Introduction to Portfolio Theory and Applications

(FE630 Portfolio Theory and Applications)
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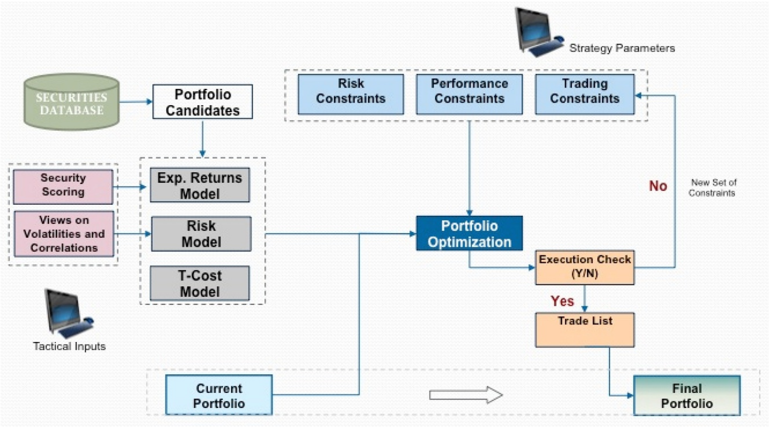
Portfolio Construction Process



Overview (I)

This course is an introduction to portfolio theory and optimization with applications.

1. It addresses
   - investor choice and market opportunities
   - optimal portfolio selection, sensitivity analysis and backtesting
2. It examines
   - security covariance and return models,
   - performance analysis, and return attribution.
3. It provides also some basic methods of robust portfolio construction.

Overview (II)

4. The course will also include a computational component in which students will
   - construct optimal portfolios
   - track their behavior and analyze their performance
5. Teaching will in a hybrid manner including lectures and Socratic discussions.
6. Evaluation based on
   - weekly assigned readings
   - quizzes, homework
   - projects (1individual and 1 group).

Expected Learning Outcomes

After successful completion of this course, students will be able to

1. Compute
   - Absolute Risk Aversion (ARA)
   - Certainty Equivalent of Risky (CER) of a risky gamble
   - Risk-premiums
2. Solve Optimal Decision Problems arising in Modern Portfolio Theory
3. Implement the solution using a high level language such as R, Matlab and Python

Expected Learning Outcomes

4. Compute Securities Expected Returns and Covariance using
   - the CAPM
   - the APT and Linear Factor Models
5. Build efficient Portfolios
   - the Two-fund theorem for Target Return
   - the One-fund Theorem for Target Return or Target Risk
6. Backtest Optimal Portfolios using historical price time series
7. Analyze the sensitivity to various inputs

References

The following books are recommended (reading assignments may come from below):

1. Francis and Kim, Modern Portfolio Theory, Wiley, 2013. ISBN: 111837052X.
2. Elton, Gruber, Brown, and Goetzman (EGBG), Modern Portfolio Theory and Investment Analysis, 9e, Wiley, 2014. ISBN: 0470388323.
3. Grinold and Kahn, Active Portfolio Management, 2e, McGraw Hill, 1999. ISBN:0070248826
4. Hubert, Essential mathematics for Market Risk Management, 2e, Wiley, 2012. ISBN 9781119979524
5. An electronic copy of Francis and Kim is available from the library.
6. Other Readings: Journal Papers or any material of interest, as needed.

Topics to be covered (1)

One-Period Utility Analysis.	Orientation. Basic ideas of investor preferences. Utility of wealth. Basic assumptions about utility. Certainty-equivalent wealth. Absolute and Relative Risk Aversion.
Computational Tools.	Review of linear and matrix algebra, matrix calculus.
Optimization Review.	Unconstrained optimization. Nonlinear optimization. Convex Constrained Optimization. KKT conditions.
The Opportunity Set.	Portfolio expected return and risk. Portfolio weights. Attainable regions of risk-return space. Risk reduction. Diversification and Markowitz.

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Efficient Frontiers.	The budget-constrained efficient frontier. Efficient frontier with short sales allowed. Efficient frontier with a risk-less security. Separation theorems.
CAPM, APT, Return Models.	Equilibrium models. Underlying Assumptions. Intuition. Price of risk. The security market line. Correlation structure of security returns. Covariance models.. Single and Multi-Index models.

Topics to be covered (3)

Robust Allocation	models of uncertainties of Expected Returns and Risk Matrices. Worst Case Optimization. Matrix Calibration. Black-Litterman Allocation.
Active Portfolios.	Active and excess return. Active weights. Information Ratio and Information Coefficient.
Bond Portfolios.	Active and Passive Bond Portfolio Management.
Dynamic Portfolio Allocation.	Risk Sensitive Asset Allocation. Maximum Principle, HJB Equation, Feedback using Riccati Equation.
Analysis.	Dynamic Allocation. Performance Measurement. Effects of cash flows. IRR. Performance Attribution.

Naive Question

- You have \$100 to invest....
- You have 2 securities you can choose between (Apple and Google)
- How do you split your initial wealth between those 2?

The Baby Portfolio Example

- Define your investment horizon
- Define your (Return) Target. , e.g. 10%,
- Define the investment "approach" you would be comfortable with - qualitative criteria ....
- Formulate an investment problem with an objective (decision) function we want to optimize
- Set some investment constraints, fully invested, long only, etc...

The Baby Portfolio Example

Model Simple price dynamics

- We know P_A^0 and P_G^0, the share prices of Apple and Google at time t=0.
- If we want to invest for 1 year, we need a model to predict P_A^1 and P_G^1, the prices at time 1.
- Simplest model on the distribution of returns: the one-year returns $r_A^{0,1}$ and $r_G^{0,1}$ are normally distributed:
  - $r_A^{0,1} = \frac{P_A^1 - P_A^0}{P_A^0} \sim \mathcal{N}(\mu_A, \sigma_A^2)$
  - $r_G^{0,1} = \frac{P_G^1 - P_G^0}{P_G^0} \sim \mathcal{N}(\mu_G, \sigma_G^2)$.

The Baby Portfolio Example

Strategy **Buy and Hold** n_A shares of Apple and n_G shares of Google at time 0

- Portfolio Value at time t=0 is $W_P^0 = n_A P_A^0 + n_G P_G^0$
- Portfolio Value at time t=1 is $W_P^1 = n_A P_A^1 + n_G P_G^1$
- Portfolio one-year Random Return $r_P^{0,1} = \frac{W_P^1 - W_P^0}{W_P^0} \sim \mathcal{N}(\mu_P, \sigma_P^2)$ with
    - $\mu_P = \omega_A \mu_A + \omega_G \mu_G$
    - $\sigma_P^2 = \omega_A^2 \sigma_A^2 + \omega_G^2 \sigma_G^2 + 2\omega_A \omega_G cov(r_A, r_G)$ where
    - ω_A and ω_G are the proportions of the initial wealth invested in Apple and Google.

The Baby Portfolio Example: Decision Model

- Choose the initial weights ω_A and ω_G, namely the optimal % of wealth to invest in Apple and Google at time 0, in order to
  1. **Minimize the variance of** $r_P^{0,1}$ under the constraints
  2. Portfolio's Target Expected Return $\mu_P = \omega_A \mu_A + \omega_G \mu_G = 10\%$
  3. Initial Wealth Fully Invested $\omega_A + \omega_G = 1$
- Mathematical Formulation

$$\begin{cases} \displaystyle\min_{\omega_A, \omega_G \in \mathbb{R}} f(\omega_A, \omega_G) & := \omega_A^2 \sigma_A^2 + \omega_G^2 \sigma_G^2 + 2\omega_A \omega_G cov(r_A, r_G) \\ & := \omega_A^2 \sigma_A^2 + \omega_G^2 \sigma_G^2 + 2\omega_A \omega_G \sigma_{A,G} \\ & \\ \text{subject to} & \omega_A \mu_A + \omega_G \mu_G = 10\% \\ & \omega_A + \omega_G = 1 \end{cases} \tag{1}$$

The Baby Portfolio Example: a Quadratic Optimization

1. Define $\omega = \begin{pmatrix} \omega_A \\ \omega_G \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_A \\ \mu_G \end{pmatrix}$, $\mathbf{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} \sigma_A^2 & \sigma_{A,G} \\ \sigma_{A,G} & \sigma_G^2 \end{pmatrix}$. Then solve

$$\begin{cases} \min\limits_{\omega \in \mathbb{R}} f(\omega) & := \omega^T \Sigma \omega \\ \text{subject to} & \mu^T \omega = 10\%, \quad \mathbf{e}^T \omega = 1 \end{cases} \tag{2}$$

   to find the optimal initial allocation of wealth ω given initial estimates of the inputs μ (Expected Returns) and Σ (Covariance Matrix)...

2. How hard is that?
3. Why settle for 10%, 15 or 20? Explore, and Measure the cost associated to each feasible Target and then choose what is acceptable for the investor?
4. Concept of Efficient Frontier and Expected Utility (a measure of satisfaction derived from an investment decision)

(Call for help) Numerical Solution of Standard Mean-Variance

- Suppose we have three Securities A, B and C with Mean Expected returns $\rho_A = 10\%$, $\rho_B = 10\%$ and $\rho_C = 15\%$. If we are given the following covariance for the returns

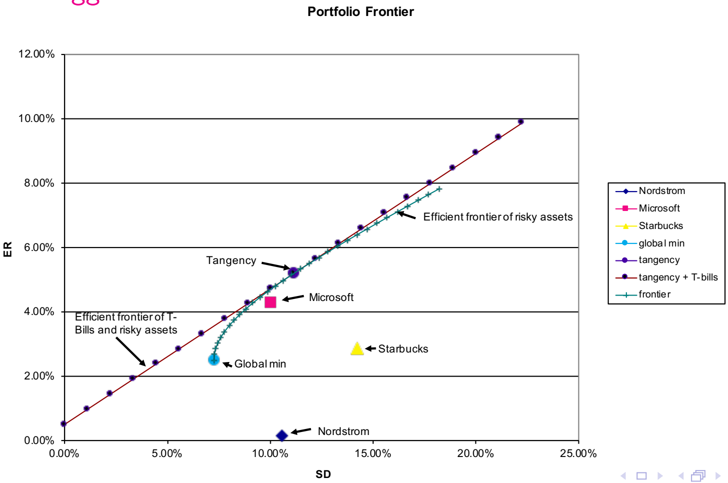
$$\begin{pmatrix} 4\% & 1\% & 0 \\ 1\% & 8\% & 2\% \\ 0 & 2\% & 16\% \end{pmatrix},$$

  what is the optimal portfolio composition that minimize the volatility for a target return of 12%? What is the meaning of Target Expected return of 10% ?
- **Class Participation Homework**: identify what packages / solvers in Python or R you would use to solve that optimization problem.
- **Open Question**: If an expected return of 12.5% was desired instead of 12%, what would be approximatively the corresponding variance?

Does the Solution to Standard Mean-Variance Exist?

- Proposed Approach
  1. Write down the optimization problem
  2. Find the requirements on the inputs (Covariance and Expected return) for existence and uniqueness of the solution
  3. Form the Lagrangian
  4. Apply the optimality condition (KKT?) to obtain the solution and the multipliers.
- Why 10%?
- Is it possible to find the optimal portfolios for all the range of attainable performance?
- We can also maximize the performance under Risk constraint...
- What if the risk is measured by the VaR, CVaR, semi-volatility, or Sharpe Ratio?

Would you put all eggs in the same basket?



Achieving Risk Reduction by Diversification (Standard Mean-Variance)

- Now consider stocks of Microsoft (MS), Nordstrom (NS) and Starbucks (ST).
- Suppose we have Expected returns ρ_{MS}, ρ_{NS} and ρ_{ST} and Covariance Σ as

$$\rho = \begin{pmatrix} \rho_{MS} \\ \rho_{NS} \\ \rho_{ST} \end{pmatrix} = \begin{pmatrix} 4.27\% \\ 0.27\% \\ 2.85\% \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 0.99\% & 0.18\% & 0.11\% \\ 0.18\% & 1.09\% & 0.26\% \\ 0.11\% & 0.26\% & 1.99\% \end{pmatrix},$$

- Clearly MS is the most attractive security (highest Exp. Return, Lowest Risk).
- But instead of betting 100% of our wealth in Microsoft, and having a highly concentrated position, can we find a Portfolio with the same Expected Return as Microsoft but with variance smaller or equal to Microsoft's variance?
- Let's try it live in Python ...

Two other examples

1. Optimal Execution
   - You have built your Portfolio
   - How do you execute it?
2. Detecting Arbitrage
   - You are given the expected return of 3 assets
   - Is there an arbitrage opportunity?

Naive Order Execution

- Suppose we have a bunch of trades (Q shares in total) we would like to execute using some brokers, Broker A and Broker B.
  - The "cost" of transaction x shares on A is ax^2
  - The cost of execution of y shares on B is by^2 where $a > 0$ and $b > 0$ are given.
  - How can we split Q while minimizing cost?
  - What happens to the cost if Q increases by $r\%$?
- What about the real case where there is a correlation between the Brokers prices?

Arbitrage Opportunity?

An investor is interested in buying shares of IBM and DELL. Today's stock prices are given in the following table.

Security	IBM	DELL
Price	\$100	\$90

Prices at time t = 0

He assumes that the dynamics the prices of the stocks in one year (time 1) can be captured using two possible states for the evolution of the economy (expansion or recession), each of these scenarios having a probability of 1/2 to occur:

Security	IBM	DELL
Scenario : Expansion (p=1/2)	\$140	\$160
Scenario Recession (p=1/2)	\$100	\$80

Prices at time t= 1.

Arbitrage Opportunity?

The National Bureau of Economics Research provides a business cycle indicator which is equal to 1 if the economy is in expansion, and equal to 0 if the economy is in recession. The investor believes that the returns of the stocks can be described with a single factor given as the business cycle factor f_{BC}.

1. What are the factor Risk β_{IBM} and β_{DELL} according to that single factor model?
2. What are the risk Premia λ_0 and λ_{BC} and why the investor would consider DELL to be riskier than IBM?
3. If we have a risk-free asset which we can buy or sell at a rate of 5%, do we have an arbitrage opportunity? If yes, how can we build an arbitrage portfolio?

Hedging Market Risk

An investor wishes to design a hedge basket to offset the Market Risk of a position she holds in stocks of Apple (Ticker: AAPL). by taking position in a Mean-variance portfolios with well-chosen target betas.

1. The market (S&P-500 index) will be represented by the SPY ETF so Market Risk may be reflected by the Beta of the position
2. Assume that you want to realize the Hedge using a basket of ETFs with tickers below:

$$\mathscr{U} = \{\text{FXE}, \text{EWJ}, \text{GLD}, \text{QQQ}, \text{SHV}, \text{DBA}, \text{USO}, \text{XBI}, \text{ILF}, \text{EPP}, \text{FEZ}\}.$$

3. How would you build a portfolio that effectively offsets the Market Risk ?

Robustness Problems may capsize the boat!

- Suppose we have three Securities A, B and C with Mean Expected returns $\rho_A = 12.13\%$, $\rho_B = 15.48\%$ and $\rho_C = 9\%$ (Large Cap, Foreign and Bond) .

- We are given a covariance matrix and design the minimum variance portfolio with Target Expected Return 13%.

- Now consider a change that makes the Large Cap Expected Return by $+10\%$.

- **How far should the new minimum variance portfolio with Target Expected Return 13% be from the previous one?**

Final Remarks . . .

Common Complaints with Portfolio Optimization

"Laundry List" when people are moving from theory to practical optimization in fund management.

- Problem 1: Portfolio optimization is too hard
- Problem 2: Portfolio optimizers suggest too much trading
- Problem 3: Expected returns are needed
- Problem 4: Mean-variance optimization is restrictive
- Problem 5: Portfolio optimization inputs are noisy estimates
- Problem 6: Transaction costs are tricky
- Problem 7: Risk and Alpha factor alignment trouble

Common Complaints with Portfolio Optimization : Laundry list....

- Problem 1: Portfolio optimization is too hard
  - True.
  - Finding the appropriate formulation that leads to a solvable problem reflecting business perspective is sometimes hard.
  - And some optimization problems are intrinsically hard...
- Problem 2: Portfolio optimizers suggest too much trading
  - $\leftarrow$ what is the decision model and the constraints set?

Common Complaints with Portfolio Optimization

- Problem 3: Expected returns are needed
  - Not always but make sense.
  - Alternative options exist:
    - Target Portfolios
    - Reverse Optimization
    - Asset Ranking

Common Complaints with Portfolio Optimization

- Problem 4: Mean-variance optimization is restrictive
  - True but allows to cover a lot of ground from Index replication to Hedging of Positions or Portfolios by replicationg its Beta
- Problem 5: Portfolio optimization inputs are noisy estimates
  - True
  - But range of tools and modeling solutions is available, from Black-Litterman approach to robust optimization, covariance reconditionning, ...

Common Complaints with Portfolio Optimization

- Problem 6: Transaction costs are tricky
  - True. A model for Trading Costs and Market market impact is needed
- Problem 7: Risk and Alpha factor alignment trouble
  - True
  - Sometimes, trouble to re-conciliate Risk factors and Performance factors

General Form of Risk minimization problem

- We may consider a problem like

$$\begin{cases} \min\limits_{\omega \in \mathbb{R}^n} f(\omega) & := \frac{1}{2}\omega^T \Sigma \omega \text{ or another Risk function} \\ \text{subject to} & A\omega = b, \quad g(\omega) \le 0 \\ & \quad \text{with } A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \rho_1 & \rho_2 & \dots & \rho_n \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ \rho_{Target} \end{pmatrix} \\ & \quad \text{and g Convex function structuring the constraints.} \end{cases}$$

  Here, g can model Risk Limits per sector, or contribution to Risk...
- Alternatively, the dual problem: maximize the return under risk constraint....

A few things we will learn how to do

- A way to price Securities in view of predicting their Expected Returns: Capital Asset Pricing Model (CAPM)
- Explain the Risk of a Security or a Portfolio using its correlation with the Market: Betas as risk Explain Portfolio Risk and decompose into Systematic and Specific Risks
- Mean Variance as a tool to diversify and reduce Portfolios Risk
- Formulate a Robust Allocation problem to deal with noise or uncertainty such as Bid-Ask spreads
- Incorporate information, Models of Uncertainty or Views in Returns or Covariance, for example in the Black-Litterman Setting

Classification of Portfolio Optimization Models

Uncertainty (Vertical) / Time (Horizontal)	
Single Period Stochastic Models Stochastic Changes Mean Variance Scenario Optimization	**Multiple Period Stochastic Models** Stochastic Changes Dynamic Allocation Stochastic Programming
Single Period Static Models Small Changes Models for Fixed Income	**Contingency Planning** Small Changes Requires Perfect Forecasting
Trivial Models (no uncertainty)	Trivial Models (no uncertainty)

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Next Lecture - Focus will be on Utility Functions.

We will cover the following topics

- Deciding How to Invest
- Decision Under Uncertainty
- Risk Aversion and Certainty Equivalent of Risky gambles
- Absolute Risk Aversion
- Expected Utility Maximization