FE621 - Homework #5

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Pledge: I pledge my honor that I have abided by the Stevens Honor System.

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Problem 1 (Portfolio Wealth Growth)

1.1 Portfolio Wealth Growth Theory

This section delves into the theoretical mathematical foundation governing the growth of portfolio wealth over time. The analysis is crucial for understanding how investments evolve under the influence of various market factors, including returns and volatility.

1.1.1 Mathematical Formulation

The wealth process $\{V_t\}_{t\geq 0}$ is modeled as a geometric Brownian motion (GBM), which is frequently used to represent stock prices and, by extension, portfolio values under stochastic environments. The stochastic differential equation (SDE) governing this process is given by:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t \tag{1}$$

Here:

- V_t represents the portfolio value at time t.
- μ is the expected return of the portfolio, expressed as a percentage of the portfolio value.
- σ is the volatility of the portfolio, which measures the standard deviation of the portfolio's returns.
- dW_t is the increment of a standard Brownian motion, which captures the random fluctuations in the market.

Interpretation

Equation (1) can be interpreted as follows:

- The term μdt captures the expected growth of the portfolio due to returns over an infinitesimally small time interval dt.

- The term σdW_t introduces randomness into the growth process, reflecting the uncertainty and risk inherent in the financial markets.

Solution to the Differential Equation

The solution to the stochastic differential equation (SDE) given in equation (1) can be expressed explicitly by integrating both sides over the interval from 0 to t:

$$\ln \frac{V_t}{V_0} = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t \tag{2}$$

where:

- V_0 is the initial value of the portfolio at time $t = 0$.
- W_t represents the standard Brownian motion at time t.

From equation (2), we can exponentiate both sides to obtain the explicit form of V_t:

$$V_t = V_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right) \tag{3}$$

Mathematical Synthesis

Equation (3) clearly shows how the portfolio value V_t evolves over time. It indicates that the portfolio value is log-normally distributed with its mean and variance increasing over time. This formulation is fundamental in finance for modeling asset prices and helps in understanding the dynamic nature of investment growth under uncertainty.

1.1.2 Expectation of Portfolio Wealth

The following section explores and delves into the expectation (first raw moment/arithmetic average) of the portfolio wealth process.

Expectation Calculation

Given the wealth process V_t which follows a geometric Brownian motion (GBM) as described by:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t \tag{1}$$

The solution to this stochastic differential equation (SDE) indicates:

$$V_t = V_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right) \tag{3}$$

To find the expectation $\mathbb{E}[V_t]$, we note that W_t is a standard Brownian motion (BM), which implies σW_t is normally distributed with mean 0 and variance $\sigma^2 t$. Thus, $\sigma W_t \sim N(0, \sigma^2 t)$, and $e^{\sigma W_t}$ follows a log-normal distribution.

We can use the moment-generating function (MGF) of a normally distributed random variable to compute the expectation of a log-normal variable. For a random variable $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, the MGF of X at s is $M_X(s) = e^{\mu_X s + \frac{1}{2}\sigma_X^2 s^2}$. Setting $s = 1$, we find:

$$\mathbb{E}[e^X] = e^{\mu_X + \frac{1}{2}\sigma_X^2} \tag{4}$$

Applying this to our case, where $\mu_X = 0$ and $\sigma_X^2 = \sigma^2 t$, we get:

$$\mathbb{E}[e^{\sigma W_t}] = e^{0 + \frac{1}{2}\sigma^2 t} = e^{\frac{1}{2}\sigma^2 t} \tag{5}$$

Now, substituting this into the solution for V_t:

$$\begin{aligned}\mathbb{E}[V_t] &= \mathbb{E}\left[V_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right)\right] \\ &= V_0 \exp\left((\mu - \frac{\sigma^2}{2})t\right)\mathbb{E}[e^{\sigma W_t}]\end{aligned} \tag{6}$$

Substituting the expectation of $e^{\sigma W_t}$:

$$\begin{aligned}\mathbb{E}[V_t] &= V_0 \exp\left((\mu - \frac{\sigma^2}{2})t\right)\exp\left(\frac{1}{2}\sigma^2 t\right) \\ &= V_0 \exp(\mu t)\end{aligned} \tag{7}$$

Thus, the expected wealth at time (t) is indeed given by:

$$\mathbb{E}[V_t] = V_0 \exp(\mu t) \tag{7}$$

This demonstrates that the expectation grows exponentially at a rate determined by the drift μ, independent of the volatility σ.