FE 621: HW4

Due date: May 5th at 11:59 pm

Problem 1 (barrier options)

The price of an up-and-out put option (UOP) with strike price K and barrier H is given by

$$P = \mathbb{E}\big[e^{-rT}(K - S_T)^+ \mathbf{1}_{\{\tau > T\}}\big], \tag{0.1}$$

where τ is the hitting time of the asset price process $(S_t)_{t\geq 0}$ to the barrier H:

$$\tau = \inf\{t > 0 : S_t \geq H\}.$$

The payoff is the same as that of a vanilla put option, unless the stock price goes above H during the life of the option, in which case the payoff is zero. Assume the process $(S_t)_{t\geq 0}$ to follow a geometric Brownian motion.

(a) Is an UOP option cheaper or more expensive than a vanilla put option? Explain.

(b) The standard Monte Carlo estimator for the price of an UOP option is given by

$$\hat{P}_{n,m} = e^{-rT}\frac{1}{n}\sum_{k=1}^{n}\big(K - \hat{S}_m(k)\big)^+ \mathbf{1}_{\{\hat{\tau}(k) > T\}}, \tag{0.2}$$

where $(\hat{S}_i(k))_{i\geq 0}$ is the k-th simulated path of GBM at times $(t_i)_{i\geq 0}$, where $t_i = i\frac{T}{m}$, and

$$\hat{\tau}(k) = \inf\{i \geq 0 : \hat{S}_i(k) > H\},$$

is the hitting time of the simulated path to the barrier H.

(i) What is the definition of $\hat{P}_{n,m}$ being an unbiased/biased high/biased low estimator for P?

(ii) Do you expect $\hat{P}_{n,m}$ to be unbiased/biased high/biased low? Explain your reasoning.

(c) Use the parameters in the table below to compute the estimator $\hat{P}_{n,m}$ along with a 95% confidence interval. Use $m = 63$ and $n = 100,000$.[1]

Initial price	$S_0 = 50$	Strike	$K = 60$
Volatility	$\sigma = 30\%$	Expiration	$T = 0.25$
Interest rate	$r = 5\%$	Barrier	$H = 55$

Given the exact option price, are your simulation results consistent with your guess about the bias in part (b)-(ii)?

[1] In the Black-Scholes model there exist explicit formulas for a variety of barrier options (see, e.g., the textbook by Björk). The exact price of the UOP option in this problem is $6.869.

Problem 2 (basket options and correlation)

Consider a basket call option with strike K, maturity T, and payoff

$$\Big(\frac{1}{d}\sum_{i=1}^{d} S_T^i - K\Big)^+,$$

where the d assets $(S_t^1, \dots, S_t^d)_{t\geq 0}$ follow a d-dimensional geometric Brownian motion. Specifically,

$$\frac{dS_t^i}{S_t^i} = r dt + \sigma^i dW_t^i,$$

where $(W_t^1, \dots, W_t^d)_{t\geq 0}$ is a d-dimensional Brownian motion with covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \rho & \rho & 1 & \dots & \rho \\ \vdots & & & \ddots & \vdots \\ \rho & \rho & \rho & \dots & 1 \end{pmatrix}.$$

In other words, $(W_t^1, \dots, W_t^d) \sim N(0, t\Sigma)$, for any $t > 0$.

(a) Compute the price of the basket option with strike $K = 100$ using Monte Carlo simulation. Explain in detail your simulation procedure.

For $d = 5$ and $d = 10$, plot in the same graph the price estimate as a function of ρ for $-\frac{1}{d-1} \leq \rho \leq 1$.

Let $r = 0.04$, $S_0^i = 100$, $\sigma^i = 0.15$, $T = 1$, and use $n = 100,000$ simulations.

(b) How do the prices in part (a) depend on the correlation ρ? How do they depend on the dimension d? Are the patterns you observe consistent with your intuition?

(c) Answer the following questions (these are questions about theoretical properties of Monte Carlo; no simulations needed):

- (i) Is the Monte Carlo estimator in part (a) biased or unbiased?
- (ii) What is the order of convergence of the Monte Carlo estimator as $n \to \infty$?
- (iii) How does the order of convergence depend on the dimension d?

(d) *(extra credit - hard)* Justify why the bound

$$-\frac{1}{d-1} \leq \rho \leq 1,$$

is needed for Σ to be a valid covariance matrix. For example, if $d = 5$ we need to have $-0.25 \leq \rho \leq 1$, and for $d = 10$ we need $-0.11 \leq \rho \leq 1$. In other words, as the number of Brownian motions gets larger, there is a lower bound on the pairwise correlation between them.

Problem 3 (variance reduction)

(a) Explain why an option with payoff $((S_T^1 \dots S_T^d)^{1/d} - K)^+$ is a good candidate as a control variate to reduce the variance of the Monte Carlo estimator for the basket option in Problem 2.

(b) *(extra credit)* Assume that $S_0^i = S_0$ and $\sigma^i = \sigma$ for $i = 1, \dots, d$. Show that

$$\mathbb{E}[e^{-rT}(S_T^1 \dots S_T^d)^{1/d} - K)^+] = C_{BS}(S_0(d), \Sigma(d), r, K, T),$$

where the right-hand side is the Black-Scholes call option formula with initial stock price and volatility given by

$$S_0(d) = S_0 e^{(\Sigma^2(d) - \sigma^2)\frac{T}{2}}, \qquad \Sigma(d) = \sigma\sqrt{\frac{1 + (d-1)\rho}{d}}.$$

Hint: For a similar derivation, see the Asian option example on pages 41–42 in the Monte Carlo notes on Canvas.

(c) Implement the variance reduction technique in part (a). Clearly explain the steps of your simulation procedure and report the estimated control variate coefficient b.

Use $d = 10$, $\rho = 0.3$, $n = 10,000$, and set other parameters to the values used in Problem 2.

Display in the same graph the convergence of the Monte Carlo estimators with and without variance reduction, along with 95% confidence intervals, for n ranging from 100 to $10,000$.[2] Comment on your findings.

[2]For clarity it may be better to plot n in increments of 100.

Problem 4 (variance reduction)

Assume the Black-Scholes model

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t.$$

(i) Consider a European call option with payoff $(S_T - K)^+$ at time T. The standard Monte Carlo estimator is

$$\widehat{P}_{Eur} = e^{-rT} \frac{1}{n} \sum_{k=1}^{n} (\hat{S}_T(k) - K)^+,$$

where $\hat{S}_T(k)$ is the k-th simulated stock price at maturity T. Explain how antithetic variates can be used to reduce the variance of the Monte Carlo estimator.

(ii) Consider an Asian call option with payoff

$$\Big(\frac{1}{m} \sum_{i=1}^{m} S_{t_i} - K\Big)^+,$$

at time T, where $t_i = iT/m$. The standard Monte Carlo estimator is given by

$$\widehat{P}_{Asian} = e^{-rT} \frac{1}{n} \sum_{k=1}^{n} (\bar{S}(k) - K)^+,$$

where $\bar{S}(k) = \frac{1}{m} \sum_{i=1}^{m} \hat{S}_i(k)$ is the average stock price over the k-th simulated path $(\hat{S}_i(k))_{1 \le i \le m}$. Explain how antithetic variates can be used to reduce the variance of the Monte Carlo estimator.

Hint: In part (i), you can simulate the stock price at time T directly (i.e., one step, requiring a single Z), but in part (ii) you need to simulate a discretized path (i.e., multiple steps, requiring a vector of Z's). Remember that the idea behind antithetic variates is to make the $(i+1)$-th simulated payoff "antithetic" to the i-th simulated payoff.