FE621 - Homework #4

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Date: May 2nd, 2024

Pledge: I pledge my honor that I have abided by the Stevens Honor System.

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Problem #1 (Barrier Options)

Price Formulation

The price of an **up-and-out put option/knock-out** (**UOP**) with strike price K and barrier H is given by:

$$P = e^{-rT}\mathbb{E}^Q[(K - S_T)_+ \mathbf{1}_{\{\tau > T\}}] \tag{1}$$

where τ is the *stopping time* of the asset price process $(S_t)_{t\geq 0}$ to the barrier H:

$$\tau = \inf\{t > 0 : S_t \geq H\} \tag{2}$$

Indicator Formulation

The **indicator function $\mathbf{1}_{\{\tau > T\}}$** in the formula for the up-and-out put option is defined as follows:

$$\mathbf{1}_{\{\tau > T\}} = \begin{cases} 1 & \text{if } \tau > T \\ 0 & \text{otherwise} \end{cases}$$

where τ is the **stopping time** defined as the first instance when the stock price S_t reaches or exceeds the barrier level H.

Indicator Definition

The **infimum function** used to define the stopping time τ for the up-and-out put option is as follows:

$$\tau = \inf\{t > 0 : S_t \geq H\}$$

In this expression:

- τ represents the **stopping time**, the earliest time at which the stock price S_t reaches or exceeds a predetermined barrier level H.
- The set $\{t > 0 : S_t \geq H\}$ includes all times t where the stock price is greater than or equal to the barrier H.

- The function $\inf\{\cdot\}$ denotes the **infimum** of a set, which is the greatest lower bound of that set. In this case, it identifies the *smallest time value* from the set of all times where S_t is at least H.

If the set $\{t > 0 : S_t \geq H\}$ is empty (i.e., the stock price never reaches or exceeds H during the option's life), τ is considered infinite, and the indicator function $\mathbf{1}_{\{\tau > T\}}$ equals 1, implying that the option behaves like a standard put option throughout its lifetime.

Payoff

The payoff is the **same** as that of a *vanilla put option*, unless the stock price goes above H during the life of the option, in which case the payoff is **zero**. Assume the process $\{S_t\}_{t\geq 0}$ to follow a GBM.

a. Is an UOP option cheaper or more expensive than a vanilla put option? Explain.

An **up-and-out put option** (UOP) is generally cheaper than a vanilla put option. This difference in pricing comes from the **additional condition** involved in the UOP, where the option becomes worthless if the stock price exceeds the barrier H before expiration. In a vanilla put option, the holder has the right to sell the stock at the strike price K **regardless of how high the stock price has climbed during the option's life**.

This restriction in the UOP **reduces the probability** of a payout compared to a vanilla put option, where there is no upper limit on the stock price affecting the payoff.

Therefore, the UOP has a **lower premium** due to its *reduced likelihood of exercising profitably*. Essentially, the risk of the option knocking out (i.e., becoming worthless if the stock price exceeds the barrier H reduces its cost.

b. The standard MC estimator for the price of an **UOP (put) option** is given by:

$$\hat{P}_{n,m} = e^{-rT}\frac{1}{N}\cdot\sum_{k=1}^{N}(K - \hat{S}_m(k))^+\mathbf{1}_{\{\hat{\tau}(k)>T\}} \tag{3}$$

where $\{\hat{S}_i(k)\}_{i\geq 0}$ is the k-th simulated path of GBM at times $\{t_i\}_{i\geq 0}$ where $t_i = i\cdot\frac{T}{m}$ and

$$\hat{\tau}(k) = \inf\{i \geq 0 : \hat{S}_i(k) > H\} \tag{4}$$

is the **stopping time** of the simulated path to the barrier H.

(i) What is the definition of $\hat{P}_{n,m}$ being an unbiased/biased high/biased low estimator for P?

The definition of $\hat{P}_{n,m}$ being an **unbiased**, **biased high**, or **biased low** estimator for P relates to its **expected value** compared to the true value P:

- **Unbiased Estimator**: $\mathbb{E}[\hat{P}_{n,m}] = P$, or the estimator *equals* to the true price.

- **Biased High Estimator**: $\mathbb{E}[\hat{P}_{n,m}] > P$, or the estimator systematically *overestimates* the true price.
- **Biased Low Estimator**: $\mathbb{E}[\hat{P}_{n,m}] < P$, or the estimator systematically *underestimates* the true price

(ii) Do you expect $\hat{P}_{n,m}$ to be biased (high/low)? Explain.

- **Potential for Low Bias**: KO put may be biased low due to the *discretization* of the path simulation. In practical settings, the simulation might not capture every peak that crosses the barrier H within the continuous monitoring of the actual path, especially if the time steps (Δt) are not small enough. This miss means some paths that should knock out (reach or exceed H)