FE621 - Homework #4

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Pledge: I pledge my honor that I have abided by the Stevens Honor System.

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Problem #1 (Barrier Options)

The price of an **up-and-out put option/knock-out** (**UOP**) with strike price K and barrier H is given by:

$$P = e^{-rT}\mathbb{E}[(K - S_T)_+ \mathbf{1}_{\{\tau > T\}}] \tag{1}$$

where τ is the *stopping time* of the asset price process $(S_t)_{t \geq 0}$ to the barrier H:

$$\tau = \inf\{t > 0 : S_t \geq H\} \tag{2}$$

The payoff is the **same** as that of a *vanilla put option*, unless the stock price goes above H during the life of the option, in which case the payoff is **zero**. Assume the process $(S_t)_{t \geq 0}$ to follow a GBM.

a. Is an UOP option cheaper or more expensive than a vanilla put option? Explain.

An **up-and-out put option** (UOP) is generally cheaper than a vanilla put option. This difference in pricing comes from the **additional condition** involved in the UOP, where the option becomes worthless if the stock price exceeds the barrier H before expiration. In a vanilla put option, the holder has the right to sell the stock at the strike price K **regardless of how high the stock price has climbed during the option's life**.

This restriction in the UOP **reduces the probability** of a payout compared to a vanilla put option, where there is no upper limit on the stock price affecting the payoff.

Therefore, the UOP has a **lower premium** due to its *reduced likelihood of exercising profitably*. Essentially, the risk of the option knocking out (i.e., becoming worthless if the stock price exceeds the barrier H reduces its cost.

b. The standard MC estimator for the price of an **UOP (put) option** is given by:

$$\hat{P}_{n,m} = e^{-rT}\frac{1}{N} \cdot \sum_{k=1}^{N}(K - \hat{S}_m(k))^+ \mathbf{1}_{\{\hat{\tau}(k) > T\}} \tag{3}$$

where $\{\hat{S}_i(k)\}_{i\geq 0}$ is the k-th simulated path of GBM at times $\{t_i\}_{i\geq 0}$ where $t_i = i \cdot \frac{T}{m}$ and

$$\hat{\tau}(k) = \inf\{i \geq 0 : \hat{S}_i(k) > H\} \tag{4}$$

is the **stopping time** of the simulated path to the barrier H.