Serie 01

Exercise 1.1

$a_1 = 2 \qquad a_{n+1} = 3a_n - 2 \quad$ für $n>1$

2, 4, 10, 28, 82, ...

$$
\begin{aligned}
a_n &= 3a_{n-1} - 2 \\
&= 3(3a_{n-2}-2)-2 \\
&= 3(3(3a_{n-3}-2)-2)-2 \\
&= 3(3(3(3a_{n-4}-2)-2)-2)-2 \\
&= 3^4 a_{n-4} - 3^3\cdot 2 - 3^2\cdot 2 - 3\cdot 2 - 2 \\
&= 3^4 a_{n-4} - 2(3^3+3^2+3^1+3^0)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad a_n &= 3^{n-1}\cdot a_1 - 2\sum_{i=0}^{n-2} 3^i \\
&= 3^{n-1}\cdot 2 - 2\sum_{i=0}^{n-2} 3^i \\
&= 2\left(3^{n-1} - \sum_{i=0}^{n-2} 3^i\right) \\
&= 2\left(3^{n-1} + \frac{1-3^{n-1}}{2}\right) \\
&= 2\cdot 3^{n-1} + 1 - 3^{n-1}
\end{aligned}
$$

$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$

$a_n = 3^{n-1} + 1$

Proof by induction:

Base Case $n \to 1$

$a_1 = 2 = 3^{1-1}+1$

I.H.

Wir nehmen an, dass $a_k = 3^{k-1}+1$ für ein $k \geq 1$

I.S. $k \to k+1$

$a_{k+1} = 3a_k - 2 \overset{I.H.}{=} 3(3^{k-1}+1) - 2 = 3^k + 3 - 2 = 3^k + 1 \quad \square$

Exercise 1.3

a) zu zeigen: $\sum_{i=1}^{n} i^k \leq n^{k+1}$

$\sum_{i=1}^{n} i^k = 1^k+2^k+3^k+\dots+n^k \leq \underbrace{n^k+n^k+n^k+\dots+n^k}_{n\text{-mal}} = \sum_{i=1}^{n} n^k = n\cdot n^k = n^{k+1} \quad \square$

b) zu zeigen: $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}} \cdot n^{k+1}$

$\sum_{i=1}^{n} i^k = 1^k+2^k+\dots+\left(\lceil\frac{n}{2}\rceil\right)^k+\left(\lceil\frac{n}{2}\rceil+1\right)^k+\dots+n^k \geq \left(\lceil\frac{n}{2}\rceil\right)^k+\left(\lceil\frac{n}{2}\rceil+1\right)^k+\dots+n^k = \sum_{i=\lceil\frac{n}{2}\rceil}^{n} i^k \geq \sum_{i=\lceil\frac{n}{2}\rceil}^{n} \left(\frac{n}{2}\right)^k = \left(n-\lceil\frac{n}{2}\rceil+1\right)\cdot\left(\frac{n}{2}\right)^k \geq \frac{n}{2}\cdot\left(\frac{n}{2}\right)^k$

$= \frac{n}{2}\cdot\left(\frac{n}{2}\right)^k = \left(\frac{n}{2}\right)^{k+1} = \frac{1}{2^{k+1}}\cdot n^{k+1} \quad \square$

$\lceil\frac{n}{2}\rceil - 1 \leq \frac{n}{2}$

$\Rightarrow n - \lceil\frac{n}{2}\rceil + 1 \geq \frac{n}{2}$

Exercise 1.4

a) $f(m) = 10m^3 - m^2$, $g(m) = 100m^3$

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{10m^3 - m^2}{100m^3} = \lim_{m\to\infty} \frac{1}{10} - \cancelto{0}{\frac{1}{100m}} = \frac{1}{10} > 0$$

b) $f(m) = 100 \cdot m^2 \cdot \log(m) + 10 \cdot m^3$, $g(m) = 5 \cdot m^3 \cdot \log(m)$

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{100 \cdot m^2 \cdot \log(m) + 10 \cdot m^3}{5m^3 \cdot \log(m)} = \lim_{m\to\infty} \cancelto{0}{\frac{20}{m}} + \cancelto{0}{\frac{2}{\log(m)}} = 0 + 0 \quad \square$$

e) $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$ $\quad \lim_{m\to\infty} \frac{g(m)}{h(m)} = 0$

$$\lim_{m\to\infty} \frac{f(m)}{h(m)} = \lim_{m\to\infty} \frac{f(m) \cdot g(m)}{h(m) \cdot g(m)} = \lim_{m\to\infty} \frac{f(m)}{g(m)} \cdot \frac{g(m)}{h(m)} = \lim_{m\to\infty} \frac{f(m)}{g(m)} \cdot \lim_{m\to\infty} \frac{g(m)}{h(m)} = 0 \quad \square$$

f) $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$ $\quad \lim_{m\to\infty} \frac{f(m)}{g(h(m))} \overset{?}{=} 0$

$f(m) = \frac{1}{m^2}$ $\quad g(m) = \frac{1}{m}$ $\quad h(m) = m^3$

$f(m) = \frac{1}{m^2}$ $\quad g(h(m)) = \frac{1}{m^3}$

O-Notation

- Eingabegrösse n
- Ops: $<, >, =, +, -, \cdot, \div$

N = Menge mögl. Eingabegrössen

Definition: Für $f: N \to \mathbb{R}^+$

$$O(f) := \{ g: N \to \mathbb{R}^+ \mid \exists C > 0 \; \forall n \in N \;\; g(n) \leq C \cdot f(n) \}$$

$\hat{=}$ Menge aller Funktionen g, die ab einem gewissen Punkt kleiner sind als f (mit konstantem Faktor C)

$\log(n) + \sqrt{n} \in O(n)$

$10 \cdot \log(n) + \log(\log(n)) \in O(\log(n))$

$1 \in O(n)$

$\log(n) \leq O(n)$

Implikationen

$g \in O(f) \iff \frac{g}{f}$ beschränkt ist (auf N)

$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty \implies g \notin O(f)$

$\lim_{n\to\infty} \frac{g(n)}{f(n)} \in \mathbb{R} \implies g \in O(f)$

Zeige: $g(n) = 10 \cdot \log(n^5) + \log(\log(n)) \leq O(\underset{f(n)}{\log(n)})$

$$\lim_{n \to \infty} \frac{10 \cdot \log(n^5) + \log(\log(n))}{\log(n)} = \lim_{n \to \infty} \frac{50 \cdot \log(n) + \log(\log n)}{\log n} = \lim_{n \to \infty} \frac{50 \cdot \cancel{\log n}}{\cancel{\log n}} + \frac{\log(\log n)}{\log n} = 50 + \lim_{y \to \infty} \frac{(\log(y))'}{(y)'} = 50 + \lim_{y \to \infty} \cancel{\frac{1}{y}}^{\to 0} = 50 \in \mathbb{R}$$

$g \in O(f)$

$$\boxed{\lim_{n \to \infty} f(n) = \lim_{n \to \infty} e^{\ln(f(n))}}$$

$\Omega(f) \hat{=}$ alle Funktionen, die schneller als f wachsen, oder gleich schnell.

$g \in O(f) \Rightarrow f \in \Omega(g)$

$\Omega(\log n) = \{\log n, 5 \cdot \log n, n, n^2, 2^n, \dots\}$

$\Theta(f) = O(f) \cap \Omega(f)$ $\hat{=}$ alle Funktionen, die gleich schnell wachsen

$\Theta(n) = \{n, 5n, 10n + \sqrt{n}, 3n + \log(n^5), \dots\}$

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

$$\binom{n}{2} = \frac{n!}{(n-2)! \cdot 2} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1}{(n-2) \cdot (n-3) \cdot \ldots \cdot 2 \cdot 1 \cdot 2} = \frac{n(n-1)}{2} = \frac{n^2 - n}{2} = \Theta(n^2)$$

$$\log(\log(n)) < \log(n) = \log(n^{100}) = \log_{123}(n^3) < \sqrt{n} < n = n + \sqrt{n} < n \cdot \log(n) < n \cdot \sqrt{n} < n^2 = 2n^2 + 3n = \binom{n}{2} < n^3 < 2^n < e^n < n! < n^n$$

$2^n \in O(e^n)$?
$2^n \notin \Theta(e^n)$?