DMath_u9_bf

9.4

In this exercise, you prove the remark in Chapter 5, Footnote 20 of the lecture notes. Consider an algebra $\langle R; +, -, 0, \cdot, 1\rangle$ such that:

i) $\langle R; +, -, 0\rangle$ is a group.
ii) $\langle R; \cdot, 1\rangle$ is a monoid.
iii) $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c, \in R$.

Prove that such an algebra satisfies Definition 5.18 in the lecture notes. Each step should consist of one or more applications of the given axioms, and the axioms used should be made explicit.

Hint: consider (1 + 1)(a + b).

Proof:

Let's consider the case $(1+1)(a+b)$.

$$\begin{aligned}
(1+1)(a+b) = (1+1)(a+b) & & \text{left and right distributive law } (5.18.\text{iii}) \\
\overset{\cdot}{\iff} (1+1)a + (1+1)b = 1(a+b) + 1(a+b) & & \text{left and right distributive law } (5.18.\text{iii}) \\
\overset{\cdot}{\iff} a+a+b+b = a+b+a+b & & \text{left cancellation law} (5.3..\text{iii}) \\
\overset{\cdot}{\iff} a+b+b = b+a+b & & \text{right cancellation law } (5.3.\text{iv}) \\
\overset{\cdot}{\iff} a+b = b+a & &
\end{aligned}$$

□

→ you are missing a step (on both sides) applying the property that 1 is the neutral element. −2

We have proven that the operation $+$ on the algebra $\langle R; +, -, 0, \cdot, 1\rangle$ is commutative. The algebra is thus proven to be a ring.

$\frac{6}{8}$

Definition 5.18.
A ring $\langle R; +, -, 0, \cdot, 1\rangle$ is an algebra for which
(i) $\langle R; +, -, 0\rangle$ is a (commutative) group
(ii) $\langle R; \cdot, 1\rangle$ is a monoid.
(iii) $a(b+c) = (ab) + (ac)$ and $(b+c)a = (ba) + (ca)$ for all $a, b, c \in R$ (left and right distributive laws).
A ring is called commutative if multiplication is commutative (ab = ba)

Lemma 5.17.
For any ring $\langle R; +, -, 0, \cdot, 1\rangle$, and for all $a, b \in R$,
(i) $0a = a0 = 0$
(ii)$(-a)b = -(ab)$
(iii) $(-a)(-b) = ab$
(iv) If R is non-trivial (i.e., if it has more than one element), then $1 \neq 0$

Definition 5.7.
A group is an algebra $\langle G; *, \hat{}, e\rangle$ satisfying the following axioms:
G1 $*$ is associative.
G2 e is a neutral element: $a * e = e * a = a$ for all $a \in G$
G3 Every $a \in G$ has an inverse element $\hat{a}$, i.e., $a * \hat{a} = \hat{a} * a = e$

Lemma 5.3.
For a group $\langle G; *, \hat{}, e\rangle$ we have for all $a, b, c \in G$:
(i) $\widehat{(\hat{a})} = a$

(ii)$\widehat{a * b} = \hat{a} * \hat{b}$

(iii) Left cancellation law: $a * b = a * c \implies b = c$

(iv) Right cancellation law: $b * a = c * a \implies b = c$

(v) The equation $a * x = b$ has a unique solution x for any a and b

So does the equation $x * a = b$