FE621 - Homework #4

Author: Sid Bhatia

Date: May 2nd, 2024

Pledge: I pledge my honor that I have abided by the Stevens Honor System.

Professor: Sveinn Olafsson

TA: Dong Woo Kim

Problem #1 (Barrier Options)

The price of an **up-and-out put option/knock-out** (**UOP**) with strike price K and barrier H is given by:

$$P = e^{-rT}\mathbb{E}[(K - S_T)_+ \mathbf{1}_{\{\tau > T\}}] \tag{1}$$

where τ is the *stopping time* of the asset price process $(S_t)_{t\geq 0}$ to the barrier H:

$$\tau = \inf\{t > 0 : S_t \geq H\} \tag{2}$$

The payoff is the **same** as that of a *vanilla put option*, unless the stock price goes above H during the life of the option, in which case the payoff is **zero**. Assume the process $(S_t)_{t\geq 0}$ to follow a GBM.

a. Is an UOP option cheaper or more expensive than a vanilla put option? Explain.

An **up-and-out put option** (UOP) is generally cheaper than a vanilla put option. This difference in pricing comes from the **additional condition** involved in the UOP, where the option becomes worthless if the stock price exceeds the barrier H before expiration. In a vanilla put option, the holder has the right to sell the stock at the strike price K **regardless of how high the stock price has climbed during the option's life**.

This restriction in the UOP **reduces the probability** of a payout compared to a vanilla put option, where there is no upper limit on the stock price affecting the payoff.

Therefore, the UOP has a **lower premium** due to its *reduced likelihood of exercising profitably*. Essentially, the risk of the option knocking out (i.e., becoming worthless if the stock price exceeds the barrier H reduces its cost.

b. The standard MC estimator for the price of an **UOP (put) option** is given by:

$$\hat{P}_{n,m} = e^{-rT}\frac{1}{N}\cdot\sum_{k=1}^{N}(K - \hat{S}_m(k))^+ \mathbf{1}_{\{\hat{\tau}(k) > T\}} \tag{3}$$

where $\{\hat{S}_i(k)\}_{i\geq 0}$ is the k-th simulated path of GBM at times $\{t_i\}_{i\geq 0}$ where $t_i = i \cdot \frac{T}{m}$ and

$$\hat{\tau}(k) = \inf\{i \geq 0 : \hat{S}_i(k) > H\} \tag{4}$$

is the **stopping time** of the simulated path to the barrier H.

(i) What is the definition of $\hat{P}_{n,m}$ being an unbiased/biased high/biased low estimator for P?

The definition of $\hat{P}_{n,m}$ being an **unbiased**, **biased high**, or **biased low** estimator for P relates to its **expected value** compared to the true value P:

- **Unbiased Estimator**: $\mathbb{E}[\hat{P}_{n,m}] = P$, or the estimator *equals* to the true price.
- **Biased High Estimator**: $\mathbb{E}[\hat{P}_{n,m}] > P$, or the estimator systematically *overestimates* the true price.
- **Biased Low Estimator**: $\mathbb{E}[\hat{P}_{n,m}] < P$, or the estimator systematically *underestimates* the true price

(ii) Do you expect $\hat{P}_{n,m}$ to be biased (high/low)? Explain.

- **Potential for Bias Low**: KO put may be biased low due to the *discretization* of the path simulation. In practical settings, the simulation might not capture every peak that crosses the barrier H within the continuous monitoring of the actual path, especially if the time steps (Δt) are not small enough. This miss means some paths that should knock out (reach or exceed H) might not actually do so in the simulation, leading to an *overestimation of paths* that contribute to the payoff sum, hence **underestimating** the true option price where more paths should knock out.

- **Discretization Error**: As the number of time steps m increases (i.e., the simulation becomes more granular), the estimator should become *less biased*. The limit of the estimator as $(m \to \infty)$, with continuous monitoring of the barrier, would theoretically be unbiased. However, for finite m and practical implementations, we might expect a bias due to discretization.

- **Random Fluctuations and Estimator Variance**: The *standard deviation* of the estimator can also play a role in perceived bias over different simulations. With a finite number N of paths, random fluctuations might cause $\hat{P}_{n,m}$ to occasionally estimate higher or lower than its expected value, but that's more generally about variance than systematic bias (which can occur for any simulation with standard errors in Bayesian statistics).