We develop a time evolution code using finite elements that treats general conservation laws in two spatial dimensions taking the following form,

$$f^0(U)_t + \sum_{k=1}^{2} f^k(U)_{x_k} = \sum_{j,k=1}^{2} \left(B^{jk}(U)U_{x_k}\right)_{x_j},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $t \in \mathbb{R}$, and $U \in \mathbb{R}^n$ with $f^j : \mathbb{R}^n \to \mathbb{R}^n$, and it is assumed that each B^{jk} has the block structure

$$B^{jk} = \begin{pmatrix} 0_{r\times r} & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & b^{jk}(U) \end{pmatrix}.$$

The domain is given by $(t, x_1, x_2) \in [0, T] \times [a, b] \times [c, d]$ where $T > 0$, $a < b$, and $c < d$. We allow for periodic boundary conditions in the spatial variables, or general Robin boundary conditions.

1 1D Time Evolution

Consider the PDE

$$f^0(U)_t + f^1(U)_x - (B(U)U_x)_x = 0$$

over the domain Ω.

We wish to approximate the solution to this PDE using finite difference in time and finite element in space.

First we substitute in the finite difference approximation of the time derivative, letting U^n correspond to the solution U at time $t = t_n$.

$$\frac{1}{\Delta t}\left(f^0(U^n) - f^0(U^{n-1})\right) + f^1(U^n)_x - (B(U^n)U^n_x)_x = 0$$

Now we derive the weak form of this system. Let $V = \{v \in C^1(\Omega) : v|_{\partial\Omega} = 0\}$. Then for all $v \in V$ we have:

$$\frac{1}{\Delta t}\int \left(f^0(U^n) - f^0(U^{n-1})\right) v + \int f^1(U^n)_x v - \int (B(U^n)U^n_x)_x v = 0$$

Now we can apply integration by parts to simiplify our problem to:

$$\frac{1}{\Delta t}\int f^0(U^n)v - \frac{1}{\Delta t}\int f^0(U^{n-1})v - \int f^1(U^n)v_x + \int B(U^n)U^n_x v_x = 0$$

Now we take a finite subspace of our solution space with basis functions $\{\phi_i\}_{i=0}^N$ and we let $v = \sum c_j\phi_j$ and $U^n = \sum A^n_i\phi_i$.

Plugging these approximations back into our weak form we get.

$$\frac{1}{\Delta t}\int f^0(\sum A_i^n\phi_i)(\sum c_j\phi_j)-\frac{1}{\Delta t}\int f^0(\sum A_i^{n-1}\phi_i)(\sum c_j\phi_j)-\int f^1(\sum A_i^n\phi_i)(\sum c_j\phi_j')+\int B(\sum A_i^n\phi_i)(\sum$$

Which is equivalent to:

$$\sum c_j\left(\frac{1}{\Delta t}\int f^0(\sum A_i^n\phi_i)\phi_j-\frac{1}{\Delta t}\int f^0(\sum A_i^{n-1}\phi_i)\phi_j-\int f^1(\sum A_i^n\phi_i)\phi_j'+\int B(\sum A_i^n\phi_i)(\sum A_i^n\phi_i')\phi_j'\right)$$

Now since this must be true for any set $\{c_j\}\subset\mathbb{R}$ it follows that for each j

$$\frac{1}{\Delta t}\int f^0(\sum A_i^n\phi_i)\phi_j-\frac{1}{\Delta t}\int f^0(\sum A_i^{n-1}\phi_i)\phi_j-\int f^1(\sum A_i^n\phi_i)\phi_j'+\int B(\sum A_i^n\phi_i)(\sum A_i^n\phi_i')\phi_j'=0$$

Now we will define

$$\phi_j(x)=\begin{cases}(x-x_{j-1})/h & x\in[x_{j-1},x_j]\\ (x_{j+1}-x)/h & x\in[x_j,x_{j+1}]\\ 0 & otherwise\end{cases}$$

Thus ϕ_j is only nonzero on the domain $[x_{j-1},x_{j+1}]$ which corresponds to the elements e_j and e_{j+1}. Due to the linearity of the integral we can write our equation equivalently as:

$$\sum_{k=0}^{N}\left(\frac{1}{\Delta t}\int_{e_k} f^0(\sum A_i^n\phi_i)\phi_j-\frac{1}{\Delta t}\int_{e_k} f^0(\sum A_i^{n-1}\phi_i)\phi_j-\int_{e_k} f^1(\sum A_i^n\phi_i)\phi_j'+\int_{e_k} B(\sum A_i^n\phi_i)(\sum A_i^n\phi_i')\phi_j'\right)$$

And then because $\int_{e_k}\phi_j=0\quad\forall k\neq j,j+1$ this simplifies to

$$\sum_{k=j}^{j+1}\left(\frac{1}{\Delta t}\int_{e_k} f^0(\sum A_i^n\phi_i)\phi_j-\frac{1}{\Delta t}\int_{e_k} f^0(\sum A_i^{n-1}\phi_i)\phi_j-\int_{e_k} f^1(\sum A_i^n\phi_i)\phi_j'+\int_{e_k} B(\sum A_i^n\phi_i)(\sum A_i^n\phi_i')\phi_j'\right)$$

Since we are only looking at the local intervals over each element e_k our approximation for U will now be $U=A_{k-1}\phi_{k-1}+A_k\phi_k$ While