Solutions to Linear Representations of Finite Groups

Bryn Elesedy*
Department of Computer Science
University of Oxford

July 19, 2020

Contents

2 Character Theory **2**
2.1 2
2.2 2
2.3 2
2.4 2
2.5 2
2.6 2
2.7 3
2.8 3
2.9 4
2.10 4

3 Subgroups, Products and Induced Representations **4**
3.1 4
3.2 4
3.3 5

*bryn@robots.ox.ac.uk

2 Character Theory

2.1

Let V and V' be the corresponding representations. Then $\chi+\chi'$ is the character of the direct sum $V\oplus V'$. The character of the alternating square is then, for any s in the group,

$$\begin{aligned}(\chi+\chi')^2_\sigma &= \frac{1}{2}((\chi(s)+\chi'(s))^2+\chi(s^2)+\chi'(s^2))\\ &= \frac{1}{2}(\chi(s)^2+\chi(s^2))+\frac{1}{2}(\chi'(s)^2+\chi'(s^2))+\chi(s)\chi'(s)\\ &= \chi^2_\sigma(s)+{\chi'_\sigma}^2(s)+\chi(s)\chi'(s).\end{aligned}$$

We can do the same thing for the symmetric square

$$\begin{aligned}(\chi+\chi')^2_\alpha &= \frac{1}{2}((\chi(s)+\chi'(s))^2-\chi(s^2)+\chi'(s^2))\\ &= \frac{1}{2}(\chi(s)^2-\chi(s^2))+\frac{1}{2}(\chi'(s)^2-\chi'(s^2))+\chi(s)\chi'(s)\\ &= \chi^2_\alpha(s)+{\chi'_\alpha}^2(s)+\chi(s)\chi'(s).\end{aligned}$$

2.2

In the permutation representation, the matrix ρ_s will be a permutation matrix, with 1 at position (i,i) if i is fixed by s and 0 on the diagonal otherwise. The character $\chi(s)$ is the trace of this matrix, so is the number of elements fixed by s.

2.3

We may choose bases $\{\boldsymbol{e}_i\}$ for V and $\{\boldsymbol{e}'_j\}$ V' such that $\langle \boldsymbol{e}_i,\boldsymbol{e}'_j\rangle=\delta_{ij}$. Then for any matrix A we have, in the given bases,

$$\langle \rho_s x, Ax'\rangle=\sum_{ij} x_i x'_j(\rho_s)_{ki}A_{kj}.$$

So the map is invariant if and only if $A=(\rho_s^{-1})^\top$. We can then define a representation by $\rho'_s=(\rho_s^{-1})^\top$, which is clearly a representation. Existence and uniqueness have therefore been established.

2.4

The map ρ_s is clearly linear and invertible, since the $\rho_{i,s}$ are linear and invertible. Further, for any f in W

$$\rho_{st}f=\rho_{2,st}f\rho^{-1}_{1,st}=\rho_{2,s}\rho_{2,t}f\rho^{-1}_{1,t}\rho^{-1}_{1,s}=\rho_s\rho_t f.$$

We can then calculate the character by expanding in a basis. Let f have components f_{ab} in a basis for W. Then (using Einstein notation)

$$(\rho_s f)_{ij}=(\rho_{2,s})_{i\alpha}(\rho^{-1}_{1s})_{\beta j}f_{\alpha\beta}.$$

We calculate the trace of ρ_s by contraction of $\rho_s\boldsymbol{e}_{ij}$ with elements $\boldsymbol{e}'_{kl}$ of the dual basis, which are such that $\langle \boldsymbol{e}_{ij},\boldsymbol{e}'_{kl}\rangle=\delta_{ij}\delta_{kl}$. The conclusion then follows from $\chi(s^{-1})=\chi(s)^*$ and that $\rho_{1,s}$ is a homomorphism.

2.5

The character of the unit representation is 1, so the number of times this representation occurs in ρ is therefore $(\chi|1)=\frac{1}{|G|}\sum_{g\in G}\chi(g)$.

2.6

a) Let $\boldsymbol{e}_x$ be the basis for the permutation representation V. From each of the c orbits, take a representative arbitrarily and denote them y_i $i=1,\dots,c$. For each orbit, we can construct an invariant in V

$$\nu_i=\sum_{g\in G}\rho_g\boldsymbol{e}_{y_i}.$$

Denote by W_i the space spanned by ν_i, which is clearly invariant under G and irreducible, W_i is fixed by G and is hence the unit representation. We can therefore decompose V into the direct sum

$$U \oplus W_1 \oplus \cdots \oplus W_c.$$

We can notice that U cannot contain the unit representation. If it did, then there would have to be an element of U that is fixed by G, but this element would form n orbit, which contradicts the hypothesis.

b) In the permutation representation ρ_s is a permutation matrix, so the character of the representation $\chi(s)$ is the number of elements of X that are fixed by s. Clearly, there must be $\chi(s)^2$ elements fixed by ρ_s in $X \times X$.

c)
- i) $\iff$ ii) G is doubly transitive on $X \times X$, so is transitive on X, so the diagonal Δ is an orbit. Since G is doubly transitive, Δ^c is also an orbit. So these are the only two orbits. These statements also work the other way around.
- ii) $\iff$ iii) is just part a)
- iii) $\iff$ iv) is given in the hint. $\chi = 1 + \psi$ where ψ is the character of θ. Then since $(\chi^2|1) = 2$ (two orbits) expanding the sqare tells us that $(\psi^2|1) = 1$. However, because ψ is real valued (all of the matrices will be real in the permutation representation) we see that $(\psi^2|1) = \frac{1}{|G|}\sum_{g\in G}\psi(s)\psi(s)^* = (\psi|\psi)$. We know that if a character η has $(\eta|\eta) = 1$ then it is the character of an irreducible representation, so θ is irreducible.

2.7

Let χ be a character of G. We know that the regular representation has character $r_G(s) = |G|\,\mathbb{1}\{s = e\}$. If χ vanishes on all elements apart from the identity then we have that $\chi(s) = \chi(e)\mathbb{1}\{s = e\}$. The number of times that χ contains the unit representation is the number of distinct orbits in G, which is clearly an integer. Thus, with 1 denoting the character of the unit representation,

$$(\chi|1) = \frac{1}{|G|}\sum_{g\in G}\chi(g) = \frac{\chi(e)}{|G|} \in \mathbb{Z}$$

so $\chi(e)$ is an integral multiple of $|G|$.

2.8

a) We can write $V_i = \bigoplus_{j=1}^{m_i} W_i$ where the W_i are irreducible representations. The map $h_i : W_i \to V_i$ can be used to define a linear map $h_i^j : W_i \to W_i$ with image the j^{th} copy of W_i by composing with a linear projection. This will of course still be linear and still commute with ρ_s because of the direct sum. By Schur's lemma we get that each h_j is a scalar multiple of the identity. For any h_i we therefore have $\dim V/\dim W_i$ choices of these scalars, so this is the dimension of H.

b) $a(h \cdot w) = (ah) \cdot w = h \cdot (aw)$ by the definition of tensor product multiplication. Hence F is linear by linearity of a and the sum given. Recall that ρ acts trivially on h so we have

$$F(\rho_s(h \cdot w)) = (\rho_s h)(\rho_s w) = \rho_s h(w) = \rho_s F(h \cdot w)$$

because h commutes with ρ_s. Clearly F is non-zero, otherwise H would be trivial. By Schur's lemma we therefore know that $H \otimes W_i \cong V_i$ and that F is a scalar multiple of the identity, hence provides us with such an isomorphism.

c) *I'm not totally clear on what this question means.* I think 'in the obvious way' means $h(\boldsymbol{w}) = \sum_\alpha h_\alpha w_\alpha$, in which case we can just apply part b) and we are done. Another possible interpretation is that $h(\boldsymbol{w}) = \sum_\alpha a_\alpha h_\alpha(w_\alpha)$ for some scalars a_α, but in this case some of the scalars could be 0 and I think that this would mean that h is not invertible and hence couldn't be an isomorphism.

2.9

$V_i = \bigoplus_{j=1}^m W_i$ and $V_{i,\alpha}$ is the image projection of the projection $p_{\alpha\alpha}$. We know from the proof of proposition 8 that the action of $p_{\alpha\alpha}$ on the basis of W_k $\{e_\alpha^k\}$ is $p_{\alpha\alpha} e_\beta^k = \mathbb{1}\{\beta = \alpha\}\, e_\alpha^k$, so $V_{i,\alpha}$ is the direct sum

$$V_{i,\alpha} = \bigoplus_{j=1}^m \operatorname{Span}\{e_\alpha^j\}.$$

We also know from 2.8 that h can be written in terms of a basis h_i where each of the $h_i = \lambda_i I$ is a scalar multiple of the identity. This means that h is surjective, since each summand of the direct sum is spanned by one element. It also means that h is injective because the sum is direct, so the elements of each of the summand are linearly independent. By linearity each h is clearly a homomorphism. The proof is complete.

2.10

Don't understand this exercise and can't finish it...

- $V(x)$ is the smallest sub rep of V that contains x. This means that $V(x)$ is the subspace generated by the orbit of x under G
$$V(x) = \operatorname{Span}\{\rho_s x : s \in G\}.$$
- We can write $V_i = \bigoplus_{j=1}^m W_i$ where the j^{th} W_i has basis $\{e_\alpha^j\}$. So $x_\alpha^1 = \sum_{j=1}^m x_{j,\alpha} e_1^j$ where $x_{j,\alpha}$ are the components of the j^{th} term in the direct summand that forms $x \in V$ in the appropriate basis.

3 Subgroups, Products and Induced Representations

3.1

Let ρ be a representation of G corresponding to the irreducible representation space V. G is Abelian, therefore for any $s, t \in G$

$$\rho_t \rho_s = \rho_{ts} = \rho_{st} = \rho_s \rho_t.$$

An application of Schur's lemma then tells us that $\rho_s = \lambda_s I$ is a homothety for any element s of G. V therefore must have dimension 1, since if it has large dimension then choosing an orthogonal basis for this space exhibits a direct sum decomposition of V into subspaces stable under this representation of G.

3.2

a) The center C of G is an Abelian subgroup, so we know from the solution to the previous exercise that ρ is a homothety for any $s \in C$. Write $\rho_s = \lambda_s I$, then $\chi(s) = n\lambda_s$ and

$$|\chi(s)|^2 = \chi(s)\chi(s)^* = \chi(s)\chi(s^{-1}) = n^2 \lambda_s \lambda_s^{-1} = n^2.$$

Note that λ_s cannot be zero because ρ must be invertible.

b)
$$|G| = \sum_{s \in G} |\chi(s)|^2 \geq \sum_{s \in C} |\chi(s)|^2 = n^2 c$$

using part a).

c)

Claim 1. ρ faithful $\implies$ ρ an injection.

Proof. If $\rho_s = \rho_g$ then $\rho_s^{-1}\rho_g = \rho_{s^{-1}}\rho_g = \rho_{s^{-1}g} = I$, which by the hypothesis means that $s^{-1}g = e$. ☐

Consider now the set of unit modulus complex scalars $\omega_s = \frac{1}{n}\chi(s)$. There are exactly $|C|$ of these because ρ is injective. All of these satisfy $\omega_s^{|C|} = 1$ because C is a group. These are therefore the roots of unity, which form a cyclic group under multiplication.

3.3

Let's take the multiplication to mean that $\chi_1\chi_2$ is the function on G with values $\chi_1(s)\chi_2(s)$. This operation is clearly commutative since it returns elements of the underlying field of the representation space. We are given that $\hat{G}$ is closed under multiplication. From the previous exercises we know that the irreducible characters of G are of the form $\chi(s) = n\lambda_s$ where $\lambda_s \neq 0$, so these are invertible. This shows that $\hat{G}$ is an Abelian group. Now, since G is Abelian we know it has exactly $|G|$ conjugacy classes and therefore $|G|$ irreducible characters. Since $\hat{G}$ is closed, this means that $|\hat{G}| = |G|$.

Write $\psi_x : \chi \mapsto \chi(x)$ for $x \in G$. We are told that this is an irreducible character of $\hat{G}$. Define multiplication on $\hat{\hat{G}}$ by $\psi_x\psi_y\chi = \chi(x)\chi(y)$. We can verify that $\psi_x\psi_y\chi = \chi(x)\chi(y) = \lambda_x\lambda_y = \lambda_{xy} = \chi(xy)$, since the irreducible representations have degree 1 because $\hat{G}$ is Abelian. This means that $\hat{\hat{G}}$ is an Abelian group.

Since $\hat{G}$ is an Abelian group of order $|G|$, we know that it has $|G|$ irreducible characters (with equality up to isomorphism). So the set $\hat{\hat{G}}$ has order at most $|G|$. If the map $s \mapsto \psi_s$ is injective then the set $\hat{\hat{G}}$ has size $|G|$, and above we checked that this is a homomorphism. An injective map between two finite sets of the same size is a bijection. Hence, checking injectivity is enough to prove isomorphism.

References

[Serre, 1977] Serre, J.-P. (1977). *Linear representations of finite groups*. Graduate texts in mathematics; 42. Springer-Verlag, New York.