n\}$ such that $a_1 \cdots a_n = 0$ but as $\mathcal{F}$ is a filter this would imply that $\mathcal{F} = \mathcal{P}(\mathbb{N})$ however $E \notin \mathcal{F}$. We really want, however, to show that $\mathcal{F} \cup E^c$ is centered. It suffices to show that if $f \in \mathcal{F}$ then $f \cap E^c \neq \varnothing$.[3] However, if $f \in \mathcal{F}$ then $f^c = \{n_1, n_2, \ldots, n_m\}$ for some $n_1, \ldots, n_m \in \mathbb{N}$. Necessarily, this means that there are an *infinite* number of odd integers $2k+1 \in f$, and hence $f \cap E^c \neq \varnothing$. Therefore, $\mathcal{F} \cup E^c$ is centered and can be extended to a filter (Exercise 3.9 LM) and then to an ultrafilter (Theorem 3.5 LM), say $\mathcal{F}_0$.

Now look at the evens. They further branch into even evens $\{4, 8, 12, \ldots\}$ and odd evens $\{2, 6, 10, \ldots\}$. Now form the union of odd evens with $\mathcal{F}$ and again extend that (by similar techniques as above) to an ultrafilter $\mathcal{F}_1$.

Now look at the even evens. They further branch into odd even evens $\{4, 12, 20, \ldots\}$ and even even evens $\{8, 16, 24, \ldots\}$. Now form the union of odd even evens with $\mathcal{F}$ and extend that to an ultrafilter $\mathcal{F}_2$.

Now look at the even even evens. They..... Continuing in this manner we have an infinite number of (distinct) ultrafilters $(\mathcal{F}_n)_n$. To see that these are distinct, note that $E^c \in \mathcal{F}_0$ but that $E^c \notin \mathcal{F}_1$. Since if $E^c \in \mathcal{F}_1$ then as the even evens are an element of $\mathcal{F}_1$ it follows the evens are as well, but then $\mathcal{F}_1$ would contain both the evens and the odds, which is impossible. More generally, suppose that $\mathcal{F}_n = \mathcal{F}_m$ for some $m > n$ then $\mathcal{F}_m$ would contain the odd even^m (where even is iterated

[3] Since $\mathcal{F}$ is closed under finite multiplication if there exists a finite set $\{f_1, \ldots, f_n, E^c\}$ such that their product is zero,$\varnothing$, this is equivalent to $f \cdot E^c = \varnothing$ where $f = f_1 \cdots f_n \in \mathcal{F}$.

m many times here) which is a subset of the even even^n (where even is iterated n many times here) and hence it would contain this element as well. But because the two sets are equal $\mathcal{F}_m$ also contains the odd evens^n since $\mathcal{F}_n$ does (by construction) but if $\mathcal{F}_m$ contains both the even even^n and the odd even^n then it contains the emptyset which is impossible. Therefore, $\mathcal{F}_n \neq \mathcal{F}_m$ whenever $n \neq m$.

6 Stone spaces

3.6.6 Let $p \in X$ and let $S \subseteq X$. Then $p \in \overline{S}$ if and only if every open neighborhood U of p has nonempty intersection with S.

Proof: We prove by contraposition. $p \notin \overline{S}$ if and only if there is a closed set C containing S with $p \notin C$ if and only if there is an open set U containing p such that $U \cap S = \varnothing$.

3.6.12 Several topological facts.

(a) Show that X is regular iff for each $x \in X$ and open neighborhood U of x, there is an open neighborhood V of x such that $\overline{V} \subset U$.

Proof: Suppose that X is regular. Fix $x \in X$ and consider an open neighborhood U of x, then $C = U^c$ is a closed set not containing x. Since X is regular there exists an open set $W \supset C$ and an open neighborhood V of x such that $V \cap W = \varnothing$. Notice that $W^c \subset C^c = U$ and $V \subset W^c$. Now $\overline{V}$ is the smallest closed set containing V but W^c is also closed therefore $\overline{V} \subset W^c \subset U$.

Conversely, suppose that X has the properties of the RHS of the theorem. Fix $x \in X$ and consider an arbitrary closed set $C \subset X$ such that $x \notin C$. Then $U = C^c$ is open and $x \in U$. Then by our assumption there exists an open neighborhood V of x such that $\overline{V} \subset U = C^c$. Define $W = (\overline{V})^c$ then W is an open set such that $W = (\overline{V})^c \supset C$ and $V \cap W = V \cap (\overline{V}^c) \subset \overline{V} \cap (\overline{V})^c = \varnothing$.

(b) Show that if $E \subseteq F$ then $\overline{E} \subseteq \overline{F}$.

Proof: The closure of a set is closed since it is the intersection of (all) closed sets (containing that given set), and from Proposition 3.6.10 we get that $\overline{F} \supset F \supset E$. $\overline{E}$ is just the *smallest* closed set containing E, therefore $\overline{E} \subset \overline{F}$.

(c) Show that $\overline{\overline{E}} = \overline{E}$.

Proof: $\overline{\overline{E}}$ is the intersection of all closed sets containing $\overline{E}$ but $\overline{E}$ is itself closed and containing $\overline{E}$ so it is necessarily the smallest closed set containing $\overline{E}$. Hence the result follows.

(d) Show that the intersection of two topologies is a topology.

Proof: Let X be the underlying space, then $\varnothing$ and X are in τ_1 and τ_2 therefore they are in the intersection. Suppose that $U, V \in \tau_1 \cap \tau_2$ then in particular $U, V \in \tau_j$ so that $U \cap V \in \tau_j$ for both $j = 1, 2$ but then $U \cap V \in \tau_1 \cap \tau_2$. Finally, assume that $\{V_j\}_{j \in J}$ is any collection of open sets in $\tau_1 \cap \tau_2$ then it is a collection in each individual topology and so the union is contained in each individual topology and therefore in the intersection of the topologies.

(e) Show that the infinite distributive law holds:

$$U \cap (\bigcup_{i \in I} V_i) = \bigcup_{i \in I} (U \cap V_i)$$

Proof: Let $x \in U \cap (\bigcup_{i \in I} V_i)$ then $x \in U$ and $\bigcup_{i \in I} V_i$. Since $x \in \bigcup_{i \in I} V_i$, it follows that $x \in V_j$ for some $j \in I$. Hence $x \in U \cap V_j \subset \bigcup_{i \in I} (U \cap V_i)$. Conversely, suppose that $x \in \bigcup_{i \in I} (U \cap V_i)$ then for some $j, x \in U \cap V_j$; hence $x \in U$ and $x \in V_j \subset \bigcup_{i \in I} V_i$.

(f) Show that a space X is Hausdorff if and only if the diagonal $\Delta = \{\langle x, x \rangle : x \in X\}$ is closed in the product topology on $X \times X$.

Proof: Suppose that X is Hausdorff and consider $(x, y) \in \Delta^c$ then $x \neq y$ so there are open disjoint sets U_x and V_y containing x and y, respectively. $U_x \times V_y$ is an open basic set in $X \times X$ and suppose that $U_x \times V_y \cap \Delta \neq \varnothing$ then there exists $(z,z) \in U_x \times V_y$ but then $U_x \cap V_y \neq \varnothing$ contradicting our construction. Therefore $U_x \times V_y \subset \Delta^c$ and hence Δ is closed.

Conversely, suppose that Δ^c is open and consider $(x,y) \in \Delta^c$ then there exists a basic open set $U_x \times V_y \subset \Delta^c$ containing (x,y). Clealry, $x \in U_x$ and $y \in V_y$ and these sets are disjoint since otherwise $U_x \times V_y \notin \Delta^c$. Therefore, X is Hausdorff.

3.6.25 More topological facts.

(a) Show that if f and g are continuous, then $g \circ f$ is continuous.

Proof: Let U be an open set then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open since f is continuous and $g^{-1}(U)$ is open as g is continuous.

(b) Suppose that $f : X \to Y$ is a surjection. Show that if E is dense in X, then $f(E)$ is dense in Y .

Proof: First we prove that if f is continuous then $f(\overline{E}) \subset \overline{f(E)}$. Let $x \in \overline{E}$ then there exists a net $x_\alpha \to x$ but as f is continuous we have $f(x_\alpha) \to f(x) \in \overline{f(E)}$.

Now we prove the main problem. Because f is surjective $f(X) = Y$ and so

$$Y = f(X) = f(\overline{E}) \subset \overline{f(E)}$$

(c) Show that $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed whenever C is closed.

Proof:

First, a needed result. Observe that $x \in f^{-1}(E^c)$ iff $f(x) \in E^c$ iff $f(x) \notin E$ iff $x \notin f^{-1}(E)$ iff $x \in [f^{-1}(E)]^c$. Therefore, $f^{-1}(E^c) = [f^{-1}(E)]^c$.

Now let C be an arbitrary closed set, then

$$f^{-1}(C) = f^{-1}(C^{cc}) = f^{-1}(C^c)^c$$

Notice C^c is our arbitrary open set and so if $f^{-1}(C)$ is closed then $f^{-1}(C^c)$ is open and therefore f is continuous; on the other hand, if f is continuous then $f^{-1}(C^c)^c = f^{-1}(C)$ is closed (since under continuous f the pre-image of an open set is open).

(d) Let Y be a Hausdorff space, and let $f, g : X \to Y$ be continuous. Show that if f and g agree on a dense subset of X, then $f = g$.

Proof: Let $E = \{x | g(x) = f(x)\}$ then $\overline{E} = X$. Since E is dense then for all $x \in X$ there exists a net $(x_\alpha)_\alpha$ from E such that $x_\alpha \to x$. Now fix $x \in X$, then there is a net converging to it and since g is continuous $f(x_\alpha) = g(x_\alpha) \to g(x)$, likewise since f is continuous $g(x_\alpha) = f(x_\alpha) \to f(x)$. So apparently there are two limits – namely $g(x)$ and $f(x)$ – however in a Hausdorff space every net converges to a unique limit, hence $f(x) = g(x)$.

3.6.26 Show that $f^{-1}(V) \subseteq U$ if and only if $V \subseteq Y \backslash f(X \backslash U)$.

Proof: First we remark that $Y \backslash f(X \backslash U) = f(X \backslash U)^c$.

If $f^{-1}(V) \subseteq U$ then $V \subseteq f(U)$. Suppose that $V \cap f(X \backslash U) \neq \varnothing$ then let y be in this intersection. As $y \in V \subseteq f(U)$ then there exists $x \in U$ such that $f(x) = y$ and there exists $z \in U^c$ such that $f(z) = y$. Then $f^{-1}(\{y\}) \cap U^c \subseteq f^{-1}(V) \cap U^c \subseteq U \cap U^c = \varnothing$. A contradiction! Therefore, $V \subseteq f(X \backslash U)^c$.

Now suppose that $V \subseteq f(X \backslash U)^c$. If $f^{-1}(V) \nsubseteq U$ then there exists $z \in f^{-1}(V) \cap U^c$, hence $f(z) \in f(X \backslash U)^c$ and $f(z) \in f(X \backslash U)$, a contradiction!

3.6.28 Let $f : X \to Y$ be continuous. Then show the following are equivalent:

(a) f is closed.

(b) For every open set $U \subseteq X$, the set $\{y \in Y | f^{-1}\{y\} \subseteq U\}$ is open.

(c) For every $y \in Y$, and every neighborhood U of $f^{-1}\{y\}$, there is a neighborhood V of y such that $f^{-1}(V) \subseteq U$.

Proof: The equivalence of 1 and 3 are shown in the text; we show here the equivalence of 2 and 3.

(2) $\implies$ (3): Let $y \in Y$ and let U be a neighbourhood of $f^{-1}\{y\}$ and let $V = \{x \in Y | f^{-1}\{x\} \subseteq U\}$ then V is open by (2). Since U is a neighbourhood of $f^{-1}\{y\}$, then $f^{-1}\{y\} \subseteq U$, so $y \in V$ and (since V is open) V is thus a neighbourhood of y. Then we have

$$\begin{aligned} f^{-1}(V) &= \{z | f(z) \in V\} \\ &= \{z | f(z) \in \{x \in Y | f^{-1}\{x\} \subseteq U\}\} \\ &= \{z | f(z) \in Y \wedge f^{-1}\{f(z)\} \subseteq U\} \\ &= \{z | f^{-1}\{f(z)\} \subseteq U\} \\ &\subseteq U \end{aligned}$$

where the final $\subseteq$ is justified by

$$\begin{aligned} & w \in \{z | f^{-1}\{f(z)\} \subseteq U\} \\ \iff & f^{-1}\{f(w)\} \subseteq U \\ \iff & \{z | f(z) = f(w)\} \subseteq U \\ \iff & \{w\} \subseteq \{z | f(z) = f(w)\} \wedge \{z | f(z) = f(w)\} \subseteq U \\ \implies & \{w\} \subseteq U \\ \iff & w \in U \end{aligned}$$

So we have that V is a neighbourhood of y such that $f^{-1}(V) \subseteq U$.

(3) $\implies$ (2): Let U be an open subset of X. For every y such that $f^{-1}\{y\} \subseteq U$, we have (by (3)) that there is a neighbourhood V_y of y such that $f^{-1}(V_y) \subseteq U$ and $f^{-1}\{y\} \subseteq U$.

Clearly

$$\{y \in Y | f^{-1}\{y\} \subseteq U\} \subseteq \bigcup \{V_y | y \in Y \wedge f^{-1}\{y\} \subseteq U\}$$

since $y \in V_y$ for every y. Conversely, if $x \in \bigcup \{V_y | y \in Y \wedge f^{-1}\{y\} \subseteq U\}$ then $x \in V_y$ for some $y \in Y$ such that $f^{-1}\{y\} \subseteq U$. It follows that $f^{-1}\{x\} \subseteq f^{-1}(V_y) \subseteq U$ and hence that $x \in \{y \in Y | f^{-1}\{y\} \subseteq U\}$.

Therefore

$$\{y \in Y | f^{-1}\{y\} \subseteq U\} = \bigcup \{V_y : y \in Y \wedge f^{-1}\{y\} \subseteq U\}$$

which shows that $\{y \in Y | f^{-1}\{y\} \subseteq U\}$ is the union of open sets (and hence open itself).

7 Stone Duality

3.7.2 Here we prove the details of the proof of Lemma 3.7.2 and some claims before it.

(a) $\mathcal{B} = \{C_a | a \in B\}$ is a basis for a topology on $\hom(B, 2)$.

Proof: Let $C_a, C_b \in \mathcal{B}$ then

$$\begin{aligned} C_a \cap C_b &= \{\phi \in \hom(B,2) | \phi(a) = 1\} \cap \{\phi \in \hom(B,2) | \phi(b) = 1\} \\ &= \{\phi \in \hom(B,2) | \phi(a) = 1 \text{ and } \phi(b) = 1\} \\ &= \{\phi \in \hom(B,2) | \phi(a) \wedge \phi(b) = 1\} \\ &= \{\phi \in \hom(B,2) | \phi(a \wedge b) = 1\} \\ &= C_{a \wedge b} \end{aligned}$$

Since $\mathcal{B}$ is closed under finite intersection it follows that it is a basis for *some* topology by Definition 3.6.3.

(b) $\mathcal{B}$ is a basis of clopen sets.

Proof: Let $C_a \in \mathcal{B}$ then

$$\begin{aligned} C_a^c &= \{\phi \in \hom(B,2) | \phi(a) = 0\} \\ &= \{\phi \in \hom(B,2) | \neg\phi(a) = 1\} \\ &= \{\phi \in \hom(B,2) | \phi(\neg a) = 1\} \\ &= C_{\neg a} \end{aligned}$$

This proves that C_a^c is open and therefore C_a is clopen.

(c) "This ultrafilter U corresponds to a $\phi : B \to 2$, we have $\phi(a) = 1$ whenever $C_a \in \mathcal{C}$."

Proof: We have that, for any $C_a \in \mathcal{C}$, $C_a \subseteq C_a$, so $a \in F$. Since F is contained in U, we thus have that $a \in U$. By Proposition 3.5.19(2) there is a homomorphism $\phi : B \to 2$ such that $\phi(x) = 1$ whenever $x \in U$. In particular, we have $\phi(a) = 1$ whenever $C_a \in \mathcal{C}$.

(d) "In other words, $\phi \in C_a$, whenever $C_a \in \mathcal{C}$."

Proof: If $C_a \in \mathcal{C}$ then by the previous claim there is a $\phi : B \to 2$ such that $\phi(a) = 1$ and so (by definition) $\phi \in C_a$.

(e) "Therefore, $\bigcap \mathcal{C}$ is nonempty, and $S(B)$ is compact."

Since $\phi \in C_a$ for every $C_a \in \mathcal{C}$ it follows that $\bigcap \mathcal{C}$ is nonempty so by Proposition 3.6.16 it follows that $S(B)$ is compact.

3.7.? In Top show that

(a) A map is monic iff it is injective.

(b) A map is epi iff it is surjective.

(c) A continuous bijection need not be an isomorphism.

3.7.5 Why does it suffice to show that θ_x (used in the proof of Lemma 3.7.5) is bijective and continuous?

PART IV

Syntactic Metalogic

1 Regimenting theories

There were no exercises in this section.

2 Logical grammar

4.2.16 Show that no Σ-formula can occur as a proper subformula of itself.

Proof: Recall that "The parse trees of formulas are finite, by definition." Now suppose BWOC there were some formula F which was a subformula of itself. If we were to produce a parse tree for F it would have to contain F as a distinct node below itself, and likewise this instance of F would have to contain F as a distinct node below itself, and so on *ad infinitum*. But this contradicts the assumption that parse trees are finite.

3 Deduction rules

There were no exercises in this section.

4 Empirical theories

There were no exercises in this section.

5 Translation

4.5.20 Suppose that $F : T \to T'$ is conservative. Show that if T is consistent then T' is consistent.

Proof: Suppose that T' is inconsistent then by Proposition 4.3.9 $T' \vdash \bot$. But $F(\bot) = \bot$ so that $T' \vdash F(\bot)$, and F is conservative so that $T \vdash \bot$ by Definition 4.5.18. We conclude that T is inconsistent by Proposition 4.3.9.

4.5.21 Suppose that T is a consistent and complete theory in signature Σ, let $\Sigma \subseteq \Sigma'$, and let T' be a consistent theory in Σ'. Show that if T' is an extension of T, then T' is a conservative extension of T.

Proof: Let $\iota : \Sigma \to \Sigma'$ be the inclusion mapping which extends to the set of sentences in the obvious way. Since T' is an extension of T, we have that $T \vdash \phi$ implies $T' \vdash \phi$ for all $\phi \in \mathrm{Sent}(\Sigma)$, but this is the same as $T' \vdash \iota(\phi)$, since ι is the inclusion map so $\iota(\phi) = \phi$. It follows that $\iota : T \to T'$ is a translation. Suppose that $T' \vdash \iota(\phi)$ but BWOC that $T \nvdash \phi$ and hence that $T \vdash \neg\phi$ since T is complete. On the one hand, since T' is an extension we have that $T' \vdash \neg\phi$. On the other hand, as $T' \vdash \iota(\phi)$, we have $T' \vdash \phi$. Thus, from $\wedge$ introduction, we have that $T' \vdash \phi \wedge \neg\phi$, and, from $\perp$ introduction, we thus have that $T' \vdash \perp$. Since T' is consistent, this is a contradiction (by Proposition 4.3.9). We thus have that $T \vdash \phi$, and hence ι is conservative so T' is a conservative extension of T.

4.5.22 Let T be the theory from the previous example, and let $\Sigma' = \{p\}$, where p is a unary predicate symbol. Which theories in Σ' are extensions of T? Which of these extensions is conservative? More difficult: classify all extensions of T in the language Σ', up to homotopy equivalence. (In other words, consider two extensions to be the same if they are homotopy equivalent. *Hint:* consider the question, "how many p are there?")

4.5.24 Let $\Sigma = \{p\}$, where p is a unary predicate, and let $\Sigma' = \{r\}$, where r is a binary relation. Let T be the empty theory in Σ, and let T' be the theory in Σ' that says that r is symmetric, i.e. $r(x,y) \to r(y,x)$. Let $F : \Sigma \to \Sigma'$ be the reconstrual that takes p to $\exists z r(x,z)$. Is F essentially surjective? (This exercise will be a lot easier to answer after Chapter 6.)

4.5.25 Let Σ be the signature with a single unary predicate symbol p, and let Σ' be the empty signature. Let T' be the theory in Σ' that says "there are exactly two things", and let T be the extension of T' in Σ that also says "there is a unique p". Is there an essentially surjective translation $F : T \to T'$? (This exercise will be a lot easier to answer after Chapter 6.)

6 Definitional extension and equivalence

There were no exercises in this section.

PART V

Syntactic Metalogic Redux

1 Many-sorted logic

5.1.8 The theory of categories can conveniently be formulated as a many-sorted theory. Let $\Sigma = \{O, A, d_0, d_1, i, \circ\}$, where O and A are sorts, $d_0 : A \to O$, $d_1 : A \to O, i : O \to A$, and $\circ$ is a relation of sort $A \times A \times A$. (The relation $\circ$ is used as the composition function on arrows – i.e., a partial function defined for compatible arrows.) We will leave it as an exercise for the reader to write down the axioms corresponding to the following ideas:

(a) For each arrow $f, d_0 f$ is the domain object, and $d_1 f$ is the codomain object. Thus, we may write $f : d_0 f \to d_1 f$. More generally, we write $f : x \to y$ to indicate that $x = d_0 f$ and $y = d_1 f$. The function $\circ$ is defined on pairs of arrows where the first arrow's domain matches the second arrow's codomain.

(b) The function $\circ$ is associative.

(c) For each object $x, i(x) : x \to x$. Moreover, for any arrow f such that $d_1 f = x$, we have $i(x) \circ f = f$. And for any arrow g such that $d_0 g = x$, we have $g \circ i(x) = g$.

2 Morita extension and equivalence

5.2.1. "One can easily verify that this is not true of Morita extensions." page 181.

5.2.2. Prove that T_1^1 and T_2^2 (from Example 5.2.5) are logically equivalent

3 Quine on the dispensability of many-sorted logic

5.3.1. Verify equation 5.3 in the proof of Theorem 5.3.1. That is, verify by induction on the complexity of ψ that

$$T_4 \vdash \psi \leftrightarrow \hat{\psi} \qquad \text{and} \qquad \hat{T}_4 \vdash \psi \leftrightarrow \hat{\psi}$$

for every Σ-sentence ψ.

5.3.2. Then use equation (5.3) to show that T_4 and $\hat{T}_4$ are logically equivalent.

4 Translation generalized

There were no exercises in this section.

5 Symmetry

5.5.1. Example 5.5.4: Suppose that T' is the theory in Σ with the single axiom $\vdash p$. Then intuitively, there should not be a symmetry of T' that takes p to q and vice versa. Prove this!

5.5.2. Prove that $F : T \to T$ in Example 5.5.5 is in fact a syntactic symmetry.

5.5.3. Suppose now that T' is the theory in Σ with the single axiom $\vdash \forall x \exists y r(x, y)$. Then there is no syntactic symmetry $F : T' \to T'$ such that $Fr = r^{op}$. Indeed, if there were such a symmetry, then we would have

$$\forall x \exists y r(x, y) \vdash \forall y x \exists y r(y, x)$$

which is intuitively not the case. Prove this!

5.5.4. Suppose now that T is the theory Σ with the single axiom

$$r(x, y) \vdash \neg r(y, x)$$

which says that r is asymmetric. This axiom can be rewritten as

$$r(x, y) \vdash \neg r^{op}(x, y).$$

Show that $Fr = r^{op}$ defines a symmetry of T.

5.5.5. Show that the theory of a partial order (Example 4.1.1) has a symmetry that maps $\leq$ to the converse relation $\geq$.

5.5.6. Show that one can, in fact, present projective geometry as a single-sorted theory R, with predicates for "is a point" and "is a line". Further, the duality of projective geometry is a syntactic symmetry F of T that exchanges these two predicates. The duality of theorems amounts to the fact that $T \vdash \psi$ iff $T \vdash F\psi$.

PART VI

Semantic Metalogic

1 The semantic turn

6.1.7 Suppose that $x = x_1, \ldots, x_n$ is the canonical context for t. Show that $FV(T) = \{x_1, \ldots, x_n\}$.

Proof: Suppose t is the variable x then by Definition 6.1.6 the canonical context for x *is* x, and by Definition 4.2.7(1) we have $FV(x) = \{x\}$.

Now suppose t is the term $f(t_1, \ldots, t_n)$ where we assume by induction that the result holds for the terms $t_1, \ldots, t_n$. If $x_j = x_{j_1}, \ldots, x_{j_{k(j)}}$ is the canonical context for t_j then we have that $FV(t_j) = \{x_{j_1}, \ldots, x_{j_{k(j)}}\}$ by the induction hypothesis. By Definition 6.1.6 the canonical context for t is $(\cdots((x_1.x_2)\cdots).x_n = (\cdots((x_{1_1} \ldots x_{1_{k(1)}}.x_{2_1} \ldots x_{2_{k(2)}})\cdots).x_{n_1} \ldots x_{n_{k(n)}}$ and by Definition 6.1.5 any repeated terms are deleted (from left to right). Further, by Definition 4.2.7(2) we have that $FV(t) = FV(t_1) \cup \ldots \cup FV(t_n)$. By the induction hypothesis every set $FV(t_j)$ contains the elements of x_j and from how sets work if there is an element $y \in FV(t_j) \cap FV(t_i)$ then it only appears once in $FV(t)$ (just as we deleted repeated terms in the canonical context).

6.1.9 Show that the canonical context for ϕ does, in fact, contain all and only those variables that are free in ϕ.

Proof: If ϕ is $\perp$ then (by Definition 4.2.8(1)) $FV(\perp) = \varnothing$ so that $\perp$ is a sentence and (by Definition 6.1.4) the zero-length string of variables is a context for sentences.

If p is a relation (possibly equality[1]) then (by Definition 6.1.8) the canonical context for $p(t_1, \ldots, t_n)$ is $(\cdots((x_1.x_2)\cdots).x_n$ while $FV(r(t_1, \ldots, t_n)) = FV(t_1) \cup \cdots \cup FV(t_n)$ and it follows proving the result in the case of relations mimics the structure of the proof of the previous exercise.

Now suppose the result holds for two formulas ψ and ϕ. Then $FV(\neg\psi) = FV(\psi)$ and the canonical context of $\neg\psi$ is the same as that of ψ. Let $\square \in \{\wedge, \vee, \to\}$ then $FV(\phi \,\square\, \psi) = FV(\phi) \cup FV(\psi)$ and if x and y are the canonical contexts for ϕ and ψ, respectively, then the canonical context for $\phi \,\square\, \psi$ is $x.y$ which shows the result holds. Finally, $FV(\exists z\phi) = FV(\phi) \backslash \{x\}$ and the canonical context for $\exists z\phi$ is the result of deleting z from x, if it occurs. Thus we have established the result by induction.

6.1.16 Describe $M(f(x,y) = f(y,x))$, and explain why it won't necessarily be the entire set $S \times S$.

Proof: Let $f : S \times S \to S$. By definition 6.1.14(2), $M(f(x,y) = f(y,x))$ is defined to be the equalizer for the functions $M(f(x,y)) \circ \pi_1$ and $M(f(y,x)) \circ \pi_2$.

Now the equalizer of $M(f(x,y)) \circ \pi_1 : S \times S \to S$ and $M(f(y,x)) \circ \pi_2 : S \times S \to S$ is the set of points

$$\{\langle a, b\rangle \in S \times S | M(f(x,y)) \circ \pi_1 = M(f(y,x)) \circ \pi_2\}$$

[1]Equality is never explicitly used in the definition of formulas despite being discussed explicitly in the definition for a canonical context, and it's pretty clear that equality behaves just like a relation.

Not sure how to finish this...

2 The semantic view of theories

There were no exercises in this section.

3 Soundness, completeness, compactness

6.3.1 Can you see where in the proof we make use of a choice principle?

6.3.4 Finish the remaining inductive steps of the proof of Lemma 6.3.4.

4 Categories of models

6.4.3 Show that the composite of elementary embeddings is an elementary embedding.

6.4.8 Show that if $h : M \to N$ is an isomorphism, then M and N are elementarily equivalent.

5 Ultraproducts

There were no exercises in this section.

6 Relations between theories

6.6.9 Prove that the following are equivalent:

(a) T is complete.

(b) $\mathrm{Cn}(T) = \mathrm{Th}(M)$ for some Σ-structure M.

(c) T has a unique model, up to elementary equivalence. i.e. if M, N are models of T, then $M \equiv N$.

(d) $\mathrm{Mod}(T)$ is directed in the sense that for any two models M_1, M_2 of T, there is a model N of T and elementary embeddings $h_i : M_i \to N$.

7 Beth's theorem and implicit definition

There were no exercises in this section.

PART VII

Semantic Metalogic Redux

1 Structures and models

There were no exercises in this section.

2 The dual functor to a translation

There were no exercises in this section.

3 Morita equivalence implies categorical equiv- alence

There were no exercises in this section.

4 From geometry to conceptual relativity

There were no exercises in this section.

5 Morita equivalence is intertranslatability

There were no exercises in this section.

6 Open questions

The questions in this section are open....

PART VIII

From Metatheory to Philosophy

There are no explicit questions in this chapter.

Appendix

Axiom 5 is Consistent

[This section included for myself when I briefly doubted axiom 5]

We have that $\mathsf{f} : 1 \to \Omega$ is a monomorphism by Exercise 2.1.18, and hence it defines a subobject. We can then apply Axiom 5.2 to yield the following diagram, which is a pullback:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "\mathsf{f}"] & 1 \arrow[d, "\mathsf{t}"] \\ \Omega \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

But what is χ? If Y^X denotes the set of functions from X to Y (see page 48) then $|Y^X| = |Y|^{|X|}$. With this in mind, we see that there are $|\Omega|^{|\Omega|} = 2^2 = 4$ elements. Clearly they are "identity", "always true", "always false", and "switch truth value".

Since this is a pullback, we can obtain the following diagram:

$$\begin{tikzcd} 1 \arrow[dr, dashed, "j"] \arrow[drr] \arrow[ddr, "\mathsf{h}"'] & & \\ & 1 \arrow[r] \arrow[d, "\mathsf{f}"] & 1 \arrow[d, "\mathsf{t}"] \\ & \Omega \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

regardless of what the choice is for h. However, the question remains.... what is χ. Well very clearly it is not the identity or the "always false" map because then the inner square fails to commute. So it is *either* "always true" or "switch truth value".

Let us consider if it is the "always true" map, $\chi = \chi_{\frownie}$. In this case, it is possible for h to be t or f. If $h = t$ then

$$\chi_{\frownie} \circ h = \chi_{\frownie} \circ t = t$$

yet $h \neq f \circ \mathrm{id}_1$, a contradiction!!! Well, it must just be that this is *not* the unique map χ generated from Axiom 3.2. Let us suppose then that $\chi = \chi_{\smiley}$, the "switch truth value" map, then again either h is true or false.

If $h = t$ then

$$\chi_{\smiley} \circ h = \chi_{\smiley} \circ t = f \neq t$$

so the pullback property is satisfied since "$fh \neq gk$". Otherwise, suppose that $h = f$ then

$$\chi_{\smiley} \circ h = \chi_{\smiley} \circ f = t$$

but we also have $h = f \circ id_1$, as required.

In any case, it must be that $\chi = \chi_{\smiley}$ the unique pullback map which switches the truth value it acts on.