Exercise sheet 04

Exercise 4.3

Algorithm 1 Bubble Sort (input: array $A[1 \dots n]$).

for $j = 1, \dots, n$ **do**
  for $i = 1, \dots, n-1$ **do**
    if $A[i] > A[i+1]$ **then**
      Swap $A[i]$ and $A[i+1]$

Base case $j=1$

$$A = [a_1\ a_2\ \dots\ a_{l-1}\ a_l\ a_{l+1}\ \dots\ a_{n-1}\ a_n]$$

I.H. Inv. gilt für ein $j=k$ $k<n$

$$A=[a_1\ a_2\ \dots\ a_{n-k-1}\ a_{n-k}\ \underbrace{a_{n-k+1}\ \dots\ a_{n-1}\ a_n}_{\checkmark}]$$

I.S. $j=k+1$

$$A[1,\dots,n-k]$$

$\Rightarrow INV(k+1)$

$INV(n)$ gilt $\Rightarrow$ Nach den ersten n Iterationen, sind die n grössten Elemente alle an der korrekten Position.

Exercise 4.4

$f(i) \geq f(j)$ $i \geq j$

kleinstes $T \in \mathbb{N}$, sodass $f(T) \geq N$

a) $f(T_{ub}) \geq N$ $T_{ub} \leq 2T$

$k = 1, 2, 3, \dots$ $T_k = 2^k$

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T ← 1
while (f(T) < N) do
    T ← T·2
return T
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$T_{ub} \leq 2T$ T_k

$T_{k-1} < N$

$T_k = 2 \cdot T_{k-1} < 2N$

$\Rightarrow \underline{T_k < 2N}$ ✓

$$T_{\lceil \log_2 T \rceil} = 2^{\lceil \log_2 T \rceil} \geq T$$

b) kleinstes $T \in \mathbb{N}$ $f(T) \geq N$

$T_{ub} \leftarrow Algo_a()$

$i_{high} \leftarrow T_{ub}$

$i_{low} \leftarrow 1$ // oder $\lfloor T_{ub}/2 \rfloor$

while ($i_{low} < i_{high}$)

  $i_{mid} \leftarrow \lceil (i_{low} + i_{high})/2 \rceil$

  if $f(i_{mid}) \geq N$ then // $T \leq i_{mid}$

    if $f(i_{mid} - 1) < N$ then // $T \geq i_{mid}$

      Return i_{mid}

    else

      $i_{high} \leftarrow i_{mid} - 1$

  else

    $i_{low} \leftarrow i_{mid}$

Return i_{low}

$i_{low} \leq T \leq i_{high}$

Exercise 4.5

a)

Algorithm 4

$i \leftarrow 1$

while $i \leq n$ **do** wenn $i > \log_2(n)$

  $j \leftarrow i$

  while $2^j \leq n$ **do** while $j \leq \log_2(n)$

    $f()$

    $j \leftarrow j + 1$

  $i \leftarrow i + 1$

$$\sum_{j=i}^{\lfloor \log_2 n \rfloor} 1 = \lfloor \log_2 n \rfloor - i + 1$$

$$\sum_{i=1}^{\lfloor \log_2 n \rfloor} (\lfloor \log_2 n \rfloor - i + 1) = \lfloor \log_2(n) \rfloor \cdot (\lfloor \log_2 n \rfloor + 1)/2 = \Theta(\log^2(n))$$

b)

Algorithm 5

function $A(n)$

  $i \leftarrow 0$

  while $i < n^2$ **do**

    $j \leftarrow n$

    while $j > 0$ **do**

      $f()$

      $f()$

      $j \leftarrow j - 1$

    $i \leftarrow i + 1$

  $k \leftarrow \lfloor \frac{n}{2} \rfloor$

  for $l = 0 \ldots 3$ **do**

    if $k > 0$ **then**

      $A(k)$

      $A(k)$

$$\sum_{j=1}^{n} 2 = 2n \qquad \sum_{i=0}^{n^2-1} 2n = 2n^3$$

$T(n) = 8T(\frac{n}{2}) + 2n^3$

Falls $k>0$: $8 \cdot T(\lfloor \frac{n}{2} \rfloor)$

Falls $k=0$: $O(1)$

$\Leftrightarrow n=1$: $O(1)$

$T(n) = 2n^3 + 8T(\lfloor \frac{n}{2} \rfloor)$ $= \Theta(n^3 \log n)$

$T(1) = 2$

(c) $\forall n' \in \mathbb{N} \quad n' \leq n \qquad T(n'+1) \geq T(n')$

B.C. $\quad T(2) = 16 + 16 \cdot T(1) = 32 \geq 2 = T(1) \quad \Rightarrow n' \in \mathbb{N} \quad n \leq 1 \quad n' = 1$

I.H. $\quad T(k'+1) \geq T(k') \quad \forall k' \in \mathbb{N} \quad k' \leq k$

I.S. $\quad T(k+2) \geq T(k+1) \qquad$ mit I.H. $\Rightarrow T(k'+1) \geq T(k') \quad \forall k' \leq k+1$

$$\left\lfloor \frac{k+1}{2} \right\rfloor \leq k$$

Nach I.H.: $\quad T\left(\left\lfloor \frac{k+2}{2} \right\rfloor\right) \geq T\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right)$

$\left\lfloor \frac{k+2}{2} \right\rfloor = \left\lfloor \frac{k+1}{2} \right\rfloor \qquad \left\lfloor \frac{k+2}{2} \right\rfloor = \left\lfloor \frac{k+1}{2} \right\rfloor + 1$

$$T(k+2) = 2(k+2)^3 + 8T\left(\left\lfloor \frac{k+2}{2} \right\rfloor\right) \geq 2(k+1)^3 + 8T\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right) = T(k+1)$$

$n' \in \mathbb{N} \quad n' \leq n \quad : T(n'+1) \geq T(n')$

$\Rightarrow T(n+1) \geq T(n)$

True/ False ?

$\frac{n}{\log n} \leq O(\sqrt{n})$ ✗

$\log(n!) \geq \Omega(n^2)$ ✗ $\qquad \log(n!) = \Theta(n \log n) \qquad \log(n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1)$

$= \log(n) + \log(n-1) + \log(n-2) + \ldots +$

$\leq \log(n) + \log(n) + \ldots = n \cdot \log(n)$

$n^k \geq \Omega(k^n)$, if $1 < k \leq O(1)$ ✗ $\qquad \log(n!) = \sum_{i=1}^{n} \log(i)$

$\log_3(n^4) = \Theta(\log_7(n^8))$ ✓ $\qquad \geq \sum_{i=\frac{n}{2}}^{n} \log(i)$

$\log_3(n^4) = \frac{4 \cdot \ln(n)}{\ln(3)} \qquad \geq \sum_{i=\frac{n}{2}}^{n} \log(\frac{n}{2})$

$= \frac{n}{2} \cdot \log(n/2)$

$$n! \neq \Theta\left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)$$

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n} = 1$$

$$n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$$

$\log_7(n^8) \overset{?}{=} \Theta(\log_3(n^{\sqrt{n}}))$ ✗

$3n^4 + n^2 + n \geq \Omega(n^2)$ ✓

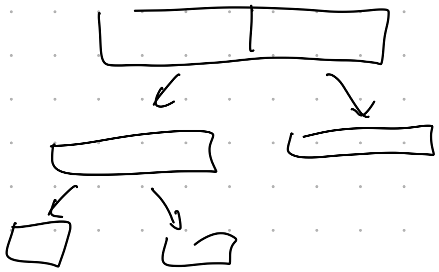
(*) $\quad n! \leq O(n^{n/2})$ ✗

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 \geq \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (\lceil n/10 \rceil)}_{n-(n/10)\text{- Terme}} \geq (\lceil n/10 \rceil) \cdot (\lceil n/10 \rceil) \cdot \ldots = (\lceil n/10 \rceil)^{0.9n}$$

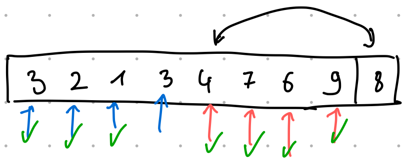
$$\lim_{n \to \infty} \frac{n^{n/2}}{(n/10)^{0.9n}} = \lim_{n \to \infty} n^{0.5n - 0.9n} \cdot 10^{0.9n} = \lim_{n \to \infty} n^{-0.4n} \cdot 10^{0.9n} = \lim_{n \to \infty} \left(\frac{10^{0.9/0.4}}{n}\right)^{0.4n} = 0$$

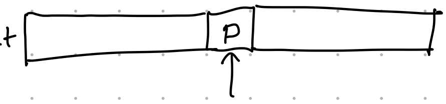
Vorlesung - Recap

	Vergleiche	Bewegungen
Bubble Sort	$O(n^2)$	$O(n^2)$
Selection	$O(n^2)$	$O(n)$
Insertion	$O(n \cdot \log n)$	$O(n^2)$
Merge	$O(n \cdot \log n)$	$O(n \cdot \log n)$



Quicksort



gut


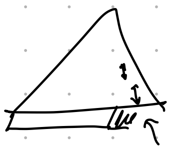
$T(n) \leq 2 \cdot T(\frac{n}{2}) + cn \leq O(n \log n)$

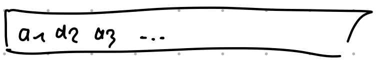
schlecht


$T(n) \leq T(n-1) + cn \leq O(n^2)$

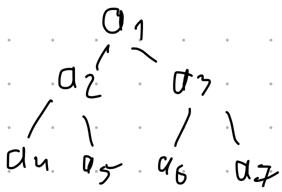
Heaps

value (knoten) $\geq$ value (kinder)





$\Rightarrow$



- Laufzeit von Codeschnipseln ||
  - $\Rightarrow$ Summenformel
- Invarianten |||
  - ↳ Präzise Formulierung
- Algorithmus-erstellen-Aufgaben /
  - ↳ Aus Inv Algo kreieren |
- (Induktions) Bew. von Ungleichungen ||
- Abschätzungen mit n! |