CSYS 300 Assignment 1

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Link to Github for Assignment Code: `https://github.com/brynchristineloftness/csys300pocs`

Question 1.

Use a back-of-an-envelope scaling argument to show that maximal rowing speed V increases as the number of oarspeople N as $V \propto N^{1/9}$.
Assume the following:
(a) Rowing shells are geometrically similar (isometric). The table below taken from McMahon and Bonner[1] shows that shell width is roughly proportional to shell length ℓ
(b) The resistance encountered by a shell is due largely to drag on its wetted surface.
(c) Drag force is proportional to the product of the square of the shell's speed (V^2) and the area of the wetted surface($\propto \ell^2$ due to shell isometry).
(d) Power $\propto$ drag force X speed (in symbols: $P \propto D_f$ X V).
(e) Volume displacement of water by a shell is proportional to the number of oarspeople N (i.e. the team's combined weight).
(f) Assume the depth of water displacement by the shell grows isometrically wih the boat length ℓ.
(g) Power is proportional to the number of oarspeople N.

Responses:
Summarizing the assumptions via a set of proportional equations, we can then deduce from substitution of variables in these proportionalities that maximal rowing speed V increases as the number of oarspeople N as $V \propto N^{1/9}$. See below for these steps.
Assumptions in the form of proportionalities:

$$(b)\ell \times b \propto \ell^2$$

$$(c)D_f \propto \ell^2 \times V^2$$

$$(d)P \propto D_f \times V$$

$$(e)D_w \propto N$$

$$(f)d \propto \ell$$

$$(g)P \propto N$$

Starting with $(b)\ell \times b \propto \ell^2$ and $(e)D_w \propto N$...

$$d \times \ell^2 \propto N$$

incorporating (f) leads to...

$$\ell \times \ell^2 \propto N$$

simplifying to...

$$\ell^3 \propto N$$

incorporating (g) leads to...

$$\ell^3 \propto P$$

next incorporating (d) leads to...

$$\ell^3 \propto D_f \times V$$

next incorporating (c) leads to...

$$\ell^3 \propto \ell^2 \times V^2 \times V$$

which can be simplified to...

$$\ell^3 \propto \ell^2 \times V^3$$

then to simplify out divide both sides by ℓ^2 to produce...

$$\ell \propto V^3$$

which ℓ can also be thought of us $N^{1/3}$ because we've shown $N \propto \ell^3$ is true. Therefore leading us to...

$$N^{1/3} \propto V^3$$

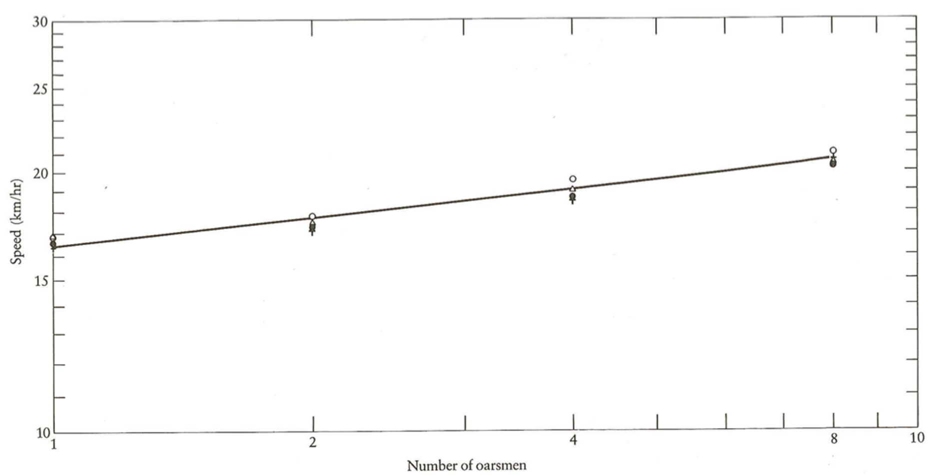
which can be simplified by taking both side to the exponent of 1/3 which leaves us at our conclusion...

$$N^{1/9} \propto V$$

This shows that maximal rowing speed V increases as the number of oarspeople N as $V \propto N^{1/9}$

Question 2.

Find the modern day world record times for 2000 metre races and see if this scaling holds up. Of course, our relationship is approximate as we have neglected numerous factors, the range is extremely small (1-8 oarspeople), and the scaling is very weak (1/9). But see what you can find. The figure below shows data from McMahon and Bonner.

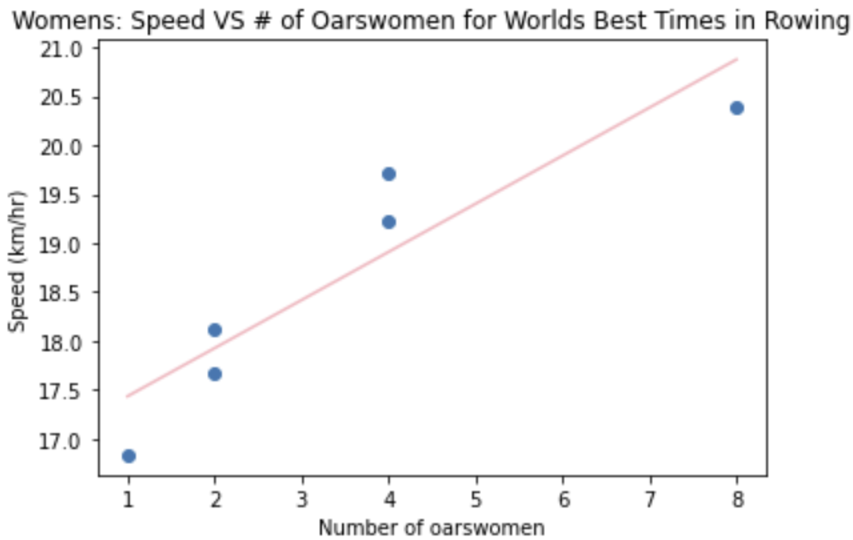


Responses:
I computed the Worlds Best Times in Rowing for Men and Women to analyze the scaling compared to the weak scaling found from McMahon and Bonner (1/9).

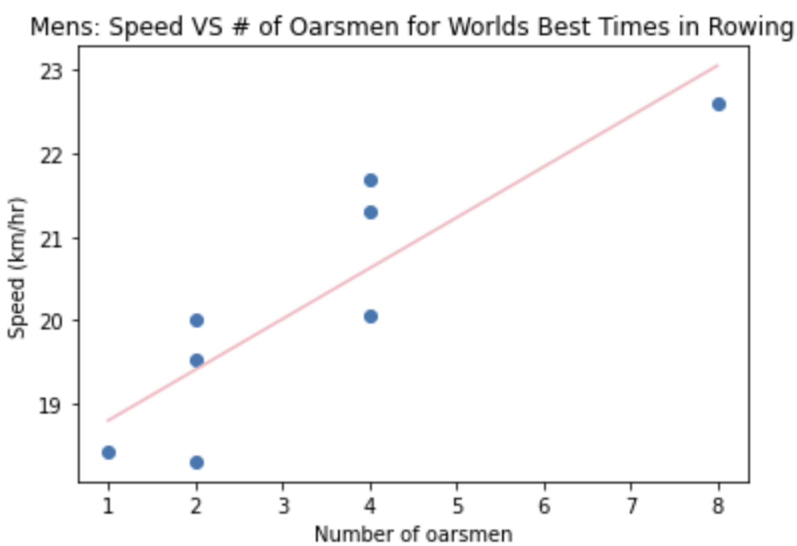
Steps:
First I found the data on Wikipedia (page linked in code document found from github linked above)
Then I converted all times from seconds/2000 meters to kilometers per hour.
I thereafter ran a Linear Regression on the Number of Oarsmen or Oarswomen and Speed to compare scaling (written in python, code can be found at the link above via my GitHub page. The results of the regressions

can be found below.

```
scaling exponent: [0.49164763]
```




```
scaling exponent: [0.60745363]
```



As can be seen from the figures above, scaling for both men and women are much stronger than what McMahon and Bonner found 1/9, which is .111111, whereas for my analyses I found scaling from the men's records regression as .60745363 and scaling from the women's records regression as .49164763.

Question 3.

Finish the calculation for the platypus on a pendulum problem to show that a simple pendulum's period τ is indeed proportional to $\sqrt{\ell/g}$.

Basic plan from lectures: Create a matrix A where ijth entry is the power of dimension i in the jth variable, and solve by row reduction to find basis null vectors. In lectures, we arrived at:

$$A\overrightarrow{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

You only have to take a few steps from here.
From lecture 3: the Buckingham π theorem (20 minutes, *link omitted*)

Responses:
We start with the matrix above...

$$A\overrightarrow{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We then perform the following steps to row reduce to find basis null vectors. First, $R3 \times (-1/2) -> R3$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

Next, $(-1) \times R3 + R1 -> R1$

$$\begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

Now that the rows are reduced and ready, we can produce the following equivalences:

$$X_1 + 1/2x_4 = 0$$

$$X_2 = 0$$

$$X_3 - 1/2x_4 = 0$$

Which can be re-written in terms of X_4 as...

$$X_1 = -1/2X_4$$

$$X_3 = 1/2X_4$$

We will then re-write X_4 using a common term y.

$$X_4 = y$$

$$X_3 = 1/2y$$

$$X_2 = 0$$

$$X_1 = -1/2y$$

This can be thus translated into our Null Vector as...

$$Null(a) = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$$

Moving forward from here, we can transition back to show a simple pendulum's period τ is indeed proportional to $\sqrt{\ell/g}$. We can remember that the original equation from the Powerpoint prior to getting to this matrix was

$$[\pi_i] = L^{x_1} \times M^{x_2} \times (LT^{-2})^{x_3} \times T^{x_4}$$

... We can now replace our values for $x_1 ... x_4$ to produce...

$$1 = L^{-1/2} \times M^0 \times (LT^{-2})^{1/2} \times T^1$$

which can be simplified to...

$$1 = L^{-1/2} \times \sqrt{LT^{-2}} \times T$$

We then can replace LT^{-2} with [g] as we saw earlier in the assumptions of the pendulum problem from the slideshow that $[g] = LT^-2$. We can also replace L for ell from the same list of assumptions ($[ell] = L$). The final replacement is T for τ based on the variable dimension assumptions given and used above...Combining with some simplification as well, and changing this to a proportionality instead of an equality, this produces...

$$1 \propto \frac{1}{\sqrt{\ell}} \times \sqrt{g} \times \tau$$

We after-fore move to isolate τ. This gives:

$$\frac{\sqrt{\ell}}{\sqrt{g}} = \tau$$

This can be simplified to...

$$\sqrt{\frac{\ell}{g}} = \tau$$

This is our final statement showing that through our row reduction, re-substitution, and solving we have found that τ is proportional to $\sqrt{\frac{\ell}{g}}$.

Question 4.

Show that the maximum speed of animals V_{max} is proportional to their length L [2] Here are the five dimensional parameters:

- V_{max}, maximum speed.
- ℓ, animal length.
- p, organismal density.
- σ, maximum speed.
- b, maximum metabolic rate per unit mass (b has the dimensions of power per unit mass).

And here are the three dimensions: L, M, and T.

Use a back-of-the-envelope calculation to express V_{max}/ℓ in terms of p, σ, and b.

Note: It's argued in [2] that these latter three parameters vary little across all organisms (we're mostly thinking about running organisms here), and so finding V_{max}/ℓ as a function of them indicates that V_{max}/ℓ is also roughly constant.

Responses:
After reading [2] we can find in Part B of Section 3 the following equations:

$$1. p \propto [M] \times [L]^{-3}$$

$$2. \sigma \propto [M] \times [L]^{-1} \times [T]^{-2}$$

$$3. \beta_m \propto [L]^2 \times [T]^{-3}$$

$$4. V_{max}/L \propto [T]^{-1}$$

$$5. V_{max}/L \propto (\beta_m \times p)/\sigma$$

$$6. L \propto \ell$$

With the goal of finding V_{max}/L in terms of p, $sigma$, and β we can do a series of substitutions to back calculate the equation to prove our proportionalities are accurate and express V_{max}/L in the requested format.

We will start with $V_{max}/L \propto (\beta_m \times p)/\sigma$ given in the paper and work backwards.

$$V_{max}/L \propto (\beta_m \times p)/\sigma$$

We will now substitute from the above equations given in [2]- the paper provided in the original homework assignment.

$$V_{max}/L \propto ([L]^2 \times [T]^{-3} \times [M] \times [L]^{-3})/[M] \times [L]^{-1} \times [T]^{-2}$$

which simplifies to...

$$V_{max}/L \propto \frac{[M]}{[T]^3 \times [L]} / \frac{[M]}{[L] \times [T]^2}$$

which can then be algebraically simplified again by dividing out $\frac{1}{[T]}$ on the left side...

$$V_{max}/L \propto \frac{1}{[T]} \times \frac{[M]}{[T]^2 \times [L]} / \frac{[M]}{[L] \times [T]^2})$$

this simplifying to...

$$\frac{V_{max}}{L} \propto \frac{1}{[T]}$$

which is the same as equation 4 given above: $V_{max}/L \propto [T]^{-1}$.

This backwards substitution producing one of our original equation thus shows that we can express V_{max}/L in terms of p, $sigma$, and β; where L can be expressed as L or ℓ by equation 6 taken from the paper [2]. These backwards substitution steps using back-of-the-envelope calculation style thus satisfy our goal.

Question 5.

Use the Buckingham π theorem to reproduce G.I. Taylor's finding the energy of an atom bomb E is related to the density of air p and the radius of the blast wave R at time t:

$$E = constant \times pR^5/t^2.$$

In constructing the matrix, order parameters as E, p, R, and t and dimensions as L, T, and M.

Responses:
Using the slides from the Scaling PowerPoint off the classroom website I was able to find equations relating to this problem. These equations are...

$$[R] = L$$

$$[t] = T$$
$$[p] = M/L^3$$
$$[E] = ML^2/T^2$$

We can then assign these values to dimensionless spaces, such as...

$$[q_1] = ML^2/T^2$$
$$[q_2] = M/L^3$$
$$[q_3] = L$$
$$[q_4] = T$$

Then, using the formula provided on the slideshow, we can rewrite these as...

$$(ML^2/T^2)^{X_1} \times (M/L^3)^{X_2} \times (L)^{X_3} \times (T)^{X_4} = 0$$

This equation can be reduced to...

$$(M^{X_1} \times L^{2X_1} \times T^{-2X_1}) \times (M^{X_2} \times L^{-3X_2}) \times L^{X_3} \times T^{X_4} = 0$$

which can be then simplified to...

$$M^{X_1+X_2} \times L^{2X_1-3X_2+X_3} \times T^{X_4-2X_1} = 0$$

Next we can begin to form the matrix. We know the following...

$$X_1 + X_2 = 0$$
$$2X_1 - 3X_2 + X_3 = 0$$
$$X_4 - 2X_1 = 0$$

Therefore we can produce the following matrix:

$$Ax = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We then perform the following steps to row reduce to find basis null vectors. First, $R3 + R2- > R2$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 1 \\ -2 & 0 & 0 & 1 \end{bmatrix}$$

Next, 2R1 + R3 -> R3

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

Followed by, 1/2R3 <-> R2

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & -3 & 1 & 1 \end{bmatrix}$$

Finally, R3 + 3R2 -> R3

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 5/2 \end{bmatrix}$$

This then produces the following equivalences:

$$X_1 + X_2 = 0$$

$$X_2 + 1/2X_4 = 0$$

$$X_3 + 5/2X_4 = 0$$

Which can be re-written in terms of X_4 as...

$$X_1 = 1/2X_4$$

$$X_2 = -1/2X_4$$

$$X_3 = -5/2X_4$$

We will then re-write X_4 using a common term a.

$$X_4 = a$$

$$X_3 = -5/2a$$

$$X_2 = -1/2a$$

$$X_1 = -1/2a$$

This can be thus translated into our Null Vector Matrix as...

$$Null(a) = \begin{bmatrix} 1/2 \\ -1/2 \\ -5/2 \\ 1 \end{bmatrix}$$

Due to this successful cancellation and production of a Null Vector Matrix, we can now re-substitute in to produce the original theorem.

We can then rewrite the original equation, $(ML^2/T^2)^{X_1} \times (M/L^3)^{X_2} \times (L)^{X_3} \times (T)^{X_4} = 0$ now equal to 1 with our found coefficients of $(X_1 ... X_4)$ such as...

$$(ML^2/T^2)^{1/2} \times (M/L^3)^{-1/2} \times (L)^{-5/2} \times (T)^1 = 1$$

We can then isolate E from this equation...

$$(ML^2/T^2)^{1/2} = L^{5/2} \times (M/L^3)^{1/2} \times (1/T)$$

We can then take the square of both sides to produce...

$$(ML^2/T^2) = L^5 \times \frac{M}{L^3} \times \frac{1}{T^2}$$

This can be simplified to...

$$(ML^2/T^2) = \frac{L^2 \times M}{T^2}$$

then by using our equations from above we remember that...

$$[R] = L$$

$$[t] = T$$
$$[p] = M/L^3$$
$$[E] = ML^2/T^2$$

Therefore, we can rewrite the above statement as...

$$E = \frac{R^2 \times (pR^3)}{t^2}$$

which now we introduce back our difference of a *constant* as we found through our null vector analysis and can simplify to...

$$E = constant \times pR^5/t^2$$

... which is a reproduced replica of the equation that was found by G.I.Taylor.

Question 6.

Use the Buckingham π theorem to derive Kepler's third law, which states that the square of the orbital period of a planet is proportional to the cube of its semi-major axis.

Let's shed some enlightenment and assume circular orbits.

Parameters:

- Planet's mass m;
- Sun's mass M;
- Orbital period T;
- Orbital period r;
- Gravitational constant G.

(a) What are the dimensions of these five quantities?
(b) You will find that there are two dimensionless parameters using the Buckingham π theorem, and that you can choose one to be $\pi_2 = m/M$. Find the other dimensionless parameter, π_1.
(c) Now argue that $T^2 \propto r^3$.
(d) For our solar system's nine (9) planets (yes, Pluto is on the team here), plot T^2 versus r^3, and using basic linear regression report on how well Kepler's third law holds up.

Responses:

a.)
We can find 3 dimensions from the five quantities as shown below...
1.) Planet's mass m > m
2.) Sun's mass M -> m
3.) Orbital period T -> t
4.) Orbital radius r -> ℓ
5.) Gravitational constant G -> $\frac{\ell^3}{\mathrm{m} \times \mathrm{t}^2}$

We can think of m as Mass, t as Time, and ℓ as Length.

b.)

$$[q_1] = \mathrm{m}$$
$$[q_2] = \mathrm{m}$$
$$[q_3] = \mathrm{t}$$
$$[q_4] = \ell$$
$$[q_5] = \ell^3 \times \mathrm{m}^{-1} \times \mathrm{t}^{-2}$$

Then, using the formula provided on the slideshow for Buckingham π theorem, we can rewrite these as...

$$(\mathrm{m})^{X_1} \times (\mathrm{m})^{X_2} \times (\mathrm{t})^{X_3} \times (\ell)^{X_4} \times (\ell^3 \times \mathrm{m}^{-1} \times \mathrm{t}^{-2})^{X_5} = 0$$

This equation can then be simplified to...

$$\mathrm{m}^{X_1+X_2-X_5} \times \mathrm{t}^{X_3-2X_5} \times \ell^{X_4+3X_5} = 0$$

Next we can begin to form the matrix. We know the following...

$$X_1 + X_2 - X_5 = 0$$
$$X_3 - 2X_5 = 0$$
$$X_4 + 3X_5 = 0$$

Therefore we can produce the following matrix:

$$Ax = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We then review the matrix to see if we need to row reduce before finding basis null vectors. After review, we see that the matrix is already in row echelon form, so no reduction is needed.

Therefore we can produce the following equivalences again:

$$X_1 + X_2 - X_5 = 0$$
$$X_3 - 2X_5 = 0$$
$$X_4 + 3X_5 = 0$$

Seeing that there are two independent (free) variables, we rewrite in terms of those variables; X_5 and X_2...

$$X_1 = X_5 - X_2$$
$$X_4 = -3X_5$$
$$X_3 = 2X_5$$

We then translate these two free variables to two separate Null Vector outputs.
First, we set

$$X_5 = 0, X_2 = 1$$

to produce...

$$Null(X_5) = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Second, we set

$$X_2 = 0, X_5 = 1$$

to produce...

$$Null(X_2) = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \\ 1 \end{bmatrix}$$

We can now re-substitute in to produce the original theorem to find π_1, we know one of the two Null Vectors above will produce $\pi_2 = m/M$, as given in the problem statement.
First we plug in the values from the $Null(X_5)$ Vector...

$$\mathrm{m}^-1 \times \mathrm{m}^1 \times \mathrm{t}^0 \times \ell^0 \times (\ell \times \mathrm{m}^-1 \times \mathrm{t}^{-2})^0$$

This can be simplified to...

$$\mathrm{m/m}$$

which is the same as what was given in the problem statement; m/M. We now can say that the $Null(X_5)$ Vector corresponds to the pi_2 equation given, and we must continue to find the pi_1 with the $Null(X_2)$ Vector.
We now start again at re-substituting to produce the original theorem to find π_1, plugging in the values from the

$$Null(X_2)$$

Vector. I substitute in the terms of original variables to produce....

$$M^1 \times m^0 \times T^2 \times r^{-3} \times G^1$$

Which can be simplified to...

$$\frac{M \times T^2 \times G}{r^3} = \pi_1$$

This is our final equation to satisfy finding the Null Vectors for the two independent (free) variables and solving for π_1 and π_2.

c.)
By using the equation found above for π_1, we can work to show that $T^2 \propto r^3$. First, we look at our equation for π_1...

$$\frac{M \times T^2 \times G}{r^3} = \pi_1$$

We then move the equation around algebraically to isolate T^2 to produce...

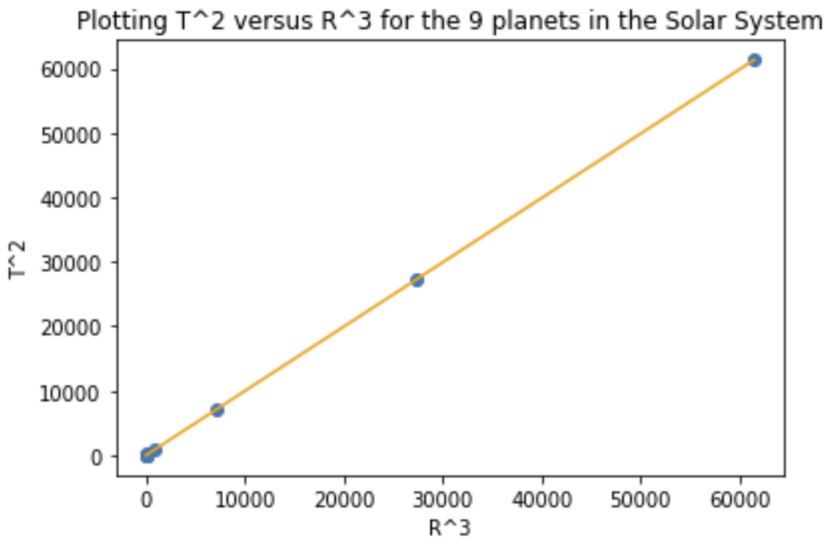
$$T^2 = \frac{r^3}{G \times M}$$

Now, knowing that G is the gravitational constant is a constant and that the Sun's Mass M is also a constant, we can replace these in the equality with a 1 and make the equality a proportionality to then produce... $T^2 \propto \frac{r^3}{1}$ which is the same as saying...

$$T^2 \propto r^3$$

These understandings of constants thus show that the above statement is an accurate proportionality.
d.) Taking the T^2 versus R^3 for the 9 planets of the solar system and plotting the regression line the following figure is produced...

```
scaling exponent: [0.99746762]
```



The extremely strong scaling exponent of .99746762 shows that Kepler's 3rd Law does hold up.