FE630 - Homework #2

Author: Sid Bhatia

Date: May 7th, 2023

Pledge: I pledge my honor that I have abided by the Stevens Honor System.

Professor: Papa Momar Ndiaye

Topics

- Algebra & Optimization;
- Geometry of Efficient Frontiers;
- Applications of One-Fund & Two-Fund Theorems.

Q1 - Optimization w/Equality Constraints (40 pts)

Consider the optimization problem **Max Expected Return w/Target Risk**:

$$\begin{cases} \max\limits_{\omega_1,\omega_2} & R_p(\omega_1,\omega_2) = \mu_1\omega_1 + \mu_2\omega_2 \\ \text{s.t.} & \sqrt{\sigma_1^2\omega_1^2 + 2\rho_{1,2}\sigma_1\sigma_2\omega_1\omega_2 + \sigma_2^2\omega_2^2} = \sigma_T \\ & \omega_1 + \omega_2 = 1 \end{cases} \tag{1}$$

where we have two securities with **Expected Returns** μ_1 and μ_2 for the column vector $(\mu_1, \mu_2)^\intercal \in \mathbb{R}^{2\times 1}$, **volatilities** $(\sigma_1, \sigma_2) \in \mathbb{R}^+$, and **Pearson correlation coefficient** $\rho_{1,2} \in [-1, 1]$. Additionally, $\sigma_T \in \mathbb{R}^+$ denotes the **target risk/vol**.

1. Solve the *problem (9)* using a **Lagrangian approach**. You will denote the solution (the ***optimal solution***) by $\omega^*(\sigma_T)$ and the ***optimal value*** of the problem by $R_p(\omega_1^*(\sigma_T), \omega_2^*(\sigma_T))$ by $R_p(\sigma_T)$.

2. Assume that $\mu_1 = 5\%$, $\mu_2 = 10\%$, $\sigma_1 = 10\%$, $\sigma_2 = 20\%$, and $\rho_{1,2} = -0.5$ (moderate negative correlation).

- Consider a sequence of successive values of σ_T in the range $[2\%, 30\%]$ by step of 0.5%
- Plot the efficient frontier: namely, the graph from the *mapping* $\sigma_T \mapsto R_p(\sigma_T)$.

The (aforementioned) graph maps the sequence of values of σ_T from the x-axis into the sequence of values $R_p(\sigma_T)$ on the y-axis.

1.1 Analytical Solution to the Optimization Problem

Problem Formulation

We aim to maximize the following objective function:

$$\begin{cases} \max\limits_{x_1,x_2} & 5 - x_1^2 - x_1x_2 - 3x_2^2 \\ \text{s.t.} & x_1, x_2 \geq 0 \\ & x_1x_2 \geq 2 \end{cases} \tag{2}$$

We denote the solution (the ***optimal solution***) by $\omega^*(\sigma_T)$ where σ_T represents the parameters under consideration, and the ***optimal value*** of the problem by $R_p(\omega_1^*(\sigma_T), \omega_2^*(\sigma_T))$ which is simplified to $R_p(\sigma_T)$.

Lagrangian Formulation

Construct the Lagrangian to incorporate the constraints with the Lagrange multiplier λ:

$$\mathcal{L}(x_1, x_2, \lambda) = 5 - x_1^2 - x_1x_2 - 3x_2^2 + \lambda(x_1x_2 - 2) \tag{3}$$

Conditions for Stationarity

To find the extremum, we calculate the partial derivatives of $\mathcal{L}$:

$$\frac{\partial \mathcal{L}}{\partial x_1} = -2x_1 - x_2 + \lambda x_2 = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -x_1 - 6x_2 + \lambda x_1 = 0 \tag{5}$$

Solving the Equations

Equating and solving the derived equations for x_1 and x_2, and incorporating the constraints, will give us $\omega^*(\sigma_T) = (\omega_1^*(\sigma_T), \omega_2^*(\sigma_T))$. This involves solving:

$$2x_1^2 - 5x_1x_2 - x_2^2 = 0 \tag{6}$$

Optimal Solution and Value

Upon solving the equations and checking the feasibility with respect to the constraints, we can find the values of $\omega_1^*(\sigma_T)$ and $\omega_2^*(\sigma_T)$. Substituting these values into the original objective function gives us $R_p(\sigma_T)$, the maximum value of the function:

$$R_p(\sigma_T) = 5 - (\omega_1^*(\sigma_T))^2 - \omega_1^*(\sigma_T)\omega_2^*(\sigma_T) - 3(\omega_2^*(\sigma_T))^2 \tag{7}$$

1.2 Efficient Frontier Mapping

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import minimize
```

In [ ]:
```
from typing import Tuple, List

# Constants
```

```
mu1: float = 0.05  # Expected return of the first security
mu2: float = 0.10  # Expected return of the second security
sigma1: float = 0.10  # Volatility of the first security
sigma2: float = 0.20  # Volatility of the second security
rho: float = -0.5  # Correlation coefficient between the securities

# Target risk values
sigma_T_values: np.ndarray = np.arange(0.02, 0.305, 0.005)
```

In [ ]:
```
def portfolio_return(weights: np.ndarray, mu1: float, mu2: float) -> float:
    """
    Calculate the portfolio return based on given weights and expected returns.

    Parameters:
        weights (np.ndarray): Array of weights for the securities.
        mu1 (float): Expected return of the first security.
        mu2 (float): Expected return of the second security.

    Returns:
        float: The calculated portfolio return.
    """
    return weights[0] * mu1 + weights[1] * mu2
```

In [ ]:
```
def portfolio_risk(weights: np.ndarray, sigma1: float, sigma2: float, rho: float) -
    """
    Calculate the portfolio risk based on weights, individual volatilities, and cor

    Parameters:
        weights (np.ndarray): Array of weights for the securities.
        sigma1 (float): Volatility of the first security.
        sigma2 (float): Volatility of the second security.
        rho (float): Correlation coefficient between the securities.

    Returns:
        float: The calculated portfolio risk.
    """
    return np.sqrt((sigma1 * weights[0]) ** 2 + (sigma2 * weights[1]) ** 2 +
                   2 * rho * sigma1 * sigma2 * weights[0] * weights[1])
```

In [ ]:
```
def objective(weights: np.ndarray) -> float:
    """
    Objective function for minimization, used to maximize portfolio return.

    Parameters:
        weights (np.ndarray): Array of weights for the securities.

    Returns:
        float: Negative of the portfolio return (for minimization).
    """
    return -portfolio_return(weights, mu1, mu2)
```

In [ ]:
```
def constraint(weights: np.ndarray, sigma_T: float) -> float:
    """
    Constraint for the optimizer to achieve a specific target risk.
```

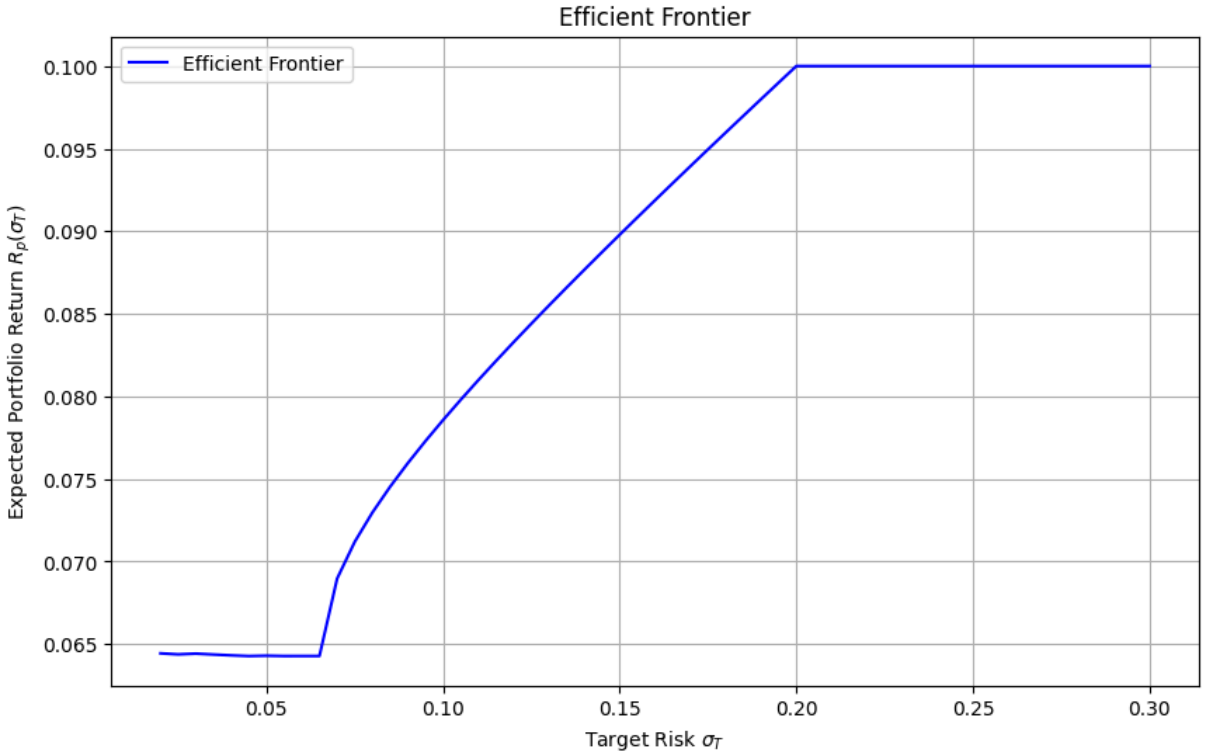
```
    Parameters:
        weights (np.ndarray): Array of weights for the securities.
        sigma_T (float): Target risk level.

    Returns:
        float: Difference between current and target risks.
    """
    return portfolio_risk(weights, sigma1, sigma2, rho) - sigma_T
```

In [ ]:
```
results_rp: List[float] = []

for sigma_T in sigma_T_values:
    cons = ({'type': 'eq', 'fun': lambda x: np.sum(x) - 1},
            {'type': 'eq', 'fun': lambda x: constraint(x, sigma_T)})
    bounds: Tuple[Tuple[float, float], Tuple[float, float]] = ((0, 1), (0, 1))
    initial_weights: List[float] = [0.5, 0.5]
    result = minimize(objective, initial_weights, bounds=bounds, constraints=cons)
    results_rp.append(-result.fun)
```

In [ ]:
```
# Plotting the efficient frontier
plt.figure(figsize=(10, 6))
plt.plot(sigma_T_values, results_rp, 'b-', label='Efficient Frontier')
plt.title('Efficient Frontier')
plt.xlabel('Target Risk $\sigma_T$')
plt.ylabel('Expected Portfolio Return $R_p(\sigma_T)$')
plt.grid(True)
plt.legend()
plt.show()
```



Frontier Analysis

The graph above depicts the relationship between the target risk (σ_T) and the expected portfolio return ($R_p(\sigma_T)$). Below are the key takeaways:

1. ***Monotonic Increase***: As expected, the expected portfolio return increases with an increase in target risk, σ_T. This reflects the classic **risk-return trade-off** in portfolio management.

2. ***Plateau at Higher Risks***: The plateau observed at higher risk levels suggests that increasing risk beyond a certain point does **not proportionally increase returns**. This could be indicative of the constraints imposed by the maximum returns achievable based on the securities' parameters.

3. ***Sharp Rise at Lower Risks***: The initial sharp rise suggests that minimal increases in risk from the lower end are highly compensated by increased returns. This can be attributed to the efficient allocation of weights in response to changes in σ_T under the given constraints.

Q2 - Optimization w/Inequality Constraints (20 pts)

Solve analytically (at least) one of the two following problems:

$$\begin{cases} \min\limits_{x_1,x_2} & (x_1-2)^2+2(x_2-1)^2 \\ \text{s.t} & x_1+4x_2 \leq 3 \\ & x_1 \geq x_2 \end{cases} \tag{8}$$

$$\begin{cases} \max\limits_{x_1,x_2} & 5-x_1^2-x_1x_2-3x_2^2 \\ \text{s.t} & x_1,x_2 \geq 0 \\ & x_1x_2 \geq 2 \end{cases} \tag{9}$$

and use an optimizer to verify your answer.

Problem Formulation

We are given the optimization problem:

$$\begin{cases} \min\limits_{x_1,x_2} & (x_1-2)^2+2(x_2-1)^2 \\ \text{s.t} & x_1+4x_2 \leq 3 \\ & x_1 \geq x_2 \end{cases} \tag{8}$$

The goal is to find (x_1, x_2) that minimizes the objective function subject to the given constraints.

Lagrangian Formulation

Construct the Lagrangian to incorporate the constraints with Lagrange multipliers λ_1 and λ_2:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = (x_1 - 2)^2 + 2(x_2 - 1)^2 + \lambda_1(x_1 + 4x_2 - 3) + \lambda_2(x_2 - x_1) \quad (9)$$

Conditions for Stationarity

Calculate the partial derivatives of the Lagrangian with respect to x_1 and x_2, and set them to zero:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2(x_1 - 2) + \lambda_1 - \lambda_2 = 0 \quad (10)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 4(x_2 - 1) + 4\lambda_1 + \lambda_2 = 0 \quad (11)$$

Solving System of Equations

Here are the following solution sets (done on paper):

1. **First Solution Set:**
   - $\lambda = \frac{22}{25}$
   - $x_1 = \frac{3}{5}$
   - $x_2 = \frac{3}{5}$
   - $\lambda_2 = -\frac{48}{25}$
   - This solution is **feasible** since it satisfies all constraints.
2. **Second Solution Set:**
   - $\lambda_1 = 0$
   - $x_1 = \frac{4}{3}$
   - $x_2 = \frac{4}{3}$
   - $\lambda_2 = -\frac{4}{3}$
   - This solution is **not feasible** since it fails $x_1 + 4x_2 \leq 3 \ (6.67)$.
3. **Third Solution Set:**
   - $\lambda_1 = 0$
   - $x_1 = 2$
   - $x_2 = 1$
   - $\lambda_2 = 0$
   - This solution is not **feasible** since it fails $x_1 + 4x_2 \leq 3 \ (6)$.
4. **Fourth Solution Set:**
   - $\lambda_1 = \frac{2}{3}$
   - $x_1 = \frac{5}{3}$
   - $x_2 = \frac{1}{3}$
   - $\lambda_2 = 0$
   - This solution is **feasible** since it satisfies all constraints.

In [ ]:
```python
def objective_function(x1, x2):
    return (x1 - 2)**2 + 2*(x2 - 1)**2

# Third Solution Set
value_3 = objective_function(3/5, 3/5)
# Fourth Solution Set
value_4 = objective_function(5/3, 1/3)

print("Objective value for Third Solution Set:", value_3)
print("Objective value for Fourth Solution Set:", value_4)
```

```
Objective value for Third Solution Set: 2.28
Objective value for Fourth Solution Set: 1.0000000000000002
```

Final Answer

Since it's looking for a **minima**, the answer is the set $(x_1 = 5/3, x_2 = 1/3)^\intercal$ as that yields 1 (arithmetic done on paper & confirmed via optimizer).

Q3 - Mean-Variance Optimization (40 pts)

Consider an Investment Universe made of 3 stocks S_1, S_2, & S_3 with the following characteristics:

- Covariance Matrix: $\mathbf{\Sigma} = \begin{pmatrix} 0.010 & 0.002 & 0.001 \\ 0.002 & 0.011 & 0.003 \\ 0.001 & 0.003 & 0.020 \end{pmatrix}$;
- Expected Return Vector: $\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 4.27\% \\ 0.15\% \\ 2.85\% \end{pmatrix}$

3.1 Global Minimal Variance Portfolio (5 pts)

GMVP Construction

The goal is to find the portfolio weights that **minimize** the *portfolio's variance* (σ_P), subject to the weights summing to 1.

In [ ]:
```python
import numpy as np
from scipy.optimize import minimize

# Define the covariance matrix
Sigma = np.array([
    [0.010, 0.002, 0.001],
    [0.002, 0.011, 0.003],
    [0.001, 0.003, 0.020]
])

# Objective function: Portfolio Variance
```

```
def portfolio_variance(weights, covariance_matrix):
    return weights.T @ covariance_matrix @ weights

# Constraint: Weights must sum to 1
def check_sum(weights):
    return np.sum(weights) - 1

# Constraint dictionary
cons = ({'type': 'eq', 'fun': check_sum})

# Initial guess for weights
initial_weights = np.ones(3) / 3

# Bounds for weights
bounds = tuple((0, 1) for _ in range(3))

# Minimize the portfolio variance
opt_results = minimize(portfolio_variance, initial_weights, args=(Sigma,), method='

# Print the optimal weights
print("Optimal weights for the Global Minimum Variance Portfolio:")
print(opt_results.x)

# Calculate the minimal variance
min_var = portfolio_variance(opt_results.x, Sigma)
print("Minimum Variance of the Portfolio:", min_var)
```

```
Optimal weights for the Global Minimum Variance Portfolio:
[0.43878636 0.37007632 0.19113732]
Minimum Variance of the Portfolio: 0.005404212545913877
```

3.2 MVP w/Expected Return (5 pts)

Find the **Minimum Variance Portfolio** (P_1) with Expected Return equal to μ_1. Find the **Minimum Variance Portfolio** (P_2) with Expected Return equal to $\frac{\mu_2+\mu_3}{2}$.

In [ ]:
```
import numpy as np
from scipy.optimize import minimize

# Define the covariance matrix
Sigma = np.array([
    [0.010, 0.002, 0.001],
    [0.002, 0.011, 0.003],
    [0.001, 0.003, 0.020]
])

# Define the expected returns vector
mu = np.array([0.0427, 0.0015, 0.0285])  # Convert percentages to decimals for calc

# Define target returns
target_return_P1 = mu[0]
target_return_P2 = (mu[1] + mu[2]) / 2

# Objective function: Portfolio Variance
def portfolio_variance(weights, covariance_matrix):
```

```
    return weights.T @ covariance_matrix @ weights

# Constraint: Expected return should match target return
def target_return_constraint(weights, target_return):
    return weights @ mu - target_return

# Constraint: Weights must sum to 1
def weights_sum_constraint(weights):
    return np.sum(weights) - 1

# Minimize the portfolio variance subject to the return constraint and sum of weigh
def find_min_variance_portfolio(target_return):
    cons = (
        {'type': 'eq', 'fun': weights_sum_constraint},
        {'type': 'eq', 'fun': lambda w: target_return_constraint(w, target_return)}
    )
    bounds = tuple((0, 1) for _ in range(len(mu)))
    initial_weights = np.array([1/len(mu)] * len(mu))
    result = minimize(portfolio_variance, initial_weights, args=(Sigma,), method='S
    return result.x, result.fun

# Calculate optimal portfolios
optimal_weights_P1, min_var_P1 = find_min_variance_portfolio(target_return_P1)
optimal_weights_P2, min_var_P2 = find_min_variance_portfolio(target_return_P2)

print("Optimal weights for Portfolio P1 (target return = {:.4f}):".format(target_re
print(optimal_weights_P1)
print("Minimum Variance of Portfolio P1:", min_var_P1)

print("\nOptimal weights for Portfolio P2 (target return = {:.4f}):".format(target_
print(optimal_weights_P2)
print("Minimum Variance of Portfolio P2:", min_var_P2)
```

```
Optimal weights for Portfolio P1 (target return = 0.0427):
[1.00000000e+00 0.00000000e+00 1.09198739e-12]
Minimum Variance of Portfolio P1: 0.009999999999980346

Optimal weights for Portfolio P2 (target return = 0.0150):
[0.22965758 0.62078287 0.14955955]
Minimum Variance of Portfolio P2: 0.006409901313492481
```