FE621 - Homework #3

Author: Sid Bhatia

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Pledge: I pledge my honor that I have abided by the Stevens Honor System.

Professor: Sveinn Olafsson

TA: Dong Woo Kim

Problem #1 (Monte Carlo Error)

Use Monte Carlo simulation to price a European call option in the Black-Scholes model with the following parameters: $S_0 = 100, \sigma = 0.30, r = 0.05, T = 1$, and $K = 100$.

a. Use (exact) simulation based on the closed-form solution of geometric Brownian motion. Use $n = 100,000$ paths.

Clearly describe the steps of your simulation procedure, and provide formulas for the Monte Carlo estimator and a corresponding 95% confidence interval. Report both the estimator and the confidence interval. Does the confidence interval contain the true price of the option?

Procedure

1. **Simulation of Stock Prices**: According to BSM, the stock process S_t at future time t is as follows:

$$S_t = S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}W_T\}$$

where

- S_0 = initial stock price
- r = rfr (e.g., 3-month UST)
- σ = vol
- T = time till maturity
- W_T = standard BM $\sim \mathcal{N}(0, T)$

2. **Payoff Calculation**: For a call option, the payoff at maturity is $(S_T - K)_+$ where K is the strike price. For puts, it's the converse $(K - S_T)_+$.

3. **MC Estimator**: The price of the option is the present value of the expected payoff under the risk-neutral measure $\mathbb{Q}$, which is estimated as the average of the discounted payoffs across all simulated paths:

$$P = e^{-rT}\mathbb{E}^{Q}[f(S_t)], \text{ where } f \text{ is the payoff function.}$$

$$\dots$$

$$\hat{C} = \exp\{(-rT)\}\frac{1}{n}\sum_{i=1}^{n} f(S_t)$$

4. **CI**: The 95% confidence interval for the true option price is given by

$$\hat{C} \pm z_{\alpha/2} \cdot SE$$

where $\alpha = 0.05$ and $SE = \text{standard error}$. Therefore,

$$\hat{C} = 1.96 \cdot \frac{\sigma_{\hat{C}}}{\sqrt{n}}$$

where $\sigma_{\hat{C}}$ is the standard deviation of the stimulated payoffs.

5. **True Price Comparison**: The true price of the option can be calculated using the BSM closed-form solution. We compare the confidence interval obtained from the Monte Carlo simulation with the true price to see if it contains the true price.