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Diskrete Mathematik

Solution 5

5.1 Computing Representations of Relations

a) We have $\rho^3 = \{(1,1),(1,3),(2,2),(4,4)\}$ and

$$M^{\rho^*} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

5.2 Operations on Relations

	Relation	reflexive	symmetric	transitive
a)	$< \circ \mid$	✗	✗	✓
b)	$\mid \cup \equiv_2$	✓	✗	✗
c)	$\mid \cup \mid^{-1}$	✓	✓	✗

a) Two numbers (a,b) are in the relation whenever there exists an x such that $a < x$ and $x \mid b$. This relation is not reflexive, since $(1,1) \notin \, < \circ \mid$. Moreover, it is not symmetric, because $(1,2) \in \, < \circ \mid$, but $(2,1) \notin \, < \circ \mid$. This relation is transitive. For any (a,b,c), assume that there exist some x and y, such that $a < x,\ x \mid b,\ b < y$ and $y \mid c$. From $x \mid b$ it follows that $x \leq b$, hence, $a < x \leq b < y$. Therefore, $a < y$ and $y \mid c$.

b) Two numbers (a,b) are in the relation whenever $a \mid b$ or $a \equiv_2 b$. This relation is reflexive, since for any a, we have $a \equiv_2 a$ (alternatively, one could use the fact that $a \mid a$). It is, however, not symmetric, because $(1,2) \in \, \mid \cup \equiv_2$, but $(2,1) \notin \, \mid \cup \equiv_2$. It is also not transitive, since $(3,1) \in \, \mid \cup \equiv_2$ and $(1,2) \in \, \mid \cup \equiv_2$, but $(3,2) \notin \, \mid \cup \equiv_2$.

c) Two numbers (a,b) are in the relation whenever $a \mid b$ or $b \mid a$. This relation is reflexive, since for any a, we have $a \mid a$. It is also symmetric, because for any (a,b), we trivially have $a \mid b$ or $b \mid a$ if and only if $b \mid a$ or $a \mid b$. The relation is, however, not transitive, since $(3,1) \in \, \mid \cup \mid^{-1}$ and $(1,2) \in \, \mid \cup \mid^{-1}$ but $(3,2) \notin \, \mid \cup \mid^{-1}$.

5.3 A False Proof

a) For an arbitrary $x \in A$, there does not always exist a $y \in A$ such that $x \; \rho \; y$.

b) Consider the following counterexample: $A = \{1,2\}$ and $\rho = \{(1,1)\}$. The relation ρ is symmetric and transitive. However, it is not reflexive, since $2 \; \rho \; 2$ does not hold.

5.4 An Equivalence Relation

a) We prove that $\sim$ satisfies all properties of an equivalence relation.

Reflexivity: For any point $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have $(x, y) \sim (x, y)$, because one can choose $\lambda = 1$ in the definition of $\sim$.

Symmetry: Let $x_1, y_1, x_2, y_2 \in \mathbb{R} \setminus \{0\}$ and assume that $(x_1, y_1) \sim (x_2, y_2)$. It follows that $x_1 = \lambda x_2$ and $y_1 = \lambda y_2$ for some $\lambda > 0$. Hence, $x_2 = \frac{1}{\lambda} x_1$ and $y_2 = \frac{1}{\lambda} y_1$, where $\frac{1}{\lambda} > 0$. Therefore, $(x_2, y_2) \sim (x_1, y_1)$.

Transitivity: Let $x_1, y_1, x_2, y_2, x_3, y_3 \in \mathbb{R} \setminus \{0\}$ and assume that $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$. This means that $(x_1, y_1) = (\lambda_1 x_2, \lambda_1 y_2)$ and $(x_2, y_2) = (\lambda_2 x_3, \lambda_2 y_3)$ for some $\lambda_1, \lambda_2 > 0$. It follows that $(x_1, y_1) = (\lambda x_3, \lambda y_3)$, where $\lambda > 0$ is defined as $\lambda_1 \lambda_2$. Hence, $(x_1, y_1) \sim (x_3, y_3)$.

b) An equivalence class $[(x, y)]_\sim$ contains all points on the ray through the origin $(0, 0)$ and the point (x, y) (excluding the origin). Note that no equivalence class can contain the origin $(0, 0)$ ($\sim$ is only defined on $\mathbb{R}^2 \setminus \{(0,0)\}$).

5.5 Order Relations on Quotient Sets

To show that $(A/\theta, \rho)$ is a partial order we need to show that the relation ρ is reflexive, antisymmetric, and transitive.

Notice that for all $a \in A$ the element $\texttt{least}([a]_\theta)$ is well-defined because $[a]_\theta \neq \varnothing$. Indeed, since θ is an equivalence relation, it is reflexive, so that $a \,\theta\, a$ and $a \in [a]_\theta$, by definition of $[a]_\theta$.

- **Reflexivity:** Let $[a]_\theta \in A/\theta$. One has

$$\begin{aligned} & [a]_\theta \ \rho\ [a]_\theta \\ \overset{\cdot}{\Longleftrightarrow}\ & \texttt{least}([a]_\theta) \leq \texttt{least}([a]_\theta) && \text{(Definition of } \rho) \\ \overset{\cdot}{\Longleftrightarrow}\ & \text{true.} && \text{(Reflexivity of } \leq) \end{aligned}$$

- **Antisymmetry:** Let $[a]_\theta, [b]_\theta \in A/\theta$. We have

$$\begin{aligned} & ([a]_\theta \ \rho\ [b]_\theta) \wedge ([b]_\theta \ \rho\ [a]_\theta) \\ \overset{\cdot}{\Longleftrightarrow}\ & \big(\texttt{least}([a]_\theta) \leq \texttt{least}([b]_\theta)\big) \wedge \big(\texttt{least}([b]_\theta) \leq \texttt{least}([a]_\theta)\big) && \text{(Definition of } \rho\text{, twice)} \\ \overset{\cdot}{\Longrightarrow}\ & \texttt{least}([a]_\theta) = \texttt{least}([b]_\theta) && \text{(Antyisymmetry of } \leq) \\ \overset{\cdot}{\Longrightarrow}\ & [a]_\theta \cap [b]_\theta \neq \varnothing && \text{(Definition of } \cap) \\ \overset{\cdot}{\Longrightarrow}\ & [a]_\theta = [b]_\theta. && \text{(Theorem 3.11)} \end{aligned}$$

- **Transitivity:** Let $[a]_\theta, [b]_\theta, [c]_\theta \in A/\theta$. We have

$$\begin{aligned} & ([a]_\theta \ \rho\ [b]_\theta) \wedge ([b]_\theta \ \rho\ [c]_\theta) \\ \overset{\cdot}{\Longleftrightarrow}\ & \big(\texttt{least}([a]_\theta) \leq \texttt{least}([b]_\theta)\big) \wedge \big(\texttt{least}([b]_\theta) \leq \texttt{least}([c]_\theta)\big) && \text{(Definition of } \rho\text{, twice)} \\ \overset{\cdot}{\Longrightarrow}\ & \texttt{least}([a]_\theta) \leq \texttt{least}([c]_\theta) && \text{(Transitivity of } \leq) \\ \overset{\cdot}{\Longrightarrow}\ & [a]_\theta \,\rho\, [c]_\theta. && \text{(Definition of } \rho) \end{aligned}$$

5.6 Lifting an Operation to Equivalence Classes

a) We define the function $\mathsf{sum} : A^2 \to A$ by

$$\mathsf{sum}((a,b),(c,d)) \stackrel{\text{def}}{=} (ad+bc, bd).$$

Observe that $bd \neq 0$ since $b \neq 0$ and $d \neq 0$.

b) f is θ-consistent if and only if

$$(b_1 \;\theta\; b_1' \text{ and } b_2 \;\theta\; b_2') \implies f(b_1,b_2) \;\theta\; f(b_1',b_2')$$

is true for all $b_1, b_2, b_1', b_2' \in B$. Alternatively (and equivalently) we could say that f is θ-consistent if and only if

$$([b_1]_\theta = [b_1']_\theta \text{ and } [b_2]_\theta = [b_2']_\theta) \implies [f(b_1,b_2)]_\theta = [f(b_1',b_2')]_\theta$$

is true for all $b_1, b_2, b_1', b_2' \in B$.

c) Let $(a,b),(a',b'),(c,d),(c',d') \in A$ be arbitrary. We have

$$\begin{aligned}
&(a,b) \;\sim\; (a',b') \text{ and } (c,d) \;\sim\; (c',d') \\
&\quad \stackrel{\cdot}{\iff} ab' = ba' \text{ and } cd' = dc' && \text{(def. } \sim\text{)} \\
&\quad \stackrel{\cdot}{\implies} ab' \cdot dd' + cd' \cdot bb' = ba' \cdot dd' + dc' \cdot bb' \\
&\quad \stackrel{\cdot}{\iff} ad \cdot b'd' + bc \cdot b'd' = bd \cdot a'd' + bd \cdot b'c' && \text{(comm.)} \\
&\quad \stackrel{\cdot}{\iff} (ad+bc) \cdot b'd' = bd \cdot (a'd' + b'c') && \text{(distr.)} \\
&\quad \stackrel{\cdot}{\iff} (ad+bc, bd) \;\sim\; (a'd'+b'c', b'd') && \text{(def. } \sim\text{)} \\
&\quad \stackrel{\cdot}{\iff} \mathsf{sum}((a,b),(c,d)) \;\sim\; \mathsf{sum}((a',b'),(c',d')). && \text{(def. } \mathsf{sum}\text{)}
\end{aligned}$$

Hence, sum is $\sim$-consistent.