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Prof. Ueli Maurer
Giovanni Deligios

Diskrete Mathematik

Solution 12

12.1 Normal Forms

a) The function table of $F = (\neg(A \to C)) \leftrightarrow (A \to B)$ is

A	B	C	$(\neg(A \to C))$	$(A \to B)$	F
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	0	1	0
0	1	1	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	1	0	1	1	1
1	1	1	0	1	0

Using the technique from the proof of Theorem 6.6, we can find an equivalent formula in CNF:

$$(A \vee B \vee C) \wedge (A \vee B \vee \neg C) \wedge (A \vee \neg B \vee C) \wedge (A \vee \neg B \vee \neg C) \wedge (\neg A \vee B \vee C) \wedge (\neg A \vee \neg B \vee \neg C)$$

and an equivalent formula in DNF:

$$(A \wedge \neg B \wedge C) \vee (A \wedge B \wedge \neg C)$$

b)
$$\begin{aligned}
& (A \wedge \neg B) \vee (\neg A \wedge (C \wedge D)) \\
\equiv\ & \big((A \wedge \neg B) \vee \neg A\big) \wedge \big((A \wedge \neg B) \vee (C \wedge D)\big) && \mid 6) \\
\equiv\ & \big(\neg A \vee (A \wedge \neg B)\big) \wedge \big((A \wedge \neg B) \vee (C \wedge D)\big) && \mid 2) \\
\equiv\ & \big((\neg A \vee A) \wedge (\neg A \vee \neg B)\big) \wedge \big((A \wedge \neg B) \vee (C \wedge D)\big) && \mid 6) \\
\equiv\ & \big((\neg A \vee A) \wedge (\neg A \vee \neg B)\big) \wedge \big(((A \wedge \neg B) \vee C) \wedge ((A \wedge \neg B) \vee D)\big) && \mid 6) \\
\equiv\ & \big((\neg A \vee A) \wedge (\neg A \vee \neg B)\big) \wedge \big((C \vee (A \wedge \neg B)) \wedge (D \vee (A \wedge \neg B))\big) && \mid 2), 2) \\
\equiv\ & (\neg A \vee A) \wedge (\neg A \vee \neg B) \wedge (C \vee A) \wedge (C \vee \neg B) \wedge (D \vee A) \wedge (D \vee \neg B) && \mid 6), 6)
\end{aligned}$$

This formula is in CNF. Using equivalences 2), 11), 2) and 9), one can find a simpler formula equivalent to G, also in CNF:

$$(\neg A \vee \neg B) \wedge (C \vee A) \wedge (C \vee \neg B) \wedge (D \vee A) \wedge (D \vee \neg B).$$

12.2 Free Variables

i) $\forall x \forall y\ (P(x,y) \vee P(x,\underline{\mathbf{z}}))$

ii) $\big(\forall x\ (\exists x\ P(x) \wedge P(x)) \vee P(\underline{\mathbf{x}})\big)$
In the first occurence of $P(x)$, x is bound by $\exists x$ and in the second occurrence it is bound by $\forall x$.

iii) There are no free variables in this formula.

12.3 Interpretations

a) **i)** $\mathcal{A}$ is a model for F, because for all positive natural numbers x, y, z we have:

$$x \mid xy \quad \wedge \quad y \mid xy \quad \wedge \quad (y \nmid x \rightarrow yz \nmid x).$$

ii) $\mathcal{A}$ is not a model for F, because there exist positive natural numbers x, y, z, for which the following does not hold:

$$x \mid x^y \quad \wedge \quad y \mid x^y \quad \wedge \quad (y \nmid x \rightarrow y^z \nmid x).$$

The counterexample is $x = 2, y = 3$ (note that $y \nmid x^y$).

iii) $\mathcal{A}$ is a model for F, because for all subsets A, B, C of $\mathbb{N}$ we have:

$$A \cap B \subseteq A \quad \wedge \quad A \cap B \subseteq B \quad \wedge \quad (A \nsubseteq B \rightarrow A \nsubseteq B \cap C).$$

b) There are many correct solutions. Below we give an example.

i) The structure $\mathcal{A}$ that defines only the universe: $U^{\mathcal{A}} = \{0\}$.

ii) The structure $\mathcal{A}$ with $U^{\mathcal{A}} = \{0\}$ and $P^{\mathcal{A}}(0,0) = 0$. $\mathcal{A}$ is not a model, because $\forall x \exists y\ P(x,y)$ is false (since $P(x,y)$ is always false).

iii) The structure $\mathcal{A}$ with $U^{\mathcal{A}} = \mathbb{Z}_3$ and $P^{\mathcal{A}}(x,y) = 1$ if and only if $x + 1 \equiv_3 y$. $\mathcal{A}$ is a model for G, because (1) for any x there exists a $y = R_3(x+1)$ such that $x+1 \equiv_3 y$ and similarly for any y there exists an $x = R_3(y-1)$ such that $x + 1 \equiv_3 y$, and (2) if $x + 1 \equiv_3 y$ then $y + 1 \equiv_3 x + 2 \not\equiv_3 x$.

12.4 Predicate Logic with Equality

a) An interpretation $\mathcal{A}$ is a model for F if and only if $|U^{\mathcal{A}}| = 1$.
If $|U^{\mathcal{A}}| = 1$, then clearly for all elements x, y of the universe, we have $x = y$ and $\mathcal{A}$ is a model for F. On the other hand, if $U^{\mathcal{A}}$ contains at least two different elements, then $\mathcal{A}$ is not a model, because there exists x and y such that $\neg(x = y)$.

b) An interpretation $\mathcal{A}$ is a model for G if and only if $|U^{\mathcal{A}}| > 1$.
If $|U^{\mathcal{A}}| > 1$, then there exist two different elements x, y of the universe and $\mathcal{A}$ is a model for G. On the other hand, if $|U^{\mathcal{A}}| = 1$, then $\mathcal{A}$ is not a model, because for all x, y, we have $x = y$.

c) An example of such formula H is $\exists x \exists y \exists z \big(\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z)\big)$.

If $\left|U^{\mathcal{A}}\right| \geq 3$, then there exist three different elements x, y, z of the universe. These elements satisfy $\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z)$.

If $\left|U^{\mathcal{A}}\right| < 3$, then, by the pigeonhole principle, at least two among three elements chosen from the universe must be equal. Hence, at least one of $\neg(x = y), \neg(y = z)$ and $\neg(x = z)$ must be false and $\mathcal{A}(H) = 0$.

12.5 A New Quantifier $\bigcirc$

a) The statement is false. Let $F = P(x)$, and let $G = (\bigcirc x F) \wedge (\bigcirc x \neg F)$.

Consider the following suitable interpretation $\mathcal{A}$ for G (notice that there are no free variables in G):

- $U^{\mathcal{A}} = \mathbb{N}$,
- $P^{\mathcal{A}}(u) = 1 \iff u$ is an even number.

We prove that $\mathcal{A}(G) = 1$ (here an informal argument would suffice, but we provide a detailed one for reference). We have

$$\begin{aligned}
& \mathcal{A}\big((\bigcirc x P(x)) \wedge (\bigcirc x \neg P(x)\big) = 1 & \\
\stackrel{\cdot}{\iff}\ & \mathcal{A}\big(\bigcirc x P(x)\big) = 1 \text{ and } \mathcal{A}\big(\bigcirc x \neg P(x)\big) = 1 & \text{(Semantics of } \wedge\text{)} \\
\stackrel{\cdot}{\iff}\ & \{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[x \to u]}(P(x)) = 1\} \sim U^{\mathcal{A}} \text{ and} & \\
& \{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[x \to u]}(\neg P(x)) = 1\} \sim U^{\mathcal{A}} & \text{(Semantics of } \bigcirc\text{)} \\
\stackrel{\cdot}{\iff}\ & \{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[x \to u]}(P(x)) = 1\} \sim U^{\mathcal{A}} \text{ and} & \\
& \{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[x \to u]}(P(x)) = 0\} \sim U^{\mathcal{A}} & \text{(Semantics of } \neg\text{)} \\
\stackrel{\cdot}{\iff}\ & \{u \in U^{\mathcal{A}} \mid P^{\mathcal{A}_{[x \to u]}}(x^{\mathcal{A}_{[x \to u]}}) = 1\} \sim U^{\mathcal{A}} \text{ and} & \\
& \{u \in U^{\mathcal{A}} \mid P^{\mathcal{A}_{[x \to u]}}(x^{\mathcal{A}_{[x \to u]}}) = 0\} \sim U^{\mathcal{A}} & \text{(Semantics of predicates)} \\
\stackrel{\cdot}{\iff}\ & \{u \in U^{\mathcal{A}} \mid P^{\mathcal{A}_{[x \to u]}}(u) = 1\} \sim U^{\mathcal{A}} \text{ and} & \\
& \{u \in U^{\mathcal{A}} \mid P^{\mathcal{A}_{[x \to u]}}(u) = 0\} \sim U^{\mathcal{A}} & \text{(Semantics of variables)} \\
\stackrel{\cdot}{\iff}\ & \{u \in \mathbb{N} \mid u \text{ is even}\} \sim \mathbb{N} \text{ and } \{u \in \mathbb{N} \mid u \text{ is odd }\} \sim \mathbb{N} & \text{(Definition of } \mathcal{A}\text{)} \\
\stackrel{\cdot}{\iff}\ & \mathsf{true} &
\end{aligned} \tag{1}$$

b) The statement is true. Let $\mathcal{A}$ be a structure which is suitable for both the left-hand-side and the right-hand-side formulas, and such that $\mathcal{A}$ is a model for $\bigcirc x F$. We have

$$\begin{aligned}
& \mathcal{A}(\bigcirc x F) = 1 & \\
\stackrel{\cdot}{\iff}\ & \{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[x \to u]}(F) = 1\} \sim U^{\mathcal{A}} & \text{(Semantics of } \bigcirc\text{)} \\
\stackrel{\cdot}{\implies}\ & \{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[x \to u]}(F) = 1\} \neq \varnothing & (U^{\mathcal{A}} \neq \varnothing) \\
\stackrel{\cdot}{\iff}\ & \mathcal{A}_{[x \to u]}(F) = 1 \text{ for some } u \in U^{\mathcal{A}} & \text{(Rewriting)} \\
\stackrel{\cdot}{\iff}\ & \mathcal{A}(\exists x F) = 1. & \text{(Semantics of } \exists\text{)}
\end{aligned} \tag{2}$$

c) The statement is false. Let $F = P(x, y)$. Consider the following suitable interpretation $\mathcal{A}$ for both $\forall x \bigcirc y F$ and $\bigcirc y \forall x F$.

- $U^{\mathcal{A}} = \mathbb{N} \setminus \{0\}$,
- $P^{\mathcal{A}}(u, v) = 1 \iff u \mid v$.

The structure $\mathcal{A}$ is a model for $\forall x \bigcirc y F$, because all non-zero natural numbers have a set of multiples which is trivially in bijection with $\mathbb{N}$. However $\mathcal{A}$ is not a model for $\bigcirc y \forall x F$. This is because the set of non-zero natural numbers which are divided by all other non-zero natural numbers is $\varnothing$, which is clearly not in bijection with $\mathbb{N}$.

12.6 Statements About Formulas

a) The statement is true.

Proof. Let $\mathcal{A}$ be any interpretation suitable for both $\forall x\ (F \wedge G)$ and $(\forall x\ F) \wedge G$, such that $\mathcal{A}\left(\forall x\ (F \wedge G)\right) = 1$. According to the semantics of $\forall$, we have $\mathcal{A}_{[x \to u]}(F \wedge G) = 1$ for all $u \in U$. According to the semantics of $\wedge$, we further have (1) $\mathcal{A}_{[x \to u]}(F) = 1$ for all $u \in U$ and (2) $\mathcal{A}_{[x \to u]}(G) = 1$ for all $u \in U$.

The fact (1) implies (3) $\mathcal{A}(\forall x\ F) = 1$, according to the semantics of $\forall$. Furthermore, note that if x appears free in G, then it also appears free in $(\forall x\ F) \wedge G$, and since $\mathcal{A}$ is suitable for $(\forall x\ F) \wedge G$, it must assign a value to x. We now define u^* as follows: if x appears free in G, then u^* is the value assigned to x by $\mathcal{A}$, else u^* is arbitrary. By the definition of u^*, we have $\mathcal{A}_{[x \to u^*]}(G) = \mathcal{A}(G)$, so by (2), we have (4) $\mathcal{A}(G) = 1$.

The facts (3) and (4) imply that $\mathcal{A}\left((\forall x\ F) \wedge G\right) = 1$.

b) The statement is false.

Counterexample. Let $F = P(x)$ and $G = Q(x)$. Let $\mathcal{A}$ be the interpretation with the universe $U^{\mathcal{A}} = \{0, 1\}$, which defines:

- $P^{\mathcal{A}}(0) = 1$ and $P^{\mathcal{A}}(1) = 1$
- $Q^{\mathcal{A}}(0) = 1$ and $Q^{\mathcal{A}}(1) = 0$
- $x^{\mathcal{A}} = 1$

We then have $\mathcal{A}(\exists x\ (P(x) \wedge Q(x))) = 1$, because $\mathcal{A}_{[x \to 0]}(P(x) \wedge Q(x)) = 1$. However, $\mathcal{A}((\exists x\ P(x)) \wedge Q(x)) = 0$, because $\mathcal{A}(Q(x)) = 0$.