Vagueness

- The Oxford Companion to Philosophy (1995):

  "Words like smart, tall, and fat are vague since in most contexts of use there is no bright line separating them from not smart, not tall, and not fat respectively …"

Vagueness

- Imprecision vs. Uncertainty:

  The bottle is about half-full.

  vs.

  It is likely to a degree of 0.5 that the bottle is full.

Fuzzy Sets

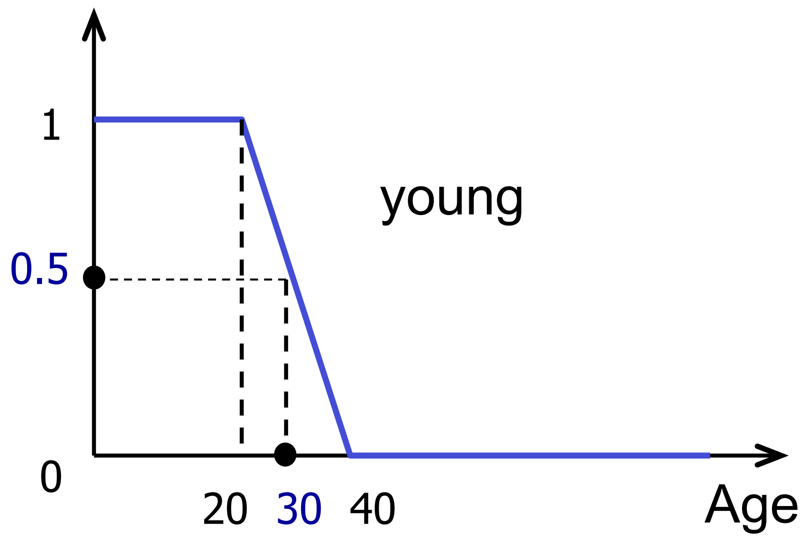
- Zadeh, L.A. (1965). Fuzzy Sets
  Journal of Information and Control

Fuzzy Set Definition

A fuzzy set is defined by a membership function that maps elements of a given domain (a crisp set) into values in [0, 1].

$\mu_A: U \rightarrow [0, 1]$

$\mu_A \leftrightarrow A$



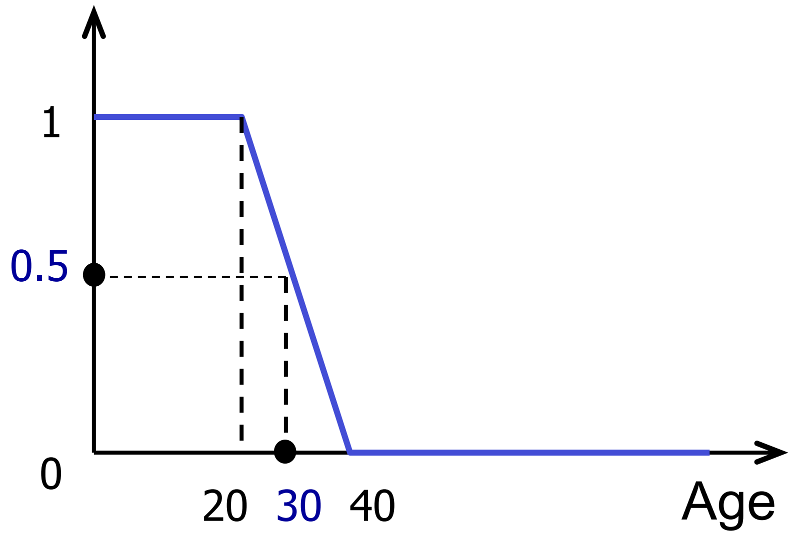
Fuzzy Set Representation

- Discrete domain:

  high-dice score: {1:0, 2:0, 3:0.2, 4:0.5, 5:0.9, 6:1}

- Continuous domain:

  $A(u) = 1$ for $u \in [0, 20]$
  $A(u) = (40 - u)/20$ for $u \in [20, 40]$
  $A(u) = 0$ for $u \in [40, 120]$



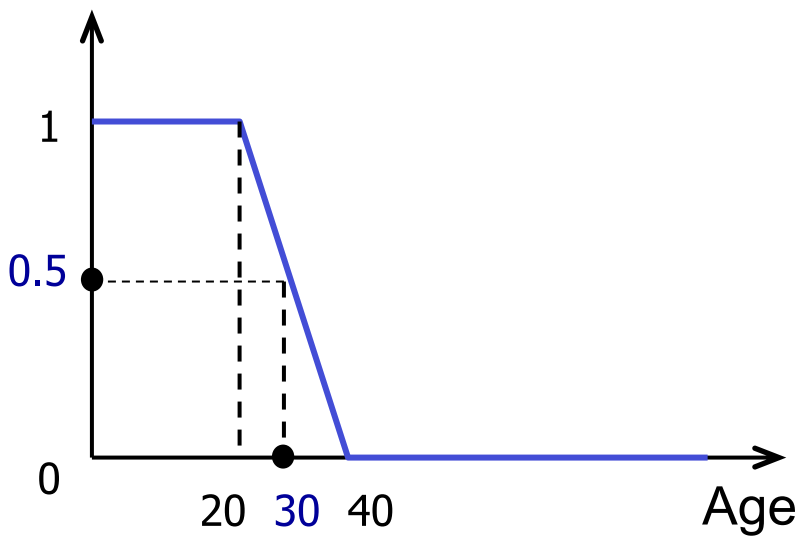
Fuzzy Set Representation

- α-cuts:

$A^{\alpha} = \{u \mid A(u) \geq \alpha\}$

$A^{\alpha+} = \{u \mid A(u) > \alpha\}$ strong α-cut

$A^{0.5} = [0, 30]$



Fuzzy Set Representation

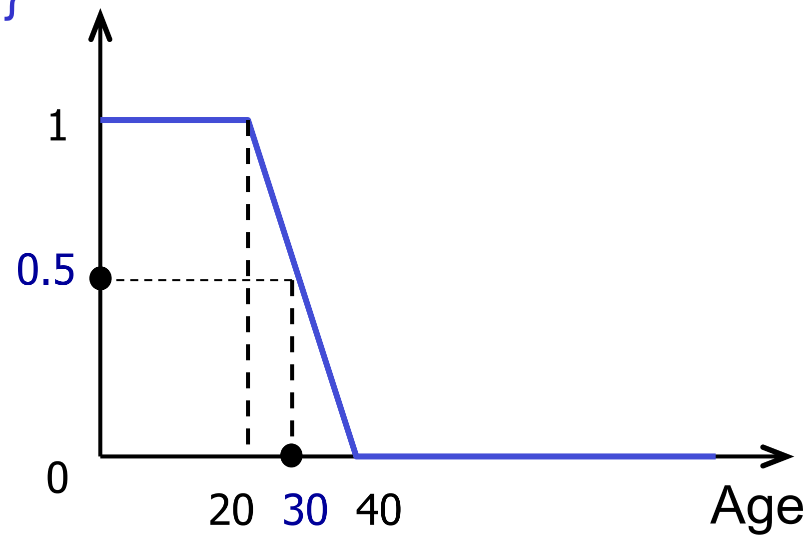
- α-cuts:

  $A^{\alpha} = \{u \mid A(u) \geq \alpha\}$

  $A^{\alpha+} = \{u \mid A(u) > \alpha\}$ strong α-cut

  $A(u) = \sup \{\alpha \mid u \in A^{\alpha}\}$

  $A^{0.5} = [0, 30]$



Fuzzy Set Representation

- **Support:**

  $\text{supp}(A) = \{u \mid A(u) > 0\} = A^{0+}$

- **Core:**

  $\text{core}(A) = \{u \mid A(u) = 1\} = A^{1}$

- **Height:**

  $h(A) = \sup_U A(u)$

Fuzzy Set Representation

- Normal fuzzy set: h(A) = 1
- Sub-normal fuzzy set: h(A) < 1

Membership Degrees

- Subjective definition

Membership Degrees

- Voting model:

  Each voter has a subset of U as his/her own crisp definition of the concept that A represents.

  A(u) is the proportion of voters whose crisp definitions include u.

  A defines a probability distribution on the power set of U across the voters.

Membership Degrees

- **Voting model:**

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
1										
2										
3	x	x								
4	x	x	x	x	x					
5	x	x	x	x	x	x	x	x	x	
6	x	x	x	x	x	x	x	x	x	x

Fuzzy Subset Relations

$A \subseteq B$ iff $A(u) \leq B(u)$ for every $u \in U$

Fuzzy Subset Relations

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A is more specific than B

Fuzzy Subset Relations

$A \subseteq B$ iff $A(u) \leq B(u)$ for every $u \in U$

A is more specific than B

"X is A" entails "X is B"

Fuzzy Set Operations

- Standard definitions:

Complement: $\overline{A}(u) = 1 - A(u)$

Intersection: $(A \cap B)(u) = \min[A(u), B(u)]$

Union: $(A \cup B)(u) = \max[A(u), B(u)]$

Fuzzy Set Operations

- Example:

    not young = $\overline{\text{young}}$

    not old = $\overline{\text{old}}$

    middle-age = not young$\cap$not old

    old = $\neg$young

Fuzzy Numbers

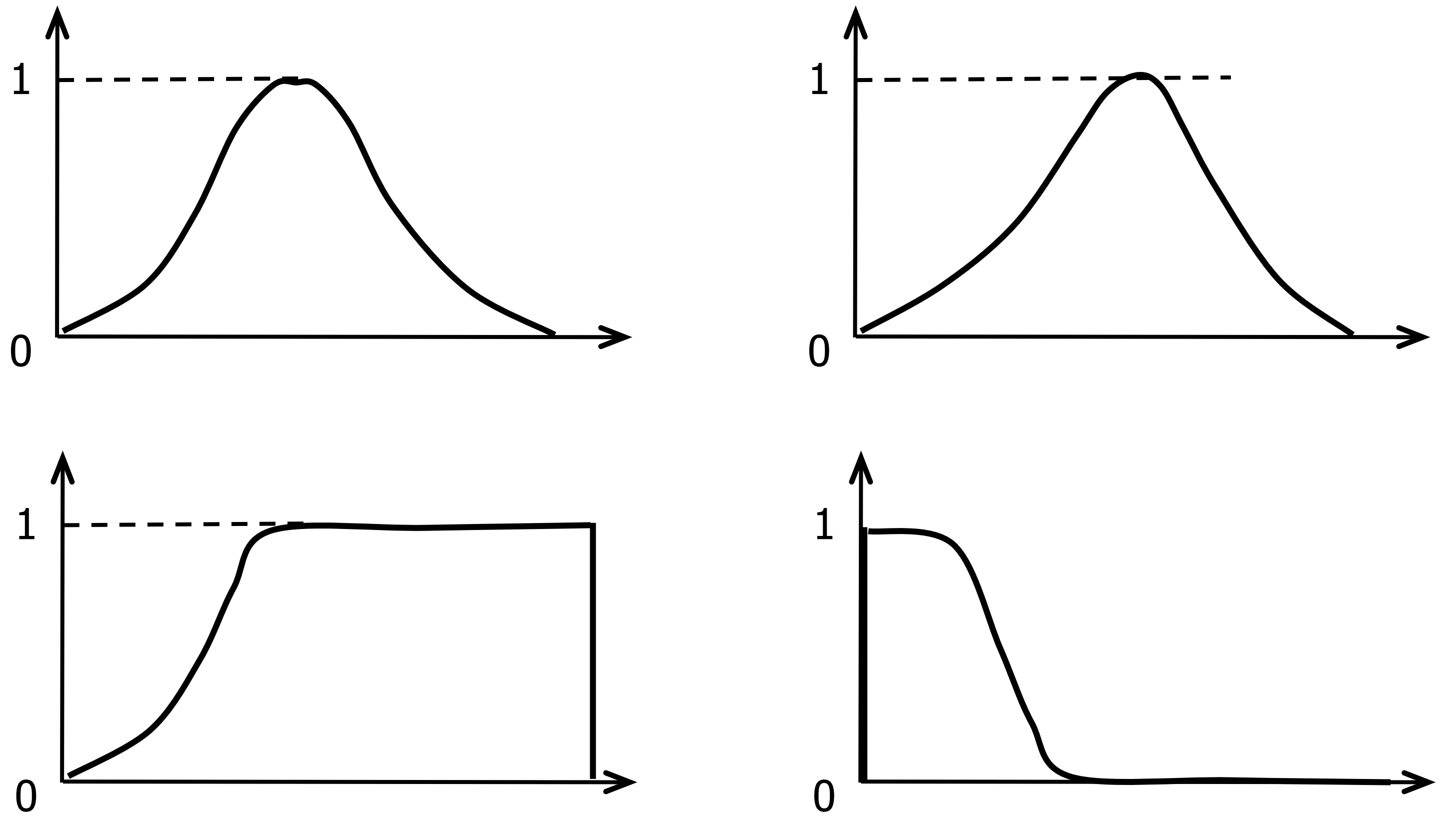
- A fuzzy number A is a fuzzy set on R:

  A must be a normal fuzzy set

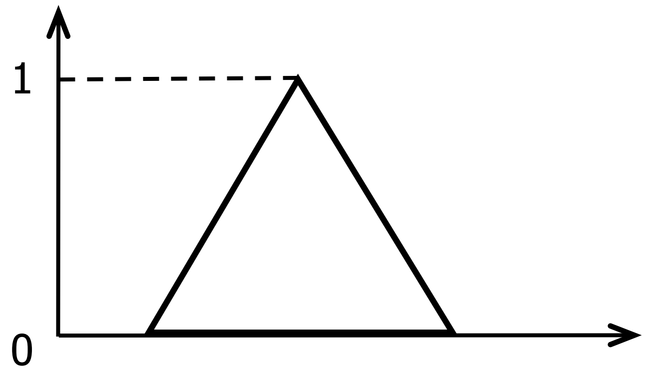
  A^{α} must be a closed interval for every $\alpha \in (0, 1]$

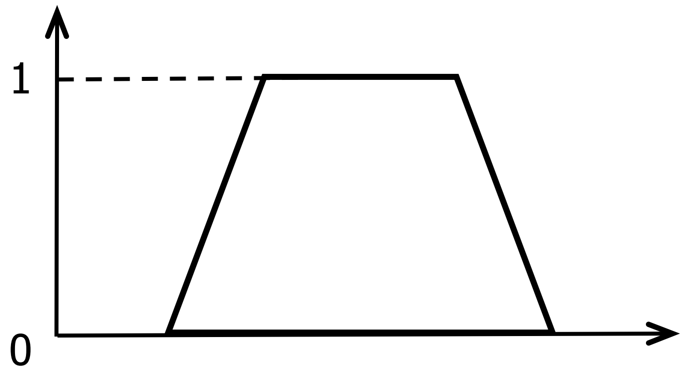
  $supp(A) = A^{0+}$ must be bounded

Basic Types of Fuzzy Numbers



Basic Types of Fuzzy Numbers





Operations of Fuzzy Numbers

- Extension principle for fuzzy sets:

  $f: U_1 \times \ldots \times U_n \rightarrow V$

  induces

  $g: \widetilde{U}_1 \times \ldots \times \widetilde{U}_n \rightarrow \widetilde{V}$

Operations of Fuzzy Numbers

- Extension principle for fuzzy sets:

$f: U_1 \times \ldots \times U_n \rightarrow V$

induces

$g: \tilde{U}_1 \times \ldots \times \tilde{U}_n \rightarrow \tilde{V}$

$[g(A_1,\ldots,A_n)](v) = \sup_{\{(u1,\ldots,un) \mid v = f(u1,\ldots,un)\}} \min\{A_1(u_1),\ldots,A_n(u_n)\}$

Operations of Fuzzy Numbers

- EP-based operations:

$$(A + B)(z) = \sup_{\{(x,y) \mid z = x+y\}} \min\{A(x),B(y)\}$$

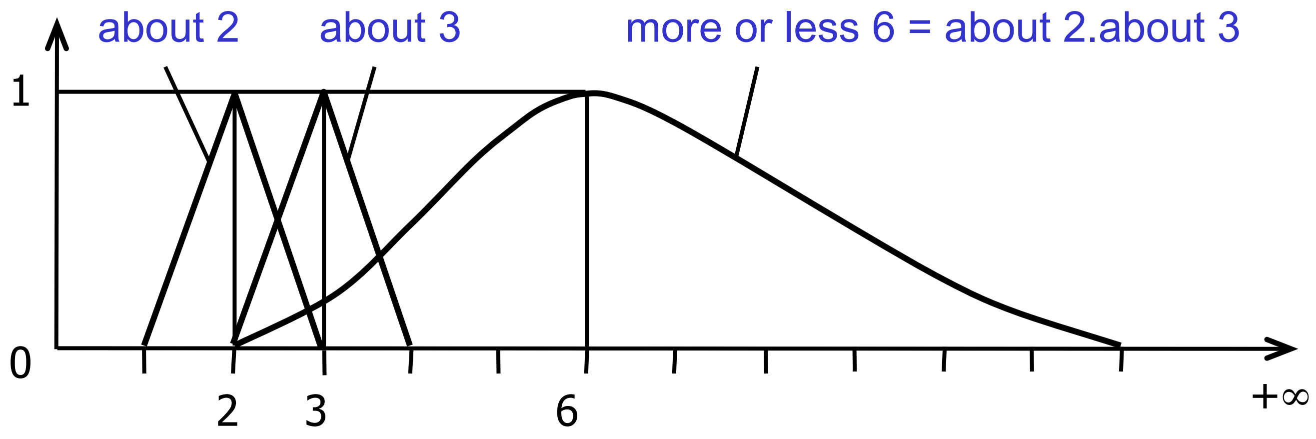
$$(A - B)(z) = \sup_{\{(x,y) \mid z = x-y\}} \min\{A(x),B(y)\}$$

$$(A * B)(z) = \sup_{\{(x,y) \mid z = x*y\}} \min\{A(x),B(y)\}$$

$$(A / B)(z) = \sup_{\{(x,y) \mid z = x/y\}} \min\{A(x),B(y)\}$$

Operations of Fuzzy Numbers

- EP-based operations:



Operations of Fuzzy Numbers

- Discrete domains:

$$A = \{x_i: A(x_i)\} \qquad B = \{y_i: B(y_i)\}$$

$$A \circ B = ?$$

Operations of Fuzzy Numbers

- Interval-based operations:

$$(A \circ B)^{\alpha} = A^{\alpha} \circ B^{\alpha}$$

Operations of Fuzzy Numbers

- Arithmetic operations on intervals:

$$[a, b] \circ [d, e] = \{f \circ g \mid a \leq f \leq b, d \leq g \leq e\}$$

Operations of Fuzzy Numbers

- Arithmetic operations on intervals:

$$[a, b] \circ [d, e] = \{f \circ g \mid a \le f \le b, d \le g \le e\}$$

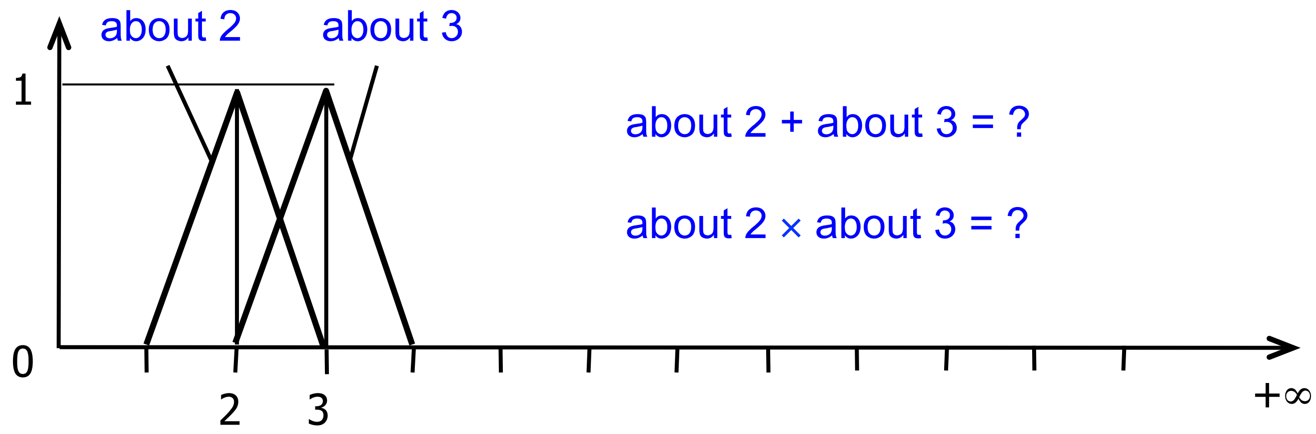
$$[a, b] + [d, e] = [a + d, b + e]$$

$$[a, b] - [d, e] = [a - e, b - d]$$

$$[a, b]*[d, e] = [\min(ad, ae, bd, be), \max(ad, ae, bd, be)]$$

$$[a, b]/[d, e] = [a, b]*[1/e, 1/d] \quad 0 \notin [d, e]$$

Operations of Fuzzy Numbers



about 2 + about 3 = ?

about 2 × about 3 = ?

Operations of Fuzzy Numbers

- Discrete domains:

$$A = \{x_i: A(x_i)\} \qquad B = \{y_j: B(y_j)\}$$

$$A \circ B = ?$$

Possibility Theory

- Possibility vs. Probability
- Possibility and Necessity

Possibility Theory

- Zadeh, L.A. (1978). Fuzzy Sets as a Basis for a Theory of Possibility

  Journal of Fuzzy Sets and Systems

Possibility Theory

- Membership degree = possibility degree

Possibility Theory

- Axioms:

$0 \leq Pos(A) \leq 1$

$Pos(\Omega) = 1$ $\qquad$ $Pos(\varnothing) = 0$

$Pos(A \cup B) = \max[Pos(A), Pos(B)]$

$Nec(A) = 1 - Pos(\overline{A})$

Possibility Theory

- Derived properties:

$\text{Nec}(\Omega) = 1 \qquad \text{Nec}(\varnothing) = 0$

$\text{Nec}(A \cap B) = \min[\text{Nec}(A), \text{Nec}(B)]$

$\max[\text{Pos}(A), \text{Pos}(\overline{A})] = 1$

$\min[\text{Nec}(A), \text{Nec}(\overline{A})] = 0$

$\text{Pos}(A) + \text{Pos}(\overline{A}) \geq 1$

$\text{Nec}(A) + \text{Nec}(\overline{A}) \leq 1$

$\text{Nec}(A) \leq \text{Pos}(A)$

Possibility Theory

Probability	Possibility
$p: U \rightarrow [0, 1]$ $Pro(A) = \sum_{u \in A} p(u)$	$r: U \rightarrow [0, 1]$ $Pos(A) = \max_{u \in A} r(u)$
Normalization: $\sum_{u \in U} p(u) = 1$	$\max_{u \in U} r(u) = 1$
Additivity: $Pro(A \cup B) = Pro(A) + Pro(B) - Pro(A \cap B)$	$Pos(A \cup B) = \max[Pos(A), Pos(B)]$
$Pro(A) + Pro(\overline{A}) = 1$	$Pos(A) + Pos(\overline{A}) \geq 1$
Total ignorance: $p(u) = 1/\lvert U \rvert$ for every $u \in U$	$r(u) = 1$ for every $u \in U$
Probability-possibility consistency principle: $Pro(A) \leq Pos(A)$	

Fuzzy Relations

- Crisp relation:

$$R(U_1, ..., U_n) \subseteq U_1 \times ... \times U_n$$

$$R(u_1, ..., u_n) = 1 \text{ iff } (u_1, ..., u_n) \in R \text{ or } = 0 \text{ otherwise}$$

Fuzzy Relations

- Crisp relation:

  $R(U_1, \ldots, U_n) \subseteq U_1 \times \ldots \times U_n$

  $R(u_1, \ldots, u_n) = 1$ iff $(u_1, \ldots, u_n) \in R$ or $= 0$ otherwise

- Fuzzy relation is a fuzzy set on $U_1 \times \ldots \times U_n$

Fuzzy Relations

- Fuzzy relation:

  U_1 = {New York, Paris}, U_2 = {Beijing, New York, London}

  R = very far

	NY	Paris
Beijing	1	.9
NY	0	.7
London	.6	.3

R = {(NY, Beijing): 1, ...}

Multivalued Logic

- Truth values are in [0, 1]

- Lukasiewicz:

$$\neg a = 1 - a$$

$$a \wedge b = \min(a, b)$$

$$a \vee b = \max(a, b)$$

$$a \Rightarrow b = \min(1, 1 - a + b)$$

Fuzzy Logic

if x is A then y is B

x is A*

----

y is B*

Fuzzy Logic

if height is TALL then shoe-size is LARGE

height is RATHER TALL

----

y is ??

Fuzzy Logic

- View a fuzzy rule as a fuzzy relation

if x is A then y is B	$R(u, v) \equiv A(u) \Rightarrow B(v)$
x is A*	$A^*(u)$
------------------------	--
y is B*	$B^*(v)$

Fuzzy Logic

- Measure similarity of A and A*

if x is A then y is B
x is A*
----
y is B*

$$B^* = B + \Delta(A/A^*)$$

Fuzzy Controller

- As special expert systems
- When difficult to construct mathematical models
- When acquired models are expensive to use

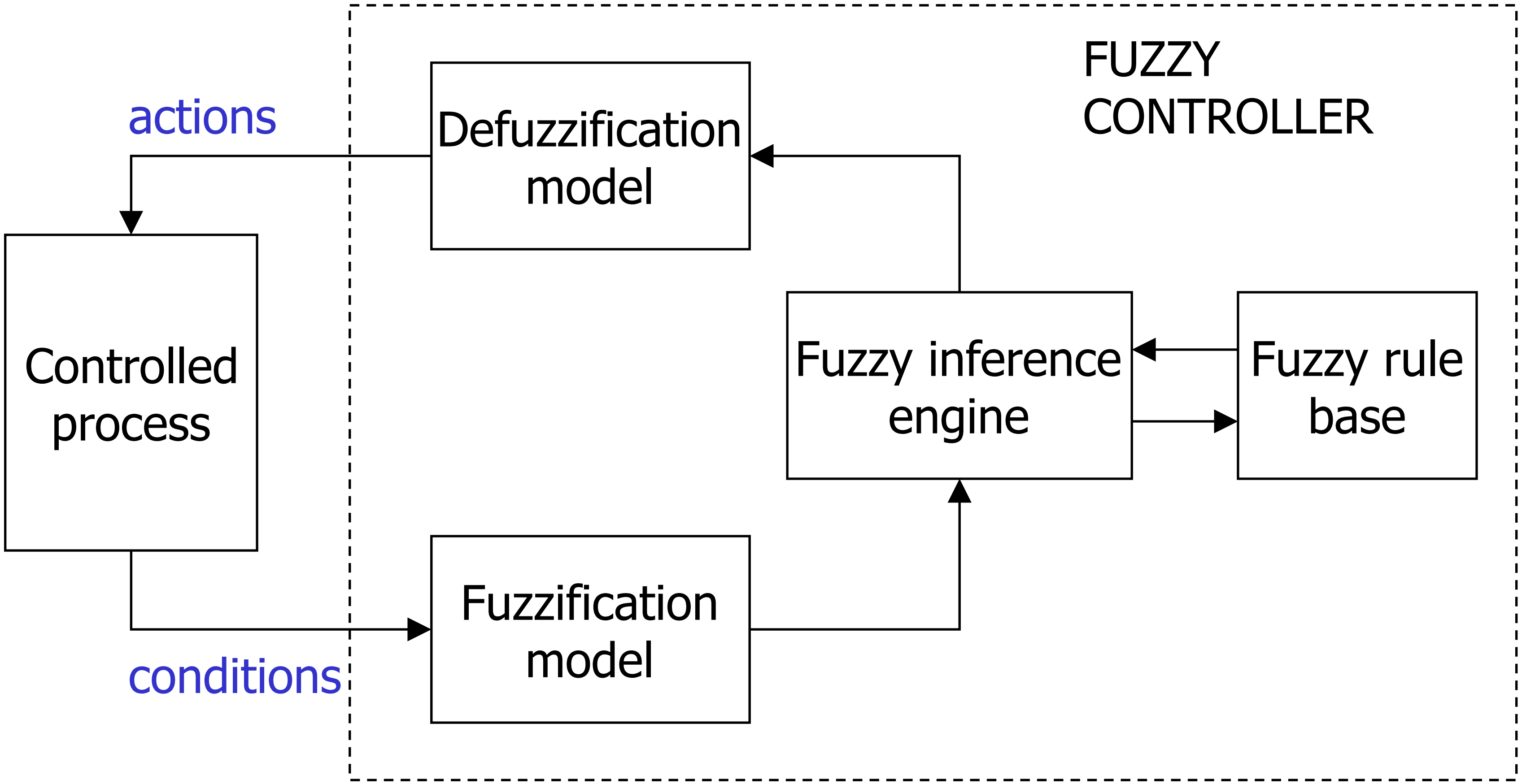
Fuzzy Controller

IF the temperature is very high

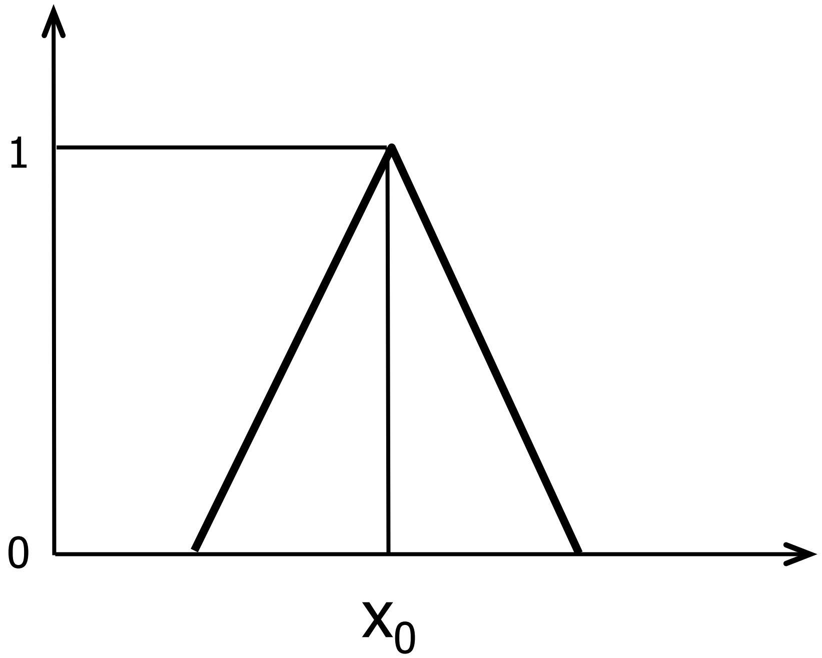
AND the pressure is slightly low

THEN the heat change should be sligthly negative

Fuzzy Controller



Fuzzification



Defuzzification

- Center of Area:

$$x = (\sum A(z).z)/\sum A(z)$$

Defuzzification

- Center of Maxima:

$$M = \{z \mid A(z) = h(A)\}$$

$$x = (\min M + \max M)/2$$

Defuzzification

- Mean of Maxima:

$$M = \{z \mid A(z) = h(A)\}$$

$$x = \sum z/|M|$$

Exercises

- In Klir's FSFL: 1.9, 1.10, 2.11, 4.5, 5.1 (a)-(b), 8.6, 12.1.