Combinational Circuits: Theory

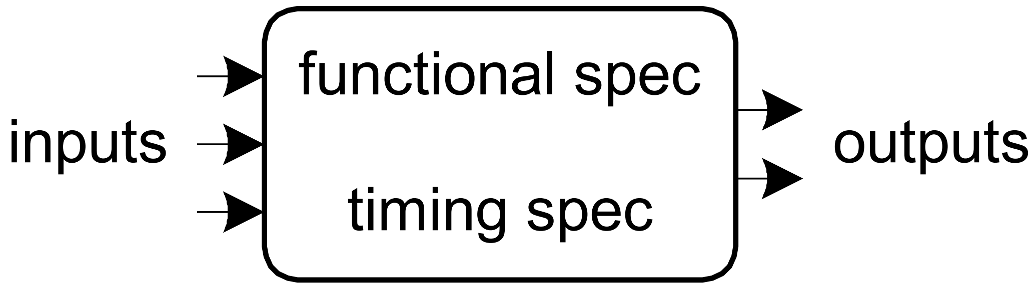
Digital Design and Computer Architecture
Mohammad Sadrosadati
Frank K. Gürkaynak

http://safari.ethz.ch/ddca

What we will learn

- Boolean Algebra
- Theorems
- Simplifying Boolean Equations
- Proving Theorems
- Bubble Pushing

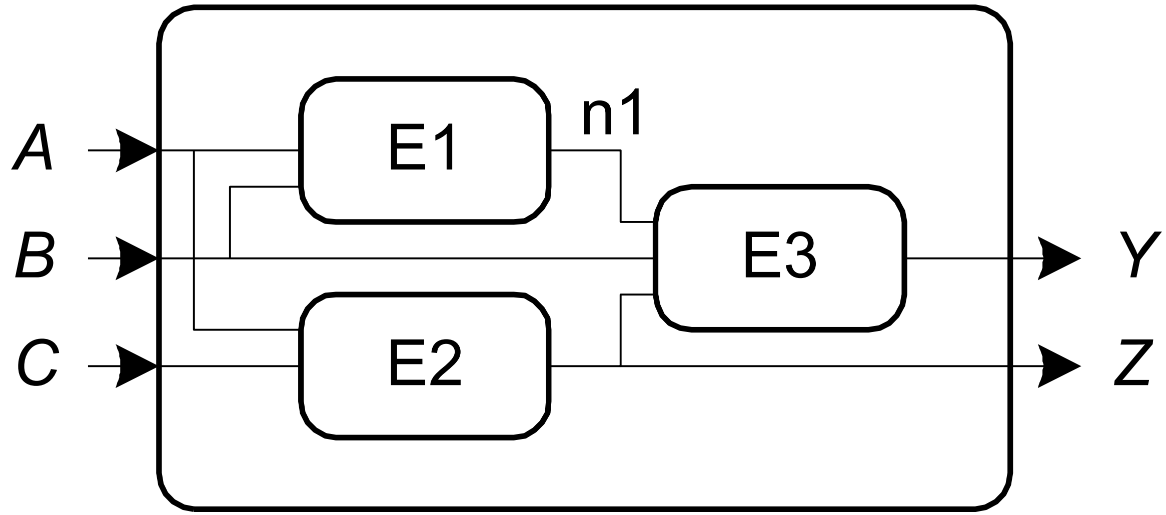
Introduction



- **A logic circuit is composed of:**
  - Inputs
  - Outputs
- ***Functional specification*** **(describes relationship between inputs and outputs)**
- ***Timing specification*** **(describes the delay between inputs changing and outputs responding)**

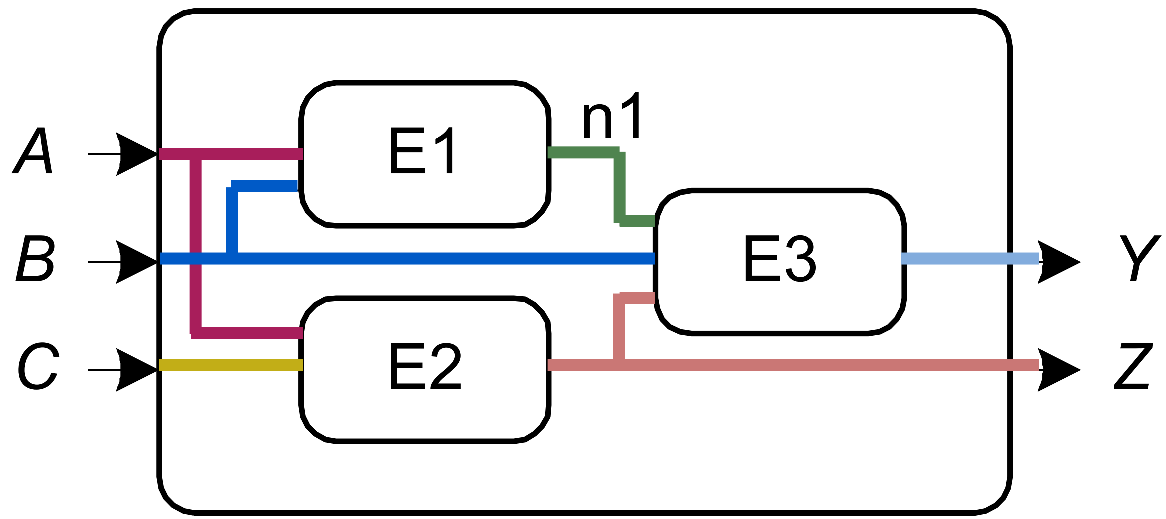
Circuits

- **Circuit elements**
  - E1, E2, E3
  - Each itself a circuit
- **Nodes (wires)**
  - Inputs: *A*, *B*, *C*
  - Outputs: *Y*, *Z*
  - Internal: n1

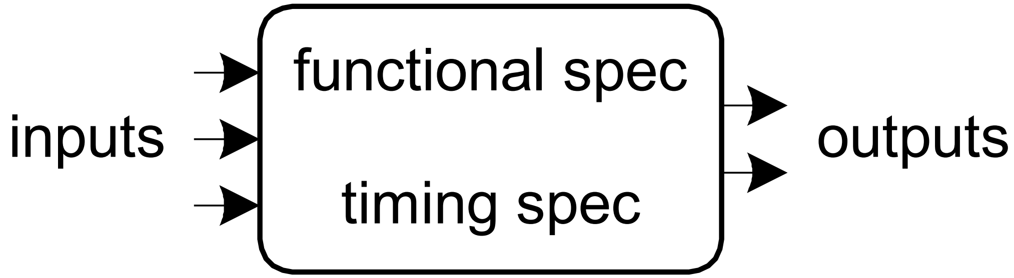


Circuits

- **Circuit elements**
  - E1, E2, E3
  - Each itself a circuit
- **Nodes (wires)**
  - Inputs: *A*, *B*, *C*
  - Outputs: *Y*, *Z*
  - Internal: n1
  - To count the nodes look at
    - *outputs of every circuit element*
    - *inputs to the entire circuit*



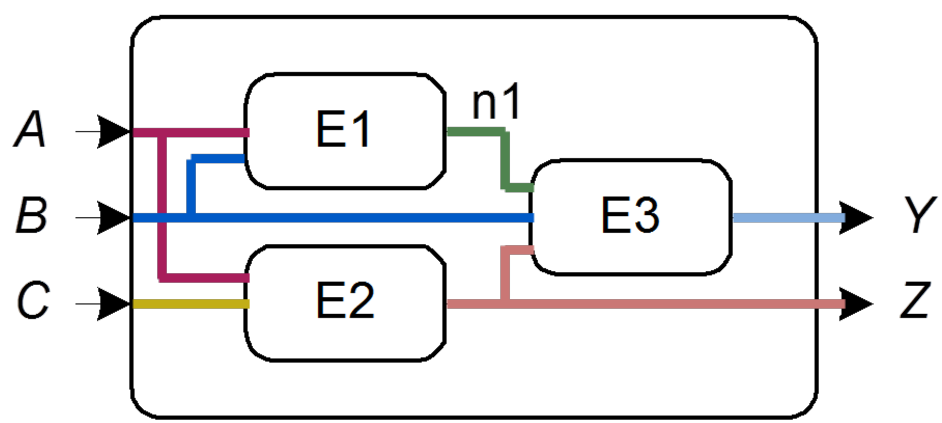
Types of Logic Circuits



- *Combinational Logic*
  - Memoryless
  - Outputs determined by current values of inputs
  - In some books called Combinatorial Logic
- *Sequential Logic*
  - Has memory
  - Outputs determined by previous and current values of inputs

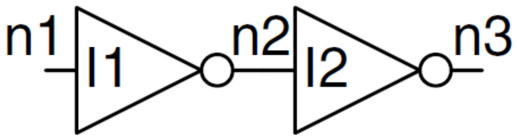
Rules of Combinational Composition

- **Every circuit element is itself combinational**
- **Every node of the circuit is either**
  - designated as an input to the circuit or
  - connects to exactly one output terminal of a circuit element
- **The circuit contains no cyclic paths: every path through the circuit visits each circuit node at most once**
- ***Example:***
  (If E1-3 combinational)

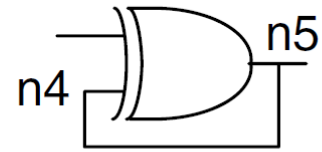


Combinational Or Not?

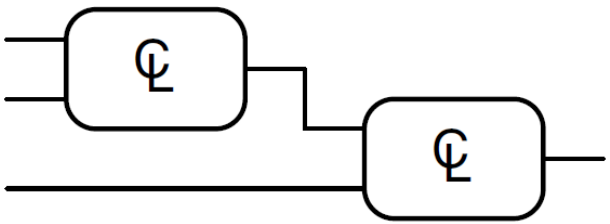
1
yes



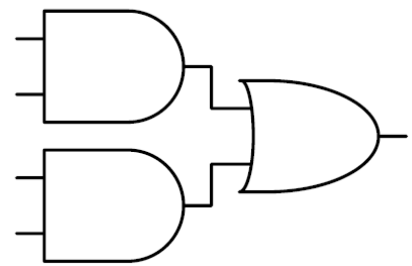
2
no



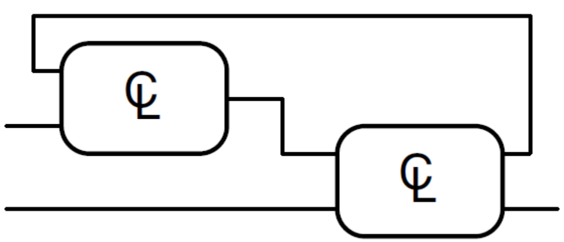
3
yes



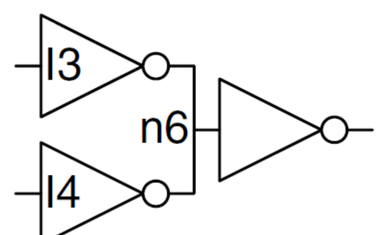
4
yes



5
no



6
no



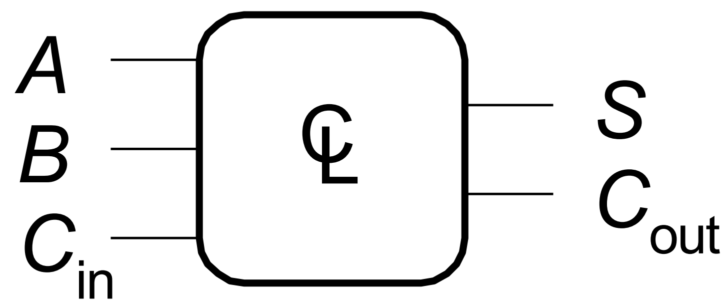
Boolean Equations

- **Functional specification of outputs in terms of inputs**

- ***Example (full adder – more later):***

$$\boldsymbol{S = F(A, B, C_{in})}$$

$$\boldsymbol{C_{out} = G(A, B, C_{in})}$$



$$S = A \oplus B \oplus C_{in}$$

$$C_{out} = AB + AC_{in} + BC_{in}$$

Boolean Algebra

- **Set of axioms and theorems to simplify Boolean equations**
- **Like regular algebra, but in some cases simpler because variables can have only two values (1 or 0)**
- **Axioms and theorems obey the principles of duality:**
  - **stay correct if**
    ANDs and ORs interchanged *and*
    0's and 1's interchanged
  - *Examples:*

$\overline{0} =$

$B \cdot \overline{B} =$

Boolean Algebra

- **Set of axioms and theorems to simplify Boolean equations**

- **Like regular algebra, but in some cases simpler because variables can have only two values (1 or 0)**

- **Axioms and theorems obey the principles of duality:**
  - **stay correct if**
    ANDs and ORs interchanged *and*
    0's and 1's interchanged
  - *Examples:*

	dual
$\overline{0} = 1$	$\overline{1} = 0$
$B \cdot \overline{B} = 0$	$B + \overline{B} = 1$

Boolean Axioms

	Axiom		Dual	Name
A1	$B = 0$ if $B \neq 1$	A1′	$B = 1$ if $B \neq 0$	Binary field
A2	$\overline{0} = 1$	A2′	$\overline{1} = 0$	NOT
A3	$0 \bullet 0 = 0$	A3′	$1 + 1 = 1$	AND/OR
A4	$1 \bullet 1 = 1$	A4′	$0 + 0 = 0$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	A5′	$1 + 0 = 0 + 1 = 1$	AND/OR

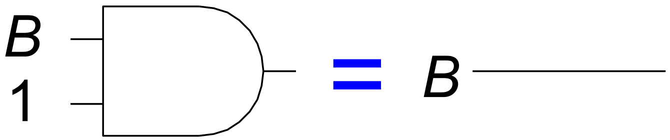
Duality: **If the symbols 0 and 1 and the operators • (AND) and + (OR) are interchanged, the statement will still be correct.**

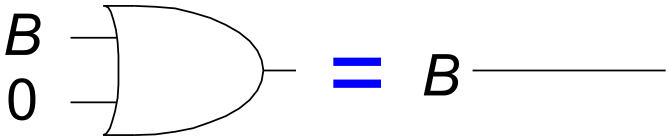
T1: Identity Theorem

- **$B \cdot 1 \quad =$**
- **$B + 0 \ =$**

T1: Identity Theorem

- $B \cdot 1 = B$
- $B + 0 = B$



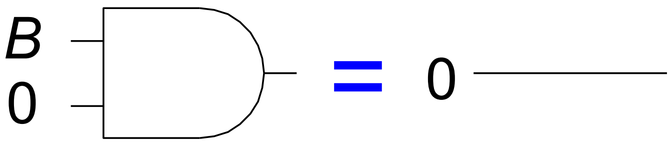


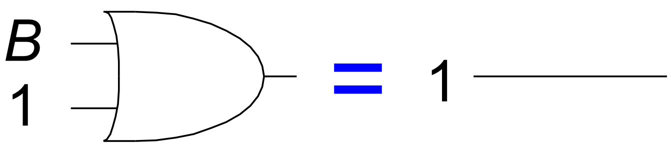
T2: Null Element Theorem

- $B \cdot 0 \; =$
- $B + 1 \; =$

T2: Null Element Theorem

- **B · 0 = 0**
- **B + 1 = 1**



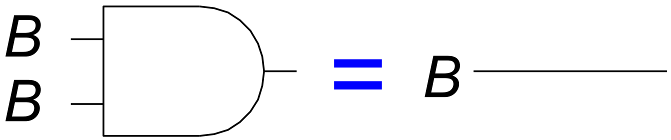


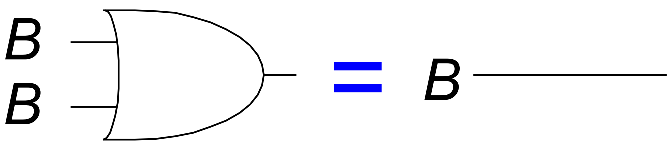
T3: Idempotency Theorem

- $B \cdot B =$
- $B + B =$

T3: Idempotency Theorem

- **B · B = B**
- **B + B = B**



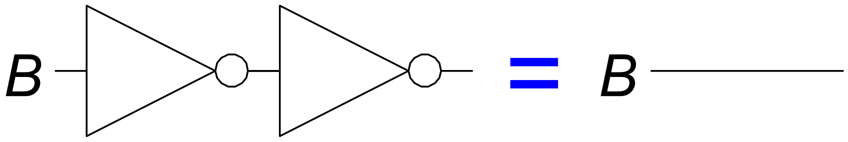


T4: Involution Theorem

- $\overline{\overline{B}}$ =

T4: Involution Theorem

- $\overline{\overline{B}} = B$

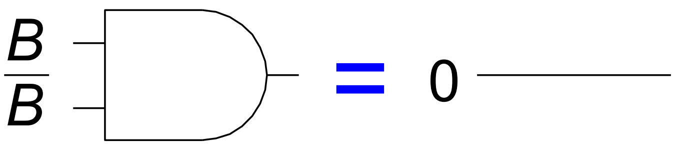


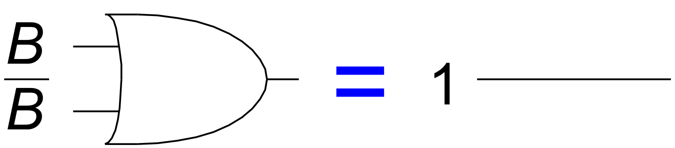
T5: Completeness Theorem

- $\mathbf{B} \cdot \overline{\mathbf{B}}$ =
- $\mathbf{B} + \overline{\mathbf{B}}$ =

T5: Completeness Theorem

- $B \cdot \overline{B} = 0$
- $B + \overline{B} = 1$





Boolean Theorems: Summary

	Theorem		Dual	Name
T1	$B \bullet 1 = B$	T1′	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	T2′	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	T3′	$B + B = B$	Idempotency
T4		$\bar{\bar{B}} = B$		Involution
T5	$B \bullet \overline{B} = 0$	T5′	$B + \overline{B} = 1$	Complements

Boolean Theorems of Several Variables

	Theorem		Dual	Name
T6	$B \bullet C = C \bullet B$	T6′	$B + C = C + B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	T7′	$(B + C) + D = B + (C + D)$	Associativity
T8	$(B \bullet C) + (B \bullet D) = B \bullet (C + D)$	T8′	$(B + C) \bullet (B + D) = B + (C \bullet D)$	Distributivity
T9	$B \bullet (B + C) = B$	T9′	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	T10′	$(B + C) \bullet (B + \overline{C}) = B$	Combining
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D)$ $= B \bullet C + \overline{B} \bullet D$	T11′	$(B + C) \bullet (\overline{B} + D) \bullet (C + D)$ $= (B + C) \bullet (\overline{B} + D)$	Consensus
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \ldots}$ $= (\overline{B_0} + \overline{B_1} + \overline{B_2} \ldots)$	T12′	$\overline{B_0 + B_1 + B_2 \ldots}$ $= (\overline{B_0} \bullet \overline{B_1} \bullet \overline{B_2})$	De Morgan's Theorem

Simplifying Boolean Expressions: Example 1

$Y = \bar{A}B + AB$

Simplifying Boolean Expressions: Example 1

$Y = \bar{A}B + AB$

$= B(\bar{A} + A)$ T8

$= B(1)$ T5'

$= B$ T1

Simplifying Boolean Expressions: Example 2

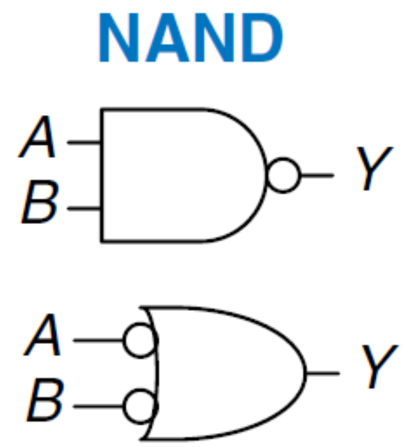
$Y = A(AB + ABC)$

Simplifying Boolean Expressions: Example 2

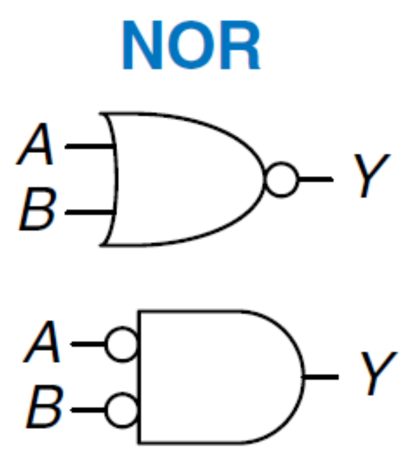
$$
\begin{aligned}
Y &= A(AB + ABC) & \\
&= A(AB(1 + C)) & \text{T8} \\
&= A(AB(1)) & \text{T2'} \\
&= A(AB) & \text{T1} \\
&= (AA)B & \text{T7} \\
&= AB & \text{T3}
\end{aligned}
$$

DeMorgan's Theorem

- $Y = \overline{A \cdot B} = \overline{A} + \overline{B}$

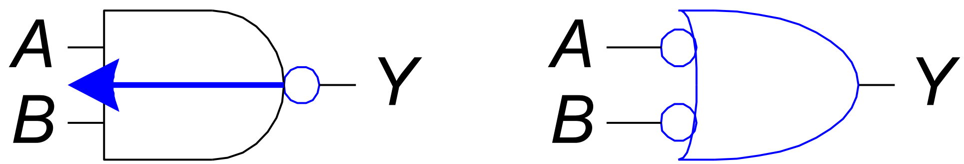


- $Y = \overline{A + B} = \overline{A} \cdot \overline{B}$



Bubble Pushing

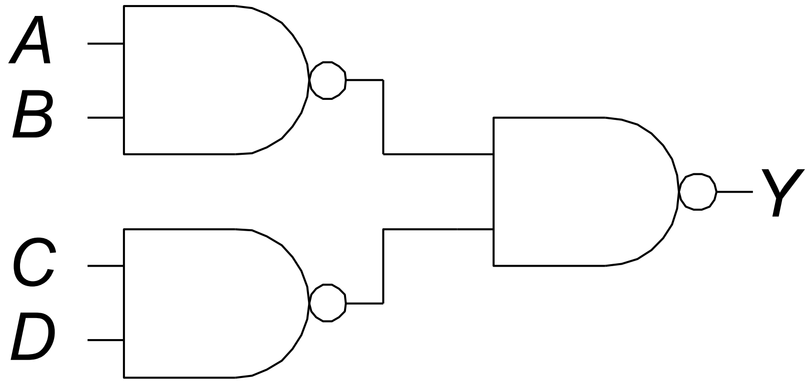
- **Pushing bubbles backward (from the output) or forward (from the inputs) changes the body of the gate from AND to OR or vice versa.**
  - Pushing a bubble from the output back to the inputs puts bubbles on all gate inputs.



  - Pushing bubbles on all gate inputs forward toward the output puts a bubble on the output and changes the gate body.

Bubble Pushing

- **What is the Boolean expression for this circuit?**



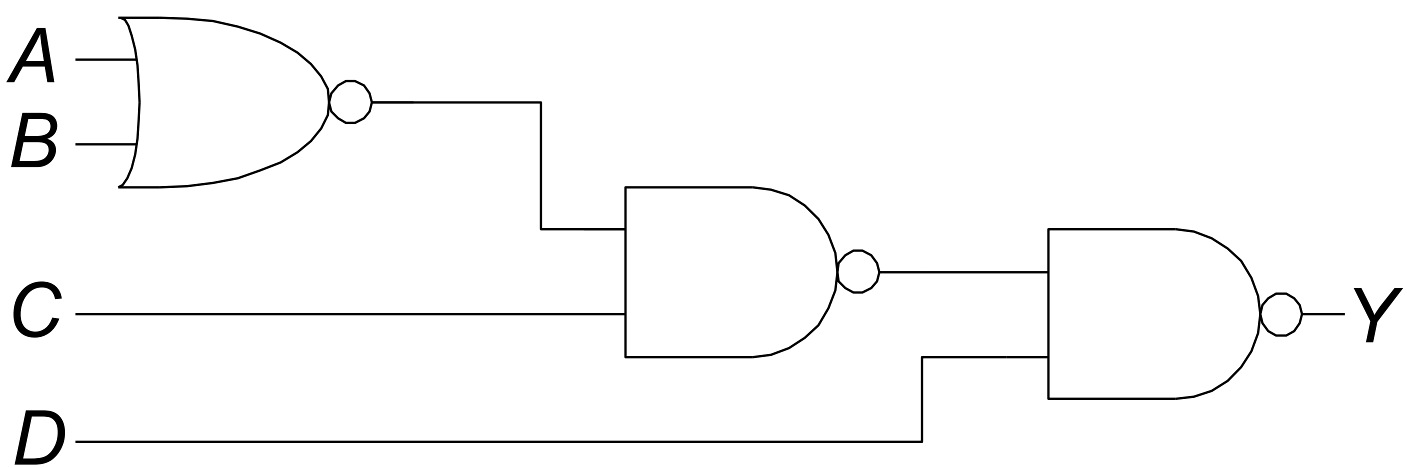
$$Y = \overline{\overline{AB} \bullet \overline{CD}}$$

$$= AB + CD$$

How to get with bubble pushing?

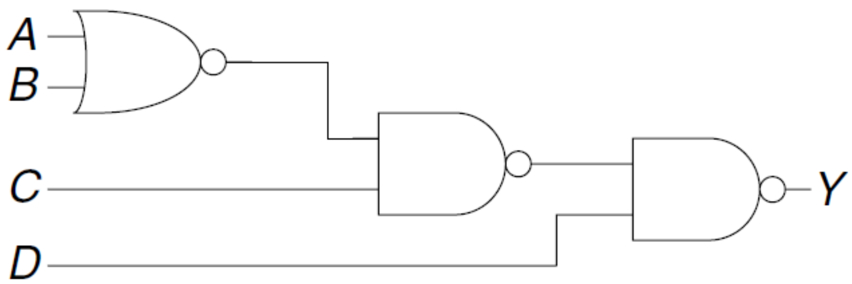
Bubble Pushing Rules

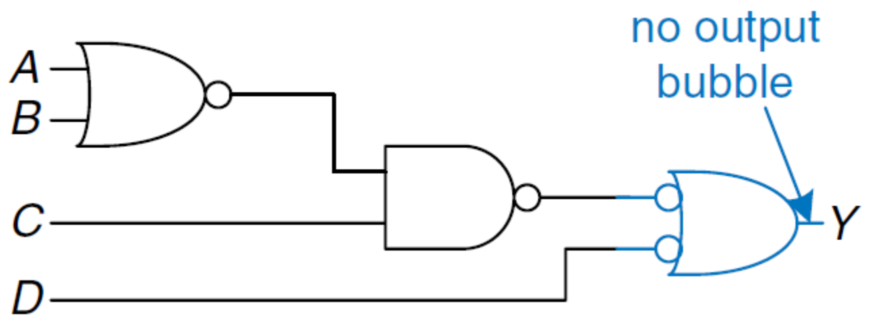
- Begin at the output of the circuit and work toward the inputs
- Push any bubbles on the final output back toward the inputs
- Draw each gate in a form so that bubbles cancel



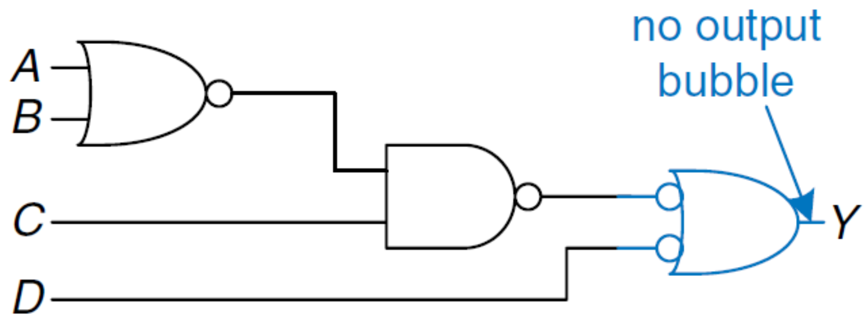
$$Y = \overline{\overline{\overline{(A+B)} \bullet C} \bullet D}$$

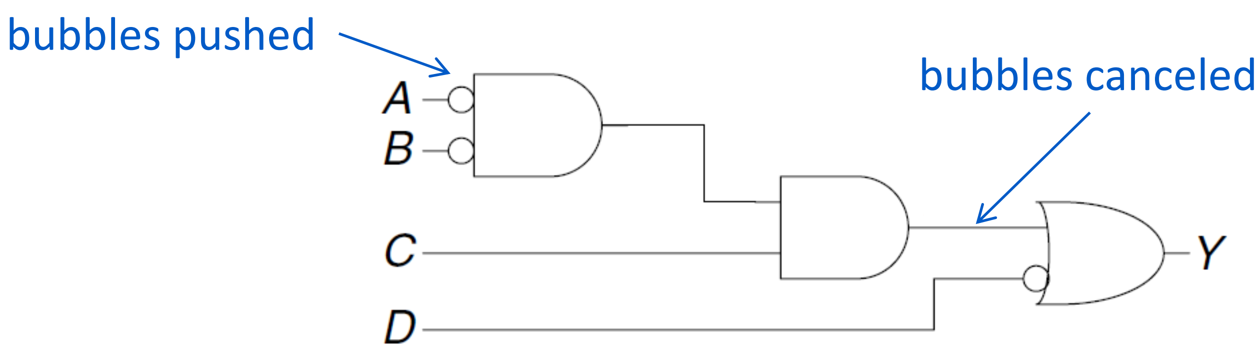
Bubble Pushing Example





Bubble Pushing Example





$$Y = \bar{A}\bar{B}C + \bar{D}$$

$$\overline{\overline{\overline{(A+B)} \bullet C} \bullet D} = \bar{A}\bar{B}C + \bar{D}$$

What have we learned

- Combinational circuit discipline
- Boolean algebra theorems
- Bubble pushing (De Morgan's Theorem)