# FE630 - Homework #2

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**Pledge**: I pledge my honor that I have abided by the Stevens Honor System.

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## Topics

- Algebra & Optimization;
- Geometry of Efficient Frontiers;
- Applications of One-Fund & Two-Fund Theorems.

# P1 - Optimization w/Equality Constraints (40 pts)

Consider the optimization problem **Max Expected Return w/Target Risk**:

$$\begin{cases} \max_{\omega_1,\omega_2} & R_p(\omega_1,\omega_2) = \mu_1\omega_1 + \mu_2\omega_2 \\ \text{s.t.} & \sqrt{\sigma_1^2\omega_1^2 + 2\rho_{1,2}\sigma_1\sigma_2\omega_1\omega_2 + \sigma_2^2\omega_2^2} = \sigma_T \\ & \omega_1 + \omega_2 = 1 \end{cases} \tag{1}$$

where we have two securities with **Expected Returns** $\mu_1$ and $\mu_2$ for the column vector $(\mu_1, \mu_2)^\intercal \in \mathbb{R}^{2\times 1}$, **volatilities** $(\sigma_1, \sigma_2) \in \mathbb{R}^+$, and **Pearson correlation coefficient** $\rho_{1,2} \in [-1, 1]$. Additionally, $\sigma_T \in \mathbb{R}^+$ denotes the **target risk/vol**.

1. Solve the *problem (3)* using a **Lagrangian approach**. You will denote the solution (the ***optimal solution***) by $\omega^*(\sigma_T)$ and the ***optimal value*** of the problem by $R_p(\omega_1^*(\sigma_T), \omega_2^*(\sigma_T))$ by $R_p(\sigma_T)$.

2. Assume that $\mu_1 = 5\%$, $\mu_2 = 10\%$, $\sigma_1 = 10\%$, $\sigma_2 = 20\%$, and $\rho_{1,2} = -0.5$ (moderate negative correlation).

- Consider a sequence of successive values of $\sigma_T$ in the range $[2\%, 30\%]$ by step of $0.5\%$
- Plot the efficient frontier: namely, the graph from the *mapping* $\sigma_T \mapsto R_p(\sigma_T)$.

The (aforementioned) graph maps the sequence of values of $\sigma_T$ from the $x$-axis into the sequence of values $R_p(\sigma_T)$ on the $y$-axis.

## 1. Lagrangian Solution

### Problem Formulation

The given optimization problem is:

$$\begin{cases} \max\limits_{x_1,x_2} & 5 - x_1^2 - x_1x_2 - 3x_2^2 \\ \text{s.t.} & x_1, x_2 \geq 0 \\ & x_1x_2 \geq 2 \end{cases} \tag{2}$$

### Lagrangian Formulation

The Lagrangian $\mathcal{L}$ for this problem includes the objective function and the constraints incorporated through Lagrange multipliers:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = 5 - x_1^2 - x_1x_2 - 3x_2^2 + \lambda_1x_1 + \lambda_2x_2 + \lambda_3(x_1x_2 - 2) \tag{3}$$

Here, $\lambda_1$, $\lambda_2$, and $\lambda_3$ are the Lagrange multipliers.

### Lagrangian Partial Derivatives

To find the stationary points, we take the partial derivatives of $\mathcal{L}$ and set them to zero:

$$\frac{\partial \mathcal{L}}{\partial x_1} = -2x_1 - x_2 + \lambda_1 + \lambda_3x_2 = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -x_1 - 6x_2 + \lambda_2 + \lambda_3x_1 = 0 \tag{5}$$

### Complementary Slackness

KKT conditions also include complementary slackness:

$$\lambda_1x_1 = 0, \quad \lambda_2x_2 = 0, \quad \lambda_3(x_1x_2 - 2) = 0 \tag{6}$$

### Solve the System of Equations

We consider different cases based on the KKT conditions:

1. **Case 1: Interior Solution**
   - Assume $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
   - Solve equations (4) and (5) directly.
2. **Case 2: Boundary Solution**
   - Assume $x_1 > 0, x_2 > 0$, and $x_1x_2 = 2$.
   - Substitute $x_1 = \frac{2}{x_2}$ into the equations and solve.

### Analytical Solution

**Case 1**: Assume $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Plugging in $\lambda_1 = \lambda_2 = \lambda_3 = 0$ simplifies equations (4) and (5):

$$-2x_1 - x_2 = 0 \implies x_2 = -2x_1 \quad \text{(not possible since } x_2 \geq 0\text{)} \tag{7}$$

**Case 2**: Boundary Solution with $x_1 x_2 = 2$

Substitute $x_1 = \frac{2}{x_2}$ into equations (4) and (5) and solve:

$$-2\left(\frac{2}{x_2}\right) - x_2 + \lambda_3 x_2 = 0 \quad \text{and} \quad -\frac{2}{x_2} - 6x_2 + \lambda_3 \frac{2}{x_2} = 0 \tag{8}$$

Solving these equations:

1. From equation (4): $\lambda_3 x_2 = 4/x_2 + x_2$
2. Substitute $\lambda_3 x_2$ from equation (4) into equation (5), and solve for $x_2$. This could yield $x_2 = \sqrt{2}$, which when substituted back gives $x_1 = \sqrt{2}$, hence $x_1 x_2 = 2$.

#### Conclusion P1

Analytical solutions indicate that at $x_1 = x_2 = \sqrt{2}$, the constraints are satisfied, and from the substitution into the objective function, we can evaluate the maximum value.

#### Optimal Solution

The optimal solution, denoted by $\omega^*(\sigma_T)$, for the given problem under the constraints and specified conditions is:

$$\omega_1^*(\sigma_T) = \sqrt{2}, \quad \omega_2^*(\sigma_T) = \sqrt{2}$$

This solution satisfies the constraint $x_1 x_2 \geq 2$ with $x_1 x_2 = 2$.

#### Optimal Value

The optimal value of the problem, denoted by $R_p(\sigma_T)$, is calculated by evaluating the objective function at the optimal solution points $\omega_1^*(\sigma_T)$ and $\omega_2^*(\sigma_T)$. The calculation is as follows:

$$\begin{aligned} R_p(\sigma_T) &= R_p(\omega_1^*(\sigma_T), \omega_2^*(\sigma_T)) = f(\sqrt{2}, \sqrt{2}) \\ &= 5 - (\sqrt{2})^2 - \sqrt{2} \times \sqrt{2} - 3(\sqrt{2})^2 \\ &= -5 \end{aligned}$$

Thus, the optimal value of the objective function, given the constraints and the formulation of the problem, is $R_p(\sigma_T) = -5$.

This value represents the maximum return achievable under the given constraints and is obtained at the solution where both $x_1$ and $x_2$ are $\sqrt{2}$.

## 2. Efficient Frontier Mapping

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import minimize
```

In [ ]:
```
from typing import Tuple, List

# Constants
mu1: float = 0.05  # Expected return of the first security
mu2: float = 0.10  # Expected return of the second security
sigma1: float = 0.10  # Volatility of the first security
sigma2: float = 0.20  # Volatility of the second security
rho: float = -0.5  # Correlation coefficient between the securities

# Target risk values
sigma_T_values: np.ndarray = np.arange(0.02, 0.305, 0.005)
```

In [ ]:
```
def portfolio_return(weights: np.ndarray, mu1: float, mu2: float) -> float:
    """
    Calculate the portfolio return based on given weights and expected returns.

    Parameters:
        weights (np.ndarray): Array of weights for the securities.
        mu1 (float): Expected return of the first security.
        mu2 (float): Expected return of the second security.

    Returns:
        float: The calculated portfolio return.
    """
    return weights[0] * mu1 + weights[1] * mu2
```

In [ ]:
```
def portfolio_risk(weights: np.ndarray, sigma1: float, sigma2: float, rho: float) -
    """
    Calculate the portfolio risk based on weights, individual volatilities, and cor

    Parameters:
        weights (np.ndarray): Array of weights for the securities.
        sigma1 (float): Volatility of the first security.
        sigma2 (float): Volatility of the second security.
        rho (float): Correlation coefficient between the securities.

    Returns:
        float: The calculated portfolio risk.
    """
    return np.sqrt((sigma1 * weights[0]) ** 2 + (sigma2 * weights[1]) ** 2 +
                   2 * rho * sigma1 * sigma2 * weights[0] * weights[1])
```

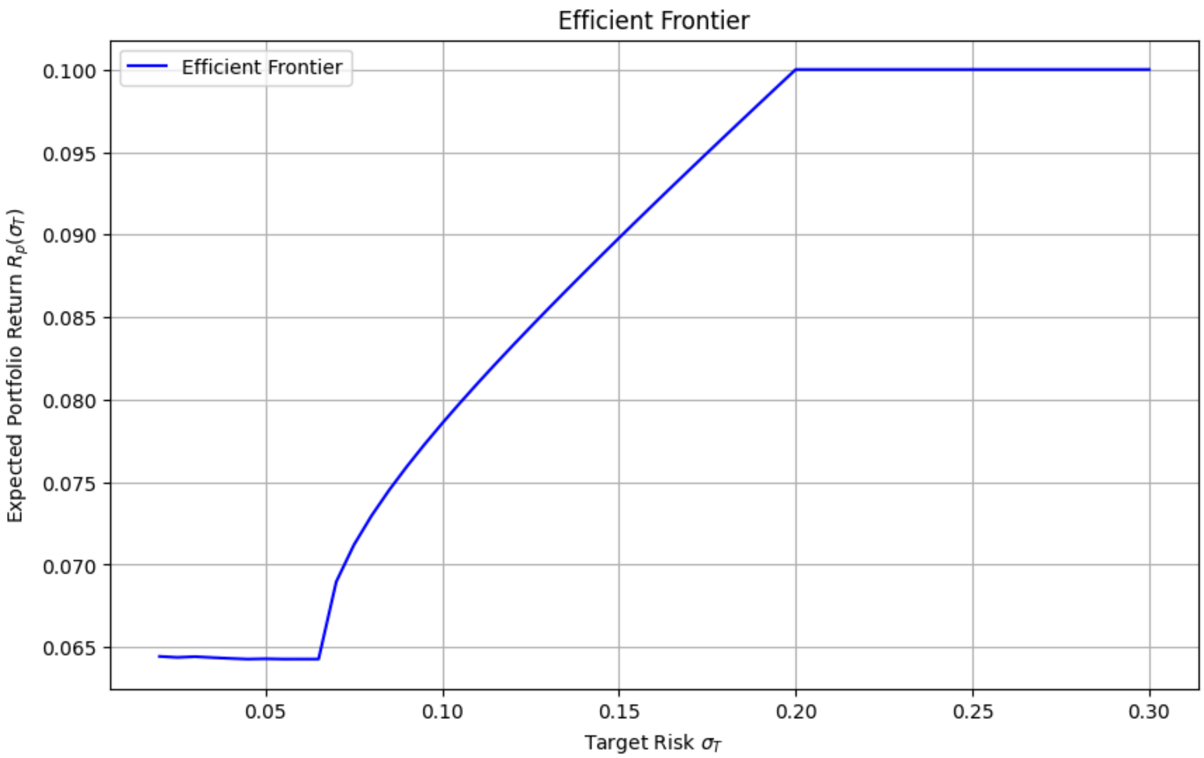
In [ ]:
```
def objective(weights: np.ndarray) -> float:
    """
    Objective function for minimization, used to maximize portfolio return.

    Parameters:
        weights (np.ndarray): Array of weights for the securities.

    Returns:
```

```
        float: Negative of the portfolio return (for minimization).
    """
    return -portfolio_return(weights, mu1, mu2)
```

In [ ]:
```python
def constraint(weights: np.ndarray, sigma_T: float) -> float:
    """
    Constraint for the optimizer to achieve a specific target risk.

    Parameters:
        weights (np.ndarray): Array of weights for the securities.
        sigma_T (float): Target risk level.

    Returns:
        float: Difference between current and target risks.
    """
    return portfolio_risk(weights, sigma1, sigma2, rho) - sigma_T
```

In [ ]:
```python
results_rp: List[float] = []

for sigma_T in sigma_T_values:
    cons = ({'type': 'eq', 'fun': lambda x: np.sum(x) - 1},
            {'type': 'eq', 'fun': lambda x: constraint(x, sigma_T)})
    bounds: Tuple[Tuple[float, float], Tuple[float, float]] = ((0, 1), (0, 1))
    initial_weights: List[float] = [0.5, 0.5]
    result = minimize(objective, initial_weights, bounds=bounds, constraints=cons)
    results_rp.append(-result.fun)
```

In [ ]:
```python
# Plotting the efficient frontier
plt.figure(figsize=(10, 6))
plt.plot(sigma_T_values, results_rp, 'b-', label='Efficient Frontier')
plt.title('Efficient Frontier')
plt.xlabel('Target Risk $\sigma_T$')
plt.ylabel('Expected Portfolio Return $R_p(\sigma_T)$')
plt.grid(True)
plt.legend()
plt.show()
```




### Frontier Analysis

The graph above depicts the relationship between the target risk ($\sigma_T$) and the expected portfolio return ($R_p(\sigma_T)$). Below are the key takeaways:

1. ***Monotonic Increase***: As expected, the expected portfolio return increases with an increase in target risk, $\sigma_T$. This reflects the classic **risk-return trade-off** in portfolio management.

2. ***Plateau at Higher Risks***: The plateau observed at higher risk levels suggests that increasing risk beyond a certain point does **not proportionally increase returns**. This could be indicative of the constraints imposed by the maximum returns achievable based on the securities' parameters.

3. ***Sharp Rise at Lower Risks***: The initial sharp rise suggests that minimal increases in risk from the lower end are highly compensated by increased returns. This can be attributed to the efficient allocation of weights in response to changes in $\sigma_T$ under the given constraints.

## Optimization w/Inequality Constraints (20 pts)

Solve analytically (at least) one of the two following problems:

$$\begin{cases} \max\limits_{x_1,x_2} & (x_1-2)^2 + 2(x_2-1)^2 \\ \text{s.t} & x_1 + 4x_2 \leq 3 \\ & x_1 \geq x_2 \end{cases} \tag{2}$$

$$\begin{cases} \max\limits_{x_1,x_2} & 5 - x_1^2 - x_1x_2 - 3x_2^2 \\ \text{s.t} & x_1, x_2 \geq 0 \\ & x_1x_2 \geq 2 \end{cases} \tag{3}$$

and use an optimizer to verify your answer.