FE630 - Homework #2

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Pledge: I pledge my honor that I have abided by the Stevens Honor System.

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Topics

- Algebra & Optimization;
- Geometry of Efficient Frontiers;
- Applications of One-Fund & Two-Fund Theorems.

P1 - Optimization w/Equality Constraints (40 pts)

Consider the optimization problem **Max Expected Return w/Target Risk**:

$$\begin{cases} \max_{\omega_1,\omega_2} & R_p(\omega_1,\omega_2) = \mu_1\omega_1 + \mu_2\omega_2 \\ \text{s.t.} & \sqrt{\sigma_1^2\omega_1^2 + 2\rho_{1,2}\sigma_1\sigma_2\omega_1\omega_2 + \sigma_2^2\omega_2^2} = \sigma_T \\ & \omega_1 + \omega_2 = 1 \end{cases} \tag{1}$$

where we have two securities with **Expected Returns** μ_1 and μ_2 for the column vector $(\mu_1, \mu_2)^\intercal \in \mathbb{R}^{2\times1}$, **volatilities** $(\sigma_1, \sigma_2) \in \mathbb{R}^+$, and **Pearson correlation coefficient** $\rho_{1,2} \in [-1, 1]$. Additionally, $\sigma_T \in \mathbb{R}^+$ denotes the **target risk/vol**.

1. Solve the *problem (3)* using a **Lagrangian approach**. You will denote the solution (the ***optimal solution***) by $\omega^*(\sigma_T)$ and the ***optimal value*** of the problem by $R_p(\omega_1^*(\sigma_T), \omega_2^*(\sigma_T))$ by $R_p(\sigma_T)$.

2. Assume that $\mu_1 = 5\%$, $\mu_2 = 10\%$, $\sigma_1 = 10\%$, $\sigma_2 = 20\%$, and $\rho_{1,2} = -0.5$ (moderate negative correlation).

- Consider a sequence of successive values of σ_T in the range $[2\%, 30\%]$ by step of 0.5%
- Plot the efficient frontier: namely, the graph from the *mapping* $\sigma_T \mapsto R_p(\sigma_T)$.

The (aforementioned) graph maps the sequence of values of σ_T from the x-axis into the sequence of values $R_p(\sigma_T)$ on the y-axis.

Efficient Frontier Mapping

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import minimize
```

In [ ]:
```
from typing import Tuple, List

# Constants
mu1: float = 0.05  # Expected return of the first security
mu2: float = 0.10  # Expected return of the second security
sigma1: float = 0.10  # Volatility of the first security
sigma2: float = 0.20  # Volatility of the second security
rho: float = -0.5  # Correlation coefficient between the securities

# Target risk values
sigma_T_values: np.ndarray = np.arange(0.02, 0.305, 0.005)
```

In [ ]:
```
def portfolio_return(weights: np.ndarray, mu1: float, mu2: float) -> float:
    """
    Calculate the portfolio return based on given weights and expected returns.

    Parameters:
        weights (np.ndarray): Array of weights for the securities.
        mu1 (float): Expected return of the first security.
        mu2 (float): Expected return of the second security.

    Returns:
        float: The calculated portfolio return.
    """
    return weights[0] * mu1 + weights[1] * mu2
```

In [ ]:
```
def portfolio_risk(weights: np.ndarray, sigma1: float, sigma2: float, rho: float) -
    """
    Calculate the portfolio risk based on weights, individual volatilities, and cor

    Parameters:
        weights (np.ndarray): Array of weights for the securities.
        sigma1 (float): Volatility of the first security.
        sigma2 (float): Volatility of the second security.
        rho (float): Correlation coefficient between the securities.

    Returns:
        float: The calculated portfolio risk.
    """
    return np.sqrt((sigma1 * weights[0]) ** 2 + (sigma2 * weights[1]) ** 2 +
                   2 * rho * sigma1 * sigma2 * weights[0] * weights[1])
```

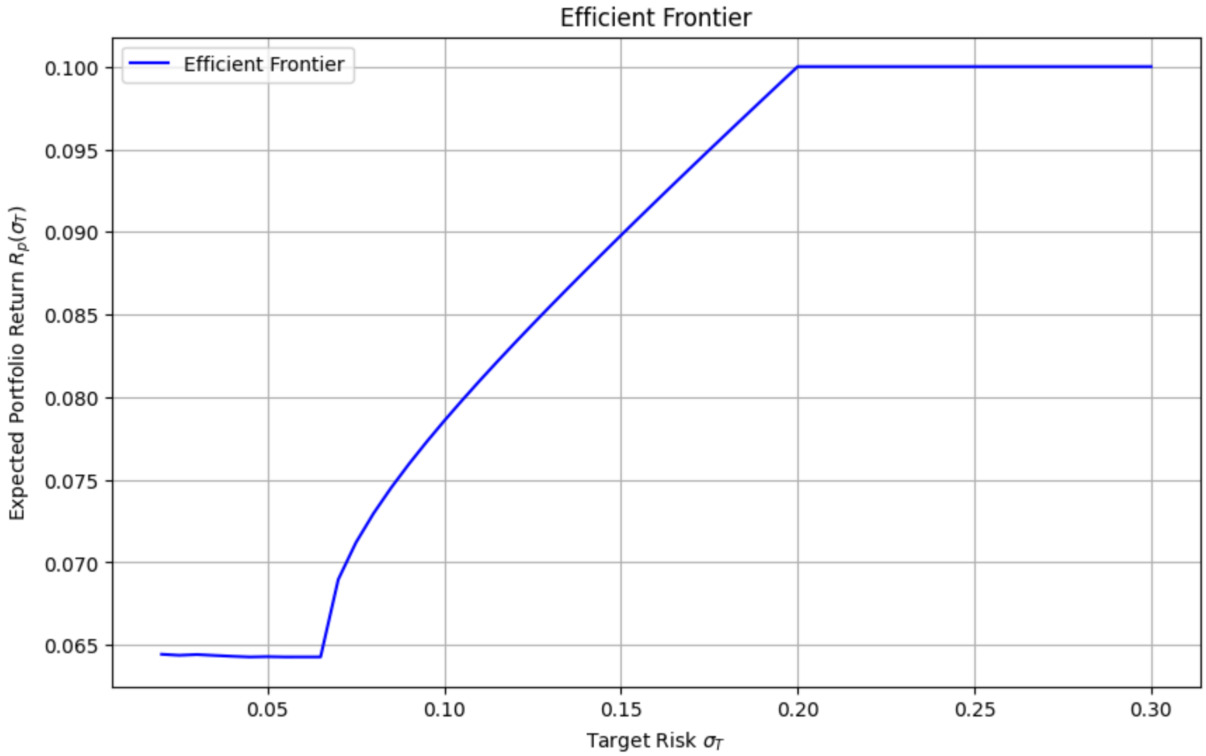
In [ ]:
```
def objective(weights: np.ndarray) -> float:
    """
    Objective function for minimization, used to maximize portfolio return.

    Parameters:
        weights (np.ndarray): Array of weights for the securities.

    Returns:
```

```
        float: Negative of the portfolio return (for minimization).
    """
    return -portfolio_return(weights, mu1, mu2)
```

In [ ]:
```
def constraint(weights: np.ndarray, sigma_T: float) -> float:
    """
    Constraint for the optimizer to achieve a specific target risk.

    Parameters:
        weights (np.ndarray): Array of weights for the securities.
        sigma_T (float): Target risk level.

    Returns:
        float: Difference between current and target risks.
    """
    return portfolio_risk(weights, sigma1, sigma2, rho) - sigma_T
```

In [ ]:
```
results_rp: List[float] = []

for sigma_T in sigma_T_values:
    cons = ({'type': 'eq', 'fun': lambda x: np.sum(x) - 1},
            {'type': 'eq', 'fun': lambda x: constraint(x, sigma_T)})
    bounds: Tuple[Tuple[float, float], Tuple[float, float]] = ((0, 1), (0, 1))
    initial_weights: List[float] = [0.5, 0.5]
    result = minimize(objective, initial_weights, bounds=bounds, constraints=cons)
    results_rp.append(-result.fun)
```

In [ ]:
```
# Plotting the efficient frontier
plt.figure(figsize=(10, 6))
plt.plot(sigma_T_values, results_rp, 'b-', label='Efficient Frontier')
plt.title('Efficient Frontier')
plt.xlabel('Target Risk $\sigma_T$')
plt.ylabel('Expected Portfolio Return $R_p(\sigma_T)$')
plt.grid(True)
plt.legend()
plt.show()
```


Efficient Frontier
Efficient Frontier
0.100
0.095
0.090
0.085
0.080
0.075
0.070
0.065
Expected Portfolio Return $R_p(\sigma_T)$
0.05
0.10
0.15
0.20
0.25
0.30
Target Risk σ_T