23. a.

$$\forall j \in \mathbb{N}, \; S_j \leq 1$$

$$S_j = \sum_{i=(2^{j-1})+1}^{2^j} \left(\frac{1}{i}\right) \quad \left.\right\} \quad \underbrace{\frac{i}{n+1} + \frac{i}{n+2} + \cdots \frac{i}{2n}}_{n \text{ terms}}$$

$2^j \rightarrow 2n$

$(2^{j-1}) \rightarrow n$

$$= \sum_{i=n+1}^{2n} \left(\frac{1}{i}\right)$$

$$\leq n \cdot \left(\frac{1}{n+1}\right) = \frac{n}{n+1}$$

$$\leq \underline{\underline{1}}$$

$\blacksquare$

2.3 b)

To prove: For any $j \in \mathbb{N}$, $\sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i} \geq \frac{1}{2}$

$n = 2^{j-1}$

$$\sum_{i=n+1}^{2n} \frac{1}{i} \geq \frac{1}{n+1} \geq \frac{1}{2^{j-1}+1} \geq \frac{1}{2}$$

2.3 c)

To prove: For any $k \in \mathbb{N}_0$:

$$\sum_{i=1}^{2^k} \frac{1}{i} \leq k+1$$

Proof:

$$1 + \sum_{j=1}^{k} S_j \leq k+1$$

$$\sum_{j=1}^{k} S_j \leq k$$

$$\sum_{j=1}^{k} S_j \leq \sum_{j=1}^{k} 1 \leq k$$

To prove: For any $k \in \mathbb{N}_0$:

$$\sum_{i=1}^{2^k} \frac{1}{i} \geq \frac{k+1}{2}$$

Proof:

$$1 + \sum_{j=1}^{k} S_j \geq \frac{k+1}{2}$$

$$2 + 2\sum_{j=1}^{k} S_j \geq k+1$$

$$2\sum_{j=1}^{k} S_j \geq k-1$$

$$2\sum_{j=1}^{k} S_j \geq 2\sum_{j=1}^{k} \frac{1}{2} \geq 2 \cdot k \cdot \frac{1}{2} \geq k \geq k-1$$

2.5.a

```
for(a in range(1,n))
    for(b in range(1,n))
        for(c in range(1,n))
            for(d in range(1,n))
                if( f(a)+f(b)+f(c) = f(d))
                    output("YES")
output("NO")
```

$n \cdot n \cdot n \cdot n = n^4$

$\therefore \; O(n^4)$

2.5b

```
array: possible_sums = [false] * n^3
for(a in range(1,n))
    for(b in range(1,n))
        for(c in range(1,n))
            sum_abc = f(a)+f(b)+f(c)
            if(sum_abc in range(1,n^3))
                possible_sums(sum_abc) = true
for(d in range(1,n)
    if(possible_sums(f(d))
        output("YES")
output("NO")
```

$(n \cdot n \cdot n) + n = O(n^3)$