Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Computational and Applied Mathematics 194 (2006) 409–424
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Finite-element method for time-dependent Maxwell's equations based on an explicit-magnetic-field scheme ☆

Changfeng Ma[a, b, *]

[a] *Department of Information and Computational Science, College of Mathematics and Physics, Zhejiang Normal University, Zhejiang 321004, PR China*
[b] *Department of Computational Science and Mathematics, Guilin University of Electronic Technology, Guangxi 541004, PR China*

Received 21 September 2003; received in revised form 26 July 2005

Abstract

This paper provides a convergence analysis of the finite-element method for time-dependent Maxwell's equations by means of an explicit-magnetic-field scheme. Error estimates in finite time are given. And it is verified that provided the time-stepsize τ is sufficiently small, the proposed algorithm yields for finite time T an error of $\mathcal{O}(h^s + \tau)$ in the L^2-norm for the electric field $\boldsymbol{E}$, the magnetic field $\boldsymbol{H}$, where h is the mesh size and $\frac{1}{2} < s \leqslant 1$. In addition, some numerical results are reported in the paper.

MSC: 65M60; 65N30; 46E35; 35Q60

Keywords: Time-dependent Maxwell's equations; Finite-element method; Decoupled scheme; Error analysis

1. Introduction

The Maxwell equations describe electromagnetic phenomena. Numerical methods for the Maxwell equations are usually referred to as computational electromagnetics. There are a wide range of

☆ Supported by the National Natural Science Foundation of China and Zhejiang Science Foundation.
* Department of Information and Computational Science, College of Mathematics and Physics, Zhejiang Normal University, Zhejiang 321004, PR China. Tel.: +8605792291074; fax: +8605792291074.
E-mail address: macf@zjnu.cn.

0377-0427/$ - see front matter
doi:10.1016/j.cam.2005.08.008

applications for the numerical solutions of Maxwell equations, for instance, electromagnetic compatibility, antenna analysis and synthesis, radar cross-section calculations, microwave ovens, and diffraction of electromagnetic wave. As is known, to solving numerically the full of system of time-dependent Maxwell's equations can be extremely costly in terms of computer time. A number of different numerical methods for the equations have been proposed [1–3,6,9,12,14]. We are interested, in this paper, in the numerical approximations to time-dependent Maxwell's equations by decoupling the electric field and magnetic field.

Maxwell equations were first formulated by James Clerk Maxwell. They are

$$\varepsilon\frac{\partial \boldsymbol{E}}{\partial t} = \nabla\times\boldsymbol{H} - \sigma\boldsymbol{E} - \boldsymbol{J}_{\mathrm{s}} \quad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\mu\frac{\partial \boldsymbol{H}}{\partial t} = -\nabla\times\boldsymbol{E} \quad \text{in } Q_T = \Omega\times(0,T), \tag{1.2}$$

where Ω is a bounded Lipschitz continuous polyhedral domain in $\Re^3$, and $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{H}(\boldsymbol{x},t)$ the electric and magnetic fields, respectively. And $\varepsilon(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ are the dielectric constant and the conductivity of the medium, respectively, while $\mu(\boldsymbol{x})$ and $\boldsymbol{J}_{\mathrm{s}}(\boldsymbol{x},t)\in L^\infty(\Omega)$ are the magnetic permeability of the material in Ω and source electric current density, respectively. We assume that the boundary of Ω, denoted by Γ, is a perfect conductor, that is,

$$\boldsymbol{n}\times\boldsymbol{E} = 0 \quad \text{on } \Sigma_T = \Gamma\times(0,T), \tag{1.3}$$

where $\boldsymbol{n}$ is the unit normal vector to Γ. We supplement Maxwell's Eqs. (1.1)–(1.2) with initial conditions:

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(\boldsymbol{x}) \quad \text{in } \Omega, \tag{1.4}$$

where $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are given functions and $\boldsymbol{H}_0$ satisfies

$$\nabla\cdot(\mu\boldsymbol{H}_0) = 0 \ \text{ in } \Omega, \quad \boldsymbol{n}\cdot\boldsymbol{H}_0 = 0 \ \text{ on } \Gamma. \tag{1.5}$$

The divergence-free condition in (1.5) together with (1.2) implies that

$$\nabla\cdot(\mu\boldsymbol{H}) = 0 \quad \text{in } Q_T \tag{1.6}$$

which is usually included with (1.1)–(1.2) in the statement of Maxwell's equations. In addition, the boundary condition in (1.5) together with (1.1) and (1.2) implies

$$\boldsymbol{n}\cdot\boldsymbol{H} = 0 \quad \text{on } \Sigma_T. \tag{1.7}$$

Furthermore, for the sake of simplicity, it is assumed that the coefficients ε, μ and σ are $L^\infty(\Omega)$ functions, that is, there exist positive constants $\underline{\varepsilon}$, $\underline{\mu}$, $\bar{\varepsilon}$, $\bar{\mu}$ and $\bar{\sigma}$ such that

$$\left.\begin{aligned} &\underline{\varepsilon} \leqslant \varepsilon(\boldsymbol{x}) \leqslant \bar{\varepsilon} < \infty,\\ &\underline{\mu} \leqslant \mu(\boldsymbol{x}) \leqslant \bar{\mu} < \infty,\\ &0 \leqslant \sigma(\boldsymbol{x}) \leqslant \bar{\sigma} < \infty, \end{aligned}\right\} \quad \text{a.e. in } \Omega. \tag{1.8}$$

One can easily find that the initial-boundary problem (1.1)–(1.4) and (1.7) is well posed under hypothesis (1.8).

Due to practical interest, there has been a great deal of work on numerical approximations to time-dependent Maxwell's equations and also on the convergence analysis of a numerical scheme for stationary Maxwell's equations and related models; see, for example, [1,5,14] and the references therein. However, there does not exist much work on the convergence analysis or error estimates for fully discrete time-dependent Maxwell's equations. For some recent work in this aspect, we refer readers to [6,8,11] for time-dependent Maxwell systems with continuous coefficients and Chen [3] with discontinuous coefficients.

In this paper we will study a finite-element approximation for time-dependent Maxwell systems by a so-called explicit-magnetic-field scheme. In Section 2, a decoupled scheme for Maxwell equations is described in detail. In Section 3, we discuss the semidiscrete approximation of the proposed scheme and give its so-called continuous error bounds. Section 4 is devoted to the spatial approximation of the semidiscrete scheme and gain so-called discrete error estimates. Finally, some numerical results are reported in Section 5.

2. A decoupled scheme for Maxwell equations

Let us first introduce the following notations used in the sequel. We define

$$H(\mathbf{curl};\Omega)=\{\boldsymbol{v}\in L^2(\Omega)^3 : \nabla\times\boldsymbol{v}\in L^2(\Omega)^3\},$$

$$H_0(\mathbf{curl};\Omega)=\{\boldsymbol{v}\in H(\mathbf{curl};\Omega) : \boldsymbol{n}\times\boldsymbol{v}|_\Gamma=\mathbf{0}\},$$

$$H^\alpha(\mathbf{curl};\Omega)=\{\boldsymbol{v}\in H^\alpha(\Omega)^3 : \nabla\times\boldsymbol{v}\in H^\alpha(\Omega)^3\},$$

where α is a nonnegative number. $H(\mathbf{curl};\Omega)$ and $H^\alpha(\mathbf{curl};\Omega)$ are equipped with the following norms:

$$\|\boldsymbol{v}\|^2_{0,\mathbf{curl}}=\|\boldsymbol{v}\|_0^2+\|\nabla\times\boldsymbol{v}\|_0^2,$$

$$\|\boldsymbol{v}\|^2_{\alpha,\mathbf{curl}}=\|\boldsymbol{v}\|_\alpha^2+\|\nabla\times\boldsymbol{v}\|_\alpha^2.$$

Here and hereafter, $\|\cdot\|_0$ will always mean the $L^2(\Omega)^3$-norm (or $L^2(\Omega)$-norm, if only scalar functions are involved). In addition, the Green's formula of integration by parts is as follows:

$$(\boldsymbol{v},\nabla\times\boldsymbol{w})-(\nabla\times\boldsymbol{v},\boldsymbol{w})=\int_\Gamma \boldsymbol{v}\times\boldsymbol{n}\cdot\boldsymbol{w},\quad \forall\boldsymbol{v}\in H(\mathbf{curl};\Omega),\quad \forall\boldsymbol{w}\in H^1(\Omega)^3. \tag{2.1}$$

Hereafter we shall repeatedly use the following discrete Gronwall inequality (see [13]):

Lemma 2.1. *Let $\delta, g_0, a_n, b_n, c_n$ and $\gamma_n (n=0,1,\ldots)$ be a sequence of nonnegative numbers so that*

$$a_n+\delta\sum_{i=0}^n b_i\leqslant\delta\sum_{i=0}^n\gamma_i a_i+\delta\sum_{i=0}^n c_i+g_0.$$

Assume that $\gamma_i\delta<1$ for all i, and set $\sigma_i=(1-\gamma_i\delta)^{-1}$. Then we obtain for all $n\geqslant 0$

$$a_n+\delta\sum_{i=0}^n b_i\leqslant\left(\delta\sum_{i=0}^n c_i+g_0\right)\exp\left(\delta\sum_{i=0}^n\sigma_i\gamma_i\right).$$

In the sequel of this paper, C will always denote a generic constant which is independent of both the time step τ and finite-element mesh h.

Let us now propose a decoupled scheme for Maxwell systems (1.1)–(1.2). We divide the times interval $(0, T)$ into N equally spaced subintervals by using nodal points

$$0 = t^0 < t^1 < \cdots < t^N = T$$

with $t^n = n\tau, n = 0, 1, \ldots, N$. For a given sequence $\{\boldsymbol{u}^n\}_{n=1}^N \subset L^2(\Omega)^3$, we introduce the first-order backward finite difference:

$$\partial_\tau \boldsymbol{u}^n = \frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\tau}. \tag{2.2}$$

Then in view of (2.2), Eqs. (1.1)–(1.2) can be written as the following backward Euler discretization:

$$\varepsilon \frac{\boldsymbol{E}^n - \boldsymbol{E}^{n-1}}{\tau} = \nabla \times \boldsymbol{H}^n - \sigma \boldsymbol{E}^n - \boldsymbol{J}_s^n, \tag{2.3}$$

$$\mu \frac{\boldsymbol{H}^n - \boldsymbol{H}^{n-1}}{\tau} = -\nabla \times \boldsymbol{E}^n. \tag{2.4}$$

Combining (2.3) and (2.4), we can infer

$$\left(\varepsilon \frac{\boldsymbol{E}^n - \boldsymbol{E}^{n-1}}{\tau} + \sigma \boldsymbol{E}^n, \boldsymbol{F}\right) + \left(\frac{\tau}{\mu} \nabla \times \boldsymbol{E}^n, \nabla \times \boldsymbol{F}\right)$$
$$= (\boldsymbol{H}^{n-1}, \nabla \times \boldsymbol{F}) - (\boldsymbol{J}_s^n, \boldsymbol{F}), \quad \forall \boldsymbol{F} \in H_0(\mathbf{curl}; \Omega), \tag{2.5}$$

$$\boldsymbol{H}^n = \boldsymbol{H}^{n-1} - \frac{\tau}{\mu} \nabla \times \boldsymbol{E}^n. \tag{2.6}$$

Remark 2.2. Schemes (2.5)–(2.6) are so-called explicit-magnetic-field schemes or the decoupled scheme. That is to say, after obtaining solution $\boldsymbol{E}^n$ of (2.5) by using a known value $\boldsymbol{H}^{n-1}$, we can explicitly gain $\boldsymbol{H}^n$ by the recurrence formula $\boldsymbol{H}^n = \boldsymbol{H}^{n-1} - v\tau \nabla \times \boldsymbol{E}^n$, for $n = 1, \ldots, N$, where v is the inverse of the magnetic permeability μ (magnetic susceptibility).

We end this section with the following stability estimate for schemes (2.5)–(2.6):

Theorem 2.3. *There exists a constant $C > 0$ independent of τ such that for all sufficiently small τ,*

$$\varepsilon \|\boldsymbol{E}^n\|_0^2 + \mu \|\boldsymbol{H}^n\|_0^2 + \frac{\tau^2}{\mu} \sum_{n=1}^N \|\nabla \times \boldsymbol{E}^n\|_0^2 \leqslant C. \tag{2.7}$$

Proof. Without loss of generality, parameters ε, σ and μ are assumed to be constants in the sequel.

(i) By Green's formula (2.1), taking the inner product of (2.5) by $2\tau \boldsymbol{E}^n \in H_0(\mathbf{curl};\Omega)$ and using the identity $2(a-b,a)=|a|^2+|a-b|^2-|b|^2$, we obtain

$$\begin{aligned}&\varepsilon\|\boldsymbol{E}^n\|_0^2+\varepsilon\|\boldsymbol{E}^n-\boldsymbol{E}^{n-1}\|_0^2-\varepsilon\|\boldsymbol{E}^{n-1}\|_0^2+2\sigma\tau\|\boldsymbol{E}^n\|_0^2\\&\quad+\frac{2\tau^2}{\mu}\|\nabla\times\boldsymbol{E}^n\|_0^2=2\tau(\boldsymbol{H}^{n-1},\nabla\times\boldsymbol{E}^n)-2\tau(\boldsymbol{J}_s^n,\boldsymbol{E}^n).\end{aligned}\tag{2.8}$$

We take the inner product of (2.3) by $2\tau\boldsymbol{H}^n\in H_0(\mathbf{curl};\Omega)$ to obtain

$$\mu\|\boldsymbol{H}^n\|_0^2+\mu\|\boldsymbol{H}^n-\boldsymbol{H}^{n-1}\|_0^2-\mu\|\boldsymbol{H}^{n-1}\|_0^2=-2\tau(\nabla\times\boldsymbol{E}^n,\boldsymbol{H}^n).\tag{2.9}$$

(ii) After summing up (2.8) and (2.9), we gain

$$\begin{aligned}&\varepsilon\|\boldsymbol{E}^n\|_0^2-\varepsilon\|\boldsymbol{E}^{n-1}\|_0^2+2\sigma\tau\|\boldsymbol{E}^n\|_0^2+\frac{2\tau^2}{\mu}\|\nabla\times\boldsymbol{E}^n\|_0^2\\&\quad+\mu\|\boldsymbol{H}^n\|_0^2+\mu\|\boldsymbol{H}^n-\boldsymbol{H}^{n-1}\|_0^2-\mu\|\boldsymbol{H}^{n-1}\|_0^2\\&\qquad\leqslant-2\tau(\boldsymbol{H}^n-\boldsymbol{H}^{n-1},\nabla\times\boldsymbol{E}^n)-2\tau(\boldsymbol{J}_s^n,\boldsymbol{E}^n).\end{aligned}\tag{2.10}$$

Note that, by the inequality $2(a,b)\leqslant\gamma|a|^2+|b|^2/\gamma$,

$$-2\tau(\boldsymbol{H}^n-\boldsymbol{H}^{n-1},\nabla\times\boldsymbol{E}^n)\leqslant\mu\|\boldsymbol{H}^n-\boldsymbol{H}^{n-1}\|_0^2+\frac{\tau^2}{\mu}\|\nabla\times\boldsymbol{E}^n\|_0^2$$

and

$$-2\tau(\boldsymbol{J}_s^n,\boldsymbol{E}^n)\leqslant C\tau\|\boldsymbol{E}^n\|_0^2+C\tau\|\boldsymbol{J}_s^n\|_0^2.$$

Then, by these bounds, it follows from (2.10) that

$$\begin{aligned}&\varepsilon\|\boldsymbol{E}^n\|_0^2-\varepsilon\|\boldsymbol{E}^{n-1}\|_0^2+2\sigma\tau\|\boldsymbol{E}^n\|_0^2\\&\quad+\frac{\tau^2}{\mu}\|\nabla\times\boldsymbol{E}^n\|_0^2+\mu\|\boldsymbol{H}^n\|_0^2-\mu\|\boldsymbol{H}^{n-1}\|_0^2\\&\qquad\leqslant C\tau\|\boldsymbol{E}^n\|_0^2+C\tau\|\boldsymbol{J}_s^n\|_0^2.\end{aligned}\tag{2.11}$$

Adding up from $n=1$ to N we have

$$\begin{aligned}&\varepsilon\|\boldsymbol{E}^N\|_0^2+\mu\|\boldsymbol{H}^N\|_0^2+\frac{\tau^2}{\mu}\sum_{n=1}^{N}\|\nabla\times\boldsymbol{E}^n\|_0^2\\&\quad\leqslant\varepsilon\|\boldsymbol{E}^0\|_0^2+\mu\|\boldsymbol{H}^0\|_0^2+C\tau\sum_{n=1}^{N}\|\boldsymbol{E}^n\|_0^2+C\int_0^T\|\boldsymbol{J}_s(t)\|_0^2\,\mathrm{d}t.\end{aligned}$$

By discrete Gronwall inequality Lemma 2.1, one can easily find that, for all sufficiently small τ, (2.7) holds. □

3. Semidiscrete approximation

In this section, we shall propose a semidiscrete decoupled scheme and give so-called continuous error estimates. We define two sequences of approximate electric field $\{\boldsymbol{E}^n \in H_0(\mathbf{curl};\Omega)\}$ and magnetic field $\{\boldsymbol{H}^n \in H(\mathbf{curl};\Omega)\}$ as follows:

- **The semidiscrete decoupled scheme**

Step 1 (Initialization): The sequences $\{\boldsymbol{E}^n \in H_0(\mathbf{curl};\Omega)\}$ and $\{\boldsymbol{H}^n \in H(\mathbf{curl};\Omega)\}$ are initialized by

$$\boldsymbol{E}^0 = \boldsymbol{E}(t=0) \quad \text{and} \quad \boldsymbol{H}^0 = \boldsymbol{H}(t=0), \tag{3.1}$$

respectively.

Step 2 (Time loop): For $1 < n < N$, seek $\{\boldsymbol{E}^n \in H_0(\mathbf{curl};\Omega)\}$

$$\left(\varepsilon \frac{\boldsymbol{E}^n - \boldsymbol{E}^{n-1}}{\tau} + \sigma \boldsymbol{E}^n, \boldsymbol{F}\right) + \left(\frac{\tau}{\mu}\nabla \times \boldsymbol{E}^n, \nabla \times \boldsymbol{F}\right)$$
$$= (\boldsymbol{H}^{n-1}, \nabla \times \boldsymbol{F}) - (\boldsymbol{J}_{\mathrm{s}}^n, \boldsymbol{F}), \quad \forall \boldsymbol{F} \in H_0(\mathbf{curl};\Omega) \tag{3.2}$$

and set

$$\boldsymbol{H}^n = \boldsymbol{H}^{n-1} - \frac{\tau}{\mu}\nabla \times \boldsymbol{E}^n. \tag{3.3}$$

We assume in the sequel that the continuous solution $(\boldsymbol{E}, \boldsymbol{H})$ of initial-boundary value problem (1.1)–(1.4) and (1.7) is unique and satisfies:

(H1) $\int_0^{\mathrm{T}} (\|\boldsymbol{E}''(t)\|_0^2 + \|\boldsymbol{H}''(t)\|_0^2)\,\mathrm{d}t \leqslant C$.

In the sequel we use C as a generic constant depending on $\boldsymbol{J}_{\mathrm{s}}, \boldsymbol{E}_0, \boldsymbol{H}_0$, ε, σ, μ and Ω, but not on the time step τ or on the mesh size h.

Let us now give the error bounds of schemes (3.2)–(3.3). We define the continuous errors (as for the spatial variables) as

$$\boldsymbol{\eta}_{\mathrm{c}}^n = \boldsymbol{E}(t^n) - \boldsymbol{E}^n, \quad \boldsymbol{\theta}_{\mathrm{c}}^n = \boldsymbol{H}(t^n) - \boldsymbol{H}^n.$$

For convenience, we introduce the notations: $\forall \boldsymbol{w} \in L^2(\Omega)^3$,

$$\|\boldsymbol{w}\|_\varepsilon^2 = (\varepsilon \boldsymbol{w}, \boldsymbol{w}), \quad \|\boldsymbol{w}\|_\sigma^2 = (\sigma \boldsymbol{w}, \boldsymbol{w}),$$
$$\|\boldsymbol{w}\|_\mu^2 = (\mu \boldsymbol{w}, \boldsymbol{w}), \quad \|\nabla \times \boldsymbol{w}\|_{\mu^{-1}}^2 = (\mu^{-1}\nabla \times \boldsymbol{w}, \nabla \times \boldsymbol{w}).$$

Then the continuous error estimates read as follows:

Theorem 3.1. *Assume that Assumption* (H1) *holds. Then, there exists a constant* $C > 0$ *independent of* τ *such that for all sufficiently small* τ,

$$\|\boldsymbol{\eta}_{\mathrm{c}}^N\|_\varepsilon^2 + \|\boldsymbol{\theta}_{\mathrm{c}}^N\|_\mu^2 + \frac{1}{2}\tau^2 \sum_{n=1}^{N} \|\nabla \times \boldsymbol{\eta}_{\mathrm{c}}^n\|_{\mu^{-1}}^2 \leqslant C\tau^2. \tag{3.4}$$

Proof. We argue in the following three steps:

Step 1: It follows from (1.1)–(1.2) for all $\boldsymbol{F} \in H_0(\mathbf{curl}; \Omega)$,

$$\begin{cases} \left(\varepsilon \dfrac{\boldsymbol{E}(t^n) - \boldsymbol{E}(t^{n-1})}{\tau}, \boldsymbol{F}\right) + (\sigma \boldsymbol{E}(t^n), \boldsymbol{F}) \\ \quad -(\boldsymbol{H}(t^n), \nabla \times \boldsymbol{F}) = (-\boldsymbol{J}_s^n + \varepsilon \boldsymbol{R}_1^n, \boldsymbol{F}), \\ \mu \dfrac{\boldsymbol{H}(t^n) - \boldsymbol{H}(t^{n-1})}{\tau} = -\nabla \times \boldsymbol{E}(t^n) + \mu \boldsymbol{R}_2^n, \end{cases} \tag{3.5}$$

where the truncation errors $\boldsymbol{R}_1^n$ and $\boldsymbol{R}_2^n$ are defined by

$$\begin{cases} \boldsymbol{R}_1^n = -\dfrac{1}{\tau} \int_{t^{n-1}}^{t^n} (t - t^{n-1}) \boldsymbol{E}''(t)\, \mathrm{d}t, \\ \boldsymbol{R}_2^n = -\dfrac{1}{\tau} \int_{t^{n-1}}^{t^n} (t - t^{n-1}) \boldsymbol{H}''(t)\, \mathrm{d}t. \end{cases} \tag{3.6}$$

By (3.5), we can deduce the following:

$$\begin{cases} \left(\varepsilon \dfrac{\boldsymbol{E}(t^n) - \boldsymbol{E}(t^{n-1})}{\tau} + \sigma \boldsymbol{E}(t^n), \boldsymbol{F}\right) + \left(\dfrac{\tau}{\mu} \nabla \times \boldsymbol{E}(t^n), \nabla \times \boldsymbol{F}\right) \\ \quad = (-\boldsymbol{J}_s^n + \varepsilon \boldsymbol{R}_1^n, \boldsymbol{F}) + (\boldsymbol{H}(t^{n-1}) + \tau \boldsymbol{R}_2^n, \nabla \times \boldsymbol{F}), \\ \boldsymbol{H}(t^n) = \boldsymbol{H}(t^{n-1}) - \dfrac{\tau}{\mu} \nabla \times \boldsymbol{E}(t^n) + \tau \boldsymbol{R}_2^n. \end{cases} \tag{3.7}$$

Subtracting (3.2) and (3.3) from the first equation and second Eq. of (3.7) and using the definitions of $\boldsymbol{\eta}_c^n$ and $\boldsymbol{\theta}_c^n$, we obtain

$$\left(\varepsilon \frac{\boldsymbol{\eta}_c^n - \boldsymbol{\eta}_c^{n-1}}{\tau}, \boldsymbol{F}\right) + (\sigma \boldsymbol{\eta}_c^n, \boldsymbol{F}) + \left(\frac{\tau}{\mu} \nabla \times \boldsymbol{\eta}_c^n, \nabla \times \boldsymbol{F}\right)$$
$$= (\varepsilon \boldsymbol{R}_1^n, \boldsymbol{F}) + (\boldsymbol{\theta}_c^{n-1}, \nabla \times \boldsymbol{F}) + \tau (\boldsymbol{R}_2^n, \nabla \times \boldsymbol{F}) \tag{3.8}$$

and

$$\boldsymbol{\theta}_c^n = \boldsymbol{\theta}_c^{n-1} - \frac{\tau}{\mu} \nabla \times \boldsymbol{\eta}_c^n + \tau \boldsymbol{R}_2^n. \tag{3.9}$$

Step 2: Taking $\boldsymbol{F} = 2\tau \boldsymbol{\eta}_c^n \in H_0(\mathbf{curl}; \Omega)$ in (3.8) and using the inequality $2(a - b, a) \geqslant |a|^2 - |b|^2$, we obtain

$$\|\boldsymbol{\eta}_c^n\|_\varepsilon^2 - \|\boldsymbol{\eta}_c^{n-1}\|_\varepsilon^2 + 2\tau \|\boldsymbol{\eta}_c^n\|_\sigma^2 + 2\tau^2 \|\nabla \times \boldsymbol{\eta}_c^n\|_v^2$$
$$\leqslant 2\tau(\boldsymbol{\theta}_c^{n-1}, \nabla \times \boldsymbol{\eta}_c^n) + 2\tau(\varepsilon \boldsymbol{R}_1^n, \boldsymbol{\eta}_c^n) + 2\tau^2 (\boldsymbol{R}_2^n, \nabla \times \boldsymbol{\eta}_c^n). \tag{3.10}$$

Noting that, by the inequality $2(a,\ b) \leqslant \gamma |a|^2 + |b|^2/\gamma$,

$$2\tau(\varepsilon \boldsymbol{R}_1^n, \boldsymbol{\eta}_c^n) \leqslant \gamma_1 \tau \|\boldsymbol{\eta}_c^n\|_\varepsilon^2 + C_{\gamma_1} \tau \|\boldsymbol{R}_1^n\|_0^2$$
$$\leqslant \gamma_1 \tau \|\boldsymbol{\eta}_c^n\|_\varepsilon^2 + C_{\gamma_1} \tau^2 \int_{t^{n-1}}^{t^n} \|\boldsymbol{E}''(t)\|_0^2 \, \mathrm{d}t$$

and

$$2\tau^2(\boldsymbol{R}_2^n, \nabla\times\boldsymbol{\eta}_c^n) \leqslant \gamma_2\tau^3\|\nabla\times\boldsymbol{\eta}_c^n\|_v^2 + C_{\gamma_2}\tau\|\boldsymbol{R}_2^n\|_0^2$$
$$\leqslant \gamma_2\tau^3\|\nabla\times\boldsymbol{\eta}_c^n\|_v^2 + C_{\gamma_2}\tau^2\int_{t^{n-1}}^{t^n}\|\boldsymbol{H}''(t)\|_0^2\,\mathrm{d}t.$$

Thereby, using these bounds, it follows from (3.10),

$$\|\boldsymbol{\eta}_c^n\|_\varepsilon^2 - \|\boldsymbol{\eta}_c^{n-1}\|_\varepsilon^2 + 2\tau\|\boldsymbol{\eta}_c^n\|_\sigma^2 + 2\tau^2\|\nabla\times\boldsymbol{\eta}_c^n\|_v^2$$
$$\leqslant 2\tau(\boldsymbol{\theta}_c^{n-1}, \nabla\times\boldsymbol{\eta}_c^n) + \gamma_1\tau\|\boldsymbol{\eta}_c^n\|_\varepsilon^2$$
$$+ \gamma_2\tau^3\|\nabla\times\boldsymbol{\eta}_c^n\|_0^2 + C\tau^2\int_{t^{n-1}}^{t^n}(\|\boldsymbol{E}''(t)\|_0^2 + \|\boldsymbol{H}''(t)\|_0^2)\,\mathrm{d}t.$$

Choose parameter $\gamma_2 \leqslant \dfrac{1}{2\tau}$; then, we have

$$\|\boldsymbol{\eta}_c^n\|_\varepsilon^2 - \|\boldsymbol{\eta}_c^{n-1}\|_\varepsilon^2 + 2\tau\|\boldsymbol{\eta}_c^n\|_\sigma^2 + \tfrac{3}{2}\tau^2\|\nabla\times\boldsymbol{\eta}_c^n\|_v^2$$
$$\leqslant 2\tau(\boldsymbol{\theta}_c^{n-1}, \nabla\times\boldsymbol{\eta}_c^n) + \gamma_1\tau\|\boldsymbol{\eta}_c^n\|_\varepsilon^2$$
$$+ C\tau^2\int_{t^{n-1}}^{t^n}(\|\boldsymbol{E}''(t)\|_0^2 + \|\boldsymbol{H}''(t)\|_0^2)\,\mathrm{d}t. \tag{3.11}$$

Step 3: We take the inner product of (3.9) by $2\mu\boldsymbol{\theta}_c^n \in L^2(\Omega)^3$ and using the identity $2(a-b,a) = |a|^2 + |a-b|^2 - |b|^2$, we obtain

$$\|\boldsymbol{\theta}_c^n\|_\mu^2 + \|\boldsymbol{\theta}_c^n - \boldsymbol{\theta}_c^{n-1}\|_\mu^2 - \|\boldsymbol{\theta}_c^{n-1}\|_\mu^2$$
$$= -2\tau(\nabla\times\boldsymbol{\eta}_c^n, \boldsymbol{\theta}_c^n) + 2\tau(\mu\boldsymbol{R}_2^n, \boldsymbol{\theta}_c^n)$$
$$\leqslant -2\tau(\nabla\times\boldsymbol{\eta}_c^n, \boldsymbol{\theta}_c^n) + C\tau\|\boldsymbol{\theta}_c^n\|_\mu^2 + C\tau^2\int_{t^{n-1}}^{t^n}\|\boldsymbol{H}''(t)\|_0^2\,\mathrm{d}t. \tag{3.12}$$

By (3.11) + (3.12) we obtain

$$\|\boldsymbol{\eta}_c^n\|_\varepsilon^2 - \|\boldsymbol{\eta}_c^{n-1}\|_\varepsilon^2 + 2\tau\|\boldsymbol{\eta}_c^n\|_\sigma^2 + \tfrac{3}{2}\tau^2\|\nabla\times\boldsymbol{\eta}_c^n\|_{\mu^{-1}}^2$$
$$+ \|\boldsymbol{\theta}_c^n\|_\mu^2 + \|\boldsymbol{\theta}_c^n - \boldsymbol{\theta}_c^{n-1}\|_\mu^2 - \|\boldsymbol{\theta}_c^{n-1}\|_\mu^2$$
$$\leqslant -2\tau(\boldsymbol{\theta}_c^n - \boldsymbol{\theta}_c^{n-1}, \nabla\times\boldsymbol{\eta}_c^n) + C\tau\|\boldsymbol{\eta}_c^n\|_\varepsilon^2$$
$$+ C\tau\|\boldsymbol{\theta}_c^n\|_\mu^2 + C\tau^2\int_{t^{n-1}}^{t^n}(\|\boldsymbol{E}''\|_0^2 + \|\boldsymbol{H}''\|_0^2)\,\mathrm{d}t.$$

Since

$$-2\tau(\boldsymbol{\theta}_c^n - \boldsymbol{\theta}_c^{n-1}, \nabla\times\boldsymbol{\eta}_c^n) = -2(\mu(\boldsymbol{\theta}_c^n - \boldsymbol{\theta}_c^{n-1}), \mu^{-1}\tau\nabla\times\boldsymbol{\eta}_c^n)$$
$$\leqslant \|\boldsymbol{\theta}_c^n - \boldsymbol{\theta}_c^{n-1}\|_\mu^2 + \tau^2\|\nabla\times\boldsymbol{\eta}_c^n\|_{\mu^{-1}}^2,$$

we have

$$
\begin{aligned}
&\|\boldsymbol{\eta}_{\mathrm{c}}^n\|_\varepsilon^2+\|\boldsymbol{\theta}_{\mathrm{c}}^n\|_\mu^2+2\tau\|\boldsymbol{\eta}_{\mathrm{c}}^n\|_\sigma^2+\tfrac{1}{2}\tau^2\|\nabla\times\boldsymbol{\eta}_{\mathrm{c}}^n\|_{\mu^{-1}}^2\\
&\quad\leqslant\|\boldsymbol{\eta}_{\mathrm{c}}^{n-1}\|_\varepsilon^2+\|\boldsymbol{\theta}_{\mathrm{c}}^{n-1}\|_\mu^2+C\tau(\|\boldsymbol{\eta}_{\mathrm{c}}^n\|_\varepsilon^2+\|\boldsymbol{\theta}_{\mathrm{c}}^n\|_\mu^2)\\
&\qquad+C\tau^2\int_{t^{n-1}}^{t^n}(\|\boldsymbol{E}''(t)\|_0^2+\|\boldsymbol{H}''(t)\|_0^2)\,\mathrm{d}t.
\end{aligned}\tag{3.13}
$$

Summing up for $n=1,\ldots,N$, we obtain

$$
\begin{aligned}
&\|\boldsymbol{\eta}_{\mathrm{c}}^N\|_\varepsilon^2+\|\boldsymbol{\theta}_{\mathrm{c}}^N\|_\mu^2+2\tau\sum_{n=1}^{N}\|\boldsymbol{\eta}_{\mathrm{c}}^n\|_\sigma^2+\frac{1}{2}\tau^2\sum_{n=1}^{N}\|\nabla\times\boldsymbol{\eta}_{\mathrm{c}}^n\|_{\mu^{-1}}^2\\
&\quad\leqslant\|\boldsymbol{\eta}_{\mathrm{c}}^0\|_\varepsilon^2+\|\boldsymbol{\theta}_{\mathrm{c}}^0\|_\mu^2+C\tau\sum_{n=1}^{N}(\|\boldsymbol{\eta}_{\mathrm{c}}^n\|_0^2+\|\boldsymbol{\theta}_{\mathrm{c}}^n\|_0^2)\\
&\qquad+C\tau^2\int_0^T(\|\boldsymbol{E}''(t)\|_0^2+\|\boldsymbol{H}''(t)\|_0^2)\,\mathrm{d}t.
\end{aligned}
$$

Using initialization condition (3.1), Assumption (H3) and discrete Gronwall inequality Lemma 2.1, it can easily be found that (3.4) follows. □

4. Fully discrete decoupled scheme

In this section, we discuss a spatial approximation of the semi-discrete (3.2)–(3.3) with Nédélec's finite-elements and gain the so-called discrete errors bound.

We first triangulate the solution domain Ω and assume that $\mathscr{T}_h$ is a regular partition of Ω into tetrahedrons with a mesh h(see [4,6,13]). An element of $\mathscr{T}_h$ is denoted by K, and the diameters of K and its inscribed ball are denoted by h_K and ρ_K, respectively. We then introduce the following Nédélec's $H(\mathbf{curl};\Omega)$ conforming finite-element space:

$$V_h=\{\boldsymbol{v}_h\in H(\mathbf{curl};\Omega);\,\boldsymbol{v}_h|_K\in\mathscr{P}_1^3,\forall K\in\mathscr{T}_h\},$$

where $\mathscr{P}_1$ is the space of linear polynomials. It was proved in Nédélec [10] that any function $\boldsymbol{v}$ in V_h can be uniquely determined by the degrees of freedom in the momentum set $M_E(\boldsymbol{v})$ on each element $K\in\mathscr{T}_h$. Here $M_E(\boldsymbol{v})$ is defined as follows:

$$M_E(\boldsymbol{v})=\left\{\int_e\boldsymbol{v}\cdot\tau\,\mathrm{d}s;\,e\text{ is any an edge of }K\right\},$$

where τ is the unit vector along the edge e. We know that the integrals required in the definition of $M_E(\boldsymbol{v})$ make sense for any $\boldsymbol{v}\in H^s(K)^3$, with $s>\frac{1}{2}$. Thus, we can define an interpolation $\mathscr{I}_h\boldsymbol{v}$ of any $\boldsymbol{v}\in H^s(K)^3$ such that $\mathscr{I}_h\boldsymbol{v}\in V_h$ and $\mathscr{I}_h\boldsymbol{v}$ have the same degrees of freedom as $\boldsymbol{v}$ on each K in $\mathscr{T}_h$.

In order to take the boundary condition $\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0}$ on $\Gamma = \partial\Omega$, we define a subspace of V_h:

$$X_h = V_h \cap H_0(\mathbf{curl}; \Omega).$$

These finite-element spaces satisfy the following approximating properties(see [6,7,9,10]):

Lemma 4.1. *There exists a constant* $C > 0$ *such that*

$$\|\boldsymbol{u} - \mathscr{I}_h \boldsymbol{u}\|_0 + \|\nabla \times (\boldsymbol{u} - \mathscr{I}_h \boldsymbol{u})\|_0 \leqslant C h^s \|\boldsymbol{u}\|_{s,\mathbf{curl}}$$

for all $\boldsymbol{u} \in H^s(\mathbf{curl}; \Omega)$ *with* $\frac{1}{2} < s \leqslant 1$.

Let us now define a fully discrete version (3.2)–(3.3). We define two sequences of approximate electric field $\{\boldsymbol{E}_h^n \in X_h\}$ and magnetic field $\{\boldsymbol{H}_h^n \in V_h\}$ as follows:

- **The fully discrete decoupled scheme**

Step 1 (Initialization): The sequences $\{\boldsymbol{E}_h^n \in X_h\}$ and $\{\boldsymbol{H}_h^n \in V_h\}$ are initialized by

$$\boldsymbol{E}_h^0 = \mathscr{I}_h^1 \boldsymbol{E}^0 \quad \text{and} \quad \boldsymbol{H}_h^0 = \mathscr{I}_h^2 \boldsymbol{H}^0, \tag{4.1}$$

respectively. Here $\mathscr{I}_h^1 \boldsymbol{E}^0$ and $\mathscr{I}_h^2 \boldsymbol{H}^0$ are finite-element interpolants of semidiscrete solutions $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$, respectively.

Step 2 (Time loop): For $1 < n < N$, seek $\{\boldsymbol{E}_h^n \in X_h\}$

$$\left(\varepsilon \frac{\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}}{\tau} + \sigma \boldsymbol{E}_h^n, \boldsymbol{F}_h\right) + \left(\frac{\tau}{\mu} \nabla \times \boldsymbol{E}_h^n, \nabla \times \boldsymbol{F}_h\right)$$
$$= (\boldsymbol{H}_h^{n-1}, \nabla \times \boldsymbol{F}_h) - (\boldsymbol{J}_{\mathrm{s}}^n, \boldsymbol{F}_h), \quad \forall \boldsymbol{F}_h \in X_h \tag{4.2}$$

and set

$$\boldsymbol{H}_h^n = \boldsymbol{H}_h^{n-1} - \frac{\tau}{\mu} \nabla \times \boldsymbol{E}_h^n. \tag{4.3}$$

We now proceed to obtain error estimates for the fully discrete electric field $\boldsymbol{E}_h^n$ (resp. the fully discrete magnetic field $\boldsymbol{H}_h^n$) as an approximation of the semi-discrete solution $\boldsymbol{E}^n$ (resp. $\boldsymbol{H}^n$) under suitable regularity assumptions on the continuous problem. We define and split the error of the method as follows:

$$\boldsymbol{\eta}^n = \boldsymbol{E}(t^n) - \boldsymbol{E}_h^n = \boldsymbol{\eta}_{\mathrm{c}}^n + (\boldsymbol{E}^n - \mathscr{I}_h^1 \boldsymbol{E}^n) + \boldsymbol{\eta}_{\mathrm{d}}^n,$$
$$\boldsymbol{\theta}^n = \boldsymbol{H}(t^n) - \boldsymbol{H}_h^n = \boldsymbol{\theta}_{\mathrm{c}}^n + (\boldsymbol{H}^n - \mathscr{I}_h^2 \boldsymbol{H}^n) + \boldsymbol{\theta}_{\mathrm{d}}^n,$$

where the discrete errors are defined as

$$\boldsymbol{\eta}_{\mathrm{d}}^n = \mathscr{I}_h^1 \boldsymbol{E}^n - \boldsymbol{E}_h^n, \quad \boldsymbol{\theta}_{\mathrm{d}}^n = \mathscr{I}_h^2 \boldsymbol{H}^n - \boldsymbol{H}_h^n,$$

where $\mathscr{I}_h^1 \boldsymbol{E}^n$ and $\mathscr{I}_h^2 \boldsymbol{H}^n$ are finite-element interpolants of semidiscrete solutions $\boldsymbol{E}^n$ and $\boldsymbol{H}^n$, respectively, and satisfying the following interpolation properties:

$$\|\boldsymbol{E}^n - \mathscr{I}_h^1 \boldsymbol{E}^n\|_0 + \|\nabla \times (\boldsymbol{E}^n - \mathscr{I}_h^1 \boldsymbol{E}^n)\|_0 \leqslant C h^s \|\boldsymbol{E}^n\|_{s,\mathbf{curl}}, \tag{4.4}$$

$$\|\boldsymbol{H}^n - \mathscr{I}_h^2 \boldsymbol{H}^n\|_0 + \|\nabla \times (\boldsymbol{H}^n - \mathscr{I}_h^2 \boldsymbol{H}^n)\|_0 \leqslant C h^s \|\boldsymbol{H}^n\|_{s,\mathbf{curl}} \tag{4.5}$$

for all $(\boldsymbol{E}^n, \boldsymbol{H}^n) \in H^s(\mathbf{curl}; \Omega) \times H^s(\mathbf{curl}; \Omega)$, $\frac{1}{2} < s \leqslant 1$.

Later on, we will need the following estimates for $B = H^s(\mathbf{curl}; \Omega)$ with $\frac{1}{2} < s \leqslant 1$ or $B = H^\alpha(\Omega)^3$ with $\alpha \geqslant 0$:

$$\|\partial_\tau \boldsymbol{E}^n\|_0^2 \leqslant \frac{1}{\tau} \int_{t^{n-1}}^{t^n} \|\boldsymbol{H}'(t)\|_0^2 \,\mathrm{d}t, \quad \forall \boldsymbol{E} \in H^1(0, T; B), \tag{4.6}$$

$$\|\partial_\tau \boldsymbol{H}^n\|_0^2 \leqslant \frac{1}{\tau} \int_{t^{n-1}}^{t^n} \|\boldsymbol{H}'(t)\|_0^2 \,\mathrm{d}t, \quad \forall \boldsymbol{H} \in H^1(0, T; B), \tag{4.7}$$

where notation $\partial_\tau \boldsymbol{p}^n = (\boldsymbol{p}^n - \boldsymbol{p}^{n-1})/\tau$.

Furthermore, we assume that

(H2) $\boldsymbol{E}_t, \boldsymbol{H}_t \in L^2(0, T; L^2(\Omega)^3)$; and

(H3) $(\boldsymbol{E}^n)_n$, $(\boldsymbol{H}^n)_n$ uniformly bounded in $H^s(curl, \Omega)$.

The following theorem is the main result of this section.

Theorem 4.2. *Let* $(\boldsymbol{E}^n, \boldsymbol{H}^n)$ *and* $(\boldsymbol{E}_h^n, \boldsymbol{H}_h^n)$ *be the solutions of the semidiscrete scheme* (3.2)–(3.3) *and the fully discrete scheme* (4.2)–(4.3), *respectively. Assumptions* (H1) *and* (H3) *hold. Assume that for* $n = 1, \ldots, N$,

$$\boldsymbol{E}^n \in H_0(\mathbf{curl}; \Omega) \cap H^s(\mathbf{curl}; \Omega) \quad \text{and} \quad \boldsymbol{H}^n \in H(\mathbf{curl}; \Omega) \cap H^s(\mathbf{curl}; \Omega),$$

with $\frac{1}{2} < s \leqslant 1$. Then, we have

$$\|\boldsymbol{\eta}_\mathrm{d}^N\|_\varepsilon^2 + \|\boldsymbol{\theta}_\mathrm{d}^N\|_\mu^2 + \frac{1}{2}\tau^2 \sum_{n=1}^{N} \|\nabla \times \boldsymbol{\eta}_\mathrm{d}^n\|_{\mu^{-1}}^2 \leqslant C h^{2s}. \tag{4.8}$$

Proof. *Step* 1: Subtracting (4.2)–(4.3) from (3.2)–(3.3), respectively, and using the definitions of $\boldsymbol{\eta}_\mathrm{d}^n$ and $\boldsymbol{\theta}_\mathrm{d}^n$, we obtain

$$\begin{aligned}
&(\varepsilon \partial_\tau \boldsymbol{\eta}_\mathrm{d}^n + \sigma \boldsymbol{\eta}_\mathrm{d}^n, \boldsymbol{F}_h) + \left(\frac{\tau}{\mu} \nabla \times \boldsymbol{\eta}_\mathrm{d}^n, \nabla \times \boldsymbol{F}_h\right) - (\boldsymbol{\theta}_\mathrm{d}^{n-1}, \nabla \times \boldsymbol{F}_h) \\
&\quad = (\varepsilon \partial_\tau (\mathscr{I}_h^1 \boldsymbol{E}^n - \boldsymbol{E}^n), \boldsymbol{F}_h) + (\sigma(\mathscr{I}_h^1 \boldsymbol{E}^n - \boldsymbol{E}^n), \boldsymbol{F}_h) \\
&\qquad + (\tau \mu^{-1} \nabla \times (\mathscr{I}_h^1 \boldsymbol{E}^n - \boldsymbol{E}^n), \nabla \times \boldsymbol{F}_h) \\
&\qquad - (\mathscr{I}_h^2 \boldsymbol{H}^{n-1} - \boldsymbol{H}^{n-1}, \nabla \times \boldsymbol{F}_h), \quad \forall \boldsymbol{F}_h \in X_h.
\end{aligned} \tag{4.9}$$

and

$$\mu \partial_\tau \boldsymbol{\theta}_\mathrm{d}^n + \nabla \times \boldsymbol{\eta}_\mathrm{d}^n = \nabla \times (\mathscr{I}_h^1 \boldsymbol{E}^n - \boldsymbol{E}^n) + \mu \partial_\tau (\mathscr{I}_h^2 \boldsymbol{H}^n - \boldsymbol{H}^n). \tag{4.10}$$

Step 2: Taking $\boldsymbol{F}_h = 2\tau\boldsymbol{\eta}_{\mathrm{d}}^n \in X_h$ in (4.9), we obtain

$$\begin{aligned}
&\|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\varepsilon^2 + \|\boldsymbol{\eta}_{\mathrm{d}}^n - \boldsymbol{\eta}_{\mathrm{d}}^{n-1}\|_\varepsilon^2 - \|\boldsymbol{\eta}_{\mathrm{d}}^{n-1}\|_\varepsilon^2 \\
&\quad + 2\tau\|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\sigma^2 + 2\tau^2\|\nabla\times\boldsymbol{\eta}_{\mathrm{d}}^n\|_\nu^2 - 2\tau(\boldsymbol{\theta}_{\mathrm{d}}^{n-1}, \nabla\times\boldsymbol{\eta}_{\mathrm{d}}^n) \\
&\qquad = 2\tau(\varepsilon\partial_\tau(\mathscr{I}_h^1\boldsymbol{E}^n - \boldsymbol{E}^n), \boldsymbol{\eta}_{\mathrm{d}}^n) + 2\tau(\sigma(\mathscr{I}_h^1\boldsymbol{E}^n - \boldsymbol{E}^n), \boldsymbol{\eta}_{\mathrm{d}}^n) \\
&\qquad\quad + 2\tau^2(\mu^{-1}\nabla\times(\mathscr{I}_h^1\boldsymbol{E}^n - \boldsymbol{E}^n), \nabla\times\boldsymbol{\eta}_{\mathrm{d}}^n) \\
&\qquad\quad - 2\tau(\mathscr{I}_h^2\boldsymbol{H}^{n-1} - \boldsymbol{H}^{n-1}, \nabla\times\boldsymbol{\eta}_{\mathrm{d}}^n).
\end{aligned} \tag{4.11}$$

We take the inner product of (4.10) by $2\tau\boldsymbol{\theta}_{\mathrm{d}}^n \in V_h$, to obtain

$$\begin{aligned}
&\|\boldsymbol{\theta}_{\mathrm{d}}^n\|_\mu^2 + \|\boldsymbol{\theta}_{\mathrm{d}}^n - \boldsymbol{\theta}_{\mathrm{d}}^{n-1}\|_\mu^2 - \|\boldsymbol{\theta}_{\mathrm{d}}^{n-1}\|_\mu^2 + 2\tau(\nabla\times\boldsymbol{\eta}_{\mathrm{d}}^n, \boldsymbol{\theta}_{\mathrm{d}}^n) \\
&\quad = 2\tau(\nabla\times(\mathscr{I}_h^1\boldsymbol{E}^n - \boldsymbol{E}^n), \boldsymbol{\theta}_{\mathrm{d}}^n) + 2\tau(\mu\partial_\tau(\mathscr{I}_h^2\boldsymbol{H}^n - \boldsymbol{H}^n), \boldsymbol{\theta}_{\mathrm{d}}^n).
\end{aligned} \tag{4.12}$$

By adding up (4.11) and (4.12) we have

$$\begin{aligned}
&\|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\varepsilon^2 + \|\boldsymbol{\eta}_{\mathrm{d}}^n - \boldsymbol{\eta}_{\mathrm{d}}^{n-1}\|_\varepsilon^2 - \|\boldsymbol{\eta}_{\mathrm{d}}^{n-1}\|_\varepsilon^2 + 2\tau\|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\sigma^2 \\
&\quad + 2\tau^2\|\nabla\times\boldsymbol{\eta}_{\mathrm{d}}^n\|_\nu^2 + \|\boldsymbol{\theta}_{\mathrm{d}}^n\|_\mu^2 + \|\boldsymbol{\theta}_{\mathrm{d}}^n - \boldsymbol{\theta}_{\mathrm{d}}^{n-1}\|_\mu^2 - \|\boldsymbol{\theta}_{\mathrm{d}}^{n-1}\|_\mu^2 \\
&\qquad = 2\tau(\varepsilon\partial_\tau(\mathscr{I}_h^1\boldsymbol{E}^n - \boldsymbol{E}^n), \boldsymbol{\eta}_{\mathrm{d}}^n) + 2\tau(\mu\partial_\tau(\mathscr{I}_h^2\boldsymbol{H}^n - \boldsymbol{H}^n), \boldsymbol{\theta}_{\mathrm{d}}^n) \\
&\qquad\quad + 2\tau(\sigma(\mathscr{I}_h^1\boldsymbol{E}^n - \boldsymbol{E}^n), \boldsymbol{\eta}_{\mathrm{d}}^n) - 2\tau(\boldsymbol{\theta}_{\mathrm{d}}^n - \boldsymbol{\theta}_{\mathrm{d}}^{n-1}, \nabla\times\boldsymbol{\eta}_{\mathrm{d}}^n) \\
&\qquad\quad - 2\tau(\mathscr{I}_h^2\boldsymbol{H}^{n-1} - \boldsymbol{H}^{n-1}, \nabla\times\boldsymbol{\eta}_{\mathrm{d}}^n) + 2\tau(\nabla\times(\mathscr{I}_h^1\boldsymbol{E}^n - \boldsymbol{E}^n), \boldsymbol{\theta}_{\mathrm{d}}^n) \\
&\qquad\quad + 2\tau^2(\mu^{-1}\nabla\times(\mathscr{I}_h^1\boldsymbol{E}^n - \boldsymbol{E}^n), \nabla\times\boldsymbol{\eta}_{\mathrm{d}}^n) =: \sum_{i=1}^{7} I_i.
\end{aligned} \tag{4.13}$$

Since, by (4.4)–(4.5) and (4.6)–(4.7)

$$\begin{aligned}
I_1 &\leqslant C\tau\|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\varepsilon^2 + C\tau\|\partial_\tau(\mathscr{I}_h^1\boldsymbol{E}^n - \boldsymbol{E}^n)\|_0^2 \\
&\leqslant C\tau\|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\varepsilon^2 + C\tau h^{2s}\|\partial_\tau\boldsymbol{E}^n\|_0^2 \\
&\leqslant C\tau\|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\varepsilon^2 + C\tau h^{2s} + Ch^{2s}\int_{t^{n-1}}^{t^n}\|\boldsymbol{E}'(t)\|_0^2\,\mathrm{d}t, \\
I_2 &\leqslant C\tau\|\boldsymbol{\theta}_{\mathrm{d}}^n\|_\mu^2 + C\tau\|\partial_\tau(\mathscr{I}_h^2\boldsymbol{H}^n - \boldsymbol{H}^n)\|_0^2 \\
&\leqslant C\tau\|\boldsymbol{\theta}_{\mathrm{d}}^n\|_\mu^2 + C\tau h^{2s}\|\partial_\tau\boldsymbol{H}^n\|_0^2 \\
&\leqslant C\tau\|\boldsymbol{\theta}_{\mathrm{d}}^n\|_\mu^2 + C\tau h^{2s} + Ch^{2s}\int_{t^{n-1}}^{t^n}\|\boldsymbol{H}'(t)\|_0^2\,\mathrm{d}t.
\end{aligned}$$

and

$$I_3 \leqslant \tau \|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\sigma^2 + C\tau \|\mathscr{I}_h^1 \boldsymbol{E}^n - \boldsymbol{E}^n\|_0^2 \leqslant \tau \|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\sigma^2 + C\tau h^{2s} \|\boldsymbol{E}^n\|_{s,curl}^2,$$

$$\begin{aligned}
I_4 &= -2(\mu(\boldsymbol{\theta}_{\mathrm{d}}^n - \boldsymbol{\theta}_{\mathrm{d}}^{n-1}), \mu^{-1}\tau\nabla \times \boldsymbol{\eta}_{\mathrm{d}}^n)\\
&\leqslant \|\boldsymbol{\theta}_{\mathrm{d}}^n - \boldsymbol{\theta}_{\mathrm{d}}^{n-1}\|_\mu^2 + \tau^2 \|\nabla \times \boldsymbol{\eta}_{\mathrm{d}}^n\|_{\mu^{-1}}^2,\\
I_5 &= -2\tau(\mathscr{I}_h^2 \boldsymbol{H}^{n-1} - \boldsymbol{H}_h^{n-1}, \nabla \times \boldsymbol{\eta}_{\mathrm{d}}^n)\\
&= -2\tau(\nabla \times (\mathscr{I}_h^2 \boldsymbol{H}^{n-1} - \boldsymbol{H}^{n-1}), \boldsymbol{\eta}_{\mathrm{d}}^n)\\
&\leqslant C\tau \|\nabla \times (\mathscr{I}_h^2 \boldsymbol{H}^{n-1} - \boldsymbol{H}^{n-1})\|_v + C\tau \|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\mu^2\\
&\leqslant C\tau h^{2s} \|\boldsymbol{H}^n\|_{s,curl}^2 + C\tau \|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\mu^2,\\
I_6 &\leqslant C\tau \|\boldsymbol{\theta}_{\mathrm{d}}^n\|_\mu + C\tau \|\nabla \times (\mathscr{I}_h^1 \boldsymbol{E}^n - \boldsymbol{E}^n)\|_{\mu^{-1}}^2\\
&\leqslant C\tau \|\boldsymbol{\theta}_{\mathrm{d}}^n\|_\mu + C\tau h^{2s} \|\boldsymbol{E}^n\|_{s,curl}^2,\\
I_7 &= 2\tau^2(\nabla \times (\mathscr{I}_h^1 \boldsymbol{E}^n - \boldsymbol{E}^n), \mu^{-1}\nabla \times \boldsymbol{\eta}_{\mathrm{d}}^n)\\
&\leqslant C\tau^2 h^{2s} \|\boldsymbol{E}^n\|_{s,curl}^2 + \tfrac{1}{2}\tau^2 \|\nabla \times \boldsymbol{\eta}_{\mathrm{d}}^n\|_{\mu^{-1}}.
\end{aligned}$$

By these bounds, it follows from (4.13) that

$$\begin{aligned}
&\|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\varepsilon^2 + \|\boldsymbol{\theta}_{\mathrm{d}}^n\|_\mu^2 + \tau \|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\sigma^2 + \tfrac{1}{2}\tau^2 \|\nabla \times \boldsymbol{\eta}_{\mathrm{d}}^n\|_v^2\\
&\quad \leqslant \|\boldsymbol{\eta}_{\mathrm{d}}^{n-1}\|_\varepsilon^2 + \|\boldsymbol{\theta}_{\mathrm{d}}^{n-1}\|_\mu^2 + C\tau(\|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\varepsilon^2 + \|\boldsymbol{\theta}_{\mathrm{d}}^n\|_\mu^2)\\
&\qquad + C\tau h^{2s}(\|\boldsymbol{E}^n\|_{s,curl}^2 + \|\boldsymbol{H}^n\|_{s,curl}^2)\\
&\qquad + Ch^{2s} \int_{t^{n-1}}^{t^n} (\|\boldsymbol{E}'(t)\|_0^2 + \|\boldsymbol{H}'(t)\|_0^2)\,\mathrm{d}t.
\end{aligned} \tag{4.14}$$

Step 3: After summing up (4.14) from $n = 1, \ldots, N$, we gain

$$\begin{aligned}
&\|\boldsymbol{\eta}_{\mathrm{d}}^N\|_\varepsilon^2 + \|\boldsymbol{\theta}_{\mathrm{d}}^N\|_\mu^2 + \tau \sum_{n=1}^N \|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\sigma^2 + \tfrac{1}{2}\tau^2 \sum_{n=1}^N \|\nabla \times \boldsymbol{\eta}_{\mathrm{d}}^n\|_v^2\\
&\quad \leqslant \|\boldsymbol{\eta}_{\mathrm{d}}^0\|_\varepsilon^2 + \|\boldsymbol{\theta}_{\mathrm{d}}^0\|_\mu^2 + C\tau \sum_{n=1}^N (\|\boldsymbol{\eta}_{\mathrm{d}}^n\|_\varepsilon^2 + \|\boldsymbol{\theta}_{\mathrm{d}}^n\|_\mu^2)\\
&\qquad + Ch^{2s}(\|\boldsymbol{E}^n\|_{s,curl}^2 + \|\boldsymbol{H}^n\|_{s,curl}^2)\\
&\qquad + Ch^{2s} \int_0^T (\|\boldsymbol{E}'(t)\|_0^2 + \|\boldsymbol{H}'(t)\|_0^2)\,\mathrm{d}t.
\end{aligned} \tag{4.15}$$

By the discrete Gronwall inequality Lemma 2.1 and initialization condition (4.1), one can easily see that

$$\|\boldsymbol{\eta}_{\mathrm{d}}^N\|_\varepsilon^2 + \|\boldsymbol{\theta}_{\mathrm{d}}^N\|_\mu^2 + \frac{1}{2}\tau^2 \sum_{n=1}^{N} \|\nabla \times \boldsymbol{\eta}_{\mathrm{d}}^n\|_\nu^2 \leqslant C h^{2s},$$

which completes the proof of this theorem. □

As a consequence of the previous results, we have the following so-called global error bounds.

Corollary 4.3. *Assume that the conditions of Theorems* 3.1 *and* 4.2 *hold. Assume also that, for* $n = 1, \ldots, N$,

$$\boldsymbol{E}^n \in H_0(\mathbf{curl}, \Omega) \cap H^s(\mathbf{curl}, \Omega) \quad \text{and} \quad \boldsymbol{H}^n \in H(\mathbf{curl}, \Omega) \cap H^s(\mathbf{curl}, \Omega), \quad \tfrac{1}{2} < s \leqslant 1$$

and that they are uniformly bounded in these spaces. Then there exists a constant $C > 0$ *independent of* τ *and* h *such that, for small enough* τ:

$$\|\boldsymbol{\eta}^N\|_\varepsilon^2 + \|\boldsymbol{\theta}^N\|_\mu^2 + \frac{1}{2}\tau^2 \sum_{n=1}^{N} \|\nabla \times \boldsymbol{\eta}^n\|_\nu^2 \leqslant C(\tau^2 + h^{2s}). \tag{4.16}$$

Proof. By the definition of $\boldsymbol{\eta}^n$ and $\boldsymbol{\theta}^n$, (4.16) follows from (3.4), (4.4)–(4.5) and (4.8). □

5. Numerical results

In this section we present some numerical test for the fully discrete decoupled scheme in Section 4. Assume that $\Omega = [0, 1]^3$, $[0, T] = [0, 1]$ and the parameters used in the test were $\varepsilon = 0.2$, $\sigma = 0.0$, $\mu = 2.5$, and the source electric current density was

$$\boldsymbol{J}_{\mathrm{s}} = (1.4\cos(2t - 3z), 1.4\cos(2t - 3x), 1.4\cos(2t - 3y))^{\mathrm{T}}.$$

This example has an exact solution

$$\boldsymbol{H} = (0.4\sin(2t - 3y), 0.4\sin(2t - 3z), 0.4\sin(2t - 3x))^{\mathrm{T}},$$

$$\boldsymbol{E} = (\sin(2t - 3z), \sin(2t - 3x), \sin(2t - 3y))^{\mathrm{T}}.$$

Error notations are introduced as follows:

$$\mathrm{Err}(\boldsymbol{H}_i^n) = \|[\boldsymbol{H}(t^n) - \boldsymbol{H}_h^n]_i\|_0, \quad \mathrm{Err}(\boldsymbol{E}_i^n) = \|[\boldsymbol{E}(t^n) - \boldsymbol{E}_h^n]_i\|_0,$$

Table 1
Error of the electric field $\boldsymbol{E}$

Mesh size	Err $(\boldsymbol{E}_1^n)$	Err $(\boldsymbol{E}_2^n)$	Err $(\boldsymbol{E}_3^n)$
$h=\frac{1}{2}$	3.7485E−01	3.7485E−01	3.7485E−01
$h=\frac{1}{4}$	1.1871E−01	1.6016E−01	1.2591E−01
$h=\frac{1}{8}$	4.3757E−02	8.8904E−02	4.3899E−02
$h=\frac{1}{16}$	1.5095E−02	4.3004E−02	1.4297E−02

(Take time stepsize $\tau = 1/32$)

Table 2
Error of the magnetic field $\boldsymbol{H}$

Mesh size	Err $(\boldsymbol{H}_1^n)$	Err $(\boldsymbol{H}_2^n)$	Err $(\boldsymbol{H}_3^n)$
$h=\frac{1}{2}$	4.5693E−01	4.5693E−01	4.5693E−01
$h=\frac{1}{4}$	2.2853E−01	2.2672E−01	2.2894E−01
$h=\frac{1}{8}$	9.3632E−02	9.3062E−02	9.2414E−02
$h=\frac{1}{16}$	4.3261E−02	4.3211E−02	4.2540E−02

(Take time stepsize $\tau = 1/32$)

Table 3
Error of the electric field $\boldsymbol{E}$

Times stepsize	Err $(\boldsymbol{E}_1^n)$	Err $(\boldsymbol{E}_2^n)$	Err $(\boldsymbol{E}_3^n)$
$\tau=\frac{1}{2}$	1.5050E−01	1.2310E−01	1.5011E−01
$\tau=\frac{1}{4}$	8.0182E−02	3.8055E−02	7.9568E−02
$\tau=\frac{1}{8}$	3.3259E−02	2.3857E−02	3.2390E−02
$\tau=\frac{1}{16}$	1.5641E−02	4.2510E−02	1.4628E−02
$\tau=\frac{1}{32}$	1.5095E−02	4.3004E−02	1.4297E−02
$\tau=\frac{1}{64}$	2.6318E−02	4.3107E−02	2.5848E−02

(Take mesh size $h = 1/16$)

where the subscript $i\,(i = 1, 2, 3)$ denotes the ith component of the electric field $\boldsymbol{E}$ or magnetic field $\boldsymbol{H}$. The numerical results are given in Tables 1–4 using different time stepsize τ or mesh size h.

Table 4
Error of the electric field $\boldsymbol{H}$

Time stepsize	Err ($\boldsymbol{H}_1^n$)	Err ($\boldsymbol{H}_2^n$)	Err ($\boldsymbol{H}_3^n$)
$\tau = \frac{1}{2}$	3.6760E−01	3.7313E−01	3.7533E−01
$\tau = \frac{1}{4}$	1.8112E−01	1.8532E−01	1.8540E−01
$\tau = \frac{1}{8}$	9.4931E−02	9.5222E−02	9.5006E−02
$\tau = \frac{1}{16}$	4.3735E−02	4.7869E−02	4.6409E−02
$\tau = \frac{1}{32}$	2.3267E−02	2.3211E−02	2.3540E−02
$\tau = \frac{1}{64}$	1.1095E−02	1.1881E−02	1.2063E−02

(Take mesh size $h = 1/16$)

References

[1] F. Assous, P. Degond, E. Heintzé, P.-A. Raviart, J. Segré, On the finite element method for solving the three-dimensional Maxwell equations, J. Comput. Phys. 109 (1993) 222–237.
[2] R. Albanese, G. Rubinacci, Formulation of the eddy-current problem, IEE Proc. 137 (1 Pt. A) (1990) 16–22.
[3] Z. Chen, Q. Du, J. Zou, Finite element methods with matching and non-matching meshes for Maxwell's equations with discontinuous coefficients, SIAM J. Numer. Anal. 37 (2000) 1542–1570.
[4] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[5] P. Ciarlet Jr., J. Zou, Finite element convergence for the Darwin model to Maxwell's equations, RAIRO Math. Modelling Numer. Anal. 31 (1997) 213–250.
[6] P. Ciarlet Jr., J. Zou, Fully discrete finite element approaches for time-dependent Maxwell's equations, Numer. Math. 82 (1999) 193–219.
[7] V. Girault, P.A. Raviart, Finite Element Method for Navier–Stokes Equations, Springer Series in Computational Mathematics, vol. 5, Springer, Berlin, New York, 1986.
[8] C. Makriddakis, P. Monk, Time-discrete finite element schemes for Maxwell equations, RAIRO Math. Modelling Numer. Anal. 29 (1995) 171–197.
[9] P. Monk, Analysis of a finite element method for Maxwell's equations, SIAM J. Numer. Anal. 29 (1992) 714–729.
[10] J.-C. Nédélec, A new family of mixed finite elements in $\mathbb{R}^3$, Numer. Math. 50 (1986) 57–81.
[11] R. Nicolaides, D.-Q. Wang, Covegence analysis of a covolume scheme for Maxwell equations in three dimensions, Math. Comput. 67 (1998) 947–963.
[12] J.T. Oden, J.N. Reddy, An Introduction to The Mathematical Theory of Finite Elements, Wiley, New York, 1974.
[13] A. Quarteroni, A. Valli, Numerical Approximation of Partial Differential Equations, Springer Series in Computational Mathematics, vol. 23, Springer, Berlin, 1994.
[14] P.-A. Raviart, Finite element approximation of the time-dependent Maxwell equations, Technical Report, Ecole Polytechnique, France, GdR SPARCH #6, 1993.