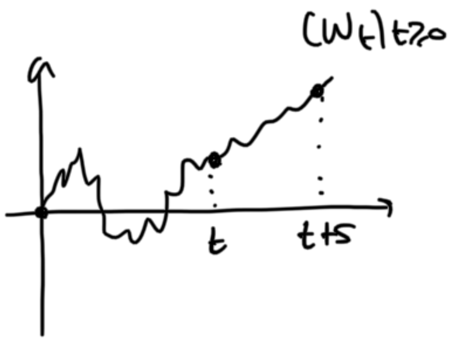
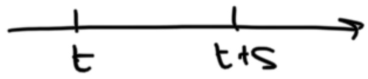
Overview of stochastic calculus and math. finance

B.M. $(W_t)_{t\geq 0}$: (i) $W_0 = 0$

(ii) Independent increments:





$W_{t+s} - W_t$ indep. of W_t

Note: W_{t+s} not indep of W_t

$$\mathrm{Cov}(W_t, W_{t+s}) = t \neq 0$$

[more generally: $\mathrm{Cov}(W_{t_1}, W_{t_2}) = \min(t_1, t_2)$]

$$\mathrm{Corr}(W_t, W_{t+s}) = \frac{\mathrm{Cov}(W_t, W_{t+s})}{\sqrt{\mathrm{Var}(W_t)}\sqrt{\mathrm{Var}(W_{t+s})}} = \frac{t}{\sqrt{t}\sqrt{t+s}}$$

$$= \sqrt{\frac{t}{t+s}} \xrightarrow[s \to \infty]{} \underline{\underline{0}}$$

(iii) Normally distributed increments:

$W_{t+s} - W_t \sim N(0, s)$

In particular: $W_t \sim N(0, t)$

(iv) $(W_t)_{t\geq 0}$ has continuous trajectories

* $(W_t)_{t\geq 0}$ is a Gaussian process

($\sum_{i=1}^{n} a_i W_{t_i}$ has a normal distribution for any constants $a_1, \dots, a_n$ and any time points $t_1, \dots, t_n$) technical definition

* $W_t \sim N(0, t)$

$\mathrm{Cov}(W_{t_1}, W_{t_2}) = t_1$

* $(W_{t_1}, W_{t_2}) \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} t_1 & t_1 \\ t_1 & t_2 \end{pmatrix}\right)$

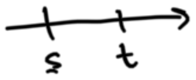
$(t_1 < t_2)$ mean-vector cov-matrix

* $(W_{t_1}, W_{t_2}, W_{t_3}) \sim N\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix} t_1 & t_1 & t_1 \\ t_1 & t_2 & t_2 \\ t_1 & t_2 & t_3 \end{pmatrix}\right)$ $(t_1 < t_2 < t_3)$

* In general, $(W_{t_1}, \dots, W_{t_n})$ has a multivariate normal dist; we will see that this makes simulation of Gaussian processes particularly easy.

* $(W_t)_{t \geq 0}$ is a martingale:

$$E[W_t \mid \mathcal{F}_s] = W_s \quad \text{where } \mathcal{F}_s = \sigma\{W_t, 0 \leq t \leq s\}$$

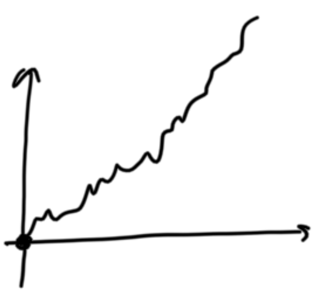


↑ filtration generated by $(W_t)_{t \geq 0}$

$$E[W_t \mid \mathcal{F}_s] = \underbrace{E[W_t - W_s \mid \cancel{\mathcal{F}_s}]}_{\underbrace{E[W_t - W_s]}_{0}} + \underbrace{E[W_s \mid \mathcal{F}_s]}_{W_s} = \underline{\underline{W_s}}$$

W_s measurable w.r.t. $\mathcal{F}_s$

$(W_t^2)_{t \geq 0}$



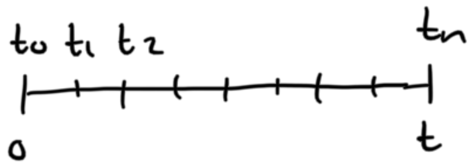
* Ex: Show (i) $(W_t^2 - t)_{t \geq 0}$ is a MG

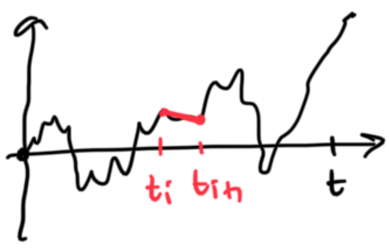
(ii) $\left(e^{\lambda W_t - \frac{\lambda^2}{2} t}\right)_{t \geq 0}$ is a MG

where $\lambda \in \mathbb{R}$ is a constant

(iii) Show $\mathrm{Cov}(W_t, W_{t+s}) = t$

Q: How do the paths of BM differ from the paths of "regular/smooth" processes?





$$TV_t = \lim_{n \to \infty} \sum_{i=0}^{n-1} |W_{t_{i+1}} - W_{t_i}| = \underline{\underline{\infty}} \quad (\text{w.p. } 1)$$

← i.e. for each path

$$QV_t = \lim_{n \to \infty} \sum_{i=0}^{n-1} \underbrace{(W_{t_{i+1}} - W_{t_i})^2}_{dW_t^2} = \underline{\underline{t}} \quad (\text{w.p } 1) \longrightarrow \underline{dW_t^2 \approx dt}$$

$(g_t)_{t\geq 0}$

For a "smooth" process $(g_t)_{t\geq 0}$:

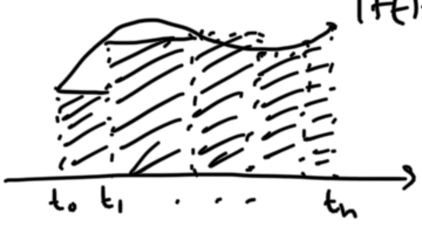
$$\begin{cases} TV_t < \infty \\ QV_t = 0 \end{cases}$$

In particular : $g_t = t \xRightarrow{QV_t = 0} dt^2 \approx 0$

Stochastic integration

$\int_0^t f_s \, ds \longrightarrow$ limit of $\sum_{i=1}^{n} f_{t_i}(t_{i+1} - t_i)$ [Riemann]

$(f_t)_{t\geq 0}$



$\int_0^t f_s \, dg_s \longrightarrow$ limit of $\sum_{i=1}^{n} f_{t_i}(g_{t_{i+1}} - g_{t_i})$ [Riemann - Stieltjes]

↳ well-defined if g has finite variation $\longrightarrow$ $(W_t)_{t\geq 0}$ does not !

How to define $\int_0^t f_s \, dW_s$?

Requires a special theory $\longrightarrow$ Itô integral

Properties of the Itô integral :

$\int_0^t f_s \, dW_s$ where $(f_t)_{t\geq 0}$ is a square-integrable process adapted to the Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$.

$\int_0^t f_s^2 \, ds < \infty$

e.g. $f_s = W_s$

$f_s = \cos(W_s)$

$f_s = s^2$ [non-random]

etc.

more generally: $\mathbb{E}\left[\int_0^t f_u dW_u \middle| \mathcal{F}_s\right] = \int_0^s f_u dW_u$

$\Rightarrow \int_0^t f_s dW_s$ is a MG

(i) $\mathbb{E}\left[\int_0^t f_s dW_s\right] = \underline{\underline{0}}$

(ii) $Var\left[\int_0^t f_s dW_s\right] = \mathbb{E}\left[\left(\int_0^t f_s dW_s\right)^2\right] = \underline{\underline{\int_0^t \mathbb{E}[f_s^2] ds}}$

$Var[X] = \mathbb{E}[X^2] - \underbrace{\mathbb{E}[X]^2}_{0}$

Itô's isometry

(iii) Do we know the dist. of $\int_0^t f_s dW_s$?

Do we have $\int_0^t f_s dW_s \sim N\left(0, \int_0^t \mathbb{E}[f_s^2] ds\right)$?

Yes, if $(f_t)_{t\geq 0}$ is non-random [e.g. $f_s = s$]

$$\left[\int_0^t f_s dW_s \approx \sum_{i=0}^{n-1} \underbrace{f_{t_i}}_{\text{constant}} \underbrace{(W_{t_{i+1}} - W_{t_i})}_{N(0,\, t_{i+1}-t_i)} \sim N(\dots)\right]$$

Itô's formula: $(W_t)_{t\geq 0}$

What is the "differential" of $f(W_t)$? $[df(W_t)]$

[f is a "nice" function]

Taylor expansion of $f(W_t + dW_t)$ around W_t:

$$f(W_t + dW_t) = f(W_t) + f'(W_t) dW_t + \frac{1}{2} f''(W_t) \underbrace{(dW_t)^2}_{dt} + \dots$$

$$\underline{\underline{df(W_t) = f'(W_t) dW_t + \frac{1}{2} f''(W_t) dt}}$$

$f(W_t + dW_t) - f(W_t)$

$$\underline{\underline{f(W_t) = f(W_0) + \int_0^t f'(W_s) dW_s + \frac{1}{2} \int_0^t f''(W_s) ds}}$$

extra term relative to regular calculus

$$df(g_t) = f'(g_t)\,dg_t \qquad [\text{"chain rule"}]$$

$(g_t)_{t\geq 0}$ smooth function

$$f(g_t) = f(g_0) + \int_0^t f'(g_s)\,dg_s$$

regular calculus

More generally:

$$f(t,W_t) = f(0,W_0) + \int_0^t f_t(s,W_s)\,ds + \int_0^t f_x(s,W_s)\,dW_s + \frac{1}{2}\int_0^t f_{xx}(s,W_s)\,ds$$

Ex: Show $W_t^2 = 2\int_0^t W_s\,dW_s + t$

$[f(t,x) = x^2]$

$$\hookrightarrow W_t^2 - t = 2\int_0^t W_s\,dW_s = \text{M.G.}$$

Ex: (i) $S_t = S_0\, e^{(\mu - \sigma^2/2)t + \sigma W_t}$

Show that $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$

$[f(t,x) = S_0\, e^{(\mu-\sigma^2/2)t + \sigma x}]$

(ii) $\dfrac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t$

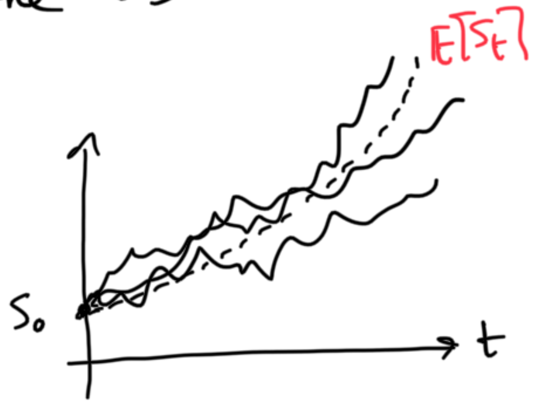
Apply Itô's formula to $\ln(S_t)$ to show that $S_t = S_0\, e^{(\mu-\sigma^2/2)t + \sigma dW_t}$.

GBM: $\dfrac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t$ and this process has solution $\underline{\underline{S_t = S_0\, e^{(\mu-\sigma^2/2)t + \sigma W_t}}}$

What is the distribution of S_t?

→ log-normal $\ln\left(\frac{S_t}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t \sim N(\dots)$

[the log-returns $\ln\left(\frac{S_t}{S_0}\right)$ are normally dist.]

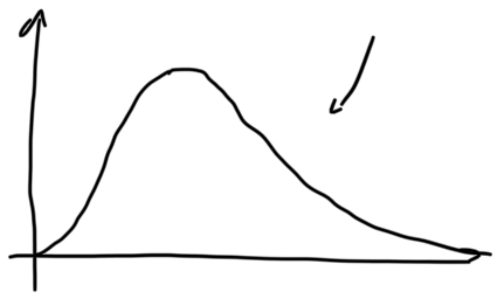


$$E[S_t] = S_0 e^{(\mu - \sigma^2/2)t} \underbrace{E[e^{\sigma W_t}]}_{e^{\sigma^2/2}}$$

$$= \underline{\underline{S_0 e^{\mu t}}}$$

$$\left(E[e^{N(\mu,\sigma^2)}] = e^{\mu + \sigma^2/2} \right)$$

long right tail



S_t has log-normal distribution

⇒ right-skewed distribution

⇒ median (S_t) < mean (S_t)

⇒ $\mathbb{P}(S_t < E[S_t]) > 0.50$

⇒ more than 50% chance of underperforming the "mean path"