FE 621: HW5

Due date: Wednesday May 15th at 11:59 pm

- This is an individual assignment, but you may discuss (and only discuss) the problems with other students.
- Important: Present your results in an organized/clear/readable manner.
- The deadline for this assignment is firm. No late days can be used.

Problem 1 (Portfolio wealth growth)

Consider a wealth process $(V_t)_{t\geq 0}$ that follows a geometric Brownian motion:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t. \tag{0.1}$$

(i) Show that the expected wealth at time t is given by $\mathbb{E}[V_t] = V_0 e^{\mu t}$.

(ii) Display in a graph 50 simulated paths of $(V_t)_{0\leq t\leq T}$ along with the expected path $(\mathbb{E}[V_t])_{0\leq t\leq T}$. Use $\mu = 0.08$, $\sigma = 0.2$, $r = 0.03$, and $T = 30$.

(iii) Use simulation to estimate the probability of V_T underperforming the expected value $\mathbb{E}[V_T]$. Specifically, for $\alpha = 1, 0.9, 0.8, \ldots, 0.1$, use a table to report 95% confidence intervals for the probabilities $\mathbb{P}(V_T \leq \alpha\mathbb{E}[V_T])$. Use the parameters in part (ii) and $n = 10,000$ simulations.

(iv) Derive a formula for the probability $\mathbb{P}(V_T \leq \alpha\mathbb{E}[V_T])$ and include the exact probabilities in the table in part (iii). Do the confidence intervals capture the exact probabilities?

(v) What does the probability $\mathbb{P}(V_T \leq \alpha\mathbb{E}[V_T])$ converge to as $T \to \infty$?

(vi) Is there an inconsistency between the behavior of the expectation $\mathbb{E}[V_T]$ and the long-run behavior of individual trajectories V_T? Discuss the difference between the two.

Problem 2 (Estimating μ and σ)

Using step size $\Delta > 0$, partition the time interval $[0, T]$ into $m = T/\Delta$ sub-intervals of equal length. Assume that the returns of a stock in these intervals are uncorrelated with distribution

$$r_i^{(\Delta)} \sim \mathcal{N}\big(\mu\Delta, \sigma^2\Delta\big), \qquad 1 \le i \le m, \tag{0.2}$$

where μ and σ are the annual drift and volatility parameters.[1]

Based on a return sample $r_1^{(\Delta)}, \ldots, r_m^{(\Delta)}$, the historical estimators for μ and σ are given by

$$\hat{\mu} = \frac{\hat{\mu}^{(\Delta)}}{\Delta}, \qquad \hat{\sigma} = \frac{\hat{\sigma}^{(\Delta)}}{\sqrt{\Delta}}, \tag{0.3}$$

where

$$\hat{\mu}^{(\Delta)} = \frac{1}{m}\sum_{i=1}^{m} r_i^{(\Delta)}, \qquad \hat{\sigma}^{(\Delta)} = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\big(r_i^{(\Delta)} - \hat{\mu}^{(\Delta)}\big)^2}.$$

Note that $\hat{\mu}^{(\Delta)}$ and $\hat{\sigma}^{(\Delta)}$ are the estimators for the mean and volatility of returns with step size Δ, and the scaling in (0.3) annualizes those estimators.

(a) Write a function that takes parameters μ, σ, T, Δ, N and simulates the returns (0.2), and then computes the estimators $\hat{\mu}$ and $\hat{\sigma}$ in (0.3). Your function should repeat this n times to compute n independent realizations of the estimators $\hat{\mu}$ and $\hat{\sigma}$.

(b) For $\Delta = 1/12$ (monthly), $\Delta = 1/52$ (weekly), and $\Delta = 1/252$ (daily), use your function in part (a) to create histograms of your simulated estimators $\hat{\mu}$ and $\hat{\sigma}$. Use parameters $\mu = 0.12$, $\sigma = 0.16$, $T = 2$, and $N = 1000$.

Note: These histograms are referred to as sampling distributions; a sampling distribution shows how a sample statistic (such as $\hat{\mu}$ and $\hat{\sigma}$) behaves for different samples.

(c) Create a table that for each value of Δ reports the estimated mean and standard deviation of the estimators $\hat{\mu}$ and $\hat{\sigma}$ (i.e., the mean and standard deviation of the sampling distributions). The table should have 12 entries.

(d) Based on your results in (b) and (c):

(i) How do the means of $\hat{\mu}$ and $\hat{\sigma}$ depend on Δ? Do $\hat{\mu}$ and $\hat{\sigma}$ appear to be unbiased estimators for μ and σ (i.e., are the mean values of the simulated estimators close to the quantities they should estimate)?

(ii) How do the standard deviations of $\hat{\mu}$ and $\hat{\sigma}$ depend on Δ?

(iii) The standard deviations of $\hat{\mu}$ and $\hat{\sigma}$ are a measure of the typical error we make when using those estimators. Based on your results, do you think $\hat{\mu}$ and $\hat{\sigma}$ are useful estimators? Do they become more useful as Δ becomes smaller (i.e., when we have more frequent observations)?

[1] Note that the mean return scales with the interval length Δ, while the return volatility scales with the square root of Δ.

Problem 3 (Jump-diffusion model)

The risk-neutral dynamics of a stock are given by

$$\frac{dS_t}{S_t} = (r - \lambda k)dt + \sigma dW_t + (J-1)dN_t,$$

where r is the risk-free rate and $(N_t)_{t\geq 0}$ is a Poisson process with jump intensity λ. If there is a jump at time t (i.e., if $dN_t = 1$), then the jump size J satisfies $\log(J) \sim \mathcal{N}(\mu_J, \sigma_J^2)$ (log-normal jumps). Finally, the constant k is given by $k = \mathbb{E}[J-1] = e^{\mu_J + \sigma_J^2/2} - 1$.[2]

Under these dynamics, the stock price at time T can be written as

$$S_T = S_0 e^{(r - \lambda k - \frac{\sigma^2}{2})T + \sigma W_T} \prod_{i=1}^{N_T} J_i, \tag{0.4}$$

where $N_T \sim Poi(\lambda T)$ is the number of jumps in the interval $[0, T]$, and $J_1, \ldots, J_{N_T}$ are the independent jump-sizes.

(a) The random variable $N_T \sim Poi(\lambda T)$ has probability mass function

$$\mathbb{P}(N_T = k) = e^{-\lambda T}\frac{(\lambda T)^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

Explain how the inverse-transform method can be used to simulate from this distribution. That is, how $U \sim \mathcal{U}[0,1]$ can be transformed to $N_T \sim Poi(\lambda T)$.

Note: You do not need to implement the algorithm.

(b) If $\lambda = 0$, the jump-diffusion model reduces to the standard Black-Scholes model. Do you think call options are more or less expensive if $\lambda > 0$ (i.e., in the presence of jumps)? What about put options? Explain your intuition for why the inclusion of a jump-component increases/decreases/does not affect the prices of options.

(c) Simulate the stock price S_T according to formula (0.4) and estimate the prices of call and put options. Use $n = 10,000$ simulations and parameters $S_0 = 100$, $K = 100$, $T = 1$, $r = 0.03$, $\sigma = 0.20$, $\mu_J = 0.02$, and $\sigma_J = 0.08$. For both $\lambda = 0$ and $\lambda = 2$, report in a table your price estimates along with 95% confidence intervals (a total of four estimates and four confidence intervals).

(d) Do your price estimates in (c) align with your intuition in part (b)?

(e) *(optional)* Show that the discounted asset price process $(e^{-rt}S_t)_{t\geq 0}$ is a martingale. That is, for S_T given in (0.4), show formally that $\mathbb{E}[S_T] = S_0 e^{rT}$.

[2] The constant k is such that the discounted asset price process $(e^{-rt}S_t)_{t\geq 0}$ is a martingale.