The Gerber statistic: a robust co-movement measure for portfolio optimization

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July 6, 2021

Abstract

The purpose of this paper is to introduce the Gerber statistic, a robust co-movement measure for covariance matrix estimation for the purpose of portfolio construction. The Gerber statistic extends Kendall's Tau by counting the proportion of simultaneous co-movements in series when their amplitudes exceed data-dependent thresholds. Since the statistic is neither affected by extremely large or extremely small movements, it is especially well-suited for financial time series, which often exhibit extreme movements as well as a great amount of noise. Operating within the mean-variance portfolio optimization framework of Markowitz (1952, 1959), we consider the performance of the Gerber statistic against two other commonly used methods for estimating the covariance matrix of stock returns: the sample covariance matrix (also called the historical covariance matrix) and shrinkage of the sample covariance matrix as formulated in Ledoit and Wolf (2004). Using a well-diversified portfolio of nine assets over a thirty year time period (January 1990-December 2020), we empirically find that, for almost all scenarios considered, the Gerber statistic's returns dominate those achieved by both historical covariance and by the shrinkage method of Ledoit and Wolf (2004).

Keywords: Gerber statistic; co-movement; robust covariance estimation; Markowitz mean-variance optimization; shrinkage method; historical covariance.

JEL Classification Codes: C13, C61, G11.

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1 Introduction

Portfolio construction (Markowitz, 1952, 1959) relies heavily on the availability of the matrix of covariances between securities' returns. Often the sample covariance matrix is used as an estimate for the actual covariance matrix (Jobson and Korkie (1980)). But as early as Sharpe (1963), various models have been used to ease the computational burden and to improve statistical properties of covariance matrix estimates. Nevertheless, a central problem still exists with many covariance matrix estimators: they employ product-moment-based estimates that inherently are not robust. This is particularly troublesome if the underlying distribution of returns is prone to containing extreme measurements or outliers. Robust estimators, based on the pioneering work of Tukey (1960), Huber and Ronchetti (2009) and Hampel (1968, 1974), have largely overcome this problem. Shevlyakov and Smirnov (2011) provide a thorough examination of modern robust methods for computing correlations.

However, financial time series have characteristics that make even standard robust techniques unsuitable. Financial time series are particularly noisy, and this noise can be easily misinterpreted as information. One consequence, for example, is that correlation matrix estimates (even ones built using robust techniques) often have non-zero entries corresponding to series that in fact have no meaningful correlation. On the other hand, the correlation estimates can also be distorted if the series contains extremely large (positive or negative) observations.

The key purpose of this paper is to introduce the Gerber statistic[1], a robust co-movement measure which ignores fluctuations below a certain threshold, while simultaneously limiting the effects of extreme movements. The Gerber statistic is designed to recognize co-movement between series when the movements are "substantial" (to be formally defined in Section 2) and to be insensitive to small co-movements that may be due to noise alone. The statistic is a generalization of Kendall's Tau (Kendall, 1938) which measures correlation between two groups of data as the ratio of the difference between the number of "concordant" and "discordant" pairs (to be formalized in Section 2) in the sets divided by the sum of the number of concordant and discordant pairs in the sets. The Gerber statistic generalizes Kendall's Tau by including thresholds such that only co-movements which exceed thresholds will be recognized as being either concordant or discordant.

We restrict our analysis in the present paper to the mean-variance optimization framework of Markowitz (1952, 1959). Within this paradigm, we compare the performance of the Gerber statistic with two commonly employed covariance matrix estimators of stock returns: (i) The sample covariance matrix (also referred to as the historical covariance matrix, or simply "historical covariance") and (ii) The shrinkage estimator of Ledoit and Wolf (2004), which shrinks the sample covariance matrix towards a structural estimator. It has been well documented that the sample covariance matrix is highly susceptible to outliers (Jobson and Korkie (1980)). A key conceptual advantage of the Gerber statistic is that, in

[1]This paper has had two previous incarnations, both of which originally appeared on SSRN but which were removed due to incompleteness. The first preprint on the Gerber Statistic (Gerber et al. (2015)) is available via www.stat.rice.edu/~pe6/Gerber2015.pdf. The second preprint on the Gerber Statistic (Gerber et al. (2019)) is available via www.stat.rice.edu/~pe6/Gerber2019.pdf. **The present paper constitutes the final version of the two aforementioned preprints**. All figures and results in this work are fully reproducible via the resources provided on the Github page https://github.com/yinsenm/gerber

contrast to the shrinkage estimator of Ledoit and Wolf (2004), it does not rely on the sample covariance matrix as input.

The remainder of this paper is organized as follows. Section 2 provides a rigorous formulation of the Gerber statistic. Section 3 introduces the dataset and the backtesting framework employed to compare the empirical performances of the methods of historical covariance, shrinkage estimation, and the Gerber statistic. Section 4 presents the results of the empirical analysis. Operating within Markowitz's mean-variance portfolio optimization framework, we find that the Gerber statistic's returns, in almost all scenarios considered, dominates those of both historical covariance and of shrinkage estimation. Section 5 discusses two lines of investigation for future study and Section 6 concludes the manuscript.

2 The Gerber statistic

The purpose of this section is to formulate the Gerber statistic and the corresponding Gerber correlation matrix, which is then converted to a Gerber covariance matrix that is inputted into the mean-variance portfolio optimizer (Section 3.3). We begin with the necessary notation for the Gerber statistic's formulation.

2.1 Notation

We consider $k = 1, \ldots, K$ securities and $t = 1, \ldots, T$ time periods. Let r_{tk} be the return of security k at time t. For each pair (i, j) of assets for each time t, we convert the return observation of pair (r_{ti}, r_{tj}) to a joint observation $m_{ij}(t)$ given by

$$m_{ij}(t) = \begin{cases} +1 \text{ if } r_{ti} \geq +H_i \text{ and } r_{tj} \geq +H_j, \\ +1 \text{ if } r_{ti} \leq -H_i \text{ and } r_{tj} \leq -H_j, \\ -1 \text{ if } r_{ti} \geq +H_i \text{ and } r_{tj} \leq -H_j, \\ -1 \text{ if } r_{ti} \leq -H_i \text{ and } r_{tj} \geq +H_j, \\ \;\;0 \text{ otherwise.} \end{cases} \tag{1}$$

In the above formulation, H_k is a threshold for security k that is calculated as

$$H_k = cs_k, \tag{2}$$

where c is some fraction (typically 1/2, but may also be increased to 7/10 or 9/10) and s_k is the sample standard deviation of the return of security k.[2] There are three key takeaways from equation (1):

- The joint observation $m_{ij}(t)$ is set to $+1$ if the series i and j simultaneously pierce their thresholds in the same direction at time t.
- The joint observation $m_{ij}(t)$ is set to -1 if the series i and j simultaneously pierce their thresholds in opposite directions at time t.

[2]More robust measures than standard deviation could of course be used for the threshold computation, but this is beyond the scope of the present work.

- The joint observation $m_{ij}(t)$ is set to 0 if at least one of the series does not pierce its threshold at time t.

We shall henceforth refer to a pair for which both components pierce their thresholds while moving in the same direction as a *concordant* pair, and to one whose components pierce their thresholds while moving in opposite directions as a *discordant* pair.

Given the above formulation, we define the Gerber statistic for a pair of assets to be

$$g_{ij} = \frac{\sum_{t=1}^{T} m_{ij}(t)}{\sum_{t=1}^{T} \left| m_{ij}(t) \right|}. \tag{3}$$

Letting n_{ij}^c be the number of concordant pairs for series i and j, and letting n_{ij}^d be the number of discordant pairs, equation (3) is immediately equivalent to

$$g_{ij} = \frac{n_{ij}^c - n_{ij}^d}{n_{ij}^c + n_{ij}^d}. \tag{4}$$

Note that the statistic in (4) is identical to Kendall's Tau if the threshold H_k is set to zero for all k.

Since the Gerber statistic in (4) relies on counts of the number of simultaneous piercings of thresholds, and not on the extent to which the thresholds are pierced, it is insensitive to extreme movements that distort product-moment-based measures. At the same time, since a series must exceed its threshold before it becomes a candidate to be "counted" (that is, it is given a value of m_{ij} that is either +1 or -1), the Gerber statistic is also insensitive to small movements that may simply be noise.

2.2 The Gerber matrix

We now turn to the formulation of the Gerber matrix $\boldsymbol{G}$, i.e., the matrix that contains the Gerber statistic g_{ij} in its i-th row and j-th column. Let us define $\boldsymbol{R} \in \mathbb{R}^{T \times K}$ to be the matrix of returns with entry r_{tk} in its t-th row and k-th column. Further, let $\boldsymbol{U}$ be an indicator matrix with the same size as $\boldsymbol{R}$ for returns exceeding the upper threshold, having entries u_{tj} such that

$$u_{tj} = \begin{cases} 1 & \text{if } r_{tj} \geq +H_j, \\ 0 & \text{otherwise.} \end{cases}$$

With this definition, the matrix of the number of samples that exceed the upper threshold is

$$\boldsymbol{N}^{\text{UU}} = \boldsymbol{U}^\top \boldsymbol{U}. \tag{5}$$

It is useful to note that the ijth element n_{ij}^{UU} of $\boldsymbol{N}^{\text{UU}}$ is the number of samples for which both time series i exceeds the upper threshold and for which time series j simultaneously exceeds the upper threshold.

Let $\boldsymbol{D}$ be an indicator matrix with the same size as $\boldsymbol{R}$ for returns falling below the lower threshold, having entries d_{tj} such that

$$d_{tj} = \begin{cases} 1 & \text{if } r_{tj} \leq -H_j, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix of the number of samples that go below the lower threshold may be written as

$$\boldsymbol{N}^{\text{DD}} = \boldsymbol{D}^\top \boldsymbol{D}. \tag{6}$$

Again, we have the useful property that the ijth element n_{ij}^{DD} of $\boldsymbol{N}^{\text{DD}}$ is the number of samples for which both time series i goes below the lower threshold and for which time series j simultaneously goes below the lower threshold.

With (5) and (6) in hand, the matrix containing the numbers of concordant pairs is

$$\boldsymbol{N}_{\text{CONC}} = \boldsymbol{N}^{\text{UU}} + \boldsymbol{N}^{\text{DD}} = \boldsymbol{U}^\top \boldsymbol{U} + \boldsymbol{D}^\top \boldsymbol{D}. \tag{7}$$

The matrix containing the numbers of discordant pairs is

$$\boldsymbol{N}_{\text{DISC}} = \boldsymbol{U}^\top \boldsymbol{D} + \boldsymbol{D}^\top \boldsymbol{U}. \tag{8}$$

We may now write the Gerber matrix $\boldsymbol{G}$ corresponding to the Gerber statistic written in (4) in the following matrix form

$$\boldsymbol{G} = (\boldsymbol{N}_{\text{CONC}} - \boldsymbol{N}_{\text{DISC}}) \oslash (\boldsymbol{N}_{\text{CONC}} + \boldsymbol{N}_{\text{DISC}}),$$

where the symbol $\oslash$ represents Hadamard (elementwise) division. The corresponding Gerber covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{GS}}$ is then correspondingly defined as

$$\boldsymbol{\Sigma}_{\boldsymbol{GS}} = \text{diag}(\boldsymbol{\sigma})\boldsymbol{G}\text{diag}(\boldsymbol{\sigma}), \tag{9}$$

where $\boldsymbol{\sigma}$ is a $N \times 1$ vector of sample standard deviation of the historical asset returns.

The covariance matrix of securities' returns must be positive semidefinite. However, when working with real data, we found that the covariance matrix in (9) was often not positive semidefinite. This led us to develop an alternative form of the Gerber statistic which gives rise to a positive semidefinite covariance matrix.

We begin our discussion of the modified form of the Gerber statistic by considering the following graphical representation for the relationship between two securities in Table 1 below. In this table, the letter U represents the case in which a security's return lies above the upper threshold (i.e., is up), the letter N represents the case in which a security's return lies between the upper and lower thresholds (i.e., is neutral), and the letter D represents the case in which a security's return lies below the lower threshold (i.e., is down). In Table 1, the rows represent categorizations of security i and the columns represent categorizations of security j. The boundaries between the rows and the columns are the chosen thresholds. For example, if at time t, the return of security i is above the upper threshold, this observation lies in the top row. If, at the same time t, the return of security j lies between the two thresholds, this observation lies in the middle column. Therefore, this specific observation would lie in the UN region.

Over the history, $t = 1, \ldots, T$, there will be observations scattered over the nine regions. Let n_{ij}^{pq} be the number of observations for which the returns of securities i and j lie in regions p and q, respectively, for $p, q \in \{U, N, D\}$. With this notation in hand, we can write an equivalent expression to the statistic presented in (4) as

$$g_{ij} = \frac{n_{ij}^{\text{UU}} + n_{ij}^{\text{DD}} - n_{ij}^{\text{UD}} - n_{ij}^{\text{DU}}}{n_{ij}^{\text{UU}} + n_{ij}^{\text{DD}} + n_{ij}^{\text{UD}} + n_{ij}^{\text{DU}}}. \tag{10}$$

Table 1: A graphical relationship between two securities.

$$\begin{array}{ccc} UD & UN & UU \\ ND & NN & NU \\ DD & DN & DU \end{array}$$

As previously noted, we must alter denominator in (4) to obtain a Gerber matrix which yields a corresponding covariance matrix in positive semidefinite form. Our alternative choice[3] motivated by the illustration of the Gerber statistic in Section 2.3 below, is

$$g_{ij} = \frac{n_{ij}^{\mathrm{UU}} + n_{ij}^{\mathrm{DD}} - n_{ij}^{\mathrm{UD}} - n_{ij}^{\mathrm{DU}}}{T - n_{ij}^{\mathrm{NN}}}. \tag{11}$$

In the empirical studies performed in Section 4, for all cases of Gerber thresholds c considered, we *always* observe the the covariance matrix corresponding to the Gerber statistic in (11) to be positive semidefinite.

We conclude this section with a more detailed discussion of the modified Gerber statistic in (11), which we shall henceforth simply refer to as the Gerber statistic. Some immediate properties of g_{ij} are as follows:

1. As with Pearson's correlation coefficient and Kendall's Tau coefficient, the value of the Gerber statistic g_{ij} must lie in the interval $[-1, 1]$.

2. By definition, the denominator in (11) is non-negative. The numerator is positive if the sum of n_{ij}^{UU} and n_{ij}^{DD} exceeds the sum of n_{ij}^{UD} and n_{ij}^{DU}, is zero if these two sums are equal, and is negative otherwise.

We now discuss a critical conceptual difference between the Gerber statistic and the standard Pearson correlation coefficient. The Pearson correlation coefficient inputs the sample covariance of assets i and j and the sample standard deviation of assets i and j (and therefore the sample means of assets i and j). By definition, the sample covariance, the sample mean, and the sample standard deviation are calculated over *all* data points, regardless of whether the points correspond to meaningful co-movement or to pure noise. This causes the Pearson correlation to be highly sensitive to small co-movements that may be due to

[3] Another alternative choice documented in the 2019 version of the present paper (Gerber et al. (2019)) but which is not employed in the present work is

$$g_{ij} = \frac{n_{ij}^{\mathrm{UU}} + n_{ij}^{\mathrm{DD}} - n_{ij}^{\mathrm{UD}} - n_{ij}^{\mathrm{DU}}}{\sqrt{n_{ij}^{(\mathrm{A})} n_{ij}^{(\mathrm{B})}}},$$

where

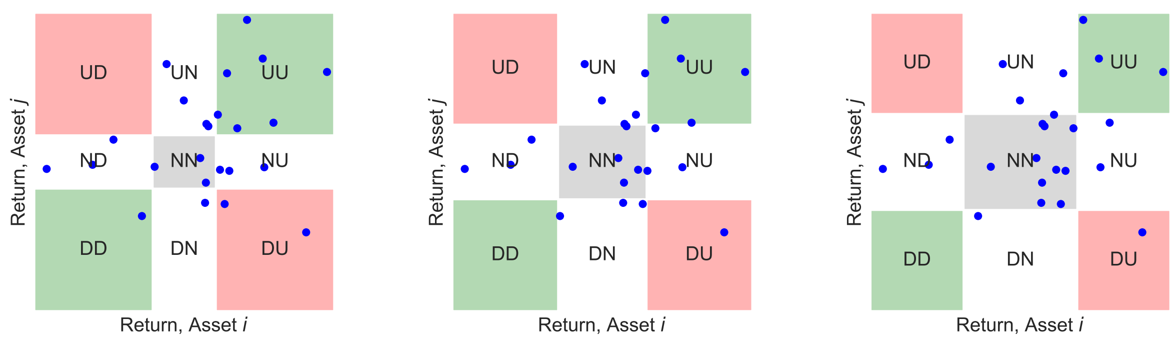
$$\begin{aligned} n_{ij}^{(A)} &= n_{ij}^{\mathrm{UU}} + n_{ij}^{\mathrm{UN}} + n_{ij}^{\mathrm{UD}} + n_{ij}^{\mathrm{DU}} + n_{ij}^{\mathrm{DN}} + n_{ij}^{\mathrm{DD}} \\ n_{ij}^{(B)} &= n_{ij}^{\mathrm{UU}} + n_{ij}^{\mathrm{NU}} + n_{ij}^{\mathrm{UD}} + n_{ij}^{\mathrm{UD}} + n_{ij}^{\mathrm{ND}} + n_{ij}^{\mathrm{DD}}. \end{aligned}$$

The corresponding covariance matrix is positive semidefinite.

noise alone. In contrast, the Gerber statistic only includes in its numerator the *subset* of the dataset containing the points which correspond to meaningful co-movement (equivalently, the Gerber statistic strips out noisy data). This is the key reason that the Gerber statistic is a more robust co-movement measure than the standard Pearson correlation. Furthermore, it is important to note that the Gerber statistic's formulation need not require any estimates of moments. Indeed, we could achieve an entirely moment-free framework for the Gerber statistic by replacing s_k in equation (2) with a more robust measure of standard deviation (as we have noted in footnote 2). We shall explore candidates for this measure in future work.

2.3 Illustration of the Gerber statistic

We now discuss Figure 1 to illustrate how the Gerber statistic in (11) is calculated between a given pair of assets. In Figure 1, we compute 24 pairwise monthly returns between the assets S&P 500 (SPX) and Gold (XAU) for the period from January 2019 to December 2020. Recalling that the threshold of H_k as defined in (2) may be altered, we consider three different values of c: $c = .5$, $c = .7$, and $c = .9$.



(a) $c = 0.5$ (b) $c = 0.7$ (c) $c = 0.9$

Figure 1: Illustration of pairwise returns for evaluating the Gerber statistics given $c = 0.5$, $c = 0.7$ and $c = 0.9$. Each pairwise monthly return appears as a blue dot. The points in the green zone correspond to the concordant pairs, whereas the points falling in the red zone are discordant pairs. The return series of assets i and j are transformed to $\{-1, 0, 1\}$ using upper and lower thresholds calculated by equation (2).

The key intuition for our choice of the Gerber statistic's denominator in (11) comes from the following observation: as c becomes larger, more data points are included in the region NN. The denominator in (11) subtracts out these points, and this leads to the statistic becoming more robust and less sensitive to noise in the data. We refer to this artifact of the Gerber statistic as "stripping noise" from the data.

We now calculate the Gerber statistic by counting the points falling into each region. The results for the three cases $c = .5$, $c = .7$, and $c = .9$ are given below. All result in Gerber statistics that differ from the standard Pearson correlation coefficient of 0.22.

(1) In the case $c = 0.5$ (Figure 1(a)), the counts for nine regions are $n_{ij}^{UD} = 0$, $n_{ij}^{UN} = 4$, $n_{ij}^{UU} = 7$, $n_{ij}^{ND} = 3$, $n_{ij}^{NN} = 3$, $n_{ij}^{NU} = 3$, $n_{ij}^{DD} = 1$, $n_{ij}^{DN} = 1$ and $n_{ij}^{DU} = 2$. Employing the definition of the Gerber statistic in (11), we obtain that

$$g_{ij} = \frac{7 + 1 - 0 - 2}{24 - 3} = \frac{2}{7} \approx 0.286.$$

(2) In the case $c = 0.7$ (Figure 1(b)), the counts for nine regions are $n_{ij}^{UD} = 0$, $n_{ij}^{UN} = 5$, $n_{ij}^{UU} = 4$, $n_{ij}^{ND} = 3$, $n_{ij}^{NN} = 6$, $n_{ij}^{NU} = 2$, $n_{ij}^{DD} = 0$, $n_{ij}^{DN} = 3$ and $n_{ij}^{DU} = 1$. Employing the definition of the Gerber statistic in (11), we obtain that

$$g_{ij} = \frac{4 + 0 - 0 - 1}{24 - 6} = \frac{1}{6} \approx 0.166.$$

(3) In the case $c = 0.9$ (Figure 1(c)), the counts for nine regions are $n_{ij}^{UD} = 0$, $n_{ij}^{UN} = 3$, $n_{ij}^{UU} = 3$, $n_{ij}^{ND} = 3$, $n_{ij}^{NN} = 11$, $n_{ij}^{NU} = 2$, $n_{ij}^{DD} = 0$, $n_{ij}^{DN} = 1$ and $n_{ij}^{DU} = 1$. Employing the definition of the Gerber statistic in (11), we obtain that

$$g_{ij} = \frac{3 + 0 - 0 - 1}{24 - 11} = \frac{2}{13} \approx 0.154.$$

3 Empirical study

In this section, we commence the empirical study of the performance of the Gerber statistic in comparison to two commonly used methods of covariance estimation for the purpose of portfolio construction: historical covariance and the shrinkage estimator of Ledoit and Wolf (2004).

3.1 Dataset

The dataset we consider is a well diversified collection of nine assets over the time period January 1988 to December 2020, as follows:

1. S&P 500 index (U.S. large-cap stocks; Ticker SPX)
2. Russell 2000 index (U.S. small-cap stocks; Ticker RTY)
3. MSCI EAFE index (captures large and mid cap equities across twenty-one developed countries excluding U.S. and Canada; Ticker MXEA)
4. MSCI Emerging Markets index (captures large and mid cap equities across twenty-seven emerging markets; Ticker MXEF)
5. Bloomberg Barclays U.S. Aggregate Bond index (includes Treasuries, government-related and corporate securities; Ticker LBUSTRUU)
6. Bloomberg Barclays U.S. Corporate High Yield Bond index; Ticker LF98TRUU

7. Real estate FTSE NAREIT all equity REITS index; Ticker FNERTR

8. Gold; Ticker XAU

9. S&P GSCI Goldman Sachs Commodity index; Ticker SPGSCI

The monthly total returns (TR) for the above nine assets for the period from January 1988 to December 2020 were obtained from the Bloomberg terminal. Each asset contains 396 observations over this time period. The TR indexes provided by Bloomberg track capital gains and account for cash distributions such as dividends or interest through asset reinvesting. As we shall detail in Section 3.3.1, our backtesting procedure for mean-variance optimization requires two years of monthly returns to initialize the first portfolio. The descriptive statistics for the monthly total returns of the nine asset portfolio in Table 2 below are therefore calculated from the period January 1990 to December 2020 rather than from the period January 1988 to December 2020.

Table 2: Descriptive statistics for the nine asset portfolio. The descriptive statistics are computed using monthly data from January 1990 to December 2020. The TR denotes the total return data, which accounts for asset appreciation from both capital gains and reinvesting of dividends.

Index	Arithmetic Return (%)	Geometric Return (%)	Annualized SD (%)
S&P 500 TR	11.90	10.47	14.63
Russel 2000 TR	11.74	10.14	19.36
MSCI EAFE TR	6.60	4.77	16.89
MSCI Emerging Market TR	12.13	7.61	22.31
U.S. Agg Bond TR	6.11	6.00	3.57
U.S. Corp High Yield Bond TR	9.35	8.36	8.74
REITS TR	11.97	10.37	18.32
Gold TR	6.04	5.03	15.26
S&P GSCI TR	5.27	2.36	21.48

Since the nine assets above are well-diversified, we do not expect to observe a strong pairwise correlation structure. This expectation is confirmed by Figure 2 below, which displays a correlation matrix of the total return series from January 1990 to December 2020 for the nine assets.

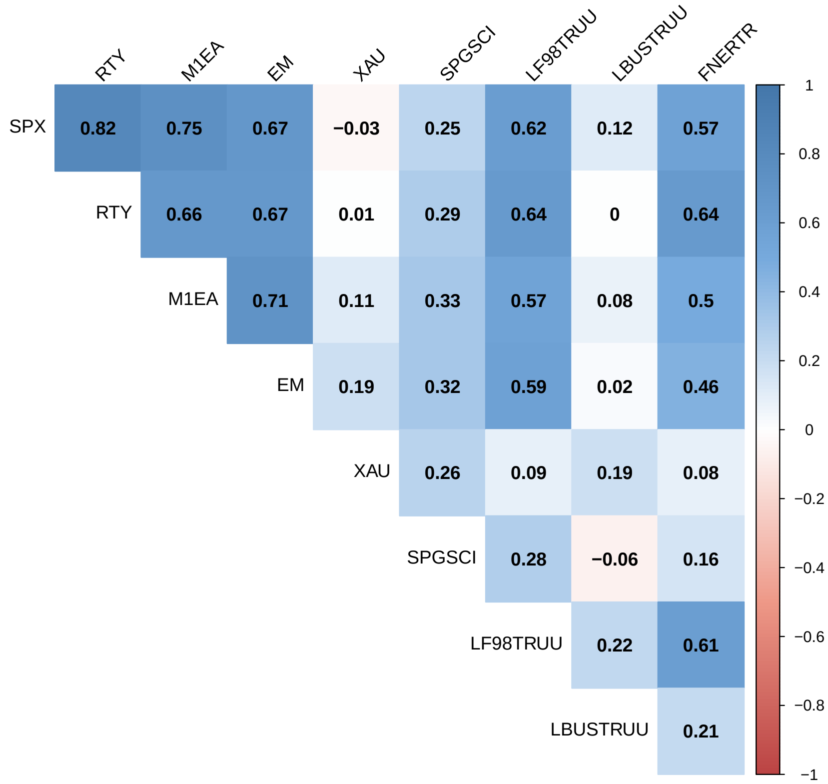


Figure 2: Heat map of the correlation matrix (given the total return series from January 1990 to December 2020) among the nine assets.

3.2 Competing methods

As previously discussed, the two competing methods to the Gerber statistic we shall consider are the historical covariance matrix and the shrinkage method of Ledoit and Wolf (2004). The historical covariance matrix (HC) is computed from the sample correlation matrix calculated via the standard Pearson correlation (Jobson and Korkie (1980)). We now briefly review the shrinkage estimator introduced by Ledoit and Wolf (2004).

Ledoit and Wolf (2004) propose a convex combination of a structure covariance matrix $\mathbf{\Sigma}_F$ and the sample historical covariance matrix $\mathbf{\Sigma}_{\mathrm{HC}}$ to obtain the shrunk matrix $\mathbf{\Sigma}_{\mathrm{SM}}$

$$\mathbf{\Sigma}_{\mathrm{SM}} = \delta\mathbf{\Sigma}_F + (1-\delta)\,\mathbf{\Sigma}_{\mathrm{HC}}, \tag{12}$$

where δ is a shrinkage constant between 0 and 1. We shall henceforth refer to this technique as the “shrinkage method” (SM). The namesake comes from the fact that the sample covariance matrix is “shrunk” towards a targeted structured estimator. In computing $\mathbf{\Sigma}_F$, Ledoit and Wolf (2004) suggest a constant correlation model, i.e. the average sample correlation of all pairs for the nondiagonal elements of the sample correlation matrix (see Appendix A of Ledoit and Wolf (2004)). They proceed to construct the corresponding covariance matrix.

In specifying a choice of δ, Ledoit and Wolf (2004) propose finding the shrinkage parameter by minimizing the Frobenius norm between the asymptotically true covariance matrix $\mathbf{\Sigma}$ and the shrinkage estimator $\mathbf{\Sigma}_{\text{SM}}$.

We conclude this section by highlighting some critical conceptual differences between the Gerber statistic and the shrinkage method of Ledoit and Wolf (2004). We see the key differences as follows:

1. As can be seen from equation (12) above, the shrinkage method (SM) directly inputs the sample covariance matrix $\mathbf{\Sigma}_{\text{HC}}$. The Gerber statistic, as given in (11) above, *does not rely* upon the sample covariance matrix as input. The Gerber statistic computes concordant and discordant pair counts, thereby extending Kendall's Tau for portfolio management.

2. The Gerber statistic's framework for considering concordant and discordant pairs of assets on one dataset (in the present paper, the dataset of historical returns) can be naturally extended to working with *multiple datasets*. For example, suppose a portfolio manager wished to consider three datasets simultaneously: historical returns, trading volume, and implied volatility. Further, suppose that this portfolio manager wished to deem two assets A and B to be concordant if the return for both assets is higher than $x\%$ *at the same time* that the trading volume increases by more than $y\%$ and the implied volatility increases by more than $z\%$, where x, y, and z are any (finite) real numbers greater than zero. The Gerber statistic provides a natural framework for designing such a rule (and more sophisticated rule-based systems) to construct Table 1 above and to compute the corresponding Gerber statistics. In contrast, moment-based methods such as shrinkage and historical covariance do not provide the same natural framework for considering such rule-based systems for determining co-movement.

As for similarities between the Gerber statistic and the shrinkage method of Ledoit and Wolf (2004), it should be noted that the shrinkage constant δ controls the degree to which the sample covariance matrix is shrunk. There is an analog for the Gerber statistic, and it is the value c (as given in equation (2)). The value of c determines the magnitude of the threshold H_k. As illustrated in Figure 1 in Section 2.3, the value of c may be increased from .5 to .7 (or .9) to strip out more noise from the dataset.

3.3 Optimization procedure

The portfolio optimization framework we shall consider is that of mean-variance optimization (MVO) (Markowitz, 1952). We briefly review the key steps and necessary notation in order to rigorously formulate the backtesting procedure to be given in Section 3.3.1.

The mean-variance optimization framework seeks to find optimal asset allocations (or portfolio weights ω_i) under certain risk and return restrictions given that future asset characteristics (i.e., the expected return μ_i, variance σ_{ii}^2 and covariance σ_{ij}) are known for each asset $i \in \{1, \cdots, K\}$. The expected return on a portfolio and its variance can then be derived as $\mu_P = \boldsymbol{\omega}^T \boldsymbol{\mu}$ and $\sigma_P^2 = \boldsymbol{w}^T \mathbf{\Sigma} \boldsymbol{w}$, where $\boldsymbol{w} = [\omega_1, \ldots, \omega_K]$ is a vector of portfolio weights for the universe of K assets, $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_K]$ is a vector of expected returns, and $\mathbf{\Sigma}$ is a covariance matrix of asset returns.

The long-only MVO representation with transaction costs is formulated as the following optimization problem

$$\begin{aligned} \text{Maximize: } & \boldsymbol{w}^T\boldsymbol{\mu} - \psi\mathbf{1}^T\left|\boldsymbol{w}-\boldsymbol{w}_0\right| \\ \text{Subject to: } & \boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w} \leq \sigma^2_{\text{target}} \\ & \boldsymbol{w}^T\mathbf{1} = \mathbf{0} \\ & 0 \leq w_k \leq 1,\ \forall k = \{1, 2, \cdots, N\}\,. \end{aligned} \tag{13}$$

The above portfolio optimization problem maximizes the expected return of a portfolio $\boldsymbol{\omega}^T\boldsymbol{\mu}$ discounted by a transaction cost or $\mathbf{1}^T\left|\boldsymbol{\omega}-\boldsymbol{\omega}_0\right|$. This transaction term tracks the proportional cost of portfolio weight changes, i.e. the absolute deviation of new weight vector $\boldsymbol{\omega}$ from the previous weight vector $\boldsymbol{\omega}_0$. The transaction penalty helps to regulate the turnover of the portfolio and reduces the associated trading costs. The symbol ψ is a fixed proportional transaction cost and is chosen to be *10 basis points* or 0.1% in the present paper. The portfolio weight vector $\boldsymbol{\omega}$ is subject to the standard constraints as given in Markowitz (1952), with no shorting permitted. One may then obtain an optimal portfolio weight $\boldsymbol{\omega}^*$ for an investor with risk constraint σ_{target} and turnover penalty ψ by solving the deterministic optimization problem as specified in (13). Given a turnover constraint ψ, the set of optimal solutions of $\boldsymbol{\omega} \in [0,1]^N$ for each risk level $\sigma_{target} \in \mathbb{R}^+$ constitute the efficient frontier. Each point on the frontier determines an efficient portfolio that yields the highest possible return given a prespecified level of risk.

What is now left is to provide estimates for $\boldsymbol{\mu}$, the vector of expected asset returns, and for $\boldsymbol{\Sigma}$, the covariance matrix of asset returns. The expected return at time t for asset i or μ_{ti} is estimated using the sample means of its historical returns given a looback window of T-months or $\mu_{ti} = \frac{1}{T}\sum_{d=t-1}^{t-T} r_{di}$. For the covariance matrix $\boldsymbol{\Sigma}$, we benchmark and compare returns among the following three competing methods:

- Historical covariance (HC);
- The shrinkage method (SM) of Ledoit and Wolf (2004)
- The Gerber statistic (GS);

The corresponding covariance matrices are denoted, respectively, by $\boldsymbol{\Sigma}_{HC}$, $\boldsymbol{\Sigma}_{SM}$ and $\boldsymbol{\Sigma}_{GS}$. We pause to recall from Section 2.2 that $\boldsymbol{\Sigma}_{GS} = \text{diag}(\boldsymbol{\sigma})\boldsymbol{G}\text{diag}(\boldsymbol{\sigma})$, where $\boldsymbol{G}$ is the Gerber matrix obtained from the Gerber statistic in (11) and $\boldsymbol{\sigma}$ is a $N \times 1$ vector of sample standard deviation of the historical asset returns.

3.3.1 Portfolio backtesting procedure

We employ the following backtesting procedure to benchmark performance among different covariance estimators under the context of portfolio optimization. From January 1990, at the beginning of each month, the monthly returns of the current list of assets from a lookback window of $T = 24$ months are utilized to estimate the expected return vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$. Thereafter, the quadratic optimizer is applied to solve for an optimal portfolio weight vector $\boldsymbol{\omega}^*$ given a risk target σ_{target}. We then rebalance our previous portfolio according to the optimal weight vector $\boldsymbol{\omega}^*$ and hold this optimized portfolio for one month. At the end of the month, the realized portfolio return is computed via $\boldsymbol{w}^{*T}\tilde{\boldsymbol{r}}$, where $\tilde{\boldsymbol{r}}$ is the vector of realized assets return for this month. In other words, our portfolios are rebalanced on a *monthly basis*. We repeat this process by moving the in-sample period one month forward and computing the updated efficient portfolio for the next month. This rolling-window investing procedure offers the advantage of being more adaptive to market structural changes and also helps to ameliorate data mining bias. Because two years worth of monthly returns are required to initialize the first portfolio, our performance evaluation ranges from the period January 1990 to December 2020.

4 Empirical results

We now arrive at the study's key empirical results. Working with dataset introduced by Section 3.1 and the transaction costs considered in Section 3.3, and employing the backtesting algorithm in Section 3.3.1, we consider the performance of the Gerber statistic (GS) in comparison to historical covariance (HC) and to the shrinkage method (SM) of Ledoit and Wolf (2004) for three different values of the threshold c given given in equation (2): $c = .5$, $c = .7$, and $c = .9$.

4.1 Gerber Statistic with $c = .5$

We first study the Gerber Statistic (GS) with a threshold value of $c = .5$. We report four key findings:

1. For all risk target levels, the Gerber statistic offers a more favorable risk-return profile than HC. With the exception of a the ultra-conservative risk target level of 3%, the Gerber statistic offers a more favorable risk-return profile than SM (see Figure 3 below).

2. For all risk targets, the Gerber statistic yields higher cumulative returns than HC. With the exception of a very conservative risk target level of 3%, the Gerber statistic yields higher cumulative returns than SM (see Figure 4 and Table 3 below).

3. With similar values of portfolio turnover, skewness, and kurtosis as both the HC and SM portfolios, the Gerber statistic enjoys higher geometric returns and higher Sharpe ratios than HC across all risk target levels (see Table 6 at the end of this section). With the exception of the very conservative risk target level of 3%, (see Table 6 at the end of this section), the Gerber statistic enjoys higher geometric returns and higher Sharpe ratios than SM across all other risk target levels.

4. For some levels of risk target, the average annualized geometric return of GS is more than 30 basis points higher than that of SM and more than 75 basis points higher than HC. The latter result is unsurprising given the limitations of HC (Jobson and Korkie (1980); Ledoit and Wolf (2004)), and so we instead focus on the advantages of GS over SM. For the 9% risk target level, the average annualized geometric return of GS is approximately 32 basis points higher than that of SM and its cumulative return is 10.16% higher than that for SM over the 1990-2020 period (see Table 6 at the end of

this section). For the 15% risk target level, the average annualized geometric return of GS is approximately 32 basis points higher than that for SM and its cumulative return is 10.32% higher than SM over the 1990-2020 period (see Table 6 at the end of this section).

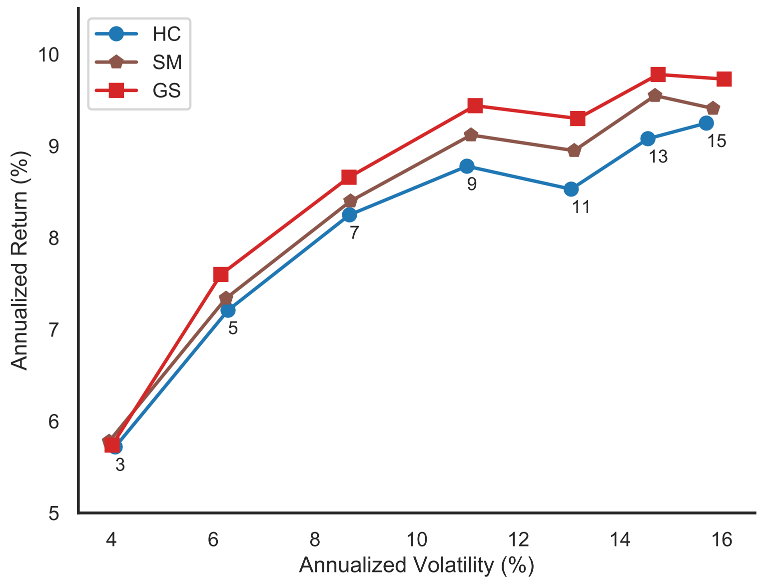


Figure 3: The realized performance in terms of annualized return and annualized volatility of portfolios (the ex-post efficient frontiers) with different risk target levels from 3% to 15%, with an increment of 2%, given the Gerber threshold $c = 0.5$. The blue frontier illustrates the ex-post performance of HC-based portfolios, the brown frontier presents the ex-post performance of SM-based portfolios, and the red frontier corresponds to the ex-post performance of the GS-based portfolios.

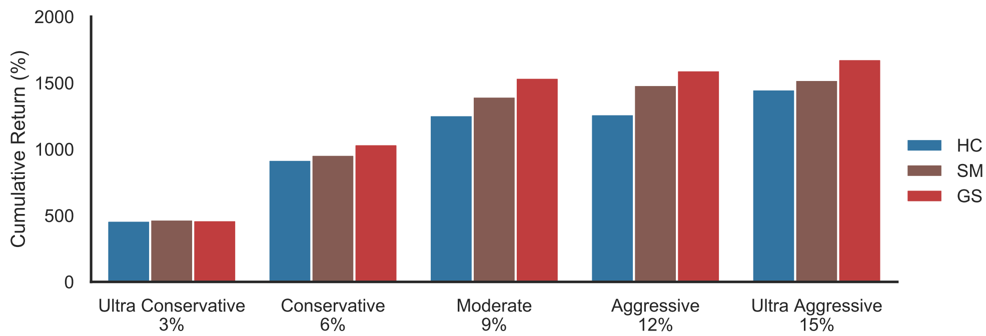


Figure 4: The cumulative returns in percentage (from 1990 to 2020) for HC-based portfolios, SM-based portfolios, and GS-based portfolios at five different annual risk target levels i.e. 3%, 6%, 9%, 12% and 15%, given the Gerber threshold $c = 0.5$. The calculation assumes that \$100K is invested in January 1990 and is left to grow according to portfolio weights determined by each covariance method and risk target level until December 2020.

Table 3: The account dollar value in December 2020 for HC-based portfolios, SM-based portfolios, and GS-based portfolios at five different annual risk target levels i.e. 3%, 6%, 9%, 12% and 15%, given the Gerber threshold $c = 0.5$. The calculation assumes that \$100K is invested in January 1990 and is left to grow according to portfolio weights determined by each covariance method and risk target level until December 2020.

Method	HC	SM	GS
Ultra Conservative (3%)	\$561,276.27	\$570,161.82	\$564,972.97
Conservative (6%)	\$1,020,099.74	\$1,058,096.16	\$1,138,042.25
Moderate (9%)	\$1,356,911.18	\$1,497,204.43	\$1,639,089.77
Aggressive (12%)	\$1,364,148.39	\$1,584,491.30	\$1,695,255.50
Ultra Aggressive (15%)	\$1,551,338.93	\$1,622,590.70	\$1,779,756.04

4.2 Gerber Statistic with $c = .7$

We proceed to study the Gerber Statistic (GS) with a value of $c = .7$. We report four key findings:

1. **For all risk target levels**, the Gerber statistic offers a more favorable risk-return profile than **both** HC and SM (see Figure 5 below).

2. **For all risk target levels**, the Gerber statistic offers superior cumulative returns to **both** HC and SM (see Figure 6 and Table 4 below).

3. **For all risk target levels**, the Gerber statistic enjoys higher geometric returns and Sharpe ratios to **both** HC and SM , and has similar values of portfolio turnover, skewness, and kurtosis to HC and SM (see Table 6 and Table 7 at the end of this section).

4. For some levels of risk target, the average annualized geometric return of GS is more than 40 basis points higher than that of SM and more than 90 basis points higher than HC. The latter is unsurprising given the limitations of HC (Jobson and Korkie (1980); Ledoit and Wolf (2004)), and so we instead focus on the advantages of GS over SM. For the 12% risk target level, the average annualized geometric return of GS is approximately 41 basis points higher than that of SM and its cumulative return is 13.12% higher than that for SM over the 1990-2020 period (see Table 6 and Table 7 at the end of this section). For the 15% risk target level, the average annualized geometric return of GS is approximately 35 basis points higher than that for SM and its cumulative return is 11.18% higher than than SM over the 1990-2020 period (see Table 6 and Table 7 at the end of this section).

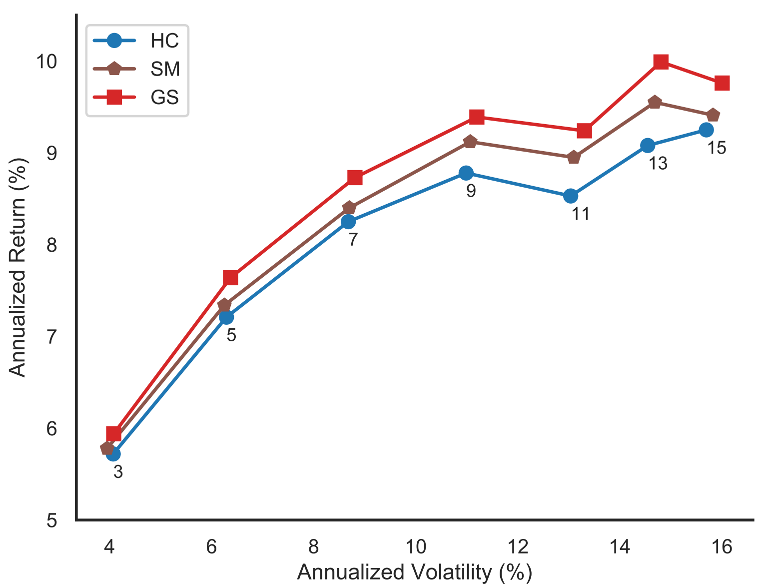


Figure 5: The realized performance in terms of annualized return and annualized volatility of portfolios (the ex-post efficient frontiers) with different risk target levels from 3% to 15%, with an increment of 2%, given the Gerber threshold $c = 0.7$. The blue frontier illustrates the ex-post performance of HC-based portfolios, the brown frontier presents the ex-post performance of SM-based portfolios, and the red frontier corresponds to the ex-post performance of the GS-based portfolios.

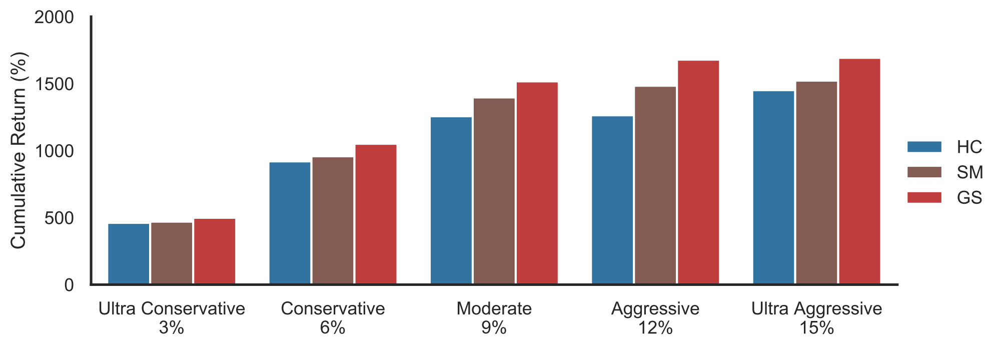


Figure 6: The cumulative returns in percentage (from 1990 to 2020) for HC-based portfolios, SM-based portfolios, and GS-based portfolios at five different annual risk target levels i.e. 3%, 6%, 9%, 12% and 15%, given the Gerber threshold $c = 0.7$. The calculation assumes that \$100K is invested in January 1990 and is left to grow according to portfolio weights determined by each covariance method and risk target level until December 2020.

Table 4: The account dollar value in December 2020 for HC-based portfolios, SM-based portfolios, and GS-based portfolios under five annual risk target levels i.e. 3%, 6%, 9%, 12% and 15%, Gerber threshold $c = 0.7$. The calculation assumes that \$100K is invested on January 1990 and is left to grow until December 2020.

Method	HC	SM	GS
Ultra Conservative (3%)	\$561,276.27	\$570,161.82	\$599,043.19
Conservative (6%)	\$1,020,099.74	\$1,058,096.16	\$1,151,575.55
Moderate (9%)	\$1,356,911.18	\$1,497,204.43	\$1,617,126.27
Aggressive (12%)	\$1,364,148.39	\$1,584,491.30	\$1,779,267.42
Ultra Aggressive (15%)	\$1,551,338.93	\$1,622,590.70	\$1,792,846.87

4.3 Gerber Statistic with $c = .9$

We proceed to study the Gerber Statistic (GS) with a value of $c = .9$. We report four key findings:

1. **For all risk target levels**, the Gerber statistic offers a more favorable risk-return profile than **both** HC and SM (see Figure 7 below).

2. **For all risk target levels**, the Gerber statistic offers superior cumulative returns to **both** HC and SM (see Figure 8 and Table 5 below).

3. **For all risk target levels**, the Gerber statistic enjoys higher geometric returns and Sharpe ratios to **both** SM and HC, and has similar values of portfolio turnover, skewness, and kurtosis to HC and SM (see Table 6 and Table 7 at the end of this section).

4. For the 3% and 6% risk target levels, the average annualized geometric return of GS is approximately 32 and 35 basis points higher than those of SM, respectively. The cumulative returns are, respectively, 12.11% and 11.67% higher than those for SM over the 1990-2020 period (see Table 6 and Table 7 at the end of this section). We also note that for the 6% risk target level, the average annualized geometric return of GS is more than 48 basis points higher than that of HC.

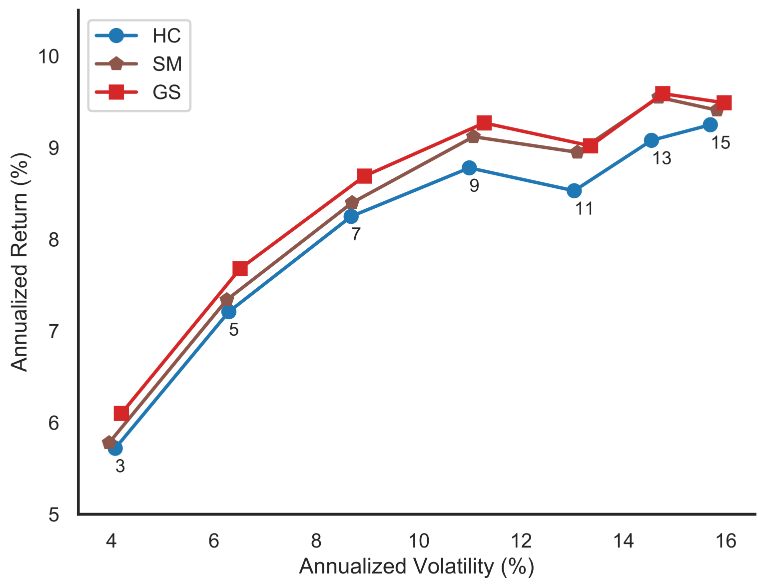


Figure 7: The realized performance in terms of annualized return and annualized volatility of portfolios (the ex-post efficient frontiers) with different risk target levels from 3% to 15%, with an increment of 2%, given the Gerber threshold $c = 0.9$. The blue frontier illustrates the ex-post performance of HC-based portfolios, the brown frontier presents the ex-post performance of SM-based portfolios, and the red frontier corresponds to the ex-post performance of the GS-based portfolios.

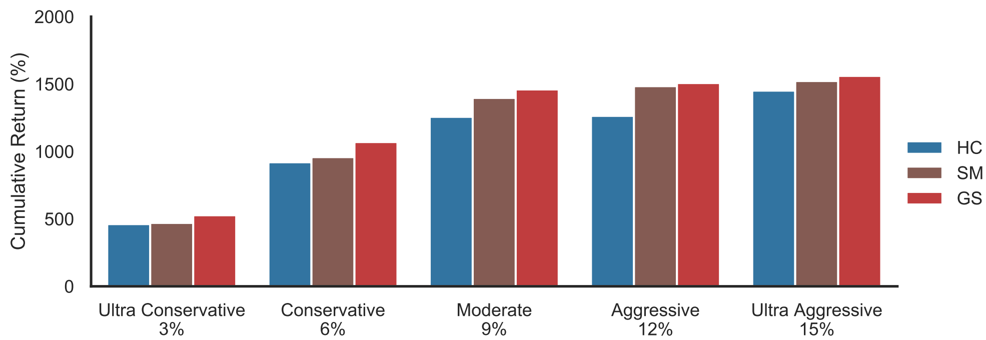


Figure 8: The cumulative returns in percentage (from 1990 to 2020) for HC-based portfolios, SM-based portfolios, and GS-based portfolios at five different annual risk target levels i.e. 3%, 6%, 9%, 12% and 15%, given the Gerber threshold $c = 0.9$. The calculation assumes that \$100K is invested in January 1990 and is left to grow according to portfolio weights determined by each covariance method and risk target level until December 2020.

Table 5: The account dollar value in December 2020 for HC-based portfolios, SM-based portfolios, and GS-based portfolios at five different annual risk target levels i.e. 3%, 6%, 9%, 12% and 15%, given the Gerber threshold $c = 0.9$. The calculation assumes that \$100K is invested in January 1990 and is left to grow according to portfolio weights determined by each covariance method and risk target level until December 2020.

Method	HC	SM	GS
Ultra Conservative (3%)	\$561,276.27	\$570,161.82	\$627,077.52
Conservative (6%)	\$1,020,099.74	\$1,058,096.16	\$1,169,911.24
Moderate (9%)	\$1,356,911.18	\$1,497,204.43	\$1,560,198.02
Aggressive (12%)	\$1,364,148.39	\$1,584,491.30	\$1,607,572.21
Ultra Aggressive (15%)	\$1,551,338.93	\$1,622,590.70	\$1,660,628.94

Table 6: This table reports the performance metrics for HC-, SM-, and GS-based portfolios at five different risk target levels, i.e. 3%, 6%, 9%, 12% and 15% for the full testing period between January 1990 and December 2020. The 3-month U.S. T-Bill rate was used as the risk free rate. The transaction cost is modeled as **10bps** of the traded volume for each rebalancing event.

	Ultra-Conservative (3%)			Conservative (6%)			Moderate (9%)			Aggressive (12%)			Ultra-aggressive (15%)		
Covariance Method	HC	SM	GS	HC	SM	GS	HC	SM	GS	HC	SM	GS	HC	SM	GS
Arithmetic Return (%)	5.83	5.90	5.85	8.10	8.22	8.46	9.41	9.77	10.09	9.69	10.32	10.50	10.54	10.78	11.17
Geometric Return (%)	5.72	5.78	5.74	7.78	7.91	8.16	8.78	9.12	9.44	8.79	9.32	9.56	9.25	9.41	9.73
Cumulative Return (%)	461.28	470.16	464.97	920.10	958.10	1038.04	1256.91	1397.20	1539.09	1264.15	1484.49	1595.26	1451.34	1522.59	1679.76
Annualized SD (%)	4.07	3.95	4.01	7.49	7.50	7.43	10.99	11.07	11.15	13.82	13.92	13.98	15.70	15.83	16.05
Annualized Skewness	-0.90	-0.90	-0.88	-0.93	-0.98	-0.99	-0.85	-0.88	-0.93	-0.93	-0.92	-0.93	-0.86	-0.86	-0.87
Annualized Kurtosis	5.21	5.37	5.31	5.53	5.85	6.07	4.97	5.11	5.60	5.27	5.30	5.56	5.34	5.42	5.55
Maximum Drawdown (%)	-10.03	-7.67	-7.73	-20.30	-19.10	-17.74	-28.83	-28.03	-26.58	-36.47	-35.22	-32.47	-41.76	-41.13	-43.01
Monthly 95% VaR (%)	-1.53	-1.52	-1.58	-2.85	-2.69	-2.68	-4.35	-4.29	-4.29	-5.99	-5.81	-5.84	-6.70	-6.59	-6.42
Sharpe Ratio	0.73	0.77	0.75	0.67	0.69	0.73	0.55	0.58	0.60	0.44	0.47	0.49	0.42	0.42	0.44
Annualized Turnover	1.80	1.39	1.58	2.78	2.53	2.49	3.69	3.40	3.37	4.45	4.27	4.19	4.46	4.33	4.23

Table 7: This table reports the performance sensitivity study of the GS-based portfolios for thresholds $c = 0.5$, $c = 0.7$ and $c = 0.9$ at five different risk target levels, i.e. 3%, 6%, 9%, 12% and 15%.

	Ultra-Conservative (3%)			Conservative (6%)			Moderate (9%)			Aggressive (12%)			Ultra-aggressive (15%)		
GS Threshold c	0.50	0.70	0.90	0.50	0.70	0.90	0.50	0.70	0.90	0.50	0.70	0.90	0.50	0.70	0.90
Arithmetic Return (%)	5.85	6.06	6.22	8.46	8.52	8.59	10.09	10.05	9.94	10.50	10.68	10.34	11.17	11.17	10.90
Geometric Return (%)	5.74	5.94	6.10	8.16	8.20	8.26	9.44	9.39	9.27	9.56	9.73	9.37	9.73	9.76	9.49
Cumulative Return (%)	464.97	499.04	527.08	1038.04	1051.58	1069.91	1539.09	1517.13	1460.20	1595.26	1679.27	1507.57	1679.76	1692.85	1560.63
Annualized SD (%)	4.01	4.08	4.19	7.43	7.62	7.76	11.15	11.20	11.28	13.98	14.02	14.15	16.05	16.01	15.97
Annualized Skewness	-0.88	-1.06	-1.04	-0.99	-1.04	-0.98	-0.93	-0.90	-0.87	-0.93	-0.94	-0.98	-0.87	-0.89	-0.86
Annualized Kurtosis	5.31	6.26	6.13	6.07	6.30	5.84	5.60	5.42	5.12	5.56	5.58	5.66	5.55	5.59	5.38
Maximum Drawdown (%)	-7.73	-9.11	-9.67	-17.74	-19.08	-20.03	-26.58	-27.75	-28.12	-32.47	-33.94	-34.01	-43.01	-43.12	-40.94
Monthly 95% VaR (%)	-1.58	-1.50	-1.55	-2.68	-2.79	-2.82	-4.29	-4.37	-4.40	-5.84	-5.78	-5.71	-6.42	-6.49	-6.53
Sharpe Ratio	0.75	0.78	0.80	0.73	0.72	0.71	0.60	0.60	0.58	0.49	0.50	0.47	0.44	0.44	0.42
Annualized Turnover	1.58	1.49	1.51	2.49	2.55	2.58	3.37	3.43	3.44	4.19	4.19	4.26	4.23	4.21	4.32

5 Future work

We summarize the two key avenues for future work that have been mentioned in the present paper. Firstly, as discussed in Section 2.2, the Gerber statistic's numerator only includes co-movements that are substantial, thereby excluding small co-movements that may be due to noise alone. The Pearson correlation coefficient inputs the sample covariance of assets i and j and the sample standard deviation of assets i and j (and therefore the sample means of assets i and j). By definition, the sample covariance, the sample mean, and the sample standard deviation are calculated over *all* data points, regardless of whether the points correspond to meaningful co-movement or to pure noise. This causes the Pearson correlation to be highly sensitive to small co-movements that may be due to noise alone. Therefore, in our future work, we shall seek to develop in a version of the existing Gerber statistic that does not require any estimates of moments. We shall do so by replacing s_k in equation (2) with a more robust measure of standard deviation. Secondly, as we have discussed in Section 3.2, the Gerber statistic's formulation provides it with a natural framework for determining co-movement of pairs of assets across multiple financial signals. Our future work shall aim to develop sophisticated rule-based systems that provide meaningful calculations of co-movement over multiple financial datasets.

6 Conclusion

This paper has introduced a co-movement measure called the Gerber statistic. The Gerber statistic is well-suited for assessing co-movement between financial time series because it is insensitive to extremely large co-movements that distort product-moment-based measures. It is also insensitive to small movements that are likely to be noise. We have studied the performance of the Gerber statistic within the mean-variance portfolio optimization framework of Markowitz (1952, 1959). In *every* scenario considered, the Gerber statistic's performance is superior to that of historical covariance. In *almost every*[4] scenario considered (see Section 4), the Gerber statistic dominates the shrinkage estimator on the key metrics of interest to any investor: cumulative return, average geometric return, and Sharpe ratio. Another advantage of the Gerber statistic lies in the fact that, unlike the shrinkage method, it does not rely upon the sample covariance matrix as input. Promising lines of future investigation for the Gerber statistic have been discussed in Section 5. Finally, we wish to emphasize that the Gerber statistic is easy to compute and is straightforward to implement in any mean-variance optimization software. Our hope is that it will become a welcome alternative to both historical covariance and to the shrinkage estimator of Ledoit and Wolf (2004).

Acknowledgments
We wish to thank David Starer and Jacques Friedman for helpful conversations.

[4]We have considered five risk targets for Gerber thresholds c=.5, c=.7, and c=.9. This amounts to fifteen scenarios in total. The shrinkage estimator of Ledoit and Wolf (2004) only dominates the Gerber statistic in one of these fifteen scenarios, corresponding to c=.5 and a very conservative risk target level of 3%.

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