FE670 Homework Assignment #2

Due Date: October 26 (Thursday).

Problem 1: Given n securities with Expected return vector $\boldsymbol{\mu}$ and Covariance matrix $\boldsymbol{\Sigma}$, the return of a portfolio with weights $\mathbf{w}$ is a random variable $R_p = \mathbf{w}'\mathbf{R}$ with expected return and variance given by

$$\mu_p = \mathbf{w}'\boldsymbol{\mu}$$
$$\sigma_p^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$$

For now, we simply assume that expected returns, $\boldsymbol{\mu}$, and their covariance matrix, $\boldsymbol{\Sigma}$, are given. To calculate the weights for one possible pair, we choose a targeted mean return, μ_0, Following Markowitz, the investor's problem is constrained minimization problem:

$$\min_{w} \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$$
$$\text{s. t.}$$
$$\mu_0 = \mathbf{w}'\boldsymbol{\mu}, \mathbf{w}'\mathbf{l}' = 1, \mathbf{l}' = [1, 1, ..., 1]$$

We refer to this version of the classical mean-variance optimization problem as the *risk minimization formulation*, and this problem is a quadratic optimization problem with equality constraints with the solution given by

$$w = \boldsymbol{\lambda}\boldsymbol{\Sigma}^{-1}\mathbf{l} + \boldsymbol{\gamma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

where

$$\boldsymbol{\lambda} = \frac{C - \mu_0 B}{\Delta}, \boldsymbol{\gamma} = \frac{\mu_0 A - B}{\Delta}$$

$$A = \mathbf{l}'\boldsymbol{\Sigma}^{-1}\mathbf{l}, B = \mathbf{l}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, C = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

$$\Delta = \boldsymbol{AC} - \boldsymbol{B}^2$$

It is easy to see that

$$\sigma_0^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$$

$$= \frac{A\mu_0^2 - 2B\mu_0 + C}{\Delta}$$

Now we take n= 2 and denote the optimal solution by $w(\sigma_0^2)$, and assume that the expected $\boldsymbol{\mu} = \begin{pmatrix} 0.05 \\ 0.10 \end{pmatrix}$ and $\boldsymbol{\Sigma} \begin{pmatrix} 0.01 & -0.01 \\ -0.01 & 0.04 \end{pmatrix}$, please answer the following questions:

(1) Please plot the efficient frontier of the portfolio, and calculate variance when $\boldsymbol{\mu_0} = 0.10$ and $\boldsymbol{\mu_0} = 0.20$ manually.

(2) If we introduce a risk free rate of 0.02, please find the **market portfolio** of these two stock portfolio, and calculate the slope of the **Capital Market Line (CML)** along with the portfolio frontier.

Problem 2. Suppose there are $N = 3$ assets s_1, s_2 and s_3 respectively. The covariance matrix and expected rates of return are

$$\boldsymbol{\Sigma} = \begin{bmatrix} 3.0 & 1.5 & 0.0 \\ 1.5 & 3.0 & 1.5 \\ 0.0 & 1.5 & 3.0 \end{bmatrix} \text{ and } \boldsymbol{\mu} = \begin{bmatrix} 0.5 \\ 0.3 \\ 0.7 \end{bmatrix}$$

One of the basic assumptions underlying the Black-Litterman model is that the expected return of a security should be consistent with market equilibrium unless the investor has a specific view on the security, and the market equilibrium can be expressed as:

$$\mathbf{\Pi} = \boldsymbol{\mu} + \boldsymbol{\epsilon}_{\mathbf{\Pi}}, \boldsymbol{\epsilon}_{\mathbf{\Pi}} \sim N(0, \tau\mathbf{\Sigma})$$

for some small parameter $\tau << 1$. We can think about $\tau\mathbf{\Sigma}$ as our confidence in how well we can estimate the equilibrium expected returns.

Formally, K views in Black-Litterman model are expressed as a vector $\mathbf{q}$ with

$$\mathbf{q} = \mathbf{P}\boldsymbol{\mu} + \boldsymbol{\epsilon}_{\boldsymbol{q}}, \boldsymbol{\epsilon}_{\boldsymbol{q}} \sim N(0, \mathbf{\Omega})$$

where $\mathbf{P}$ is a $K \times N$ matrix (explained in the following example) and $\mathbf{\Omega}$ is a $K \times K$ matrix expressing the confidence in the views.

Let us assume that the asset universe that we consider the three stocks and that an investor has the following two views ($K = 2$): (1) s_1 will have a return of 2.0%. (2)s_3 will outperform s_2 by 4.0%. Mathematically, we express the two views together as

$$\begin{bmatrix} 2.0\% \\ 4.0\% \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

We also assume a higher confidence in the views, and conversely we have

$$\mathbf{\Omega} = \begin{bmatrix} 5\%^2 & 0 \\ 0 & 6\%^2 \end{bmatrix}$$

According to Blacklitterman, we will have

$$\hat{\boldsymbol{\mu}}_{BL} = [(\tau\Sigma)^{-1} + \mathbf{P}'\Omega^{-1}\mathbf{P}]^{-1}[(\tau\Sigma)^{-1}\mathbf{\Pi} + \mathbf{P}'\Omega^{-1}\mathbf{q}]$$

where $\mathbf{\Pi}$ is the equilibrium risk premium over the risk free rate (3×1 vector) which can be calculated as $\boldsymbol{\mu}$ over the risk free rate of $r_f = 0.02$. Please answer the following questions using the data provided:

(1) Find the minimum variance portfolio - MVP.

(2) If the risk free rate is r_f, find an efficient portfolio risky assets, and plot the efficient portfolio frontier.

(3) Assume we have $\tau = 0.03$, please calculate the expected return conditional on the views $\hat{\boldsymbol{\mu}}_{BL}$.

(4) Based on $\hat{\boldsymbol{\mu}}_{BL}$ and recalculate the minimum variance portfolio. Please provide comments on the differences between the original MVP and the adjusted MVP.

Problem 3: You are given a dataset for the S&P500 stock index which consists of the high, low, opening, and closing price of each of the 500 stocks as well as the volume of each stock for each day (in all_stocks_5yr.csv). Stocks vary in the length for which they have historical data, as some companies have been public longer than other. This dataset includes all data from 2013 to 2018. Rather than considering the actual price of the stock (since some stock prices are much higher or lower than others), the change in stock price from one closing bell to the next is considered for all 500 stocks. You can get S&P 500 index returns for the same period from Yahoo finance using Python Pandas DataReader package. **In this problem, we would select 20 stocks from the S&P 500: 10 with the highest volatility and 10 with the lowest volatility.** Please use answer the following questions using the dataset:

(1) Use data from 2013 to 2016 data to build the global minimum variance portfolio (allowing for short sales). And compare the portfolio daily return for the period from 2017 to 2018 with the S&P 500 index daily return using Sharpe ratio.

(2) Now let's put constraints on the weight of all 20 stocks. Assume that the portfolio weights add up to 1, and we want to make sure each stock has minimum weight of 0.0001. Compare this new portfolio daily return with the unconstrained portfolio in the last question for the period from 2017 and 2018.

Homework Honor Policy: You are allowed to discuss the problems between yourselves, but once you begin writing up your solution, you must do so independently, and cannot show one another any parts of your written solutions.