Isoradial Reduction Existence Proof

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Lemma 1: The eigenvalues of a matrix are continuous in the entries of the matrix.
Proved in http://people.sc.fsu.edu/ jpeterson/Eigen

Definition: Let Let G be a graph on n vertices with adjacency matrix $M = [m_{ij}]$. We will call $\Sigma(M)$ the isoradial reduction of M, computed as

$$\Sigma(M)_{s=(i,j)} = M_{ss} - M_{s\bar{s}}\left(M_{\bar{s}\bar{s}} - \lambda I\right)^{-1} M_{\bar{s}s} \text{ for all } i, j \in \{1, 2, 3, ..., n\}.$$

Lemma 2: Let $A = [a_{ij}]$ be irreducible and nonnegative. Then $\frac{\delta\rho(A)}{\delta a_{ij}} > 0$ for all i and j.

Lemma 3: Let A be an adjacency matrix of a graph G and let $\{A_1, A_2, ..., A_k\}$ be the adjacency matrices of the strongly connected components of G. Then $\sigma(A) = \bigcup_{i=1}^{k} \sigma(A_i)$.

Theorem: Let G be a graph on n vertices with adjacency matrix $M = [m_{ij}]$ and let $\Sigma(M)$ be the isoradial reduction of M. If M is nonnegative and irreducible, then $\Sigma(M)$ exists.

Proof: Let G be a graph on n vertices with irreducible and nonnegative adjacency matrix $M = [m_{ij}]$. For each pair of nodes $i, j \in \{1, 2, 3, ..., n\}$, $M_{\bar{s}\bar{s}}$ is an $n - 2 \times n - 2$ matrix created by removing the ith and jth rows and columns from M. Let $\epsilon > 0$. We will remove these rows and columns by setting the nonzero entries equal to ϵ and then taking the limit as $\epsilon \to 0$. Since $\epsilon > 0$, the matrix will continue to be nonnegative and irreducible as the limit is taken. Thus by Lemma 2, the spectral radius will strictly degrease as $\epsilon \to 0$ since $\frac{\delta\rho(M)}{\delta M_{ij}} > 0$ for all i and j.

Let $\hat{M}^{\epsilon}_{\bar{s}\bar{s}}$ be an $n \times n$ matrix created by setting the ith and jth rows and columns to ϵ. Then it follows from Lemma 1 that

$$\lim_{\epsilon \to 0} \hat{M}^{\epsilon}_{\bar{s}\bar{s}} = \hat{M}^{\epsilon}_{\bar{s}\bar{s}}|_{\epsilon=0} = \hat{M}^{0}_{\bar{s}\bar{s}}.$$

Combining this with the result of Lemma 2, we have $\rho(\hat{M}^{0}_{\bar{s}\bar{s}}) < \rho(M)$. Now consider the graph corresponding to $\hat{M}^{0}_{\bar{s}\bar{s}}$, with the exception of the isolated nodes i and j, this graph has the same strongly connected components as the graph corresponding to $\hat{M}_{\bar{s}\bar{s}}$. Thus by Lemma 3 we have that $\rho(\hat{M}_{\bar{s}\bar{s}}) = \rho(\hat{M}^{0}_{\bar{s}\bar{s}}) < \rho(M)$.

Now assume by way of contradiction that $\Sigma(M)$ does not exist. By definition, this implies that there exists $s = (i, j) \in 1, 2, ...n$ such that $M_{\bar{s}\bar{s}} - \lambda I$ is not invertible and thus $det(M_{\bar{s}\bar{s}} - \lambda I) = 0$ which by definition implies that λ is an eigenvalue for $M_{\bar{s}\bar{s}}$. Thus

$$\lambda \leq \rho(M_{\bar{s}\bar{s}}) < \rho(M) = \lambda \implies \lambda < \lambda$$

which is a contradiction. Thus $\Sigma(M)$ exists. $\square$