Guide To Conway Notation

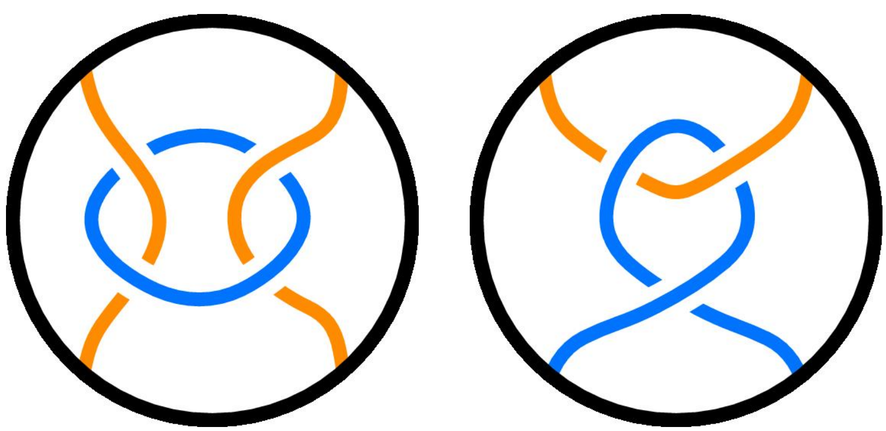
Zach Bryhtan

February 21, 2023

1 Introduction

This is a comprehensive guide to Conway's notation, originally for knots and links, introduced in his 1970 article [Con70], all diagrams generated with the help of KnotPlot[Sch22]. The notation is used constructively, relying on the notion of *tangles* before using a *numerator closure* to produce the desired knot or link. For this reason, this guide will be based on almost exclusively on its application to tangles. In an effort to make this document self-contained, we will cover several basic definitions.

Definition 1.1. *For some $n = 1, 2, 3...$, an n-tangle is a collection of n non-intersecting arcs embedded in a ball with endpoints fixed to the boundary of the ball with a (possibly empty) set of interior loops.*



(a) **(b)**

Figure 1: 2-Tangle with (a) and without (b) an interior loop

Unless otherwise stated, we will be working with 2-tangles. Moreover, we will assume that the four endpoints are fixed to the NW, NE, SW, and SE 'corners' of the bounding ball (or the enclosing circle when viewing in 2D).

(a) Reidemeister I move **(b)** Reidemeister II move **(c)** Reidemeister III move

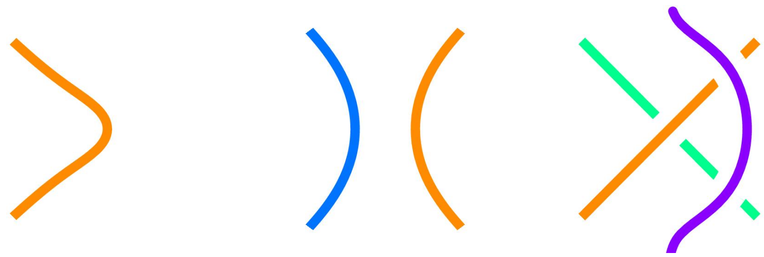


Figure 2: Reidemeister moves

Definition 1.2. *Two tangles T_1 and T_2 are* <u>*equivalent*</u> *if T_1 can be continuously deformed to produce T_2.*

We generally navigate tangle equivalences using the three Reidemeister moves, but there are some instances when a horizontal or vertical flype can be visualized more easily (but flypes can be reproduced as a sequence of Reidemeister moves).

Definition 1.3. *The* <u>*numerator closure*</u> *of a tangle T is denoted by $N(T)$ and performed by connecting the NW and SW endpoints of T to the NE and SE endpoints of T respectively, and produces either a knot or a link.*

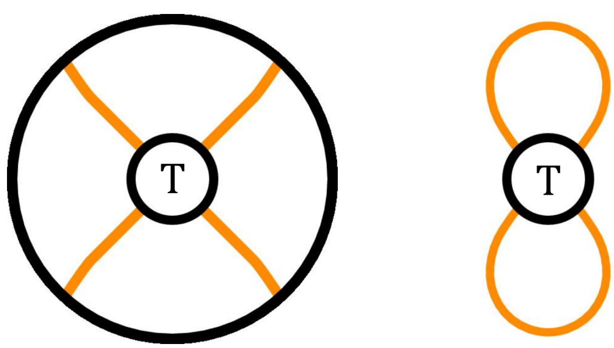


Figure 3: Numerator closure of some tangle T

The numerator closure is the standard choice, but there are more closure types including the *denominator closure*. This is similar to the numerator closure except we attach endpoints along the sides of a tangle rather than along the top and bottom.

Definition 1.4. *The sum of two tangles T_1 and T_2 is denoted $T_1 + T_2$ and performed by connecting the NE and SE endpoints of T_1 to the NW and SW endpoints of T_2 respectively.*

In practice, a tangle sum is the merging of two tangles horizontally in the easiest way.

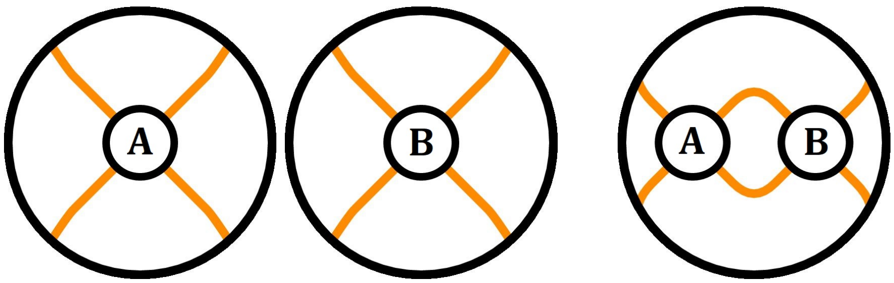


Figure 4: The sum $A + B$

Definition 1.5. *The product of two tangles T_1 and T_2 is denoted by T_1 and T_2 is denoted $T_1 * T_2$ and performed by connecting the SW and SE endpoints of T_1 to the NW and NE endpoints of T_2 respectively.*

Similar to the tangle sum, the tangle product is the merging of two tangles vertically in the easiest way. One thing to note about these operations is that they are not necessarily commutative, so pay close attention to the order in which terms appear.

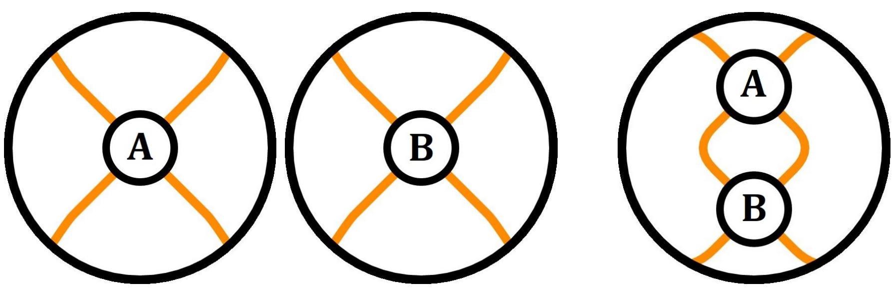
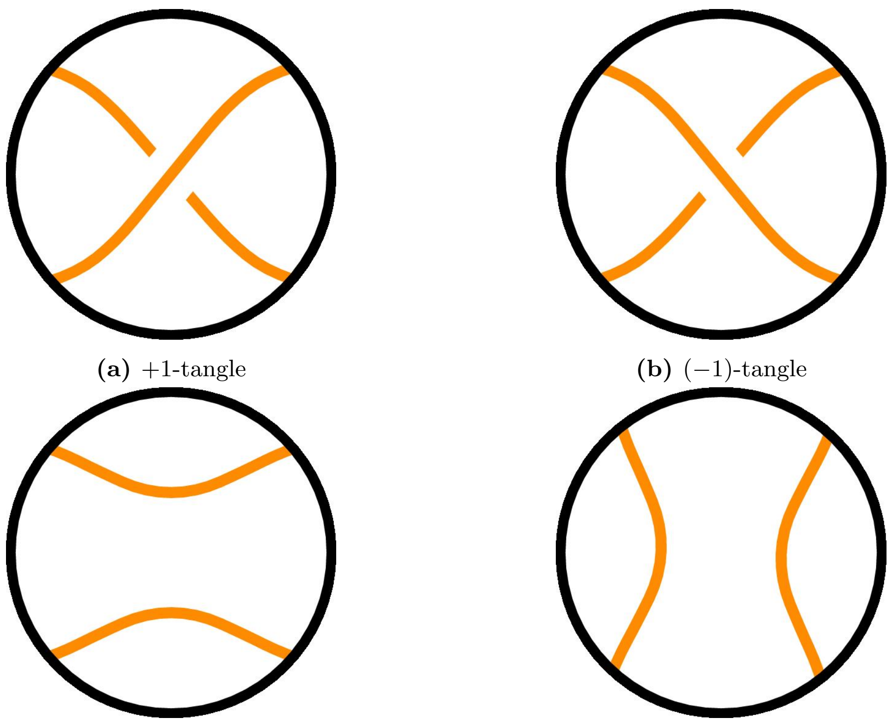


Figure 5: The product $A * B$

1.1 Basic Tangles and Fixing Conventions

We will be following the mathematical convention for labelling the four basic tangles which assigns a label corresponding to the slope of the overstrand. The basic tangles are composed of exactly two 0-crossing tangles and exactly two 1-crossing tangles. To make a 0-crossing, we may only have two horizontal arcs, or two vertical arcs. Since horizontal lines have a slope of 0, and vertical lines have an undefined slope (or a slope of ∞), we name these tangles the 0-tangle and ∞-tangle respectively. To make a 1-crossing tangle we either have the overstrand running from SW to NE or running from NW to SE. The diagonal line connecting SW to NE has a slope of 1, and the diagonal line connecting NW to SE has a slope of -1, so we name these tangles the 1-tangle and (-1)-tangle respectively.



(a) +1-tangle

(b) (-1)-tangle

(c) 0-tangle

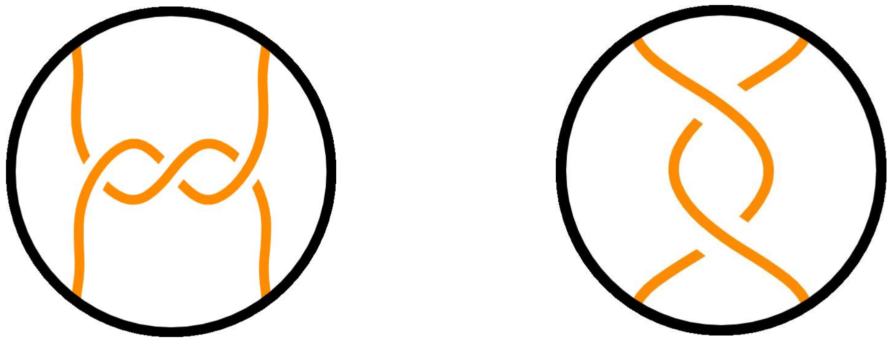
(d) ∞-tangle

Figure 6: The four basic tangles

Definition 1.6. *A horizontal (or integer) tangle is made as a combination of the* 1, -1, *and* 0 *tangles using addition exclusively.*

Definition 1.7. *A vertical tangle is made as a combination of the* 1, -1, *and* ∞ *tangles using multiplication exclusively.*

The horizontal tangles then intuitively take the shape of a series of twists arranged horizontally and similarly, vertical tangles are a series of twists arranged vertically. We assign horizontal tangles the fraction $\frac{n}{1}$, and vertical tangles the fraction $\frac{1}{n}$. In both cases, n carries the same information: $|n|$ gives the total number of crossings, and the sign of n (positive versus negative) indicates whether the crossings are all the 1 or -1 tangle.



(a) Horizontal tangle $\frac{3}{1}$

(b) Vertical tangle $\frac{1}{-2}$

Figure 7: Example of a horizontal tangle and vertical tangle

Definition 1.8. *The mirror image of a tangle T is denoted by $-T$ and performed by changing the sign of each single crossing inside T. That is, every $(+1)$-crossing becomes a (-1)-crossing and similarly every (-1)-crossing becomes a $(+1)$-crossing.*

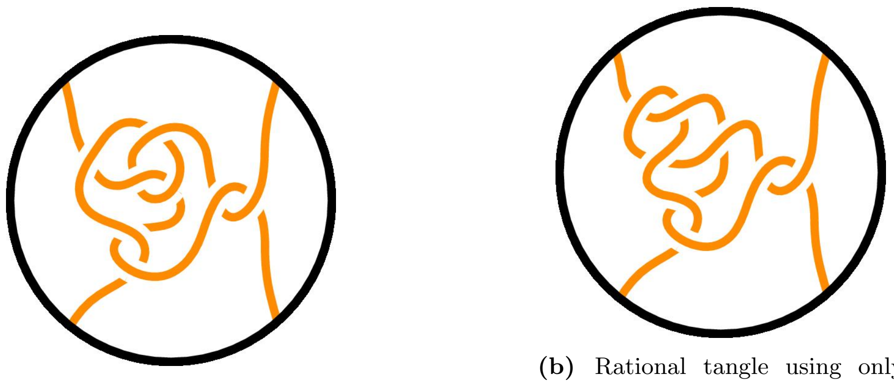
1.2 Rational Tangles and Twist Vectors

Definition 1.9. *A rational tangle is made of an 'alternating combination' of adding horizontal tangles and multiplying vertical tangles.*

This alternating pattern requires that if you started with a horizontal tangle, or must recently added a horizontal tangle, you must then multiply by a vertical tangle. Similarly, if you started with a vertical tangle, or most recently multiplied by a vertical tangle, you must then add a horizontal tangle. These operations may of course be written as an algebraic expression using the operations introduced and the fraction notation for horizontal and vertical tangles, but this kind of expression can be made more compact if we set certain requirements.

Using a series of Reidemeister moves, a rational tangles can be rearranged so that horizontal tangles are always added to the right, and vertical tangles are always added to the bottom. From here we introduce a shorthand notation which relies on this particular ordering as well as one additional assumption.

Definition 1.10. *A twist vector for a rational tangle is a sequence of integers $(x_1, ..., x_n)$ representing an alternating pattern of right and bottom twists. It is assumed that x_n always represents a horizontal twist (horizontal tangle).*



(a) Rational tangle $\left(\left(\frac{1}{1}+\left(\frac{1}{1}*\frac{2}{1}\right)\right)*\frac{1}{-2}\right)+\frac{2}{1}$

(b) Rational tangle using only right and bottom twists $\left(\left(\left(\frac{2}{1}*\frac{1}{1}\right)+\frac{1}{1}\right)*\frac{-2}{1}\right)+\frac{2}{1}$

Figure 8: Examples of rational tangles

Using the twist vector, we then have two arising algebraic expressions depending on whether n is even or odd:

$$(x_1,...,x_n)=\begin{cases}\left(\left(\cdots\left(\frac{x_1}{1}*\frac{1}{x_2}\right)+\cdots\right)*\frac{1}{x_{n-1}}\right)+\frac{x_n}{1}; & n \text{ odd}\\ \left(\left(\cdots\left(\frac{1}{x_1}+\frac{x_2}{1}\right)*\cdots\right)*\frac{1}{x_{n-1}}\right)+\frac{x_n}{1}; & n \text{ even}\end{cases}$$

In the event that a rational tangle does not end with horizontal twists, we let $x_n = 0$. With the twist vector in hand, we may assign a fraction to every rational tangle.

Definition 1.11. *For a rational tangle T with twist vector $(x_1,...,x_n)$, the continued fraction of T is*

$$\frac{a}{b}=x_n+\cfrac{1}{x_{n-1}+\cfrac{1}{\ddots+\frac{1}{x_1}}}$$

On the other hand, a twist vector can be derived from a continued fraction for a rational tangle using the Euclidean algorithm. There are infinitely many ways to decompose a fraction and obtain a unique twist vector, but any two rational tangles with equal continued fractions are *equivalent*. To reduce ambiguity, we will designate a canonical form for twist vectors to ensure a unique way of decomposing each continued fraction.

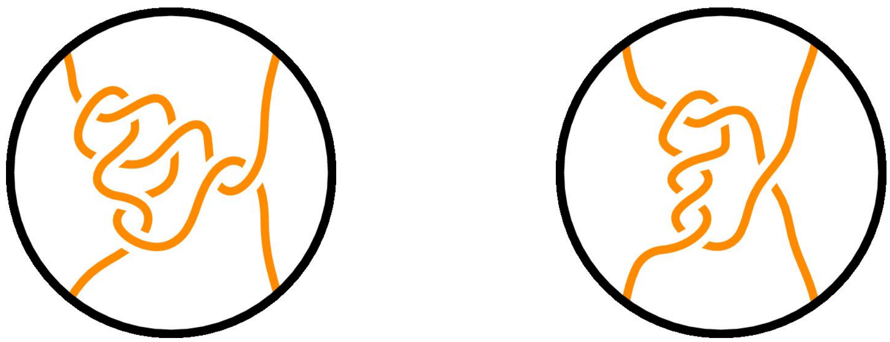
Example. *The tangle in Figure 8(b) can be made using the algebraic expression* $\left(\left(\left(\frac{2}{1}*\frac{1}{1}\right)+\frac{1}{1}\right)*\frac{-2}{1}\right)+\frac{2}{1}$. *This is translated into a twist vector as* $(2,1,1,-2,2)$.

From here we compute the continued fraction as

$$\frac{a}{b} = 2 + \cfrac{1}{-2 + \frac{1}{1+\frac{1}{1+\frac{1}{2}}}} = \frac{9}{7}.$$

Using this same fraction, we can find an alternate twist vector describing an equivalent rational tangle.

$$\frac{9}{7} = 1 + \frac{2}{7} = 1 + \frac{1}{7/2} = 1 + \frac{1}{3+\frac{1}{2}} \implies \textit{twist vector} = (2, 3, 1).$$



(a) $(2, 1, 1, -2, 2)$ **(b)** $(2, 3, 1)$

Figure 9: Equivalent rational tangles with different twist vectors

Definition 1.12. *A twist vector* $(x_1, ..., x_n)$ *is in (a short) canonical form if* n *is minimal such that:*

(a) All integers $x_1, ..., x_{n-1}$ *are non-zero*

(b) All integers $x_1, ..., x_{n-1}, x_n$ *have the same sign (with the exception that* x_n *may be zero)*

(c) The first term x_1 *has a magnitude of at least 2 (*$|x_1| \geq 2$*).*

We mark this canonical form as a *short* form to distinguish it from other common forms such as the odd-length and even-length canonical forms. For the odd-length canonical form, the only changes to our definition above is in the first sentence "...n is minimal *and odd* such that...," then in part (c) the rule widens to allow $|x_1| \geq 1$. The even-length canonical form is similar, but requiring that n be *even*.

2 Conway Notation

The following few sections will cover the symbols and conventions used in writing Conway notation for tangles. For rational tangles, the Conway notation and twist vector notation function identically. Just as we designated a canonical form for twist vectors, we designate the same canonical form for Conway notation for rational tangle. The main visual difference between Conway notation and twist vector notation is that it is more common to see Conway notation written inside brackets $[x_1, ..., x_n]$.

Remark. *In many cases, Conway notation is also written* ***omitting the separating commas****, so* $[x_1, ..., x_n]$ *would be written* $[x_1 \cdots x_n]$*. When commas are omitted, values of the sequence are generally all single digits, but it is left to context (or spacing) whether certain values should be more than one digit long. To avoid ambiguity, commas will always be included to separate values.*

Definition 2.1. *The* <u>*Conway notation*</u> *for a rational tangle is a sequence of integers* $[x_1, ..., x_n]$ *representing an alternating pattern of right and bottom twists (see* <u>*twist vector*</u>*).*

Remark. *In twist vector notation, no special treatment was given to negative integers appearing in the sequence. In Conway notation, negative integers are commonly written with a bar instead* ($-x_i \longrightarrow \overline{x_i}$)*. For example, the twist vector (not in canonical form)* $(1, 4, -3, 4, -5)$ *would be written as* $[1, 4, \overline{3}, 4, \overline{5}]$ *in Conway notation.*

The Conway notation $[x_1, ..., x_n]$ for a rational tangle is in canonical form if the corresponding twist vector $(x_1, ..., x_n)$ is in canonical form.

2.1 Periods and Colons

Periods and colons may be used in Conway notation to stand for a section of the value 1 repeated. You simply replace the total number of 1's with the same number of dots using a combination of a period and colons. When doing this replacement, we do not include any commas between or around the period and colons.

Example. *The Conway notation* $[3, 1, 1, 1, 2, 1, 4]$ *has a cluster of three* 1*s between* 3 *and* 2*. I can make three dots using one period and one colon, so this may be rewritten as* $[3 : .2, 1, 4]$*. There is another 'cluster' of a single* 1 *between the* 2 *and* 4 *so we could replace it with a single period. Making this change, the Conway notation would be* $[3 : .2.4]$*, but we generally only make this change if there are more than one copies of the value* 1*.*

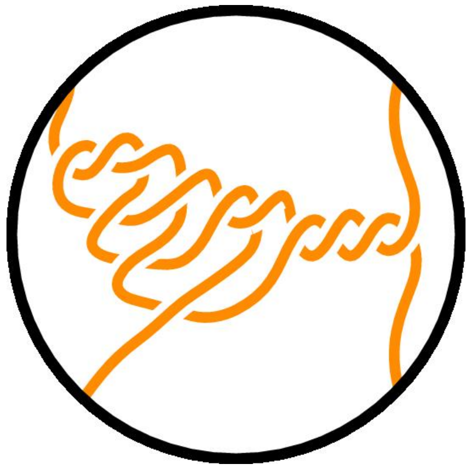


Figure 10: The rational tangle $[3:.2,1,4] \Rightarrow \left(\left(\left(\left(\left(\frac{3}{1}*\frac{1}{1}\right)+\frac{1}{1}\right)*\frac{1}{1}\right)+\frac{2}{1}\right)*\frac{1}{1}\right)+\frac{4}{1}$

2.2 Semicolons

Semicolons are used to separate two or more sections of Conway notation for a rational tangle. Once each section has been drawn according to the Conway notation, the semicolons should be read to indicate a tangle sum in the order given. Tangles made in this way (as a sum of rational tangles), make up a new category of tangles called *Montesinos tangles*.

Definition 2.2. *A Montesinos tangle M is a finite sum of rational tangles $T_1, ..., T_n$ with continued fractions $\frac{a_1}{b_1}, ..., \frac{a_n}{b_n}$ (no $\frac{a_i}{b_i} = \frac{1}{0}$) respectively, and assigned the expression*

$$M = \frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}.$$

Through a sequence of Reidemeister moves, if any rational tangle used to create a Montesinos tangle ends in a horizontal twist, these horizontal twists can be pushed through the sum and collected at the far right. Following this process ensures that each rational tangle always end in a vertical twist. For now, assume that each rational tangle component ended in a vertical twist (remaining horizontal twists are handled in the next section).

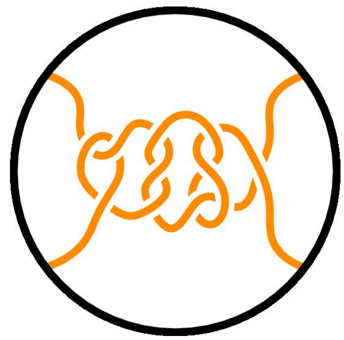
Remark. *We extend the canonical form for Conway notation so that $[A_1; \cdots ; A_n]$ $(A_i = x_{i,1}, ..., x_{i,m_i})$ is in canonical form if:*

(a) Each sequence A_i corresponds to Conway notation for a rational tangle in canonical form

(b) Each sequence A_i ends in a 0 $(x_{i,m_i} = 0)$

(c) *All sequences A_i have the same sign (each A_i is a sequence of all positive or all negative values).*

(a) Sum of rational tangles allowing horizontal twists

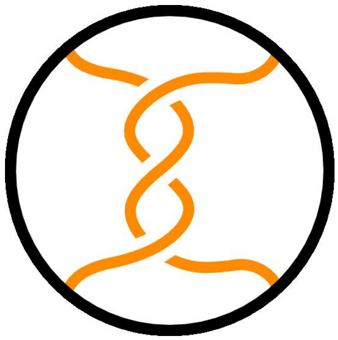


(b) Sum of rational tangles with all horizontal twists pushed to the end

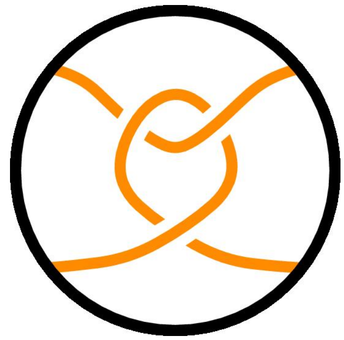
Figure 11: Montesinos tangles

Remark. *Since each pieces separated by a semicolon should end in a* 0 *(as long as it is in canonical form), these zeros are often omitted.*

Example. *A Montesinos tangle has Conway notation (in canonical form)* $[3; 2, 1; 2, 2]$. *We know zeros have been omitted, so we may first write the Conway notation as* $[3, 0; 2, 1, 0; 2, 2, 0]$. *This tangle is made up of the three rational tangles* $T_1 = [3, 0]$, $T_2 = [2, 1, 0]$, *and* $T_3 = [2, 2, 0]$. *We then find the intended Montesinos tangle by performing the sum* $(T_1 + T_2) + T_3$.



(a) $T_1 = [3, 0] = \frac{1}{3}$



(b) $T_2 = [2, 1, 0] = \frac{2}{1} * \frac{1}{1}$

(c) $T_3 = [2, 2, 0] = \frac{2}{1} * \frac{1}{2}$

(d) Result $T = [3; 2, 1; 2, 2] = (T_1 + T_2) + T_3$

Figure 12: Constructing the Montesinos tangle $[3; 2, 1; 2, 2]$

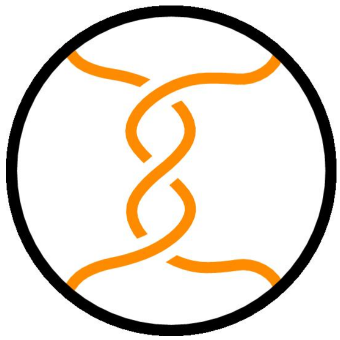
2.3 Terminating Plus and Minus Signs

Any plus or minus signs at the end of Conway notation for a Montesinos tangle indicate the combined number and sign of horizontal twists pushed to the far right end of the sum.

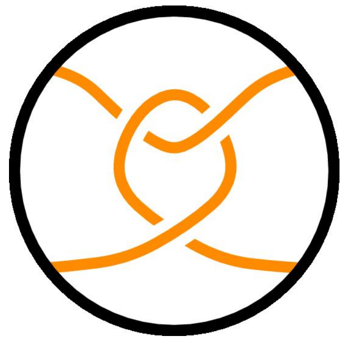
Remark. *If there are no more than two total crossings at the far right end, it is common to add one + or − per crossing (depending on the sign of the crossings). If there are more than two crossings, we instead write $+n$ or $-n$ where n is the total number of crossings.*

The canonical form for Conway notation of Montesinos tangles easily extends to include this new symbol. Since these ending horizontal twists are directly attached to the last rational tangle, these twists are pushed into its Conway notation (replacing the zero).

Example. *A Montesinos tangle has Conway notation (in canonical form)* $[3; 2, 1; 2, 2 - -]$. *We know zeros have been omitted and the two minus signs indicate two negative horizontal twists included into the third piece, so we write* *the Conway notation as* $[3,0;2,1,0;2,2,\overline{2}]$ *(recall* $\overline{n} = -n$*). This tangle is made up of the three rational tangles* $T_1 = [3,0]$*,* $T_2 = [2,1,0]$*,* $T_3 = [2,2,\overline{2}]$*. We then find the intended Montesinos tangle by performing the sum* $(T_1 + T_2) + T_3$*.*



(a) $T_1 = [3,0] = \frac{1}{3}$



(b) $T_2 = [2,1,0] = \frac{2}{1} * \frac{1}{1}$

(c) $T_3 = [2,2,\overline{2}] = (\frac{2}{1} * \frac{1}{2}) + \frac{-2}{1}$

(d) Result $T = [3;2,1;2,2--] = (T_1 + T_2) + T_3$

Figure 13: Constructing the Montesinos tangle $[3;2,1;2,2]$

2.4 Parentheses

Parentheses in Conway notation group portions of Conway notation for Montesinos tangles. There are several cases these parentheses may be encountered. We will go through the cases as listed by Caudron[Cau82].

2.4.1 Case 1. $[(a;b),(c;d)]$

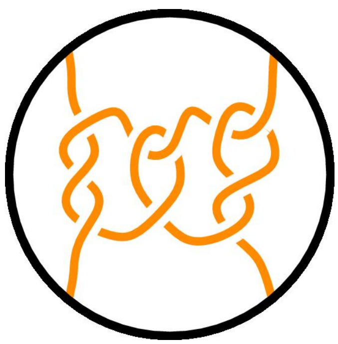
Suppose we have Conway notation with two or more Montesinos tangles (represented by two or more groupings of parentheses) separated by a comma, say $[(M_1),...,(M_n)]$. We create the tangle by following a similar alternating pattern as described in reading twist vectors. For rational tangles we alternated between adding horizontal tangles and multiplying by something similar, but with an ‘opposite orientation’: vertical tangles. We expect Montesinos tangles to be made of a *sum* of rational tangles which end in *vertical twists* (except maybe the last term). A similar but opposite orientation is then a tangle made of a *product* of rational tangles which end in *horizontal twists* (except maybe the last term).

Remark. *If M is some instance of a Montesinos tangle grouped with parentheses in Conway notation, the related tangle with opposite orientation will be denoted $M^\dagger$. Then if we have Conway notation $[(M_1), ..., (M_n)]$, we can find the resulting tangle as*

$$\left(\left((M_1 * M_2^\dagger) + \cdots\right) * M_n^\dagger\right) + M_n; \qquad n \text{ odd}$$

$$\left(\left((M_1^\dagger + M_2) * \cdots\right) * M_n^\dagger\right) + M_n; \qquad n \text{ even}$$

To emphasize the alternating orientations, in the following example we will consider Montesinos tangles which are only made up of vertical tangles.



(a) $M = [3; 2, 1; 2, 2]$

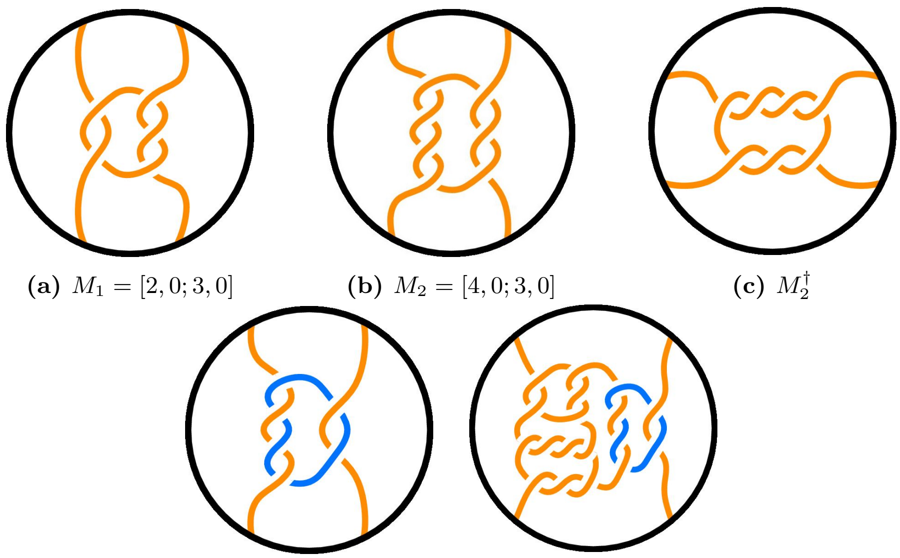
(b) $M^\dagger$

Figure 14: Comparison of orientations M vs. $M^\dagger$

Example. *A tangle has Conway notation $[(2;3), (4;3), (4;2)]$. The three Montesinos tangles are $M_1 = [2;3]$, $M_2 = [4;3]$, and $M_3 = [4;2]$. We know they must be combined in the alternating pattern*

$$(M_1 * M_2^\dagger) + M_3.$$

Both M_1 and M_3 are oriented normally, but M_2 has the opposite orientation. Then using what we know about omitted zeros, $M_1 = [2,0;3,0]$, $M_2 = [4,0;3,0]$, and $M_3 = [4,0;2,0]$. To find $M_2^\dagger$, we delete those ending zeros and read the semicolon as a product (instead of a sum) so that $M_2^\dagger = T_1 * T_2$ where T_1 is given by the Conway notation $[4]$ and T_2 is given by the Conway notation $[3]$.



(a) $M_1 = [2, 0; 3, 0]$ **(b)** $M_2 = [4, 0; 3, 0]$ **(c)** $M_2^\dagger$

(d) $M_3 = [4, 0; 2, 0]$ **(e)** $(M_1 * M_2^\dagger) + M_3$

Figure 15: Constructing the tangle $[(2; 3), (4; 3), (4; 2)]$

2.4.2 Case 2. $[(a; b), c, (d; e)]$

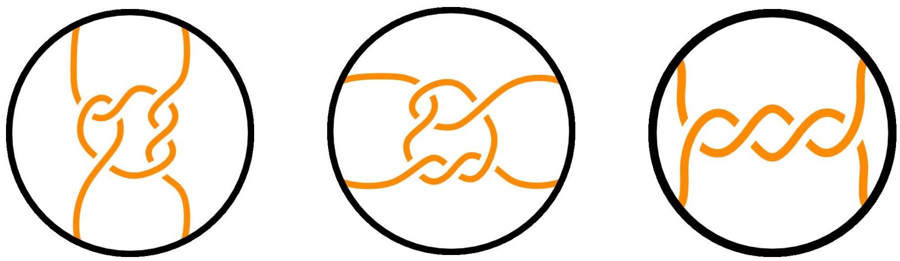
In this case, we still follow this more generalized alternating pattern but single integers are treated as in Conway notation for a rational tangle.

Example. *A tangle has Conway notation* $[(2, 1; 3), 4, 2, (4; 3, 1)]$. *The two Montesinos tangles are* $M_1 = [2, 1; 3]$ *(or* $[2, 1, 0; 3, 0]$*) and* $M_2 = [4; 3, 1]$ *(or* $[4, 0; 3, 1, 0]$*). Just as we marked Montesinos tangles* $M^\dagger$ *to indicate an opposite orientation, let* H *stand for a horizontal tangle and* $H^\dagger$ *be the vertical tangle made with the same sign and number of crossings. Then we have two single integers occurring in the sequence* $H_1 = [4]$ *and* $H_2 = [2]$. *We know these must be combined in the alternating pattern*

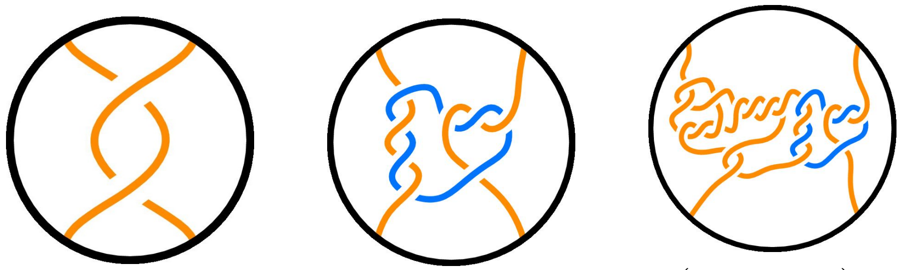
$$\Big((M_1^\dagger + H_1) * H_2^\dagger\Big) + M_2.$$

From $M_1 = [2, 1, 0; 3, 0]$, *we have* $M_1^\dagger = T_1 * T_2$ *where* T_1 *is given by Conway notation* $[2, 1]$ *and* T_2 *is given by the Conway notation* $[3]$. *As horizontal tangles* $H_1 = \frac{4}{1}$ *and* $H_2 = \frac{2}{1}$, *so* $H_2^\dagger = \frac{1}{2}$. *Including these fractions into the previous expression, we would have*

$$\Big((M_1^\dagger + \frac{4}{1}) * \frac{1}{2}\Big) + M_1.$$



(a) $M_1 = [2,1,0;3,0]$ **(b)** $M_1^\dagger$ **(c)** $H_1 = [4]$



(d) $H_2^\dagger = [2,0]$ **(e)** $M_2 = [4,0;3,1,0]$ **(f)** $\left((M_1^\dagger + H_1) * H_2^\dagger\right) + M_2$

Figure 16: Constructing the tangle $[(2,1;3),4,2,(4;3,1)]$

2.4.3 Case 3. $[(a;b);c;(e;f)]$

Just as before, semicolons encountered are used to indicate the sum of the pieces it separates. When this was used to describe Montesinos tangles specifically, we set the expectation that all rational pieces end with a zero *and* these zeros are omitted. This expectation is still followed here so that:

(a) Each section $[\cdots; x_1, ..., x_n; \cdots] = [\cdots; x_1, ..., x_n, 0; \cdots]$ and handled is a rational tangle

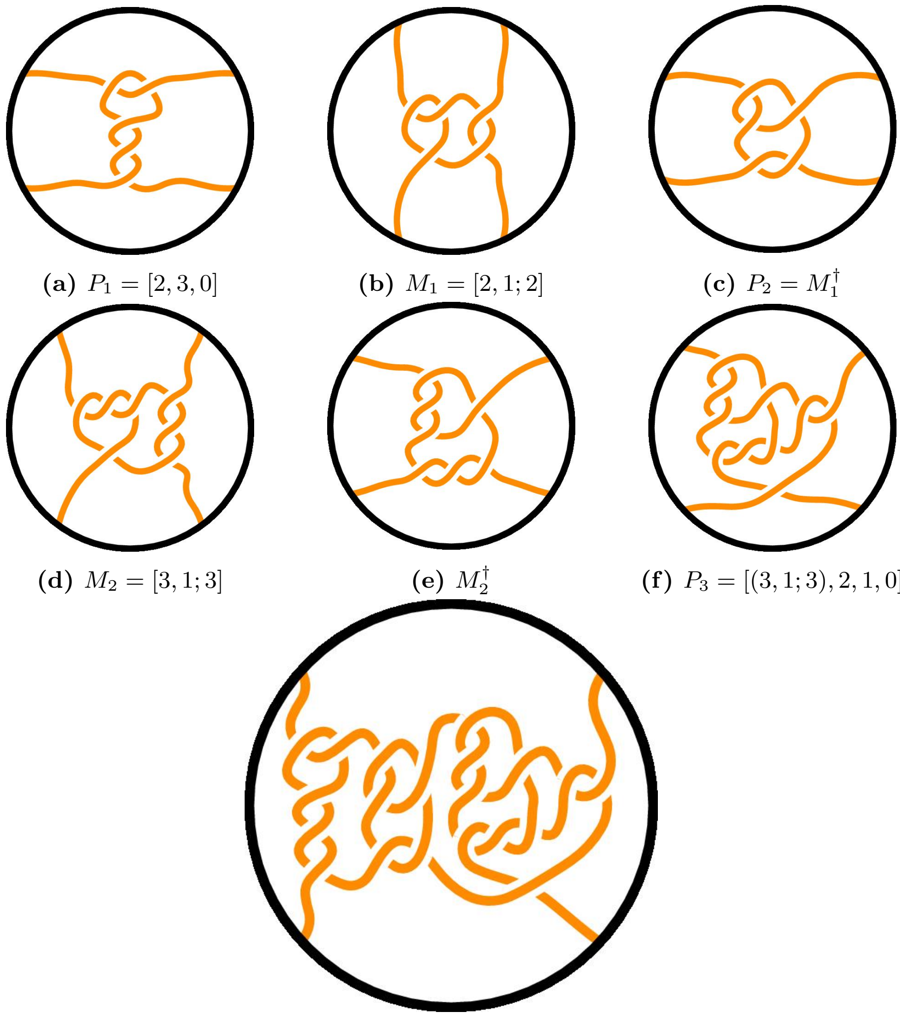
(b) Each section $[\cdots; (M), x_1, ..., x_n; \cdots] = [\cdots; (M), x_1, ..., x_n, 0; \cdots]$ and handled as in **Case 2**

(c) Each section $[\cdots; (M_1), x, (M_2); \cdots] = [\cdots; (M_2), x, (M_2), 0; \cdots]$ and handles as in **Case 2**.

Example. *A tangle has Conway notation* $[2,3;(2,1;2);(3,1;3),2,1]$. *This has three pieces* $P_1 = [2,3]$ *(or* $[2,3,0]$*),* $P_2 = [2,1;2]$ *(or* $[(2,1;2),0]$*), and* $P_3 = [(3,1;3),2,1]$ *(or* $[(3,1;3),2,1,0]$*). The first piece just corresponds to a rational tangle, but* P_2 *and* P_3 *must be handled as in* ***Case 2****. Then following those steps we can write* $P_2 = M_1^\dagger + \frac{0}{1} = M_1^\dagger$, $M_1^\dagger = T_1 * T_2$, $T_1 = [2,1]$, $T_2 = [2]$. *Similarly,* $P_3 = \left((M_2^\dagger + \frac{2}{1}) * \frac{1}{1}\right) + \frac{0}{1} = (M_2^\dagger + \frac{2}{1}) * \frac{1}{1}$, $M_2^\dagger = T_3 * T_4$, $T_3 = [3,1]$, $T_4 = [3]$.

Putting all these pieces together we find the resulting tangle as

$$P_1 + P_2 + P_3 = P_1 + M_1^\dagger + \left((M_2^\dagger + \frac{2}{1}) * \frac{1}{1}\right).$$



(a) $P_1 = [2, 3, 0]$ **(b)** $M_1 = [2, 1; 2]$ **(c)** $P_2 = M_1^\dagger$

(d) $M_2 = [3, 1; 3]$ **(e)** $M_2^\dagger$ **(f)** $P_3 = [(3, 1; 3), 2, 1, 0]$

(g) $P_1 + P_2 + P_3$

Figure 17: Constructing the tangle $[2, 3; (2, 1; 2); (3, 1; 3), 2, 1]$

2.5 Asterisks

In his original article, Conway built up his notation referencing tangles so that he could construct knots and links using certain diagrams. The notation referencing these diagrams will always be in the context of knots and links (while all previous sections could be generalized to tangles). There is a similar way to describe algebraic tangles using *constellations* as discussed by Caudron and Connolly[Cau82][Con21], but this will not be covered here.

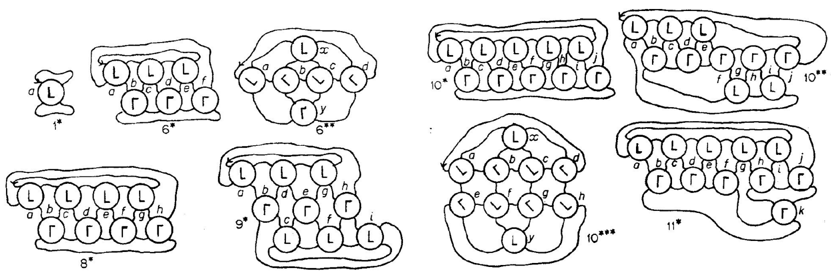


Figure 18: Conway's polyhedral diagrams

In general, asterisks are encountered at the beginning of some Conway notation as $[n^* \cdots]$ (or $[n^{**} \cdots]$ as needed) for some integer n. Looking at these diagrams, each of the n circled L's indicate a place where any kind of tangle should be inserted to fill the diagram and produce a knot or link.

Remark. *In the original article, Conway separates the tangles by a period. On the KnotInfo site[LM23], there are neither periods nor commas to separate tangles. To avoid ambiguity here, I will be separating tangles with a comma, using square brackets to group more complex tangles. For example, if I wanted to fill the* 6^* *diagrams with* $a = [2]$, $b = [3]$. $c = (3;2)$, $d = [(3;2), 1]$ $e = [2]$ $f = [1]$ *I would use the notation*

$$[6^* a, b, c, d, e, f] = [6^* 2, 3, [3;2], [(3;2), 1], 2, 1].$$

Remark. *It is common to* <u>*omit*</u> 1*s that appear at the end of this type of Conway notation. For example,* $[8^* 3, 1, 2]$ *must be made using the* 8^* *diagram which has eight positions to fill. The Conway notation we have will only fill three of those positions, so there must be five omitted* 1*s making our Conway notation* $[8^* 3, 1, 2, 1, 1, 1, 1, 1]$ *(or if using periods and colons* $[8^* 3, 1, 2 :: .]$*).*

Remark. *Certain prefixes* n^* *are also* <u>*omitted*</u> *and left to context. For example, everything discussed until this section could be written with the prefix* 1^*. *The prefix* 6^{**} *is omitted, but recognized if Conway notation begins with at least one dot. The prefix* 6^* *is omitted, but recognized if Conway notation contains at least one dot and no dots at the beginning.*

Figure 19: The link $[3, 2 : 2] = [6^*3, 2, 1, 1, 2, 1]$

Example. *A link has Conway notation* $[3, 2 : 2]$*. Since there is at least one dot, but none of them are at the beginning we must use the* 6^* *diagram and* $[3, 2 : 2] = [6^*3, 2 : 2]$*. If we replace each dot with a* 1*, then we have* $[6^*3, 2, 1, 1, 2]$ *which only has five pieces when we are expecting six so one ending occurrence of the value* 1 *has also been omitted. Then the complete Conway notation is* $[6^*3, 2, 1, 1, 2, 1]$ *and assembled according to the diagram.*

References

[Con70] John H. Conway. "An enumeration of knots and links, and some of their algebraic properties". In: 1970.

[Cau82] Alain Caudron. "Classification des noeuds et des enlacements". In: 1982.

[Con21] Nicholas Connolly. *Classification and tabulation of 2-string tangles: the astronomy of subtangle decompositions*. 2021.

[Sch22] Robert Scharein. *The KnotPlot Site*. URL: `knotplot.com`. Feb. 2022.

[LM23] Charles Livingston and Allison H. Moore. *KnotInfo: Table of Knot Invariants*. URL: `knotinfo.math.indiana.edu`. Feb. 2023.