Solutions to Linear Representations of Finite Groups

Bryn Elesedy*
Department of Computer Science
University of Oxford

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*bryn@robots.ox.ac.uk

2 Character Theory

2.1

Let V and V' be the corresponding representations. Then $\chi + \chi'$ is the character of the direct sum $V \oplus V'$. The character of the alternating square is then, for any s in the group,

$$
\begin{aligned}
(\chi + \chi')^2_\sigma &= \frac{1}{2}((\chi(s) + \chi'(s))^2 + \chi(s^2) + \chi'(s^2)) \\
&= \frac{1}{2}(\chi(s)^2 + \chi(s^2)) + \frac{1}{2}(\chi'(s)^2 + \chi'(s^2)) + \chi(s)\chi'(s) \\
&= \chi^2_\sigma(s) + {\chi'_\sigma}^2(s) + \chi(s)\chi'(s).
\end{aligned}
$$

We can do the same thing for the symmetric square

$$
\begin{aligned}
(\chi + \chi')^2_\alpha &= \frac{1}{2}((\chi(s) + \chi'(s))^2 - \chi(s^2) + \chi'(s^2)) \\
&= \frac{1}{2}(\chi(s)^2 - \chi(s^2)) + \frac{1}{2}(\chi'(s)^2 - \chi'(s^2)) + \chi(s)\chi'(s) \\
&= \chi^2_\alpha(s) + {\chi'_\alpha}^2(s) + \chi(s)\chi'(s).
\end{aligned}
$$

2.2

In the permutation representation, the matrix ρ_s will be a permutation matrix, with 1 at position (i, i) if i is fixed by s and 0 on the diagonal otherwise. The character $\chi(s)$ is the trace of this matrix, so is the number of elements fixed by s.

2.3

We may choose bases $\{e_i\}$ for V and $\{e'_j\}$ V' such that $\langle e_i, e'_j \rangle = \delta_{ij}$. Then for any matrix A we have, in the given bases,

$$
\langle \rho_s x, A x' \rangle = \sum_{ij} x_i x'_j (\rho_s)_{ki} A_{kj}.
$$

So the map is invariant if and only if $A = (\rho_s^{-1})^\top$. We can then define a representation by $\rho'_s = (\rho_s^{-1})^\top$, which is clearly a representation. Existence and uniqueness have therefore been established.

2.4

The map ρ_s is clearly linear and invertible, since the $\rho_{i,s}$ are linear and invertible. Further, for any f in W

$$
\rho_{st} f = \rho_{2,st} f \rho_{1,st}^{-1} = \rho_{2,s} \rho_{2,t} f \rho_{1,t}^{-1} \rho_{1,s}^{-1} = \rho_s \rho_t f.
$$

We can then calculate the character by expanding in a basis. Let f have components f_{ab} in a basis for W. Then (using Einstein notation)

$$
(\rho_s f)_{ij} = (\rho_{2,s})_{i\alpha} (\rho_{1s}^{-1})_{\beta j} f_{\alpha\beta}.
$$

We calculate the trace of ρ_s by contraction of $\rho_s e_{ij}$ with elements e'_{kl} of the dual basis, which are such that $\langle e_{ij}, e'_{kl} \rangle = \delta_{ij} \delta_{kl}$. The conclusion then follows from $\chi(s^{-1}) = \chi(s)^*$ and that $\rho_{1,s}$ is a homomorphism.

2.5

The character of the unit representation is 1, so the number of times this representation occurs in ρ is therefore $(\chi|1) = \frac{1}{|G|} \sum_{g \in G} \chi(g)$.

2.6

a) Let e_x be the basis for the permutation representation V. From each of the c orbits, take a representative arbitrarily and denote them y_i $i = 1, \ldots, c$. For each orbit, we can construct an invariant in V

$$
\nu_i = \sum_{g \in G} \rho_g e_{y_i}.
$$

Denote by W_i the space spanned by ν_i, which is clearly invariant under G and irreducible, W_i is fixed by G and is hence the unit representation. We can therefore decompose V into the direct sum

$$U \oplus W_1 \oplus \cdots \oplus W_c.$$

We can notice that U cannot contain the unit representation. If it did, then there would have to be an element of U that is fixed by G, but this element would form n orbit, which contradicts the hypothesis.

b) In the permutation representation ρ_s is a permutation matrix, so the character of the representation $\chi(s)$ is the number of elements of X that are fixed by s. Clearly, there must be $\chi(s)^2$ elements fixed by ρ_s in $X \times X$.

c) - i) $\iff$ ii) G is doubly transitive on $X \times X$, so is transitive on X, so the diagonal Δ is an orbit. Since G is doubly transitive, Δ^c is also an orbit. So these are the only two orbits. These statements also work the other way around.

   - ii) $\iff$ iii) is just part a)

   - iii) $\iff$ iv) is given in the hint. $\chi = 1 + \psi$ where ψ is the character of θ. Then since $(\chi^2|1) = 2$ (two orbits) expanding the sqare tells us that $(\psi^2|1) = 1$. However, because ψ is real valued (all of the matrices will be real in the permutation representation) we see that $(\psi^2|1) = \frac{1}{|G|} \sum_{g \in G} \psi(s)\psi(s)^* = (\psi|\psi)$. We know that if a character η has $(\eta|\eta) = 1$ then it is the character of an irreducible representation, so θ is irreducible.

References

[Serre, 1977] Serre, J.-P. (1977). *Linear representations of finite groups.* Graduate texts in mathematics; 42. Springer-Verlag, New York.