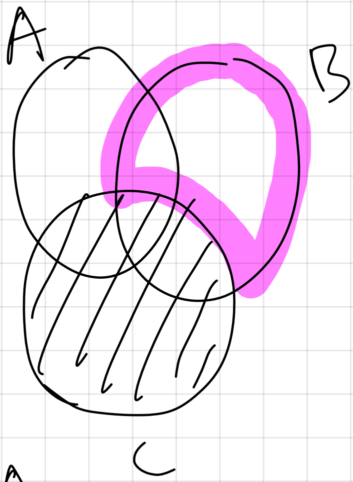
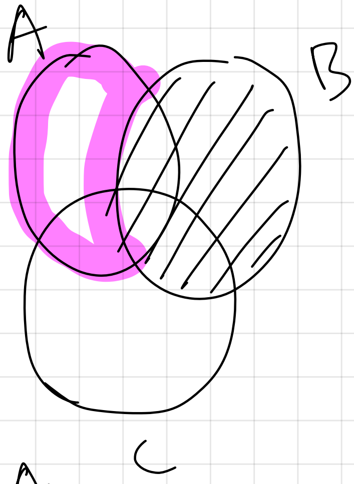
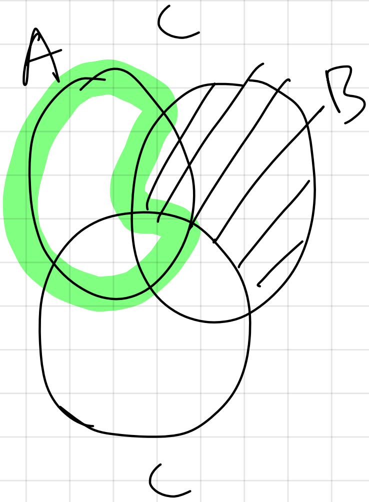
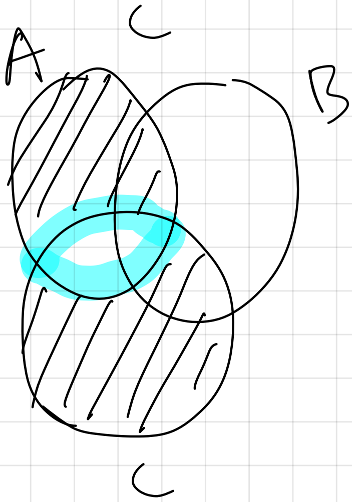
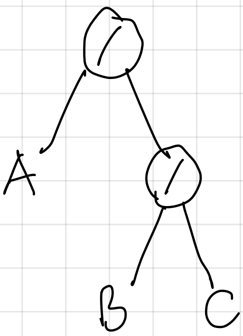
$$A\setminus(B\setminus C) = (A\setminus B)\cup(A\cap C)$$

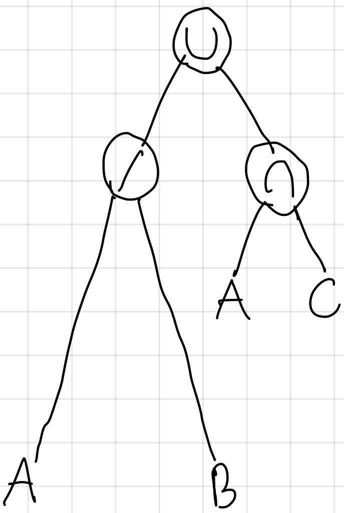


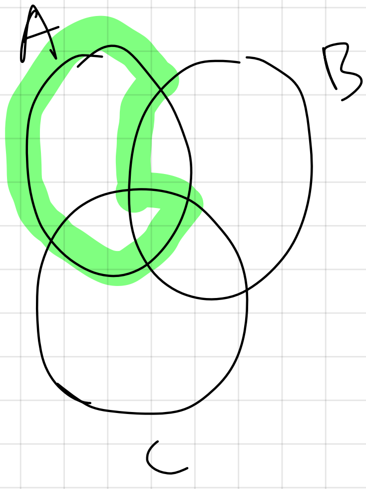












$$A\setminus B \overset{def.}{=} \{x\in A \mid x\notin B\}$$

$$A\cup B \overset{def.}{=} \{x \mid x\in A \vee x\in B\}$$

$$A\cap B \overset{def.}{=} \{x \mid x\in A \wedge x\in B\}$$

4.5 Proving/Disproving Set Properties ($\star\star$) *(8 Points)*

Prove or disprove the following statements. Argue using the definitions. You are **not allowed** to invoke properties of Theorem 3.4.

a) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$ for any sets A, B, C.

b) If $|\mathcal{P}(A) \cap \mathcal{P}(B)| = 2$ then $|A \cap B| = 1$ for any sets A, B.

c) If $B \subseteq A$ and $C \cap B \neq \varnothing$ then $C \subseteq A$ for any sets A, B, C.

a) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$ for any sets A, B, C.

$$S_L = \{x \in A \mid x \notin \{y \in B \mid y \notin C\}\}$$

$$S_R = \{x \mid x \in \{y \in A \mid y \notin B\} \vee x \in \{z \in A \wedge z \in C\}\}$$

$$(A \setminus B) \cup (A \cap C) = A \setminus (B \setminus C)$$

$(A \setminus B) \cup (A \cap C) =$ | def. $\setminus$ | def. $\cap$

$= (x \in A \land x \notin B) \lor (x \in A \land x \in C)$

$A \setminus (B \setminus C) =$ | def. $\setminus$

$A \setminus (x \in B \land x \notin C)$ | def. $\setminus$

$x \in A \land \neg(x \in B \land x \notin C)$ | de Morgan's rules

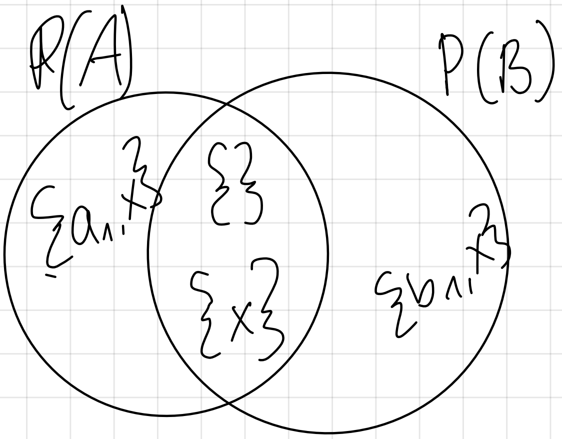
$x \in A \land (x \notin B \lor \neg\neg x \in C)$ | double negation

$x \in A \land (x \notin B \lor x \in C)$ | 1st distributive law

$(x \in A \land x \notin B) \lor (x \in A \land x \in C)$

$\square$ Proven by equivalence transformation using definitions

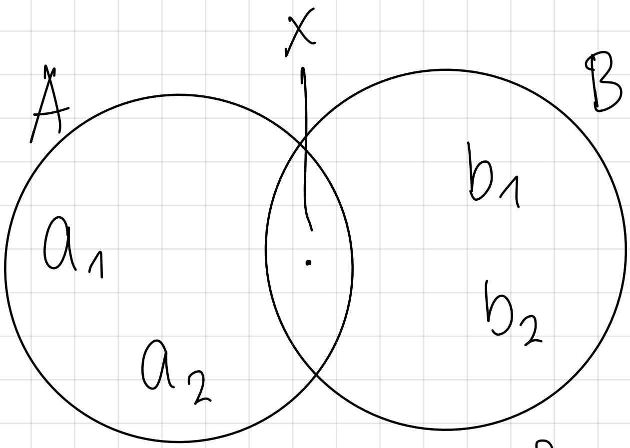
b) If $|\mathcal{P}(A) \cap \mathcal{P}(B)| = 2$ then $|A \cap B| = 1$ for any sets A, B.



$$|P(A) \cap P(B)| = 2$$

$$P(A) = \{\emptyset, \{x\}, \{a_1\}, \dots\}$$

$$P(B) = \{\emptyset, \{x\}, \{b_1\}, \dots\}$$



$$|A \cap B| = 1$$

$$\left.\begin{aligned} A &= \{x, a_1, a_2, \dots\} \\ B &= \{x, b_1, b_2, \dots\} \end{aligned}\right\} \exists x \mid x \in A \wedge x \in B$$

$$\boxed{\begin{aligned} |Z| &= k \\ |P(k)| &= 2^k \end{aligned}}$$

$$2^{|A \cap B|} = 2^{|A|} \cdot 2^{|B|}$$

$$|P(A \cap B)| \Leftrightarrow |P(A) \cap P(B)|$$

$$2^{|A \cap B|} \Leftrightarrow |P(A) \cap P(B)|$$

$$P(A) = \{X \mid X \subseteq A\} \quad | \text{def. } P$$

$$P(B) = \{Y \mid Y \subseteq B\} \quad | \text{def. } P$$

$$P(A) \cap P(B) = \{X \mid X \subseteq A\} \cap \{Y \mid Y \subseteq B\} \quad | \text{ def. } \cap$$

$$= \{Z \mid Z \subseteq A \cap B\}$$

$$P(A \cap B) = \{W \mid W \subseteq (A \cap B)\}$$

$$\therefore \quad P(A \cap B) = P(A) \cap P(B)$$

If $|P(A) \cap P(B)| = 2$, $A \cap B = 1$

since $P(A) \cap P(B) = P(A \cap B)$ we can write this as

If $|P(A \cap B)| = 2$, $A \cap B = 1$

Let's define a set S:

$S = \{A \cap B\}$

Since $|P(S)| = 2^{|S|}$

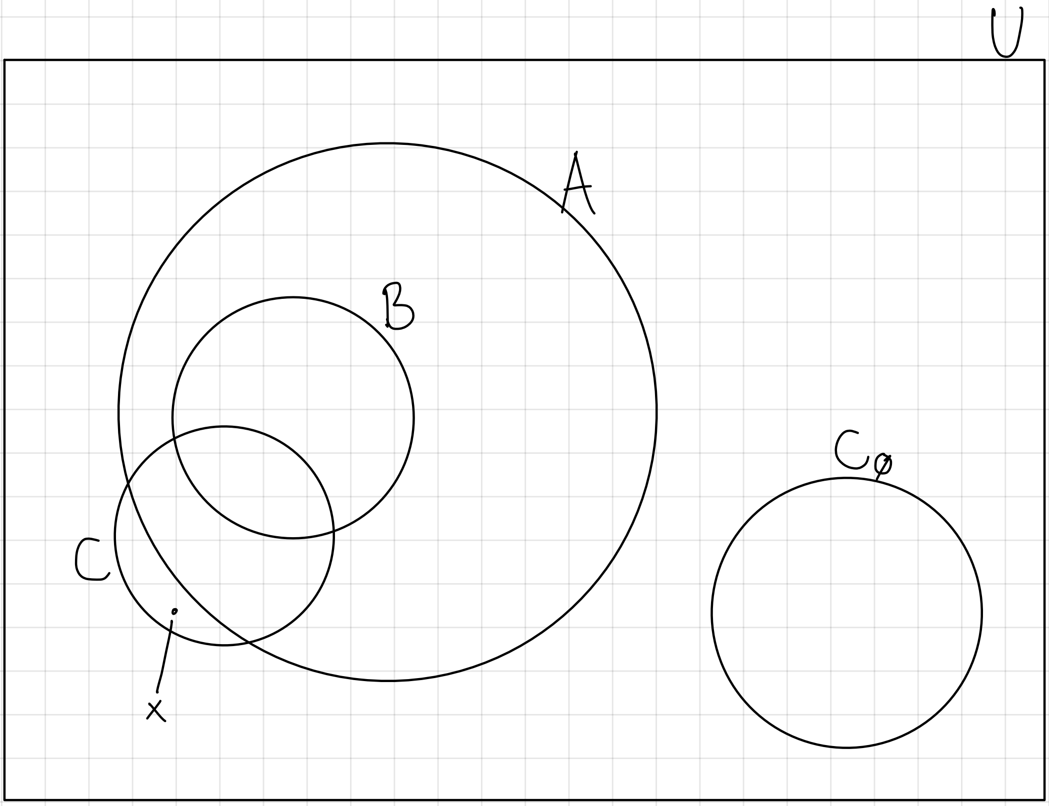
$|P(S)| = 2^1 \quad \Big| \log_2$

$|S| = 1$

$|A \cap B| = 1$

$\square$

c) If $B \subseteq A$ and $C \cap B \neq \varnothing$ then $C \subseteq A$ for any sets A, B, C.



$B \subseteq A$

$C \cap B \neq \varnothing$

$C_\varnothing \cap B = \varnothing$

$A \subseteq B \ \underline{\text{def.}}\ \forall x \{ x \in A \rightarrow x \in B \}$

$A = \varnothing \quad \text{if} \quad \forall x \neg (x \in A)$

Disproven by counterexample:

$\exists x \mid ((x \in C) \wedge (x \notin A))$

Thus the statement is disproven! It would hold with the addition of:

$C \setminus A = \varnothing$

$\square$