FE 621: HW2

Due date: Mar 15th at 11:59 pm

- Make sure that your solutions are presented in an organized and readable manner. Points are subtracted for solutions that are unclear and difficult to follow or understand.

Problem 1

(a) Provide the formulas for the parameters p, u, d, of a Cox-Ross-Rubinstein (CRR) binomial tree that approximates the dynamics of a stock following geometric Brownian motion with drift r and volatility σ.

(b) Write code that takes S_0, σ, r, K, T and N as inputs and uses an N-step CRR binomial tree to compute the prices of call and put options in the Black-Scholes model.

(c) Analyze the convergence of binomial tree option prices to Black-Scholes option prices as the number of time steps gets larger. Specifically, for both call and put options, plot the Black-Scholes price P^{BS} and the N-step binomial tree price P_N^{Tree} as a function of N. Also plot the absolute relative pricing error $\frac{|P_N^{Tree} - P^{BS}|}{P^{BS}}$ as a function of N. Comment on your findings. Does the error decrease smoothly to zero as N grows? How large does N have to be (i.e., how small does the step size Δt have to be) for satisfactory convergence?

Use parameters $S_0 = 100$, $K = 90$, $T = 0.5$, $\sigma = 0.2$ and $r = 0.04$.

Problem 2

(i) Prove that the log-return of the stock price in the Black-Scholes model satisfies

$$\log\left(\frac{S_T}{S_0}\right) = \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T,$$

where $W_T \sim \mathcal{N}(0, T)$. Plot the PDF of the distribution of the log-return.

(ii) Prove that the log-return of the stock price in an N-step CRR binomial tree satisfies

$$\log\left(\frac{S_N}{S_0}\right) = N\log(d) + X\log\left(\frac{u}{d}\right),$$

where $X \sim Bin(N, q)$. Plot the PMF of the distribution of the log-return for $N = 6$ (monthly), $N = 26$ (weekly), $N = 126$ (daily), and the parameters specified in Problem 1(c).

(iii) According to theory, S_N converges in distribution to S_T as $N \to \infty$. Do your results in (i) and (ii) align with this? To examine this, plot the CDF of S_N (for $N = 6, 26, 126$) and S_T in the same figure.

Remark: S_N converging in distribution to S_T as $N \to \infty$ means that $\mathbb{P}(S_N \le s) \longrightarrow \mathbb{P}(S_T \le s)$ for any value of s. In other words, the CDF of S_N converges to the CDF of S_T.

Problem 3

(a) Explain in simple terms the concept of exercise boundary in American option pricing.

(b) Extend your code in Problem 1 to handle American options and consider a put option with $K = 100$, $T = 5$, and $r = 0.04$. For each of the volatility levels $\sigma = 0.10$, $\sigma = 0.30$, and $\sigma = 0.50$, visualize the exercise boundary of the put option using a binomial tree with $N = 500$ steps.[1]

(c) How does the exercise boundary depend on the volatility σ? Explain intuitively in financial terms.

(d) (extra credit) A callable option is such that the issuer has the right to buy it back at any time at a predetermined *call price*. Upon issuer call, the option holder can choose to either exercise the option or receive the call price.

Write the dynamic programming equation for a callable American put option with strike price K and call price H. Explain in words the meaning of each term in your equation.

Problem 4

(a) Write the payoff function of a down-and-in barrier call option with barrier H and strike price K.

(b) Assume that $H < S_0 < K$. What needs to happen for the payoff of the barrier option to be positive?

(c) Write the dynamic programming equation for the valur of the barrier option. Explain in words the meaning of each term in your equation.

(d) When pricing barrier options using binomial trees, do you expect the convergence to be faster or slower than when pricing regular (non-barrier) options? That is, do you think a smaller or larger number of steps is required to attain a comparable level of accuracy? Explain your reasoning.

(e) (extra credit) Barrier options are *one-touch options* that are knocked in/out as soon as the barrier is hit. When the stock price is close to the barrier, the prices and deltas of such options are very sensitive to small changes in the stock price. *Parisian options*, on the other hand, require the stock price to remain beyond the barrier for $d > 1$ periods for the option to be knocked in/out.

  (i) Explain why the prices and deltas of Parisian options are less sensitive than those of one-touch options when the stock price is near the barrier.

  (ii) Explain how the state variable approach discussed in class can used to price a Parisian down-and-in call options in a binomial tree. Provide pseudocode for your approach.

[1] To check the correctness of your code, i.e., whether it prices American options correctly, you can use an online option price calculator to find the exact price of the option.