3.2 From Natural Language to a Formula (*)

Consider the universe $U = \mathbb{R}$. Express each of the following statements with a formula in predicate logic, in which the only predicates appearing are less(x, y), equals(x, y), and integer(x). Instead of less(x, y) and equals(x, y) you can write x < y and x = y. You can also use the symbols + and · to denote the addition and multiplication functions, and you can use constants (e.g., 0, 1, . . .). Do not use division. No justification is required.

---

i) There exists two positive integers whose sum is negative.

not defined – you should use the predicate integer(·) and the predicate less(·) to obtain positive integers.

$\exists x,\, y \in \mathbb{Z}^{+} (x + y) < 0$

---

ii) Any real number is not greater than all rational numbers.

not defined

$\forall x \in \mathbb{Q} \neg\exists y \in \mathbb{R},\, x < y$

---

iii) If for every pair of real numbers there exist an integer which is smaller than one of the two and larger than the other, then all real numbers are greater than zero.

also needs to capture $(x<i) \wedge (i<y)$

$\forall (x,\, y \in \mathbb{R})\, \exists i \in \mathbb{Z},\, (y < i < x) \to (\forall r \in \mathbb{R},\, 0 < r)$

not defined.

this is only defined as a binary predicate so you need to write $(y<i) \wedge (i<x)$.

---

iv) All integers whose sum is odd have different parity.

$\forall (x,\, y \in \mathbb{R}),\, \exists a \in \mathbb{R},\, (x + y = 2a + 1),\, ((x = 2a + 1) \wedge (y = 2a)) \vee ((x = 2a) \wedge (y = 2a + 1))$

---

missing quantifier

integer(a) $\wedge$

here you need a $\to$. and you must define new variables instead of reusing a. Otherwise, this states that $x+y=2a+1$ and $x=2a+1$ and $y=2a$ all the same a.

0/4

Use instead $\exists b \exists c$ (integer(b) $\wedge$ integer(c)) $\wedge$ $(((x=2b+1) \wedge (y=2c)) \vee ((x=2b) \wedge (y=2c+1)))$

3.7 For each of the following proof patterns, prove or disprove that it is sound.

---

a) To prove a statement S, find two appropriate statements T_1 and T_2. Assume that S is false and show (from this assumption) that one between the statements T_1 and T_2 is true. Then show that one statement between T_1 and T_2 is false.

1. Assume $\neg S$
2. Show that, from this assumption, one between T_1 and T_2 is true.
3. Show that one between T_1 and T_2 is false.

Almost → this should translate to $(\neg S \to (T_1 \vee T_2)) \wedge \neg(T_1 \wedge T_2)$

S	T_1	T_2	$\neg S$	$(T_1 \vee T_2) \wedge \neg(T_1 \wedge T_2)$	$\neg S \to ((T_1 \vee T_2) \wedge \neg(T_1 \wedge T_2))$ ×
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	1	1	1
0	1	1	1	0	0
1	0	0	0	0	1
1	0	1	0	1	1
1	1	0	0	1	1
1	1	1	0	0	1

1/2

Thus shows, that the proof pattern holds, except for cases where S is false and one of T_1 and T_2 is true, the other one being false. In those cases it produces false positives.

---

b) To prove an implication $S \implies T$, find an appropriate statement R. Assume that S is true and T is false, and prove that (from these assumptions) R is true. Then show that R is false.

1. Assume S and $\neg T$
2. From this assumption, prove that R is true.
3. Show that R is false.

✓

S	T	R	$(S \wedge \neg T) \to R$	$\neg R$	"Proof" of $S \implies T$	Actual $S \implies T$
0	0	0	1	1	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	0	0	1
1	0	0	0	1	0	0
1	0	1	1	0	0	0
1	1	0	1	1	1	1
1	1	1	1	0	0	1

The truth table illustrates, that the proof pattern holds in all cases, except for X

a) when S and T are false, but R is true.

b) when S is false but T and R are true. 1/2

c) when S, T and R are all true.

You only care about cases where the proof technique evaluates

---

to 1, because you are proving logical consequence.

2/4