FE670 Homework Assignment #3

Due Date: In class on Nov 23 (Thursday).

Problem 1: Shrinkage estimator is a form of averaging different estimators. The shrinkage estimator typically consists of three components: (1) an estimator with little or no structure (like the sample mean above); (2) an estimator with a lot of structure (the shrinkage target); and (3) the shrinkage intensity.

$$\hat{\mu}_{JS} = (1 - w)\hat{\mu} + w\mu_0 \mathbf{l} \tag{1}$$

The most well-known shrinkage estimator used to estimate expected returns in the financial literature is the one proposed by Jorion and Stein (1), where the shrinkage target $\mu_0 \mathbf{l}$ is given by $\mu_g \mathbf{l}$ with

$$\mu_g = \frac{\mathbf{l}'\Sigma^{-1}\hat{\boldsymbol{\mu}}}{\mathbf{l}'\boldsymbol{\Sigma}^{-1}\mathbf{l}} \tag{2}$$

and

$$w = \frac{N+2}{N + 2 + T(\hat{\boldsymbol{\mu}} - \mu_g \mathbf{l})'\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\mu}} - \mu_g \mathbf{l})}$$

The shrinkage estimator for the covariance matrix takes the form

$$\hat{\Sigma}_{LW} = w\hat{\Sigma}_{CC} + (1 - w)\hat{\Sigma}$$

where $\hat{\Sigma}$ is the sample covariance matrix, and $\hat{\Sigma}_{CC}$ is the sample covariance matrix with constant correlation.

First, we decompose the sample covariance matrix according to $\hat{\Sigma} = \Lambda C \Lambda'$, where Λ is a diagonal matrix of the standard deviation of returns and C is the sample correlation matrix.

The correlation matrix C can be written as

$$C = \begin{bmatrix} 1 & \hat{\rho}_{12} & \dots & \hat{\rho}_{1N} \\ \hat{\rho}_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{\rho}_{N-1N} \\ \hat{\rho}_{N1} & \dots & \hat{\rho}_{NN-1} & 1 \end{bmatrix}$$

We then replace the sample correlation matrix with the constant correlation matrix

$$C_{CC} = \begin{bmatrix} 1 & \hat{\rho} & \dots & \hat{\rho} \\ \hat{\rho} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{\rho} \\ \hat{\rho} & \dots & \hat{\rho} & 1 \end{bmatrix}$$

where $\hat{\rho}$ is the average of all the sample correlations, in other words

$$\hat{\rho} = \frac{2}{(N-1)N} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \hat{\rho}_{ij}$$

Finally, we have the sample covariance matrix with the constant correlation matrix as $\hat{\Sigma}_{CC} = \Lambda C_{CC} \Lambda'$, where Λ is a diagonal matrix of the standard deviation of returns and C is the sample correlation matrix.

We use the classical mean-variance optimization as the *risk minimization* formulation, and this problem is a quadratic optimization problem with equality constraints with the solution given by

$$w = \boldsymbol{\lambda} \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1} + \boldsymbol{\gamma} \hat{\boldsymbol{\Sigma}}^{-1} \boldsymbol{\mu}$$

$$\text{where}$$

$$\boldsymbol{\lambda} = \frac{C - \mu_0 B}{\Delta}, \boldsymbol{\gamma} = \frac{\mu_0 A - B}{\Delta}$$

$$A = \mathbf{1}' \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}, B = \mathbf{1}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\mu}}, C = \hat{\boldsymbol{\mu}}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\mu}}$$

$$\Delta = \boldsymbol{AC} - \boldsymbol{B}^2$$

It is easy to see that

$$\sigma_0^2 = \mathbf{w}' \hat{\boldsymbol{\Sigma}} \mathbf{w}$$

$$= \frac{\boldsymbol{A}\mu_0^2 - 2\boldsymbol{B}\mu_0 + \boldsymbol{C}}{\boldsymbol{\Delta}}$$

Now we take n= 2, T = 10,000, and denote the optimal solution by $w(\sigma_0^2)$, and assume that the sample estimator $\hat{\boldsymbol{\mu}} = \begin{pmatrix} 0.05 \\ 0.10 \end{pmatrix}$ and $\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 0.01 & -0.01 \\ -0.01 & 0.04 \end{pmatrix}$, please answer the following questions:

(1) Please use sample estimator $\hat{\mu}$ and $\hat{\Sigma}$ to plot the efficient frontier of the portfolio, and calculate variance when $\boldsymbol{\mu_0} = 0.10$ manually (please do not use simulation method).

(2) Please use shrinkage target and intensity in equations (2) and (3) to calculate the shrinkage estimator $\hat{\mu}_{JS}$ and $\hat{\Sigma}_{LW}$ to build a new efficient frontier and overlay it with the efficient frontier from the previous question, and calculate variance when $\boldsymbol{\mu_0} = 0.10$ manually.

(3) Please make comments on the differences of the two different estimation approaches in portfolio decisions.

Problem 2: The classical mean-variance problem can be formulated as

$$\max_{w} \, [w' \boldsymbol{\mu}]$$
$$\text{s. t.}$$
$$\mathbf{w}' \boldsymbol{\Sigma} \mathbf{w} \leq \sigma_0^2, \mathbf{w}'\mathbf{l}' = 1, \mathbf{l}' = [1, 1, ..., 1]$$

where $\mathbf{l} = [1, 1, .., 1]'$. In this optimization problem $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, and $\mathbf{w}$ denote the expected return, asset return covariance matrix, and portfolio weights, respectively.

The robust formulation of the mean-variance problem under the preceding assumption on $\hat{\boldsymbol{\mu}}$ is

$$\max_{w} \boldsymbol{\mu}'\mathbf{w} - \boldsymbol{\delta}'|\mathbf{w}|$$
$$\text{s. t.}$$
$$\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \leq \sigma_0^2, \mathbf{w}'\mathbf{l}' = 1, \mathbf{l}' = [1, 1, ..., 1]$$

If the weight of asset i in the portfolio is negative, the worst-case expected return for asset i is $\mu_i + \delta_i$ (we lose the largest amount possible). If the weight of asset i in the portfolio is positive, then the worst-case expected return for asset i is $\mu_i - \delta_i$ (we gain smallest amount possible).

Assets whose mean return estimates are less accurate (have a larger estimation error δ_i) are penalized in the objective function, and will tend to have smaller weights in the optimal portfolio allocation.

An easy way to incorporate uncertainty caused by estimation errors is to require that the investor be protected if the estimated expected return $\hat{\boldsymbol{\mu}}_i$ for each asset is around the true expected return μ_i. The error from the estimation can be assumed to be not larger than some small number $\delta_i > 0$. A simple choice for the *uncertainty set* for $\boldsymbol{\mu}$ is

$$U_\delta(\hat{\boldsymbol{\mu}}_i) = \{\boldsymbol{\mu} || \mu_i - \hat{\mu}_i| \leq \delta_i, i = 1, ..., N\}$$

You are given a dataset that contains the daily closing price of 20 stocks for the period of 2017-2021 (in SP20-2017-2021.csv). Please use RSOME python package to complete this assignment. An example of how to formulate a robust optimization problem is provided in Python project file (portfolio_robust_optimization.ipynb). Please answer the following questions using the dataset:

(1) Use data from 2017-01-01 to 2020-12-31 to build a global maximum portfolio with variance of portfolio less than $\sigma^2 < 0.00005$ (not allowing for short sales). Use the allocation and test the performance of the portfolio for the remaining data in 2021. Please plot the unrealized cumulative return of the portfolio for the period in 2021 and calculate the portfolio Sharpe ratio.

(2) Now let's use robust portfolio optimization approach on these 20 stocks and build a global maximum portfolio with variance of portfolio less than $\sigma^2 < 0.00005$ and $\boldsymbol{\delta} = 1.5 \cdot \mathbf{l}'$ where $\mathbf{l}' = [1, 1, ..., 1]$. Please use RSOME package *maxmin* function along with an uncertainty set on the expected return. Assume that the portfolio weights add up to 1, and we do not allow for short sales. Plot this new portfolio unrealized cumulative return with the portfolio constructed in the last question for the same period in 2021 in the same plot. Please calculate the portfolio Sharpe ratio and compare it with the result from the last question, and make comments

about the differences you observe.

Homework Honor Policy: You are allowed to discuss the problems between yourselves, but once you begin writing up your solution, you must do so independently, and cannot show one another any parts of your written solutions.