Problem #2 (Basket Options & Correlation)

Basket Call Options Overview

Basket call options are financial derivatives that derive their value from the performance of a *basket* of underlying assets. These options allow investors to bet on the **average performance** of several assets, rather than just one, providing a way to *diversify risk* across multiple securities.

Payoff Formula

The payoff of a basket call option at maturity can be described by the formula:

$$\left(\frac{1}{d}\sum_{i=1}^{d} S_T^i - K\right)^+, \tag{5}$$

where:

- S_T^i is the price at maturity T of the i-th asset in the basket.
- d is the total number of assets in the basket.
- K is the strike price of the option.

This payoff formula calculates the *average* of the final prices of the d assets, subtracts the strike price, and applies a **positive part function**, which ensures that the payoff is **non-negative**.

Asset Dynamics

The underlying assets in the basket are assumed to follow a d-dimensional geometric Brownian motion (GBM), represented by the stochastic differential equation (SDE):

$$\frac{dS_t^i}{S_t^i} = r dt + \sigma^i dW_t^i, \tag{6}$$

where:

- r is the risk-free interest rate, assumed to be constant over time.
- σ^i is the volatility of the i-th asset.
- W_t^i represents the i-th component of a d-dimensional Brownian motion.

Correlation Structure

The assets in the basket are correlated, as reflected by their movements being driven by a common set of Brownian motions $\{W_t^1, \ldots, W_t^d\}$. The correlation among these assets is captured by the covariance matrix:

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{pmatrix}, \tag{7}$$

where:

- ρ represents the pairwise Pearson correlation coefficient between any two different assets.

Statistical Properties

The Brownian motions are assumed to have a multivariate normal distribution across time with the mean vector zero and the covariance matrix scaled by time:

$$\{W_t^1, \ldots, W_t^d\} \sim \mathcal{N}(\mu = 0, \sigma^2 = t \cdot \mathbf{\Sigma}),\ \forall t \in \mathbb{R}^+, \tag{8}$$

where $t \cdot \mathbf{\Sigma}$ indicates that the variance of each Brownian motion component grows linearly with time, and the correlation structure remains constant over time.

Synthesis

Basket call options are complex financial instruments that require a deep understanding of stochastic processes, correlation, and risk management. This framework provides a way to price such options and evaluate their potential payoff in a multi-asset setting.