Solutions to Linear Representations of Finite Groups

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2 Character Theory

2.1

Let V and V' be the corresponding representations. Then $\chi+\chi'$ is the character of the direct sum $V\oplus V'$. The character of the alternating square is then, for any s in the group,

$$\begin{aligned}(\chi+\chi')^2_\sigma &= \frac{1}{2}((\chi(s)+\chi'(s))^2+\chi(s^2)+\chi'(s^2))\\ &= \frac{1}{2}(\chi(s)^2+\chi(s^2))+\frac{1}{2}(\chi'(s)^2+\chi'(s^2))+\chi(s)\chi'(s)\\ &= \chi^2_\sigma(s)+{\chi'_\sigma}^2(s)+\chi(s)\chi'(s).\end{aligned}$$

We can do the same thing for the symmetric square

$$\begin{aligned}(\chi+\chi')^2_\alpha &= \frac{1}{2}((\chi(s)+\chi'(s))^2-\chi(s^2)+\chi'(s^2))\\ &= \frac{1}{2}(\chi(s)^2-\chi(s^2))+\frac{1}{2}(\chi'(s)^2-\chi'(s^2))+\chi(s)\chi'(s)\\ &= \chi^2_\alpha(s)+{\chi'_\alpha}^2(s)+\chi(s)\chi'(s).\end{aligned}$$

2.2

In the permutation representation, the matrix ρ_s will be a permutation matrix, with 1 at position (i,i) if i is fixed by s and 0 on the diagonal otherwise. The character $\chi(s)$ is the trace of this matrix, so is the number of elements fixed by s.

2.3

We may choose bases $\{\boldsymbol{e}_i\}$ for V and $\{\boldsymbol{e}'_j\}$ V' such that $\langle \boldsymbol{e}_i, \boldsymbol{e}'_j\rangle = \delta_{ij}$. Then for any matrix A we have, in the given bases,

$$\langle \rho_s x, Ax'\rangle = \sum_{ij} x_i x'_j (\rho_s)_{ki} A_{kj}.$$

So the map is invariant if and only if $A=(\rho_s^{-1})^\top$. We can then define a representation by $\rho'_s=(\rho_s^{-1})^\top$, which is clearly a representation. Existence and uniqueness have therefore been established.

2.4

The map ρ_s is clearly linear and invertible, since the $\rho_{i,s}$ are linear and invertible. Further, for any f in W

$$\rho_{st}f=\rho_{2,st}f\rho^{-1}_{1,st}=\rho_{2,s}\rho_{2,t}f\rho^{-1}_{1,t}\rho^{-1}_{1,s}=\rho_s\rho_t f.$$

We can then calculate the character by expanding in a basis. Let f have components f_{ab} in a basis for W. Then (using Einstein notation)

$$(\rho_s f)_{ij}=(\rho_{2,s})_{i\alpha}(\rho^{-1}_{1s})_{\beta j}f_{\alpha\beta}.$$

We calculate the trace of ρ_s by contraction of $\rho_s\boldsymbol{e}_{ij}$ with elements $\boldsymbol{e}'_{kl}$ of the dual basis, which are such that $\langle \boldsymbol{e}_{ij},\boldsymbol{e}'_{kl}\rangle=\delta_{ij}\delta_{kl}$. The conclusion then follows from $\chi(s^{-1})=\chi(s)^*$ and that $\rho_{1,s}$ is a homomorphism.

2.5

The character of the unit representation is 1, so the number of times this representation occurs in ρ is therefore $(\chi|1)=\frac{1}{|G|}\sum_{g\in G}\chi(g)$.

2.6

a) Let $\boldsymbol{e}_x$ be the basis for the permutation representation V. From each of the c orbits, take a representative arbitrarily and denote them y_i $i=1,\ldots,c$. For each orbit, we can construct an invariant in V

$$\nu_i=\sum_{g\in G}\rho_g\boldsymbol{e}_{y_i}.$$

Denote by W_i the space spanned by ν_i, which is clearly invariant under G and irreducible, W_i is fixed by G and is hence the unit representation. We can therefore decompose V into the direct sum

$$U \oplus W_1 \oplus \cdots \oplus W_c.$$

We can notice that U cannot contain the unit representation. If it did, then there would have to be an element of U that is fixed by G, but this element would form n orbit, which contradicts the hypothesis.

b) In the permutation representation ρ_s is a permutation matrix, so the character of the representation $\chi(s)$ is the number of elements of X that are fixed by s. Clearly, there must be $\chi(s)^2$ elements fixed by ρ_s in $X \times X$.

c)
- i) $\iff$ ii) G is doubly transitive on $X \times X$, so is transitive on X, so the diagonal Δ is an orbit. Since G is doubly transitive, Δ^c is also an orbit. So these are the only two orbits. These statements also work the other way around.
- ii) $\iff$ iii) is just part a)
- iii) $\iff$ iv) is given in the hint. $\chi = 1 + \psi$ where ψ is the character of θ. Then since $(\chi^2|1) = 2$ (two orbits) expanding the sqare tells us that $(\psi^2|1) = 1$. However, because ψ is real valued (all of the matrices will be real in the permutation representation) we see that $(\psi^2|1) = \frac{1}{|G|}\sum_{g\in G}\psi(s)\psi(s)^* = (\psi|\psi)$. We know that if a character η has $(\eta|\eta) = 1$ then it is the character of an irreducible representation, so θ is irreducible.

2.7

Let χ be a character of G. We know that the regular representation has character $r_G(s) = |G|\,\mathbb{1}\{s = e\}$. If χ vanishes on all elements apart from the identity then we have that $\chi(s) = \chi(e)\mathbb{1}\{s = e\}$. The number of times that χ contains the unit representation is the number of distinct orbits in G, which is clearly an integer. Thus, with 1 denoting the character of the unit representation,

$$(\chi|1) = \frac{1}{|G|}\sum_{g\in G}\chi(g) = \frac{\chi(e)}{|G|} \in \mathcal{Z}$$

so $\chi(e)$ is an integral multiple of $|G|$.

2.8

References

[Serre, 1977] Serre, J.-P. (1977). *Linear representations of finite groups.* Graduate texts in mathematics; 42. Springer-Verlag, New York.