DMath_U7_bf

7.3

Prove that for all positive integers a, b, c:

$$gcd(a,\, lcm(b,\, c)) = lcm(gcd(a,\, b),\, gcd(a,\, c))$$

we define the variables $a,\, b,\, c$ as products of primes to the powers $e_i,\, f_i$ and g_i respectively:

$$a = \prod_i p_i^{e_i}$$

$$b = \prod_i p_i^{f_i}$$

$$c = \prod_i p_i^{g_i}$$

this gives us the definitions for gcd(a, b, c) and lcm(a, b, c):

$$gcd(a,\, b) = \prod_i p_i^{min(e_i,\, f_i)}$$

$$lcm(a,\, b) = \prod_i p_i^{max(e_i,\, f_i)}$$

since $e_i,\, f_i,\, g_i$ are the powers of the prime number p_i at index i the equation to prove resolves to:

$$\prod_i p_i^{min(e_i,\, max(f_i,\, g_i))} = \prod_i p_i^{max(min(e_i,\, f_i),\, min(e_i,\, g_i))}$$

we will prove the equation using case distinction. we only need to concern ourselves with these cases:

case 1:

$e_i \leq f_i,\, g_i$

If e_i is less than or equal to both f_i and g_i, then the minimum of e_i with anything will be e_i.

$min(e_i, max(f_i, g_i)) = min(e_i, g_i) = e_i$
$max(min(e_i, f_i), min(e_i, g_i)) = max(e_i, e_i) = e_i$

So, both sides of the equation will be equal.

case 2:

$e_i \geq f_i,\, g_i$
If e_i is greater than or equal to both f_i and w, then the maximum of f_i and g_i will be either f_i or g_i (whichever is greater).

$min(e_i, max(f_i, g_i)) = max(f_i, g_i)$
$max(min(e_i, f_i), min(e_i, g_i)) = max(f_i, g_i)$

In both cases, both sides of the equation will be equal.

case 3:

$f_i \leq e_i \leq g_i$

If e_i is between f_i and w, then the maximum of f_i and w will be g_i, and the minimum of e_i with g_i will be e_i.

$min(e_i, max(f_i, g_i)) = min(e_i, g_i) = e_i$
$max(min(e_i, f_i), min(e_i, g_i)) = max(e_i, e_i) = e_i$

So, both sides of the equation will be equal.

Thus, the statement is proven for all $e_i,\ f_i,\ g_i$ which corresponds to
$gcd(a,\ lcm(b,\ c)) = lcm(gcd(a,\ b),\ gcd(a,\ c))$

for all positive integers $a,\ b,\ c$
$\square$

$d = gcd(a,\ b) \overset{\cdot}{\implies} \exists u,\ v \in \mathbb{Z} \,|\, d = u \cdot a + v \cdot b$

$\forall\, a,\ b,\ u,\ v\ \in \mathbb{Z}/\{0\} \,|\, u \cdot a + v \cdot b = 1 \overset{\cdot}{\implies} gcd(a,\ b) = 1$