DMath_u6_bf

6.5

a)

Prove:
Let A, B be sets. If A is uncountable and A $\succeq$ B then B is uncountable.

Proof:
(using Lemma 3.15.(ii): The relation $\succeq$ is transitive: $A \succeq B \land B \succeq C \implies A \succeq C$)
(using Definition 3.42.(iii): A set A is called countable if $A \preceq N$, and *uncountable* otherwise)

$\mathbb{N} \succeq A \land A \succeq B \implies \mathbb{N} \succeq B$
■

b)

Prove:
The set $S = \{\text{functions } \{0,\, 1\} \rightarrow \{0,\, 1\}^{\infty}\}$ is uncountable.

Proof:
We will prove this using contradiction. Let's assume the set S is countable, so $S \sim \mathbb{N}$. This means, that there is a one to one mapping onto each unique value (bijection) between functions f_n to $\mathbb{N}$. Let us define f_n as follows:

$f_n \stackrel{def}{=} \beta_{n,\,0},\ \beta_{n,\,1},\ \beta_{n,\,2},\ \beta_{n,\,3},\ \ldots$
For some $n \in \mathbb{N}$

Let $\beta_{n,\,i}$ be the i-th bit in the n-th sequence f_n where for convenience we begin numbering the bits with $i = 0$.

Let $\overline{b}$ be the complement of a bit $b \in \{0,\, 1\}$.

We define a new semi-infinite binary sequence α as follows:
$\alpha \stackrel{def}{=} \overline{\beta_{n,\,0}},\ \overline{\beta_{n,\,1}},\ \overline{\beta_{n,\,2}},\ \overline{\beta_{n,\,3}},\ \ldots$

Obviously, $\alpha \in \{0,\, 1\}^{\infty}$ but there is no $n \in \mathbb{N}$ such that $\alpha = f_n$ since α is constructed so as to disagree in at least one bit (actually the i-th bit) with every sequence f_n for $n \in \mathbb{N}$. This shows that there cannot be an bijection from f_n to $\mathbb{N}$, which concludes the proof. We have shown that $\mathbb{N} \succeq S$ and S is thus uncountable using Cantor's diagonalization argument.
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