Tangle Tabulation

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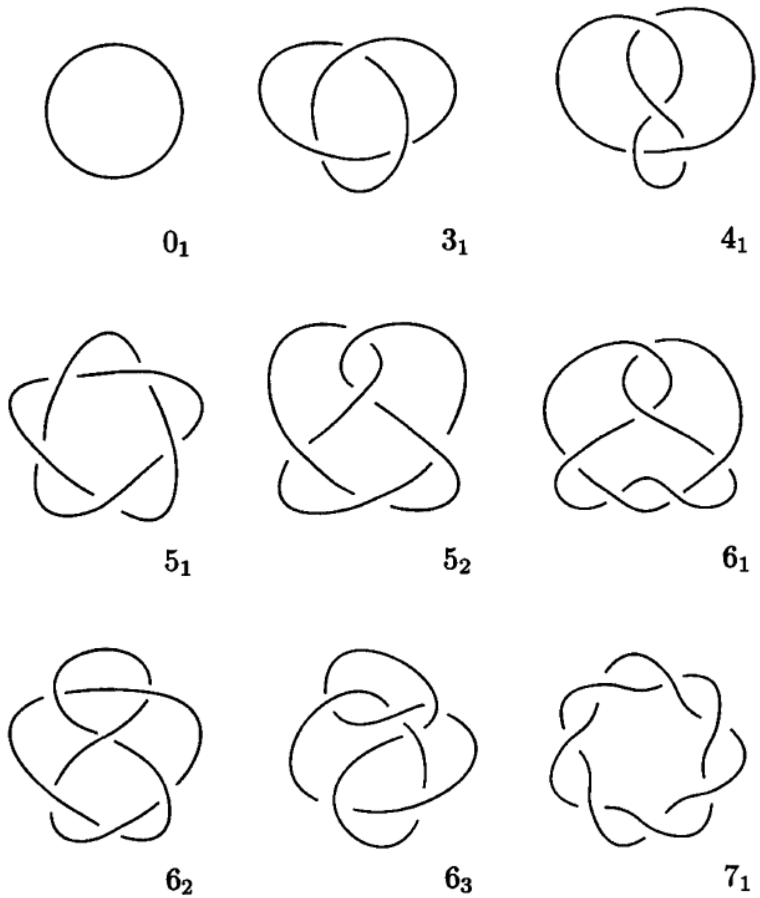
[1]University of Iowa

Introduction

Knot and Link Tabulation

- Lord Kelvin
- P. G. Tait
- C. N. Little
- M. Dehn
- J. Alexander
- K. Reidemeister
- H. Seifert
- J. H. Conway

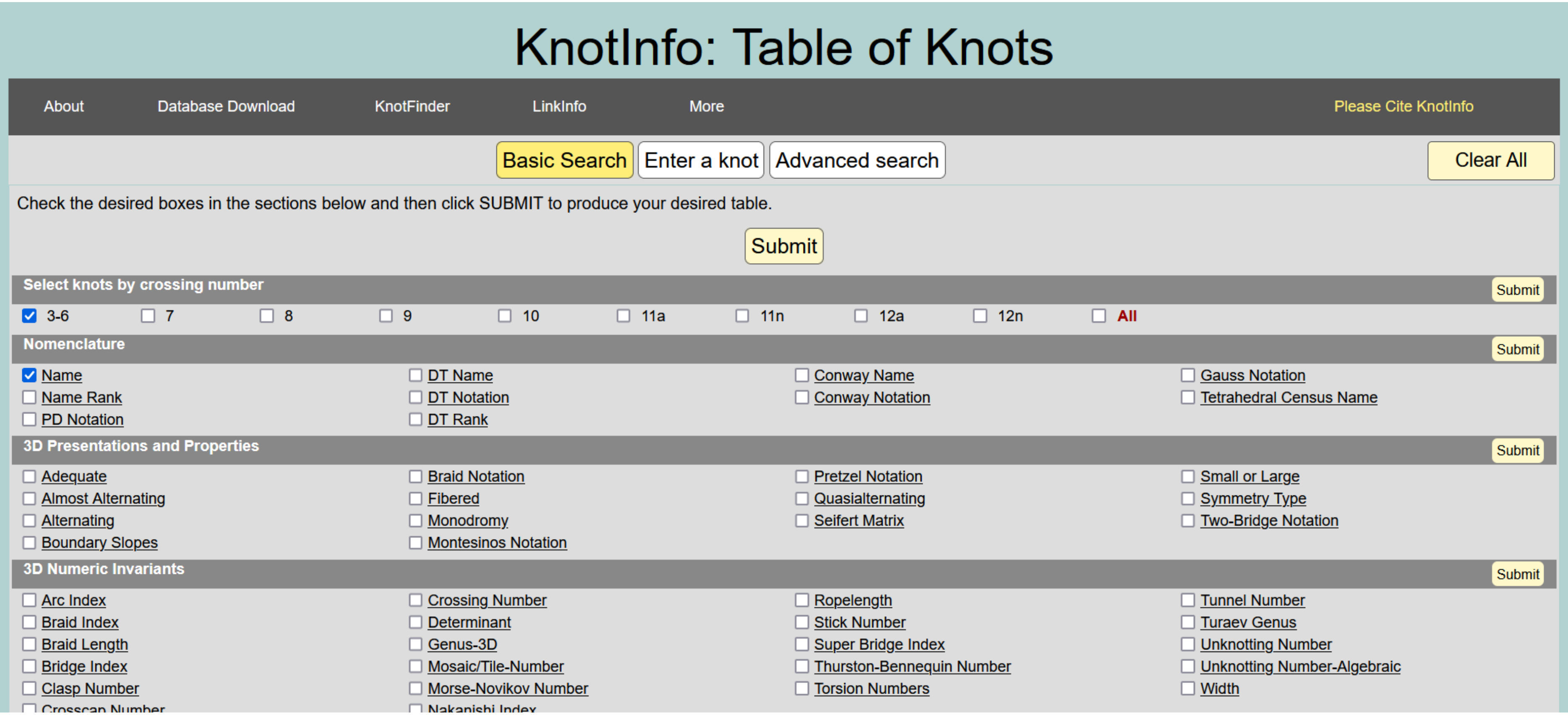
Knot and Link Tabulation



Alexander Polynomial

$\mathbf{3_1}$	$t^2 - t + 1$
$\mathbf{4_1}$	$t^2 - 3t + 1$
$\mathbf{5_1}$	$t^4 - t^3 + t^2 - t + 1$
$\mathbf{5_2}$	$2t^2 - 3t + 2$
$\mathbf{6_1}$	$2t^2 - 5t + 2$
$\mathbf{6_2}$	$t^4 - 3t^3 + 3t^2 - 3t + 1$
$\mathbf{6_3}$	$t^4 - 3t^3 + 5t^2 - 3t + 1$
$\mathbf{7_1}$	$t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$

Knot and Link Tabulation



https://knotinfo.math.indiana.edu/

Knot and Link Tabulation

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- Hoste-Thistlethwaite Table (11 crossings)
- Link Table
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- KnotTheory` Manual
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Knot Atlas:About

Contents [hide]

What is the Knot Atlas

It's editable

Almost everything in the Knot Atlas is user editable; anyone anywhere can add or change almost everything so we know it can and will grow very informative and complete (see Help:Editing). Yet mechanisms are in place to ensure that the information in the Knot Atlas will remain quite reliable (though see the Disclaimers).

It's a knot atlas

Every knot and link up to some size has a page with much data about it: words, pictures, and values of many knot invariants. Click Random Page here or on the navigation sidebar on the left, and, with a high probability, this will take you to one of those knot/link pages.

It's a knot theory database

Most available knot invariants are stored as individual database entry, under titles such as Data:5 2/Bridge Index (so this entry contains the bridge index of the knot 5_2, which happens to be 2). It is easy to read, write and create new data base entries. This can be done by hand, just like editting any other page on the wiki. Programs are also provided, and further programs will be provided, to allow machine reading and writing of database entries. Reading and writing happens via http and is not tied to any language or platform.

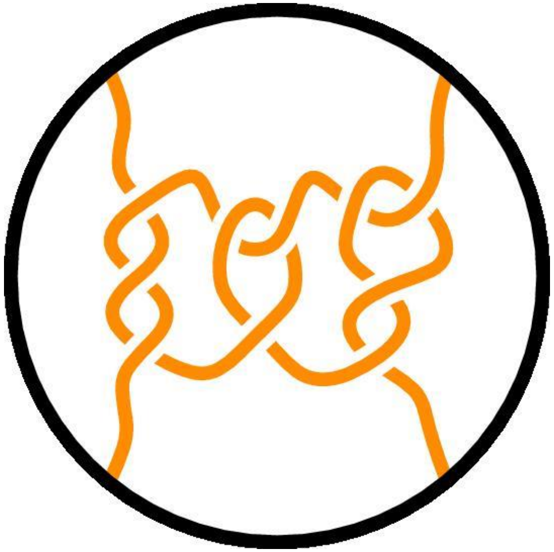
It's a knot theory knowledge base

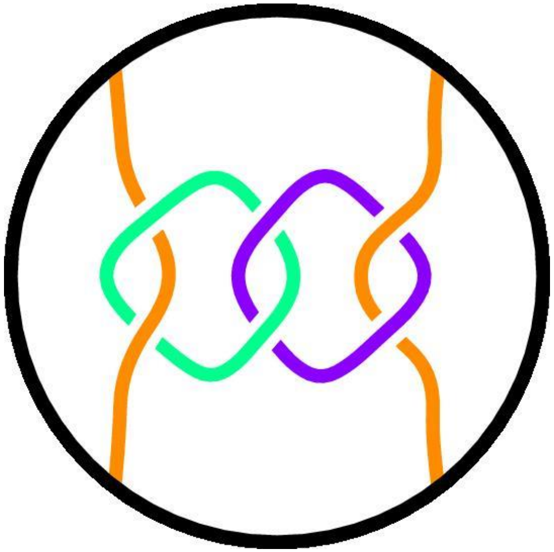
In addition to numerical, polynomial and other "math-valued" invariants, there's room for anyone easily contribute from her/his knowledge of any given knot or link, and it is very easy to add and link more general articles. Thus we expect that in time, the Knot Atlas will grow to be a central, or the central, knot theory knowledge base. Looking for a link with vanishing Multivariable Alexander Polynomial? Wondering if there's anything special about 10_124? It

http://katlas.org/wiki/Knot_Atlas:About

2-String Tangles

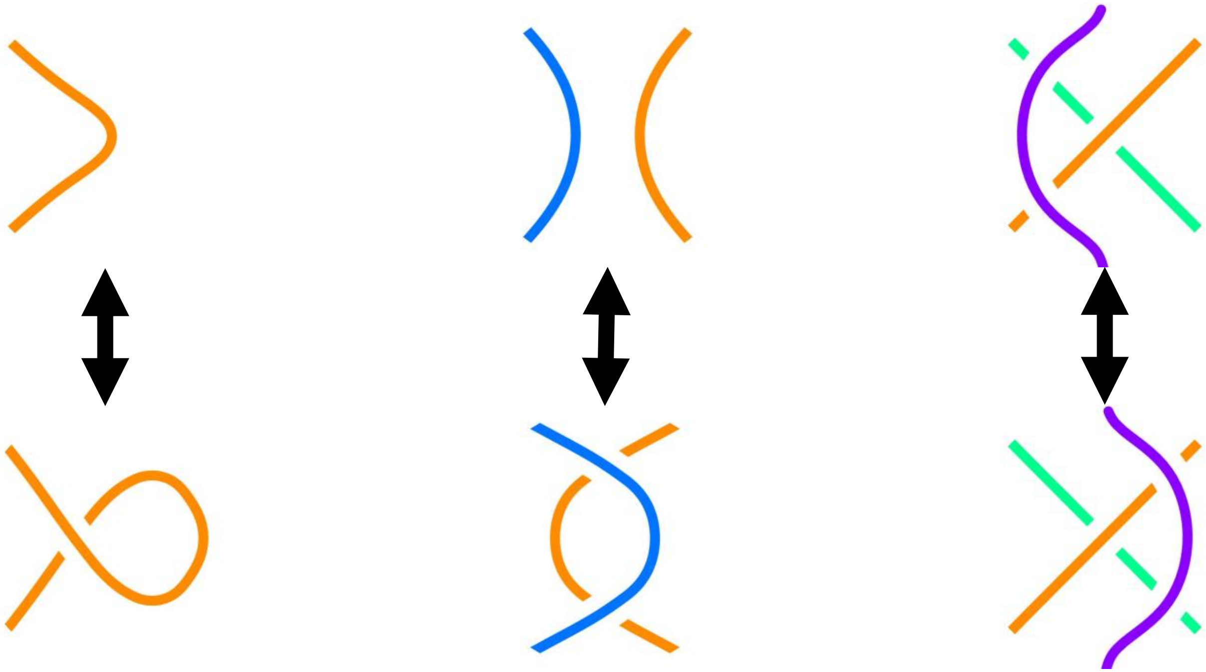
Definition. A 2-tangle is a ball containing 2 disjoint properly embedded arcs with endpoints fixed on the boundary of the ball along with a possibly non-empty set of interior loops.



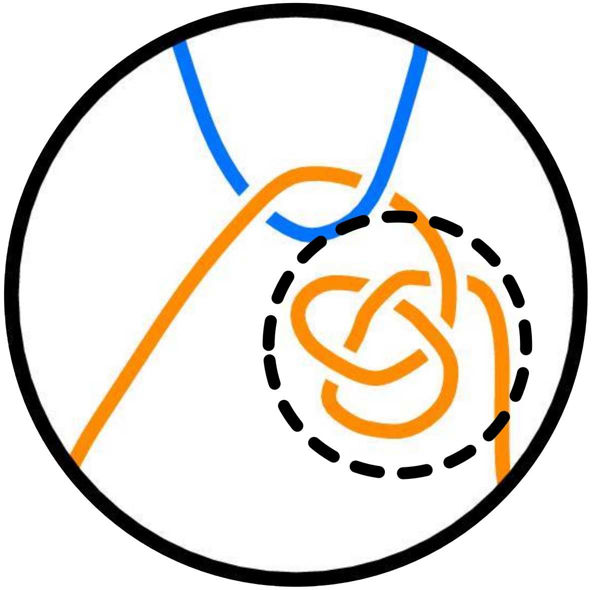


2-String Tangles

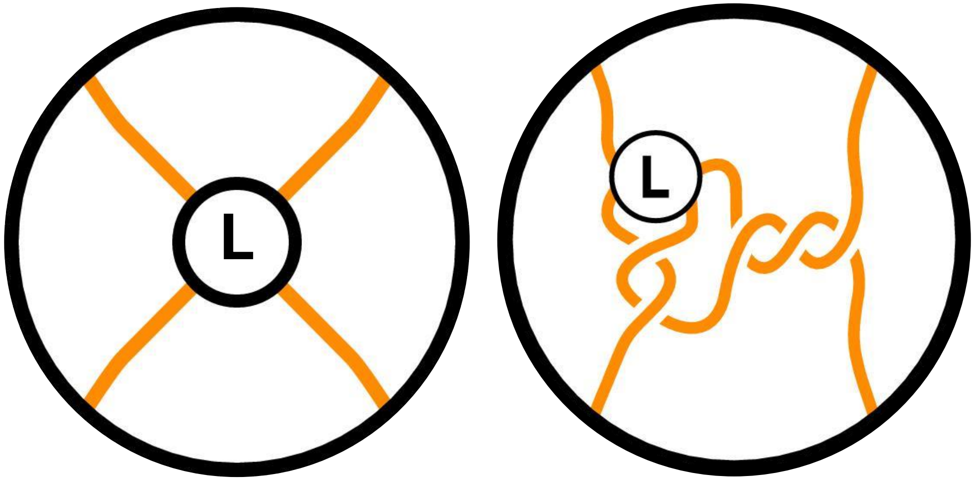
Definition. Any two 2-tangles are equivalent if one can be made from the other using Reidemeister moves.



2-String Tangles

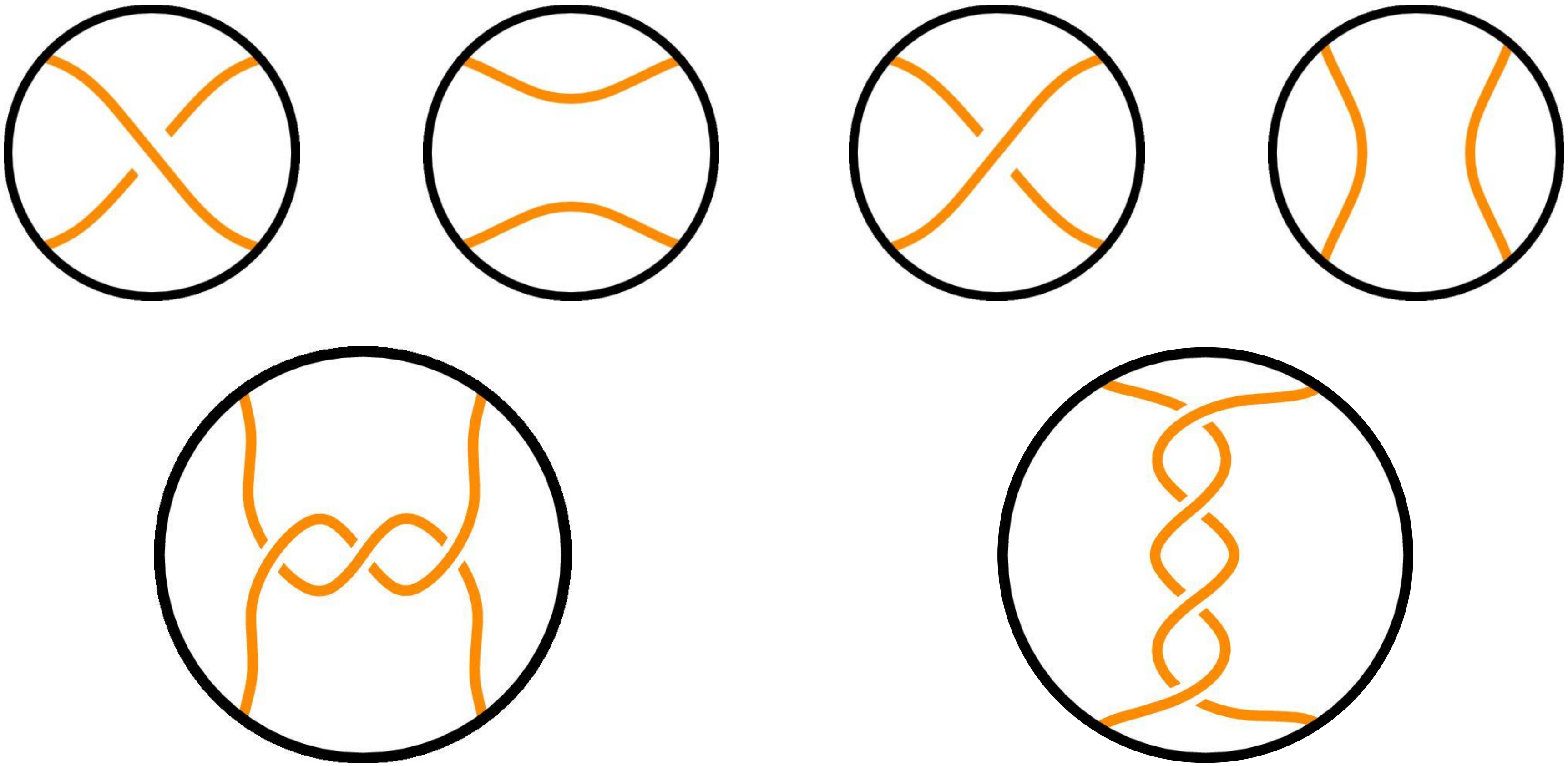


No local knots

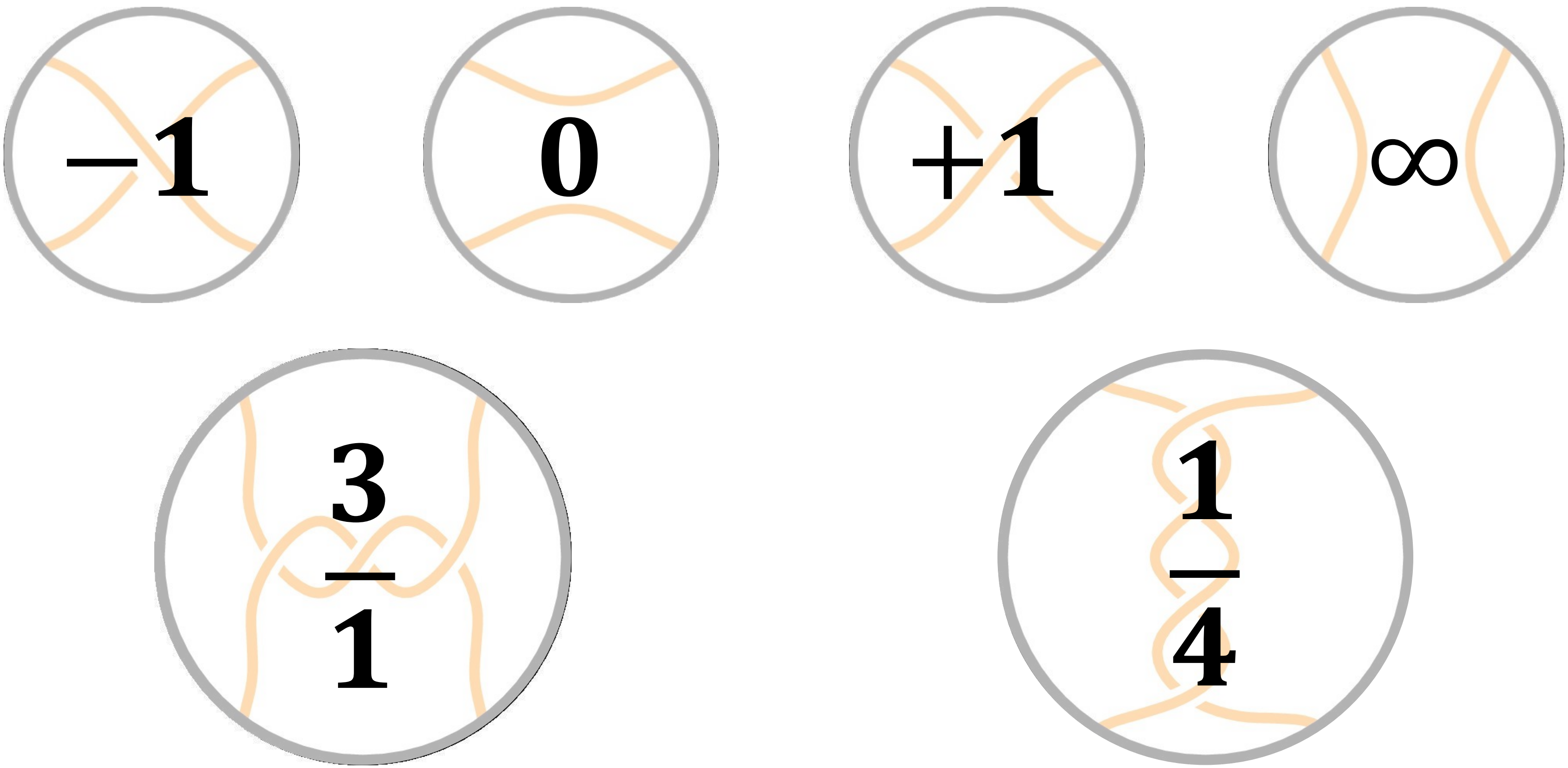


Consider both free
and fixed endpoints

2-String Tangles

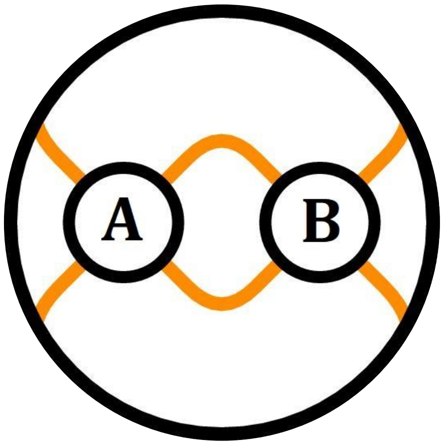


2-String Tangles

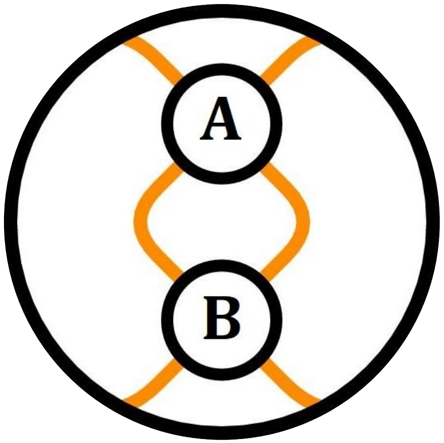


2-String Tangles

Horizontal Sum. Given two tangle A and B their horizontal sum $A + B$ is

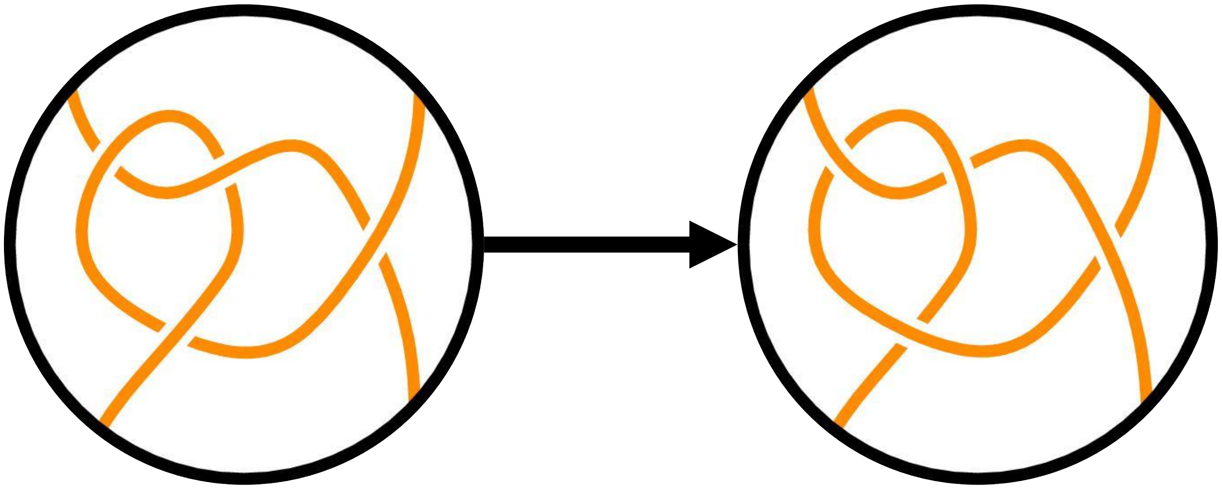


Vertical Sum. Given two tangle A and B their vertical sum $A \vee B$ is



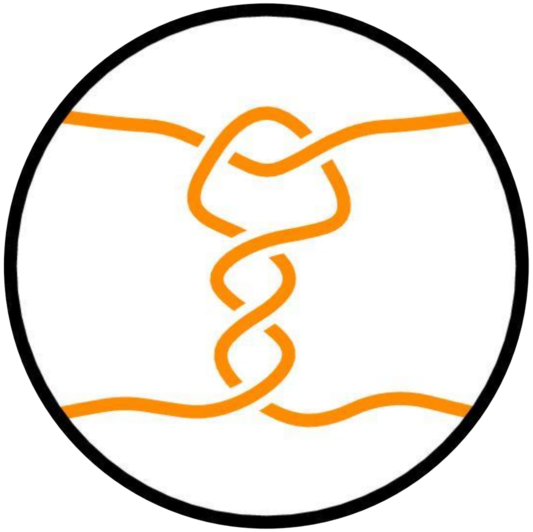
2-String Tangles

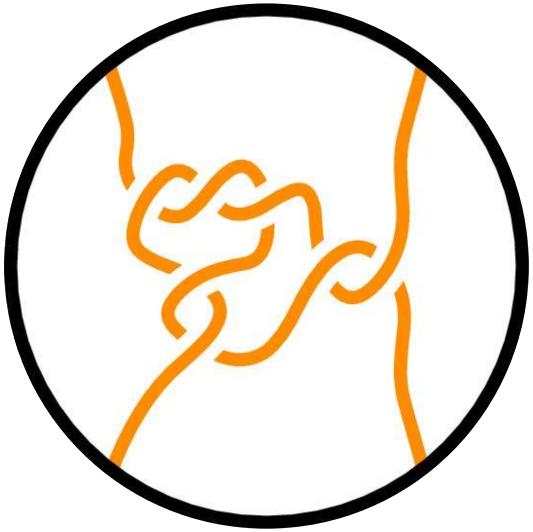
Mirror Image. Given a tangle L, the mirror image of L, denoted $-L$, is obtained be swapping all the crossings.



L $\qquad\qquad$ $-L$

Conway Notation for Rational Tangles

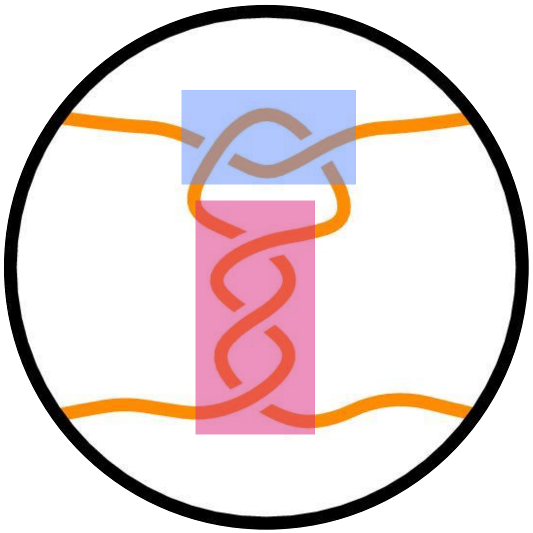


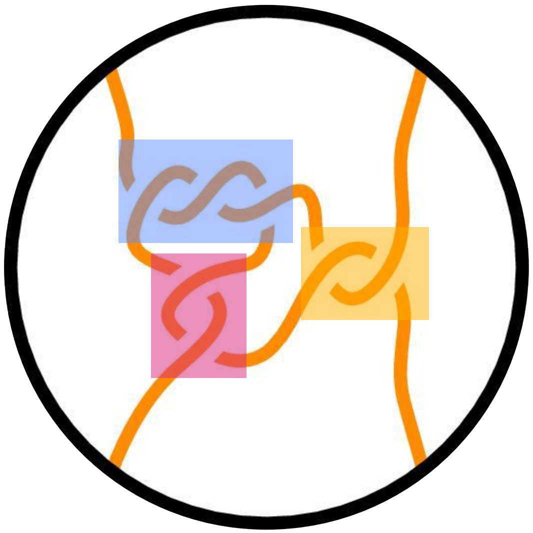


$$[2,3,0] = \frac{2}{1} \vee \frac{1}{3}$$

$$[3,2,2] = \frac{3}{1} \vee \frac{1}{2} + \frac{2}{1}$$

Conway Notation for Rational Tangles

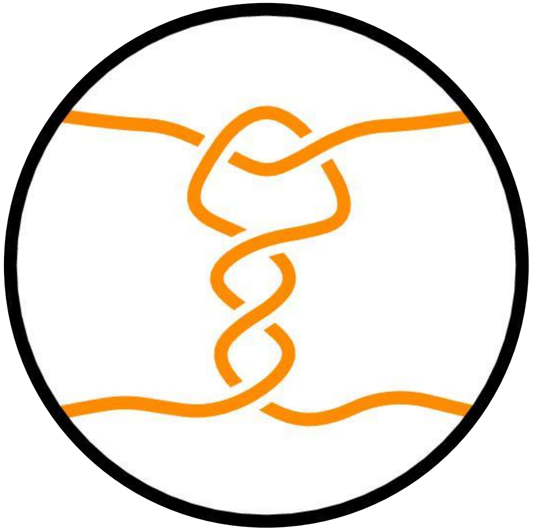




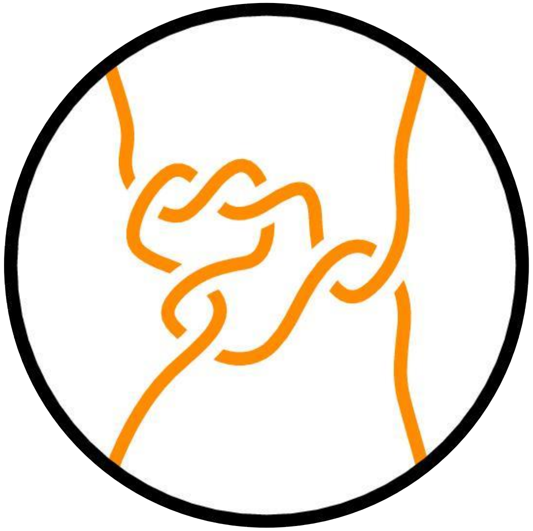
$$[2,3,0] = \frac{2}{1} \vee \frac{1}{3} + \frac{0}{1}$$

$$[3,2,2] = \frac{3}{1} \vee \frac{1}{2} + \frac{2}{1}$$

Continued Fraction for a Rational Tangle



$\neq$



$[2, 3, 0]$

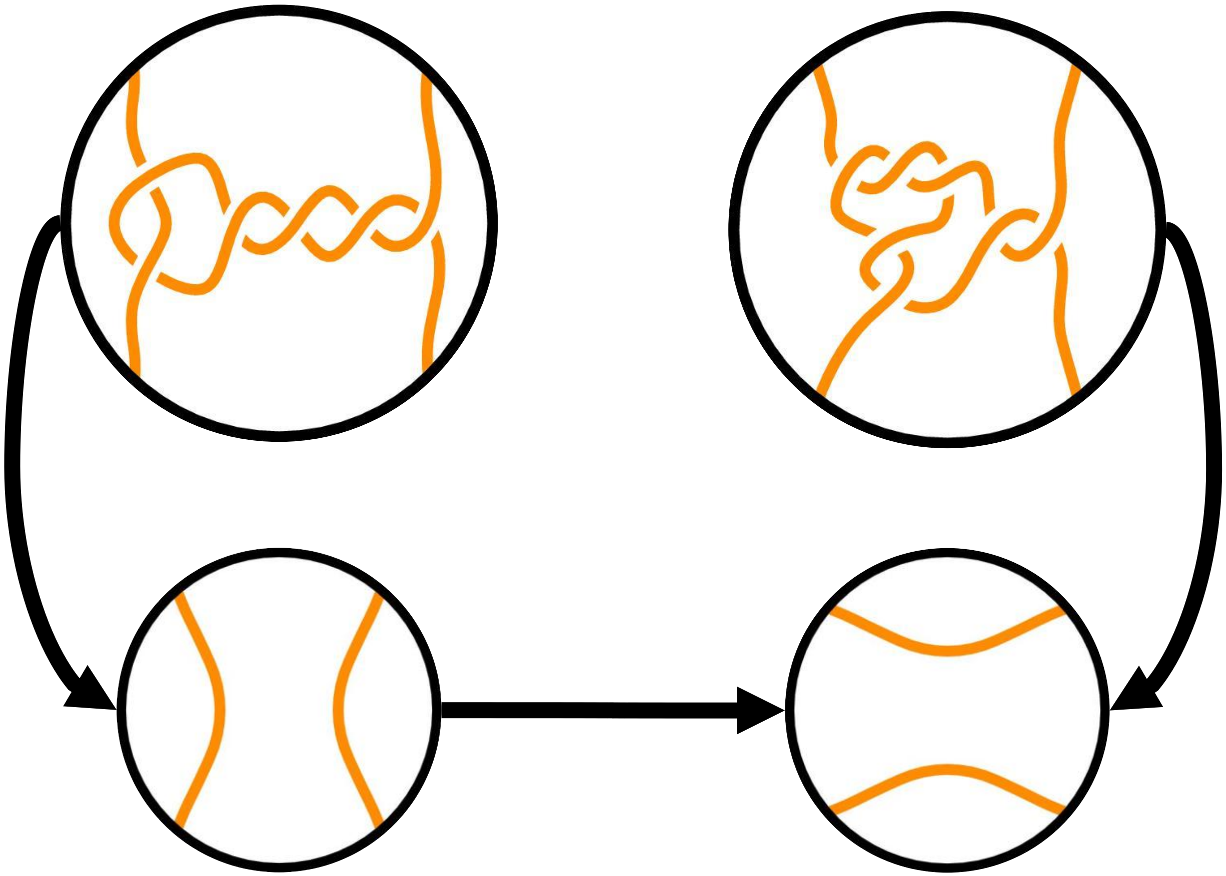
$[3, 2, 2]$

$$0 + \cfrac{1}{3 + \cfrac{1}{2}} = \frac{2}{7}$$

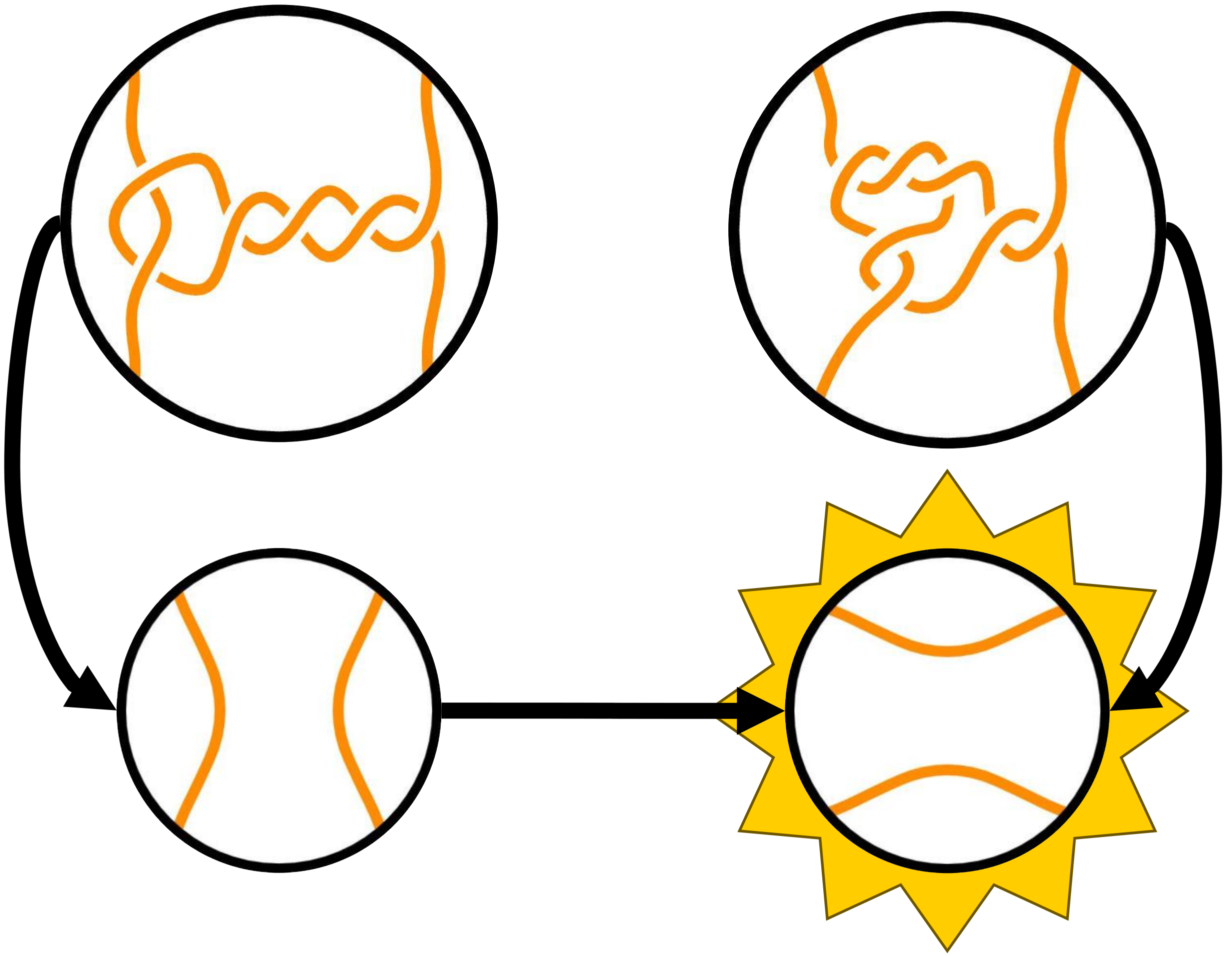
$$2 + \cfrac{1}{2 + \cfrac{1}{3}} = \frac{17}{7}$$

Counting and Generating Algebraic Tangles

Free Boundary; The Only Rational Tangle



Free Boundary; The Only Rational Tangle



Fixed Boundary; Rational Tangles

Definition. Conway notation $[\boldsymbol{x_1}, \dots, \boldsymbol{x_k}]$ for a rational tangle is in minimal length canonical form if

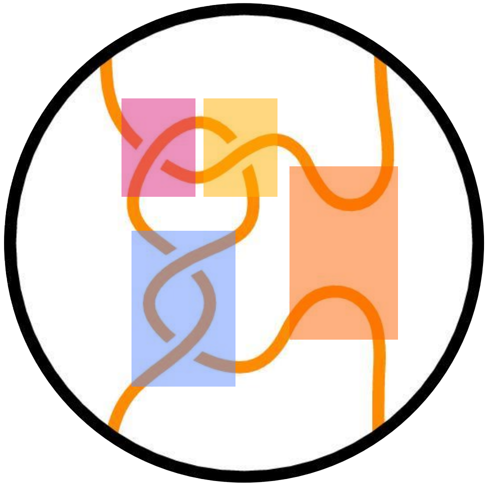
- $\boldsymbol{k}$ is minimal (implies $|\boldsymbol{x_1}| \geq \mathbf{2}$)
- $\boldsymbol{x_i}$ are all positive or negative except possibly $\boldsymbol{x_k} = \mathbf{0}$

There are odd length and even length canonical forms.

$$[\mathbf{1}, \boldsymbol{x_1} - \mathbf{1}, \dots, \boldsymbol{x_k}] = [\boldsymbol{x_1}, \dots, \boldsymbol{x_k}]$$

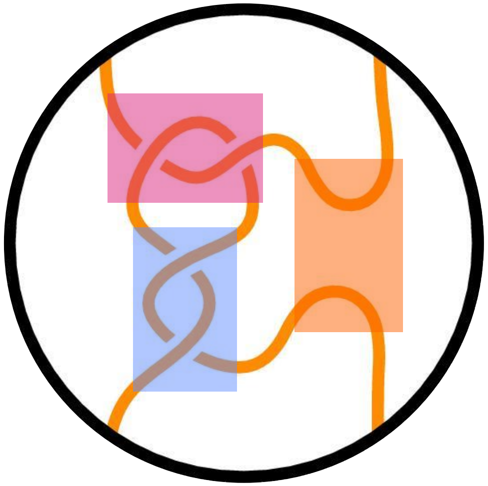
Fixed Boundary; Rational Tangles

$$[1, x_1 - 1, \ldots, x_k] = [x_1, \ldots, x_k]$$



$[1, 1, 2, 0]$

$$\frac{1}{1} + \frac{1}{1} \vee \frac{1}{2} + \frac{0}{1}$$



$[2, 2, 0]$

$$\frac{2}{1} \vee \frac{1}{2} + \frac{0}{1}$$

Fixed Boundary; Rational Tangles

Definition. Conway notation $[\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}]$ for a rational tangle is in minimal length canonical form if

- $\boldsymbol{k}$ is minimal (implies $|\boldsymbol{x_1}| \geq \mathbf{2}$)
- $\boldsymbol{x_i}$ are all positive or negative except possibly $\boldsymbol{x_k} = \mathbf{0}$

There are odd length and even length canonical forms.

$$[\mathbf{1}, \boldsymbol{x_1} - \mathbf{1}, \ldots, \boldsymbol{x_k}] = [\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}]$$

Fixed Boundary; Rational Tangles

Definition. A $\boldsymbol{k}$-composition of a natural number $\boldsymbol{N}$ is a sequence of $\boldsymbol{k}$ natural numbers $(\boldsymbol{x_1}, \dots, \boldsymbol{x_k})$ whose sum is $\boldsymbol{N}$ $(\boldsymbol{x_1} + \cdots + \boldsymbol{x_k} = \boldsymbol{N})$.

$$\mathbf{1 + 2 + 4 + 3 = 10}$$
$$\mathbf{(1, 2, 4, 3)}$$

$$\mathbf{2 + 2 + 1 + 3 + 2 = 10}$$
$$\mathbf{(2, 2, 1, 3, 2)}$$

Note. Non-trivially permuting the terms of a composition creates a new composition.

$$\mathbf{(1, 2, 4, 3) \neq (2, 4, 1, 3)}$$

Fixed Boundary; Rational Tangles

Theorem [S. Heubach, T. Mansour]. For natural numbers $\boldsymbol{N}$ and $\boldsymbol{k}$, there are $\binom{\boldsymbol{N}-\mathbf{1}}{\boldsymbol{k}-\mathbf{1}}$ many $\boldsymbol{k}$-compositions of $\boldsymbol{N}$.

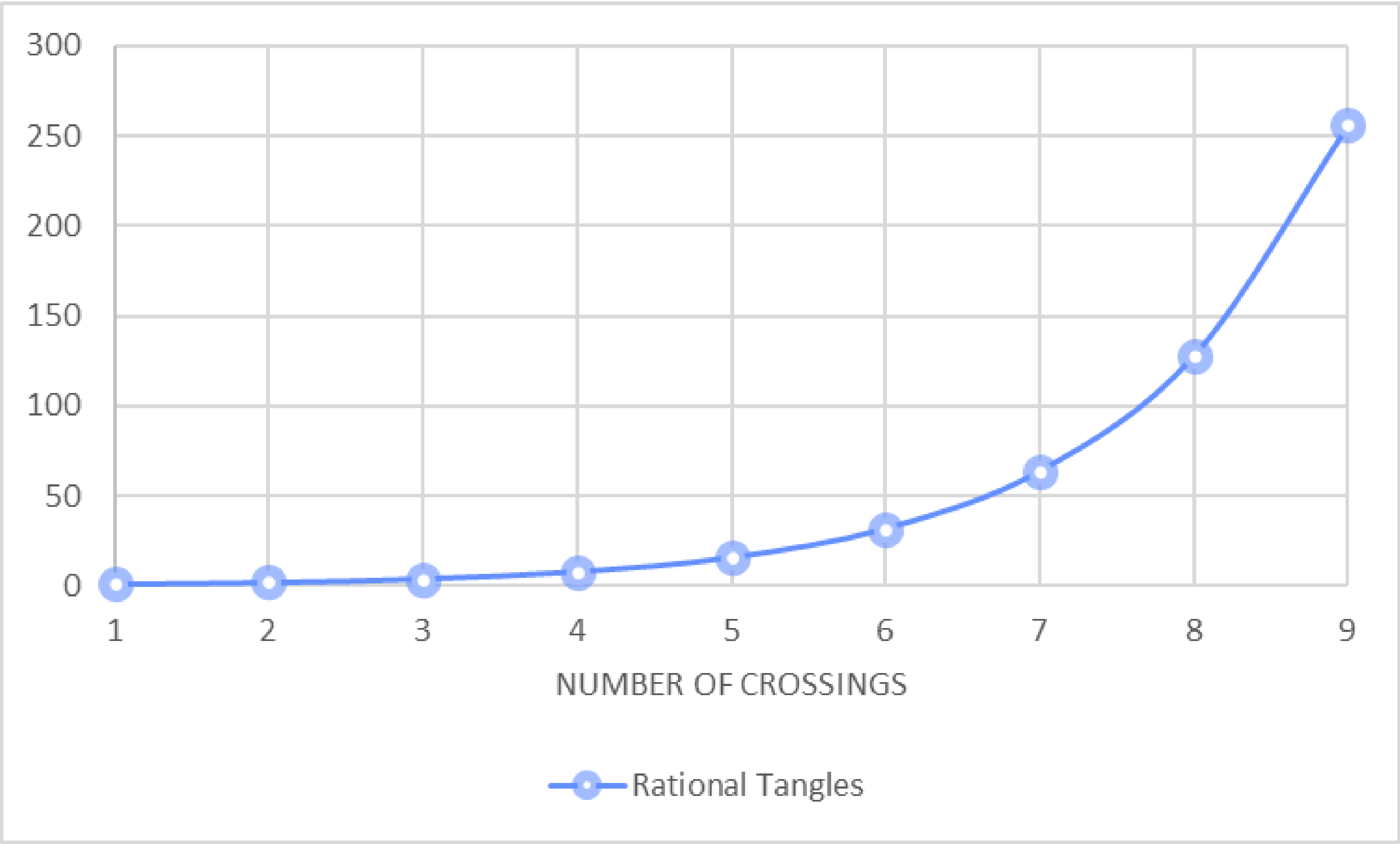
Theorem [S. Heubach, T. Mansour]. There are $\mathbf{2}^{\boldsymbol{N}-\mathbf{1}}$ total compositions of a natural number $\boldsymbol{N}$.

Fixed Boundary; Rational Tangles

Theorem. There are 2^{N-1} unique rational tangles $\frac{a}{b} > 0$ with N crossings.

Number of Crossings	1	2	3	4	5	6	7	8
Unique Rational Tangles	1	2	4	8	16	32	64	128

Fixed Boundary; Rational Tangles



Fixed Boundary; Rational Tangles

$$[1, x_1 - 1, \dots, x_k] = [x_1, \dots, x_k]$$

All 16 Compositions of 5			
(5)	$(4,1)$	$(3,2)$	$(3,1,1)$
$(2,3)$	$(2,2,1)$	$(2,1,2)$	$(2,1,1,1)$
$(1,4)$	$(1,3,1)$	$(1,2,2)$	$(1,2,1,1)$
$(1,1,3)$	$(1,1,2,1)$	$(1,1,1,2)$	$(1,1,1,1,1)$

Fixed Boundary; Rational Tangles

All 16 Unique Rational Tangles With 5 Crossings			
$[5]$	$[4, 1]$	$[3, 2]$	$[3, 1, 1]$
$[2, 3]$	$[2, 2, 1]$	$[2, 1, 2]$	$[2, 1, 1, 1]$
$[5, 0]$	$[4, 1, 0]$	$[3, 2, 0]$	$[3, 1, 1, 0]$
$[2, 3, 0]$	$[2, 2, 1, 0]$	$[2, 1, 2, 0]$	$[2, 1, 1, 1, 0]$

Montesinos Tangles

Definition. A Montesinos tangle is a horizontal sum of at least two rational tangles which are not the ∞-tangle.

Free Boundary; Montesinos Tangles

- A Montesinos tangle with free boundary in canonical form is expressed as a sum $M = \frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}$ where $0 < a_i < b_i$ for all i.

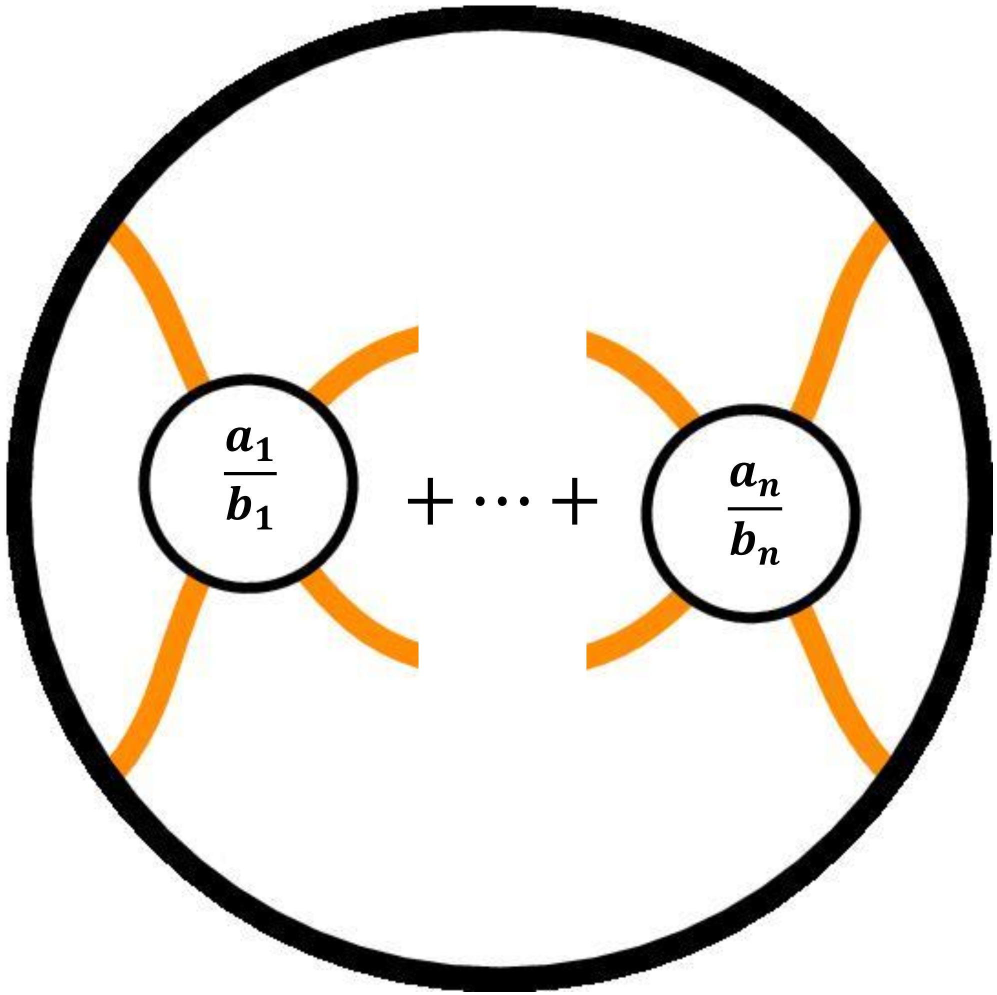
$$M = \frac{1}{3} + \frac{2}{3} + \frac{2}{5}$$

$$[3, 0] + [2, 1, 0] + [2, 2, 0]$$

Free Boundary; Montesinos Tangles

If M has
$N = c_1 + \cdots + c_n$
crossings

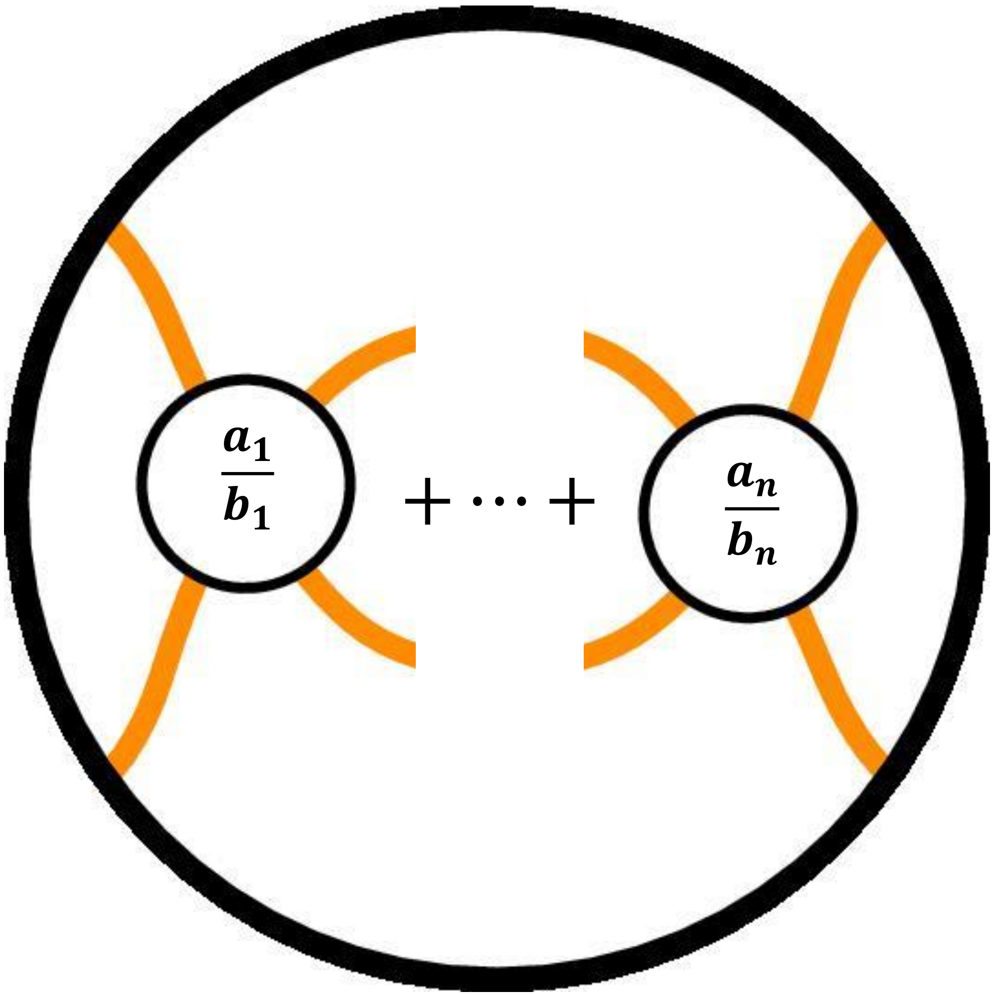
where $\frac{a_i}{b_i}$ has c_i
crossings



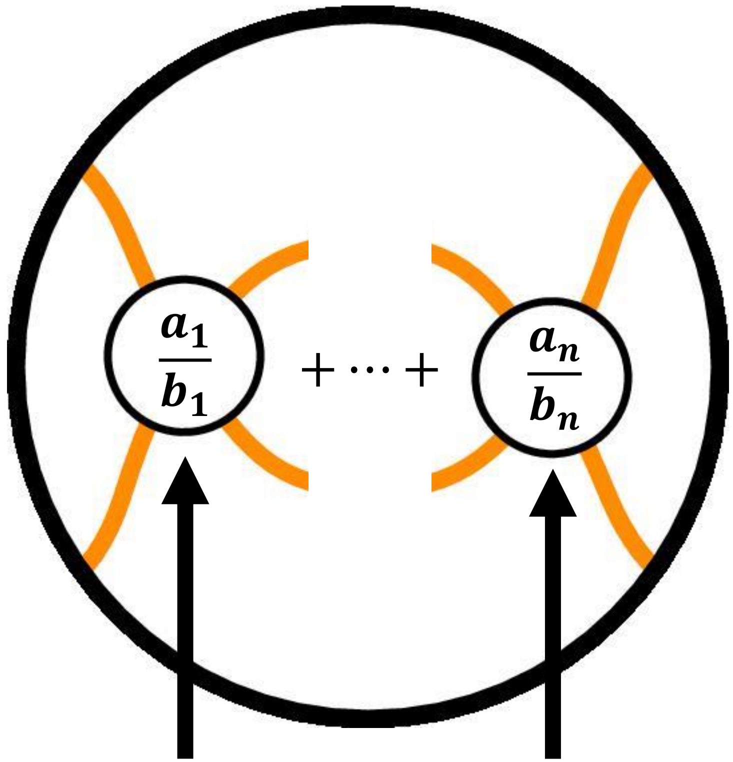
Free Boundary; Montesinos Tangles

There are $2^{c_i - 1}$ rational tangles with c_i crossings

Only half ($2^{c_i - 2}$) end with vertical twists



Free Boundary; Montesinos Tangles



$$2^{c_1-2} \ldots 2^{c_n-2}$$

Free Boundary; Montesinos Tangles

Theorem [R. Stanley]. For a natural number $\boldsymbol{N}$, there are $\boldsymbol{F_{N-1}}$ compositions of $\boldsymbol{N}$ into parts greater than 1 where $\boldsymbol{F_{N-1}}$ is the $(\boldsymbol{N-1})^{\boldsymbol{th}}$ term in the Fibonacci sequence.

Fibonacci Sequence						
F_0	F_1	F_2	F_3	F_4	F_5	F_6
0	1	1	2	3	5	8

Free Boundary; Montesinos Tangles

Theorem [B]. The number of unique Montesinos tangles with free boundary and N crossings is given by the sum

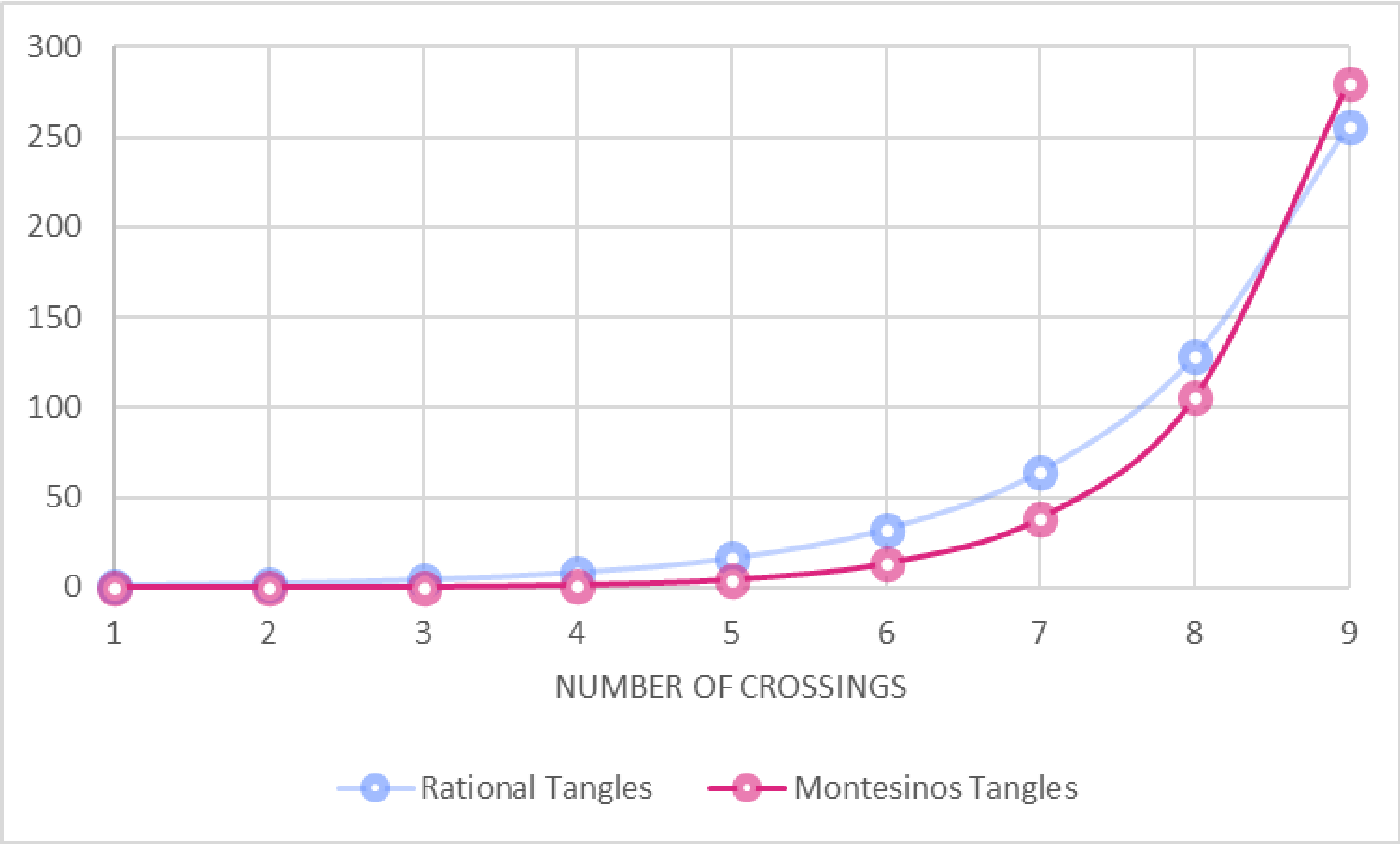
$$\sum_{C_j=\left(c_1,\ldots,c_{n_j}\right)} \left(\prod_{i=1}^{n_j} 2^{c_i-2}\right)$$

over all $F_{N-1}-1$ compositions C_j of $N \geq 4$ into at least two parts greater than 1.

Free Boundary; Montesinos Tangles

Number of Crossings	4	5	6	7	8	9
Unique Montesinos Tangles	1	4	13	38	105	280

Free Boundary; Montesinos Tangles



Free Boundary; Montesinos Tangles

- There are $F_5 = 5$ compositions of 6 into parts greater than 1.

- We only want compositions with at least 2 parts.

All Compositions of 6 Into Parts Greater Than 1				
(6)	$(4, 2)$	$(3, 3)$	$(2, 4)$	$(2, 2, 2)$

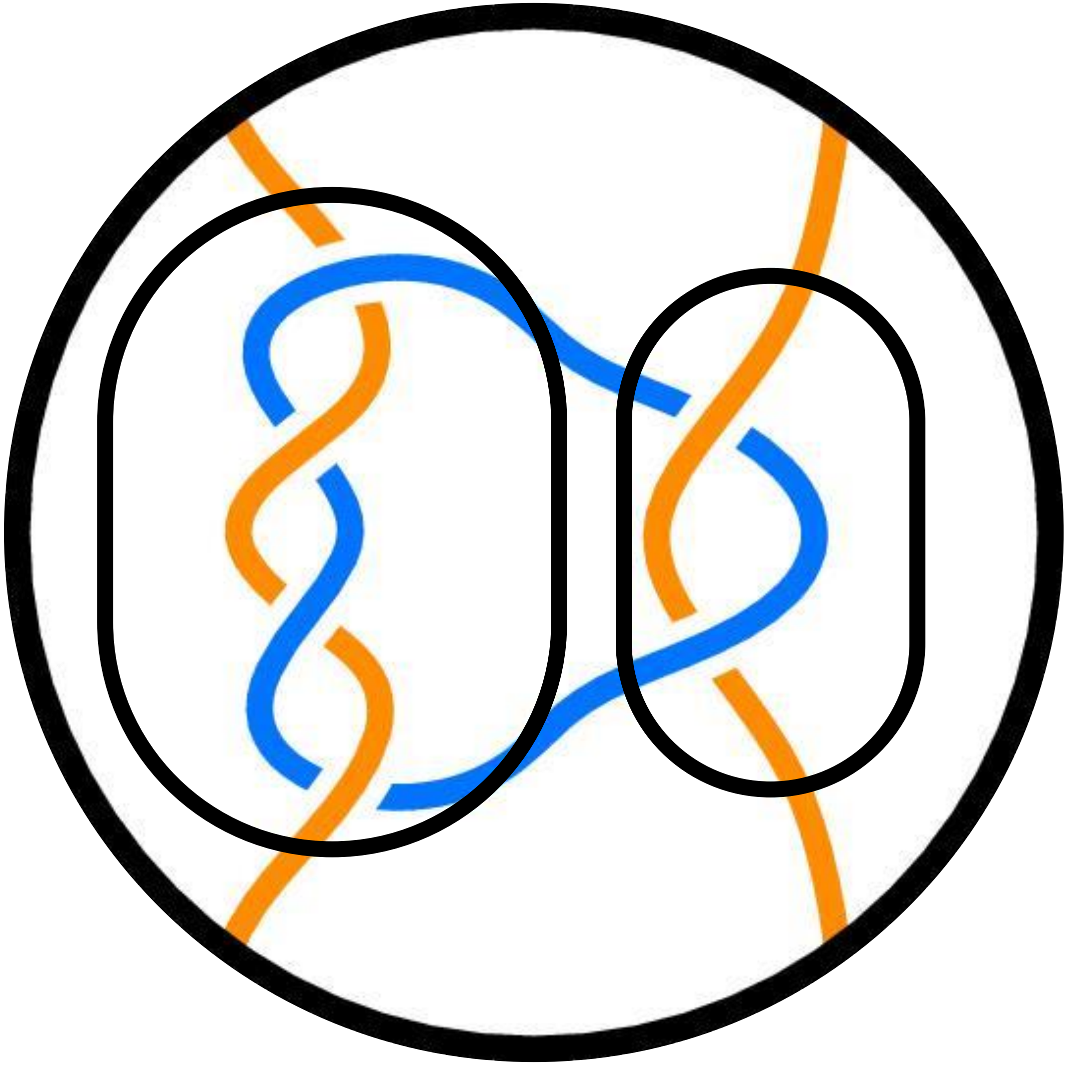
Free Boundary; Montesinos Tangles

$$\sum_{C_j=\left(c_1,\ldots,c_{n_j}\right)} \left(\prod_{i=1}^{n_j} 2^{c_i-2}\right)$$

$$C_1 = (4, 2)$$

$$2^2 \cdot 2^0 = 4$$

1 of 4



Free Boundary; Montesinos Tangles

$$\sum_{C_j=(c_1,\ldots,c_{n_j})} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$

$$C_1 = (4, 2)$$

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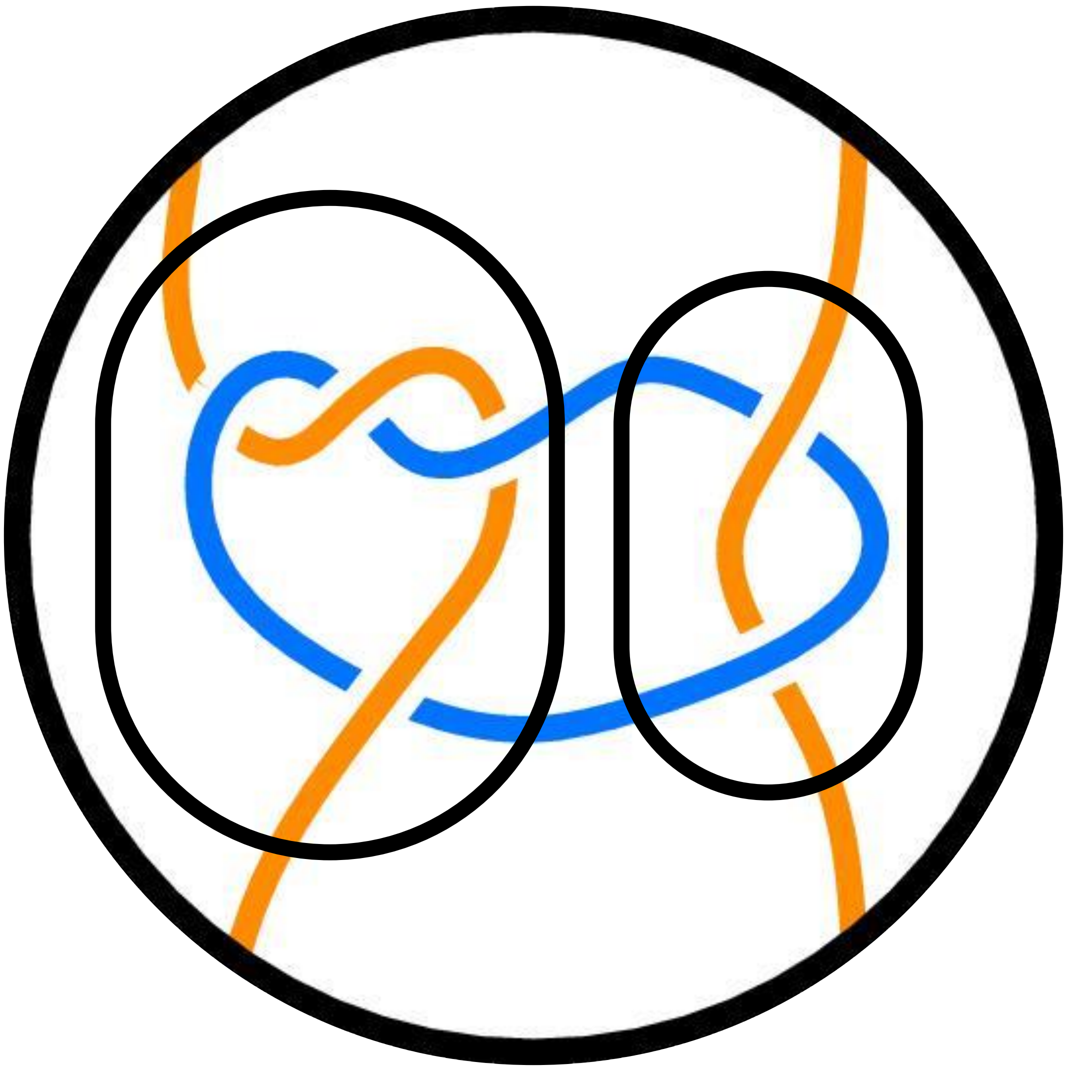
2 of 4



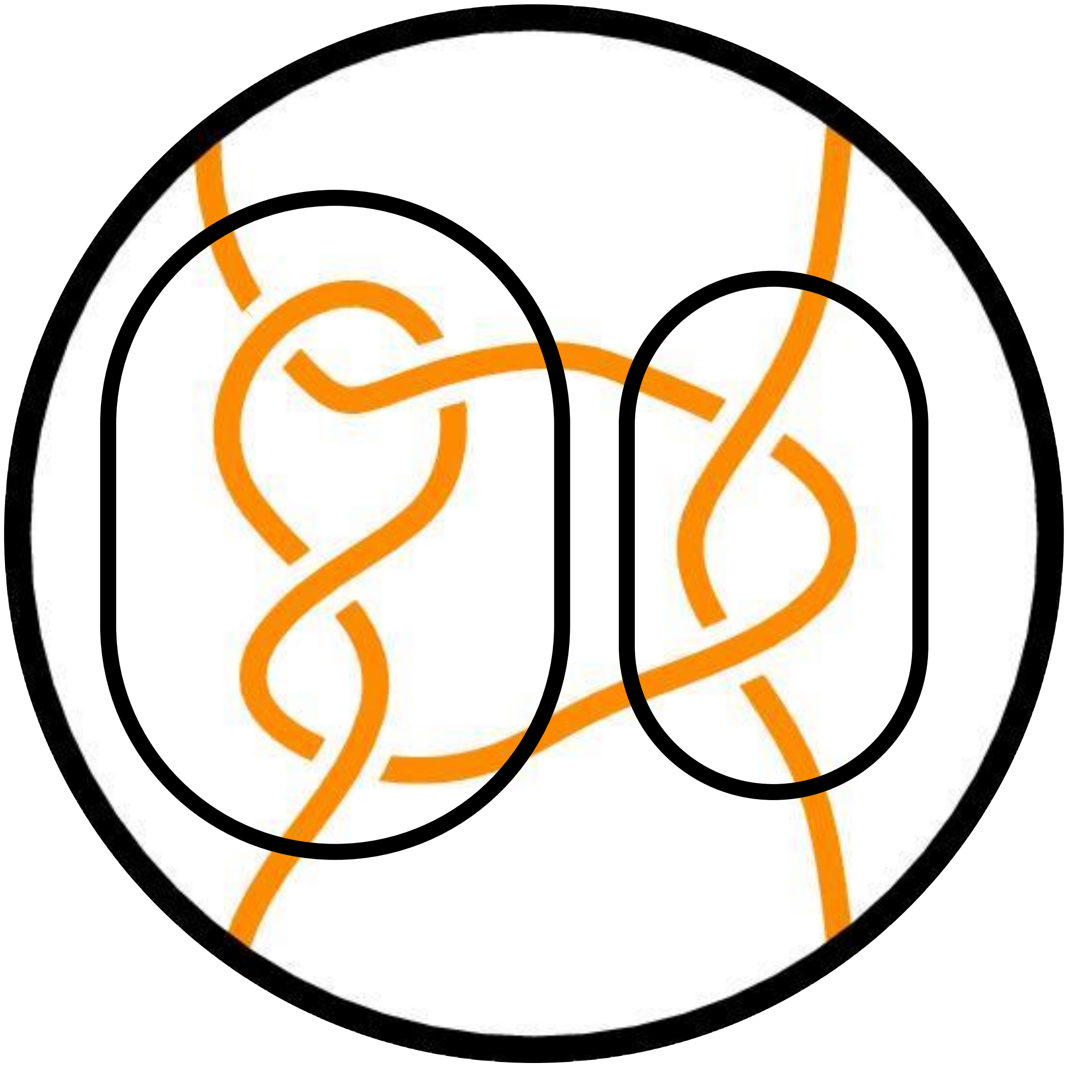
Free Boundary; Montesinos Tangles

$$\sum_{C_j=\left(c_1,\dots,c_{n_j}\right)} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$

$$C_1 = (4, 2)$$

$$2^2 \cdot 2^0 = 4$$

3 of 4



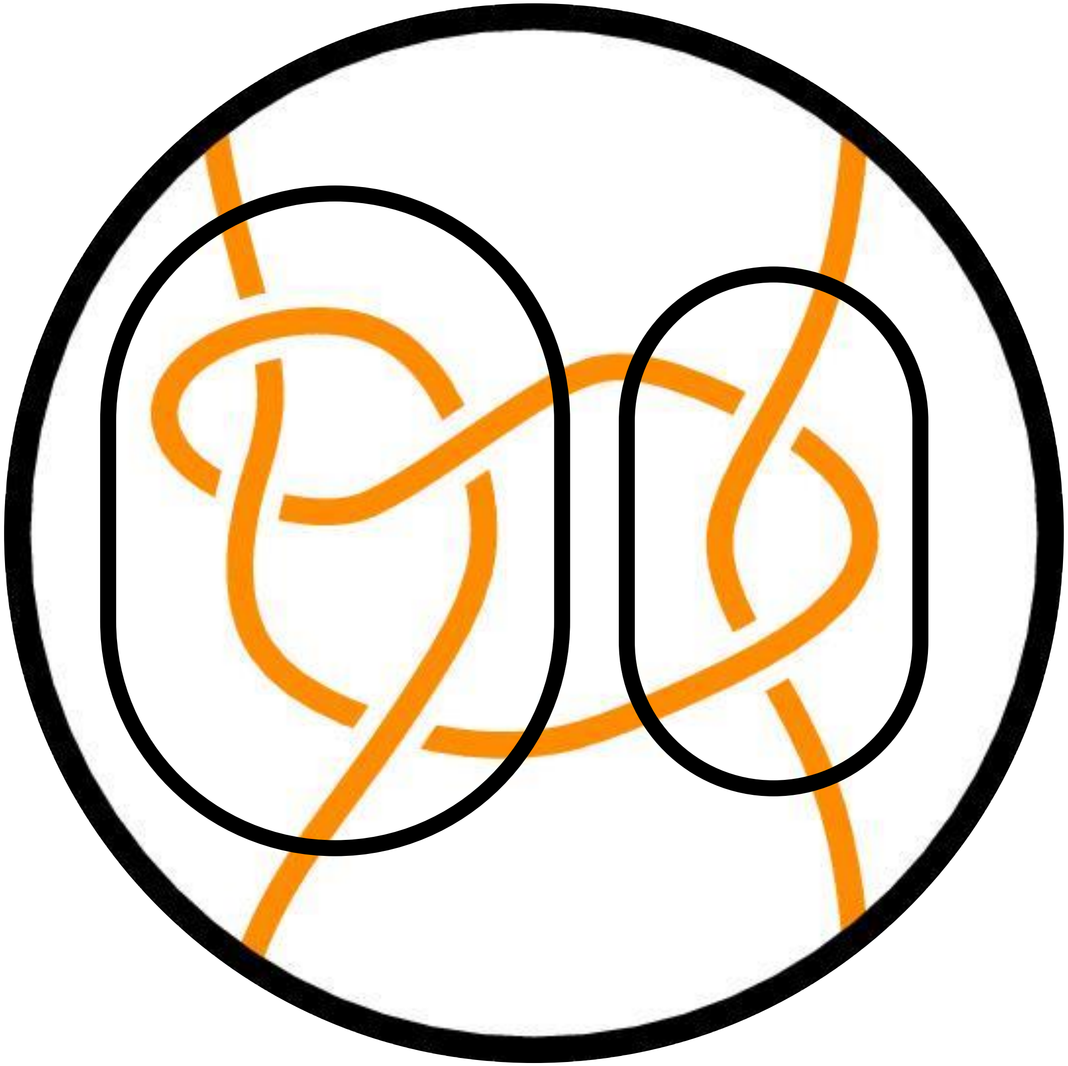
Free Boundary; Montesinos Tangles

$$\sum_{C_j=\left(c_1,\ldots,c_{n_j}\right)} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$

$$C_1 = (4, 2)$$

$$2^2 \cdot 2^0 = 4$$

4 of 4



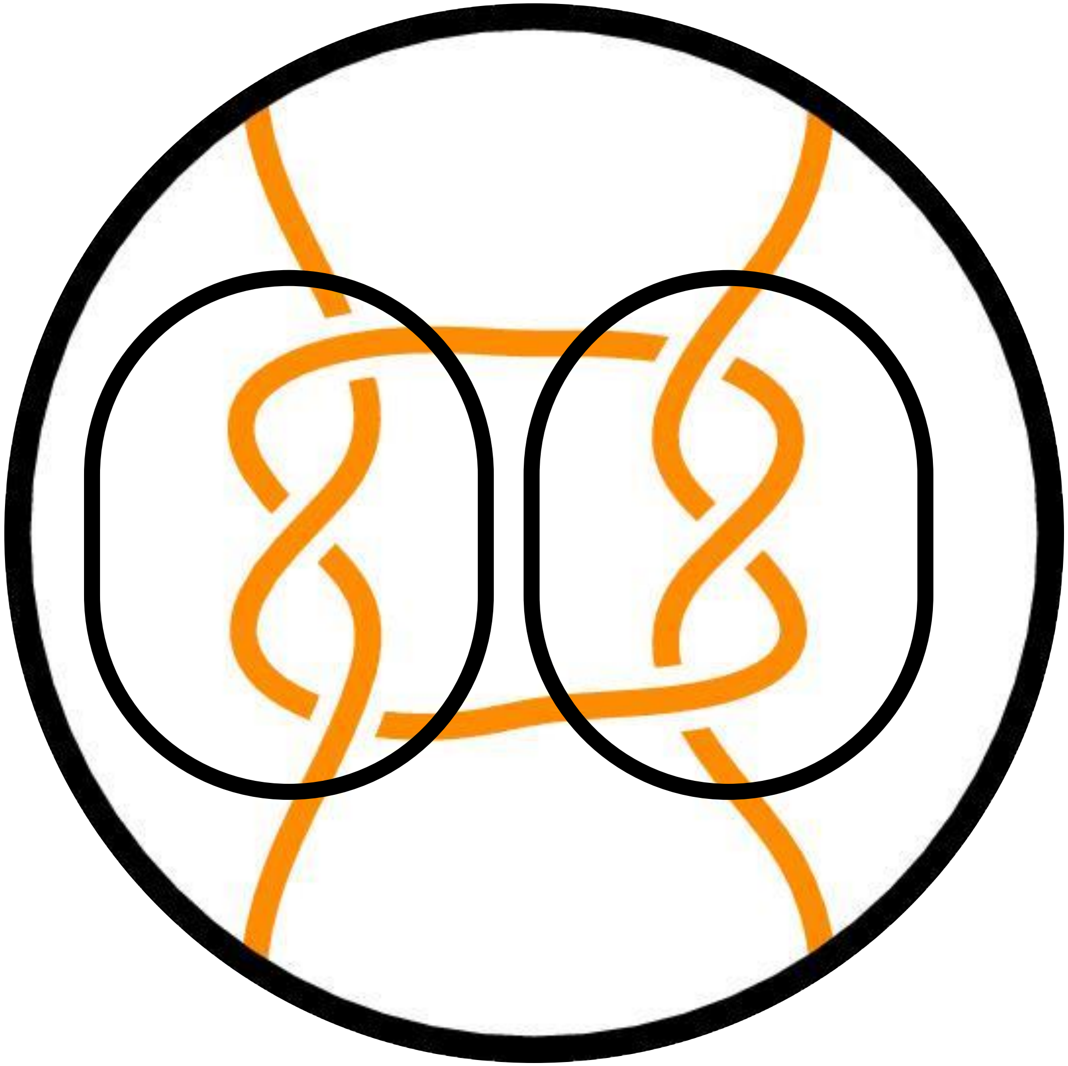
Free Boundary; Montesinos Tangles

$$\sum_{C_j=(c_1,\ldots,c_{n_j})} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$

$$C_2 = (3,3)$$

$$2^1 \cdot 2^1 = 4$$

1 of 4



Free Boundary; Montesinos Tangles

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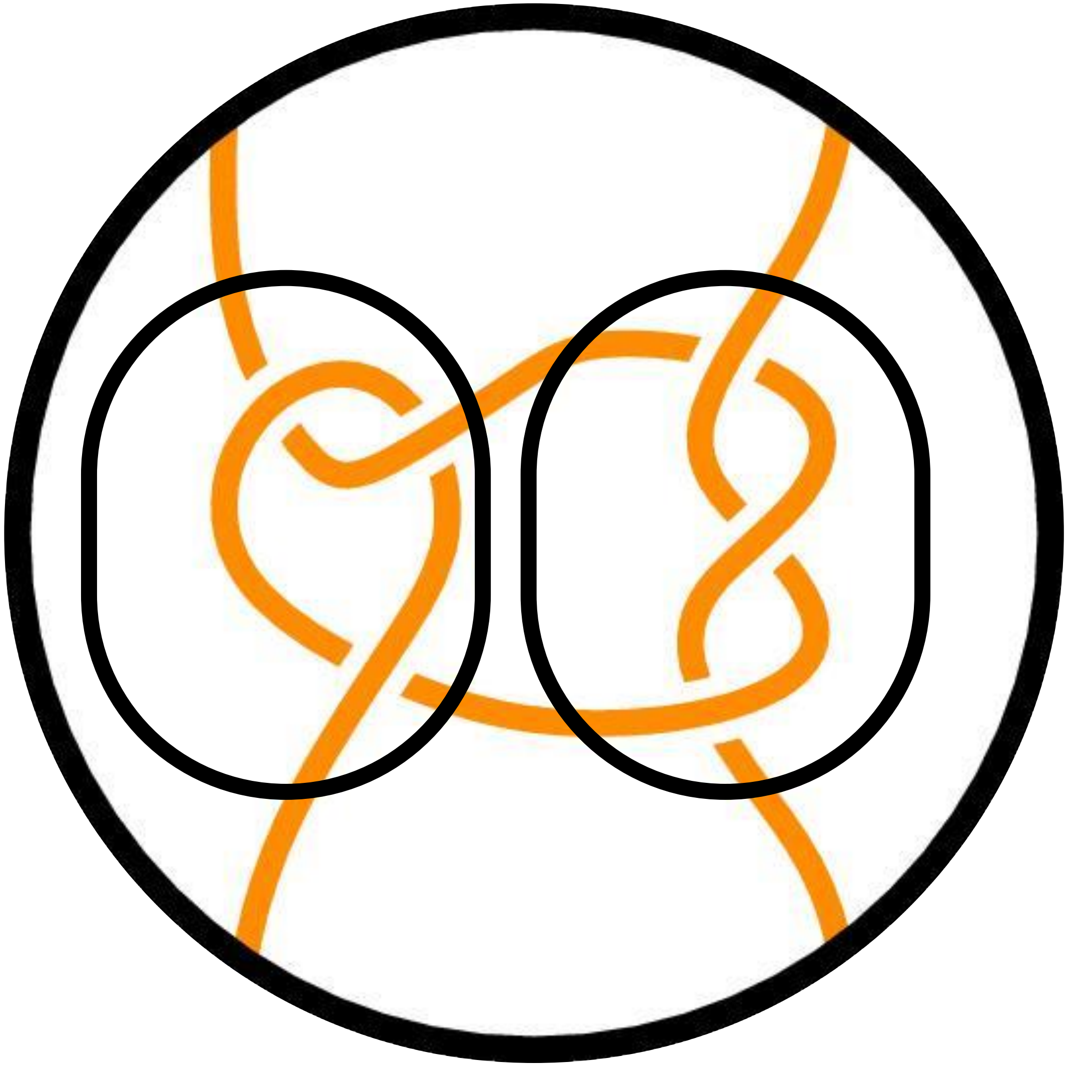
$$C_2 = (3,3)$$

$$2^1 \cdot 2^1 = 4$$

2 of 4



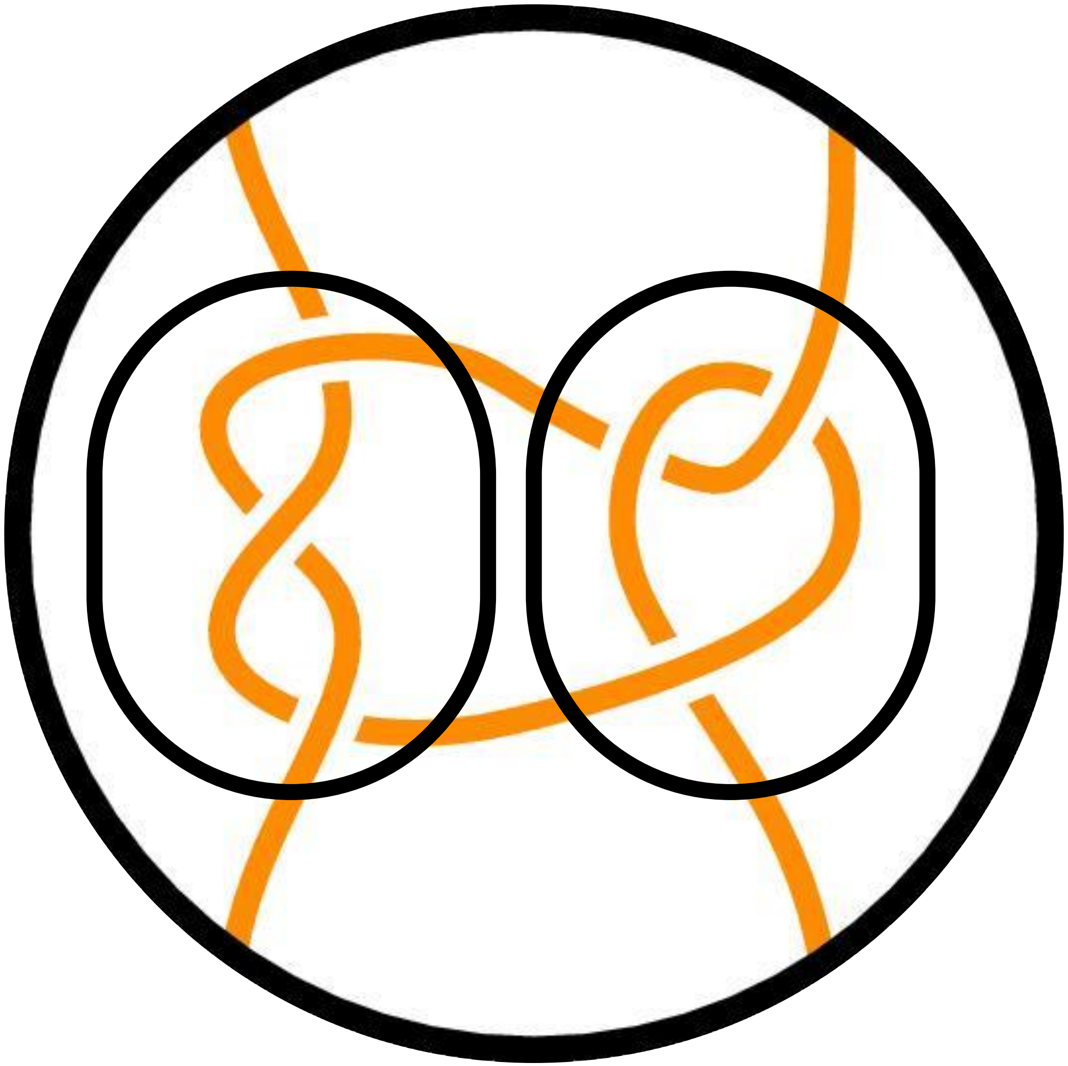
Free Boundary; Montesinos Tangles

$$\sum_{C_j=\left(c_1,\ldots,c_{n_j}\right)} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$

$$C_2 = (3, 3)$$

$$2^1 \cdot 2^1 = 4$$

3 of 4



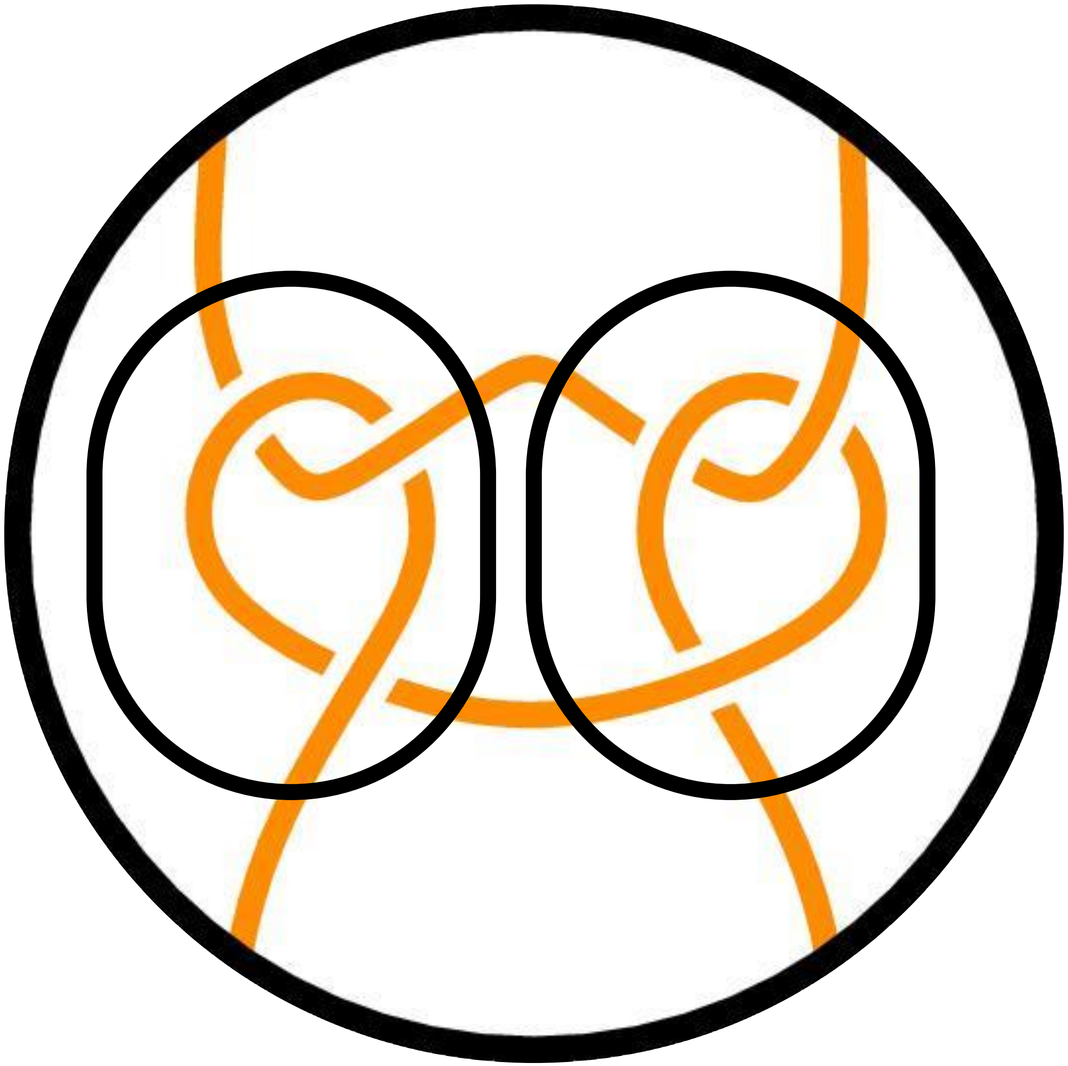
Free Boundary; Montesinos Tangles

$$\sum_{C_j=(c_1,\ldots,c_{n_j})} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$

$$C_2 = (3,3)$$

$$2^1 \cdot 2^1 = 4$$

4 of 4



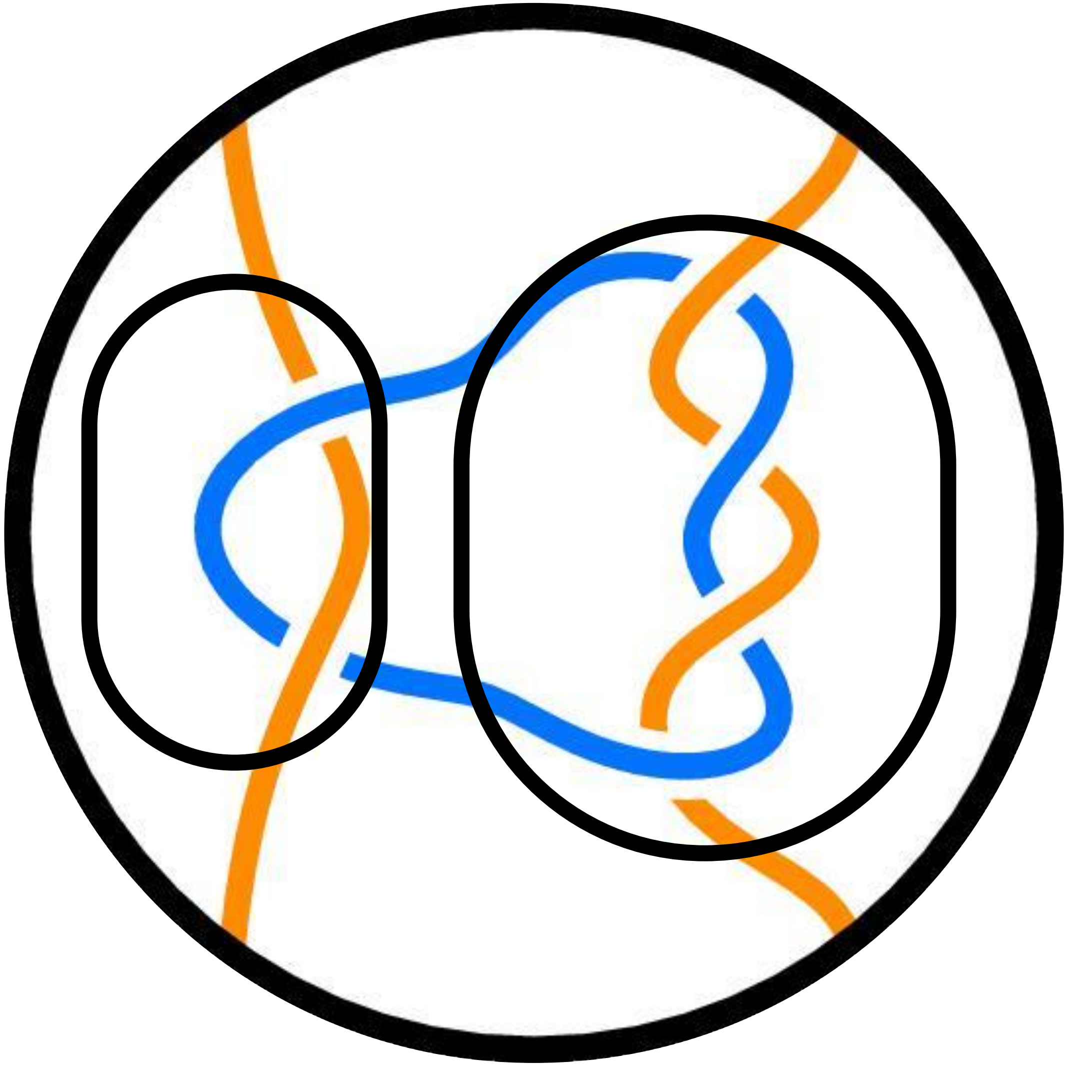
Free Boundary; Montesinos Tangles

$$\sum_{C_j=\left(c_1,\dots,c_{n_j}\right)} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$

$$C_3 = (2, 4)$$

$$2^0 \cdot 2^2 = 4$$

1 of 4



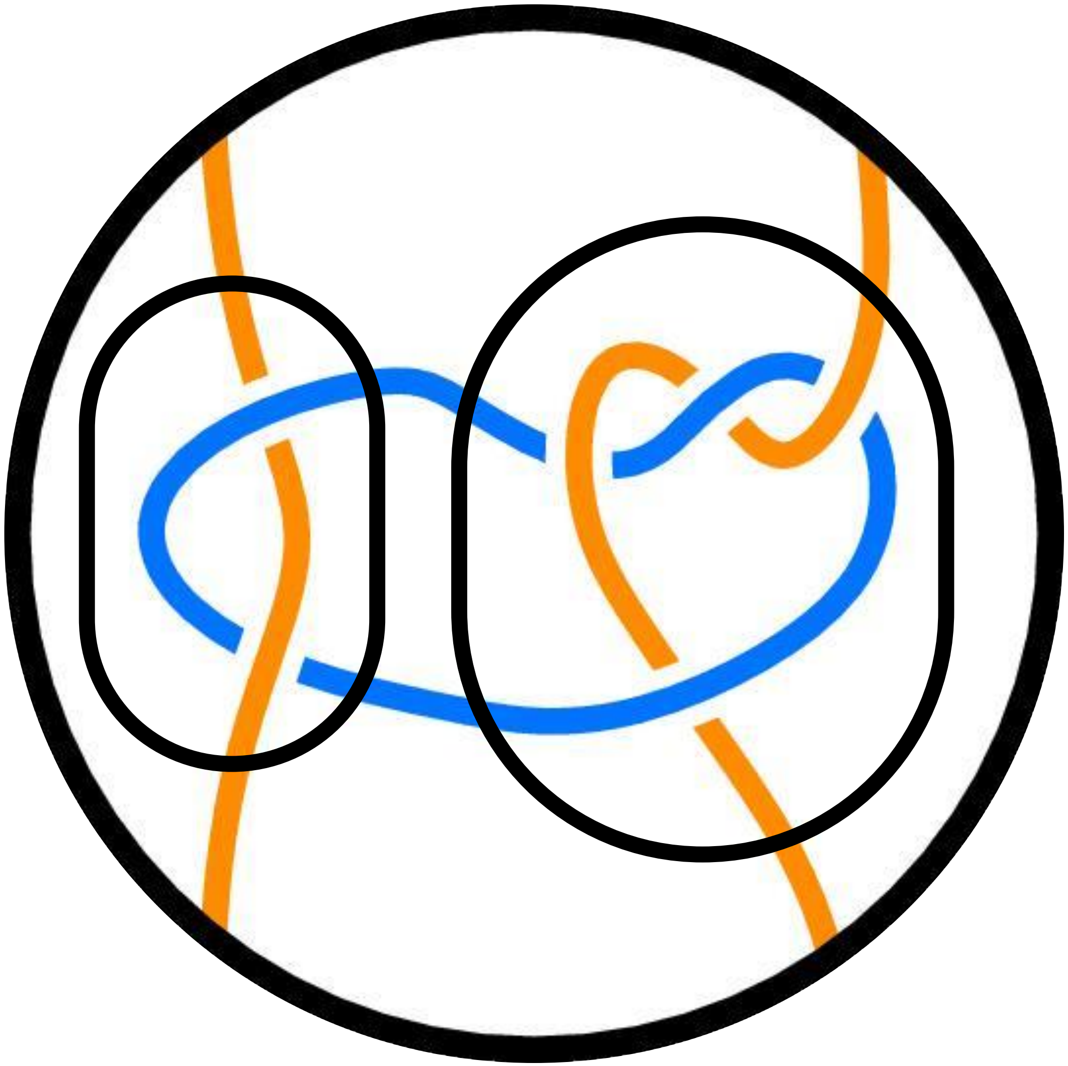
Free Boundary; Montesinos Tangles

$$\sum_{C_j=\left(c_1,\ldots,c_{n_j}\right)} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$

$$C_3 = (2, 4)$$

$$2^0 \cdot 2^2 = 4$$

2 of 4



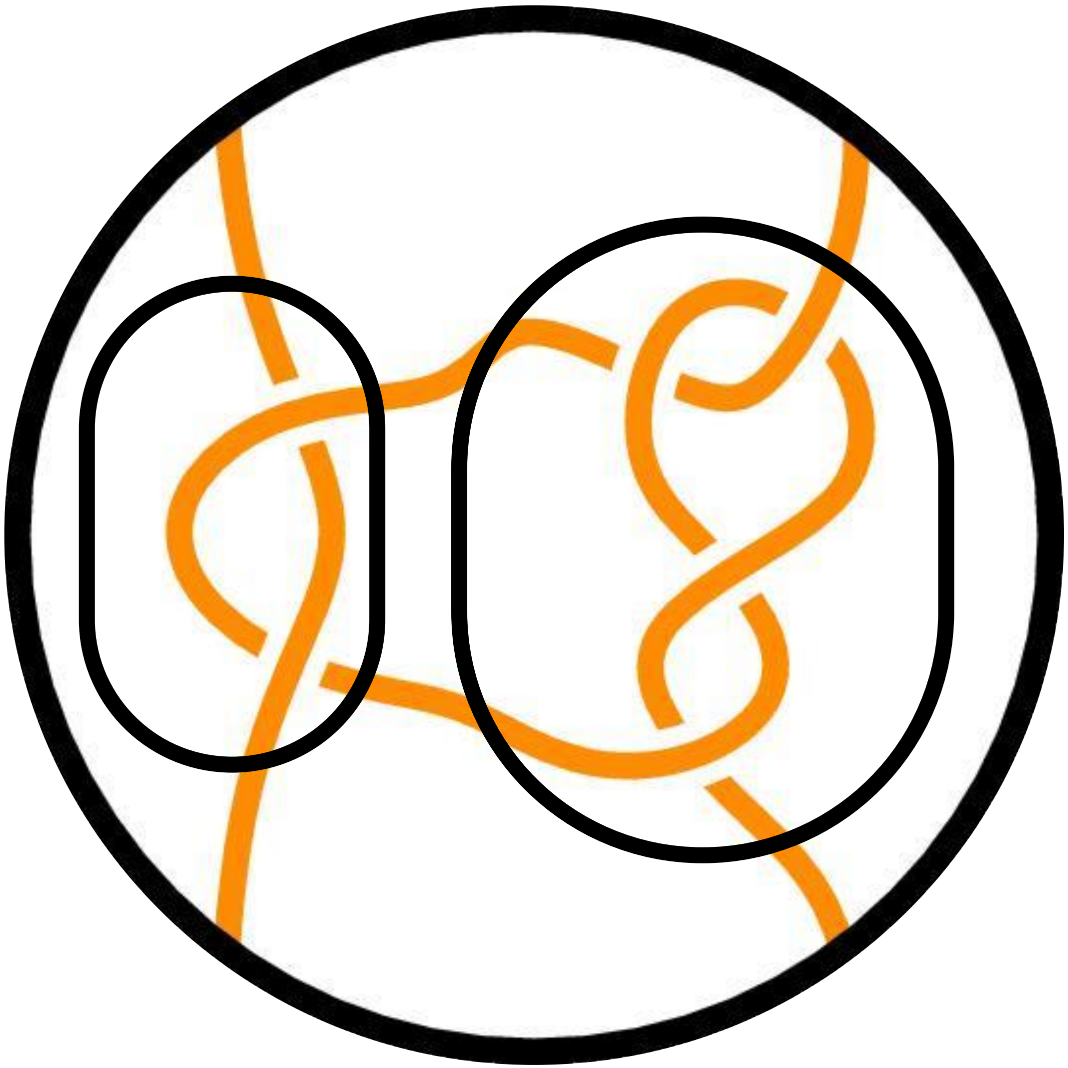
Free Boundary; Montesinos Tangles

$$\sum_{C_j=(c_1,\ldots,c_{n_j})} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$

$$C_3 = (2, 4)$$

$$2^0 \cdot 2^2 = 4$$

3 of 4



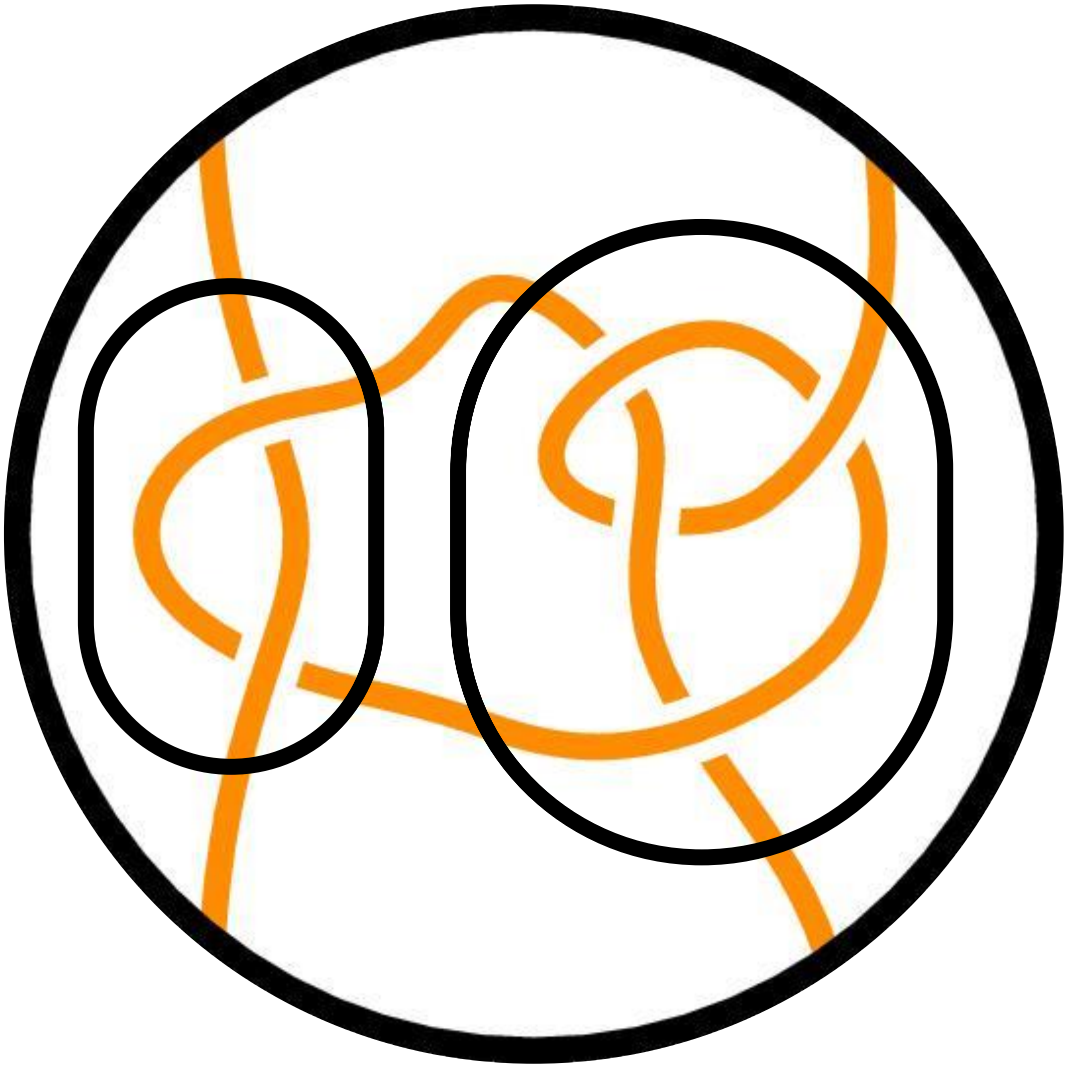
Free Boundary; Montesinos Tangles

$$\sum_{C_j=\left(c_1,\ldots,c_{n_j}\right)} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$

$$C_3 = (2, 4)$$

$$2^0 \cdot 2^2 = 4$$

4 of 4



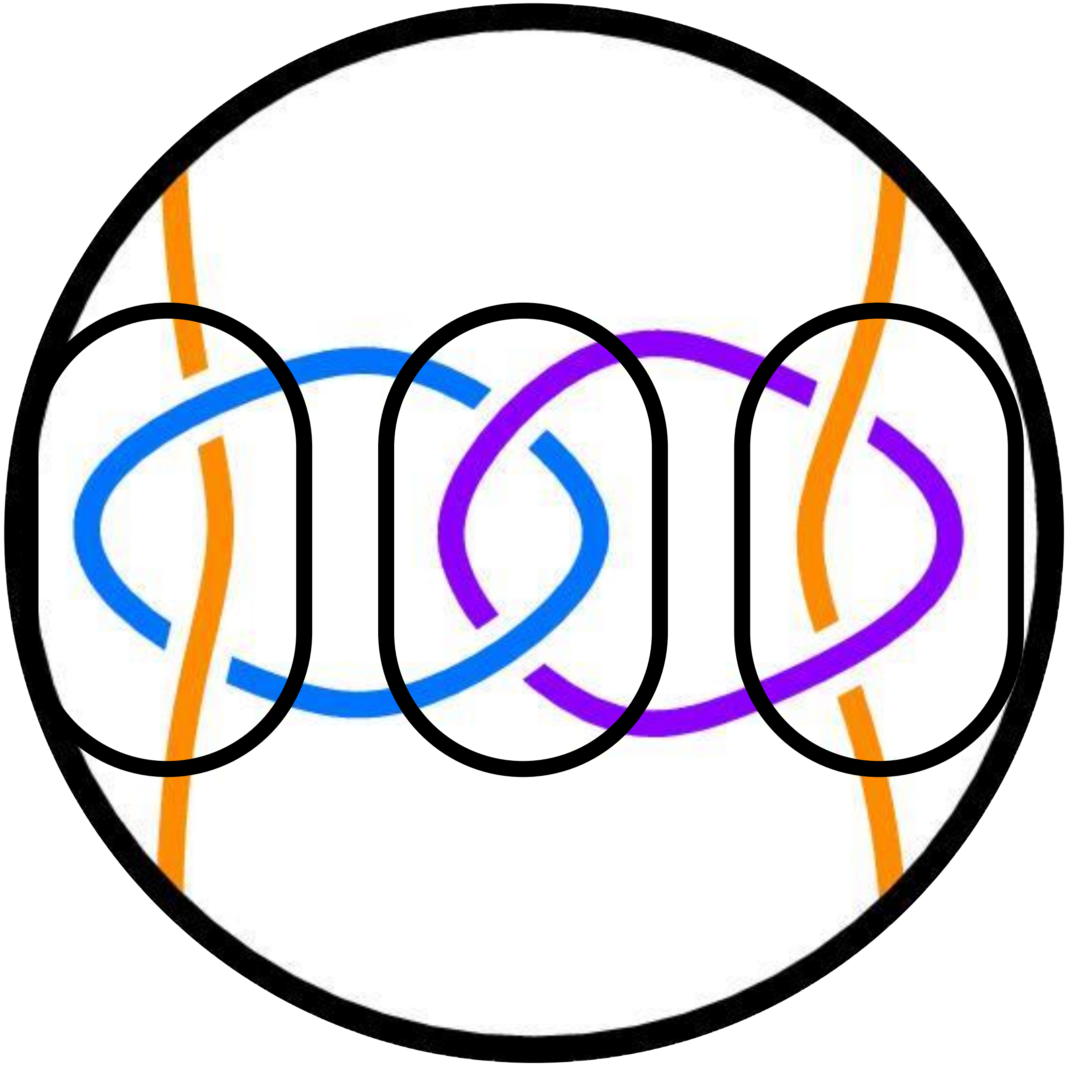
Free Boundary; Montesinos Tangles

$$\sum_{C_j=\left(c_1,\ldots,c_{n_j}\right)} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$

$$C_4 = (2,2,2)$$

$$2^0 \cdot 2^0 \cdot 2^0 = 1$$

1 of 1

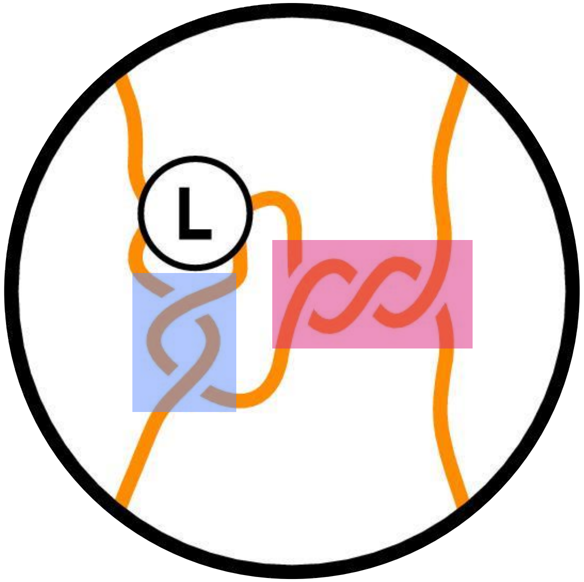


Fixed Boundary; Circle Product

Definition. The circle product $\boldsymbol{L} \circ (\boldsymbol{x_1}, \dots, \boldsymbol{x_k})$ of a tangle $\boldsymbol{L}$ and twist vector $(\boldsymbol{x_1}, \dots, \boldsymbol{x_k})$ is the performed as follows.

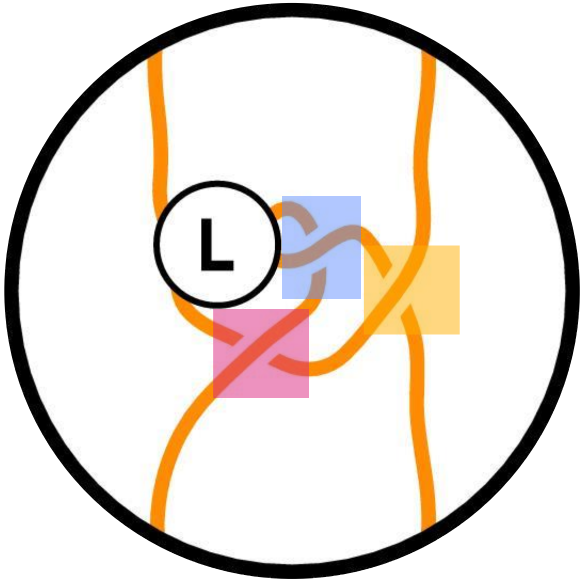
$$\boldsymbol{L} \circ (\boldsymbol{x_1}, \dots, \boldsymbol{x_k}) = \begin{cases} \boldsymbol{L} + \dfrac{\boldsymbol{x_1}}{\boldsymbol{1}} \vee \dfrac{\boldsymbol{1}}{\boldsymbol{x_2}} + \cdots \vee \dfrac{\boldsymbol{1}}{\boldsymbol{x_{k-1}}} + \dfrac{\boldsymbol{x_k}}{\boldsymbol{1}} & \boldsymbol{k = odd} \\ \boldsymbol{L} \vee \dfrac{\boldsymbol{1}}{\boldsymbol{x_1}} + \dfrac{\boldsymbol{x_2}}{\boldsymbol{1}} \vee \cdots \vee \dfrac{\boldsymbol{1}}{\boldsymbol{x_{k-1}}} + \dfrac{\boldsymbol{x_k}}{\boldsymbol{1}} & \boldsymbol{k = even} \end{cases}$$

Fixed Boundary; Circle Product



$$L \circ (2,3) =$$

$$L \vee \frac{1}{2} + \frac{3}{1}$$



$$L \circ (1,1,1) =$$

$$L + \frac{1}{1} \vee \frac{1}{1} + \frac{1}{1}$$

Generalized Montesinos Tangles

Theorem [H. Moon, I. Darcy]. Suppose that $n \leq 2$, $0 < |a_i| < b_i$ for $1 \leq i \leq n$ and $x_j \neq 0$ for $2 \leq j \leq k-1$. Suppose also that $a_i > 0$ and $x_j \geq 0$ for all i, j or $a_i < 0$ and $x_j \leq 0$ for all i, j. A generalized Montesinos tangle which is not rational is <u>uniquely represented</u> as one of the following <u>minimal crossing diagrams</u>:

1. $\left(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}\right) \circ (x_1, \ldots, x_k)$ $\quad k = \text{odd}$

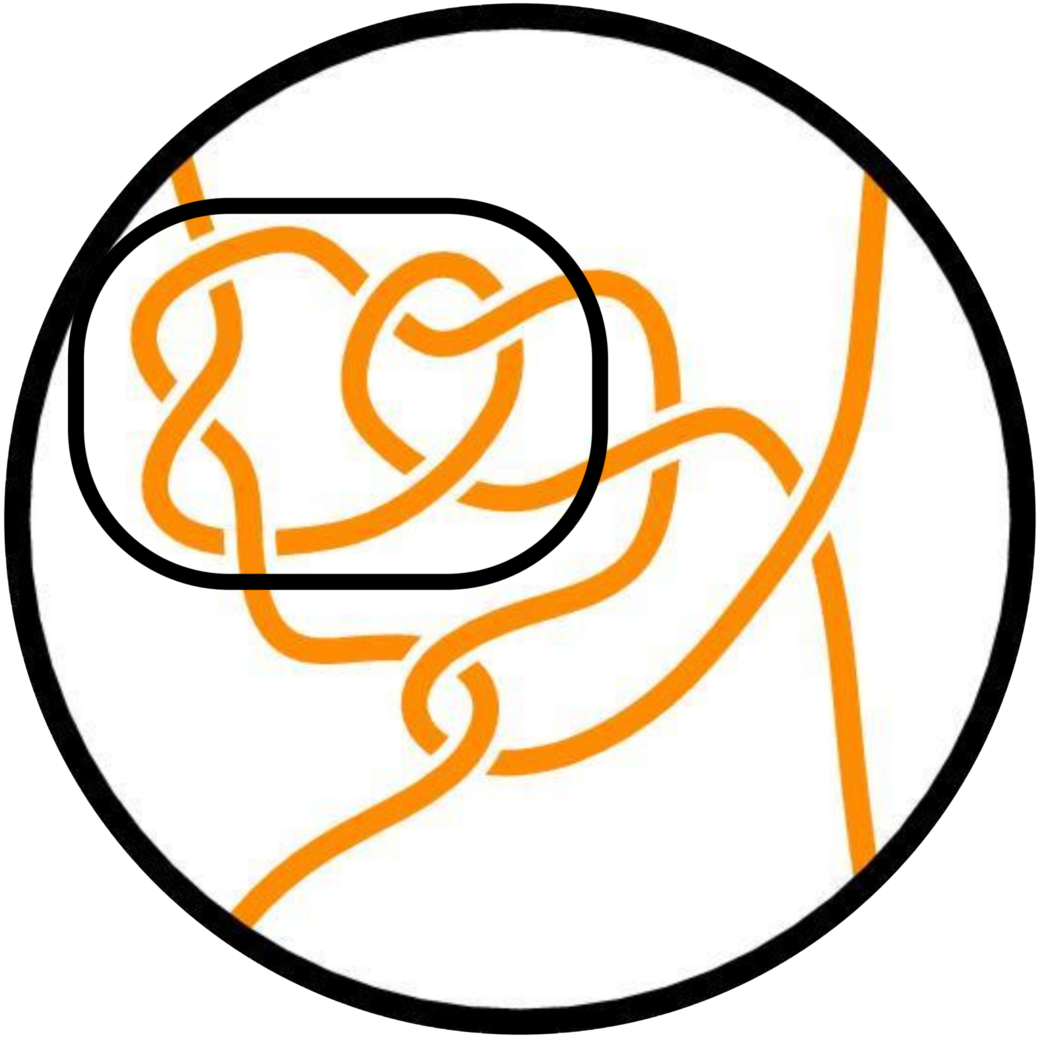
2. $\left(\frac{a_1}{b_1} + \cdots + \frac{a_{i-1}}{b_{i-1}} + \frac{-a_i}{b_i} + \cdots + \frac{-a_n}{b_n}\right) \circ (0, x_2, \ldots, x_k)$ $\quad k = \text{odd}$

3. $\left(\frac{b_1}{a_1} \vee \cdots \vee \frac{b_n}{a_n}\right) \circ (x_1, \ldots, x_k)$ $\quad k = \text{even}$

4. $\left(\frac{b_1}{a_1} \vee \cdots \vee \frac{b_{i-1}}{a_{i-1}} \vee \frac{-b_i}{a_i} \vee \cdots \vee \frac{-b_n}{a_n}\right) \circ (0, x_2, \ldots, x_k)$ $\quad k = \text{even}$

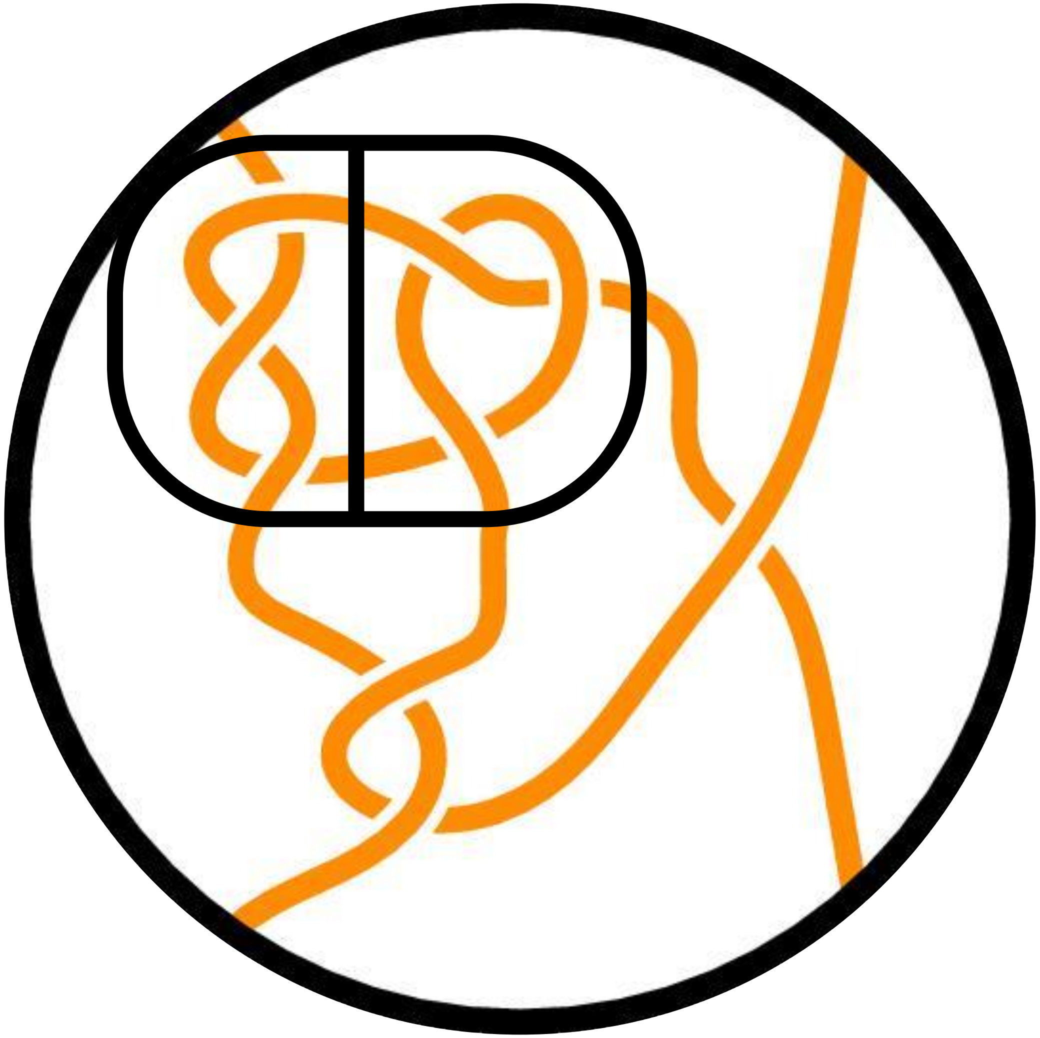
Type 1 Generalized Montesinos Tangles

$$\left(\frac{1}{3}+\frac{2}{3}\right)\circ(1,2,1)$$



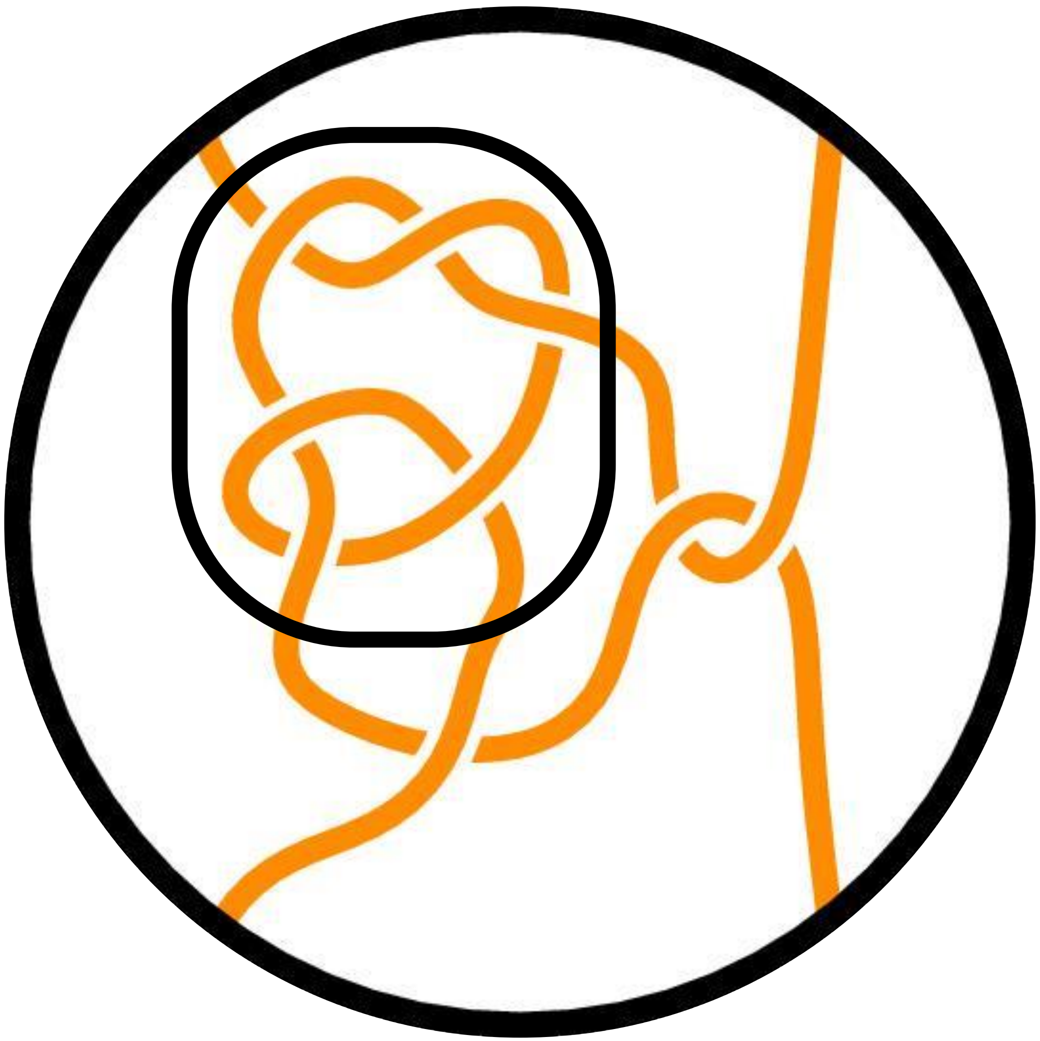
Type 2 Generalized Montesinos Tangles

$$\left(\frac{1}{3}+\frac{-2}{3}\right)\circ(0,2,1)$$



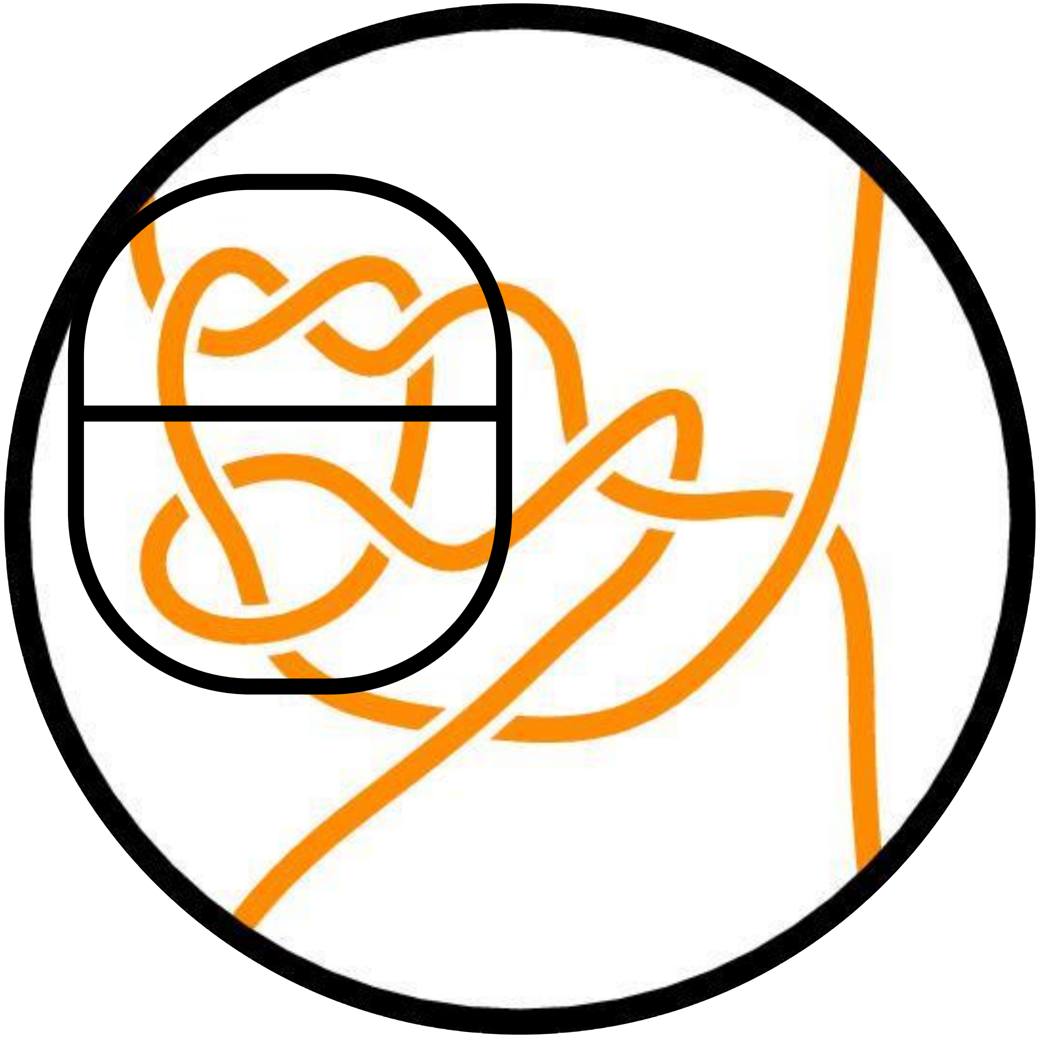
Type 3 Generalized Montesinos Tangles

$$\left(\frac{3}{1} \vee \frac{3}{2}\right) \circ (1,2)$$



Type 4 Generalized Montesinos Tangles

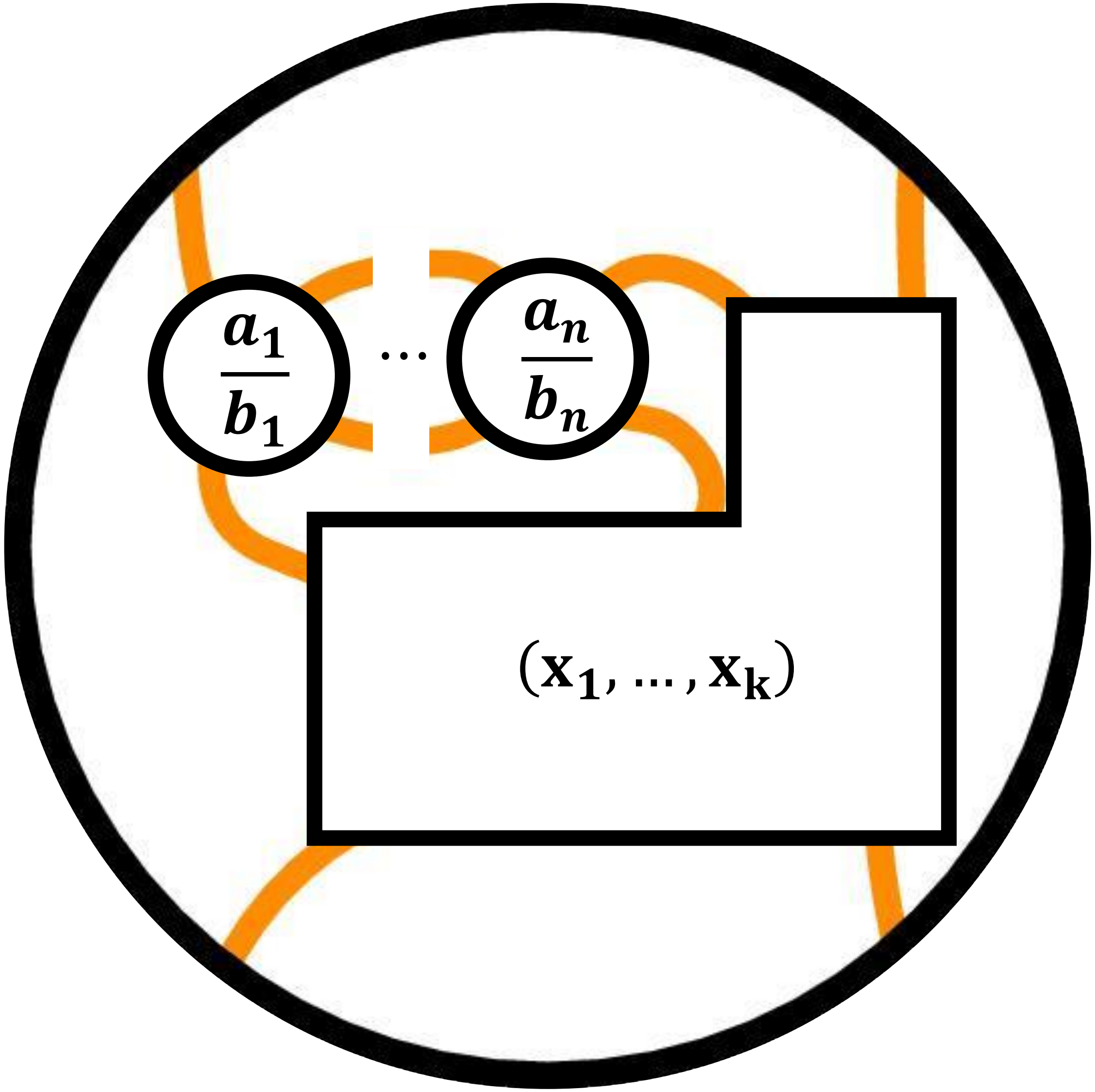
$$\left(\frac{3}{1} \vee \frac{-3}{2}\right) \circ (0, 2, 1, 1)$$



Type 1 Generalized Montesinos Tangles

If the tangle has $N = N_1 + N_2$ crossings,

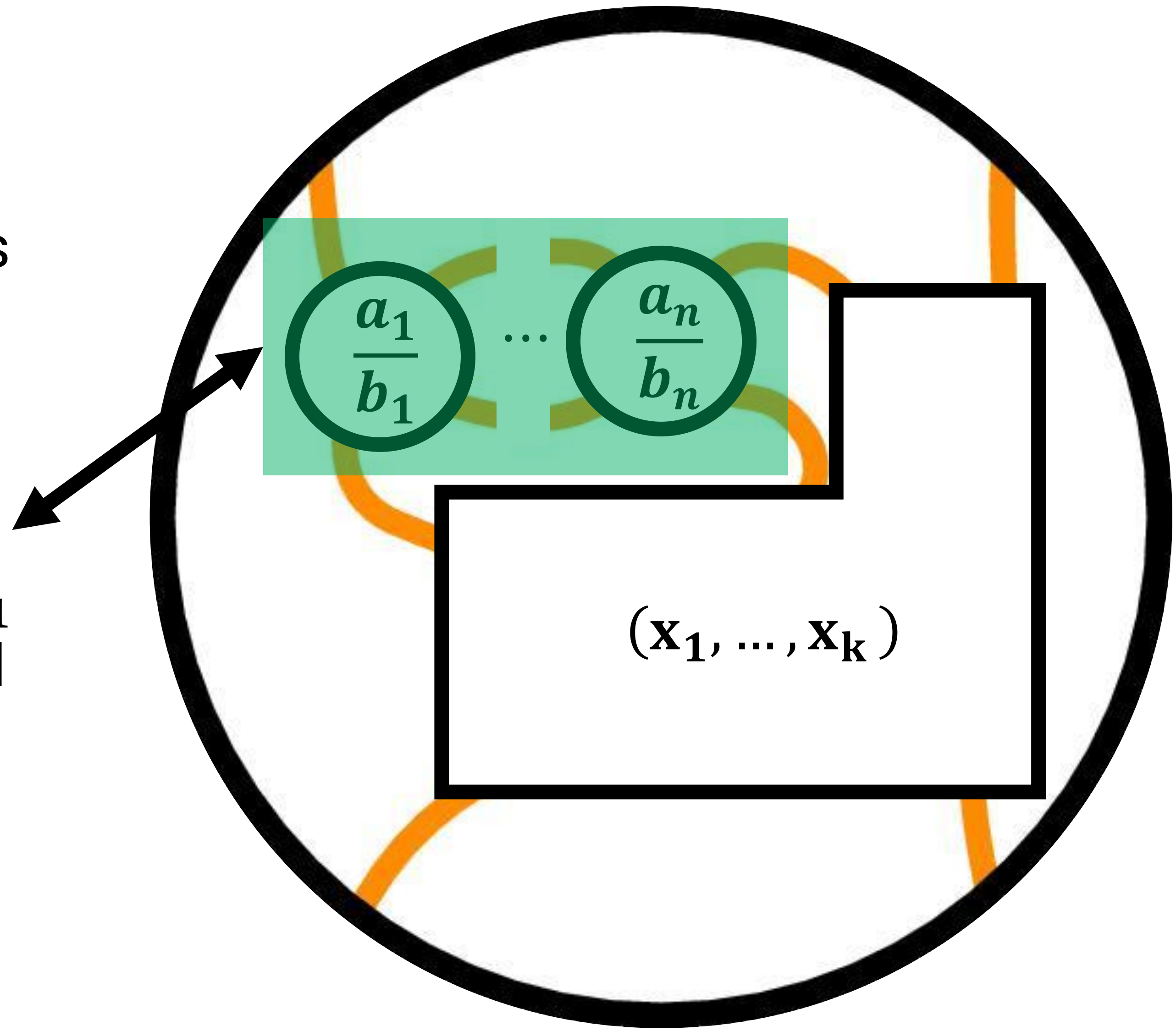
there are N_1 crossings in $\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}$ and N_2 crossings in $(x_1, \ldots, x_k)$



Type 1 Generalized Montesinos Tangles

$N = N_1 + N_2$
total crossings

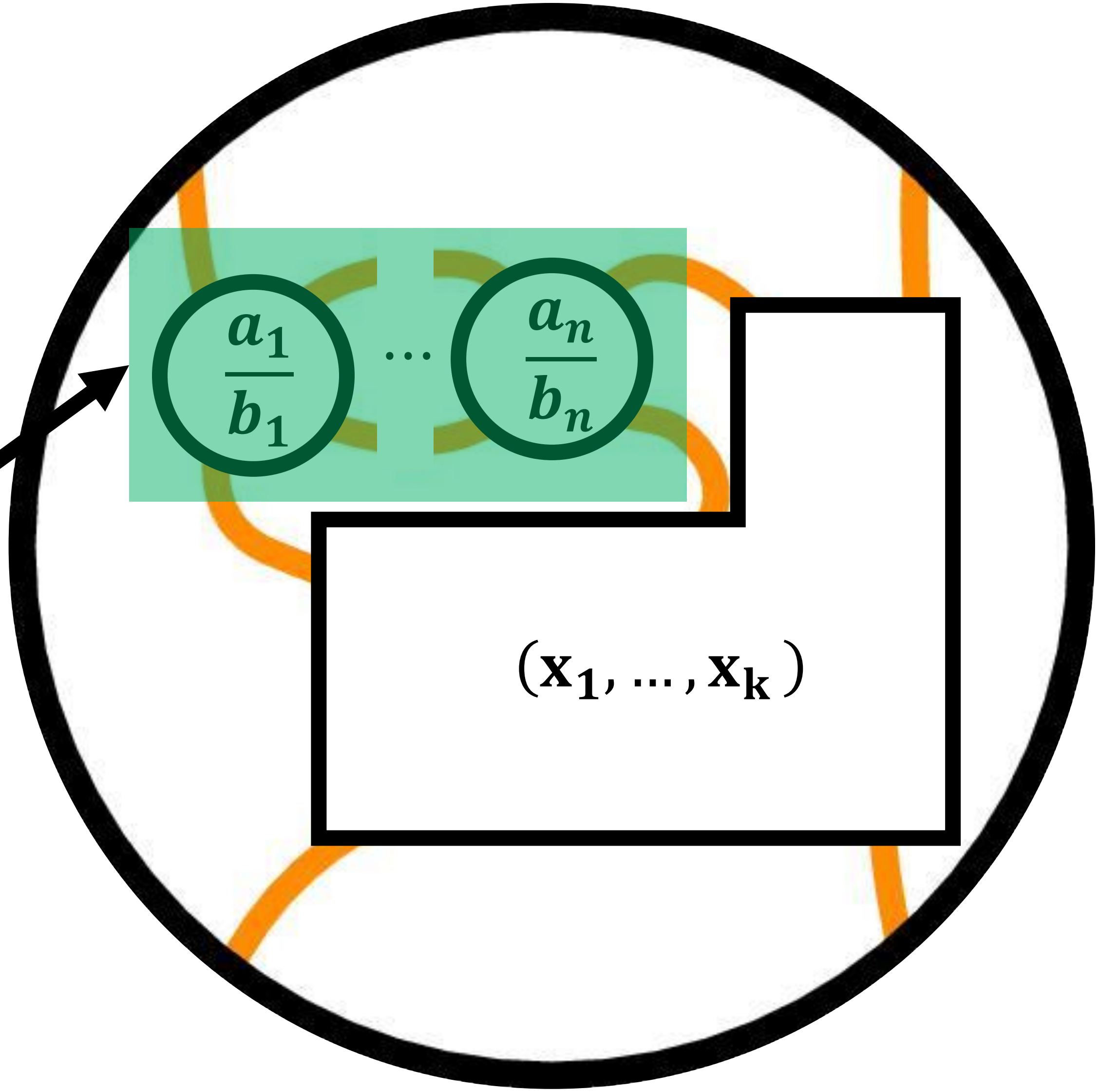
Montesinos tangle with N_1 crossings and free boundary



Type 1 Generalized Montesinos Tangles

$N = N_1 + N_2$
total crossings

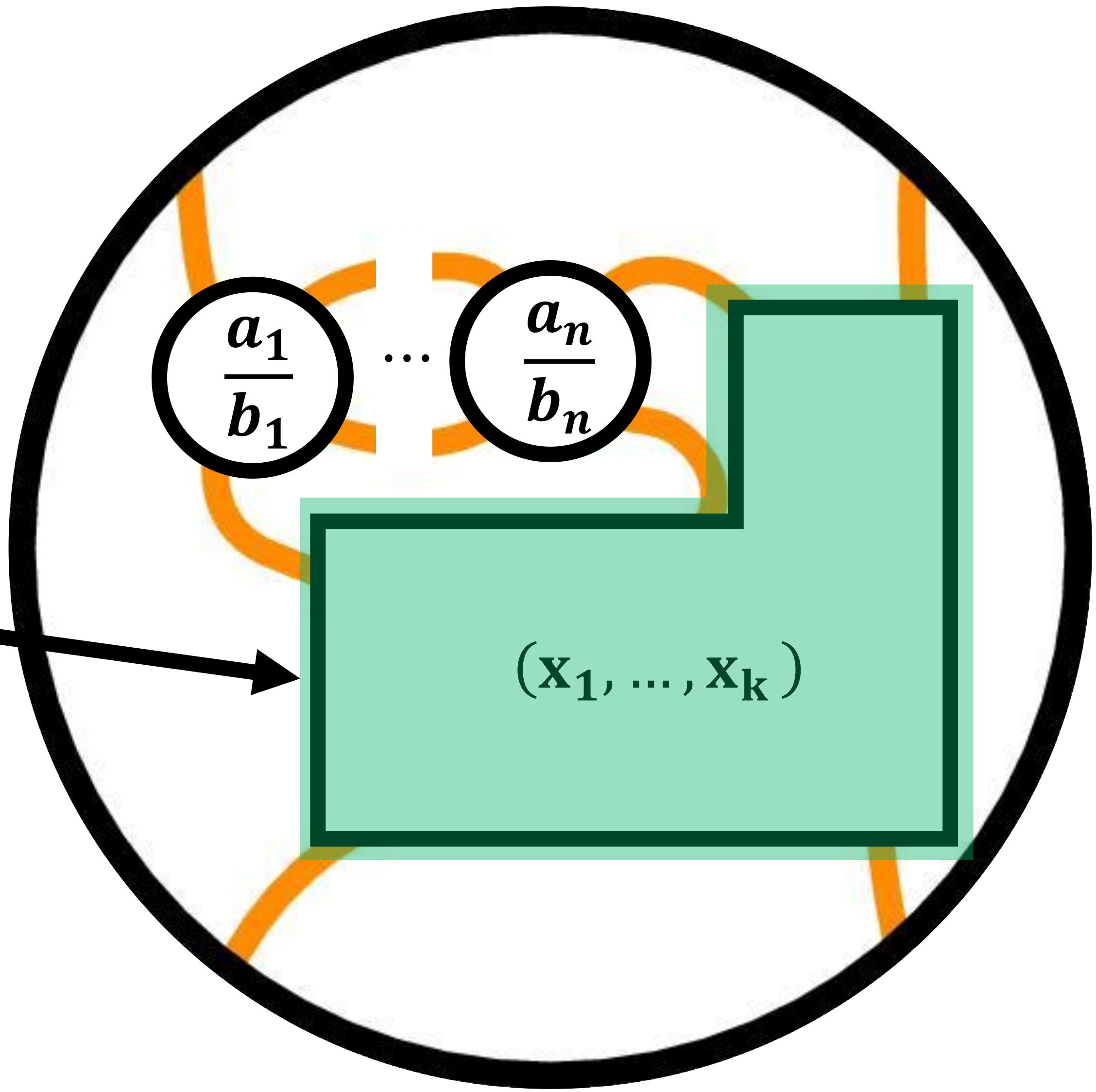
$$\sum_{c_j=(c_{1},\ldots,c_{n_j})} \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$



Type 1 Generalized Montesinos Tangles

$N = N_1 + N_2$
total crossings

Compositions of N_2 which may include zeros



Type 1 Generalized Montesinos Tangles

- Twist vectors must have **odd length** $(x_1, \dots, x_k)$.

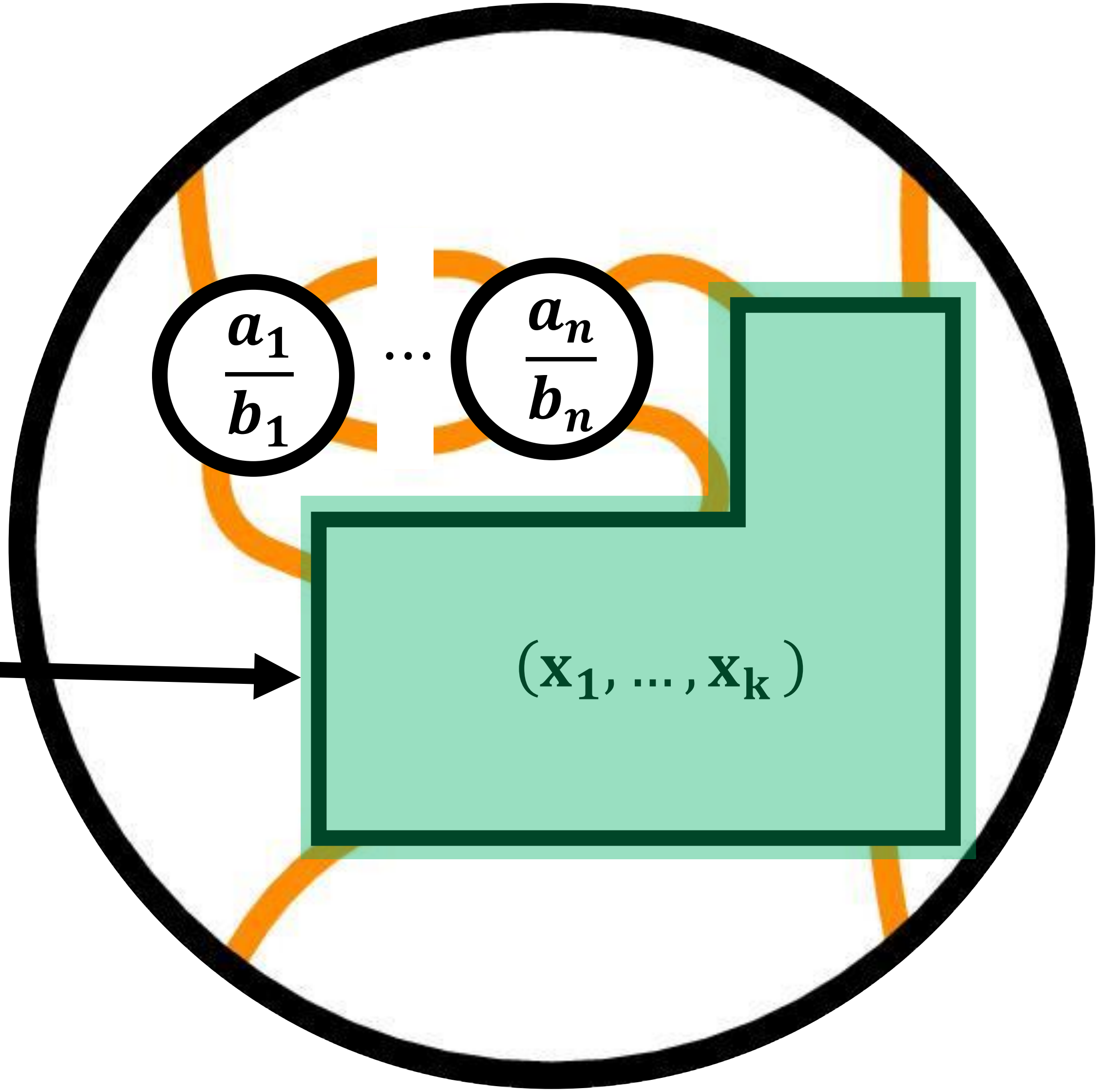
- They may be made from all **odd length** $***$ compositions of N_2 in the following forms
  - $(***)$ $\qquad 2^{N_2-2}$ many
  - $(\mathbf{0}, ***, \mathbf{0})$ $\qquad 2^{N_2-2}$ many

- They may be made from all **even length** $**$ compositions of N_2 in the following forms
  - $(\mathbf{0}, **)$ $\qquad 2^{N_2-2}$ many
  - $(**, \mathbf{0})$ $\qquad 2^{N_2-2}$ many

Type 1 Generalized Montesinos Tangles

$N = N_1 + N_2$
total crossings

$4\left(2^{N_2-2}\right)$
$= \ 2^{N_2}$



Type 1 Generalized Montesinos Tangles

Theorem [B]. The number of Type 1 generalized Montesinos tangles with $\boldsymbol{N}$ crossings are counted by the sum

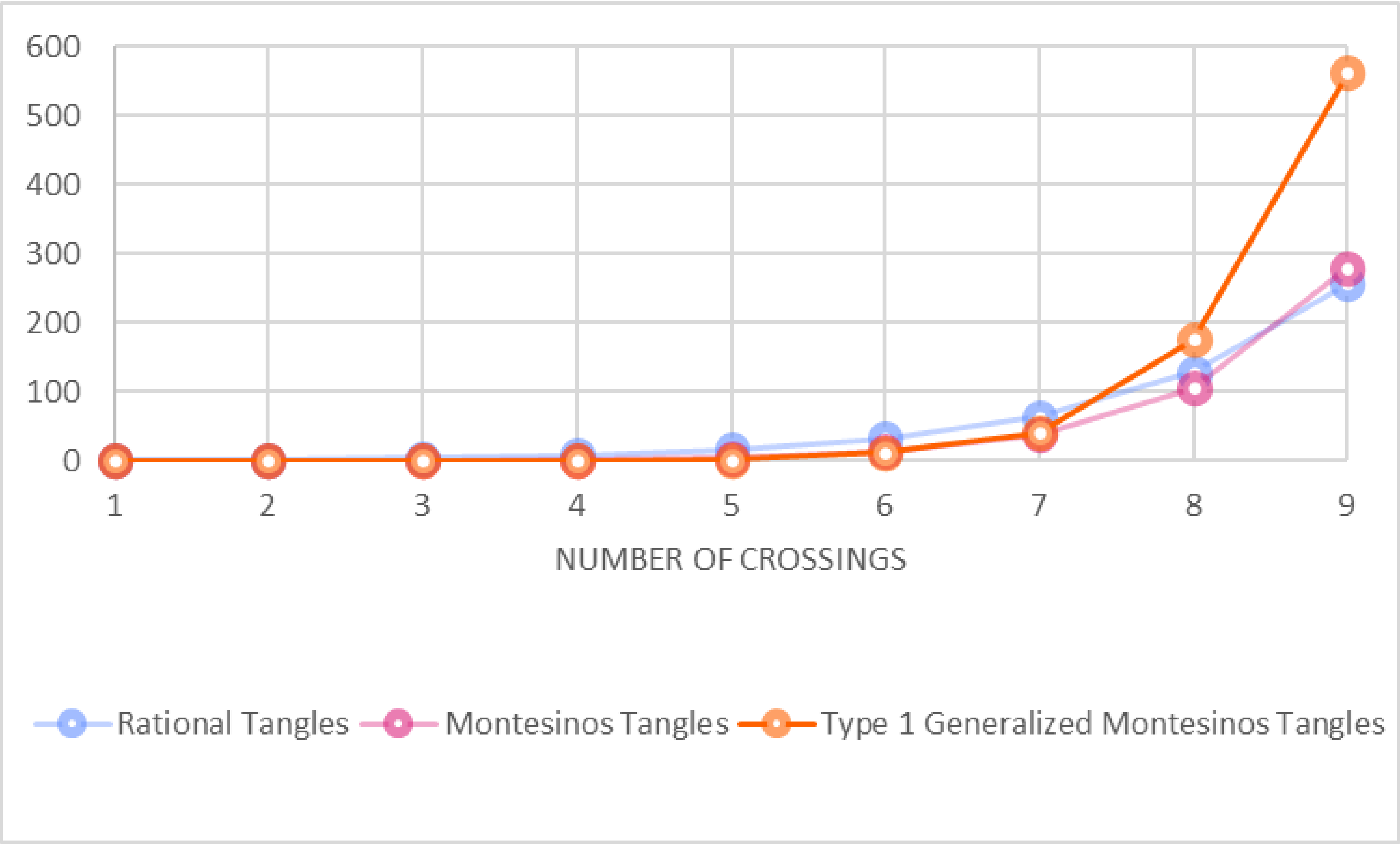
$$\sum_{(N_1,N_2)} \left(2^{N_2} \left(\sum_{C_j=\left(c_1,\ldots,c_{n_j}\right)} \prod_{i=1}^{n_j} 2^{c_i-2} \right) \right)$$

over all $\boldsymbol{N-4}$ compositions $(\boldsymbol{N_1}, \boldsymbol{N_2})$ of $\boldsymbol{N \geq 5}$ into two parts with $\boldsymbol{N_1 \geq 4}$ and over all $\boldsymbol{F_{N_1-1} - 1}$ compositions $\boldsymbol{C_j}$ of $\boldsymbol{N_1}$ into at least two parts greater than one.

Type 1 Generalized Montesinos Tangles

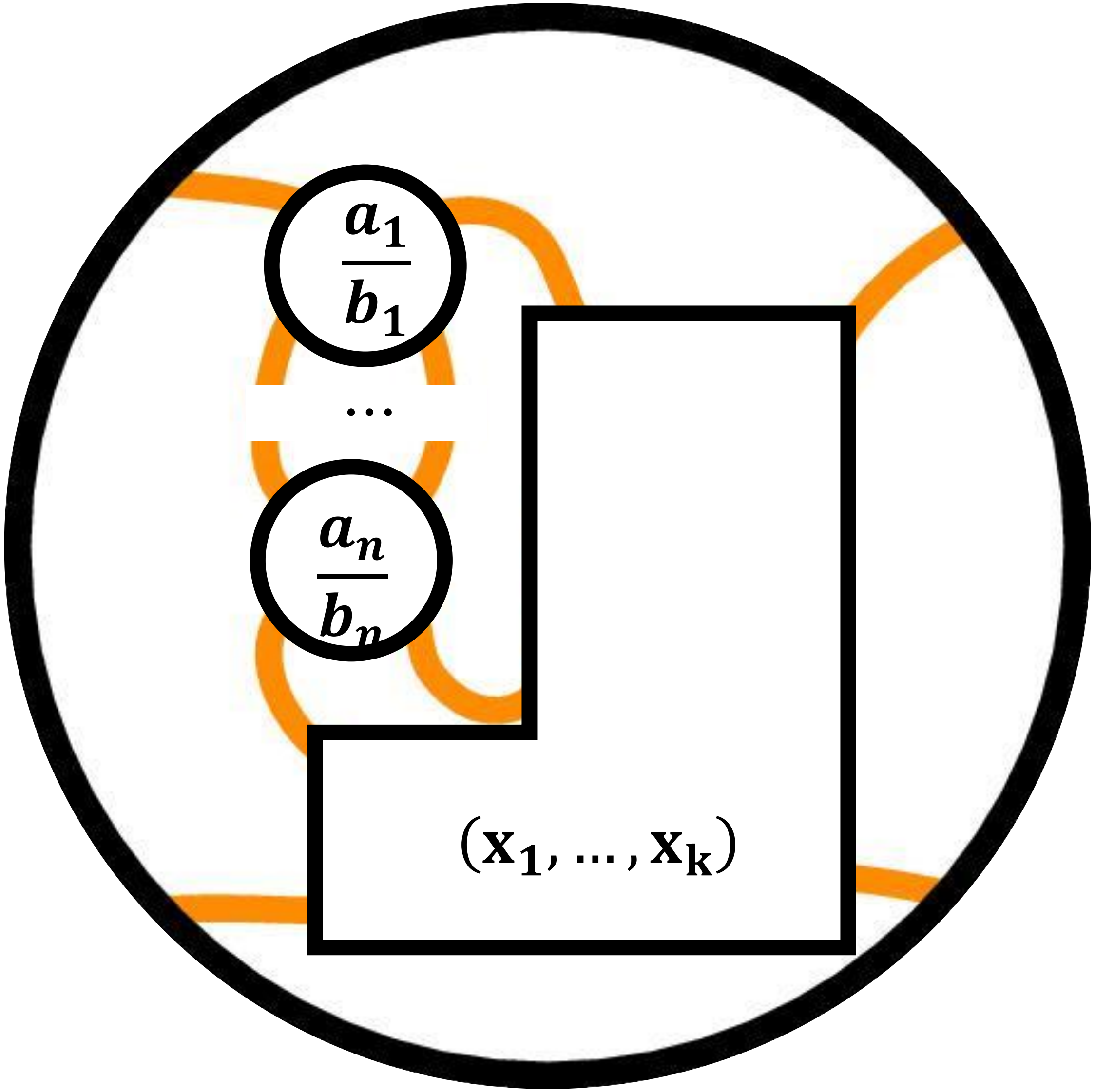
Number of Crossings	5	6	7	8	9
Unique Type 1 Generalized Montesinos Tangles	2	12	40	176	562

Type 1 Generalized Montesinos Tangles



Type 3 Generalized Montesinos Tangles

Type 3 generalized Montesinos tangles are similar to Type 1 generalized Montesinos tangles but twist vectors have **even length**.



Type 3 Generalized Montesinos Tangles

- Twist vectors must have **even length** $(x_1, \ldots, x_k)$.

- They may be made from all **even length** $**$ compositions of N_2 in the following forms
  - $(**)$ $\quad 2^{N_2-2}$ many
  - $(\mathbf{0}, **, \mathbf{0})$ $\quad 2^{N_2-2}$ many

- They may be made from all **odd length** $***$ compositions of N_2 in the following forms
  - $(\mathbf{0}, ***)$ $\quad 2^{N_2-2}$ many
  - $(***, \mathbf{0})$ $\quad 2^{N_2-2}$ many

Type 3 Generalized Montesinos Tangles

Theorem [B]. The number of Type 3 generalized Montesinos tangles with $\boldsymbol{N}$ crossings are counted by the sum

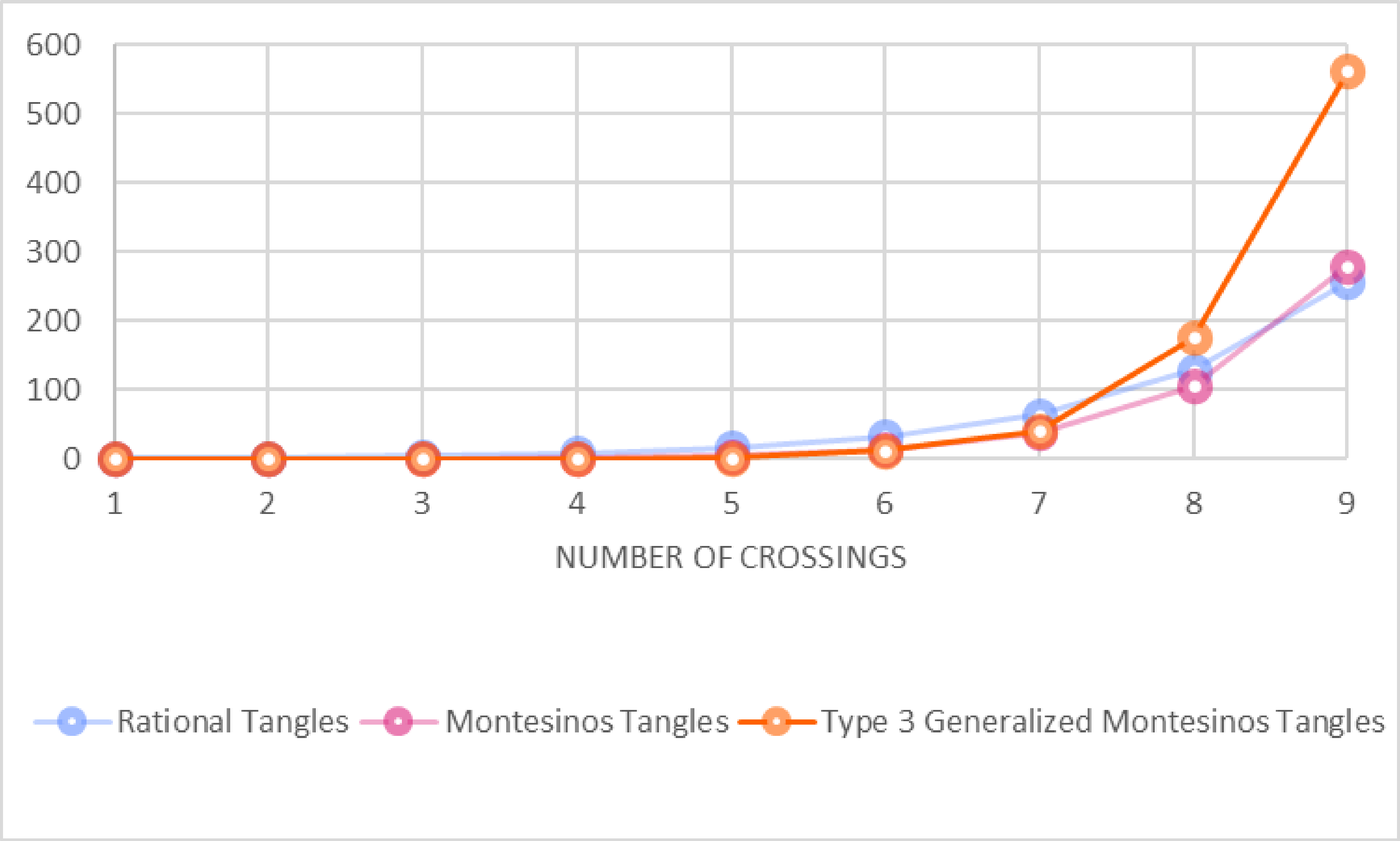
$$\sum_{(N_1,N_2)} \left(2^{N_2} \left(\sum_{C_j=\left(c_1,\ldots,c_{n_j}\right)} \prod_{i=1}^{n_j} 2^{c_i-2} \right) \right)$$

over all $\boldsymbol{N-4}$ compositions $(\boldsymbol{N_1},\boldsymbol{N_2})$ of $\boldsymbol{N \geq 5}$ into two parts with $\boldsymbol{N_1 \geq 4}$ and over all $\boldsymbol{F_{N_1-1}-1}$ compositions $\boldsymbol{C_j}$ of $\boldsymbol{N_1}$ into at least two parts greater than one.

Type 3 Generalized Montesinos Tangles

Number of Crossings	5	6	7	8	9
Unique Type 3 Generalized Montesinos Tangles	2	12	40	176	562

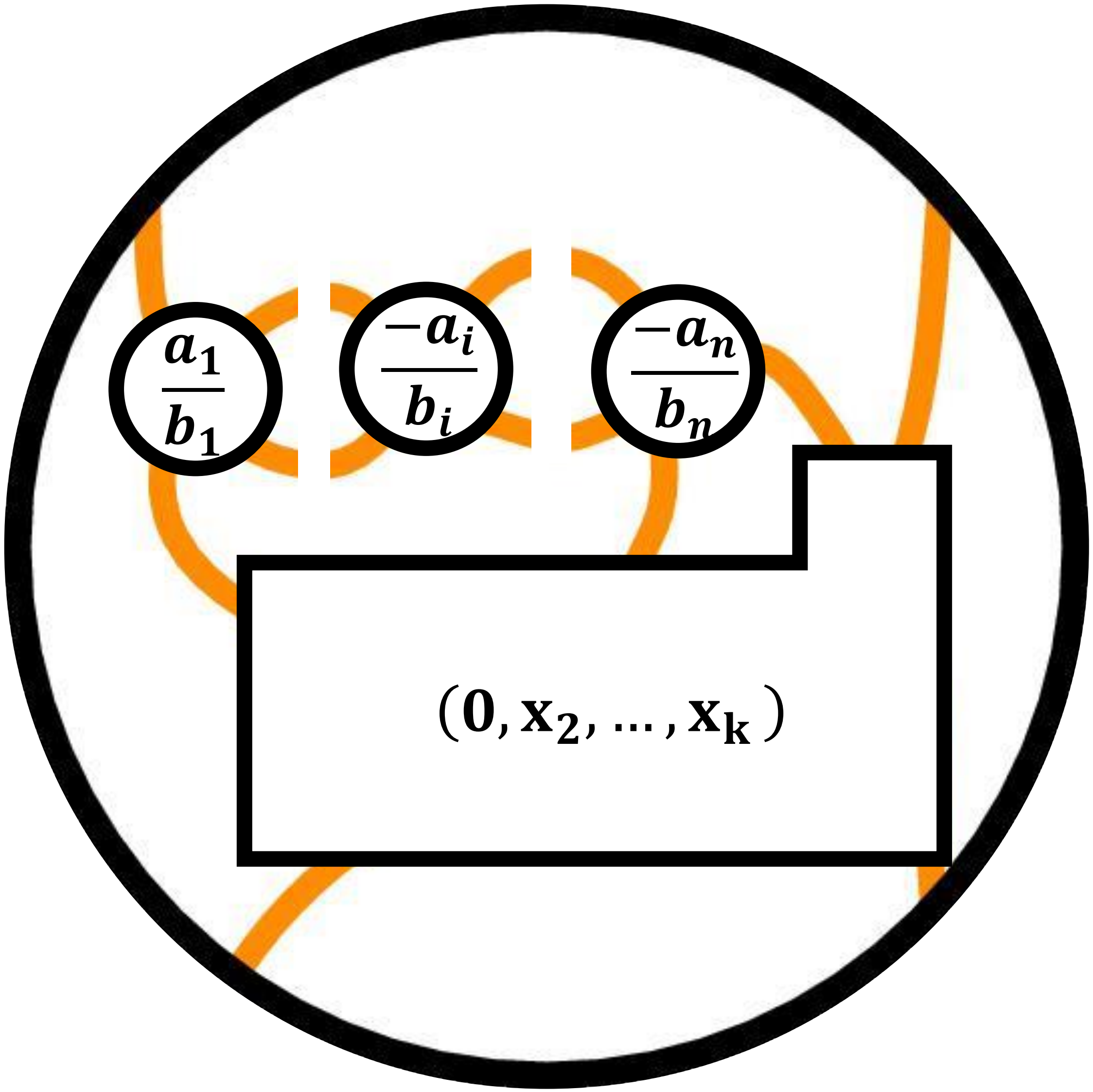
Type 3 Generalized Montesinos Tangles



Type 2 Generalized Montesinos Tangles

If the tangle has $N = N_1 + N_2$ crossings,

there are N_1 crossings in $\frac{a_1}{b_1} + \cdots + \frac{a_{i-1}}{b_{i-1}} + \frac{-a_i}{b_i} + \cdots + \frac{a_n}{b_n}$ and N_2 crossings in $(x_1, \dots, x_k)$

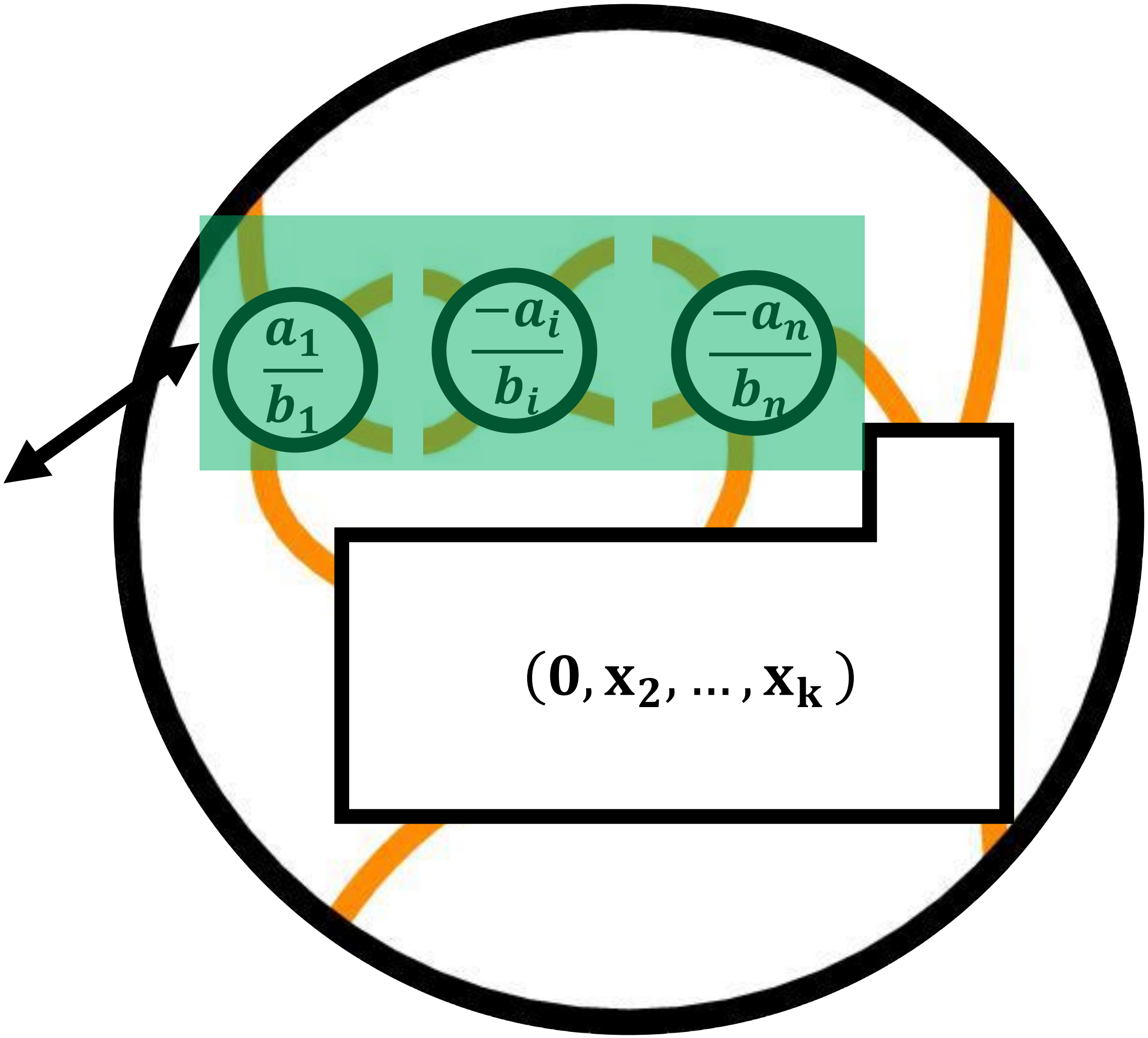


Type 2 Generalized Montesinos Tangles

$N = N_1 + N_2$
total crossings

Montesinos-like tangle with N_1 crossings and free boundary.

n places to set i.

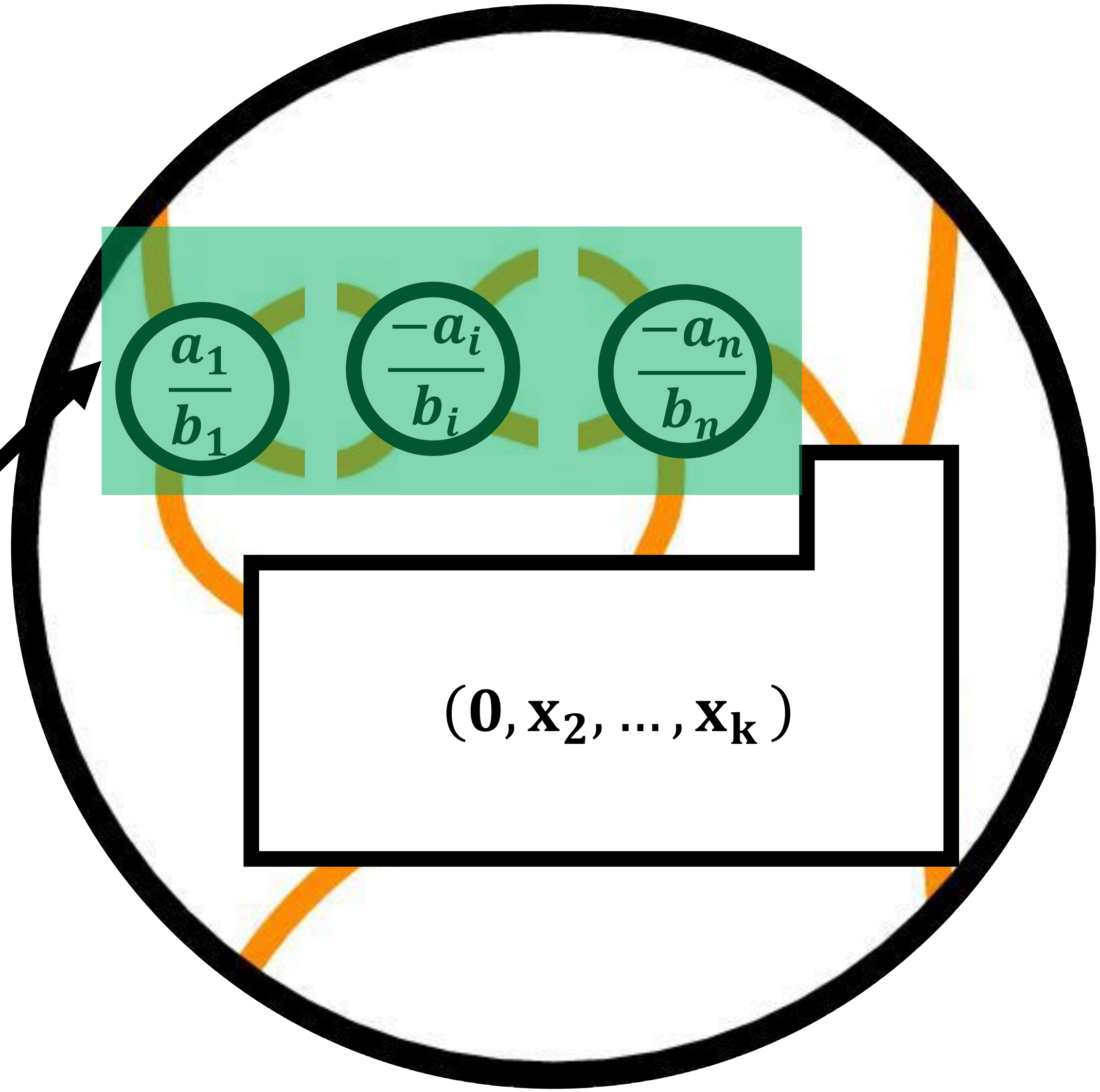


Type 2 Generalized Montesinos Tangles

$$N = N_1 + N_2$$

total crossings

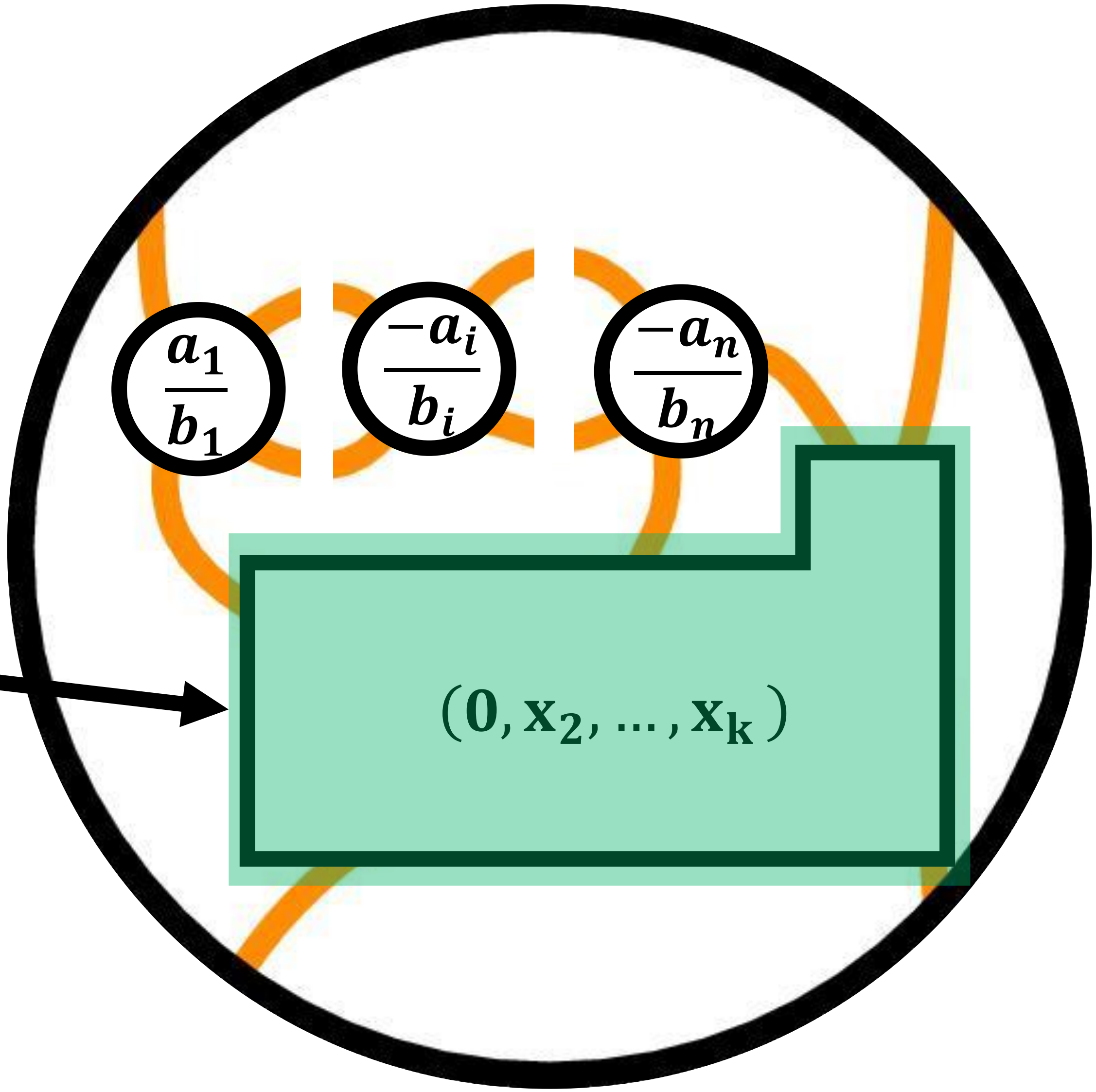
$$\sum_{C_j=\left(c_1,\dots,c_{n_j}\right)} n_j \left(\prod_{i=1}^{n_j} 2^{c_i-2} \right)$$



Type 2 Generalized Montesinos Tangles

$N = N_1 + N_2$
total crossings

Compositions of N_2 which may include zeros



Type 2 Generalized Montesinos Tangles

- Twist vectors must have **odd length** $(\mathbf{0}, \boldsymbol{x_2}, \dots, \boldsymbol{x_k})$.

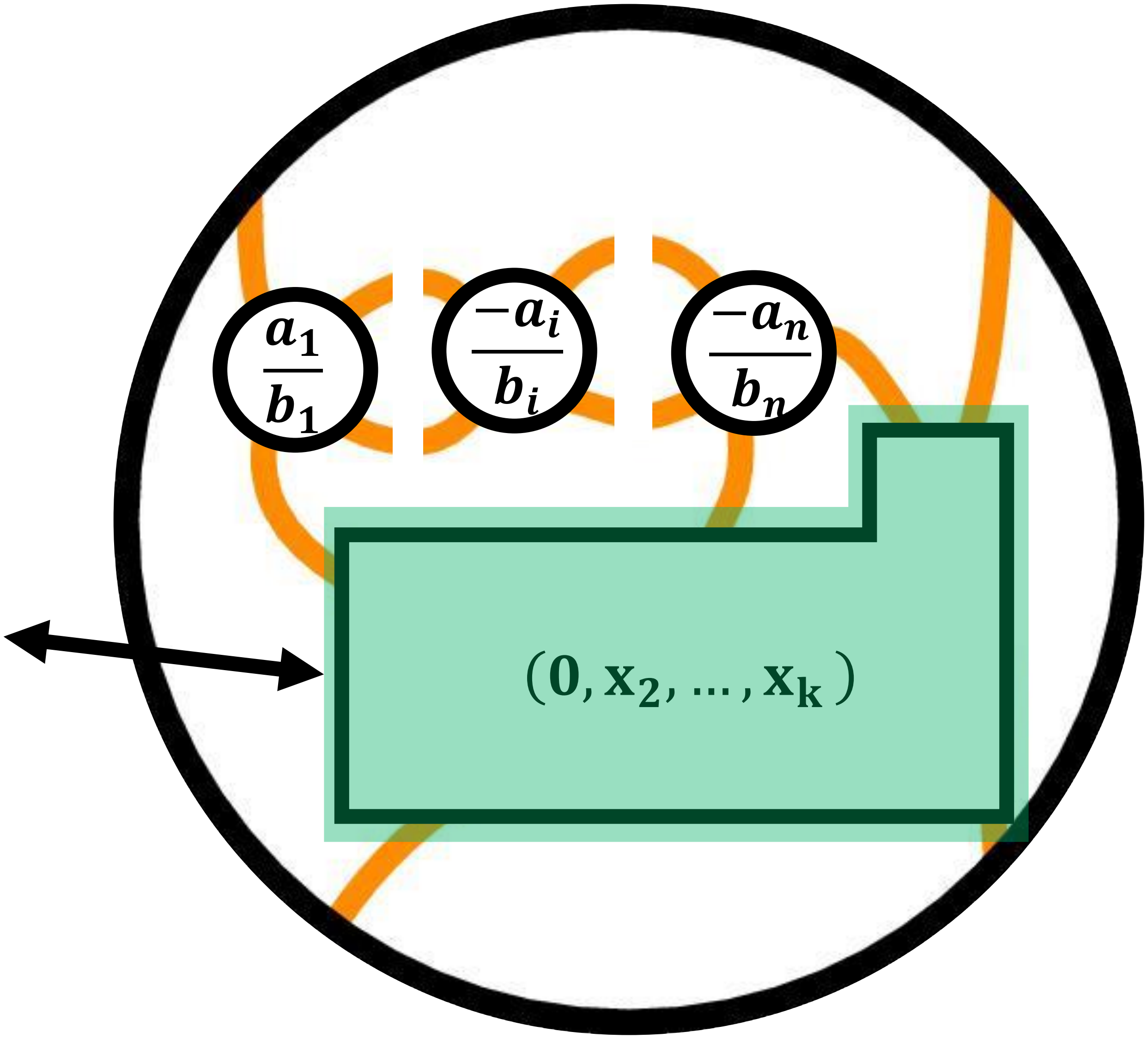
- They may be made from all **even length** $**$ compositions of $\boldsymbol{N_2}$ in the following form
  - $(\mathbf{0}, **)$ $\quad\quad$ $\mathbf{2^{N_2-2}}$ many

- They may be made from all **odd length** $***$ compositions of $\boldsymbol{N_2}$ in the following forms
  - $(\mathbf{0}, ***, \mathbf{0})$ $\quad\quad$ $\mathbf{2^{N_2-2}}$ many

Type 2 Generalized Montesinos Tangles

$N = N_1 + N_2$
total crossings

$2\left(2^{N_2-2}\right)$
$= 2^{N_2-1}$



Type 2 Generalized Montesinos Tangles

Theorem [B]. The number of Type 2 generalized Montesinos tangles with $\boldsymbol{N}$ crossings are counted by the sum

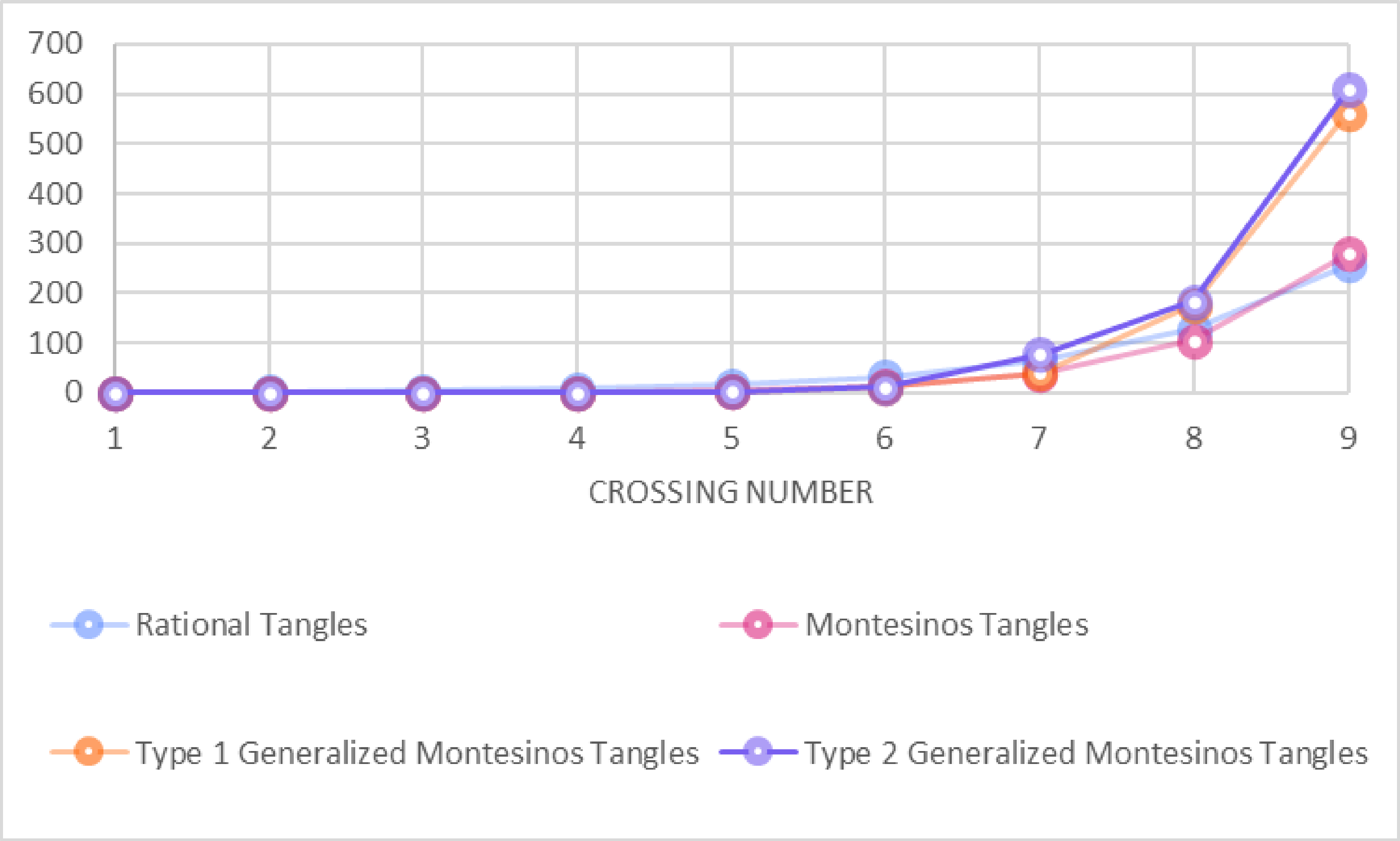
$$\sum_{(N_1,N_2)} \left(2^{N_2-1} \left(\sum_{C_j=\left(c_1,\ldots,c_{n_j}\right)} n_j \prod_{i=1}^{n_j} 2^{c_i-2} \right) \right)$$

over all $\boldsymbol{N-4}$ compositions $(\boldsymbol{N_1}, \boldsymbol{N_2})$ of $\boldsymbol{N \geq 5}$ into two parts with $\boldsymbol{N_1 \geq 4}$ and over all $\boldsymbol{F_{N_1-1}-1}$ compositions $\boldsymbol{C_j}$ of $\boldsymbol{N_1}$ into at least two parts greater than one.

Type 2 Generalized Montesinos Tangles

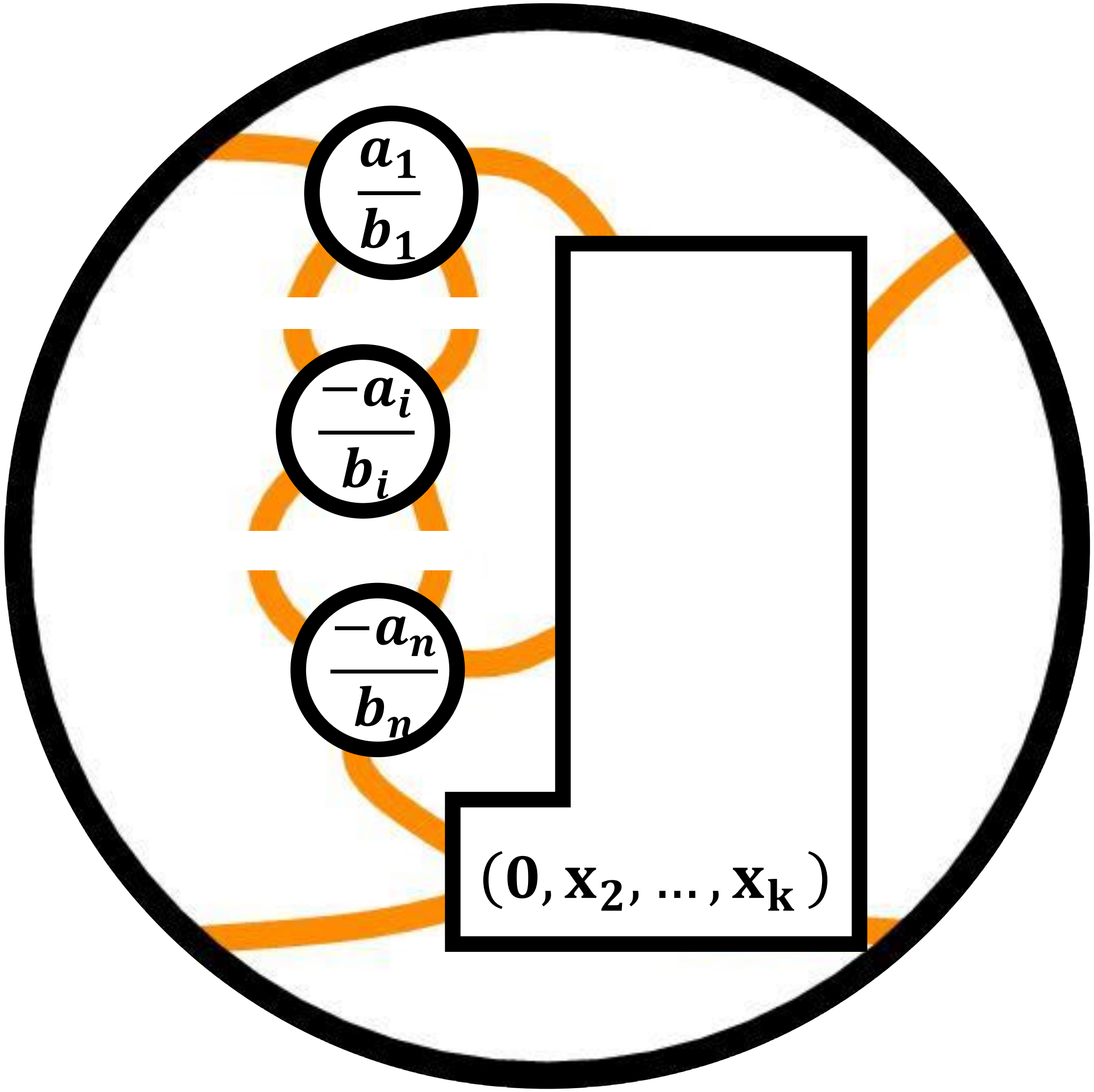
Number of Crossings	5	6	7	8	9
Unique Type 2 Generalized Montesinos Tangles	2	12	78	184	608

Type 2 Generalized Montesinos Tangles



Type 4 Generalized Montesinos Tangles

Type 4 generalized Montesinos tangles are similar to Type 2 generalized Montesinos tangles but twist vectors have **even length**.



Type 4 Generalized Montesinos Tangles

- Twist vectors must have **even length** $(\mathbf{0}, x_2, \dots, x_k)$.

- They may be made from all **odd length** $***$ compositions of N_2 in the following form
  - $(\mathbf{0}, ***)$ $\quad$ 2^{N_2-2} many

- They may be made from all **even length** $**$ compositions of N_2 in the following forms
  - $(\mathbf{0}, **, \mathbf{0})$ $\quad$ 2^{N_2-2} many

Type 4 Generalized Montesinos Tangles

Theorem [B]. The number of Type 4 generalized Montesinos tangles with $\boldsymbol{N}$ crossings are counted by the sum

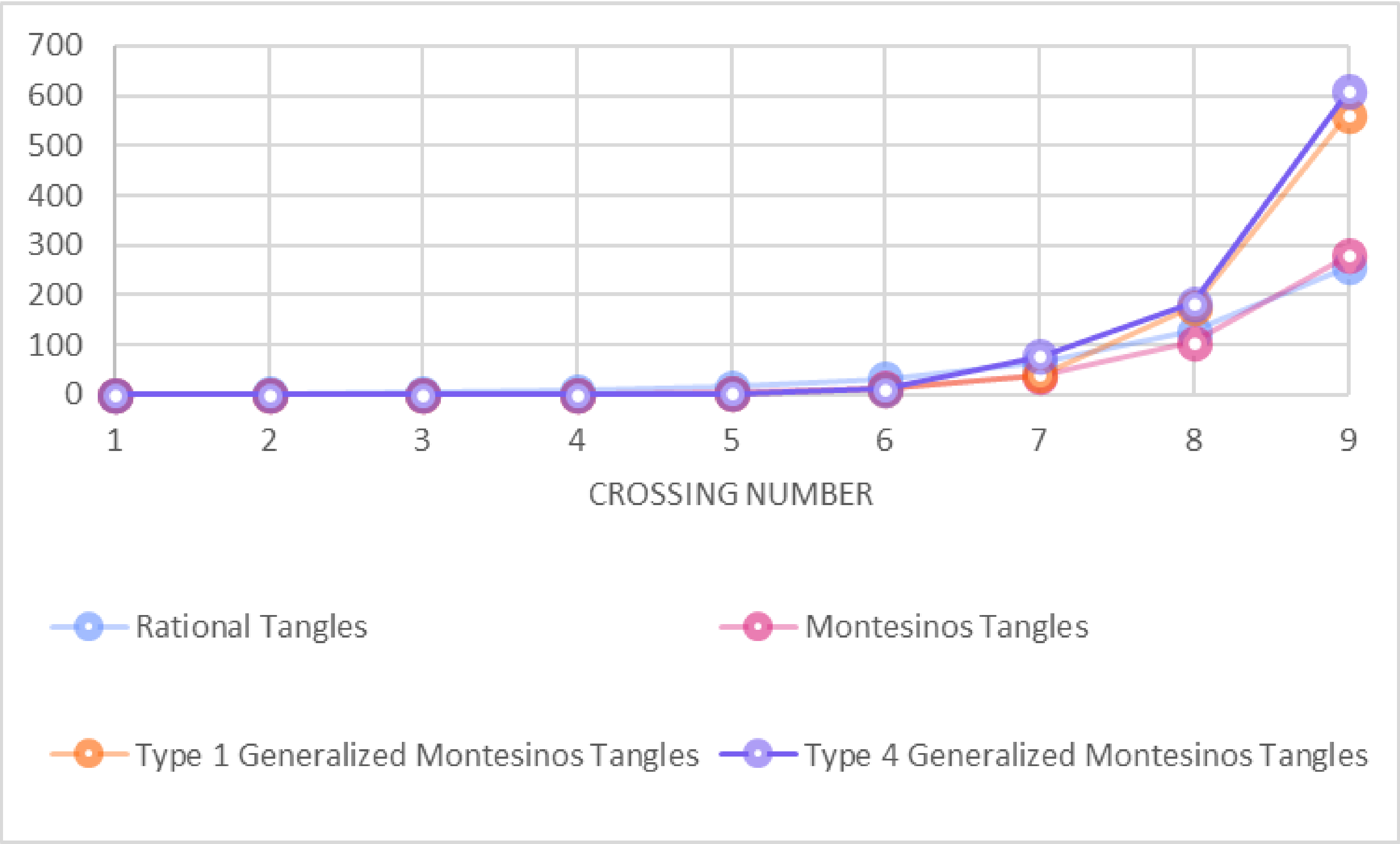
$$\sum_{(N_1,N_2)}\left(2^{N_2-1}\left(\sum_{C_j=\left(c_1,\dots,c_{n_j}\right)} n_j\prod_{i=1}^{n_j}2^{c_i-2}\right)\right)$$

over all $\boldsymbol{N-4}$ compositions $(\boldsymbol{N_1},\boldsymbol{N_2})$ of $\boldsymbol{N}\geq\boldsymbol{5}$ into two parts with $\boldsymbol{N_1}\geq\boldsymbol{4}$ and over all $\boldsymbol{F_{N_1-1}-1}$ compositions $\boldsymbol{C_j}$ of $\boldsymbol{N_1}$ into at least two parts greater than one.

Type 4 Generalized Montesinos Tangles

Number of Crossings	5	6	7	8	9
Unique Type 4 Generalized Montesinos Tangles	2	12	78	184	608

Type 4 Generalized Montesinos Tangles

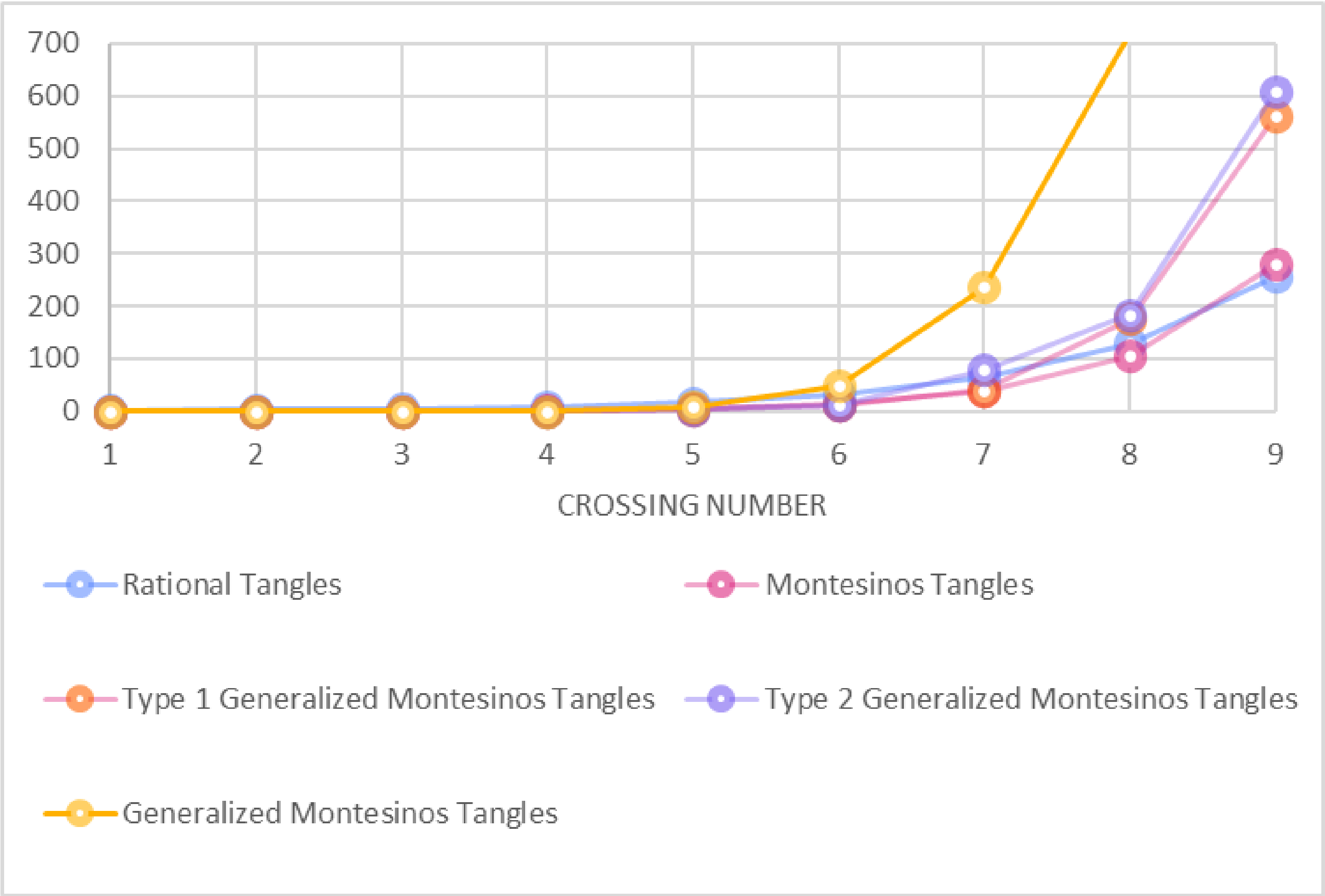


IOWA

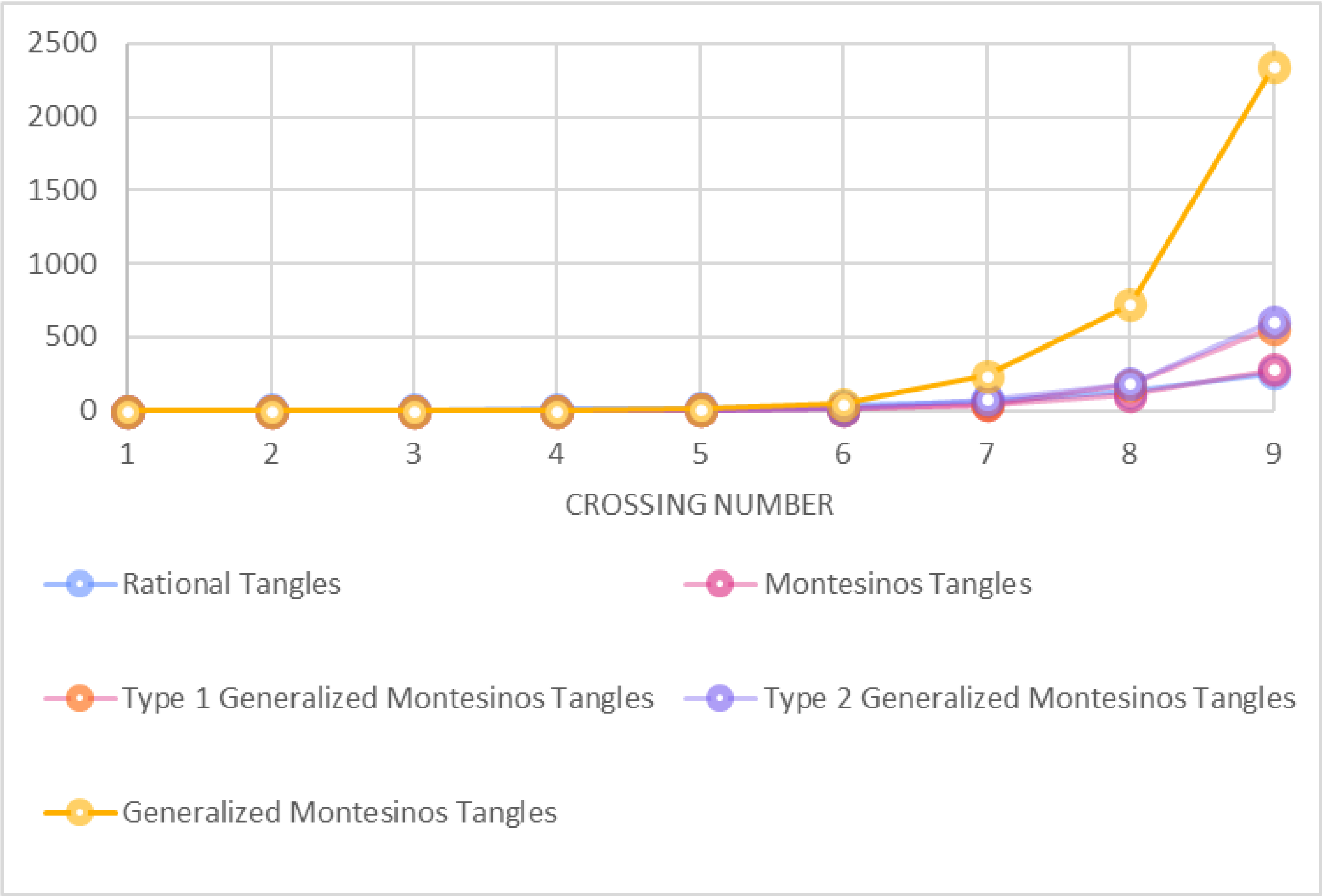
Generalized Montesinos Tangles

Number of Crossings	5	6	7	8	9
Unique Generalized Montesinos Tangles	8	48	236	750	2340

Generalized Montesinos Tangles



Generalized Montesinos Tangles



Sources

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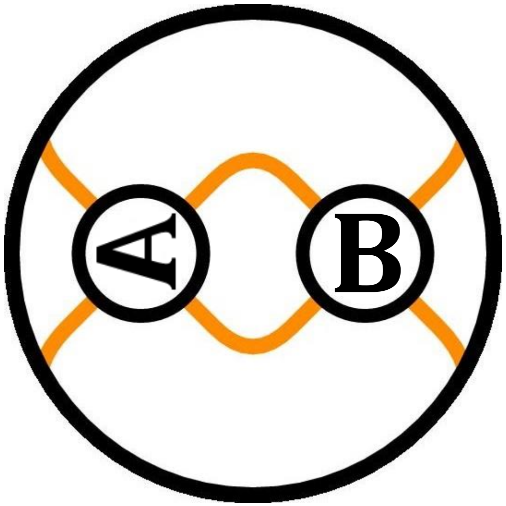
Operations

Product (Conway). Given two tangle A and B their product AB is

The tangle A has been rotated $90°$, vertically flipped, and mirrored.



Conway Notation

Conway's Definition. The sequence of integers $[a_1 \cdots a_n]$ is treated as the product $a_1 a_2 \cdots a_n$ of tangles with a_i crossings arranged horizontally.

Adjusted Definition. The sequence of integers $[a_1 \cdots a_n]$ is treated as one of the following sums.

$$\frac{a_1}{1} \vee \frac{1}{a_2} + \cdots \vee \frac{1}{a_{n-1}} + \frac{a_n}{1} \text{ when } n = \text{odd}$$

$$\frac{1}{a_1} + \frac{a_2}{1} \vee \cdots \vee \frac{1}{a_{n-1}} + \frac{a_n}{1} \text{ when } n = \text{even}$$