DMath_U11_bf

11.4

Let $\Sigma = (\mathcal{S}, \mathcal{P}, \tau, \phi)$ be a proof system. Consider the proof system. $\overline{\Sigma} = (\mathcal{S}, \mathcal{P}, \overline{\tau}, \overline{\phi})$, where for all $s \in \mathcal{S}$ and $p \in \mathcal{P}$ we define

$$\overline{\tau}(s) = 1 \iff \tau(s) = 0,$$
$$\overline{\phi}(s,p) = 1 \iff \phi(s,p) = 0.$$

Prove or disprove the following statements.
a) If Σ is sound, then $\overline{\Sigma}$ is complete.
b) If Σ is complete, then $\overline{\Sigma}$ is sound.

Through tertium non datur we can assume;

$$\overline{\tau}(s) = 0 \iff \tau(s) = 1,$$
$$\overline{\phi}(s,p) = 0 \iff \phi(s,p) = 1.$$

> **Definition 6.2.**
>
> A Proof System is sound if no false statement has a proof, i.e. for all statements for which there exists a proof $p \in \mathcal{P}$ such that the verification function returns true $\phi(s,p) = 1$, the statement must be true $\tau(s) = 1$.

> **Definition 6.3.**
>
> A Proof System is complete if every true statement has a proof, i.e. for all statements $s \in \mathcal{S}$ that are true, there exists a proof $p \in \mathcal{P}$ such that the verification function returns true $\phi(s,p) = 1$.

a)

For the scope of this exercise, we assume that $\mathcal{S} \neq \varnothing$ and $\mathcal{P} \neq \varnothing$.

If Σ is sound that means "for all statements $s \in \mathcal{S}$ for which there exists a $p \in \mathcal{P}$ with $\phi(s,p) = 1$ we have $\tau(s) = 1$", which is the same as to say that "there does not exist an $s \in \mathcal{S}$ with $\tau(s) = 0$ such that there exists a $p \in \mathcal{P}$ with $\phi(s,p) = 1$".

Which is the same as to say that "there does not exist an $s \in \mathcal{S}$ with $\overline{\tau}(s) = 1$ such that there exists a $p \in \mathcal{P}$ with $\overline{\phi}(s,p) = 0$". Since $\overline{\tau}(s)$ and $\overline{\phi}(s,p)$ in $\overline{\Sigma}$ are, per definition the opposite of $\tau(s)$ and $\phi(s,p)$ in Σ.

Which is to say that "for all $s \in \mathcal{S}$ with $\overline{\tau}(s) = 1$ there exists a $p \in \mathcal{P}$ such that $\overline{\phi}(s,p) = 1$". All this essentially tells us, that if Σ is sound, there is no $s \in \mathcal{S}$ in $\overline{\Sigma}$ for which $\overline{\tau}(s) = 1$ and $\overline{\phi}(s,p) = 0$.

So all true statements in $\overline{\Sigma}$ have a proof $p \in \mathcal{P}$ for which $\overline{\phi}(s,p) = 1$. Thus, if Σ is sound $\overline{\Sigma}$ is complete (if we assume $\mathcal{S}, \mathcal{P} \neq \varnothing$).
$\square$

If we consider $\mathcal{S}$ and $\mathcal{P}$ to possibly be the empty set, the implication could be disproven by contradiction as follows:

> Let $\Sigma = \{\mathcal{S}, \mathcal{P}, \tau, \phi\}$ where $\mathcal{S} = \{0\}$, $\mathcal{P} = \varnothing$, $\tau(0) = 0$ and $\phi : \mathcal{S} \times \mathcal{P} \to \{0,1\}$
> (Definition of cartesian product between set and empty set implies $\mathcal{S} \times \varnothing = \varnothing$)
>
> Let $\overline{\Sigma} = \{\mathcal{S}, \mathcal{P}, \overline{\tau}, \overline{\phi}\}$ where $\mathcal{S} = \{0\}$, $\mathcal{P} = \varnothing$, $\overline{\tau}(0) = 1$ and $\overline{\phi} : \mathcal{S} \times \mathcal{P} \to \{0,1\}$

Σ is sound, as there is no $s \in \mathcal{S}$ such that $\tau(s) = 1$ (Definition of Sound). Let's assume, for the sake of contradiction, that the implication holds, i.e. for all $\overline{\tau}(s) = 1$ there exists a $p \in \mathcal{P}$ such that $\overline{\phi}(s,p) = 1$. However, we arrive at a contradiction, as we defined $\mathcal{P} = \varnothing$. Thus there exists no $p \in \mathcal{P}$ such that $\overline{\phi}(s,p) = 1$ if we consider $\mathcal{P}$ to be the empty set.

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b)

For the scope of this exercise we again assume that $\mathcal{S} \neq \varnothing$ and $\mathcal{P} \neq \varnothing$.

If Σ is complete that means "for all statements $s \in \mathcal{S}$ with $\tau(s) = 1$ there exists a proof $p \in \mathcal{P}$ such that $\phi(s, p) = 1$".

Which (per definition) is the same as to say "for all statements $s \in \mathcal{S}$ with $\overline{\tau}(s) = 0$ there exists a proof $p \in \mathcal{P}$ such that $\overline{\phi}(s, p) = 0$ in $\overline{\Sigma}$".

Which gives us no further information on the soundness of the proof system $\overline{\Sigma}$, as all we know is that for all false statements there exists a proof, such that the verification of that statement with that proof is false.

Let's disprove the implication with a counterexample:

> Let $\Sigma = \{\mathcal{S}, \mathcal{P}, \tau, \phi\}$ where $\mathcal{S} = \{0, 1\}$, $\mathcal{P} = \{0, 1\}$, $\tau(1) = 1,\ \tau(0) = 0$ and $\phi(1, 0) = 0,\ \phi(1, 1) = 1,\ \phi(0, 0) = 0,\ \phi(0, 1) = 0.$
>
> Let $\overline{\Sigma} = \{\mathcal{S}, \mathcal{P}, \tau, \phi\}$ where $\mathcal{S} = \{0, 1\}$, $\mathcal{P} = \{0, 1\}$, $\overline{\tau}(1) = 0,\ \overline{\tau}(0) = 1$ and $\overline{\phi}(1, 0) = 1,\ \overline{\phi}(1, 1) = 0.$

As we can clearly see, Σ is complete (per definition of complete) but $\overline{\Sigma}$ is not sound (since there exist a $s \in \mathcal{S}$ such that $\overline{\tau}(s) = 0$ but $\overline{\phi}(s, p) = 1$, i.e. a false statement has a proof). The implication is thus disproven by counterexample.

$\square$