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1 Probability Theory

Theorem 1 (Laws of Set Algebra). For any three sets A, B, C all subsets of S , we have that the operators \cup and \cap are distributive, commutative, associative and satisfy DeMorgan's Laws:

$$\begin{aligned}(A \cup B)^c &= A^c \cap B^c \\ (A \cap B)^c &= A^c \cup B^c\end{aligned}$$

Definition 1 (Sigma Algebra). A collections of subsets \mathcal{B} is called a *sigma algebra* (or *Borel Field*), denoted by \mathcal{B} , if it has the following three properties:

- a. $\emptyset \in \mathcal{B}$
- b. $A \in \mathcal{B} \implies A^c \in \mathcal{B}$
- c. If $A_i \in \mathcal{B}$ for i in \mathcal{I} then $\cup_{i \in \mathcal{I}} A_i \in \mathcal{B}$, where \mathcal{I} is countable. So \mathcal{B} is closed under countable union.

Note that from DeMorgan's laws we have

$$\left(\bigcup_{i=1}^{\infty} A_i^c \right)^c = \bigcap_{i=1}^{\infty} A_i \quad (1)$$

which means that, using b. we get \mathcal{B} is also closed under countable intersections $\cap_{i=1}^{\infty} A_i \in \mathcal{B}$.

Definition 2 (Kolmogorov Axioms). Given a sample space S and an associated sigma algebra \mathcal{B} , a *probability function* is a function \mathbb{P} with domain \mathcal{B} that satisfies

1. $\mathbb{P}(A) \geq 0 \quad \forall A \in \mathcal{B}$
2. $\mathbb{P}(S) = 1$
3. If $A_1, A_2, \dots \in \mathcal{B}$ are pairwise disjoint, then $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

The following result makes it a bit easier to find probability functions.

Theorem 2. Let $S = \{s_1, s_2, \dots\}$ be a countable set. Let \mathcal{B} be any sigma algebra of subsets of S . Let p_1, p_2, \dots be nonnegative numbers that sum to 1. For any $A \in \mathcal{B}$, define $\mathbb{P}(A)$ by

$$\mathbb{P}(A) = \sum_{\{i: s_i \in A\}} p_i.$$

Then \mathbb{P} is a probability function on \mathcal{B} .

Definition 3 (Random Variable). A *random variable* is a function from a sample space S into the real numbers.

Definition 4 (cdf). The *cumulative distribution function* of *cdf* of a random variable X is defined by

$$F_X(x) = \mathbb{P}_X(X \leq x) \quad \forall x$$

Theorem 3. The function $F(x)$ is a cdf if and only if:

1. $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$.
2. $F(x)$ is a non-decreasing function of x .
3. $F(x)$ is right-continuous, that is, for every number x_0 , $\lim_{x \downarrow x_0} F(x) = F(x_0)$

We say that a random variable is continuous if its cdf is continuous, and we say that it is discrete if its cdf is a step function.

Theorem 4. The following statements are equivalent:

1. X and Y are identically distributed
2. $F_X(x) = F_Y(x) \quad \forall x$

Definition 5 (probability mass function). The *probability mass function* or *pmf* of a discrete random variable X is

$$f_X(x) = P(X = x) \quad \forall x$$

Definition 6 (probability density function). The *probability density function* or *pdf* of a continuous random variable X is the function $f_X(x)$ that satisfies

$$F_X(x) = \int_{-\infty}^x f_X(t) dt \quad \forall x$$

Theorem 5. A function $f_X(x)$ is a pdf (of pmf) of a random variable X if and only if

- a. $f_X(x) \geq 0 \quad \forall x$
- b. $\sum_x f_X(x) = 1$ (pmf) or $\int_x f_X(x) dx = 1$ (pdf).

2 Transformations and Expectations

2.1 Transformations

If X and Y are discrete random variables, with $Y = g(X)$, then

$$f_Y(y) = \sum_{\{x: g(x)=y\}} f_X(x). \quad (2)$$

For the remainder of this section we will take X and Y to be continuous random variables, with $Y = g(X)$. As such, we will define the following sets $\mathcal{X} = \{x : f_X(x) > 0\}$ and $\mathcal{Y} = \{y : y = g(x) \mid x \in S \subseteq \mathcal{X}\}$.

Theorem 6. Let X have cdf $F_X(s)$, let $Y = g(X)$ and let \mathcal{X} and \mathcal{Y} be defined as above. Then

- a. If g is increasing on \mathcal{X} , then $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.
- b. If g is decreasing on \mathcal{X} , then $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.

Theorem 7. Let X have pdf $f_X(x)$ and $Y = g(X)$, where g is monotone. Define \mathcal{X} and \mathcal{Y} as above. Suppose $f_X(x)$ is continuous on \mathcal{X} and $g^{-1}(y)$ has continuous first derivative on \mathcal{Y} . Then the pdf of Y is given by:

$$f_Y(y) = \mathbf{1}_{\{y \in \mathcal{Y}\}} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

If g is not globally monotone, then we just partition \mathcal{X} into subsets on which g is continuous and monotone and sum the results. If such a partition doesn't exist, then we have technical problems.

Theorem 8 (Probability integral transform). Let X have continuous cdf $F_X(x)$ and define the random variable Y by $Y = F_X(X)$. Then Y is uniformly distributed on $(0, 1)$.

If $F_X(x) = y$ is constant on some interval then we define the inverse by

$$F_X^{-1}(y) = \inf\{x : F_X(x) = y\}.$$

2.2 Expectations

Definition 7 (Expected value). The *expected value* of a random variable $g(X)$, denoted $\mathbb{E}[X]$ is defined by:

- $\mathbb{E}[X] = \int_{\mathbb{R}} g(x) f_X(x) dx$ if X is continuous,
- $\mathbb{E}[X] = \sum_{x \in \mathcal{X}} g(x) \mathbb{P}(X = x)$ if X is discrete.

Theorem 9 (Properties of expectation). a. Linearity.

- b. $g_1(x) \geq g_2(x) \forall x \implies \mathbb{E}[g_1(X)] \geq \mathbb{E}[g_2(X)]$
- c. $a \leq g(x) \leq b \forall x \implies a \leq \mathbb{E}[g(X)] \leq b$
- d. $\operatorname{argmin}_c \mathbb{E}[(X - c)^2] = \mathbb{E}[X]$

2.3 Moments

Definition 8 (Moment). For integer n , the n^{th} moment of X is

$$\mu'_n = \mathbb{E}[X^n].$$

The n^{th} central moment, μ_n is

$$\mu_n = \mathbb{E}[(X - \mu)^n].$$

Where $\mu = \mu_1 = \mathbb{E}[X]$

2.3.1 Variance

Definition 9 (Variance). The variance of a random variable X , written $\text{Var}[X]$ is the second central moment of X ,

$$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

The standard deviation of X , denoted σ_X , is given by $\sigma_X = \sqrt{\text{Var}[X]}$.

Theorem 10 (Properties of variance). If X has finite variance then:

- a. $\text{Var}[aX + b] = a^2 \text{Var}[X]$
- b. $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

2.3.2 Moment Generating Functions

Definition 10 (Moment generating function). Let X be a random variable with cdf F_X . The moment generating function (mgf) of X , denoted $M_X(t)$, is given by

$$M_X(t) = \mathbb{E}[e^{tX}]$$

provided that the expectation exists for t in some (open) neighbourhood of 0 (otherwise we say the mgf does not exist).

Remark 1. The mgf is the Laplace transform of the pdf.

Theorem 11. If X has mgf $M_X(t)$ then

$$\mathbb{E}[X^n] = \left. \frac{d}{dt} M_X(t) \right|_{t=0}$$

The mgf can be used to calculate moments, but its principal utility is in characterising a distribution. This relationship can run into some technical difficulties. If the mgf exists, it characterises an infinite set of moments. However, it is possible for two distinct random variables to give rise to the same set of moments.

The problem of uniqueness of moments does not occur if the random variables have bounded support (in this case an infinite sequence of moments uniquely determines the distribution). Further, if the mgf exists in a neighbourhood of 0 then it uniquely determines the distribution, no matter the support. Thus, existence of an infinite set of moments is not equivalent to the existence of the mgf. We have the following theorem, describing when the mgf determines the distribution.

Theorem 12 (When mgf determines distribution). Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exist.

- a. If X and Y have bounded support then $F_X(u) = F_Y(u) \forall u$ if and only if $\mathbb{E}[X^r] = \mathbb{E}[Y^r] \forall r \in \mathbb{N}$. (So the cdfs are equal if and only if all the moments agree.)

b. If the mgfs exist and are identical in some neighbourhood of 0 then the cdfs are equal.

Theorem 13 (Convergence of mgfs near 0 implies convergence of cdfs). Suppose $\{X_i, i = 1, 2, \dots\}$ is a sequence of random variables, each with mgf $M_{X_i}(t)$. Suppose also that for all t in a neighbourhood of 0

$$\lim_{i \rightarrow \infty} M_{X_i} = M_X(t)$$

where $M_X(t)$ is an mgf. Then there is a unique cdf F_X whose moments are determined by $M_X(t)$ and, for all x at which $F_X(x)$ is continuous, we have

$$\lim_{i \rightarrow \infty} F_{X_i}(x) = F_X(x).$$

Remark 2. The convergence of a sequence of moments is not enough to show the convergence of random variables. We need the moment sequence to be unique too. However, if the mgfs converge in a neighbourhood of 0 as above, then we know that the random variables converge. Convergence of mgfs is therefore a sufficient, but not necessary, condition for convergence of the random variables.

Theorem 14. For any constants a and b

$$M_{aX+b}(t) = e^{bt} M_X(at)$$

2.4 Other Generating Functions

Definition 11 (Cumulant generating function). The *cumulant generating function* is $\log(M_X(t))$. The *cumulants* of X are defined as the coefficients of the Taylor series of this function.

Definition 12 (Factorial moment generating function). The *factorial moment generating function* is $\mathbb{E}[t^X]$. The name comes from

$$\left. \frac{d^r}{dt^r} \mathbb{E}[t^X] \right|_{t=1} = \mathbb{E}[X(X-1) \cdots (X-r+1)].$$

For discrete distributions this is the *probability generating function* and the coefficients of the power series give the probabilities

$$\left. \frac{1}{k!} \frac{d^k}{dt^k} \mathbb{E}[t^X] \right|_{t=0} = \mathbb{P}(X = k).$$

Definition 13 (Characteristic function). The *characteristic function* of a random variable X is

$$\phi_X(t) = \mathbb{E}[e^{itX}]$$

Remark 3. The characteristic function is the most useful of the generating functions. Every cdf has a unique characteristic function. When the moments of the cdf exist, the characteristic function can be used to calculate them.

3 Common Families of Distributions

Lots of this chapter is standard definitions of distributions, so is omitted

3.1 Some Distributions

Definition 14 (Normal Distribution). If $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ then \mathbf{X} has pdf

$$f_{\mathbf{X}}(x_1, \dots, x_k | \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^k \det \boldsymbol{\Sigma}}} \exp \left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right)$$

3.1.1 Chi-Squared Distribution

Definition 15. The *chi-squared distribution with p degrees of freedom* has pdf

$$\chi_p^2 \sim \frac{1}{\Gamma(p/2) 2^{p/2}} x^{(p/2)-1} e^{-x/2}, \quad 0 < x < \infty.$$

Theorem 15 (Some facts).

- a. If $Z \sim \mathcal{N}(0, 1)$ then $Z^2 \sim \chi_1^2$
- b. If X_1, \dots, X_n are independent $X_i \sim \chi_{p_i}^2$ then $X_1 + \dots + X_n \sim \chi_{p_1 + \dots + p_n}^2$.

3.1.2 Student's t -Distribution

Definition 16 (Student's t -distribution). $T \sim t_p$, a *t -distribution with p degrees of freedom* if it has pdf

$$f_T(t) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{\sqrt{p\pi}} \frac{1}{(1 + t^2/p)^{(p+1)/2}}, \quad t \in \mathbb{R}$$

If $p = 1$ then this is the Cauchy distribution.

Remark 4. If X_1, \dots, X_n are a random sample from $\mathcal{N}(\mu, \sigma^2)$ then

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}.$$

This is often taken as the definition. Note that the denominator is independent of the numerator.

Lemma 1 (Moments and mgf of t -distribution). Student's t has no mgf because it does not have moments of all orders: t_p has only $p - 1$ moments. If $T_p \sim t_p$ then

$$\begin{aligned} \mathbb{E}[T_p] &= 0 \quad p > 1 \\ \text{Var}[T_p] &= \frac{p}{p-2} \quad p > 2 \end{aligned}$$

3.1.3 Snedcor's F -Distribution

Definition 17 (Snedcor's F -distribution). A random variable $X \sim F_{p,q}$ has *F -distribution with p and q degrees of freedom* if its pdf is

$$f_X(x) = \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{q}{2}\right)} (p/q)^{p/2} \frac{x^{(p/2)-1}}{(1 + px/q)^{(p+q)/2}}, \quad 0 < x < \infty.$$

Remark 5. If X_1, \dots, X_n is a random sample from $\mathcal{N}(\mu_X, \sigma_X^2)$ and Y_1, \dots, Y_m is an independent random sample from $\mathcal{N}(\mu_Y, \sigma_Y^2)$, then

$$\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim F_{n-1, m-1}.$$

This is often taken as the definition.

Theorem 16 (Some facts).

- a. $X \sim F_{p,q} \implies 1/X \sim F_{q,p}$
- b. $X \sim t_q \implies X^2 \sim F_{1,q}$
- c. $X \sim F_{p,q} \implies \frac{(p/q)X}{1+(p/q)X} \sim \text{beta}(p/2, q/2)$

3.1.4 Multinomial Distribution

Definition 18 (Multinomial Distribution). Let m and n be positive integers and let $p_1, \dots, p_n \in [0, 1]$ satisfy $\sum_{i=1}^n p_i = 1$. Then the random vector (X_1, \dots, X_n) has *multinomial distribution with m trials and cell probabilities p_1, \dots, p_n* if the joint pmf of (X_1, \dots, X_n) is

$$f(x_1, \dots, x_n) = \frac{m!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n} = m! \prod_{i=1}^n \frac{p_i^{x_i}}{x_i!}$$

on the set of (x_1, \dots, x_n) such that each x_i is a nonnegative integer and $\sum_{i=1}^n x_i = m$.

Remark 6. The marginal distributions have $X_i \sim \text{binomial}(m, p_i)$.

Theorem 17 (Multinomial Theorem). Let m and n be positive integers and let \mathcal{A} be the set of vectors $\mathbf{x} = (x_1, \dots, x_n)$ such that each x_i is a nonnegative integer and $\sum_{i=1}^n x_i = m$. Then for any real numbers p_1, \dots, p_n ,

$$(p_1 + \cdots + p_n)^m = \sum_{\mathbf{x} \in \mathcal{A}} \frac{m!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n}.$$

3.2 Exponential Families

Definition 19 (Exponential family 1). A family of pmfs/pdfs is called an *exponential family* if it can be expressed

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta}) \exp \left(\sum_{i=1}^k w_i(\boldsymbol{\theta}) t_i(x) \right)$$

where $h(x) \geq 0$, the t_i are real valued functions of the observation x that do not depend on $\boldsymbol{\theta}$ and $c(\boldsymbol{\theta}) \geq 0$ and the $w_i(\boldsymbol{\theta})$ are real valued functions of $\boldsymbol{\theta}$ that do not depend on x .

Theorem 18. If X is a random variable from an exponential family distribution then

$$\mathbb{E} \left[\sum_{i=1}^k \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X) \right] = -\frac{\partial}{\partial \theta_j} \log c(\boldsymbol{\theta})$$

and

$$\text{Var} \left[\frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X) \right] = -\frac{\partial^2}{\partial \theta_j^2} \log c(\boldsymbol{\theta}) - \mathbb{E} \left[\sum_{i=1}^k \frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(X) \right]$$

Definition 20 (Exponential family 2). We can write another parameterisation of the exponential family

$$f(x|\boldsymbol{\eta}) = h(x)c^*(\boldsymbol{\eta})\exp(\boldsymbol{\eta} \cdot \mathbf{t}(x))$$

where $\boldsymbol{\eta}$ is called the *natural parameter* and the set $\mathcal{H} = \{\boldsymbol{\eta} : \int_{\mathbb{R}} f(x|\boldsymbol{\eta})dx < \infty\}$ is called the *natural parameter space* and is convex.

Remark 7. $\{\boldsymbol{\eta} : \boldsymbol{\eta} = \mathbf{w}(\boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta\} \subseteq \mathcal{H}$. So there may be more parameterisations here than previously.

The natural parameter provides a convenient mathematical formulation, but sometimes lacks simple interpretation.

Definition 21 (Curved exponential family). A *curved exponential family* distribution is one for which the dimension of $\boldsymbol{\theta}$ is $d < k$. If $d = k$ then we have a *full exponential family*.

3.3 Location and Scale Families

Definition 22 (Location family). Let $f(x)$ be any pdf. The family of pdfs $f(x - \mu)$ for $\mu \in \mathbb{R}$ is called the *location family with standard pdf* $f(x)$ and μ is the *location parameter* of the family.

Definition 23 (Scale family). Let $f(x)$ be any pdf. For any $\sigma > 0$ the family of pdfs $\frac{1}{\sigma}f(x/\sigma)$ is called the *scale family with standard pdf* $f(x)$ and σ is the *scale parameter* of the family.

Definition 24 (Location-Scale family). Let $f(x)$ be any pdf. For $\mu \in \mathbb{R}$ and $\sigma > 0$ the family of pdfs $\frac{1}{\sigma}f(\frac{x-\mu}{\sigma})$ is called the *location-scale family with standard pdf* $f(x)$; μ is the *location parameter* and σ is the *scale parameter*.

Theorem 19 (Standardisation). Let f be any pdf, $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_{>0}$. Then X is a random variable with pdf $\frac{1}{\sigma}f(\frac{x-\mu}{\sigma})$ if and only if there exists a random variable Z with pdf $f(z)$ and $X = \sigma Z + \mu$.

Remark 8. Probabilities of location-scale families can be computed in terms of their standard variables Z

$$\mathbb{P}(X \leq x) = \mathbb{P}\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

3.4 Inequalities and Identities

Theorem 20 (Chebychev's inequality). Let X be a random variable and let $g(x)$ be a nonnegative function. Then, for any $r > 0$,

$$\mathbb{P}(g(X) \geq r) \leq \frac{\mathbb{E}[g(X)]}{r}.$$

Remark 9. This bound is conservative and almost never attained.

Remark 10 (Markov inequality). The Markov inequality is the special case with $g = \mathbb{I}$.

Theorem 21. Let $X_{\alpha,\beta}$ denote a gamma(α, β) random variable with pdf $f(x|\alpha, \beta)$, where $\alpha > 1$. Then for any constants a and b :

$$\mathbb{P}(a < X_{\alpha,\beta} < b) = \beta(f(a|\alpha, \beta) - f(b|\alpha, \beta)) + \mathbb{P}(a < X_{\alpha-1,\beta} < b)$$

Lemma 2 (Stein's Lemma). Let $X \sim \mathcal{N}(\theta, \sigma^2)$ and let g be a differentiable function with $\mathbb{E}[g'(x)] < \infty$. Then

$$\mathbb{E}[g(X)(X - \theta)] = \sigma^2 \mathbb{E}[g'(X)]$$

The proof is just integration by parts.

Remark 11. Stein's lemma is useful for moment calculations

Theorem 22. Let χ_p^2 denote a chi squared distribution with p degrees of freedom. For any function $h(x)$,

$$\mathbb{E}[h(\chi_p^2)] = p \mathbb{E} \left[\frac{h(\chi_{p+2}^2)}{\chi_{p+2}^2} \right]$$

provided the expressions exist.

Theorem 23. Let $g(x)$ be a function that is bounded at -1 and has finite expectation, then

a. If $X \sim \text{Poisson}(\lambda)$,

$$\mathbb{E}[\lambda g(X)] = \mathbb{E}[X g(X - 1)].$$

b. If $X \sim \text{negative-binomial}(r, p)$,

$$\mathbb{E}[(1 - p)g(X)] = \mathbb{E} \left[\frac{X}{r + X - 1} g(X) \right].$$

4 Multiple Random Variables

4.1 Facts

- + RVs are independent if and only if their pdfs factorise
- + Functions of independent RVs are independent
- + Expectations of products (and hence mgfs, etc. of sums) of independent RVs factor
- + Variance of sum of independent RVs is sum of variances.
- + Independent RVs have vanishing covariance/correlation, but the converse is not true in general.

4.2 Bivariate Relations

Theorem 24 (Conditional Expectation).

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$$

provided the expectations exist.

Theorem 25 (Conditional variance).

$$\text{Var}[X] = \mathbb{E}[\text{Var}[X|Y]] + \text{Var}[\mathbb{E}[X|Y]]$$

provided the expectations exist.

Definition 25 (Covariance).

$$\text{Cov}[X, Y] = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$$

Theorem 26.

$$\text{Cov}[X, Y] = \mathbb{E}[XY] - \mu_X\mu_Y$$

Theorem 27.

$$\text{Var}[aX + bY] = a^2\text{Var}[X] + b^2\text{Var}[Y] + 2ab\text{Cov}[X, Y]$$

Definition 26 (Correlation).

$$\rho_{XY} = \frac{\text{Cov}[X, Y]}{\sigma_X\sigma_Y}$$

Remark 12. The correlation measures the strength of *linear* relation between two RVs. It is possible to have strong non-linear relationships but with $\rho = 0$.

We can use an argument similar to the standard proof of Cauchy-Schwarz to show the following

Theorem 28. Let X and Y be any RVs, then

- a. $-1 \leq \rho_{XY} \leq 1$,
- b. $|\rho_{XY}| = 1$ if and only if there are constants $a \neq 0, b$ such that $\mathbb{P}(Y = aX + b) = 1$. If $|\rho_{XY}| = 1$ then $\text{sign}(\rho) = \text{sign}(a)$.

4.3 Inequalities

4.3.1 Numerical Inequalities

Theorem 29. Let a and b be any positive numbers and let $p, q > 1$ satisfy $1/p + 1/q = 1$, then

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab$$

with equality if and only if $a^p = b^q$.

Theorem 30 (Hölder's Inequality). Let X and Y be any random variables and let $p, q > 1$ satisfy $1/p + 1/q = 1$, then

$$|\mathbb{E}[XY]| \leq \mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{1/p} \mathbb{E}[|Y|^q]^{1/q}$$

Corollary 1.

- Cauchy-Schwarz is the special case $p = q = 2$
- $\text{Cov}[X, Y]^2 \leq \sigma_X^2 \sigma_Y^2$
- $\mathbb{E}[|X|] \leq \mathbb{E}[|X|^p]^{1/p}$
- *Liapounov's Inequality* $\mathbb{E}[|X|^r]^{1/r} \leq \mathbb{E}[|X|^s]^{1/s}$ where $1 < r < s < \infty$.

4.3.2 Functional Inequalities

Definition 27 (Convex Function). A function $g(x)$ is *convex* on a set S if for all $x, y \in S$ and $0 < \lambda < 1$

$$g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y).$$

Strictly convex is when the inequality is strict. g is *concave* if $-g$ is convex.

Lemma 3. $g(x)$ is convex on S if $g''(x) \geq 0 \forall x \in S$.

Theorem 31 (Jensen's Inequality). If $g(x)$ is convex, then for any random variable X

$$\mathbb{E}[g(X)] \leq g(\mathbb{E}[X]).$$

Equality holds if and only if, for every line $a + bx$ that is tangent to $g(x)$ at $x = \mathbb{E}[X]$, $\mathbb{P}\{g(X) = a + bX\} = 1$. (So if and only if g is affine with probability 1.)

Corollary 2.

- $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$
- $\mathbb{E}[1/X] \geq 1/\mathbb{E}[X]$

5 Properties of a Random Sample

Definition 28 (Random Sample). A collection of random variables X_1, \dots, X_n is a *random sample of size n from population $f(x)$* if they are iid with pdf/pmf $f(x)$.

Definition 29 (Statistic). $Y = T(X_1, \dots, X_n)$ is a *statistic* if the domain of T contains the sample space of (X_1, \dots, X_n) . The distribution of Y is the *sampling distribution of Y* .

Remark 13. A statistic is any function of the data. The only restriction is that the statistic is not also a function of some other parameters.

Definition 30 (Sample Mean and Sample Variance). The *sample mean* \bar{X} and *sample variance* S^2 of a random sample X_1, \dots, X_n are, respectively,

- $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$
- $S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$.

The *sample standard deviation* is $S = \sqrt{S^2}$.

Theorem 32 (Some facts). Let X_1, \dots, X_n be a random sample from a population with mean μ and finite variance σ^2 , then

- a. $\mathbb{E}[\bar{X}] = \mu$
- b. $\text{Var}[\bar{X}] = \frac{\sigma^2}{n}$
- c. $\mathbb{E}[S^2] = \sigma^2$.

Theorem 33. Take $x_1, \dots, x_n \in \mathbb{R}$ and let \bar{x} be their mean. Then,

- a. $\min_a \sum_{i=1}^n (x_i - a)^2 = \sum_{i=1}^n (x_i - \bar{x})^2$
- b. $(n-1)s^2 = \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n x_i^2 - n\bar{x}^2$.

Theorem 34. Let X_1, \dots, X_n be a random sample from population $f(x|\theta)$ belonging to an exponential family

$$f(x|\theta) = h(x)c(\theta) \exp \left(\sum_{i=1}^k w_i(\theta)t_i(x) \right).$$

Define the statistics

$$T_i(X_1, \dots, X_n) = \sum_{j=1}^n t_i(X_j), \quad i = 1, \dots, k.$$

Then if the set $\{(w_1(\theta), \dots, w_k(\theta), \theta \in \Theta)\}$ contains an open subset of \mathbb{R}^k then the distribution of (X_1, \dots, X_n) is an exponential family of the form

$$f(u_1, \dots, u_n|\theta) = H(u_1, \dots, u_n)c(\theta)^n \exp \left(\sum_{i=1}^k w_i(\theta)u_i \right).$$

Remark 14. The open condition eliminates curved exponential families from this result.

5.1 Sampling from the Normal Distribution

Theorem 35. Let X_1, \dots, X_n be a random sample from a $\mathcal{N}(\mu, \sigma)$ distribution. Then,

- \bar{X} and S^2 are independent
- $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$
- $(n-1)S^2/\sigma^2 \sim \chi_{n-1}^2$.

Lemma 4 (Covariance and independence). In the case of samples from a multivariate normal

- Independence \iff vanishing covariance
- Pairwise independence \iff independence

5.2 Convergence Concepts

5.2.1 Convergence in Probability

Definition 31. A sequence of random variables $\{X_i : i \in \mathbb{N}\}$ *converges in probability* to a random variable X if $\forall \epsilon > 0$

$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| \geq \epsilon) = 0$$

(or equivalently if $\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| < \epsilon) = 1$).

Theorem 36 (Weak Law of Large Numbers). Let X_1, X_2, \dots be iid RVs with mean μ and $\mathbb{E}[|X_i|] < \infty$. Define $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n$. Then the sequence \bar{X}_n converges in probability to μ . That is, $\forall \epsilon > 0$

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\bar{X}_n - \mu| < \epsilon) = 1$$

Proof is using Chebychev's inequality.

Theorem 37. If X_1, X_2, \dots converges in probability to X and h is a continuous function, then $h(X_1), h(X_2), \dots$ converges in probability to $h(X)$.

5.2.2 Almost Sure Convergence

Definition 32. A sequence of random variables X_1, X_2, \dots *converges almost surely* to a random variable X if $\forall \epsilon > 0$

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} |\bar{X}_n - X| < \epsilon\right) = 1.$$

Comments.

+ Almost sure convergence is much stronger than convergence in probability. Convergence in probability states that the sequence of measures of the sets on which the sequence has finite difference from its the limit converges to 0. Almost sure convergence states that any place where the sequence has finite difference from its limit must have measure 0. It's like a sequence of integrals converging vs. whether the integrands converge.

+ Almost sure convergence implies convergence in probability but not the other way around.

Theorem 38. If a sequence converges in probability then it is possible to find a subsequence that converges almost surely.

Theorem 39 (Strong Law of Large Numbers). Let X_1, X_2, \dots be iid RVs with mean μ and $\mathbb{E}[|X_i|] < \infty$. Define $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n$. Then the sequence \bar{X}_n converges almost surely to μ . That is, $\forall \epsilon > 0$

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} |\bar{X}_n - \mu| < \epsilon\right) = 1.$$

5.2.3 Convergence in Distribution

Definition 33. A sequence of random variables X_1, X_2, \dots *converges in distribution* to a random variable X if

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x)$$

at all points where F_X is continuous.

Remark 15. Here it is really the cdfs that converge, rather than the random variables. In this way convergence in distribution differs from the previous two concepts.

Theorem 40. Convergence in probability implies convergence in distribution

Theorem 41 (Central Limit Theorem). Let X_1, X_2, \dots be a sequence of iid random variables with $\mathbb{E}[X_i] = \mu$ and finite variance $\text{Var}[X_i] = \sigma^2 < \infty$. Define $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Let $G_n(x)$ denote the cdf of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$. Then $\forall x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} G_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy.$$

That is, $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ converges in distribution to the standard normal.

Theorem 42 (Slutsky's Theorem). If $X_n \rightarrow X$ in distribution and $Y_n \rightarrow a$ in probability with a constant, then

- a. $Y_n X_n \rightarrow aX$ in distribution
- b. $X_n + Y_n \rightarrow X + a$ in distribution

Remark 16. This tells us, for instance, that

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{S_n} \rightarrow \mathcal{N}(0, 1)$$

in distribution, since we know that $S_n \rightarrow \sigma$ in probability.

5.2.4 The Delta Method

If we are interested in the convergence of some function of a sequence of RVs, rather than the RVs themselves, then we can use the Delta Method (follows from an application of Taylor's theorem and Slutsky's theorem).

Theorem 43 (Delta Method). Let Y_n be a sequence of random variables that satisfies $\sqrt{n}(Y_n - \theta) \rightarrow \mathcal{N}(0, \sigma^2)$ in distribution. For a given function g and a specific value of θ , suppose that $g'(\theta)$ exists and is non-zero. Then

$$\sqrt{n}[g(Y_n) - g(\theta)] \rightarrow \mathcal{N}(0, \sigma^2 g'(\theta)^2)$$

in distribution.

Remark 17. There exists a corresponding multivariate result.

If $g'(\theta) = 0$ then we take the next term in the Taylor series.

Theorem 44 (Second Order Delta Method). Let Y_n be a sequence of random variables that satisfies $\sqrt{n}(Y_n - \theta) \rightarrow \mathcal{N}(0, \sigma^2)$ in distribution. For a given function g and a specific value of θ , suppose that $g'(\theta) = 0$ and $g''(\theta)$ exists and is non-zero. Then

$$\sqrt{n}[g(Y_n) - g(\theta)] \rightarrow \mathcal{N}(0, \sigma^2 g''(\theta)^2)$$

in distribution.

5.3 Generating A Random Sample

Definition 34 (Direct Method). A *Direct Method* of generating a random sample uses the probability integral transform to map draws from a $\text{uniform}(0, 1)$ random variable to draws from the distribution of interest.

The Probability Integral Transform states that if X has continuous cdf $F_X(x)$ then

$$F_X(X) \sim \text{uniform}(0, 1).$$

Definition 35 (Accept-Reject Algorithm). Let $Y \sim f_Y(y)$ and $V \sim f_V(v)$ where f_Y and f_V have common support with

$$M = \sup_y f_Y(y)/f_V(y) < \infty.$$

To generate a random variable $Y \sim f_Y$:

- a. Generate $U \sim \text{uniform}(0, 1)$, $V \sim f_V$ independent.
- b. If $U < \frac{1}{M} f_Y(V)/f_V(V)$, return V as a sample of Y ; otherwise go back to (a.).

Remark 18.

- It is typical to call V the *candidate density* and Y the *target density*.
- One would normally try to choose a candidate density with heavier tails than the target density (e.g. Cauchy and normal) to ensure that the tails of the target are well represented. If the target has heavy tails, however, it can be hard to find a candidate that results in finite M . In this case people turn to MCMC methods.
- Note that $\mathbb{P}(\text{terminate}) = 1/M$. The number of trials to generate one sample of Y is therefore $\text{geometric}(1/M)$, with M the expected number of trials.
- The intuition behind this algorithm is that if we consider placing the density of a random variable Y in a box (2d for simplicity) with coordinates (v, u) , we express the cdf of Y using $V, U \sim \text{uniform}(0, 1)$

$$\mathbb{P}(Y \leq y) = \mathbb{P}(V \leq y | U \leq \frac{1}{c} f_Y(V))$$

where $c = \sup_y f_Y(y)$. In the actual algorithm we take $U \sim \text{uniform}(0, 1)$ and V to be an RV that has common support with Y .