# DMath_U8_bf

[V14_körper]

## 8.3

Let $\langle G, *, \hat{}, e\rangle$ be a group, and let $S$ be a set. Assume that $f : G \to S$ is a bijection, and consider

- the binary operation $\star$ on $S$ given by $s \star s\prime \overset{def}{=} f(f^{-1}(s) * f^{-1}(s\prime))$
- the unary operation $\tilde{}$ on $S$ given by $\tilde{s} \overset{def}{=} f(\widehat{f^{-1}(s)})$.

  Prove the following statement.
  a) Axiom G1 ($\star$ is associative) holds for $\langle S, \star, \tilde{}, f(e)\rangle$

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To prove that **G1** holds for $\langle S; \star, \tilde{}, f(e)\rangle$, we will show that the operation $\star$ is associative.

Let $s_1,\ s_2,\ s_3 \in S$
We need to show that $(s_1 \star s_2) \star s_3 = s_1 \star (s_2 \star s_3)$.
Since we have

$$(s_1 \star s_2) \star s_3 = f(f^{-1}(f(f^{-1}(s_1) * f^{-1}(s_2))) * f^{-1}(s_3)) = f((f^{-1}(s_1) * f^{-1}(s_2)) * f^{-1}(s_3))$$

and

$$s_1 \star (s_2 \star s_3) = f(f^{-1}(s_1) * f^{-1}(f(f^{-1}(s_2) * f^{-1}(s_3)))) = f(f^{-1}(s_1) * (f^{-1}(s_2) * f^{-1}(s_3)))$$

given by the definition of $\star$ and the operation $*$ is associative in $G$, we have

$$((f^{-1}(s_1) * f^{-1}(s_2)) * f^{-1}(s_3) = f^{-1}(s_1) * (f^{-1}(s_2) * f^{-1}(s_3))$$
$$f(f^{-1}(s_1) * (f^{-1}(s_2) * f^{-1}(s_3))) = f^{-1}(s_1) * (f^{-1}(s_2) * f^{-1}(s_3))$$

Thus,

$$(s_1 \star s_2) \star s_3 = s_1 \star (s_2 \star s_3)$$

$\square$

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## 8.4

c) Prove that $\langle \mathbb{Z}^*_{15}, \odot_{15}\rangle \simeq \langle \mathbb{Z}^*_{16}, \odot_{16}\rangle$.

For two groups $\langle G; *, \hat{}, e\rangle$ and $\langle H; \star, \tilde{}, e'\rangle$, a function $\psi : G \to H$ is called a group homomorphism if, for all $a$ and $b$,
$\psi(a * b) = \psi(a) \star \psi(b)$
If $\psi$ is a bijection from $G$ to $H$, then it is called an isomorphism, and we say that $G$ and $H$ are isomorphic and write $G \simeq H$. (Definition 5.10.)

$\langle \mathbb{Z}^*_{15}, \odot_{15}\rangle$

| $\odot_{15}$ | **1** | **2** | **4** | **8** | **7** | **11** | **13** | **14** |
|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 2 | 4 | 8 | 7 | 11 | 13 | 14 |
| **2** | 2 | 4 | 8 | 1 | 14 | 7 | 11 | 13 |
| **4** | 4 | 8 | 1 | 2 | 13 | 14 | 7 | 11 |
| **8** | 8 | 1 | 2 | 4 | 11 | 13 | 14 | 7 |
| **7** | 7 | 14 | 13 | 11 | 4 | 2 | 1 | 8 |
| **11** | 11 | 7 | 14 | 13 | 2 | 1 | 8 | 4 |
| **13** | 13 | 11 | 7 | 14 | 1 | 8 | 4 | 2 |

| $\odot_{15}$ | **1** | **2** | **4** | **8** | **7** | **11** | **13** | **14** |
|---|---|---|---|---|---|---|---|---|
| **14** | 14 | 13 | 11 | 7 | 8 | 4 | 2 | 1 |

$\langle \mathbb{Z}_{16}^*, \odot_{16} \rangle$

| $\odot_{16}$ | **1** | **3** | **9** | **11** | **5** | **7** | **13** | **15** |
|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 3 | 9 | 11 | 5 | 7 | 13 | 15 |
| **3** | 3 | 9 | 11 | 1 | 15 | 5 | 7 | 13 |
| **9** | 9 | 11 | 1 | 3 | 13 | 15 | 5 | 7 |
| **11** | 11 | 1 | 3 | 9 | 7 | 13 | 15 | 5 |
| **5** | 5 | 15 | 13 | 7 | 9 | 3 | 1 | 11 |
| **7** | 7 | 5 | 15 | 13 | 3 | 1 | 11 | 9 |
| **13** | 13 | 7 | 5 | 15 | 1 | 11 | 9 | 3 |
| **15** | 15 | 13 | 7 | 5 | 11 | 9 | 3 | 1 |

We define a function $\psi : \mathbb{Z}_{15}^* \to \mathbb{Z}_{16}^*$ as follows:

$\psi(1) = 1$
$\psi(2) = 3$
$\psi(4) = 9$
$\psi(8) = 11$
$\psi(7) = 5$
$\psi(11) = 7$
$\psi(13) = 13$
$\psi(14) = 15$

Obviously the function is bijective, as it maps each element onto one unique element (injective) and each element has an inverse (surjective).

Now we will prove that it is a group homomorphism on $\langle \mathbb{Z}_{15}^*, \odot_{15} \rangle$

To do this, we must prove, that for all $a, b \in \mathbb{Z}_{15}^*$

$$\psi(a \odot_{15} b) = \psi(a) \odot_{16} \psi(b)$$

We do this by case distinction:

$\psi(1 \odot_{15} 1) = 1 = \psi(1) \odot_{16} \psi(1)$
$\psi(1 \odot_{15} 2) = 3 = \psi(1) \odot_{16} \psi(2)$
$\psi(1 \odot_{15} 4) = 9 = \psi(1) \odot_{16} \psi(4)$
$\psi(1 \odot_{15} 7) = 5 = \psi(1) \odot_{16} \psi(7)$
$\psi(1 \odot_{15} 8) = 11 = \psi(1) \odot_{16} \psi(8)$
$\psi(1 \odot_{15} 11) = 7 = \psi(1) \odot_{16} \psi(11)$
$\psi(1 \odot_{15} 13) = 13 = \psi(1) \odot_{16} \psi(13)$
$\psi(1 \odot_{15} 14) = 15 = \psi(1) \odot_{16} \psi(14)$
$\psi(2 \odot_{15} 1) = 3 = \psi(2) \odot_{16} \psi(1)$
$\psi(2 \odot_{15} 2) = 9 = \psi(2) \odot_{16} \psi(2)$
$\psi(2 \odot_{15} 4) = 11 = \psi(2) \odot_{16} \psi(4)$
$\psi(2 \odot_{15} 7) = 15 = \psi(2) \odot_{16} \psi(7)$
$\psi(2 \odot_{15} 8) = 1 = \psi(2) \odot_{16} \psi(8)$
$\psi(2 \odot_{15} 11) = 5 = \psi(2) \odot_{16} \psi(11)$
$\psi(2 \odot_{15} 13) = 7 = \psi(2) \odot_{16} \psi(13)$
$\psi(2 \odot_{15} 14) = 13 = \psi(2) \odot_{16} \psi(14)$
$\psi(4 \odot_{15} 1) = 9 = \psi(4) \odot_{16} \psi(1)$
$\psi(4 \odot_{15} 2) = 11 = \psi(4) \odot_{16} \psi(2)$
$\psi(4 \odot_{15} 4) = 1 = \psi(4) \odot_{16} \psi(4)$
$\psi(4 \odot_{15} 7) = 13 = \psi(4) \odot_{16} \psi(7)$
$\psi(4 \odot_{15} 8) = 3 = \psi(4) \odot_{16} \psi(8)$
$\psi(4 \odot_{15} 11) = 15 = \psi(4) \odot_{16} \psi(11)$
$\psi(4 \odot_{15} 13) = 5 = \psi(4) \odot_{16} \psi(13)$
$\psi(4 \odot_{15} 14) = 7 = \psi(4) \odot_{16} \psi(14)$
$\psi(7 \odot_{15} 1) = 5 = \psi(7) \odot_{16} \psi(1)$
$\psi(7 \odot_{15} 2) = 15 = \psi(7) \odot_{16} \psi(2)$
$\psi(7 \odot_{15} 4) = 13 = \psi(7) \odot_{16} \psi(4)$
$\psi(7 \odot_{15} 7) = 9 = \psi(7) \odot_{16} \psi(7)$

$\psi(7 \odot_{15} 8) = 7 = \psi(7) \odot_{16} \psi(8)$
$\psi(7 \odot_{15} 11) = 3 = \psi(7) \odot_{16} \psi(11)$
$\psi(7 \odot_{15} 13) = 1 = \psi(7) \odot_{16} \psi(13)$
$\psi(7 \odot_{15} 14) = 11 = \psi(7) \odot_{16} \psi(14)$
$\psi(8 \odot_{15} 1) = 11 = \psi(8) \odot_{16} \psi(1)$
$\psi(8 \odot_{15} 2) = 1 = \psi(8) \odot_{16} \psi(2)$
$\psi(8 \odot_{15} 4) = 3 = \psi(8) \odot_{16} \psi(4)$
$\psi(8 \odot_{15} 7) = 7 = \psi(8) \odot_{16} \psi(7)$
$\psi(8 \odot_{15} 8) = 9 = \psi(8) \odot_{16} \psi(8)$
$\psi(8 \odot_{15} 11) = 13 = \psi(8) \odot_{16} \psi(11)$
$\psi(8 \odot_{15} 13) = 15 = \psi(8) \odot_{16} \psi(13)$
$\psi(8 \odot_{15} 14) = 5 = \psi(8) \odot_{16} \psi(14)$
$\psi(11 \odot_{15} 1) = 7 = \psi(11) \odot_{16} \psi(1)$
$\psi(11 \odot_{15} 2) = 5 = \psi(11) \odot_{16} \psi(2)$
$\psi(11 \odot_{15} 4) = 15 = \psi(11) \odot_{16} \psi(4)$
$\psi(11 \odot_{15} 7) = 3 = \psi(11) \odot_{16} \psi(7)$
$\psi(11 \odot_{15} 8) = 13 = \psi(11) \odot_{16} \psi(8)$
$\psi(11 \odot_{15} 11) = 1 = \psi(11) \odot_{16} \psi(11)$
$\psi(11 \odot_{15} 13) = 11 = \psi(11) \odot_{16} \psi(13)$
$\psi(11 \odot_{15} 14) = 9 = \psi(11) \odot_{16} \psi(14)$
$\psi(13 \odot_{15} 1) = 13 = \psi(13) \odot_{16} \psi(1)$
$\psi(13 \odot_{15} 2) = 7 = \psi(13) \odot_{16} \psi(2)$
$\psi(13 \odot_{15} 4) = 5 = \psi(13) \odot_{16} \psi(4)$
$\psi(13 \odot_{15} 7) = 1 = \psi(13) \odot_{16} \psi(7)$
$\psi(13 \odot_{15} 8) = 15 = \psi(13) \odot_{16} \psi(8)$
$\psi(13 \odot_{15} 11) = 11 = \psi(13) \odot_{16} \psi(11)$
$\psi(13 \odot_{15} 13) = 9 = \psi(13) \odot_{16} \psi(13)$
$\psi(13 \odot_{15} 14) = 3 = \psi(13) \odot_{16} \psi(14)$
$\psi(14 \odot_{15} 1) = 15 = \psi(14) \odot_{16} \psi(1)$
$\psi(14 \odot_{15} 2) = 13 = \psi(14) \odot_{16} \psi(2)$
$\psi(14 \odot_{15} 4) = 7 = \psi(14) \odot_{16} \psi(4)$
$\psi(14 \odot_{15} 7) = 11 = \psi(14) \odot_{16} \psi(7)$
$\psi(14 \odot_{15} 8) = 5 = \psi(14) \odot_{16} \psi(8)$
$\psi(14 \odot_{15} 11) = 9 = \psi(14) \odot_{16} \psi(11)$
$\psi(14 \odot_{15} 13) = 3 = \psi(14) \odot_{16} \psi(13)$
$\psi(14 \odot_{15} 14) = 1 = \psi(14) \odot_{16} \psi(14)$
$\square$