Finite Element MHD Solver using deal.II

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Outline

Magnetohydrodynamics

MHD describes behavior of conducting fluids by coupling

- Navier-Stoke equations for fluid dynamics with
- Maxwell's equation of electromagnetism.

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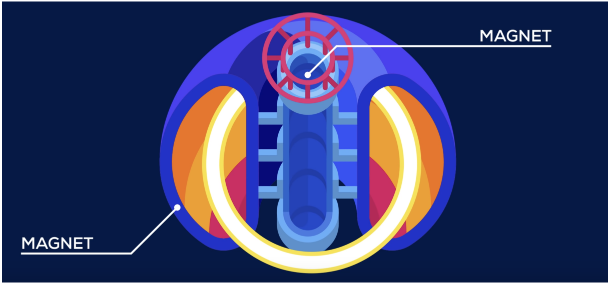
MHD has applications in developing tokamak nuclear fusion reactor for power generation.



Magnetohydrodynamics

Equations for 2-D MHD in Eulerian Coordinates:

$$\begin{aligned} \rho_t + \mathrm{div}(\rho u) &= 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u - h \otimes h) + \nabla q &= \mu \Delta u + (\eta + \mu)\nabla \mathrm{div} u, \\ h_t - \nabla \times (u \times h) &= \nu \Delta h, \\ (\rho E + \frac{1}{2}h^2)_t + \mathrm{div}(A) &= \mathrm{div}\left(\sum u\right) + \kappa \Delta T + \nu\ \mathrm{div}(B), \\ \mathrm{div}(h) &= 0, \end{aligned}$$

where $\sum := \eta \mathrm{div}(u) I + \mu(\nabla u + (\nabla u)^t)$,
$A = (\rho E + p)u + h \times (u \times h)$, $B = h \times (\nabla \times h)$,
$E := e + u_1^2/2 + u_2^2/2$, and $q = p + \frac{|h|^2}{2}$; see [2, 3, 4, 5, 6].

Magnetohydrodynamics: β-Model

In order to guarentee consistent splitting, we use the β-model of our system:

$$\begin{aligned}
\rho_t + \operatorname{div}(\rho u) &= 0, \\
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h_t - \nabla \times (u \times h) + \beta \operatorname{div}(h) e_1 &= \nu \Delta h, \\
(\rho E + \frac{1}{2} h^2)_t + \operatorname{div}(A) &= \operatorname{div}\left(\sum u\right) + \kappa \Delta T + \nu \operatorname{div}(B), \\
\operatorname{div}(h) &= 0,
\end{aligned}$$

where β is a real valued parameter and $e_1 = (1 \quad 0)^T$; see [1].

Simplified MHD

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With these simplifications our system is given by:

$$\begin{aligned}
\mathrm{div}(u) &= 0, \\
\rho(u_t + u \cdot \nabla u) + \nabla p &= \mu \Delta u + h \cdot \nabla h - \frac{1}{2}\nabla h^2, \\
h_t - \nabla \times (u \times h) + \beta \mathrm{div}(h) e_1 &= \nu \Delta h, \\
\mathrm{div}(h) &= 0.
\end{aligned}$$

Maxwell's Equations

The Maxwell's equations relevant to MHD are given by

$$\nabla \times \boldsymbol{h} = \mu J, \quad \nabla \cdot J = 0, \quad \nabla \times E = -\frac{\partial \boldsymbol{h}}{\partial t}, \quad \nabla \cdot \boldsymbol{h} = 0,$$

where

$$J = \sigma(E + \boldsymbol{u} \times \boldsymbol{h}),$$

and F is the Lorenz force given by

$$F = J \times \boldsymbol{h}.$$

This system can be simplified to:

$$\frac{\partial \boldsymbol{h}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{h}) + \nu \nabla \times (\nabla \times \boldsymbol{h}) = 0,$$
$$\nabla \cdot \boldsymbol{h} = 0.$$

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$$\frac{\partial \boldsymbol{h}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{h}) + \nu \nabla \times (\nabla \times \boldsymbol{h}) + \nabla q + \beta (\nabla \cdot \boldsymbol{h}) \boldsymbol{e}_1 = \boldsymbol{f},$$

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The weak form corresponding to the test function $\phi = [\boldsymbol{v} \quad w]^\top$ is given by:

$$\left(\boldsymbol{v}, \frac{\partial \boldsymbol{h}}{\partial t}\right)_\Omega - (\nabla \times \boldsymbol{v}, \boldsymbol{u} \times \boldsymbol{h})_\Omega + \nu\,(\nabla \times \boldsymbol{v}, \nabla \times \boldsymbol{h})_\Omega - (\nabla \cdot \boldsymbol{v}, q)_\Omega$$
$$+ \beta\,(\boldsymbol{v}, (\nabla \cdot \boldsymbol{h})\boldsymbol{e}_1)_\Omega - (w, \nabla \cdot \boldsymbol{h})_\Omega = (\boldsymbol{v}, \boldsymbol{f})_\Omega.$$

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- Using backward Euler, at each time step, we end up with a system of the form

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{h} \\ q \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

  where A is composed of the mass matrix, the velocity term, and the curl curl operator. B is the divergence operator. F is the right hand side.

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  where A is composed of the mass matrix, the velocity term, and the curl curl operator. B is the divergence operator. F is the right hand side.
- Solve using Schur decomposition with a direct solve for A and the lagrange mass matrix as the preconditioner for the Schur complement.

Maxwell Results

Test problem:

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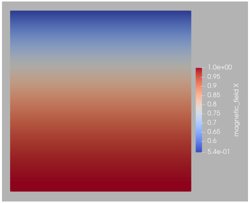
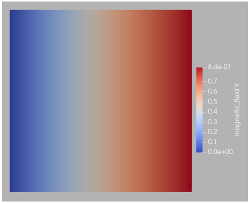
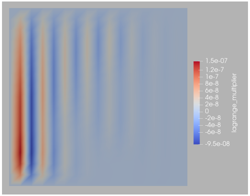


Magnetic field - x



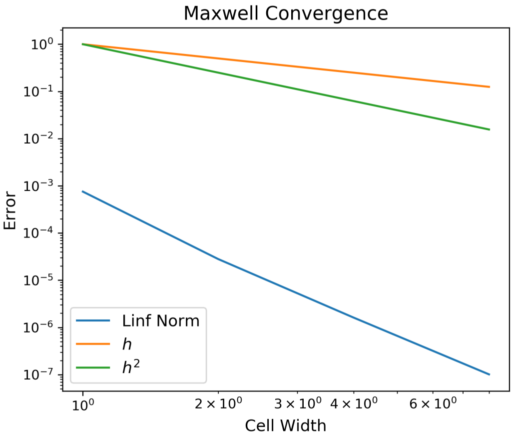
Magnetic field - y



Lagrange Multiplier

Maxwell Convergence

Second order convergence:



Requires extremely small time step for stability.

Incompressible Navier-Stokes

Our equations for incompressible Navier-Stokes are given by:

$$-\nabla \cdot \boldsymbol{u} = 0,$$
$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) + \nabla p = \mu \Delta \boldsymbol{u} + \boldsymbol{f},$$

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\end{aligned}
$$

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$$
\begin{aligned}
\rho \left(\boldsymbol{v}, \frac{\partial \boldsymbol{u}}{\partial t} \right)_\Omega + \rho(\boldsymbol{v}, \boldsymbol{u} \cdot \nabla \boldsymbol{u})_\Omega + \mu(\nabla \boldsymbol{v}, \nabla \boldsymbol{u})_\Omega \\
- (\nabla \cdot \boldsymbol{v}, p)_\Omega - (w, \nabla \cdot \boldsymbol{u})_\Omega = (\boldsymbol{v}, \boldsymbol{f})_\Omega.
\end{aligned}
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  Even with Grad-Div stabilization, the condition number of the coupled system is super bad.
- Projection method, or essentially decoupling the pressure and velocity.

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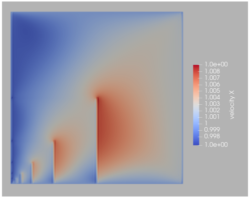
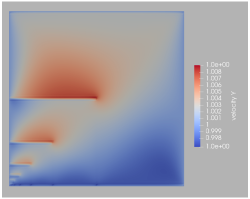
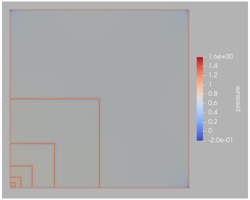


Velocity field - x



Velocity field - y



Pressure

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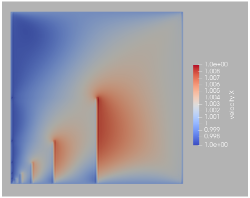

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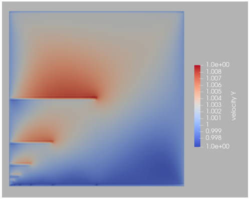

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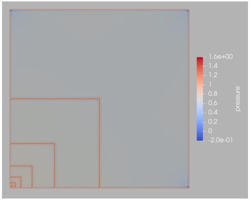
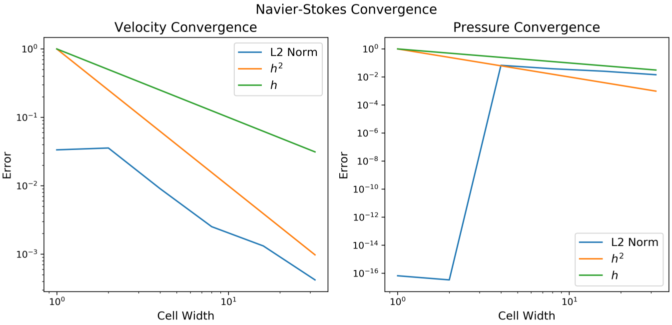


Pressure

Note: the velocity solve is what is lagging the convergence.

Navier-Stokes Convergence

First order convergence in both velocity and pressure:



Still stable with larger time step.

Coupling the Two Solvers

Our coupled system is given by:

$$\frac{\partial \boldsymbol{h}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{h}) + \nu \nabla \times (\nabla \times \boldsymbol{h}) + \nabla q + \beta (\nabla \cdot \boldsymbol{h}) \boldsymbol{e}_1 = 0,$$

$$\nabla \cdot \boldsymbol{h} = 0,$$

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Solving the Coupled System

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```
for (; current_time <= 1; current_time += time_step, ++time_step_number)
{
    // solve for h^1 and h^2 without updating velocity field
    solve_maxwell(time_step_number > 2); // this boolean = don't update velocity

    // solve for u^k and p^k for all k >= 2
    if (time_step_number > 1)
        solve_ns();

}
```

Testing the Coupled Solver

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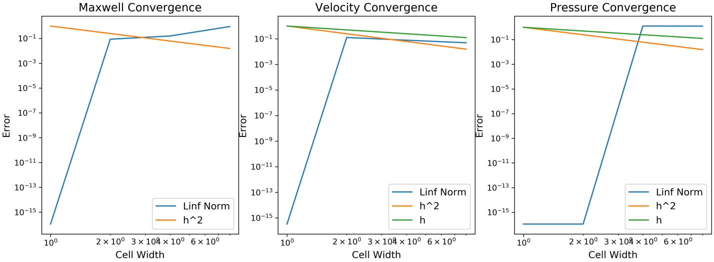
  To start, we test with a constant solution (constant magnetic field, velocity field, and pressure) to make sure the coupling is set-up correctly.

Testing the Coupled Solver

Convergence results for the constant solution:



Testing the Coupled Solver

Identifying the weakness in the solver.

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```
Number of active cells:       1
Number of degrees of freedom: 50
Timestep size: 1
                             maximum m error = 2.22045e-16
                             maximum u error = 3.33067e-16
                             maximum p error = 1.11022e-16
Number of active cells:       4
Number of degrees of freedom: 152
Timestep size: 0.1
                             maximum m error = 1.93623e-13
                             maximum u error = 0.13072
                             maximum p error = 1.11022e-16
Number of active cells:       16
Number of degrees of freedom: 524
Timestep size: 0.01
                             maximum m error = 2.28758e-11
                             maximum u error = 0.079528
                             maximum p error = 1.25707
```

```
Number of active cells:       1
Number of degrees of freedom: 50
Timestep size: 1
                            maximum m error = 1.11022e-16
                            maximum u error = 3.33067e-16
                            maximum p error = 1.11022e-16
Number of active cells:       4
Number of degrees of freedom: 152
Timestep size: 0.1
                            maximum m error = 1.91036e-12
                            maximum u error = 3.33067e-16
                            maximum p error = 1.11022e-16
Number of active cells:       16
Number of degrees of freedom: 524
Timestep size: 0.01
                            maximum m error = 9.25832e-10
                            maximum u error = 3.33067e-16
                            maximum p error = 5.37126e-13
```

Exact Magnetic field

Exact Velocity field

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- Add in temperature equation.
- Test with traveling wave stability with stationary traveling wave.

That's It!

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