DMath_U12_bf

12.5

We extend predicate logic with a new quantifier $\bigcirc$ (read: for many) as follows:

Syntax: If F is a formula, then for any variable symbol x_i, $\bigcirc x_i F$ is a formula.
Semantics: $\mathcal{A}(\bigcirc x_i F) = 1 \iff \{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[x_i \to u]}(F) = 1\} \sim U^{\mathcal{A}}$.

Using the semantics of predicate logic extended in this way, prove or disprove the following statements, where F is an arbitrary formula.

a) The formula $(\bigcirc x F) \wedge (\bigcirc x \neg F)$ is unsatisfiable.
b) $\bigcirc x F \models \exists x F$
c) $\forall x \bigcirc y F \models \bigcirc y \forall x F$

Expectation: If the statement is true, your proof should use the definitions of the semantics. In each step, at most one definition (e.g., the semantics of $\bigcirc$) should be applied. If the statement is not true, you should provide a counterexample: make sure to define everything needed for a suitable interpretation.

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a) The formula $(\bigcirc x F) \wedge (\bigcirc x \neg F)$ is unsatisfiable.

Disproven by counterexample. We want to show that there exists an interpretation $\mathcal{A}$ such that $(\bigcirc x F \wedge (\bigcirc x \neg F) = 1$, making the formula satisfiable.

Let F be the formula $P(x)$

Let $\mathcal{A} = (U, \phi, \psi, \xi)$ be a suitable interpretation for the formula.

- Let $U^{\mathcal{A}} = \mathbb{N}^*$
- Let $\phi = \varnothing$ (there are no functions in our formula)
- Let $\psi = \{P^{\mathcal{A}}(x) = 1 \iff x \equiv_2 0, \text{ i.e. } P^{\mathcal{A}}(x) = 1 \text{ if and only if x is even}\}$
- Let $\xi = \varnothing$ (there are no free variables in our formula)

$$
\begin{aligned}
&\mathcal{A}((\bigcirc x F \wedge (\bigcirc x \neg F) \\
\stackrel{\cdot}{\Longrightarrow}\ &\mathcal{A}((\bigcirc x P(x)) \wedge (\bigcirc x \neg P(x))) && \text{(Definition of } F) \\
\stackrel{\cdot}{\Longrightarrow}\ &\mathcal{A}((\bigcirc x P(x)))\ \ and\ \ \mathcal{A}((\bigcirc x \neg P(x))) && \text{(Definition 6.24; Semantics of } \wedge\text{)} \\
\stackrel{\cdot}{\Longrightarrow}\ &\{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[x \to u]}(P(x)) = 1\} \sim \mathbb{N}\ \ and\ \ \{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[x \to u]}(\neg P(x)) = 1\} \sim \mathbb{N} && \text{(Semantics of } \bigcirc x_i) \\
\stackrel{\cdot}{\Longrightarrow}\ &\{u \in U^{\mathcal{A}} \mid \mathcal{A}(P(u) = 1)\} \sim \mathbb{N}\ \ and\ \ \{u \in U^{\mathcal{A}} \mid \mathcal{A}(\neg P(u) = 1)\} \sim \mathbb{N} && ([x \to u]) \\
\stackrel{\cdot}{\Longrightarrow}\ &\{u \in U^{\mathcal{A}} \mid \mathcal{A}(P(u) = 1)\} \sim \mathbb{N}\ \ and\ \ \{u \in U^{\mathcal{A}} \mid \mathcal{A}(P(u) = 0)\} \sim \mathbb{N} && \text{(Definition 6.24; Semantics of } \neg) \\
\stackrel{\cdot}{\Longrightarrow}\ &\{u \in U^{\mathcal{A}} \mid u \text{ is even})\} \sim \mathbb{N}\ \ and\ \ \{u \in U^{\mathcal{A}} \mid u \text{ is not even}\} \sim \mathbb{N} && \text{(Interpretation of } P^{\mathcal{A}}(x)) \\
\stackrel{\cdot}{\Longrightarrow}\ &\{u \in \mathbb{N} \mid u \text{ is even})\} \sim \mathbb{N}\ \ and\ \ \{u \in \mathbb{N} \mid u \text{ is not even}\} \sim \mathbb{N} && \text{(Interpretation of } U^{\mathcal{A}})
\end{aligned}
$$

All that remains is to show is that $\{u \in \mathbb{N} \mid \text{u is even})\} \sim \mathbb{N}$ and $\{u \in \mathbb{N} \mid \text{u is not even}\} \sim \mathbb{N}$.

We define two sets $S = \{u \in \mathbb{N} \mid u \text{ is even}\}$ and $T = \{u \in \mathbb{N} \mid u \text{ is not even}\}$ and construct a bijection between them and $\mathbb{N}^*$
Let $f : \mathbb{N} \to S$ be defined as $f(x) = 2 \cdot x$. It's easy to see that, for every $x \in \mathbb{N}, f(x) \in S$.
Let $g : \mathbb{N} \to T$ be defined as $g(x) = 2 \cdot x + 1$. Since, for every $x \in \mathbb{N}, f(x) \in T$ as $f(x) \equiv_2 1$ for all $x \in \mathbb{N}$.

Thus both parts are satisfiable under some interpretation $\mathcal{A}$, disproving the statement.
$\square$

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b) $\bigcirc x F \models \exists x F$

We proof this by showing that any interpretation that's suitable for both sides of the formula and is a model for $\bigcirc x F$ is also a model for $\exists x F$.

Since any $U^{\mathcal{A}}$ (where $\mathcal{A}$ is a model for the LHS) is equinumerous to the set $\{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[x_i \to u]}(F) = 1\}$ and no $U^{\mathcal{A}}$ can be empty (as per Definition 6.34.), that same $\mathcal{A}$ is also a model for $\exists x F$, by definition 6.36. $\mathcal{A}(\exists x F)$

$$
\begin{aligned}
&\mathcal{A}(\bigcirc x_i F) = 1 \\
&\overset{\cdot}{\Longrightarrow} \{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[x_i \to u]}(F) = 1\} \sim U^{\mathcal{A}} \text{ and } U^{\mathcal{A}} \neq \varnothing && \text{(Semantics of } \bigcirc x_i\text{, Definition 6.34. } U \neq \varnothing) \\
&\overset{\cdot}{\Longrightarrow} \mathcal{A}_{[x \to u]}(F) = 1 \text{ for some } u \in U && \text{Definition 6.36. } \mathcal{A}(\exists x F) \\
&\overset{\cdot}{\Longrightarrow} \mathcal{A}(\exists x F) = 1
\end{aligned}
$$

As was to be shown.
□

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c) $\forall x \bigcirc y F \models \bigcirc y \forall x F$

We will disprove this statement with a counterexample.

Let F be the formula $P(x, y)$

Let $\mathcal{A} = (U, \phi, \psi, \xi)$ be a suitable interpretation for the formula.

- Let $U^{\mathcal{A}} = \mathbb{N}^*$
- Let $\phi = \varnothing$ (there are no functions in our formula)
- Let $\psi = \{P^{\mathcal{A}}(x) = 1 \iff y > x, \text{ i.e. } P^{\mathcal{A}}(x) = 1 \text{ if and only if y is greater than x}\}$
- Let $\xi = \varnothing$ (there are no free variables in our formula)

$$
\begin{aligned}
&\mathcal{A}(\forall x \bigcirc y F) \models \mathcal{A}(\bigcirc y \forall x F) \\
&\overset{\cdot}{\Longrightarrow} \mathcal{A}(\forall x \bigcirc y P(x, y)) \models \mathcal{A}(\bigcirc y \forall x P(x, y)) && (1) \\
&\overset{\cdot}{\Longrightarrow} \mathcal{A}_{[x \to u]}(\bigcirc y P(x, y)) = 1 \text{ for all } u \in U^{\mathcal{A}} \models \mathcal{A}(\bigcirc y \forall x P(x, y)) && (2) \\
&\overset{\cdot}{\Longrightarrow} \mathcal{A}_{[x \to u]}(\bigcirc y P(x, y)) = 1 \text{ for all } u \in U^{\mathcal{A}} \models \{u \in U^{\mathcal{A}} \mid \{\mathcal{A}_{[x \to u]}(\forall x P(x, y)) = 1\} \sim U^{\mathcal{A}} && (3) \\
&\overset{\cdot}{\Longrightarrow} \mathcal{A}_{[x \to u]}(\{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[y \to v]}(P(x, y) = 1\} \sim U^{\mathcal{A}}) = 1 \text{ for all } v \in U^{\mathcal{A}} \models \{u \in U^{\mathcal{A}} \mid \{\mathcal{A}_{[y \to v]}(\forall x P(x, y)) = 1\} \sim U^{\mathcal{A}} && (4) \\
&\overset{\cdot}{\Longrightarrow} \mathcal{A}_{[x \to u]}(\{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[y \to v]}(P(x, y) = 1\} \sim U^{\mathcal{A}}) = 1 \text{ for all } v \in U^{\mathcal{A}} \models \{u \in U^{\mathcal{A}} \mid \{\mathcal{A}_{[y \to v]}(\mathcal{A}_{[x \to u]}(P(x, y)) = 1 \text{ for all } v \in U^{\mathcal{A}}) = 1\} \sim U^{\mathcal{A}} && (5) \\
&\overset{\cdot}{\Longrightarrow} \mathcal{A}(\{u \in U^{\mathcal{A}} \mid \mathcal{A}_{[y \to v]}(P(y, u) = 1\} \sim U^{\mathcal{A}}) = 1 \text{ for all } v \in U^{\mathcal{A}} \models \{u \in U^{\mathcal{A}} \mid \{\mathcal{A}_{[y \to u]}(\mathcal{A}(P(u, y)) = 1 \text{ for all } v \in U^{\mathcal{A}}) = 1\} \sim U^{\mathcal{A}} && (6) \\
&\overset{\cdot}{\Longrightarrow} \mathcal{A}(\{u \in U^{\mathcal{A}} \mid \mathcal{A}(P(v, u) = 1\} \sim U^{\mathcal{A}}) = 1 \text{ for all } v \in U^{\mathcal{A}} \models \{u \in U^{\mathcal{A}} \mid \{\mathcal{A}(\mathcal{A}(P(u, v)) = 1 \text{ for all } v \in U^{\mathcal{A}}) = 1\} \sim U^{\mathcal{A}} && (7)
\end{aligned}
$$

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\begin{array}{lr}
(1) & \text{(Definition of } F) \\
(2) & \text{(Definition 6.36. } \forall x F) \\
(3) & \text{(Semantics of } \bigcirc x F) \\
(4) & \text{(Semantics of } \bigcirc x F) \\
(5) & \text{(Definition 6.36. } \forall x F) \\
(6) & [x \to u] \\
(7) & [y \to v]
\end{array}
$$

What this is saying in words is, that our chosen interpretation $\mathcal{A}$ is a model for the LHS (since, for all natural numbers $x \in \mathbb{N}^*$, there is a set of numbers $y_i \in \mathbb{N}^*$ equinumerous to the set of natural nubers ($\{y_i \in \mathbb{N}^*\} \sim \mathbb{N}^*$) that is larger than than x) but not a model for the RHS (since, not for every number $x_i \in \mathbb{N}$ all numbers $y_i \in \mathbb{N}$ are larger than that number).

We have shown there exists some interpretation $\mathcal{A}$ that is suitable for both sides of the formula and is a model for $\forall x \bigcirc y F$ but not for $\bigcirc y \forall x F$.
□