FE630 Portfolio Theory and Applications - Assignment #2
Deadline for submission: May 7th, 2023

Prof. Papa Momar Ndiaye

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1 Instructions

- **Please read these instructions carefully and follow them precisely.**
- **Independence:** All students must work independently.
- **Submission:** Submit your answer document via Canvas.
- **Answer Document:** Your answer document MUST be in the form of a single pdf file that contains all of your answers including code printouts and graphs. Do not submit your answer document in any format other than pdf. Any answer document that does not comprise a single pdf file complete with all answers will receive not be graded.
- **Cover Sheet:** When required, your answer document must include a cover sheet that states the course name, the homework number, the date, and your name.
- **Legibility and Logical Presentation:** Answer documents that are not easily legible, or not logically presented, or have a non-professional appearance will not be graded.
- **Source Code Requirement:** Your submission should also contain a separate set of source code files for all of your solutions. I may run your source code to ensure that it provides the results that you claim.
- **Permissible Computer Languages:** You can use any matrix-oriented computer programming language (Python, R or Matlab), but do not use any spreadsheets. Problems solved with spreadsheets will receive no credit. item **Permissible Computer Languages:** You can use any matrix-oriented computer programming language (Python, R or Matlab for example), but do not use any spreadsheets. Problems solved with spreadsheets will receive no credit.
- **Late Submission Policy:** If answer documents and source code files are not submitted by the due date and time, they will incur an immediate penalty of 20%. An additional 10% of penalty will be applied for each additional late day.

Topics for this assignment are the following:

- Algebra and Optimization;
- Geometry of Efficient Frontiers;
- Applications of one fund and two-fund theorems.

Please submit via Canvas a PDF file with your answers and also a zip file with the code used for computations and graphic display, if applicable. **Deadline for submission: April 17th, 2023**

2 Optimization with Equality Constraints (40 points)

Consider the optimization problem (Max Expected Return with Target Risk)

$$\begin{cases} \max\limits_{\omega_1,\omega_2} & R_P(\omega_1,\omega_2) = \rho_1\omega_1 + \rho_2\omega_2 \\ \text{subject to} & \\ & \sqrt{\sigma_1^2\omega_1^2 + \sigma_2^2\omega_2^2 + 2\rho_{12}\sigma_1\sigma_2\omega_1\omega_2} = \sigma_T \\ & \omega_1 + \omega_2 = 1 \end{cases} \tag{1}$$

where we are given two securities with Expected Returns ρ_1 and ρ_2, volatilities σ_1 and σ_2 and correlation ρ_{12}. In equation (3), σ_T is positive number called the target risk.

1. Solve the problem (3) using a Lagrangian approach. You will denote the solution the optimal solution by $\omega^*(\sigma_T)$ and the optimal value of the problem by $R_p(\omega_1^*(\sigma_T), \omega_2^*(\sigma_T))$ by $R_p(\sigma_T)$.
2. Assume that $\rho_1 = 5\%, \rho_2 = 10\%, \sigma_1 = 10\%, \sigma_2 = 20\%, \rho_{12} = -0.5$. Consider a sequence of successive values of σ_T in the range $[2\%, 30\%]$ by step of 0.5%, plot the efficient frontier, namely the graph of the mapping $\sigma_T \longmapsto R_p(\sigma_T)$. That graphs maps the sequence of values of σ_T from the x-axis into the sequence of values $R_p(\sigma_T)$ on the y-axis.

3 Optimization with Inequality Constraints (20 points)

Solve analytically at least one of the two following problems

$$\begin{cases} \min\limits_{x_1,x_2} & (x_1-2)^2 + 2(x_2-1)^2 \\ \text{subject to} & \\ & x_1 + 4x_2 \leq 3 \\ & x_1 \geq x_2 \end{cases} \tag{2}$$

$$\begin{cases} \max\limits_{x_1,x_2} & 5 - x_1^2 - x_1x_2 - 3x_2^2 \\ \text{subject to} & \\ & x_1, x_2 \geq 0 \\ & x_1x_2 \geq 2 \end{cases} \tag{3}$$

and use an optimizer to verify your answer.

4 Mean-Variance Optimization (40 pts)

Consider an Investment Universe made of 3 stocks S_1, S_2 and S_3 with the following characteristics:

- Covariance matrix: $\Sigma = \begin{pmatrix} 0.010 & 0.002 & 0.001 \\ 0.002 & 0.011 & 0.003 \\ 0.001 & 0.003 & 0.020 \end{pmatrix}$;
- Expected Return vector: $\rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} 4.27\% \\ 0.15\% \\ 2.85\% \end{pmatrix}$.

1. (5 points) Find the Global Minimal Variance Portfolio.

2. (5 points) Find the Minimum Variance Portfolio (P_1) with Expected Return equal to ρ_1. Find the Minimum Variance Portfolio (P_2) with Expected Return equal to $\dfrac{\rho_2 + \rho_3}{2}$.

3. (5 points) Using the Portfolios (P_1) and (P_2) previously found, apply the Two-fund Theorem to find the Minimum Variance Portfolio with Expected Return equal to 4%.

4. (5 points) Apply the Two-fund Theorem to generate and plot the Mean-Variance efficient frontier (the graph should also display the Expected Returns and Volatilities of securities S_1, S_2 and S_3).

5. Assume now that we add a Riskless Asset S_0 with return $\rho_0 = 1\%$.

   (a) (5 points) Find the Tangent Portfolio P_T. Add the the new Efficient Frontier to the graph generated in question 4. Does that efficient frontier intersect with the one obtained with risky asset only? Explain why.

   (b) (5 points) Using the One-fund Theorem, find the Efficient Portfolio (P_3) with target Expected Return equal to 7%.

   (c) (5 points) Using the One-fund Theorem, find the Efficient Portfolio (P_4) with with target volatility equal to 2%.

   (d) (5 points) Add the Portfolios (P_3) and (P_4) to the graph from question 5(b).