insurance JCR

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Let agent i have distribution of insured losses F_i and a CARA(σ_i).

The contract $y_i = (R_i, P_i)$ where reimbursement R_i is a function of L and P_i is the premium. There is a no-insurance option $y_0 = \emptyset = (0, 0)$.

The certainty equivalent of a contract $y = (R, P)$ for i is $V_i(R) - P$, with

$$V_i(R) = -\frac{1}{\sigma_i} \log \int \exp(\sigma_i(L - R(L)))dF_i(L)$$

The participation constraint is $V_i(R_i) - P_i \geq V_i(0)$, so define

$$U_i = V_i(R_i) - P_i - V_i(0)$$

so that $P_i = V_i(R_i) - V_i(0) - U_i$.

We have by (IC):

$$\begin{aligned} U_i &\geq V_i(R_j) - P_j - V_i(0) \\ &= (V_i(R_j) - V_i(0)) - (V_j(R_j) - V_j(0) - U_j), \end{aligned}$$

that is $U_i - U_j \geq (V_i(R_j) - V_i(0)) - (V_j(R_j) - V_j(0))$. The RHS should be $b_i(R_j) - b_j(R_j)$ so we define $b_i(y) = V_i(R) - V_i(0)$ for $y = (R, P)$.

The monopoly insurer's objective function on agent i is, given a loading factor l:

$$P_i - (1 + l) \int R_i(L)dF_i(L) = S_i(y) - U_i$$

where the first-best surplus is

$$S_i(y) = V_i(R) - V_i(0) - (1 + l) \int R_i(L)dF_i(L) = b_i(y) - (1 + l) \int R_i(L)dF_i(L),$$

so we take $\lambda = 1$.

the algorithm

We need

$$\Lambda_{ij}(y) = b_i(y_j) - b_j(y_j)$$

and

$$\left((\Lambda'(y))^* u\right)_j = \sum_i (b_i'(y_j) - b_j'(y_j))u_{ij}.$$

$$y_i^{(k+1)} = \mathrm{prox}_{-\tau f_i S_i}\left(y_i^{(k)} - \tau\left(\left(\Lambda'(y^{(k)})\right)^* v^{(k)}\right)_i\right)$$

then

$$\tilde{y}^{(k+1)} = 2y^{(k+1)} - y^{(k)}$$

and

$$v^{(k+1)} = \mathrm{proj}_K\left(v^{(k)} + c\tilde{y}^{(k+1)}\right).$$

We need $c\tau M^2 < 1$, with M an upper bound on the Lipschitz constant of Λ, that is the sup of $|b_i'(y) - b_j'(y)|$ over i, j, and y.

the proximal

$\mathrm{prox}_{-\tau f_i S_i}(z_i)$ is obtained by minimizing

$$-S_i(y_i) + \frac{1}{2\tau f_i}\|y_i - z_i\|^2.$$

Here $z_i = y_i^{(k)} - \tau \sum_j (b_j'(y_i^{(k)}) - b_i'(y_i^{(k)}))v_{ji}^{(k)}$.

Now denote the first best y_i^*. By definition, at least close to y_i^* the derivative $\frac{\partial S_i}{\partial y_i}$ has positive (negative) components on every axis to the left (right) of the corresponding component of y_i^*. Since $S_i'(y_i) = (y_i - z_i)/(\tau f_i)$, each component of $(y_i - z_i)$ must be positive (negative) to the left (right) of y_i^*.

It follows that (assuming concavity of S_i) each component of y_i is between the corresponding components of y_i^* and z_i.

the projection

To get $v = \mathrm{proj}_K(w)$, we define the function

$$v_{ij}(\beta) = \max(0, w_{ij} - \beta_i + \beta_j)$$

for $\beta \in \mathbb{R}^N$.

The solution v is $v(\beta)$ for the β that minimizes

$$\frac{1}{2}\sum_{i,j} v_{ij}(\beta)^2 + f\cdot\beta.$$

The objective is C^1, with gradient wrt β_i

$$\sum_{k,l} \max(0, w_{kl} - \beta_k + \beta_l) 1(w_{kl} > \beta_k - \beta_l) \left(1(l = i) - 1(k = i)\right) + f_i,$$

that is

$$\sum_k \left(v_{ki}(\beta) 1(w_{ki} > \beta_k - \beta_i) - v_{ik}(\beta) 1(w_{ik} > \beta_i - \beta_k)\right) + f_i.$$

We can get the minimizing β by Nesterov AGD:

- we start from some $\beta^{(0)} = \bar{\beta}^{(0)} \in \mathbb{R}^N$ and $M \geq$ the largest eigenvalue of DD^*, with

$$(Du)_{ij} = u_i - u_j;\ (D^* v)_i = \sum_j (v_{ij} - v_{ji})$$

$$(DD^* v)_{ij} = \sum_k (v_{ik} - v_{ki} - v_{jk} + v_{kj})$$

so that

$$(DD^*)_{ij,kl} = 1(k = i) - 1(l = i) - 1(k = j) + 1(l = j).$$

The largest eigenvalue of DD^* is $2N - 2$ if $n_0 = n_1$.

- then we iterate

$$\beta^{(k+1)} = \max\left(0, \beta^{(k)} - \frac{1}{M}(f - D^* v(\bar{\beta}^{(k)}))\right)$$

then

$$\bar{\beta}^{(k+1)} = \beta^{(k+1)} + \frac{t^{(k)} - 1}{t^{(k+1)}}(\beta^{(k+1)} - \beta^{(k)})$$

with $t^{(k+1)} = \left(1 + \sqrt{1 + 4(t^{(k)})^2}\right)/2$ and $t^{(0)} = 0$.

the algorithm in `proj_K` works better.

0-L (degenerate) example

If F_i has mass $(1 - p_i)$ in 0 and mass p_i in L, then with a straight deductible contract $y = D$ (with $D \leq L$) we get for agent i with ARA(σ_i):

$$
\begin{aligned}
V_i(D) &= -\frac{1}{\sigma_i}\log(1-p_i+p_i\exp(\sigma_i D))\\
b_i(y) &= \frac{1}{\sigma_i}\log\frac{1-p_i+p_i\exp(\sigma_i L)}{1-p_i+p_i\exp(\sigma_i D)}\\
S_i(y) &= b_i(y)-p_i(L-D).
\end{aligned}
$$

Only one instrument here so not that interesting (is it even covered in the paper?)

general CARA

Suppose agent i has type (σ_i,δ_i) with $F_i(L)=F(L,\delta_i)$; and R_i has parameters y_i: $R_i(L)=R(L,y_i)$, with $R(L,\emptyset)\equiv 0$ for the no-insurance contract $\emptyset$.

Then

$$
b_{\sigma,\delta}(y) = -\frac{1}{\sigma}\log\int\exp(\sigma(L-R(L,y)))dF(L,\delta)+\frac{1}{\sigma}\log\int\exp(\sigma L)dF(L,\delta)
$$

and

$$
S_{\sigma,\delta}(y)=b_{\sigma,\delta}(y)-\int R(L,y)dF(L,\delta).
$$

The proximal projector requires solving

$$
y-\tau f_{\sigma,\delta}S'_{\sigma,\delta}(y)=z,
$$

that is

$$
y-\tau f_{\sigma,\delta}\int\frac{\partial R}{\partial y}(L,y)\left(\frac{\exp(\sigma(L-R(L,y))}{\int\exp(\sigma(t-R(t,y))dF(t,\delta)}-(1+l)\right)dF(L,\delta)=z
$$

since

$$
b'_{\sigma,\delta}(y)=\int\frac{\partial R}{\partial y}(L,y)\frac{\exp(\sigma(L-R(L,y))}{\int\exp(\sigma(t-R(t,y))dF(t,\delta)}dF(L,\delta).
$$

We need to study the integral

$$
\begin{aligned}
I_{\sigma,\delta}(y) \equiv & \int \exp(\sigma(L - R(L,y))dF(L,\delta) \\
&= F(0,\delta)\exp(-\sigma R(0,y)) \\
&+ \int_{0+}^{y_0} \exp(\sigma L)dF(L,\delta) \\
&+ \int_{y_0}^{\infty} \exp\left(\sigma(y_1 L + y_0(1-y_1))\right) dF(L,\delta),
\end{aligned}
;
$$

and $b_{\sigma,\delta}(y) = \left(\log I_{\sigma,\delta}(\emptyset) - \log I_{\sigma,\delta}(y)\right)/\sigma$, where $\emptyset$ denotes no insurance.

fixed deductible + proportional copay

Suppose $y = (y_0, y_1)$ and we look at contracts with a fixed deductible y_0 and proportional copay y_1 above the deductible: $R(L,y) = (1-y_1)\max(L-y_0, 0)$.

No insurance is $y_1 = 1$; full insurance is $y_0 = y_1 = 0$.

Then $L - R = L$ if $L < y_0$ and $L - R = y_1 L + y_0(1-y_1)$ if $L > y_0$. Moreover

$$
\frac{\partial R}{\partial y_0} = (y_1 - 1)\mathbf{1}(L > y_0)
$$

and

$$
\frac{\partial R}{\partial y_1} = \min(y_0 - L, 0).
$$

censored normal losses

Let's go crazy: $L = \max(0, N(\delta, s^2))$ for type (σ,δ), so that $L = 0$ with probability $\Phi(-\delta/s)$ and at $L > 0$, the pdf is $\phi((L-\delta)/s)/s$.

Lemma 1:

$$
\int_{\alpha}^{\beta} \exp(\gamma u + \nu)\phi(u)du = \left(\Phi\left(\beta - \gamma\right) - \Phi\left(\alpha - \gamma\right)\right) \times \exp(\gamma^2/2 + \nu).
$$

Lemma 2:

$$
\begin{aligned}
\int_a^b \exp(cL + d)\frac{1}{s}\phi\left(\frac{L-\delta}{s}\right) dL &= \int_{(a-\delta)/s}^{(b-\delta)/s} \exp(csu + c\delta + d)\phi(u)du \\
&= \left(\Phi\left(\frac{b-\delta}{s} - cs\right) - \Phi\left(\frac{a-\delta}{s} - cs\right)\right) \\
&\times \exp(c^2 s^2/2 + c\delta + d).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
I_{\sigma,\delta}(y) &= F(0,\delta)\times 1 + \int_{0^+}^{y_0}\exp(\sigma L)dF(L,\delta) + \int_{y_0}^{\infty}\exp\left(\sigma(y_1 L + y_0(1-y_1))\right)dF(L,\delta)\\
&\equiv A_{\sigma,\delta} + B_{\sigma,\delta}(y) + C_{\sigma,\delta}(y)\\
&= \Phi(-\delta/s)\\
&+ \left(\Phi((y_0-\delta)/s-\sigma s) - \Phi(-\delta/s-\sigma s)\right)\times\exp(\sigma^2 s^2/2+\sigma\delta)\\
&+ \Phi(\sigma y_1 s - (y_0-\delta)/s)\times\exp(\sigma^2 y_1^2 s^2/2 + \sigma y_1\delta + \sigma y_0(1-y_1)).
\end{aligned}
$$

The no-insurance case is

$$
I_{\sigma,\delta}(0,1) = A_{\sigma,\delta} + C_{\sigma,\delta}(0,1) = \Phi(-\delta/s) + (1-\Phi(-\delta/s-\sigma s))\exp(\sigma^2 s^2/2+\sigma\delta).
$$

This gives us $b_{\sigma,\delta}(y) = \left(\log I_{\sigma,\delta}(0,1) - \log I_{\sigma,\delta}(y)\right)/\sigma$ and its derivatives:

$$
\frac{\partial b_{\sigma,\delta}}{\partial y_0} = -\frac{1}{\sigma I_{\sigma,\delta}}\frac{\partial(B_{\sigma,\delta}+C_{\sigma,\delta})}{\partial y_0}
$$

and

$$
\frac{\partial b_{\sigma,\delta}}{\partial y_1} = -\frac{1}{\sigma I_{\sigma,\delta}}\frac{\partial C_{\sigma,\delta}}{\partial y_1}.
$$

Define $d = (\delta - y_0)/s$ and $d_1 = \sigma s y_1 + d$. We have

$$
B = (\Phi(\delta/s+\sigma s) - \Phi(d+\sigma s))\exp(\sigma^2 s^2/2+\sigma\delta)
$$

and

$$
C = \Phi(d_1)\exp(\sigma^2 y_1^2 s^2/2 + \sigma(y_0 + s d y_1)).
$$

We calculate

$$
\begin{aligned}
\frac{\partial B_{\sigma,\delta}}{\partial y_0} &= \phi(d+\sigma s)\ \exp(\sigma^2 s^2/2+\sigma\delta)/s;\\
\frac{\partial C_{\sigma,\delta}}{\partial y_0} &= (\sigma(1-y_1)\Phi(d_1) - \phi(d_1)/s)\\
&\quad\times\exp(\sigma^2 y_1^2 s^2/2+\sigma(y_0+s d y_1));
\end{aligned}
$$

and, denoting $H(x)\equiv\phi(x)+x\Phi(x)$,

$$
\frac{\partial C_{\sigma,\delta}}{\partial y_1} = \sigma s H(d_1)\times\exp(\sigma^2 y_1^2 s^2/2+\sigma(y_0+s d y_1))
$$

Finally, we have

$$
\begin{aligned}
D_{\sigma,\delta}(y) &\equiv \int R(L,y)dF(L,\delta) \\
&= (1-y_1)\int_{y_0}^{\infty}(L-y_0)\phi((L-\delta)/s)dL/s \\
&= (1-y_1)\int_{(y_0-\delta)/s}^{\infty}(su+\delta-y_0)\phi(u)du \\
&= (1-y_1)\left[-s\phi(u)+(\delta-y_0)\Phi(u)\right]_{(y_0-\delta)/s}^{\infty} \\
&= s(1-y_1)H(d)
\end{aligned}
$$

and $S_{\sigma,\delta}(y) = b_{\sigma,\delta}(y) - (1+l)D_{\sigma,\delta}(y)$.

Note that

$$
\frac{\partial D_{\sigma,\delta}}{\partial y_0} = -(1-y_1)\Phi(d)
$$

and

$$
\frac{\partial D_{\sigma,\delta}}{\partial y_1} = -sH(d).
$$

the first best

The first best maximizes $S = b - (1+l)D$. We know that it is a straight deductible contract, with $y_1 = 0$ and an $y_0 > 0$ if $l > 0$.

At $y_0 = y_1 = 0$, there is full insurance so $I = 1$ and the value of $\partial S/\partial y_0$ is

$$
\frac{1}{\sigma s}\left(\phi(\delta/s) - \phi(-\delta/s-\sigma s)\ \exp(\sigma^2 s^2/2+\sigma\delta)\right) + l\Phi(\delta/s).
$$

Since $\phi(-\delta/s-\sigma s)\ \exp(\sigma^2 s^2/2+\sigma\delta) = \phi(\delta/s)$ this gives $l\Phi(\delta/s)$ which is always positive, hence $y_0 > 0$.

The first-best y_0 is given by

$$
\sigma(1+l)\Phi(\delta/s)I(y_0,0) = \frac{\partial(B+C)}{\partial y_0}(y_0,0).
$$

calibration

We take a loading factor $l = 0.25$. Consider the ratio $r = \delta/s$. The probability of an accident is $\Phi(r)$; it should be between 0.03 and 0.1 (per year), which means $-1.9 < r < -1.2$. Then the expected positive loss $E(L|L>0)$ is $s \times (\phi(r)/\Phi(r)+r)$ which is between $0.4s$ and $0.5s$. We want it to be about 2 (in k-euros) so we take $s=4$; this gives us δ between -8 and -5.

For risk-aversion σ, we note that with no copay, the first-best deductible when we have a 0-1 loss with probability p is given by

$$\sigma D^* = \log \frac{(1-p)(1+l)}{1-p(1+l)}.$$

for small p, this gives $\sigma D^* \simeq \log(1+l)$, which in our case is 0.22. Say we want D^* between 0.5 and 2 k-euros; then we need σ between 0.1 and 0.5.

intro

It is nonlinear; it is not clear what the second derivatives $\frac{\partial b_i}{\partial i \partial y}$ look like, so even less so what the implications of

$$\frac{\partial b_i}{\partial i_k \partial y_0}\frac{\partial y_0}{\partial i_l} + \frac{\partial b_i}{\partial i_k \partial y_1}\frac{\partial y_1}{\partial i_l} \gg 0$$

for the properties of $i \to (y_0, y_1)$ might be.

If we compare two nearby types with the same value of y_1, then we have

$$\frac{\partial b_i}{\partial i_k \partial y_0}\frac{\partial y_0}{\partial i_l} \gg 0,$$

a bit better.

On the other hand, there is a clear "top": when risk aversion and risk are at their highest the WTP for insurance is maximal and there we have SB=FB, a straight deductible contract.

to do

plots and stats

Compute the expected claims $(1-y_1)E_{\sigma,\delta}\max(L-y_0,0)$ under both FB and SB; the surplus loss $S_{SB}-S_{FB}$.

Get the informational rents U_i. Start from $U_i^{(0)}=0$ and iterate

$$U_i^{(k+1)} = \max_j (U_j^{(k)} + \Lambda_{ij}(y))$$

where $\Lambda_{ij}(y) = b_i(y_j) - b_j(y_j)$. Plot the U_i, S_i, and $S_i - U_i$.

IR is binding for i iff $U_i=0$; IC $i \to j$ is binding iff $U_i - U_j = \Lambda_{ij}(y)$.

try straight deductible contracts

$R(L, y) = 0$ for $L \leq y_0$ and $R(L, y) = L - y_0$ above; like the 2-parameter contract with $y_1 = 0$.

A and B are unchanged; C becomes

$$\Phi((\delta - y_0)/s) \exp(\sigma y_0)$$

and D becomes

$$s\phi((y_0 - \delta)/s) + (\delta - y_0)\Phi((\delta - y_0)/s).$$

try more contract parameters

Zero bracket: $R(L, y) = 0$ for $L \leq y_0$.

For contracts of dimension $2K$: choose $(y_0 <)\ y_2 < \ldots < y_{2K-2}$, $0 < y_1, \ldots, y_{2K-1} < 1$.

Bracket $k = 1, \ldots, K$ has $y_{2k-2} \leq L \leq y_{2k}$; there $R(L, y) = A_k(y) + (1 - y_{2k-1})(L - y_{2k-2})$ with

$$A_k(y) = \sum_{l=1}^{k-1} (1 - y_{2l-1})(y_{2l} - y_{2l-2}).$$

This gives $2K$ parameters. The 2-dimensional contract has $K = 1$.

maybe optimize further

imposed penalties

We need interior optima, so allow y_0, y_1 to take any values but penalize $y_0 < 0, y_1 < 0$, y_0 large, $y_1 > 1$. Still, sometimes we get stuck with $y_0 = y_1 = 0$ for some type. To avoid that, pemalty on $y_0 + y_1 < 0.1$.

fix at the top

Easy: just fix the SB at the first B and run the proximal part only on the lower types.

interpolate

If we solved for an (n, n) grid, then we can interpolate linearly any point in the square, on a $(n+k,n+k) grid for instance, and start from there.

new calibration

January 4, 2024: keep $s = 4$, make $\delta = -8$ to -4 so probabilities of accident are between 0.02 and 0.16.
The resutst are better, in `n0xn1new`.

optimized `JLambda` to `JLambda_j`

It was the most costly piece by far. Now it runs 10 times faster.

todo

- how should the algorithm change with bounds on contract variables?
- make it general (number of types/contract vars, utilities)
- test it on the linear examples of the paper.
- try reasonable distributions (not uniform in square)
- play with slack coefficients on the IC constraints
  That is:

$$U_i - U_j \geq b_i(y_j) - b_j(y_j) - K$$

with K starting large and then decreasing.

- play with range of (σ, δ)
  Start small and increase.