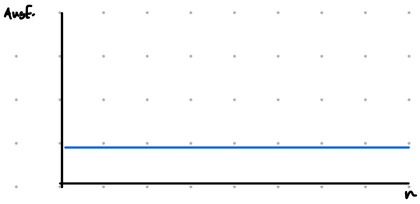
Komplexitätsklassen

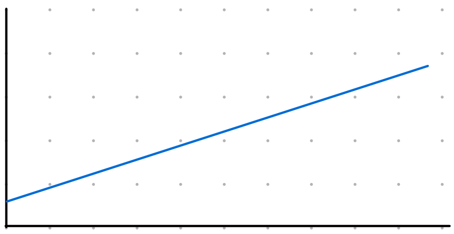
n := Anzahl der Eingabeelemente

Konstanter Aufwand - $O(1)$



Beispiel:
- Aufschlagen der ersten Seite
- Zugriff auf Array Element, z.B. array[index]

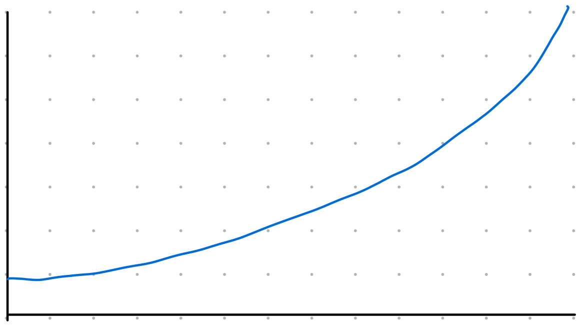
Linearer Aufwand - $O(n)$



Beispiel:
- Finden des Namens einer Tel-Nummer
- Summe eines Arrays

$2n+4 \in O(n)$

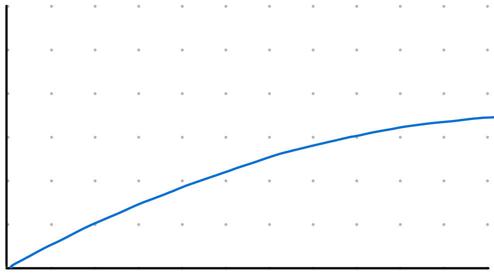
Quadratischer Aufwand - $O(n^2)$



Beispiel:
- Korrigieren jeder der n Seiten von n Büchern

Logarithmischer Aufwand - $O(\log n)$

- Verdopplung von n → konstanter Zuwachs des Aufwands



Beispiel:
- Binäre Suche, z.B. Nummer von Person XY im Tel-Buch finden

Serie 02

Exercise 2.1 z.z.: $2^n > n^2$ für $n \geq 5$, $n \in \mathbb{N}$

a) Base Case $n \to 5$

$$2^5 = 32 > 25 = 5^2$$

I.H.

Wir nehmen an, $2^k > k^2$, gilt für ein $k \geq 5, k \in \mathbb{N}$.

I.S. $k \to k+1$

$$2^{k+1} > (k+1)^2$$
$$\Downarrow$$
$$3 > 1$$

$$\begin{aligned}
2^{k+1} = 2 \cdot 2^k &\overset{\text{I.H.}}{>} 2 \cdot k^2 \\
&= k^2 + k^2 \\
&= k^2 + k \cdot k && \mid k \geq 5 \, ! \\
&\geq k^2 + 5 \cdot k \\
&= k^2 + 2k + 3k && \mid k \geq 5 \\
&\geq k^2 + 2k + 15 \\
&> k^2 + 2k + 1 \\
&= (k+1)^2 \qquad \square
\end{aligned}$$

b) z.z.: $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot x^i$, $\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$, $\underbrace{\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}}_{(1)}$

B.C. $n \to 1$

$$(1+x)^1 = 1 + x = x^0 + x^1 = \binom{1}{0} \cdot x^0 + \binom{1}{1} \cdot x^1 = \sum_{i=0}^{1} \binom{n}{i} x^i \quad \checkmark$$

I.H.

Wir nehmen an, $(1+x)^k = \sum_{i=0}^{k} \binom{k}{i} x^i$ gilt für ein $k \geq 1$.

I.S. $k \to k+1$

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)\underbrace{(1+x)^k}_{\text{I.H.}} \\
&\overset{\text{I.H.}}{=} (1+x) \cdot \sum_{i=0}^{k} \binom{k}{i} x^i \\
&= \sum_{i=0}^{k} \binom{k}{i} \cdot x^i + \sum_{i=0}^{k} \binom{k}{i} \cdot x^{i+1} \\
&= \sum_{i=0}^{k} \binom{k}{i} \cdot x^i + \sum_{i=1}^{k+1} \binom{k}{i-1} \cdot x^i \\
&= \binom{k}{0} \cdot x^0 + \sum_{i=1}^{k} \binom{k}{i} \cdot x^i + \sum_{i=1}^{k} \binom{k}{i-1} \cdot x^i + \binom{k}{k} \cdot x^{k+1} \\
&= \binom{k}{0} \cdot x^0 + \sum_{i=1}^{k} \left(\binom{k}{i} \cdot x^i + \binom{k}{i-1} \cdot x^i \right) + \binom{k}{k} \cdot x^{k+1} \\
&= \binom{k+1}{0} \cdot x^0 + \sum_{i=1}^{k} \binom{k+1}{i} \cdot x^i + \binom{k+1}{k+1} x^{k+1} \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} \cdot x^i \qquad \blacksquare
\end{aligned}$$

$\binom{k}{0} = \frac{k!}{(k-0)!\, 0!} = \frac{k!}{k!} = 1$

$\binom{k+1}{0} = \frac{(k+1)!}{(k+1-0)! \cdot 0!} = \frac{(k+1)!}{(k+1)!} = 1$

Exercise 2.2

$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty \Rightarrow g \notin O(f)$

$\lim_{n\to\infty} \frac{g(n)}{f(n)} = C \in \mathbb{R} \Rightarrow g \in O(f)$

(1) $2n^5 + 10n^2 \overset{?}{\leq} O(\frac{1}{100} n^6)$ ✓

$$\lim_{n\to\infty} \frac{2n^5+10n^2}{\frac{1}{100} n^6} = \lim_{n\to\infty} \frac{2n^5}{\frac{1}{100} n^6} + \frac{10n^2}{\frac{1}{100} n^6} = \lim_{n\to\infty} 200 \cdot \underbrace{\frac{1}{n}}_{\to 0} + 1000 \cdot \underbrace{\frac{1}{n^4}}_{\to 0} = 0$$

(2) $n^{10} + 2n^2 + 7 \leq O(100 n^9)$ ✗

$$\lim_{n\to\infty} \frac{n^{10}+2n^2+7}{100 n^9} = \lim_{n\to\infty} \underbrace{\frac{1}{100} n}_{\to\infty} + \frac{1}{50} \cdot \underbrace{\frac{1}{n^7}}_{\to 0} + \frac{7}{100} \cdot \underbrace{\frac{1}{n^9}}_{\to 0} = \infty$$

$\rightarrow n^{10} + 2n^2 + 7 \notin O(100 n^9)$

(3) $e^{1.2n} \leq O(e^n)$ ✗

$$\lim_{n\to\infty} \frac{e^{1.2 \cdot n}}{e^n} = \lim_{n\to\infty} e^{1.2n - n} = \lim_{n\to\infty} e^{0.2n} = \infty$$

(4) $n^{\frac{2n+3}{n+1}} \leq O(n^2)$ ✓

"∞^0" $\qquad e^{\ln(x)} = x$

$$\lim_{n\to\infty} \frac{n^{\frac{2n+3}{n+1}}}{n^2} = \lim_{n\to\infty} n^{\frac{2n+3}{n+1} - 2 \cdot \frac{(n+1)}{(n+1)}} = \lim_{n\to\infty} n^{\frac{2n+3-2n-2}{n+1}} = \lim_{n\to\infty} n^{\frac{1}{n+1}} = \lim_{n\to\infty} e^{\ln(n^{\frac{1}{n+1}})} = \lim_{n\to\infty} e^{\frac{1}{n+1} \cdot \ln(n)}$$

$$= \lim_{n\to\infty} e^{\overbrace{\frac{\ln(n)}{n+1}}^{\to 0}}$$

$$= e^0 = 1 \in \mathbb{R}$$

(b) $f \leq O(g)$ $\qquad \lim_{n\to\infty} \frac{f(n)}{g(n)}$ existiert nicht

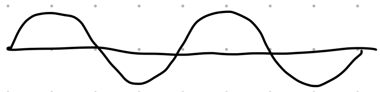
$g(x) = 1 \qquad f(x) = \sin(x) + 1$

$$\lim_{n\to\infty} \frac{\sin(x)+1}{1} = \lim_{n\to\infty} \sin(x) + 1$$

$2 \cdot g \geq f$

$-1 \leq \sin(x) \leq 1$

+1



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Exercise 2.3

$$\sum_{i=1}^{n} \frac{1}{i} \le O(\log n) \qquad \log(n) \le O\left(\sum_{i=1}^{n} \frac{1}{i}\right) \qquad \Rightarrow \sum_{i=1}^{n} \frac{1}{i} \in \Theta(\log n)$$

$n = 2^k$

$$S_j = \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i}$$

(a)

$S_j \le 1$

$$S_j = \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i} = \overset{\frac{1}{2^{j-1}} \ge}{\frac{1}{2^{j-1}+1}} + \overset{\frac{1}{2^{j-1}} \ge}{\frac{1}{2^{j-1}+2}} + \dots + \frac{1}{2^j} \le \underbrace{\frac{1}{2^{j-1}} + \frac{1}{2^{j-1}} + \dots + \frac{1}{2^{j-1}}}_{2^{j-1} \text{ Terme}} = 2^{j-1} \cdot \frac{1}{2^{j-1}} = 1 \checkmark$$

(b) z.z. $S_j \ge \frac{1}{2}$

$$S_j = \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i} = \underset{\ge \frac{1}{2^j}}{\frac{1}{2^{j-1}+1}} + \underset{\ge \frac{1}{2^j}}{\frac{1}{2^{j-1}+2}} + \dots + \frac{1}{2^j} \ge \underbrace{\frac{1}{2^j} + \frac{1}{2^j} + \dots + \frac{1}{2^j}}_{2^{j-1}} = 2^{j-1} \cdot \frac{1}{2^j} = \frac{1}{2} \checkmark$$

(c) z.z.: $\frac{k+1}{2} \le \sum_{i=1}^{2^k} \frac{1}{i} \le k+1$

Hint: $\sum_{i=1}^{2^k} \frac{1}{i} = 1 + \sum_{j=1}^{k} S_j$

$$\sum_{i=1}^{2^k} \frac{1}{i} = 1 + \sum_{j=1}^{k} \underbrace{S_j}_{\le 1} \le 1 + \sum_{j=1}^{k} 1 = 1+k \quad // \quad \checkmark$$

$$\sum_{i=1}^{2^k} \frac{1}{i} = 1 + \sum_{j=1}^{k} \underbrace{S_j}_{\ge \frac{1}{2}} \ge 1 + \sum_{j=1}^{k} \frac{1}{2} = 1 + \frac{k}{2} \ge \frac{1}{2} + \frac{k}{2} = \frac{k+1}{2} \quad // \quad \checkmark$$

(d) $\frac{\log_2(n)}{2} \le \sum_{i=1}^{n} \frac{1}{i} \le \log_2(n) + 2$

$k_1 = \lceil \log_2(n) \rceil$

$k_2 = \lfloor \log_2(n) \rfloor$

$$\boxed{\begin{matrix} x \le \lceil x \rceil < x+1 \\ x \ge \lfloor x \rfloor > x-1 \end{matrix}}$$

$$\sum_{i=1}^{n} \frac{1}{i} \le \sum_{i=1}^{2^{\lceil \log_2(n) \rceil}} \frac{1}{i} = \sum_{i=1}^{2^{k_1}} \frac{1}{i} \le 1 + k_1 = 1 + \lceil \log_2(n) \rceil \le 1 + \log_2(n) + 1 = \log_2(n) + 2 \quad \square$$

$k_2 = \lfloor \log_2(n) \rfloor$

$$\sum_{i=1}^{n} \frac{1}{i} \ge \sum_{i=1}^{2^{k_2}} \frac{1}{i} \ge \frac{k_2+1}{2} = \frac{\lfloor \log_2(n) \rfloor + 1}{2} \ge \frac{\log_2(n) - 1 + 1}{2} = \frac{\log_2(n)}{2} \checkmark \square$$

Exercise 2.5

f $\qquad f(a)+f(b)+f(c)=f(d)$

$1 \le a,b,c,d \le n$

(a)

```
for a=1 to n
    for b=1 to n
        for c=1 to n
            for d=1 to n
                if (f(a)+f(b)+f(c) == f(d))
                    return "YES"
return "NO"
```

(b) $f(k) \le k^3 \quad \Rightarrow O(n^3)$

```
array[] = new array[n^3]
for d=1 to n                                  } O(n)
    array[f(d)] = true
for a=1 to n
    for b=1 to n
        for c=1 to n                          } O(n^3)
            if (f(a)+f(b)+f(c) <= n^3)
                if (array[f(a)+f(b)+f(c)] == true)
                    return "YES"
return "NO"
```

$n+n^3 \le O(n^3)$

(c) $f(k) \le k^2 \quad \Rightarrow O(n^2)$

$f(a)+f(b)+f(c)=f(d)$

$\Longleftrightarrow f(a)+f(b)=f(d)-f(c)$

```
array[] = new array[2·n^2]                    } O(2n^2) ≤ O(n^2)
for a=1 to n
    for b=1 to n                              } O(n^2)
        arr[f(a)+f(b)] = true

for c=1 to n
    for d=1 to n                              } O(n^2)
        if (f(d)-f(c) >= 0)
            if (arr[f(d)-f(c)])
                return "YES"
return "NO"
```

$\Rightarrow 2n^2+n^2+n^2 = 4n^2 \le O(n^2)$

a) for $i=1$ to n
  for $j=1$ to n
    for $k=1$ to n
      f()

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}1 = \sum_{i=1}^{n}\sum_{j=1}^{n}n = \sum_{i=1}^{n}n^2 = n^3 \leq O(n^3)$$

b) for $i=1$ to n
  $k=\min(i,100)$
  for $j=1$ to k
    f()

$$\sum_{i=1}^{n}\sum_{j=1}^{\min(i,100)}1 = \sum_{i=1}^{n}\min(i,100) \leq \sum_{i=1}^{n}100 = 100\cdot n \leq O(n)$$

c) for $i=1$ to n
  if $i^2 \leq n$ ← $i^2 \leq n \Rightarrow i \leq \lfloor\sqrt{n}\rfloor$
    for $k=i$ to n
      f()
      f()
      f()

$$\sum_{i=1}^{\lfloor\sqrt{n}\rfloor}\sum_{k=i}^{n}3 = \sum_{i=1}^{\lfloor\sqrt{n}\rfloor}3(n-i+1) = \sum_{i=1}^{\lfloor\sqrt{n}\rfloor}3n-3i+3 \leq \sum_{i=1}^{\lfloor\sqrt{n}\rfloor}3n = 3n\cdot\lfloor\sqrt{n}\rfloor \leq 3n\sqrt{n} \leq O(n\sqrt{n})$$

$n\sqrt{n} \notin O(n)$

$$\boxed{\lim_{n\to\infty}\frac{n\sqrt{n}}{n} = \lim_{n\to\infty} n = \infty}$$

Peer Grading: 2.5

$3n+\sqrt{n} \in O(n)$ ✓

$$\lim_{n\to\infty}\frac{3n+\sqrt{n}}{n} = \lim_{n\to\infty} 3 + \cancel{\frac{1}{\sqrt{n}}}^{0} = 3 \in \mathbb{R}$$