FE 621: HW3

Due date: April 5th at 11:59 pm

Problem 1 (Monte Carlo error)

Use Monte Carlo simulation to price a European call option in the Black-Scholes model with the following parameters: $S_0 = 100$, $\sigma = 0.30$, $r = 0.05$, $T = 1$, and $K = 100$.

(a) Use (exact) simulation based on the closed-form solution of geometric Brownian motion. Use $n = 100,000$ paths.

Clearly describe the steps of your simulation procedure, and provide formulas for the Monte Carlo estimator and a corresponding 95% confidence interval. Report both the estimator and the confidence interval. Does the confidence interval contain the true price of the option?

(b) Use (biased) simulation based on the Euler discretization scheme for geometric Brownian motion. Use a discretization with $m = 5$ steps and $n = 100,000$ paths.

Clearly describe the steps of your simulation procedure, and provide formulas for the Monte Carlo estimator and a corresponding 95% confidence interval. Report both the estimator and the confidence interval. Does the confidence interval contain the true price of the option?

(c) In a single plot, display the evolution of the Monte Carlo estimators in parts (a) and (b) as the sample size increases. Specifically, plot the value of the estimators for sample sizes $k = 50, 100, \ldots, n$. Also include a horizontal line representing the true price of the option.

(d) What are the two sources of error that result in the Monte Carlo estimators in parts (a) and (b) not being equal to the true price of the option? How can each of those errors be reduced?

(e) In a single plot, display overlapping histograms for the simulated stock prices at time $T = 1$ in parts (a) and (b). How do the two simulated distributions differ? Is their difference consistent witht the simulation bias observed in part (c)?

Problem 2 (Delta hedging)

In this problem we use simulation to estimate the hedging error resulting from discrete portfolio rebalancing. We sell a 3-month European call option and hedge our position by holding "delta" shares of the underlying stock. Assume that we can borrow and deposit money at a constant interest rate.

- At initiation of the contract, we receive the option premium from the client and buy delta shares of the stock. We may need to borrow money to set up the hedging portfolio.
- At each time step, the stock price has evolved from the previous step and the hedge must be adjusted. Depending on how the delta has changed, we need either to buy or sell shares. We also pay or earn interest on any money borrowed or deposited over the previous period.
- At maturity of the contract, we close our position. This means selling our shares of the stock, closing our cash account, and paying $(S_T - K)^+$ to the client. What is left after that is our profit or loss (PnL).

Assume the underlying asset to follow a geometric Brownian motion with the parameters specified below. Black-Scholes theory says that if we sell the option at the Black-Scholes price and continuously hedge using the Black-Scholes delta, then the hedging error (PnL) goes to zero as the rebalancing frequency increases. However, continuous trading is impossible in practice, and any discrete hedging strategy results in a nonzero hedging error.

Initial price: $S_0 = 50$,	Rate of return: $\mu = 10\%$
Volatility: $\sigma = 30\%$,	Interest rate: $r = 5\%$
Strike: $K = 50$,	Expiration: $T = 0.25$

(a) Simulate $n = 10,000$ paths of the stock price and report the mean and standard deviation of the discounted hedging error with weekly ($m = 13$) and daily ($m = 63$) rebalancing. Create histograms of the distribution of the hedging error as a fraction of the price of the option. Comment on your findings.

Note: For each simulated path, there is one hedging error (PnL). Each histograms will thus based on n values.

(b) Consider values of μ in the range r to 1, and plot the mean and standard deviation of the hedging error as a function of μ. Use $n = 10,000$ and consider weekly ($m = 13$) and daily ($m = 63$) rebalancing. How does the value of μ impact the hedging errors? Comment on your findings.

(c) Let Δt denote the rebalancing interval. We know that according to theory, the hedging error goes to zero as $\Delta t \to 0$. What does your simulation indicate about the order of this convergence? Does the hedging error appear to be of order $(\Delta t)^\alpha$ for some $\alpha > 0$, and if so, what α?

Hint: Let $m = 13$, $m = 63$, $m = 252$, $m = 1008$ and draw a log-log plot of the hedging error as a function of $\Delta t = T/m$. Then note that

$$error \sim (\Delta t)^\alpha \quad \Longrightarrow \quad \log(error) \sim \alpha \log(\Delta t),$$

so the order of convergence can be estimated by the slope of the log-log curve. For the hedging error you can either use the standard deviation or the root-mean-squared-error $RMSE = \sqrt{Bias^2 + Var}$.

Problem 3 (Hedging volatility different from realized volatility)

In this problem we consider what happens if future realized volatility is different from the implied volatility of the option. Specifically, consider the setup of Problem 2 and assume that the option is sold at implied volatility $\sigma_I = 0.3$, but that future realized volatility is equal to $\sigma_R = 0.2$.

(a) Repeat part (a) of Problem 2 where the option is sold and hedged at implied volatility $\sigma_I = 0.3$, but realized volatility is $\sigma_R = 0.2$.[1]

Note: The only thing that changes from Problem 2 is that the stock price evolves with volatility σ_R. The option premium and the deltas are computed using the implied volatility $\sigma_I = 0.3$.

(b) How do your results for the hedging error change? How do they depend on the hedging frequency?

(c) Assume that you have a reason to believe that future realized volatility will be lower than the quoted implied volatility of an option. Based on your results, propose an investment strategy that you can use to benefit from your information.

[1] The results would be qualitatively the same if you hedged the option at the realized volatility σ_R rather than σ_I.