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Diskrete Mathematik

Solution 2

2.1 Logical Consequence

a) We first construct the function table for the formula $A \wedge (A \to B)$.

A	B	$A \wedge (A \to B)$
0	0	0
0	1	0
1	0	0
1	1	1

The above table shows that the truth value of $A \wedge (A \to B)$ is 1 only for the truth assignment in the last row. Clearly, B is also true for that assignment. Thus, B is the logical consequence of $A \wedge (A \to B)$ and the statement holds.

b) The statement is false. There exists a truth assignment, namely one in which A is false and B is true, for which $A \to B$ is true, but $\neg A \to \neg B$ is false.

Thus, $\neg A \to \neg B$ is not a logical consequence of $A \to B$.

c) The statement is true. One way to show this is to construct a function table for $F = (A \to B) \vee (B \to A)$. We present a different proof: assume, by contradiction, that some truth assignment of the propositional symbols A and B makes F false. Then, under this truth assignment, both the formula $(A \to B)$ and the formula $(B \to A)$ are false, because otherwise their conjunction would be true. The only truth assignment for which $A \to B$ is false is one where A is true but B is false. For this truth assignment, $B \to A$ is true, which is a contradiction.

d) We construct the function table for both formulas: $(A \to B) \wedge (B \to C)$ and $A \to C$.

A	B	C	$A \to B$	$B \to C$	$(A \to B) \wedge (B \to C)$	$A \to C$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	0	1	0	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Analogously to Subtask a), we can show that the statement holds.

2.2 Satisfiability and Tautologies

a) This formula is satisfiable, since it is true for the assignment $A = 0, B = 1$. It is, however, not a tautology, since it is false for the assignment $A = 0, B = 0$.

b) This formula is unsatisfiable (hence, it is not a tautology). In order to prove this, let $F = \big((A \to B) \wedge (B \to C)\big) \wedge \neg(A \to C)$. We notice that

$$\begin{aligned} \neg F \;&\equiv\; \neg\big((A \to B) \wedge (B \to C)\big) \vee (A \to C) && \text{(de Morgan, double negation)} \\ &\equiv\; (A \to B) \wedge (B \to C) \;\to\; (A \to C) && \text{(def. } \to\text{)} \end{aligned}$$

From Task 2.1 c), we know that $(A \to B) \wedge (B \to C) \;\models\; (A \to C)$ is true. From this fact, together with Lemma 2.3, it follows that $\neg F$ is a tautology. Hence, by Lemma 2.2, F is unsatisfiable.

2.3 Simplifying a Formula

We choose the formula $G = A$. In the following, we prove that $F \equiv G$:

$$\begin{aligned} &\big((\neg A \vee \neg B) \to (A \wedge \neg B)\big) \wedge (C \vee A) \\ \equiv\; &\big(\neg(\neg A \vee \neg B) \vee (A \wedge \neg B)\big) \wedge (C \vee A) && \text{(definition of } \to\text{)} \\ \equiv\; &\big(((\neg\neg A) \wedge (\neg\neg B)) \vee (A \wedge \neg B)\big) \wedge (C \vee A) && \text{(de Morgan's Rules)} \\ \equiv\; &\big((A \wedge (\neg\neg B)) \vee (A \wedge \neg B)\big) \wedge (C \vee A) && \text{(double negation)} \\ \equiv\; &\big((A \wedge B) \vee (A \wedge \neg B)\big) \wedge (C \vee A) && \text{(double negation)} \\ \equiv\; &\big(A \wedge (B \vee \neg B)\big) \wedge (C \vee A) && \text{(first distributive law)} \\ \equiv\; &(A \wedge \top) \wedge (C \vee A) && (F \vee \neg F \equiv \top) \\ \equiv\; &A \wedge (C \vee A) && (A \wedge \top \equiv A) \\ \equiv\; &A \wedge (A \vee C) && \text{(commutativity of } \vee\text{)} \\ \equiv\; &A && \text{(absorption)} \end{aligned}$$

2.4 Knights and Knaves

Let A be the proposition "The left road leads to the village." and let B be the proposition "The islander is a knight.". We want to ask the islander about the truth value of a formula F in A and B in order to determine whether A is true.

In order to be guaranteed to learn whether A is true or not, we have to receive a fixed answer (say, "Yes") from the islander in case A is true, and the opposite (say, "No") in case A is false. This has to hold *independently* of whether the islander is a knight or a knave (since we have no information about that).

If the islander is a knight (B is true) the answer will be the truth value of F (since knights always tell the truth). However, if the islander is a knave (B is false) the answer will be the truth value of $\neg F$ (since knaves always lie).

Hence, we derive the following partial function table:

A	B	F	$\neg F$
0	0		0
0	1	0	
1	0		1
1	1	1	

This partial function table can be completed (uniquely) to the following function table:

A	B	F	$\neg F$
0	0	1	0
0	1	0	1
1	0	0	1
1	1	1	0

From the function table we obtain a possible formula $F = (\neg A \wedge \neg B) \vee (A \wedge B)$. Formulated as a question: "Does the left road lead to the jungle and you are a knave, or is it the case that the left road leads to the village and you are a knight?".

2.5 Quantifiers and Predicates

a) **i)** $\forall m\, \forall n\, \big(0 < m \cdot n \rightarrow (0 < m \vee 0 < n)\big)$

This statement is false. For example, $(-2) \cdot (-2) = 4$.

ii) $\forall m\, \big(-1 < m \rightarrow \exists n \big(-1 < n \wedge m < n \wedge (\exists k\; n = 3 \cdot k)\big)\big)$

This statement is true. For any n, one of the numbers $n+1, n+2, n+3$ must be divisible by 3.

It is also allowed to drop the condition $-1 < n$, since it is implied by $m < n$.

iii) $\forall n\, \big(((\exists k\; n = 2 \cdot k) \wedge 2 < n) \rightarrow \exists p\, \exists q\, (\texttt{prime}(p) \wedge \texttt{prime}(q) \wedge n = p + q)\big)$

This statement is known as the (strong) Goldbach conjecture. It is not known whether it is true.

b) There are many equally good ways to describe given formulas using words. We only give examples:

i) "For every integer x, there exists an integer y, such that xy is equal to 1."

An alternative solution would be "Each integer has a multiplicative inverse."

This statement is false. For example, there is no integer that will give 1 when multiplied by 5.

ii) "There exists an integer x, such that for all integers y, the product xy is not equal to 1, and such that there exists an integer greater than 0."

This statement is true. For $x = 0$, we have that for any integer y, the product xy is not equal to 1, and that there exists a positive integer, namely 42.

Be careful, the following interpretation is *not* correct (Why?): "There exists an integer x, such that for all integers y, the product xy is not equal to 1 and y is positive."

2.6 Finding an Interpretation for a Formula

a) $U = \mathbb{Z}$ and $P(x, y) = 1 \iff x < y$.

b) $U = \{0, \ldots, n-1\}$ and $P(x, y) = 1 \iff (x < n-1 \wedge y = x+1) \vee (x = n-1 \wedge y = 0)$.