1 Multiple Random Variables

1.1 Facts

+ RVs are independent if and only if their pdfs factorise
+ Functions of independent RVs are independent
+ Expectations (and hence mgfs, etc.) of independent RVs factor
+ Independent RVs have vanishing covariance/correlation, but the converse is not true in general.

1.2 Bivariate Relations

Theorem 1 (Conditional Expectation).

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$$

provided the expectations exist.

Theorem 2 (Conditional variance).

$$\mathrm{Var}[X] = \mathbb{E}[\mathrm{Var}[X|Y]] + \mathrm{Var}[\mathbb{E}[X|Y]]$$

provided the expectations exist.

Definition 1 (Covariance).

$$\mathrm{Cov}[X,Y] = \mathbb{E}[(X-\mu_X)(Y-\mu_Y)]$$

Theorem 3.

$$\mathrm{Cov}[X,Y] = \mathbb{E}[XY] - \mu_X\mu_Y$$

Theorem 4.

$$\mathrm{Var}[aX+bY] = a^2\mathrm{Var}[X] + b^2\mathrm{Var}[Y] + 2ab\mathrm{Cov}[X,Y]$$

Definition 2 (Correlation).

$$\rho_X Y = \frac{\mathrm{Cov}[X,Y]}{\sigma_X\sigma_Y}$$

Remark 1. The correlation measures the strength of *linear* relation between two RVs. It is possible to have strong non-linear relationships but with $\rho = 0$.

We can use an argument similar to the standard proof of Cauchy-Schwarz to show the following

Theorem 5. Let X and Y be any RVs, then

a. $-1 \leq \rho_{XY} \leq 1$,

b. $|\rho_{XY}| = 1$ if and only if there are constants $a \neq 0, b$ such that $\mathbb{P}(Y = aX + b) = 1$. If $|\rho_{XY}| = 1$ then $\mathrm{sign}(\rho) = \mathrm{sign}(a)$.

1.3 Inequalities

1.3.1 Numerical Inequalities

Theorem 6. Let a and b be any positive numbers and let $p, q > 1$ satisfy $1/p + 1/q = 1$, then

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab$$

with equality if and only if $a^p = b^q$.

Theorem 7 (Hölder's Inequality)**.** Let X and Y be any random variables and let $p, q > 1$ satisfy $1/p + 1/q = 1$, then

$$|\mathbb{E}[XY]| \leq \mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{1/p}\mathbb{E}[|Y|^q]^{1/q}$$

Corollary 1.

- Cauchy-Schwarz is the special case $p = q = 2$
- $\mathrm{Cov}[X, Y]^2 \leq \sigma_X^2 \sigma_Y^2$
- $\mathbb{E}[|X|] \leq \mathbb{E}[|X|^p]^{1/p}$
- *Liapounov's Inequality* $\mathbb{E}[|X|^r]^{1/r} \leq \mathbb{E}[|X|^s]^{1/s}$ where $1 < r < s < \infty$.

1.3.2 Functional Inequalities

Definition 3 (Convex Function)**.** A function $g(x)$ is *convex* on a set S if for all $x, y \in S$ and $0 < \lambda < 1$

$$g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y).$$

Strictly convex is when the inequality is strict. g is *concave* if $-g$ is convex.

Lemma 1. $g(x)$ is convex on S if $g''(x) \geq 0 \ \forall x \in S$.

Theorem 8 (Jensen's Inequality)**.** If $g(x)$ is convex, then for any random variable X

$$\mathbb{E}[g(X)] \leq g(\mathbb{E}[X]).$$

Equality holds if and only if, for every line $a + bx$ that is tangent to $g(x)$ at $x = \mathbb{E}[X]$, $\mathbb{P}(g(X) = a + bX) = 1$. (So if and only if g is affine with probability 1.)

Corollary 2.

- $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$
- $\mathbb{E}[1/X] \geq 1/\mathbb{E}[X]$