FE621 - Homework #5

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Pledge: I pledge my honor that I have abided by the Stevens Honor System.

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Problem 1 (Portfolio Wealth Growth)

1.1 Portfolio Wealth Growth Theory

This section delves into the theoretical mathematical foundation governing the growth of portfolio wealth over time. The analysis is crucial for understanding how investments evolve under the influence of various market factors, including returns and volatility.

1.1.1 Mathematical Formulation

The wealth process $\{V_t\}_{t\geq 0}$ is modeled as a geometric Brownian motion (GBM), which is frequently used to represent stock prices and, by extension, portfolio values under stochastic environments. The stochastic differential equation (SDE) governing this process is given by:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t \tag{1}$$

Portfolio Wealth Growth Simulation vs. Expectation Here:

- V_t represents the portfolio value at time t.
- μ is the expected return of the portfolio, expressed as a percentage of the portfolio value.
- σ is the volatility of the portfolio, which measures the standard deviation of the portfolio's returns.
- dW_t is the increment of a standard Brownian motion, which captures the random fluctuations in the market.

Interpretation

Equation (1) can be interpreted as follows:

- The term μdt captures the expected growth of the portfolio due to returns over an infinitesimally small time interval dt.

- The term σdW_t introduces randomness into the growth process, reflecting the uncertainty and risk inherent in the financial markets.

Solution to the Differential Equation

The solution to the stochastic differential equation (SDE) given in equation (1) can be expressed explicitly by integrating both sides over the interval from 0 to t:

$$\ln \frac{V_t}{V_0} = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t \tag{2}$$

where:

- V_0 is the initial value of the portfolio at time $t = 0$.
- W_t represents the standard Brownian motion at time t.

From equation (2), we can exponentiate both sides to obtain the explicit form of V_t:

$$V_t = V_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right) \tag{3}$$

Mathematical Synthesis

Equation (3) clearly shows how the portfolio value V_t evolves over time. It indicates that the portfolio value is log-normally distributed with its mean and variance increasing over time. This formulation is fundamental in finance for modeling asset prices and helps in understanding the dynamic nature of investment growth under uncertainty.

1.1.2 Expectation of Portfolio Wealth

The following section explores and delves into the expectation (first raw moment/arithmetic average) of the portfolio wealth process.

Expectation Calculation

Given the wealth process V_t which follows a geometric Brownian motion (GBM) as described by:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t \tag{1}$$

The solution to this stochastic differential equation (SDE) indicates:

$$V_t = V_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right) \tag{3}$$

To find the expectation $\mathbb{E}[V_t]$, we note that W_t is a standard Brownian motion (BM), which implies σW_t is normally distributed with mean 0 and variance $\sigma^2 t$. Thus, $\sigma W_t \sim N(0, \sigma^2 t)$, and $e^{\sigma W_t}$ follows a log-normal distribution.

We can use the moment-generating function (MGF) of a normally distributed random variable to compute the expectation of a log-normal variable. For a random variable $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, the MGF of X at s is $M_X(s) = e^{\mu_X s + \frac{1}{2}\sigma_X^2 s^2}$. Setting $s = 1$, we find:

$$\mathbb{E}[e^X] = e^{\mu_X + \frac{1}{2}\sigma_X^2} \tag{4}$$

Applying this to our case, where $\mu_X = 0$ and $\sigma_X^2 = \sigma^2 t$, we get:

$$\mathbb{E}[e^{\sigma W_t}] = e^{0 + \frac{1}{2}\sigma^2 t} = e^{\frac{1}{2}\sigma^2 t} \tag{5}$$

Now, substituting this into the solution for V_t:

$$\begin{aligned}\mathbb{E}[V_t] &= \mathbb{E}\left[V_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right)\right] \\ &= V_0 \exp\left((\mu - \frac{\sigma^2}{2})t\right)\mathbb{E}[e^{\sigma W_t}]\end{aligned} \tag{6}$$

Substituting the expectation of $e^{\sigma W_t}$:

$$\begin{aligned}\mathbb{E}[V_t] &= V_0 \exp\left((\mu - \frac{\sigma^2}{2})t\right)\exp\left(\frac{1}{2}\sigma^2 t\right) \\ &= V_0 \exp(\mu t)\end{aligned} \tag{7}$$

Thus, the expected wealth at time (t) is indeed given by:

$$\mathbb{E}[V_t] = V_0 \exp(\mu t) \tag{7}$$

This demonstrates that the expectation grows exponentially at a rate determined by the drift μ, independent of the volatility σ.

1.2 Portfolio Wealth Growth Implementation

The following section implements applications of factors governing the growth of portfolio wealth over time in Python.

1.2.1 True vs. Expected Path Simulation

This Python code snippet simulates 50 paths of a portfolio's wealth process $\{V_t\}_{t\in[0,T]}$ modeled as a geometric Brownian motion (GBM) alongside the expected (arithmetic average) path $\{\mathbb{E}[V_t]\}_{t\in[0,T]}$. We use the parameters $\mu = 0.08$, $\sigma = 0.2$, $T = 30$ years, and an initial portfolio value $V_0 = 100$.

Code Breakdown

Step 1: Import Libraries:

- `numpy` for numerical operations.
- `matplotlib.pyplot` for plotting the results.

```
import numpy as np
import matplotlib.pyplot as plt
```

Step 2: Set Parameters:

- `mu` : the expected return rate of the portfolio.
- `sigma` : the volatility or standard deviation of returns.
- `T` : the total time horizon for the simulation (30 years).
- `dt` : the time increment for each step in the simulation.
- `V0` : the initial portfolio value.
- `N` : the number of time steps calculated as the total time divided by the increment.
- `num_paths` : the number of simulation paths.

```
mu = 0.08       # drift coefficient
sigma = 0.2     # volatility coefficient
T = 30          # time horizon
dt = 0.01       # time increment
V0 = 100        # initial wealth
N = int(T/dt)   # number of time steps
num_paths = 50  # number of paths to simulate
```

Step 3: Simulate Paths:

- Generate multiple paths of the GBM using random normal distributions to simulate daily returns.
- Calculate the portfolio value over time for each path based on the GBM formula.

```
np.random.seed(42)  # for reproducibility
paths = np.zeros((num_paths, N))
for i in range(num_paths):
    dB = np.sqrt(dt) * np.random.normal(size=N-1)
    W = np.cumsum(dB)
    W = np.insert(W, 0, 0)  # insert the initial condition W_0 = 0
    paths[i] = V0 * np.exp((mu - 0.5 * sigma**2) * t + sigma * W)
```

Step 4: Calculate Expected Path:

- Compute the expected path using the deterministic part of the GBM formula.

```
expected_path = V0 * np.exp(mu * t)
```

Step 5: Plot the Results:

- Plot all simulated paths and the expected path to visualize the potential variance around the expected growth.

```
plt.figure(figsize=(12, 8))
for path in paths:
    plt.plot(t, path, 'r', linewidth=0.5, alpha=0.5)  # red lines for
```

```
simulated paths
plt.plot(t, expected_path, 'b', linewidth=2.5, label='Expected Path
$\mathbb{E}[V_t]$')  # blue line for the expected path
plt.title('Simulation of Portfolio Wealth Growth and Expected Path')
plt.xlabel('Time (years)')
plt.ylabel('Portfolio Value')
plt.legend()
plt.grid(True)
plt.show()
```

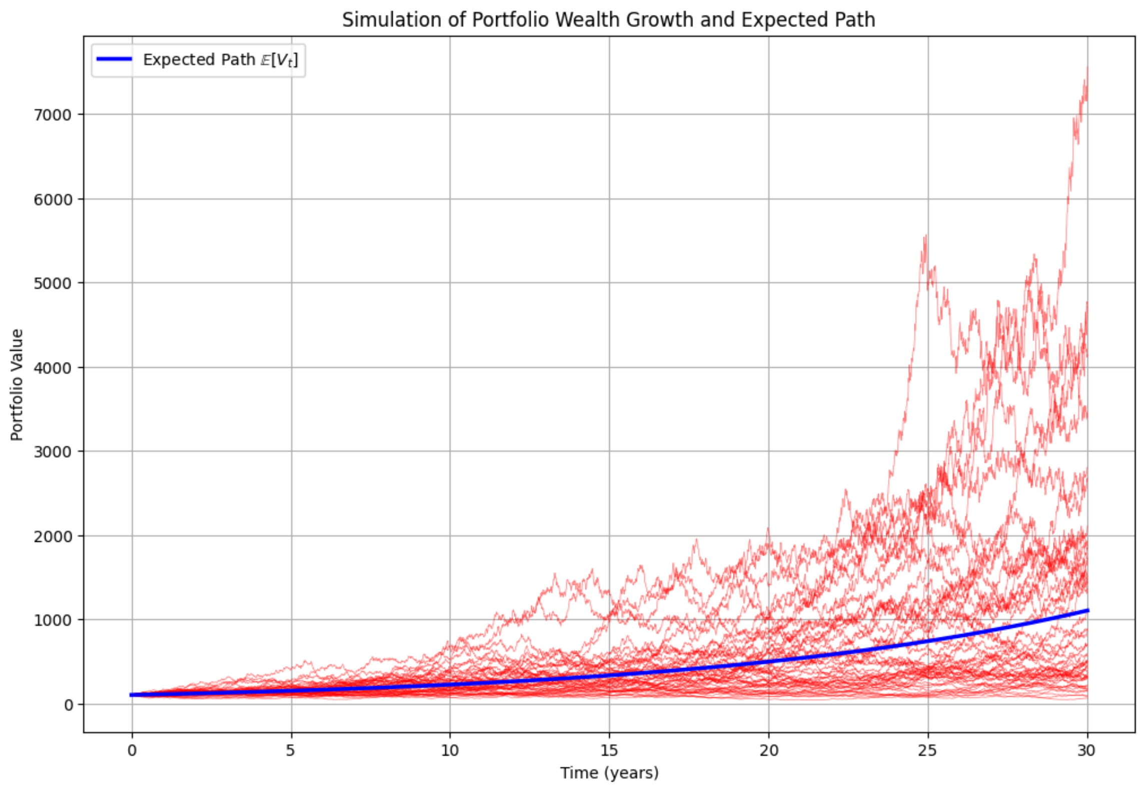

Figure 1 - Portfolio Wealth Growth Simulation vs. Expectation

1.2.2 Underperformance Confidence Interval Simulation

In this section, we estimate the probability that the terminal wealth V_T underperforms the expected terminal wealth $\mathbb{E}[V_T]$ by varying degrees specified by α. We will compute the 95% confidence intervals for these probabilities.

Code Breakdown

Step 1: Library Importation

First, we need to import the necessary Python libraries for calculations and data handling.

```
import numpy as np
import scipy.stats as stats
```

Step 2: Parameter Definition

We will define the parameters for the simulation, including the number of simulations n, and setup the range for α.

```
# Parameters
mu = 0.08         # drift coefficient
sigma = 0.2       # volatility coefficient
T = 30            # time horizon
V0 = 100          # initial wealth
n = 10000         # number of simulations
alphas = np.arange(1, 0, -0.1)  # range of alpha from 1 to 0.1
```

Step 3: Terminal Wealth Value Simulation

Simulate the terminal wealth values V_T using the geometric Brownian motion model.

```
np.random.seed(42)  # for reproducibility
terminal_values = V0 * np.exp((mu - 0.5 * sigma**2) * T + sigma *
np.sqrt(T) * np.random.normal(size=n))
```

Step 4: Underperformance Probability Computation

For each α_i, calculate the probability that V_T is less than or equal to $\alpha \times \mathbb{E}[V_T]$. The expectation $\mathbb{E}[V_T]$ is computed based on its analytical expression.

```
expected_VT = V0 * np.exp(mu * T)  # calculate expected V_T
probabilities = [np.mean(terminal_values <= alpha * expected_VT) for alpha
in alphas]
```

Step 5: Confidence Interval Computation

Calculate the 95% confidence intervals for the probabilities of underperformance using the normal approximation.

```
confidence_intervals = [stats.norm.interval(0.95, loc=p, scale=np.sqrt((p*
(1-p))/n)) for p in probabilities]
```

Step 6: Result Output

Finally, output the results in a structured format.

```
print("Alpha\tProbability\t95% Confidence Interval")
for alpha, p, ci in zip(alphas, probabilities, confidence_intervals):
    print(f"{alpha:.1f}\t{p:.4f}\t{ci}")
```

1.2.3 Underperformance Confidence Interval Simulation Results

The table below presents the estimated probabilities of underperformance $\mathbb{P}(V_T \leq \alpha\mathbb{E}[V_T])$ at various levels of α along with their 95% confidence intervals, based on $n = 10,000$ simulations.

Alpha	Probability	95% Confidence Interval
1.0	0.7103	(0.7014, 0.7192)
0.9	0.6763	(0.6671, 0.6855)
0.8	0.6373	(0.6279, 0.6467)
0.7	0.5888	(0.5792, 0.5984)
0.6	0.5334	(0.5236, 0.5432)
0.5	0.4651	(0.4553, 0.4749)
0.4	0.3871	(0.3776, 0.3966)
0.3	0.2913	(0.2824, 0.3002)
0.2	0.1797	(0.1722, 0.1872)
0.1	0.0600	(0.0553, 0.0647)

Table 1 - Probabilities of Terminal Wealth Underperformance at Various Thresholds with 95% Confidence Intervals

This table helps visualize the varying degrees of underperformance risk associated with different thresholds of expected wealth, providing a statistical outlook on potential investment outcomes over time.

1.3 Formulaic Derivations & Theory of Terminal Wealth Underperformance

This section explores the formulaic derivations and the theoretical aspects of the underperformance of terminal wealth V_T, defined as $\mathbb{P}(V_T \leq \alpha \mathbb{E}[V_T])$.

1.3.1 Formula Derivation for Underperformance

We begin by rigorously defining and deriving the formula for the probability $\mathbb{P}(V_T \leq \alpha \mathbb{E}[V_T])$. To do this, consider that V_T, the terminal wealth, follows a log-normal distribution due to its dependence on a geometric Brownian motion (GBM) process:

$$V_T = V_0 \exp\left((\mu - \frac{1}{2}\sigma^2)T + \sigma W_T\right) \tag{8}$$

where:

- V_0 is the initial wealth.
- μ is the expected return.
- σ is the volatility.
- W_T is a standard Brownian motion at time T.

Given that W_T follows a normal distribution $W_T \sim \mathcal{N}(\mu = 0, \sigma^2 = T)$, the variable σW_T is normally distributed as $\mathcal{N}(0, \sigma^2 T)$. Thus, we can transform V_T to a standard normal variable by logarithmic transformation and normalization:

$$\ln V_T = \ln V_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma W_T \tag{9}$$

Let's define random variable Z as:

$$Z = \frac{\ln V_T - \ln V_0 - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \sim \mathcal{N}(0, 1) \tag{10}$$

We want to find the probability that $V_T \leq \alpha \mathbb{E}[V_T]$. The expected terminal wealth $\mathbb{E}[V_T]$ is given by:

$$\mathbb{E}[V_T] = V_0 \exp(\mu T) \tag{11}$$

Thus, the inequality $V_T \leq \alpha \mathbb{E}[V_T]$ becomes:

$$V_T \leq \alpha V_0 \exp(\mu T) \tag{12}$$

Taking logarithms:

$$\ln V_T \leq \ln(\alpha V_0 \exp(\mu T)) = \ln(\alpha) + \ln V_0 + \mu T \tag{13}$$

Now, substituting Z:

$$\ln V_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z \leq \ln(\alpha) + \ln V_0 + \mu T \tag{14}$$

Simplifying and solving for Z:

$$Z \leq \frac{\ln(\alpha) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \tag{15}$$

This transformation yields a probability:

$$\mathbb{P}(V_T \leq \alpha \mathbb{E}[V_T]) = \Phi\left(\frac{\ln(\alpha) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \tag{16}$$

where Φ is the cumulative distribution function (CDF) of the standard normal distribution. In the next section, we will compare these theoretical probabilities with those estimated in Table 1, and discuss the extent to which our calculated confidence intervals capture these exact, theoretical probabilities.

1.3.2 Comparative Analysis of Theoretical and Empirical Probabilities of Wealth Underperformance

In this section, we compare the theoretical probabilities calculated for the underperformance of terminal wealth V_T against the empirical estimates obtained from simulations, as presented in Table 1. The goal is to evaluate how well our simulated data aligns with the theoretical expectations derived from the probability density function of a log-normally distributed variable.

Theoretical Probabilities Calculation

Using the derived formula from the previous section:

$$\mathbb{P}(V_T \leq \alpha \mathbb{E}[V_T]) = \Phi\left(\frac{\ln(\alpha) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \tag{16}$$

we calculate the theoretical probabilities for each α using the cumulative distribution function (CDF) Φ of the standard normal distribution. Here, $\sigma = 0.2$, $T = 30$, and the different values of α range from 1.0 to 0.1. Please note that these theoretical probabilities were computed in Python.

Comparison with Empirical Estimates

The table below compares the theoretical probabilities with the empirical estimates:

Alpha	Empirical Probability	Theoretical Probability	95% Confidence Interval
1.0	0.7103	0.7081	(0.7014, 0.7192)
0.9	0.6763	0.6742	(0.6671, 0.6855)
0.8	0.6373	0.6346	(0.6279, 0.6467)
0.7	0.5888	0.5879	(0.5792, 0.5984)
0.6	0.5334	0.5324	(0.5236, 0.5432)
0.5	0.4651	0.4661	(0.4553, 0.4749)
0.4	0.3871	0.3864	(0.3776, 0.3966)
0.3	0.2913	0.2907	(0.2824, 0.3002)
0.2	0.1797	0.1784	(0.1722, 0.1872)
0.1	0.0600	0.0601	(0.0553, 0.0647)

Table 2 - Comparison of Empirical and Theoretical Probabilities of Terminal Wealth Underperformance

Discussion

From the table above, we can observe how closely the empirical probabilities and their confidence intervals align with the theoretical probabilities. This comparison helps validate our simulation methodology and provides insight into the accuracy and reliability of our model in capturing the dynamics of terminal wealth underperformance.

1.3.3 Probability of Underperformance Convergence

This section explores the theoretical convergence of the probability $\mathbb{P}(V_T \leq \alpha\mathbb{E}[V_T])$ as the time horizon T approaches infinity. Understanding this convergence provides insights into the long-term risk profile of the investment strategy.

Theoretical Background

Given the stochastic model for terminal wealth:

$$V_T = V_0 \exp\left((\mu - \frac{1}{2}\sigma^2)T + \sigma W_T\right) \tag{8}$$

where W_T is a standard Brownian motion, the variable V_T is log-normally distributed. The expected value of V_T is $V_0 \exp(\mu T)$.

Convergence Analysis

As T increases, the term σW_T—which has a mean of 0 and a standard deviation of $\sigma\sqrt{T}$—becomes dominant in influencing the distribution of $\ln V_T$. This leads us to consider the behavior of the ratio $\frac{V_T}{\mathbb{E}[V_T]}$:

$$\frac{V_T}{\mathbb{E}[V_T]} = \exp\left(-\frac{1}{2}\sigma^2 T + \sigma W_T\right) \tag{17}$$

As $T \to \infty$, the distribution of $\ln\left(\frac{V_T}{\mathbb{E}[V_T]}\right) = -\frac{1}{2}\sigma^2 T + \sigma W_T$ shifts towards $-\infty$, because the deterministic component $-\frac{1}{2}\sigma^2 T$ dominates. Thus, regardless of the value of α (as long as $\alpha > 0$):

$$\lim_{T\to\infty} \mathbb{P}\left(\frac{V_T}{\mathbb{E}[V_T]} \leq \alpha\right) = \lim_{T\to\infty} \mathbb{P}\left(-\frac{1}{2}\sigma^2 T + \sigma W_T \leq \ln(\alpha)\right) = 1$$

This implies that the probability that V_T underperforms $\alpha \times \mathbb{E}[V_T]$ approaches 1 as T becomes very large.

Implications

This result has significant implications for long-term investment strategies. It suggests that, under the assumed model, the relative underperformance compared to the growing expectation becomes almost certain in the long run. This highlights the importance of considering the effects of volatility and the time horizon in financial planning and risk assessment.