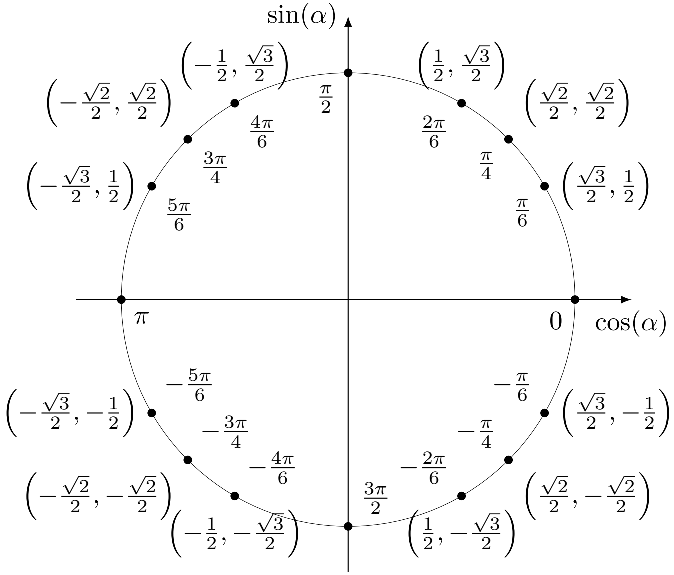

Vectorspaces

$\mathbb{F}$-Vectorspace $\langle V; +, 0, -, \cdot, 1\rangle$

(i) Additive albenian group $\langle V; +, 0\rangle$ where $+ : V \times V \to V$, $0 :\to V$ and $- : V \to V$
(ii) Scalar algebra $\langle V, \mathbb{F}; \cdot, 1\rangle$ where $\cdot : \mathbb{F} \times V \to V$ and $1 :\to \mathbb{F}$

Subspace U of V Must be closed under

(i) addition $(\forall u, v \in U \quad u + v \in U)$
(ii) scalar multiplication $(\forall v \in U, c \in \mathbb{F} \quad cv \in U)$

Fundamental Subspaces

$C(A) = \{Ax | x \in V\} = C(AA^\top)$ span of the column vectors.
$R(A) = C(A^\top) = C(A^\top A)$ span of the row vectors.
$N(A) = \{x \in V | Ax = 0\}$ nullspace, $N(A^\top)$ left nullspace.

Complement Subspaces have no vectors incommon except for 0 and when combined are equivalent to the whole vectorspace.

Orthogonal Subspaces are subspaces where all vectors of one subspace is orthogonal to all other vectors in all other orthogonal subspaces.
$N(A)/N(A^\top)$ orthogonal complement to $C(A^\top)/C(A)$.

Determinants

$\det(A) = 0 \iff A$ is sigular. $C(A) \neq \mathbb{F}^n$
$\det(A) \neq 0 \iff A$ is regular. $C(A) = \mathbb{F}^n$
$\det(A)$ is linear in each row.
$|\det(Q)| = 1$ for any orthogonal matrix Q.
$\det(cA) = c^n \det(A)$.
A diagonal or triangular $\implies \det(A) = \prod_{i=1}^n a_{ii}$.
$\det(AB) = \det(A)$,$\det(A^\top) = \det(A)$

Swapping rows of A changes the sign of $\det(A)$.
Non swapping row ops on A do not change $\det(A)$.

$$\det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \det(A)\det(B)$$

Choose row j then $\det(A) = \sum_{i=1}^n a_{ji} \det(C_{ji})$
where C_{ji} is A without row j and column i.

Solving LSEs $A^{m\times n}x = b$

Overdetermined $m < n$
Underdetermined $m > n$
Number of solutions

#of sol.	$r = n$	$r < n$
$r = m$	1	∞
$r < m$	0 / 1	0 / ∞

$PA = LU$ Decomposition
P is the permutation of the rows during gauss.
L is a lower triangular containing the gauss operations.
U is a upper triangular being the gaussed matrix.

Solving LSEs with $PA = LU$
solve $Lc = Pb$ for c and then $Ux = c$ for x.

$A = CR$ Decomposition
C are the linear independent column of A. $C(A) = C(C)$
R is the rref form of A.

Projections

$\text{proj}_{\text{span}\{a\}}(b) = \frac{aa^\top}{a^\top a} b$
$\text{proj}_{C(A)}(b) = A\tilde{x}$ where $A^\top A\tilde{x} = A^\top b$ (normal eq)
if $A^\top A$ is invertible $\text{proj}_{C(A)}(b) = A\left(A^\top A\right)^{-1} A^\top b$

LeastSquare
fitting $(x_{1,*}$,$x_{n,*}$,$y_*)$ to $\alpha_1 x_1 + \dots \alpha_n x_n = y$ for $0 \leq * \leq m$

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

fitting $\alpha_0 + \alpha_1 t = b$

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} m & \Sigma_{k=1}^m t_k \\ \Sigma_{k=1}^m t_k & \Sigma_{k=1}^m t_k^2 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_{k=1}^m b_k \\ \Sigma_{k=1}^m t_k b_k \end{bmatrix} = \begin{bmatrix} \frac{1}{m}\sum_{k=1}^m b_k \\ \frac{\sum_{k=1}^m t_k b_k}{\sum_{k=1}^m t_k^2} \end{bmatrix}$$

$A = QR$ Decomposition
R is a upper triangular matrix given by $R = Q^\top A$.
Q is a orthogonal matrix.

Algorithm 1 Gram-Schmidt

1: $q_1 = \frac{a_1}{\|a_1\|}$
2: **for** $k = 2, \dots, n$ **do**
3: $\quad q_k' = a_k - \sum_{i=1}^{k-1} (a_k^\top q_i) q_i$
4: $\quad q_k = \frac{q_k'}{\|q_k'\|}$
5: **end for**

Solving LeastSquare with $A = QR$
$R\hat{x} = Q^\top b$

Pseudoinverse $A^\dagger$
$\text{rank}(A) = n$ then $A^\dagger A = I$ where $A^\dagger = (A^\top A)^{-1} A^\top$.
$\text{rank}(A) = m$ then $AA^\dagger = I$ where $A^\dagger = A(AA^T)^{-1}$.
$A = CR$ then $A^\dagger = R(RR^\top)^{-1}(C^\top C)^{-1} C^\top$

Eigenvalues and -vectors $Av = \lambda v$

Calculating λ: $\det(A - \lambda I) = 0$
Calculating v of λ: $(A - \lambda I)v = 0$

A with distinct $\lambda_1, \dots, \lambda_k$

$v_1, \dots, v_k$ linearly independent.
$\{v_1, \dots, v_k\}$ is a basis for $C(A)$.
are the same as the ones of $A^\top$.

Geometric Multiplicity of λ: $\dim(N(A - \lambda I))$

Similar Matrices $C = T^{-1}AT$
$\mathrm{Tr}(A) = \mathrm{Tr}(C), \det(A) = \det(C)$
$\mathrm{Tr}(A) = \sum_{i=1}^n \lambda_i, \quad \det(A) = \prod_{i=1}^n \lambda_i$

Spectral Theorem
Any symmetric matrix $A \in \mathbb{R}^{n \times n}$ has n real eigenvalues and an orthonormal basis of eigen-vectors of A.

$A = V\Lambda V^\top$
The columns of V are the eigen-vectors of A,
Λ is a diagonal matrix with corresponding eigen-values.
$A^m = V\Lambda^m V^\top$

Raylight Quotient $R(x) = \frac{x^\top Ax}{x^\top x}$

Positive (Semi-) Definite
eigen-values $> / \geq 0 \iff x^\top Ax > / \geq 0$

Gram-Matrix $A^\top A$
$A^\top A$ and $AA^\top$ have the same non-zero eigen-values.

Cholesky Decomposition $M = C^\top C$
M symmetric PSD, C upper trianular.

SVD $A = U\Sigma V^\top$

$U \in \mathbb{R}^{m \times m}$ orthogonal ($U^\top U = I$) eigenvectors of $AA^\top$
$V \in \mathbb{R}^{n \times n}$ orthogonal ($V^\top V = I$) eigenvectors of $A^\top A$
$\Sigma \in \mathbb{R}^{m \times n}$ diagonal $\Sigma_{ii} = \sigma_i$ sigular value.
Singular values are the square roots of the non-zero eigen-values of $AA^\top$ / $A^\top A$

Calculate SVD

(i) Caclulate $A^\top A / AA^\top$

(ii) Find eigenvalues $\lambda_1, \dots, \lambda_{n/r}$ of $A^\top A / AA^\top$

(iii) $\Sigma_r = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n/r} \end{bmatrix}$

(iv) $\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}$

(v) calculate eigenvectors $v_1, \dots, v_{n/r}$ of $A^\top A$

(vi) norm eigenvectors $\frac{v_i}{\|v_i\|}$

(vii) write V with the eigenvectors as columns

(viii) Solve $U_r = AV\Sigma^{-1}$

(ix) If U_r does not have the right dimensions, we have to apply gram-schmidt

(x) $A = U\Sigma V^\top$