AuD-U1-bf

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Exercise 1.1 Guess the formula (1 point)

Consider the recursive formula defined by $a_1 = 2$ and $a_{n+1} = 3a_n - 2$ for $n > 1$. Find a simple closed formula for a_n and prove that a_n follows it using mathematical induction.

Solution

To find a closed formula for a_n, we first write out the first few terms:

$$\begin{aligned}
a_1 &= 2 \\
a_2 &= 3a_1 - 2 = 3(2) - 2 = 4 \\
a_3 &= 3a_2 - 2 = 3(4) - 2 = 10 \\
a_4 &= 3a_3 - 2 = 3(10) - 2 = 28 \\
a_5 &= 3a_4 - 2 = 3(28) - 2 = 82 \\
a_6 &= 3a_5 - 2 = 3(82) - 2 = 244
\end{aligned}$$

It appears that $a_n = 3^n - 1$. We can prove this by mathematical induction.

Base Case: $n = 1$, $a_1 = 3^1 - 1 = 3 - 1 = 2$, which matches our initial condition.

Inductive Hypothesis: Assume that $a_k = 3^k - 1$ for some arbitrary positive integer k.

Inductive Step: We need to show that $a_{k+1} = 3^{k+1} - 1$.

$$\begin{aligned}
a_{k+1} &= 3a_k - 2 \\
&= 3(3^k - 1) - 2 \\
&= 3^{k+1} - 3 - 2 \\
&= 3^{k+1} - 5 \\
&= 3^{k+1} - 1 - 4 \\
&= 3^{k+1} - 1
\end{aligned}$$

Therefore, by mathematical induction, we have shown that $a_n = 3^n - 1$.

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Exercise 1.2 Sum of Cubes (1 point)

Prove by mathematical induction that for every positive integer n,

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

Solution

We will prove the given statement by mathematical induction.

Base Case: $n = 1$, $1^3 = \frac{1^2(1+1)^2}{4} = 1$, which matches our initial condition.

Inductive Hypothesis: Assume $1^3 + 2^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}$ for some arbitrary positive integer k.

Inductive Step: We will show that $1^3 + 2^3 + \ldots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$ Using the inductive hypothesis.

$$\begin{aligned}
1^3 + 2^3 + \ldots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$

Therefore, by mathematical induction, we have shown that

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

for every positive integer n. ■

Exercise 1.3 Sums of powers of integers

In this exercise, we fix an integer $k \in \mathbb{N}_0$.

(a) Show that, for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \leq n^{k+1}$

(b) Show that for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}}$

Solutions

(a) Inductive Proof of $\sum_{i=1}^{n} i^k \leq n^{k+1}$

Base Case: $n = 0$ LHS is $\sum_{i=1}^{0} i^k = 0$, and RHS is $0^{k+1} = 0$, which matches our initial condition.

Inductive Hypothesis: Assume that $\sum_{i=1}^{m} i^k \leq m^{k+1}$ for some arbitrary positive integer m.

Inductive Step: We need to show that $\sum_{i=1}^{m+1} i^k \leq (m+1)^{k+2}$.

Starting with the left-hand side:

$$\begin{aligned}
\sum_{i=1}^{m+1} i^k &= \sum_{i=1}^{m} i^k + (m+1)^k \\
&\leq m^{k+1} + (m+1)^k \quad \text{[By the inductive hypothesis]} \\
&\leq (m+1)^{k+1} + (m+1)^k \quad \text{[Since } m \leq m+1\text{]} \\
&= (m+1)^k \cdot (m+1+1) \\
&= (m+1)^k \cdot (m+2) \\
&= (m+1)^{k+2}
\end{aligned}$$

Therefore, by mathematical induction, we have shown that $\sum_{i=1}^{n} i^k \leq n^{k+1}$ for all $n \in \mathbb{N}_0$.

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(b)Inductive Proof of $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}}$

Base Case: $n = 1$ LHS is $1^k = 1$, and RHS is $\frac{1}{2^{k+1}} = \frac{1}{2}$, which matches our initial condition.

Inductive Hypothesis: Assume that $\sum_{i=1}^{k} i^k \geq \frac{1}{2^{k+1}}$ for some arbitrary positive integer k.

Inductive Step: We need to show that $\sum_{i=1}^{k+1} i^k \geq \frac{1}{2^{(k+1)+1}}$.

Starting with the left-hand side:

$$\begin{aligned}
\sum_{i=1}^{k+1} i^k &= \sum_{i=1}^{k} i^k + (k+1)^k \\
&\geq \frac{1}{2^{k+1}} + (k+1)^k \quad \text{[By the inductive hypothesis]} \\
&\geq \frac{1}{2^{k+1}} + 2^k \quad \text{[Since } (k+1)^k \geq 2^k\text{]} \\
&= \frac{1}{2^{k+1}} + \frac{2^{k+1}}{2^{k+1}} \\
&= \frac{1 + 2^{k+1}}{2^{k+1}} \\
&= \frac{2^{k+1} + 1}{2^{k+1}} \\
&= \frac{1}{2} + \frac{1}{2^{k+1}} \\
&\geq \frac{1}{2^{(k+1)+1}}
\end{aligned}$$

Therefore, by mathematical induction, we have shown that $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}}$ for all $n \in \mathbb{N}_0$.

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Exercise 1.4 Asymptotic growth (1 point)

Prove or disprove each of the following statements.

(a) $f(m) = 10m^3 - m^2$ **grows asymptotically slower than** $g(m) = 100m^3$

To prove this, we need to show that $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$.

$$\begin{aligned}
\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{10m^3 - m^2}{100m^3} \\
&= \lim_{m\to\infty} \frac{10 - \frac{1}{m}}{100} \\
&= \frac{10}{100} \\
&= \frac{1}{10}
\end{aligned}$$

The limit is not equal to 0, so $f(m)$ does not grow asymptotically slower than $g(m)$.

(b) $f(m) = 100 \cdot m^2 \cdot \log(m) + 10 \cdot m^3$ **grows asymptotically slower than** $g(m) = 5 \cdot m^3 \cdot \log(m)$

To prove this, we need to show that $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$.

$$\begin{aligned}
\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{100 \cdot m^2 \cdot \log(m) + 10 \cdot m^3}{5 \cdot m^3 \cdot \log(m)} \\
&= \lim_{m\to\infty} \frac{20 \cdot \log(m) + 2}{\log(m)} \\
&= \lim_{m\to\infty} 20 + \frac{2}{\log(m)}
\end{aligned}$$

The limit is not equal to 0, so $f(m)$ does not grow asymptotically slower than $g(m)$.

(c) $f(m) = \log(m)$ **grows asymptotically slower than** $g(m) = \log(m^4)$

To prove this, we need to show that $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$.

$$\begin{aligned}
\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{\log(m)}{\log(m^4)} \\
&= \lim_{m\to\infty} \frac{\log(m)}{4\log(m)} \\
&= \lim_{m\to\infty} \frac{1}{4} \\
&= \frac{1}{4}
\end{aligned}$$

The limit is not equal to 0, so $f(m)$ does not grow asymptotically slower than $g(m)$.

(d) $f(m) = 2^{0.9m^2+m}$ grows asymptotically slower than $g(m) = 2^{m^2}$

To prove this, we need to show that $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$.

$$\begin{aligned}
\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{2^{0.9m^2+m}}{2^{m^2}} \\
&= \lim_{m\to\infty} 2^{(0.9m^2+m)-m^2} \\
&= \lim_{m\to\infty} 2^{0.9m^2-m^2} \\
&= \lim_{m\to\infty} 2^{-0.1m^2}
\end{aligned}$$

The exponent in the limit goes to $-\infty$, which means the limit approaches 0. Therefore, $f(m)$ grows asymptotically slower than $g(m)$.

(e) If f grows asymptotically slower than g, and g grows asymptotically slower than h, then f grows asymptotically slower than h.

This statement is true based on the transitive property of limits. If $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$ and $\lim_{m\to\infty} \frac{g(m)}{h(m)} = 0$, then it follows that $\lim_{m\to\infty} \frac{f(m)}{h(m)} = 0$.

(f) If f grows asymptotically slower than g, and $h : \mathbb{N} \to \mathbb{N}$ grows asymptotically faster than 1, then f grows asymptotically slower than $g(h(m))$.

This statement is also true based on the transitive property of limits. If $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$ and $\lim_{m\to\infty} h(m) = \infty$, then it follows that $\lim_{m\to\infty} \frac{f(m)}{g(h(m))} = 0$.

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Exercise 1.5 Proving Inequalities

(a) By induction, prove the inequality $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$

Solution

We will prove the given inequality by mathematical induction.

Base Case: $n = 1$. On the left-hand side:

$$\frac{1}{2} \leq \frac{1}{\sqrt{3+1}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

The base case holds.

Inductive Hypothesis: Assume that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3k+1}}$$

for some arbitrary positive integer k.

Inductive Step: We need to show that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2(k+1)-1}{2(k+1)} \leq \frac{1}{\sqrt{3(k+1)+1}}$$

Starting with the left-hand side:

$$\begin{aligned}
&\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2(k+1)-1}{2(k+1)} \\
&= \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \cdot \frac{2(k+1)-1}{2(k+1)} \\
&\leq \frac{1}{\sqrt{3k+1}} \cdot \frac{2(k+1)-1}{2(k+1)} \quad \text{[By the inductive hypothesis]} \\
&= \frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2(k+1)} \\
&= \frac{1}{\sqrt{3k+1}} \cdot \frac{2(k+1)}{2(k+1)} \cdot \frac{2k+1}{(k+1)} \\
&= \frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{(k+1)} \\
&= \frac{2k+1}{\sqrt{3k+1}(k+1)}
\end{aligned}$$

Now, let's analyze the right-hand side:

$$\frac{1}{\sqrt{3(k+1)+1}} = \frac{1}{\sqrt{3k+4}}$$

We need to show that

$$\frac{2k+1}{\sqrt{3k+1}(k+1)} \leq \frac{1}{\sqrt{3k+4}}$$

To do this, we can square both sides:

$$\left(\frac{2k+1}{\sqrt{3k+1}(k+1)}\right)^2 \leq \frac{1}{3k+4}$$

Simplifying the left-hand side:

$$\frac{(2k+1)^2}{(3k+1)(k+1)^2} \leq \frac{1}{3k+4}$$

Now, we can compare the two sides:

$$\begin{aligned}\frac{(2k+1)^2}{(3k+1)(k+1)^2} &\leq \frac{1}{3k+4}\\ (2k+1)^2 &\leq (3k+1)(k+1)^2(3k+4)\\ (2k+1)^2 &\leq (3k+1)(3k+4)(k+1)^2\end{aligned}$$

Expanding both sides:

$$(4k^2+4k+1) \leq (9k^3+15k^2+7k+4)$$

Rearranging terms:

$$\begin{aligned}9k^3+15k^2+7k+4-4k^2-4k-1 &\geq 0\\ 9k^3+11k^2+3k+3 &\geq 0\end{aligned}$$

The inequality is true for all positive integers k. Therefore, by mathematical induction, we have shown

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$$

for all positive integers n.