Hierarchical Risk Parity: A First-Principles Explanation

Gemini

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Abstract

This document provides a mathematical explanation of the Hierarchical Risk Parity (HRP) portfolio allocation algorithm. We explain the foundational statistical concepts, break down the HRP algorithm, and discuss its place in the quantitative finance community, including common criticisms.

1 Fundamental Concepts

Before diving into the HRP algorithm, it is essential to understand the statistical building blocks of portfolio theory.

1.1 Variance, Covariance, and Correlation

Variance measures the risk of a single asset. Covariance measures how two assets move together. Correlation standardizes this measure to be between -1 and 1, making it easy to interpret and the perfect input for clustering.

1.2 The Covariance Matrix and its Inverse

The Feynman Explanation: The covariance matrix is a grid holding the variance and covariance for every pair of assets. Traditional optimizers (like Markowitz) must **invert** this matrix to find a solution. The inverted matrix is called the **Precision Matrix**. This inversion is notoriously unstable; small estimation errors in the input data can be massively amplified, leading to nonsensical results. **This instability is the primary problem that HRP is designed to solve.**

2 The HRP Algorithm: A Step-by-Step Breakdown

The HRP algorithm is a three-stage process that deliberately avoids inverting the covariance matrix.

2.1 Step 1: Tree Clustering

An agglomerative clustering algorithm uses the correlation-distance matrix to group assets into a hierarchical tree (a dendrogram).

2.2 Step 2: Quasi-Diagonalization

This step reorders the covariance matrix based on the hierarchy from the clustering step, creating a block-diagonal structure.

2.3 Step 3: Recursive Bisection

This top-down allocation step starts at the root of the tree and recursively splits capital between subclusters. At each split, it assigns more capital to the cluster with lower risk, following the principle of inverse variance.

3   An Intuitive Walkthrough: HRP in a Supermarket

A helpful analogy is filling a shopping cart with a fixed budget. You first allocate money between large aisles (Produce vs. Meats) based on their overall price volatility. Then, you take the budget for an aisle and subdivide it further (Fruits vs. Vegetables), again based on relative volatility. This continues until every item has a budget, ensuring risk is balanced at every level.

4   Common Criticisms of HRP

Despite its innovative approach, HRP is not without its critics in the quantitative finance community. The method involves trade-offs, and its suitability depends heavily on the specific goals and constraints of the portfolio manager.

4.1   Horrible Scaling and Computational Complexity

The most cited technical drawback of HRP is its poor scaling performance. The hierarchical clustering step is computationally expensive.

- Calculating the distance matrix between all pairs of N assets is an $O(N^2)$ operation.

- The standard agglomerative clustering algorithm has a time complexity of $O(N^2 \log N)$ or even $O(N^3)$ for certain linkage methods.

For a small universe of assets (e.g., a few dozen ETFs or stocks), this is trivial. However, for institutional funds dealing with thousands of signals or individual stocks, this computational burden can become a significant bottleneck. In contrast, the quadratic programming solvers used for traditional Mean-Variance Optimization are often highly efficient and can handle very large-scale problems more gracefully.

4.2   No Use of Returns Estimation

HRP is a purely risk-based methodology. It allocates capital based entirely on the historical correlation and variance of assets, deliberately ignoring any forecasts about future returns. This is a key point of philosophical contention.

The HRP Argument: Proponents of HRP argue that this is a feature, not a bug. Expected returns are notoriously difficult to predict, and basing an allocation on noisy or inaccurate forecasts can lead to instability and poor out-of-sample performance. By ignoring returns, HRP aims for a more robust and structurally balanced portfolio.

The Critic's Argument: For a quantitative fund that invests heavily in research and believes it has a genuine predictive edge (alpha), HRP is unsatisfying. It provides no mechanism to tilt the portfolio towards assets the fund expects to perform well. As the critique notes, if you have "good returns estimations," a method like Minimum Variance ("minvol") or a full Mean-Variance Optimization is more appropriate, as it directly translates those return views into portfolio weights. The choice is thus between a robust-but-agnostic approach (HRP) and a potentially higher-return-but-less-stable approach that relies on forecasting skill (MVO).

5   Conclusion

By first organizing assets based on their correlation structure and then allocating capital based on risk at each level of the hierarchy, HRP avoids the pitfalls of covariance matrix inversion, making the resulting allocation more robust. However, its computational demands and disregard for return forecasts mean it is not a universal solution. It represents a specific, and debated, choice in the spectrum of portfolio construction techniques.

References

[1] Marcos López de Prado, "Building Diversified Portfolios that Outperform Out-of-Sample," *The Journal of Portfolio Management*, 42(4), 59-69, 2016.