Stability Analysis and Controller Synthesis for a Class of Piecewise Smooth Systems

The Oral Examination
for the Degree of Doctor of Philosophy

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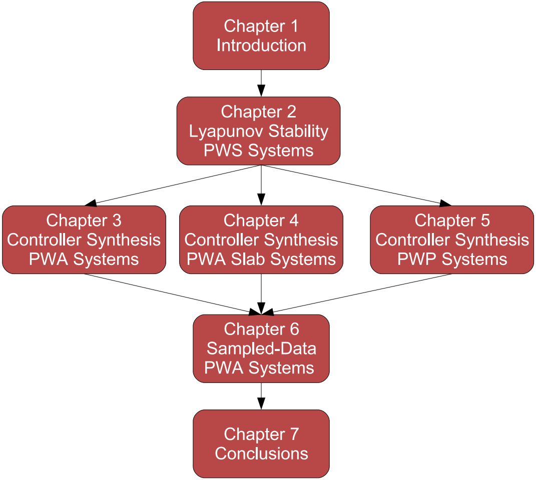
Outline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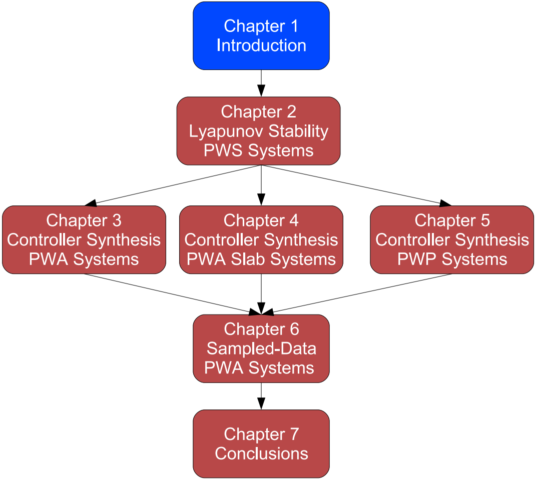
Introduction



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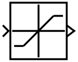
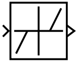
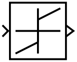
Memoryless Nonlinearities



Saturation



Dead Zone



Coulomb &
Viscous Friction

Theoretical Motivation

- *Richard Murray, California Institute of Technology: "*

Rank	Top Ten Research Problems in Nonlinear Control
10	Building representative experiments for evaluating controllers
9	Convincing industry to invest in new nonlinear methodologies
8	Recognizing the difference between regulation and tracking
7	**Exploiting special structure to analyze and design controllers**
6	**Integrating good linear techniques into nonlinear methodologies**
5	Recognizing the difference between performance and operability
4	Finding nonlinear normal systems for control
3	**Global robust stabilization and local robust performance**
2	Magnitude and rate saturation
1	**Writing numerical software for implementing nonlinear theory**

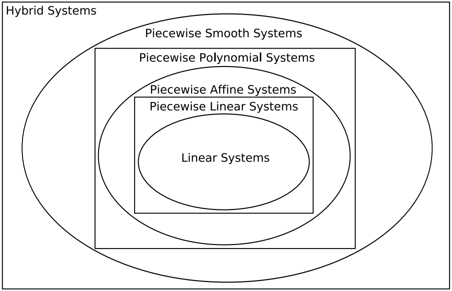
- ...This is more or less a way for me to think online, so I wouldn't take any of this too seriously."

Special Structure

- Piecewise smooth system:

$$\dot{x} = f(x) + g(x)u$$

where $f(x)$ and $g(x)$ are piecewise continuous and bounded functions of x.



Objective

To develop a **computational tool** to design controllers for **piecewise smooth systems** using **convex optimization** techniques.

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- Why convex optimization?
  - There are numerically efficient tools to solve convex optimization problems.
  - Linear Matrix Inequalities (LMI)
  - Sum of Squares (SOS) programming

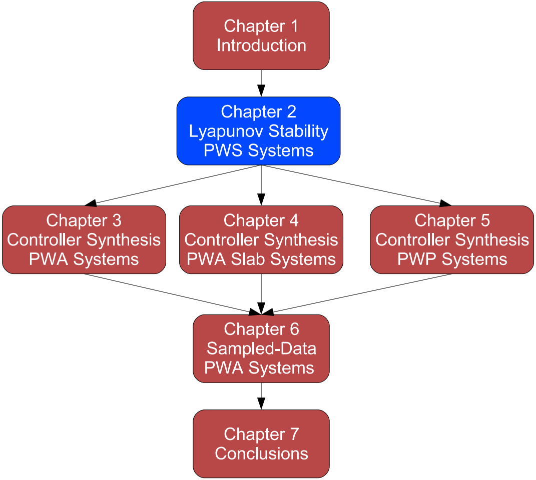
Literature

- Hassibi and Boyd (1998) - Quadratic stabilization and control of piecewise linear systems - Limited to piecewise linear controllers for PWA slab systems
- Johansson and Rantzer (2000) - Piecewise linear quadratic optimal control - No guarantee for stability
- Feng (2002) - Controller design and analysis of uncertain piecewise linear systems - All local subsystems should be stable
- Rodrigues and How (2003) - Observer-based control of piecewise affine systems - Bilinear matrix inequality
- Rodrigues and Boyd (2005) - Piecewise affine state feedback for piecewise affine slab systems using convex optimization - Stability analysis and synthesis using parametrized linear matrix inequalities

Major Contributions

1. To propose a two-step controller synthesis method for a class of uncertain nonlinear systems described by PWA differential inclusions.
2. To introduce for the first time a duality-based interpretation of PWA systems. This enables controller synthesis for PWA slab systems to be formulated as a convex optimization problem.
3. To propose a nonsmooth backstepping controller synthesis for PWP systems.
4. To propose a time-delay approach to stability analysis of sampled-data PWA systems.

Lyapunov Stability for Piecewise Smooth Systems



Lyapunov Stability for Piecewise Smooth Systems

Theorem (2.1)

For nonlinear system $\dot{x}(t) = f(x(t))$, if there exists a continuous function $V(x)$ such that

$$\begin{aligned}
&V(x^\star) = 0\\
&V(x) > 0 \text{ for all } x \neq x^\star \text{ in } \mathcal{X}\\
&t_1 \leq t_2 \Rightarrow V(x(t_1)) \geq V(x(t_2))
\end{aligned}$$

then $x = x^\star$ is a stable equilibrium point. Moreover if there exists a continuous function $W(x)$ such that

$$\begin{aligned}
&W(x^\star) = 0\\
&W(x) > 0 \text{ for all } x \neq x^\star \text{ in } \mathcal{X}\\
&t_1 \leq t_2 \Rightarrow V(x(t_1)) \geq V(x(t_2)) + \int_{t_1}^{t_2} W(x(\tau))d\tau
\end{aligned}$$

and

$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$

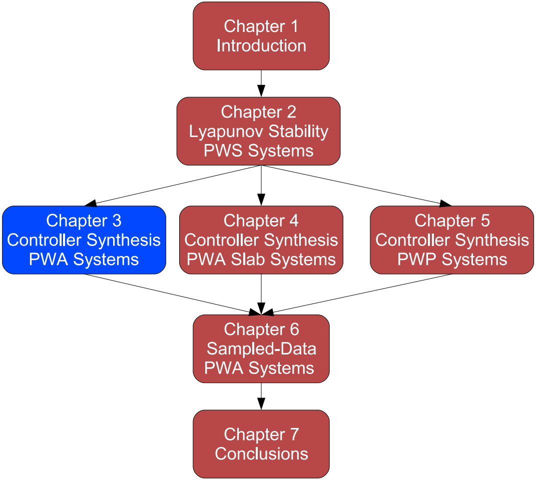
then all trajectories in $\mathcal{X}$ asymptotically converge to $x = x^\star$.

Why nonsmooth analysis?

- Discontinuous vector fields
- Piecewise smooth Lyapunov functions

Lyapunov Function	**Vector Field**	
	Continuous	Discontinuous
Smooth	☺	☺
Piecewise Smooth	☺	☹

Extension of local linear controllers to global PWA controllers for uncertain nonlinear systems



Extension of linear controllers to PWA controllers

Objectives:

- **Global robust stabilization** and **local robust performance**
- Integrating good **linear techniques** into **nonlinear methodologies**

PWA Differential Inclusions

- Consider the following **uncertain** nonlinear system

$$\dot{x} = f(x) + g(x)u$$

Let

$$\dot{x} \in \mathbf{Conv}\{\sigma_1(x,u), \ldots, \sigma_{\mathcal{K}}(x,u)\}$$

where

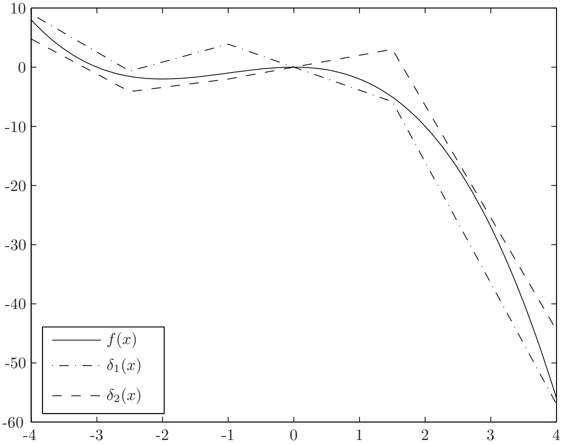
$$\sigma_\kappa(x,u) = A_{i\kappa}x + a_{i\kappa} + B_{i\kappa}u, x \in \overline{\mathcal{R}}_i,$$

with

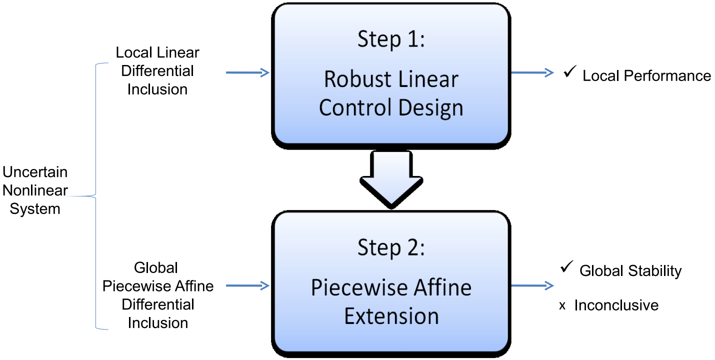
$$\mathcal{R}_i = \{x | E_i x + e_i \succ 0\}, \text{ for } i = 1, \ldots, M$$

$\succ$ represents an elementwise inequality.

PWA bounding envelope for a nonlinear function



Extension of linear controllers to PWA controllers



Extension of linear controllers to PWA controllers

Theorem (3.2)

Let there exist matrices $\bar{P}_i = \bar{P}_i^T$, $\bar{K}_i$, Z_i, $\bar{Z}_i$, $\Lambda_{i\kappa}$ and $\bar{\Lambda}_{i\kappa}$ that verify the conditions for all $i = 1, \dots, M$, $\kappa = 1, \dots, \mathcal{K}$ and for a given decay rate $\alpha > 0$, desired equilibrium point $x^\star$, linear controller gain $\bar{K}_{i^\star}$ and $\epsilon > 0$, then ***all trajectories of the nonlinear system*** *in $\mathcal{X}$ asymptotically converge to $x = x^\star$.*

Extension of linear controllers to PWA controllers

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- If the conditions are feasible, the resulting PWA controller provides global robust stability and local robust performance.

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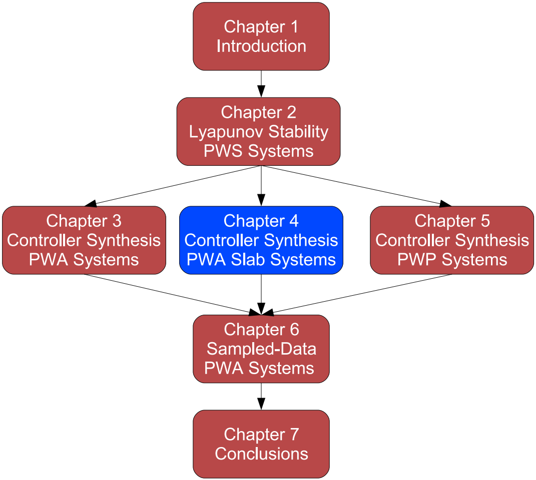
- If the conditions are feasible, the resulting PWA controller provides global robust stability and local robust performance.
- The synthesis problem includes a set of Bilinear Matrix Inequalities (BMI). In general, it is not convex.

Controller synthesis for PWA slab differential inclusions: a duality-based convex optimization approach



Controller synthesis for PWA slab differential inclusions: a duality-based convex optimization approach

Objective:

- To formulate controller synthesis for **stability** and $\mathbf{L_2}$ **gain performance** of piecewise affine **slab** differential inclusions as a set of LMIs.

- PWA slab differential inclusion:

$$\begin{aligned}\dot{x} \in& \textbf{Conv}\{A_{i\kappa}x + a_{i\kappa} + B_{w_{i\kappa}}w, \kappa = 1,2\},\ (x,w) \in \mathcal{R}_i^{\mathcal{X}\times\mathcal{W}} \\ y \in& \textbf{Conv}\{C_{i\kappa}x + c_{i\kappa} + D_{w_{i\kappa}}w, \kappa = 1,2\} \\ \mathcal{R}_i^{\mathcal{X}\times\mathcal{W}} =& \{(x,w) \mid \|L_i x + l_i + M_i w\| < 1\}\end{aligned}$$

- Parameter set:

$$\Phi = \left\{ \begin{bmatrix} A_{i\kappa_1} & a_{i\kappa_1} & B_{w_{i\kappa_1}} \\ L_i & l_i & M_i \\ C_{i\kappa_2} & c_{i\kappa_2} & D_{w_{i\kappa_2}} \end{bmatrix} \middle| \ i = 1, \ldots, M,\ \kappa_1 = 1,2,\ \kappa_2 = 1,2 \right\}$$

Dual Parameter Set

$$\Phi^{\mathsf{T}} = \left\{ \begin{bmatrix} A_{i\kappa_1}^{\mathsf{T}} & L_i^{\mathsf{T}} & C_{i\kappa_2}^{\mathsf{T}} \\ a_{i\kappa_1}^{\mathsf{T}} & l_i & c_{i\kappa_2}^{\mathsf{T}} \\ B_{w_{i\kappa_1}}^{\mathsf{T}} & M_i^{\mathsf{T}} & D_{w_{i\kappa_2}}^{\mathsf{T}} \end{bmatrix} \,\middle|\, i = 1, \ldots, M,\ \kappa_1 = 1, 2,\ \kappa_2 = 1, 2 \right\}$$

Parameter Set

$$P > 0,$$

$$\begin{bmatrix} A_{i\kappa_1}^{\mathrm{T}} P + P A_{i\kappa_1} + C_{i\kappa_2}^{T} C_{i\kappa_2} & * \\ B_{w_{i\kappa_1}}^{\mathrm{T}} P + D_{w_{i\kappa_2}}^{\mathrm{T}} C_{i\kappa_2} & -\gamma^2 I + D_{w_{i\kappa_2}}^{\mathrm{T}} D_{w_{i\kappa_2}} \end{bmatrix} < 0$$

for $i \in \mathcal{I}(0,0)$, $\kappa_1 = 1,2$ and $\kappa_2 = 1,2$ and $\lambda_{i\kappa_1\kappa_2} < 0$

$$\begin{bmatrix} \begin{pmatrix} A_{i\kappa_1}^{\mathrm{T}} P + P A_{i\kappa_1} \\ + C_{i\kappa_2}^{T} C_{i\kappa_2} \\ + \lambda_{i\kappa_1\kappa_2} L_i^{\mathrm{T}} L_i \end{pmatrix} & * & * \\ a_{i\kappa_1}^{T} P + c_{i\kappa_2}^{T} C_{i\kappa_2} + \lambda_{i\kappa_1\kappa_2} l_i L_i & \lambda_{i\kappa_1\kappa_2}(l_i^2 - 1) + c_{i\kappa_2}^{T} c_{i\kappa_2} & * \\ \begin{pmatrix} B_{w_{i\kappa_1}}^{\mathrm{T}} P \\ + D_{w_{i\kappa_2}}^{\mathrm{T}} C_{i\kappa_2} \\ + \lambda_{i\kappa_1\kappa_2} M_i^{\mathrm{T}} L_i \end{pmatrix} & D_{w_{i\kappa_2}}^{\mathrm{T}} c_{i\kappa_2} + \lambda_{i\kappa_1\kappa_2} l_i M_i^{\mathrm{T}} & \begin{pmatrix} -\gamma^2 I \\ + D_{w_{i\kappa_2}}^{\mathrm{T}} D_{w_{i\kappa_2}} \\ + \lambda_{i\kappa_1\kappa_2} M_i^{\mathrm{T}} M_i \end{pmatrix} \end{bmatrix} < 0$$

for $i \notin \mathcal{I}(0,0)$, $\kappa_1 = 1,2$ and $\kappa_2 = 1,2$

Dual Parameter Set

$$Q > 0,$$

$$\begin{bmatrix} A_{i\kappa_1}Q + QA_{i\kappa_1}^T + B_{w_{i\kappa_1}}B_{w_{i\kappa_1}}^T & * \\ C_{i\kappa_2}Q + D_{w_{i\kappa_2}}B_{w_{i\kappa_1}}^T & -\gamma^2 I + D_{w_{i\kappa_2}}D_{w_{i\kappa_2}}^T \end{bmatrix} < 0$$

for $i \in \mathcal{I}(0,0)$, $\kappa_1 = 1,2$ and $\kappa_2 = 1,2$ and $\mu_{i\kappa_1\kappa_2} < 0$

$$\begin{bmatrix} \begin{pmatrix} A_{i\kappa_1}Q + QA_{i\kappa_1}^T \\ +B_{w_{i\kappa_1}}B_{w_{i\kappa_1}}^T \\ +\mu_{i\kappa_1\kappa_2}a_{i\kappa_1}a_{i\kappa_1}^T \end{pmatrix} & * & * \\ L_iQ + M_iB_{w_{i\kappa_1}}^T + \mu_{i\kappa_1\kappa_2}l_ia_{i\kappa_1}^T & \mu_{i\kappa_1\kappa_2}(l_i^2 - 1) + M_iM_i^T & * \\ \begin{pmatrix} C_{i\kappa_2}Q + D_{w_{i\kappa_2}}B_{w_{i\kappa_1}}^T \\ +\mu_{i\kappa_1\kappa_2}c_{i\kappa_2}a_{i\kappa_1}^T \end{pmatrix} & D_{w_{i\kappa_2}}M_i^T + \mu_{i\kappa_1\kappa_2}l_ic_{i\kappa_2} & \begin{pmatrix} -\gamma^2 I \\ +D_{w_{i\kappa_2}}D_{w_{i\kappa_2}}^T \\ +\mu_{i\kappa_1\kappa_2}c_{i\kappa_2}c_{i\kappa_2}^T \end{pmatrix} \end{bmatrix} < 0$$

for $i \notin \mathcal{I}(0,0)$, $\kappa_1 = 1,2$ and $\kappa_2 = 1,2$

PWA L_2 gain controller synthesis

- The goal is to limit the L_2 gain from w to y using the following PWA controller:

$$u = K_i x + k_i, \quad x \in \mathcal{R}_i$$

- New variables:

$$\begin{aligned} Y_i &= K_i Q \\ Z_i &= \mu_i k_i \\ W_i &= \mu_i k_i k_i^T \end{aligned}$$

- Problem: W_i is not a linear function of the unknown parameters μ_i, Y_i and Z_i.

Proposed solutions:

- *Convex relaxation*: Since $W_i = \mu_i k_i k_i^\mathsf{T} \leq 0$, if the synthesis inequalities are satisfied with $W_i = 0$, they are satisfied with any $W_i \leq 0$. Therefore, the synthesis problem can be made convex by omitting W_i.

A duality-based convex optimization approach

Proposed solutions:

- *Convex relaxation*: Since $W_i = \mu_i k_i k_i^{\mathsf{T}} \leq 0$, if the synthesis inequalities are satisfied with $W_i = 0$, they are satisfied with any $W_i \leq 0$. Therefore, the synthesis problem can be made convex by omitting W_i.
- *Rank minimization*: Note that $W_i = \mu_i k_i k_i^{\mathsf{T}} \leq 0$ is the solution of the following rank minimization problem:

$$\begin{aligned} &\min \textbf{Rank } X_i \\ &\text{s.t. } X_i = \begin{bmatrix} W_i & Z_i \\ Z_i^{\mathsf{T}} & \mu_i \end{bmatrix} \leq 0 \end{aligned}$$

Rank minimization is also not a convex problem. However, trace minimization works practically well as a heuristic solution

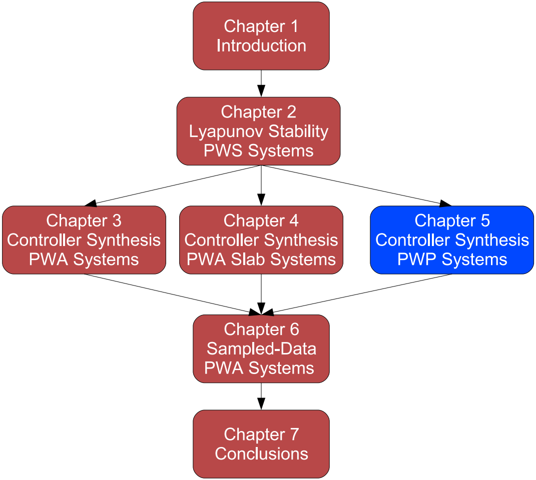
$$\min \textbf{Trace } X_i, \text{ s.t. } X_i = \begin{bmatrix} W_i & Z_i \\ Z_i^{\mathsf{T}} & \mu_i \end{bmatrix} \leq 0$$

A duality-based convex optimization approach

The following problems for PWA slab differential inclusions with PWA outputs were formulated as a set of LMIs:

- Stability analysis (Propositions 4.1 and 4.2)
- L_2 gain analysis (Propositions 4.3 and 4.4)
- Stabilization using PWA controllers (Propositions 4.5 and 4.6)
- PWA L_2 gain controller synthesis (Propositions 4.7 and 4.8)

Backstepping Controller Synthesis for PWP Systems: A Sum of Squares Approach



Backstepping Controller Synthesis for PWP Systems: A Sum of Squares Approach

Objective:

- To formulate controller synthesis for a class of piecewise polynomial systems as a Sum of Squares (SOS) programming.

Backstepping Controller Synthesis for PWP Systems: A Sum of Squares Approach

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- To formulate controller synthesis for a class of piecewise polynomial systems as a Sum of Squares (SOS) programming.
- SOSTOOLS, a MATLAB toolbox that handles the general SOS programming, was developed by S. Prajna, A. Papachristodoulou and P. Parrilo.

$$p(x) = \sum_{i=1}^{m} f_i^2(x) \geq 0$$

Backstepping Controller Synthesis for PWP Systems: A Sum of Squares Approach

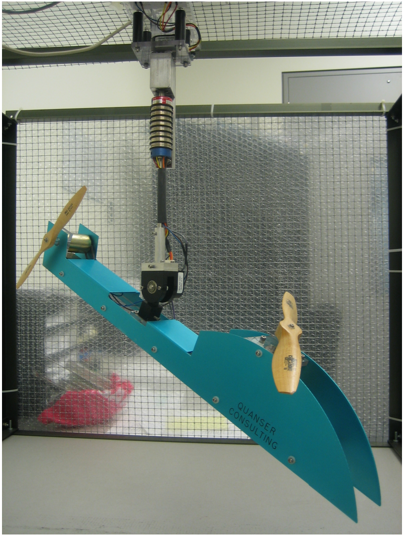
- PWP system in *strict feedback from*

$$
\begin{cases}
\dot{x}_1 = f_{1i_1}(x_1) + g_{1i_1}(x_1)x_2, & \text{for } x_1 \in \mathcal{P}_{1i_1} \\
\dot{x}_2 = f_{2i_2}(x_1, x_2) + g_{2i_2}(x_1, x_2)x_3, & \text{for } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{P}_{2i_2} \\
\vdots & \\
\dot{x}_k = f_{ki_k}(x_1, x_2, \ldots, x_k) + g_{ki_k}(x_1, x_2, \ldots, x_k)u, & \text{for } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \in \mathcal{P}_{ki_k}
\end{cases}
$$

where

$$
\mathcal{P}_{ji_j} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_j \end{bmatrix} \middle| E_{ji_j}(x_1, \ldots, x_j) \succ 0 \right\}
$$

Motivation: Toycopter, a 2 DOF helicopter model



Example:

- Pitch model of the experimental helicopter:

$$\begin{aligned}\dot{x}_1 &= x2\\ \dot{x}_2 &= \frac{1}{I_{yy}}(-m_{heli}l_{cgx}g\cos(x_1) - m_{heli}l_{cgz}g\sin(x_1) - F_{kM}\,\mathrm{sgn}(x_2)\\ &\quad - F_{vM}x_2 + u)\end{aligned}$$

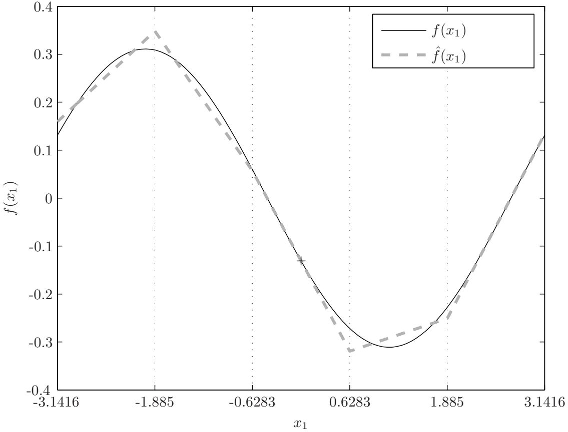
where x_1 is the pitch angle and x_2 is the pitch rate.

- Nonlinear part:

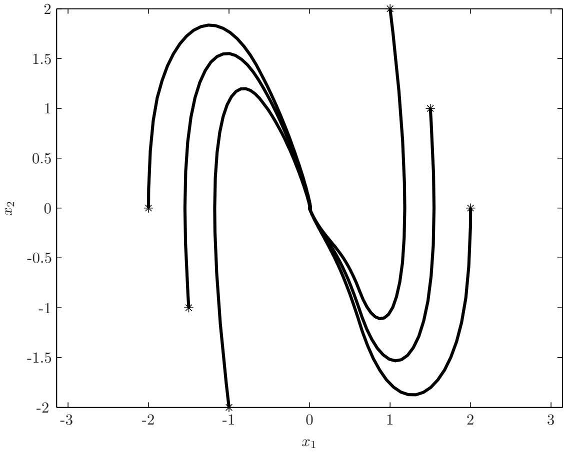
$$f(x_1) = -m_{heli}l_{cgx}g\cos(x_1) - m_{heli}l_{cgz}g\sin(x_1)$$

- PWA part:

$$f(x_2) = -F_{kM}\,\mathrm{sgn}(x_2)$$

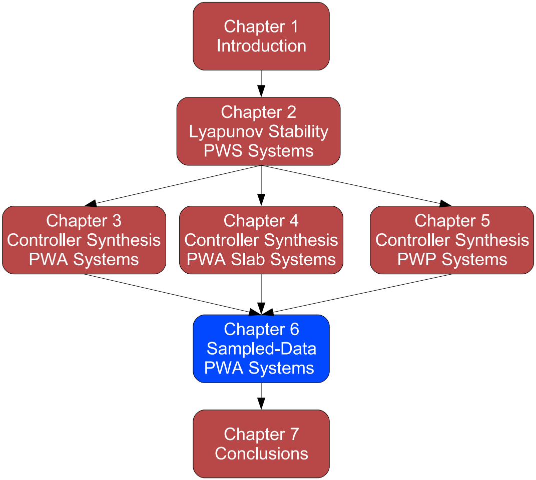


PWA approximation - Helicopter model



Continuous time PWA controller

Sampled-Data PWA Systems: A Time-Delay Approach



- Sampled-data PWA controller

$$u(t) = K_i x(t_k) + k_i, \;\; x(t_k) \in \mathcal{R}_i$$

- The closed-loop system can be rewritten as

$$\dot{x}(t) = A_i x(t) + a_i + B_i(K_i x(t_k) + k_i) + B_i w,$$

for $x(t) \in \mathcal{R}_i$ and $x(t_k) \in \mathcal{R}_j$ where

$$w(t) = (K_j - K_i)x(t_k) + (k_j - k_i), \;\; x(t) \in \mathcal{R}_i, \;\; x(t_k) \in \mathcal{R}_j$$

The input $w(t)$ is a result of the fact that $x(t)$ and $x(t_k)$ are not necessarily in the same region.

Theorem (6.1)

For the sampled-data PWA system, assume there exist symmetric positive matrices P, R, X and matrices N_i for $i = 1, \dots, M$ such that the conditions are satisfied and let there be constants Δ_K and Δ_k such that

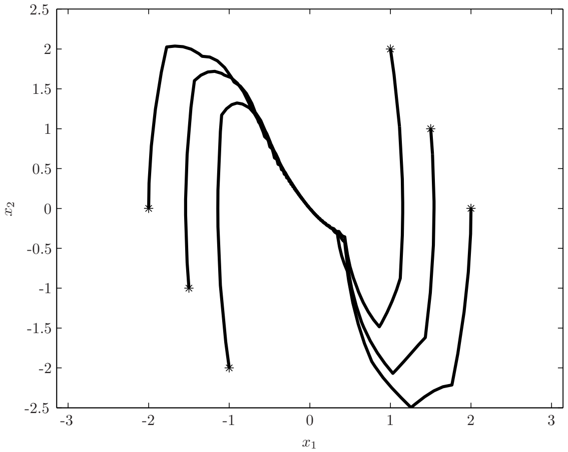
$$\|w\| \leq \Delta_K \|x(t_k)\| + \Delta_k$$

Then, all the trajectories of the sampled-data PWA system in $\mathcal{X}$ converge to the following invariant set

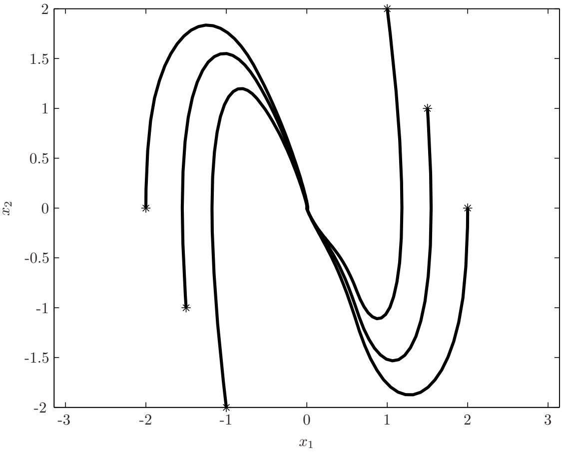
$$\Omega = \{x_s | \ V(x_s, \rho) \leq \sigma_a \mu_\theta^2 + \sigma_b\}$$

Solving an optimization problem to maximize τ_M subject to the constraints of the main theorem and $\eta > \gamma > 1$ leads to

$$\tau_M^\star = 0.2193$$



Sampled data PWA controller for $T_s = 0.2193$



Continuous time PWA controller

Breaking News

- The day before yesterday, I almost had a heart attack when...

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- Good news is that fixing the mistake slightly changes the LMI conditions for PWA systems and that does not affect the numerical results much. For example for the helicopter model, τ_M was 0.2193. Now, it is 0.2175.

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- Bad news is that for linear systems I get exactly the LMI conditions of [82]. Therefore, the result for linear systems and the linear example are no longer valid.

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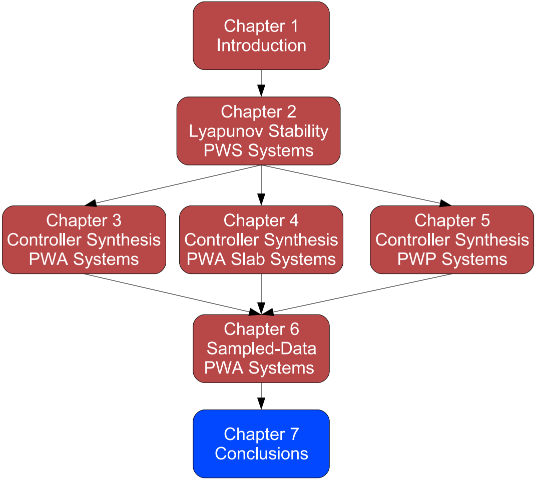
- Bad news is that for linear systems I get exactly the LMI conditions of [82]. Therefore, the result for linear systems and the linear example are no longer valid.
- Good news is that I can still keep my friendship with my undergrad classmate who is the first author of [82].

Conclusions



Summary of Major Contributions

1. To propose a two-step controller synthesis method for a class of uncertain nonlinear systems described by PWA differential inclusions.
2. To introduce for the first time a duality-based interpretation of PWA systems. This enables controller synthesis for PWA slab systems to be formulated as a convex optimization problem.
3. To propose a nonsmooth backstepping controller synthesis for PWP systems.
4. To propose a time-delay approach to stability analysis of sampled-data PWA systems.

Publications

1. B. Samadi and L. Rodrigues, "Extension of local linear controllers to global piecewise affine controllers for uncertain nonlinear systems," accepted for publication in the *International Journal of Systems Science*.
2. B. Samadi and L. Rodrigues, "Controller synthesis for piecewise affine slab differential inclusions: a duality-based convex optimization approach," under second revision for publication in *Automatica*.
3. B. Samadi and L. Rodrigues, "Backstepping Controller Synthesis for Piecewise Polynomial Systems: A Sum of Squares Approach," in preparation
4. B. Samadi and L. Rodrigues, "Sampled-Data Piecewise Affine Systems: A Time-Delay Approach," to be submitted.

Publications

1. B. Samadi and L. Rodrigues, "Backstepping Controller Synthesis for Piecewise Polynomial Systems: A Sum of Squares Approach," submitted to *the 46th Conference on Decision and Control*, cancun, Mexico, Dec. 2008.
2. B. Samadi and L. Rodrigues, "Sampled-Data Piecewise Affine Slab Systems: A Time-Delay Approach," in *Proc. of the American Control Conference,* Seattle, WA, Jun. 2008.
3. B. Samadi and L. Rodrigues, "Controller synthesis for piecewise affine slab differential inclusions: a duality-based convex optimization approach," in *Proc. of the 46th Conference on Decision and Control*, New Orleans, LA, Dec. 2007.
4. B. Samadi and L. Rodrigues, "Backstepping Controller Synthesis for Piecewise Affine Systems: A Sum of Squares Approach," in *Proc. of the IEEE International Conference on Systems, Man, and Cybernetics (SMC 2007)*, Montreal, Oct. 2007.
5. B. Samadi and L. Rodrigues, "Extension of a local linear controller to a stabilizing semi-global piecewise-affine controller," *7th Portuguese Conference on Automatic Control*, Lisbon, Portugal, Sep. 2006.

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Questions

Open Problems

- Can general PWP/PWA controller synthesis be converted to a convex problem?
- What is the dual of a PWA system?

Special Structure

Type	$f(x)$	$g(x)$
Piecewise Smooth	Piecewise Continuous	Piecewise Continuous
Piecewise Polynomial	Piecewise Polynomial	Piecewise Polynomial
Piecewise Affine	Piecewise Affine	Piecewise Constant
Piecewise Linear	Piecewise Linear	Piecewise Constant
Linear	Linear	Constant

- "In fact the great watershed in optimization is not between **linearity** and **nonlinearity**, but **convexity** and **nonconvexity**." (Rockafellar, SIAM review, 1993)
- The hard part is to find out if a problem can be formulated as a convex optimization problem.

- Piecewise Quadratic Lyapunov function

$$V(x) = x^T P_i x + 2q_i^{\mathrm{T}} x + r_i, \text{ for } x \in \overline{\mathcal{R}}_i$$

Extension of linear controllers to PWA controllers

- Conditions on the PWA controller:

$$\bar{K}_i = \bar{K}_{i^\star}, \text{ if } x^\star \in \overline{\mathcal{R}}_i$$
$$(\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)\bar{x}^\star = 0, \text{ if } x^\star \in \overline{\mathcal{R}}_i$$
$$(\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)\bar{F}_{ij} = (\bar{A}_{j\kappa} + \bar{B}_{j\kappa}\bar{K}_j)\bar{F}_{ij}, \text{ if } \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset$$

Extension of linear controllers to PWA controllers

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\end{aligned}
$$

- Continuity of the Lyapunov function:

$$
\bar{F}_{ij}^T(\bar{P}_i - \bar{P}_j)\bar{F}_{ij} = 0, \text{ if } \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset
$$

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- Continuity of the Lyapunov function:

$$\bar{F}_{ij}^T(\bar{P}_i - \bar{P}_j)\bar{F}_{ij} = 0, \text{ if } \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset$$

- Positive definiteness of the Lyapunov function:

$$\bar{P}_i\bar{x}^\star = 0, \text{ if } x^\star \in \overline{\mathcal{R}}_i$$
$$P_i > \epsilon I, \text{if } x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i \neq 0$$
$$\begin{cases} Z_i \in \mathbb{R}^{n\times n}, Z_i \succeq 0 \\ P_i - E_i^{\mathsf{T}} Z_i E_i > \epsilon I \end{cases}, \text{if } x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i = 0$$
$$\begin{cases} \bar{Z}_i \in \mathbb{R}^{(n+1)\times(n+1)}, \bar{Z}_i \succeq 0 \\ \bar{P}_i - \bar{E}_i^T \bar{Z}_i \bar{E}_i > \epsilon \bar{I} \end{cases}, \text{if } x^\star \notin \overline{\mathcal{R}}_i$$

Extension of linear controllers to PWA controllers

- Monotonicity of the Lyapunov function:

$$\text{for } i \text{ such that } x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i \neq 0,$$
$$P_i(A_{i\kappa} + B_{i\kappa} K_i) + (A_{i\kappa} + B_{i\kappa} K_i)^T P_i < -\alpha P_i$$

$$\text{for } i \text{ such that } x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i = 0,$$
$$\begin{cases} \Lambda_{i\kappa} \in \mathbb{R}^{n\times n}, \;\; \Lambda_{i\kappa} \succeq 0 \\ P_i(A_{i\kappa} + B_{i\kappa} K_i) + (A_{i\kappa} + B_{i\kappa} K_i)^\mathrm{T} P_i + E_i^\mathrm{T} \Lambda_{i\kappa} E_i < -\alpha P_i \end{cases}$$

$$\text{for } i \text{ such that } x^\star \notin \overline{\mathcal{R}}_i,$$
$$\begin{cases} \bar{\Lambda}_{i\kappa} \in \mathbb{R}^{(n+1)\times(n+1)}, \;\; \bar{\Lambda}_{i\kappa} \succeq 0 \\ \bar{P}_i(\bar{A}_{i\kappa} + \bar{B}_{i\kappa} \bar{K}_i) + (\bar{A}_{i\kappa} + \bar{B}_{i\kappa} \bar{K}_i)^T \bar{P}_i + \bar{E}_i^T \bar{\Lambda}_{i\kappa} \bar{E}_i < -\alpha \bar{P}_i \end{cases}$$

Extension of linear controllers to PWA controllers

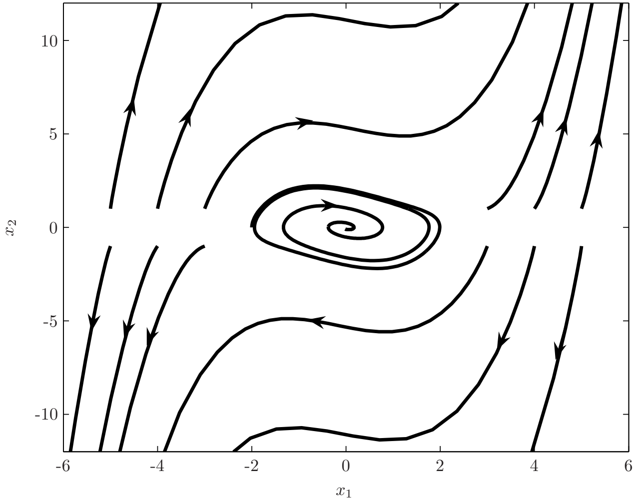
Consider the following second order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + 0.5x_2 - 0.5x_1^2 x_2 + u \end{aligned}$$

with the following domain:

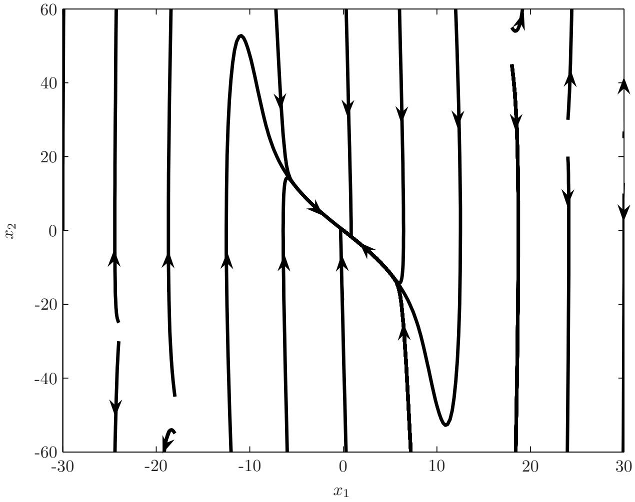
$$\mathcal{X} = \left\{ \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] \middle| -30 < x_1 < 30,\ -60 < x_2 < 60 \right\}$$

Extension of linear controllers to PWA controllers



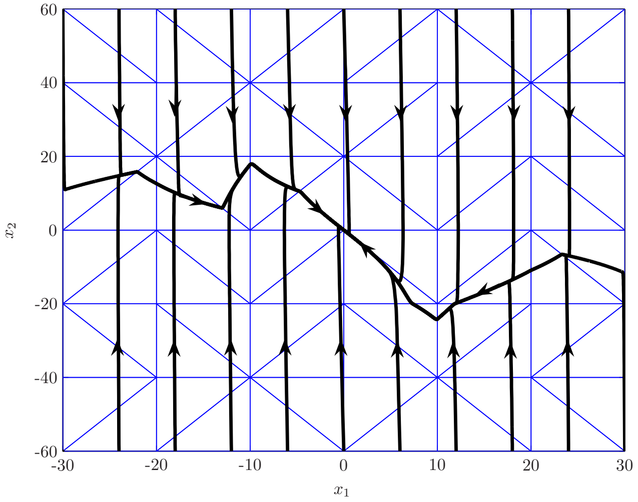
Trajectories of the open loop system

Extension of linear controllers to PWA controllers



Trajectories of the closed-loop system for the linear controller

Extension of linear controllers to PWA controllers



Trajectories of the closed-loop system for the PWA controller

Extension of linear controllers to PWA controllers

- *The main limitation:* The controller synthesis is formulated as a set of Bilinear Matrix Inequalities (BMI).
- *Linear systems analogy:* Consider the linear system $\dot{x} = Ax + Bu$ with a candidate Lyapunov function $V(x) = x^{\mathsf{T}} P x$ and a state feedback controller $u = Kx$. Sufficient conditions for the stability of this system are:
  - Positivity: $V(x) = x^{\mathsf{T}} P x > 0$ for $x \neq 0$
  - Monotonicity: $\dot{V}(x) = \dot{x}^{\mathsf{T}} P x + x^{\mathsf{T}} P \dot{x} < 0$

$$P > 0$$

$$(A^{\mathsf{T}} + K^{\mathsf{T}} B^{\mathsf{T}}) P + P(A + BK) < 0$$

- BMIs are nonconvex problems in general.

Extension of linear controllers to PWA controllers

Contributions:

- To propose a two-step controller synthesis method for a class of uncertain nonlinear systems described by PWA differential inclusions.
  - The proposed method has two objectives: global stability and local performance. It thus enables to use well known techniques in linear control design for local stability and performance while delivering a global PWA controller that is guaranteed to stabilize the nonlinear system.
  - Differential inclusions are considered. Therefore the controller is robust in the sense that it can stabilize any piecewise smooth nonlinear system bounded by the differential inclusion.
  - Stability is guaranteed even if sliding modes exist.

A duality-based convex optimization approach

Linear systems analogy:

- Consider the dual system

$$\dot{z} = (A + BK)^{\mathsf{T}} z$$

  with a candidate Lyapunov function

$$V(x) = x^{\mathsf{T}} Q x$$

- Sufficient conditions for the stability of this system are:
  - Positive definiteness: $Q > 0$
  - Monotonicity: $(A + BK)Q + Q(A^{\mathsf{T}} + K^{\mathsf{T}} B^{\mathsf{T}}) < 0$

  Define $Y = KQ$.
  - Positive definiteness: $Q > 0$
  - Monotonicity: $AQ + BY + QA^{\mathsf{T}} + Y^{\mathsf{T}} B^{\mathsf{T}} < 0$

  Then $K = YQ^{-1}$.

- PWA slab system:

$$\begin{aligned} \dot{x} =& A_i x + a_i, \;\; x \in \mathcal{R}_i \\ \mathcal{R}_i =& \{x | \; \|L_i x + l_i\| < 1\} \end{aligned}$$

- Literature
  - Hassibi and Boyd (1998)
  - Rodrigues and Boyd (2005)

A duality-based convex optimization approach

- Sufficient conditions for stability

$$P > 0,$$

$$A_i^{\mathsf{T}} P + P A_i + \alpha P < 0, \quad \text{for } 0 \in \overline{\mathcal{R}}_i,$$

$$\begin{cases} \lambda_i < 0, \\ \begin{bmatrix} A_i^{\mathsf{T}} P + P A_i + \alpha P + \lambda_i L_i^{\mathsf{T}} L_i & P a_i + \lambda_i l_i L_i^{\mathsf{T}} \\ a_i^{\mathsf{T}} P + \lambda_i l_i L_i & \lambda_i (l_i^2 - 1) \end{bmatrix} < 0, & \text{for } 0 \notin \overline{\mathcal{R}}_i \end{cases}$$

- Hassibi and Boyd (1998)

A duality-based convex optimization approach

- Sufficient conditions for stability

$$Q > 0,$$

$$A_i Q + Q A_i^T + \alpha Q < 0, \quad \text{for } 0 \in \overline{\mathcal{R}}_i,$$

$$\begin{cases} \mu_i < 0, \\ \begin{bmatrix} A_i Q + Q A_i^T + \alpha Q + \mu_i a_i a_i^T & Q L_i^T + \mu_i l_i a_i \\ L_i Q + \mu_i l_i a_i^T & \mu_i (l_i^2 - 1) \end{bmatrix} < 0, \end{cases} \text{for } 0 \notin \overline{\mathcal{R}}_i$$

- Hassibi and Boyd (1998)

A duality-based convex optimization approach

- Parameter set:

$$\Omega = \left\{ \left[\begin{array}{cc} A_i & a_i \\ L_i & l_i \end{array} \right] \middle| \, i = 1, \ldots, M \right\}$$

- Dual parameter set

$$\Omega^{\mathsf{T}} = \left\{ \left[\begin{array}{cc} A_i^{\mathsf{T}} & L_i^{\mathsf{T}} \\ a_i^{\mathsf{T}} & l_i \end{array} \right] \middle| \, i = 1, \ldots, M \right\}$$

A duality-based convex optimization approach

- PWA slab system:

$$\begin{aligned} \dot{x} =& A_i x + a_i + B_{w_i} w, \\ x \in& \mathcal{R}_i = \{x| \; \|L_i x + l_i\| < 1\}, \\ y =& C_i x + D_{w_i} w, \end{aligned}$$

- Parameter set:

$$\Phi = \left\{ \begin{bmatrix} A_i & a_i & B_{w_i} \\ L_i & l_i & 0 \\ C_i & 0 & D_{w_i} \end{bmatrix} \middle| \; i = 1, \ldots, M \right\}$$

- Hassibi and Boyd (1998)

A duality-based convex optimization approach

Summary:

- Introducing PWA slab differential inclusions
- Introducing the dual parameter set
- Extending the L_2 gain analysis and synthesis to PWA slab differential inclusions with PWA outputs
- Extending the definition of the regions of a PWA slab differential inclusion
- Proposing two methods to formulate the PWA controller synthesis for PWA slab differential inclusions as a convex problem

A sum of squares program is a convex optimization program of the following form:

$$\text{Minimize} \quad \sum_{j=1}^{J} w_j \alpha_j$$

$$\text{subject to} \quad f_{i,0} + \sum_{j=1}^{J} \alpha_j f_{i,j}(x) \text{ is SOS, for } i = 1, \ldots, I$$

where the α_j's are the scalar real decision variables, the w_j's are some given real numbers, and the $f_{i,j}$ are some given multivariate polynomials.

Sum of Squares Programming

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where the α_j's are the scalar real decision variables, the w_j's are some given real numbers, and the $f_{i,j}$ are some given multivariate polynomials.

- SOSTOOLS, a MATLAB toolbox that handles the general SOS programming, was developed by S. Prajna, A. Papachristodoulou and P. Parrilo.

- Consider the following PWP system:

$$
\begin{aligned}
\dot{x} &= f_i(x) + g_i(x)z, \;\; x \in \mathcal{P}_i \\
\dot{z} &= u
\end{aligned}
$$

where

$$
\mathcal{P}_i = \{x | E_i(x) \succ 0\}
$$

where $E_i(x) \in \mathbb{R}^{p_i}$ is a vector polynomial function of x and $\succ$ represents an elementwise inequality.

Backstepping Controller Synthesis for PWP Systems: A Sum of Squares Approach

Backstepping as a Lyapunov function construction method:

- Consider $\dot{x} = f_i(x) + g_i(x)z, \;\; x \in \mathcal{P}_i$

Backstepping as a Lyapunov function construction method:

- Consider $\dot{x} = f_i(x) + g_i(x)z, \;\; x \in \mathcal{P}_i$
- Assume that there exists a polynomial control $z = \gamma(x)$ and $V(x)$ is an SOS Lyapunov function for the closed loop system verifying

$$\begin{cases} V(x) - \lambda(x) \text{ is SOS} \\ -\nabla V(x)^{\mathsf{T}}(f_i(x) + g_i(x)\gamma(x)) - \Gamma_i(x)^{\mathsf{T}} E_i(x) - \alpha V(x) \text{ is SOS} \end{cases}$$

for $i = 1, \ldots, M$ and any $\alpha > 0$, where $\lambda(x)$ is a positive definite SOS polynomial, $\Gamma_i(x)$ is an SOS vector function

Backstepping as a Lyapunov function construction method:

- Consider $\dot{x} = f_i(x) + g_i(x)z, \;\; x \in \mathcal{P}_i$
- Assume that there exists a polynomial control $z = \gamma(x)$ and $V(x)$ is an SOS Lyapunov function for the closed loop system verifying

$$\begin{cases} V(x) - \lambda(x) \text{ is SOS} \\ -\nabla V(x)^{\mathsf{T}}(f_i(x) + g_i(x)\gamma(x)) - \Gamma_i(x)^{\mathsf{T}} E_i(x) - \alpha V(x) \text{ is SOS} \end{cases}$$

for $i = 1, \ldots, M$ and any $\alpha > 0$, where $\lambda(x)$ is a positive definite SOS polynomial, $\Gamma_i(x)$ is an SOS vector function

- Consider now the following candidate Lyapunov function

$$V_\gamma(x, z) = V(x) + \frac{1}{2}(z - \gamma(x))^{\mathsf{T}}(z - \gamma(x))$$

Note that $V_\gamma(x, z)$ is a positive definite function.

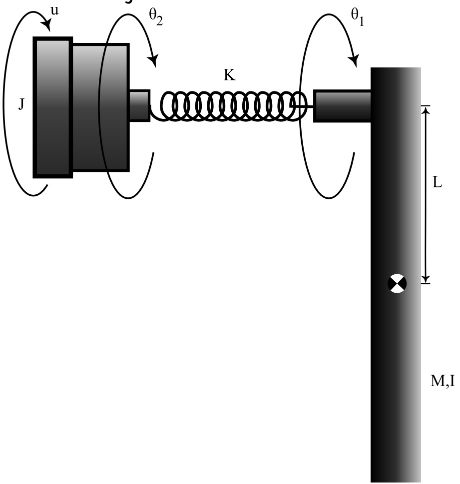
- *Controller synthesis*: The synthesis problem can be formulated as the following SOS program.

$$
\begin{aligned}
\text{Find} \quad & u = \gamma_2(x,z) \text{ and } \Gamma(x,z) \\
\text{s.t.} \quad & -\nabla_x V_\gamma(x,z)^\mathsf{T}(f_i(x) + g_i(x)z) \\
& -\nabla_z V_\gamma(x,z)^\mathsf{T} u \\
& -\Gamma(x,z)^\mathsf{T} E_i(x,z) - \alpha V_\gamma(x,z) \text{ is SOS}, \\
& \Gamma(x,z) \text{ is SOS} \\
& \gamma_2(0,0) = 0
\end{aligned}
$$

where $i_2 = 1, \ldots, M_2$ and $\gamma_2(x_1, x_2)$ is a polynomial function of x_1 and x_2.

Backstepping Controller Synthesis for PWP Systems: A Sum of Squares Approach

Example: Single link flexible joint robot:



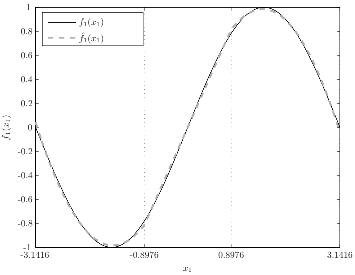
Example: Single link flexible joint robot:

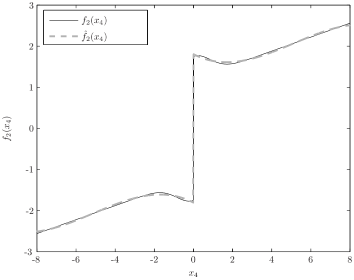
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{MgL}{I}\sin(x_1) - \frac{K}{I}(x_1 - x_3)\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= -\frac{f_2(x_4)}{J} + \frac{K}{J}(x_1 - x_3) + \frac{1}{J}u\end{aligned}$$

where $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$ and $x_4 = \dot{\theta}_2$. u is the motor torque and $f_2(x_4)$ denotes the motor friction which is described by

$$f_2(x_4) = b_m x_4 + \mathrm{sgn}(x_4)\left(F_{cm} + (F_{sm} - F_{cm})\exp(-\frac{x_4^2}{c_m^2})\right)$$

PWP approximation:



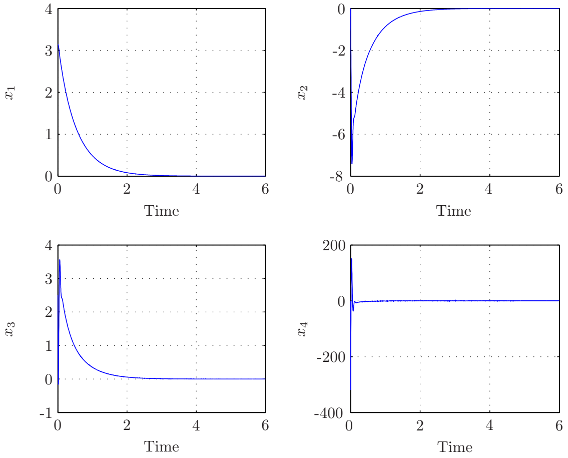


Backstepping Controller Synthesis for PWP Systems: A Sum of Squares Approach

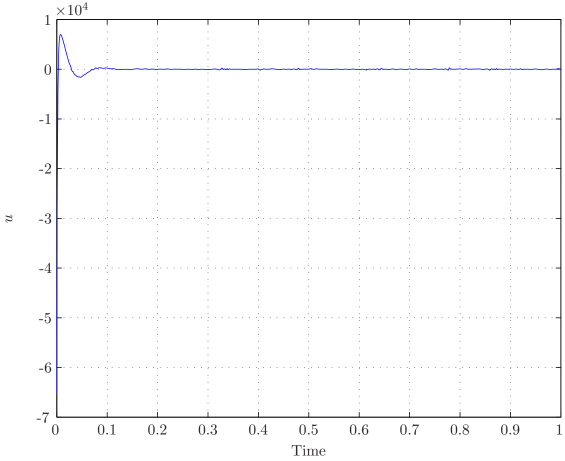
Typical structure of the regions of a PWP system in strict feedback form

$$
\begin{tikzcd}
x_1 & & & & \mathcal{P}_{11} \arrow[dll] \arrow[d] \arrow[drr] & & & \\
(x_1, x_2) & & \mathcal{P}_{21} \arrow[d] & & \mathcal{P}_{22} \arrow[d] & & \mathcal{P}_{23} \arrow[d] & \\
(x_1, x_2, x_3) & & \mathcal{P}_{31} \arrow[dl] \arrow[d] & & \mathcal{P}_{32} \arrow[dl] \arrow[dr] & & \mathcal{P}_{33} \arrow[d] \arrow[dr] & \\
(x_1, x_2, x_3, x_4) & \mathcal{P}_{41} & \mathcal{P}_{44} & \mathcal{P}_{42} & & \mathcal{P}_{45} & \mathcal{P}_{43} & \mathcal{P}_{46}
\end{tikzcd}
$$

State variables of the nonlinear model - PWA controller:

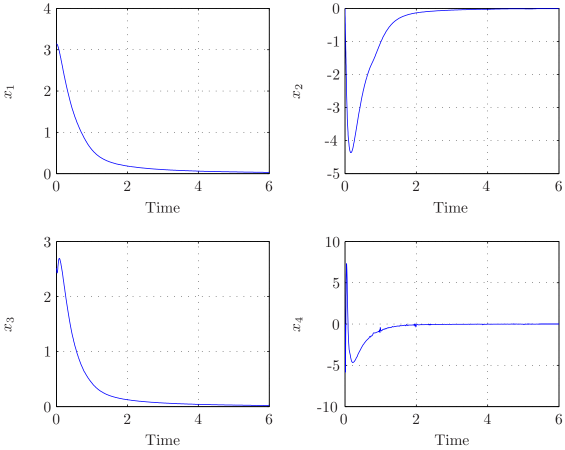


Control input - PWA controller:



Backstepping Controller Synthesis for PWP Systems: A Sum of Squares Approach

State variables of the nonlinear model - PWP controller:



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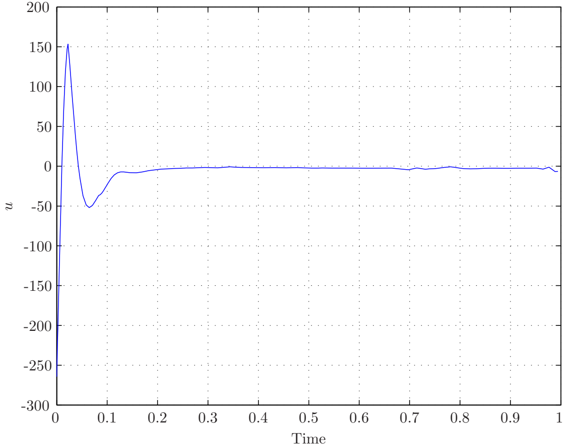

Summary of the contributions:

- Introducing PWP systems in *strict feedback form*
- Formulating backstepping controller synthesis for PWP systems as a convex optimization problem
    - Polynomial Lyapunov functions for PWP systems with discontinuous vector fields
    - PWP Lyapunov functions for PWP systems with continuous vector fields
    - Numerical tools such as SOSTOOLS and Yalmip/SeDuMi

- PWA system

$$\dot{x} = A_i x + a_i + B_i u, \quad \text{for } x \in \mathcal{R}_i$$

with the region $\mathcal{R}_i$ defined as

$$\mathcal{R}_i = \{x | E_i x + e_i \succ 0\},$$

- PWA system

$$\dot{x} = A_i x + a_i + B_i u, \quad \text{for } x \in \mathcal{R}_i$$

with the region $\mathcal{R}_i$ defined as

$$\mathcal{R}_i = \{x | E_i x + e_i \succ 0\},$$

- Continuous-time PWA controller

$$u(t) = K_i x(t) + k_i, \; x(t) \in \mathcal{R}_i$$

- Lyapunov-Krasovskii functional:

$$V(x_s, \rho) := V_1(x) + V_2(x_s, \rho) + V_3(x_s, \rho)$$

where

$$x_s(t) := \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}, \quad t_k \leq t < t_{k+1}$$

$$\begin{aligned} V_1(x) &:= x^{\mathrm{T}} P x \\ V_2(x_s, \rho) &:= \int_{-\tau_M}^{0} \int_{t+r}^{t} \dot{x}^{\mathrm{T}}(s) R \dot{x}(s) ds dr \\ V_3(x_s, \rho) &:= (\tau_M - \rho)(x(t) - x(t_k))^{\mathrm{T}} X (x(t) - x(t_k)) \end{aligned}$$

and P, R and X are positive definite matrices.

- for all i such that $0 \in \overline{\mathcal{R}}_i$,

$$\begin{bmatrix} \Psi_i + \tau_M M_{1i} & \begin{bmatrix} P \\ 0 \end{bmatrix} B_i + \tau_M \begin{bmatrix} I \\ -I \end{bmatrix} X B_i \\ B_i^{\mathrm{T}} \begin{bmatrix} P & 0 \end{bmatrix} + \tau_M B_i^{\mathrm{T}} X \begin{bmatrix} I & -I \end{bmatrix} & -\gamma I \end{bmatrix} < 0$$

$$\begin{bmatrix} \Psi_i + \tau_M M_{2i} & \tau_M \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B_i^{\mathrm{T}} \end{bmatrix} R B_i + \begin{bmatrix} P \\ 0 \end{bmatrix} B_i & \tau_M N_i \\ \tau_M B_i^{\mathrm{T}} R \begin{bmatrix} A_i & B_i K_i \end{bmatrix} + B_i^{\mathrm{T}} \begin{bmatrix} P & 0 \end{bmatrix} & \tau_M B_i^{\mathrm{T}} R B_i - \gamma I & 0 \\ \tau_M N_i^{\mathrm{T}} & 0 & -\frac{\tau_M}{2} R \end{bmatrix} < 0$$

- for all i such that $0 \notin \overline{\mathcal{R}}_i$, $\bar{\Lambda}_i \succ 0$,

$$\left[\begin{array}{cc} \overline{\Psi}_i + \tau_M \overline{M}_{1i} & \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} B_i + \tau_M \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} X B_i \\ B_i^T \begin{bmatrix} P & 0 & 0 \end{bmatrix} + \tau_M B_i^T X \begin{bmatrix} I & -I & 0 \end{bmatrix} & -\gamma I \end{array}\right] < 0$$

$$\left[\begin{array}{c|c|c} \overline{\Psi}_i + \tau_M \overline{M}_{2i} & \begin{array}{c} \tau_M \begin{bmatrix} A_i^T \\ K_i^T B_i^T \\ k_i^T B_i^T + a_i^T \end{bmatrix} R B_i \\ + \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} B_i \end{array} & \tau_M \begin{bmatrix} N_i \\ 0 \end{bmatrix} \\ \hline \begin{array}{c} \tau_M B_i^T R \begin{bmatrix} A_i & B_i K_i & B_i k_i + a_i \end{bmatrix} \\ + B_i^T \begin{bmatrix} P & 0 & 0 \end{bmatrix} \end{array} & \tau_M B_i^T R B_i - \gamma I & 0 \\ \hline \tau_M \begin{bmatrix} N_i^T & 0 \end{bmatrix} & 0 & -\frac{\tau_M}{2} R \end{array}\right] < 0$$

Summary of the contributions

- Formulating stability analysis of sampled-data PWA systems as a convex optimization problem

Example: Active suspension with a nonlinear damper