Term project 2: Modelling a Stellar Core

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Abstract

In this report, this, that and more of this is discussed. All source material related to this report can be found at the projects GitHub repository [1].

I Introduction

When trying to understand the mechanics of a star the most logical place to start is with what you can observe. Let's take a look at the properties of a star that we can measure with relative ease.

Firstly, we can look at how much radiation energy hits one particular area, i.e. a solar panel on earth. By assuming that the star radiates equally in all directions we can calculate the total amount of energy sent out by the star, the so-called luminosity (denoted as L).

In addition, we can find the surface temperature (denoted T) by looking at the color of the light emitted from the star and using Wien's displacements law.

Lastly, we might be able to look at the orbits of all the planets and do some number crunching and figure out the total mass of the star (denoted M). This last part may be easier said than done, for instance in the case of a far away solar system where we can't even see the planets orbiting, just their minuscule effect on the stars orbit. However, for the purposes of this report, let's assume that we are able to do so.

But that's about it for what we can figure out by just looking at the outside effects. There are many other properties we would like to know about, e.g. the radius (R), density (ρ) and pressure (P) of the star. In particular, we would like to have all of these properties as a function of the radius, so that we can get an idea of what the cross section of the star looks like. In order to achieve this, we need to deploy many different branches of physics, including thermodynamics, hydrostatics and nuclear physics. And of course, a good number of assumptions to simplify things for us.

II The Governing Equations

In this project, we will only be looking at the radiative core of the star. This means that we will not look at the outer convection layer, and assume that all energy transport is done through photon radiation. For reference, this corresponds to the inner $\sim 70\,\%$ of the sun. Furthermore, we will not consider time evolution; we look at one particular moment in time.

The following four differential equations govern the internal structure of radiative core of our star[1]:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{1}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \tag{2}$$

$$\frac{\partial L}{\partial m} = \epsilon \tag{3}$$

$$\frac{\partial T}{\partial m} = -\frac{3\kappa L}{256\pi^2 \sigma r^4 T^3} \tag{4}$$

The first thing to notice about these equations are that they are differentials with respect to mass, and not radius. Remember that what we want in the end is all the properties

[1] Derivations of the equations are not given, as they are shown in the lecture notes [2]. A qualitative description of their meaning is given instead.

of the star as a function of radius, so it would make sense to write the equations with respect to r instead[2]. It turns out that it is more numerically stable to use dm compared to dr, so we are going to treat the radius as $r = r(m)$. When we plot the data later, however, we will return to using r as the variable. Eq. (1) is simply stating the relation between taking an infinitesimal step in r, compared to an infinitesimal step in m.

Eq. (2) is the assumption of hydrostatic equilibrium, and states that if the gas in the star is to be at rest, then the outward pressure must exactly balance the force of gravity acting on the gas.

The ϵ in Eq. (3) represents the amount of energy produced by nuclear fusion per time and mass. This quantity will be treated more in a later section (ref??).

Eq. (4) is the temperature gradient need to transport all the produced energy out of the star. I should be noted that this is only the correct temperature gradient when all the energy is transported by radiation.

II.1 Equation of State

In the equations (1-4) we use T, ρ and P. If we take a look at how many unknowns we have, we find that we need one more equation. To this end, we are going to assume that we can treat the gas in the star as an ideal gas, thus getting an equation for the gas pressure through the ideal gas law:

$$\begin{aligned} P_g V &= N k_B T \\ \Rightarrow P_g &= \frac{N}{V} k_B T = \frac{\rho}{\mu\, m_u}, \end{aligned} \tag{5}$$

where P_g is the gas pressure, V is the volume, N is the number of particles and T is the temperature of the gas. The quantity $\mu\, m_u$ is the average mass of all particles in the gas. m_u is just a constant giving the average mass of a nucleon. μ is slightly more complicated, and represents the mass of the average particle in units of m_u (μ is unit less). It can be calculated a couple of ways. From the above equation we have

$$\mu = \frac{\rho}{m_u}\frac{V}{N} = \frac{\rho}{m_u n_{\text{tot}}} \tag{6}$$

where n_{tot} is the total number density of particles in the gas. n_{tot} can be taken as the sum of n_i for all particles i,

$$n_{\text{tot}} = \sum_i n_i = n_e \sum_i \frac{X_i \rho}{C_i m_u} \tag{7}$$

where X_i and C_i is the mass fraction, and number of core elements of particle i, respectively. For instance, ${}^4_2\text{He}$ gives $n_{{}^4_2\text{He}} = Y\rho/4m_u$, when Y is the mass fraction of Helium with four nucleons. The number density of the electron, n_e is written separate and should only consider the number of *free* electrons in the gas. In the case of only fully ionized ${}^4_2\text{He}$, $n_e = 2n_{{}^4_2\text{He}}$ because each ionized Helium core gives two electrons. I similar consideration should be taken for each species present in the gas.

Calculating n_{tot} is not necessarily so straight forward, but if we combine all the steps we have to take to compute μ, we can end up with a convenient formula:

$$\mu = \frac{1}{\sum(\text{particles provided per nucleus} \times \text{mass fraction/nucleons}} \tag{8}$$

This is the method used in the code, as a framework for summing over all particles is central to the structure of the code base.

Let's now return to the original problem of finding an extra equation. Eq. (5) gives us an additional equation, but it also introduces a new parameter, P_g. In addition to the gas pressure, we have radiation pressure P_{rad}, so that the total pressure is $P = P_g + P_{\text{rad}}$. Luckily, the expression for radiation pressure is quite simple, and depends only on temperature,

$$P_{\text{rad}} = \frac{a}{3} T^4 \tag{9}$$

[2] I will use lower case r and m when referring to the radius and mass as variables, and upper case R and M for the total radius and mass.

where $a = 4\sigma/c$ is a constant. This gives us finally the additional equation of state we need;

$$P = \frac{\rho}{\mu\, m_u} k_B T + \frac{a}{3} T^4 \tag{10}$$

$$\Leftrightarrow \rho = (P - \frac{a}{3} T^4) \frac{\mu\, m_u}{k_B T}. \tag{11}$$

I have listed the equation with respect to density as well; this relation will be used to find P given ρ and T, and vice versa.

II.2 Energy Production

In Eq. (3), the quantity ϵ represents the amount of energy produced per time and mass. We derive the value of ϵ by looking at the reactions that produce energy inside a star. It is given by

$$\epsilon = \sum_{ij} r_{ij} Q_{ij}, \tag{12}$$

where r_{ij} is the reaction rate for particles i and j (per time and mass), and Q_{ij} is the energy produced for each such reaction. The exact values of Q_{ij} can be found in the lecture notes [2]. More interesting is the rates, which are given as

$$r_{ij} = \frac{n_i n_j}{\rho(1 + \delta_{ij})} \lambda_{ij} \tag{13}$$

where the symbols have their normal meaning (including the Kronecker delta δ_{ij}). λ_{ij} is a complicated function of temperature, and we will use tabulated expressions for these functions, found once again in the lecture notes [2, Table 3.1].

There are a few different ways that fusion can happen, depending on the conditions of the star. I our case we will simplify things a bit and only consider the PPI and PPII chains (Proton-Proton based fusion).

One important thing to mention when calculating ϵ is that no reaction should happen more often (have larger value for r_{ij}) than the reaction(s) that produced the reactant(s) of the first reaction.

The last assumption we add is that all Deuterium produced by Proton-Proton fusion, immediately fuses to ${}^3_2\mathrm{He}$, thus effectively merging the two first steps in the PP chains into one reaction.

III Assumptions

Up until this point, we've made several assumptions. In order to remember the limitations of our results, I have summarized them in a list:

- Assume something.

IV Solving the Equations

At this point we are ready to start solving the governing equations. For simplicity, I have chosen to deploy the standard Forward Euler method. One should then note that the simplicity comes at the cost of an global error proportional to the step size, dm. We will try to be smart about this step size in order to minimize the effects of this.

As a reminder, Forward Euler solves a differential equation $du/dx = f(u, x)$ by

$$u_{i+i} = u_i + du = u_i + f(u_i, x_i)\, dx.$$

In our case, with four differential equations, this corresponds to the following:

$$r_{i+1} = r_i + \frac{1}{4\pi r^2 \rho}\, dm$$

$$P_{i+1} = P_i - \frac{Gm}{4\pi r^4}\, dm$$

$$L_{i+1} = L_i + \epsilon\, dm$$

$$T_{i+1} = T_i - \frac{3\kappa L}{256\pi^2 \sigma r^4 T^3}\, dm$$

IV.1 Choosing a Suitable Step Size dm

When ever a numerical solution to a differential equation is tried, one has to think about what step size dm is suitable. In a perfect world, we would like to set it to an infinitesimal number, but the constraints of reality forces us to make a compromise between accuracy and speed. We need to find a value for dm that gives an accurate enough result, but does so in a reasonable time.

In order to evaluate the effect of different step sizes, figure 1 shows $r(m)$ calculated for a number of different values of dm. From the figure we can see a large variation in $r(m)$ for the lower values of dm. The solution seems to converge when $dm \geq M_0 \times 10^{-3}$, but even there it is not perfectly aligned. It turns out that the code runs very fast anyway (in large part because it is written in C++), so it is no problem to crank the step size down to $dm = M_0 \times 10^{-5}$, and still have do so in under a second.

This is quite neat, but we could still try to be a bit smarter about this than just using brute force. The downside to setting a fixed value for dm is that we might spend a lot of computational time on a part of our function(s) that is very stable, and on the other side, we might not have enough precision for where the function(s) is very steep. The alternative is *dynamic step size*, which involves defining the step size in such a way so that the value of the function(s) does not change by more than a fixed percentage. In the case of a diff.eq. $du/dx = f(u, x)$, we define the step size in the following way:

$$\frac{du}{u} \leq p \Leftrightarrow \frac{f\,dx}{u} \leq p \tag{14}$$

$$\Rightarrow dx = \frac{pu}{f}, \tag{15}$$

where p is the fraction that the solution is allowed to change by after one step. In our case we have more than one equation, so we impose a similar requirement to all equations and then simply choose the smallest required step size.

The only thing we then have to define is what the value of p should be. As previously stated, the code is quite fast so we can afford to be quite strict. The code used in all subsequent figures is set to allow a maximum change of 1‰. Figure 2 shows the same plot as figure 1, but this time including dynamic step size as well. The figure shows that DSS gives good results.

V Changing the Parameters

In this numerical model, we have the option to change the initial parameters to our liking. In particular, we would like to find the set of initial parameters that give the most realistic model of an actual star. To this end we are now going to look at how the solution is affected by changing R_0, M_0, P_0 etc.

V.1 Changing R_0

In order to see the effects of changing the initial radius, figure 3 has been produced. From the figure we can conclude with the following for increasing initial radius:

- m dies off faster
- P becomes *much* smaller
- L remains constant for much longer (the core becomes a smaller fraction of the star)
- T becomes smaller
- ρ is similar to T, only that ρ becomes *much* smaller with increasing R_0

V.2 Changing T_0

Figure 4 shows a similar comparison for a change in the initial temperature T_0. We conclude that for increasing initial temperature we have that:

- m falls off less rapidly
- T is smaller (relative to T_0)
- L falls of more rapidly, but at constant radius.

The plots for ρ and P are less obvious and does not yield as clear conclusions.

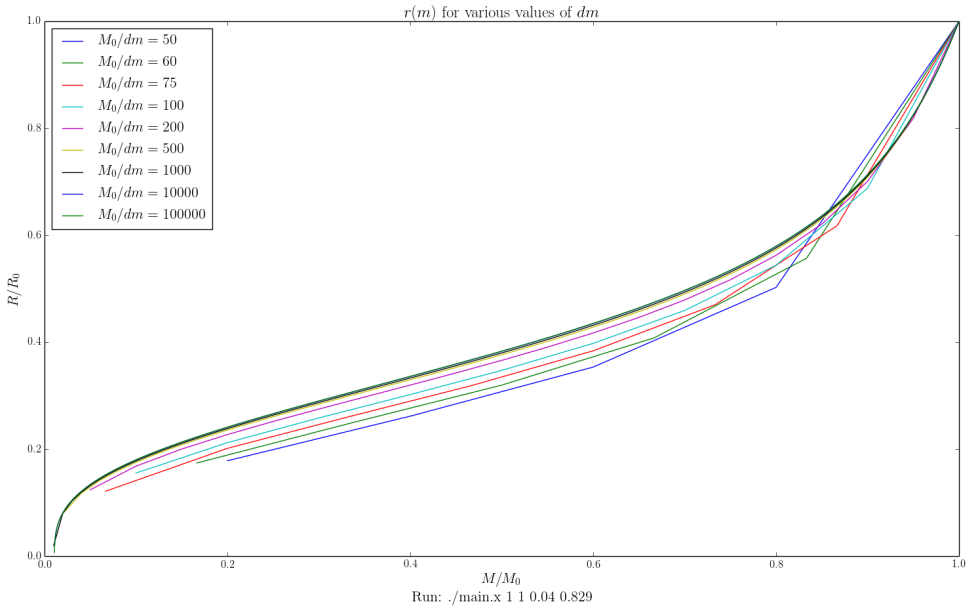

Figure 1: Plots of $r(m)$ using a range of different values for dm. We see that we need quite a large number of steps in order for the solution to converge.

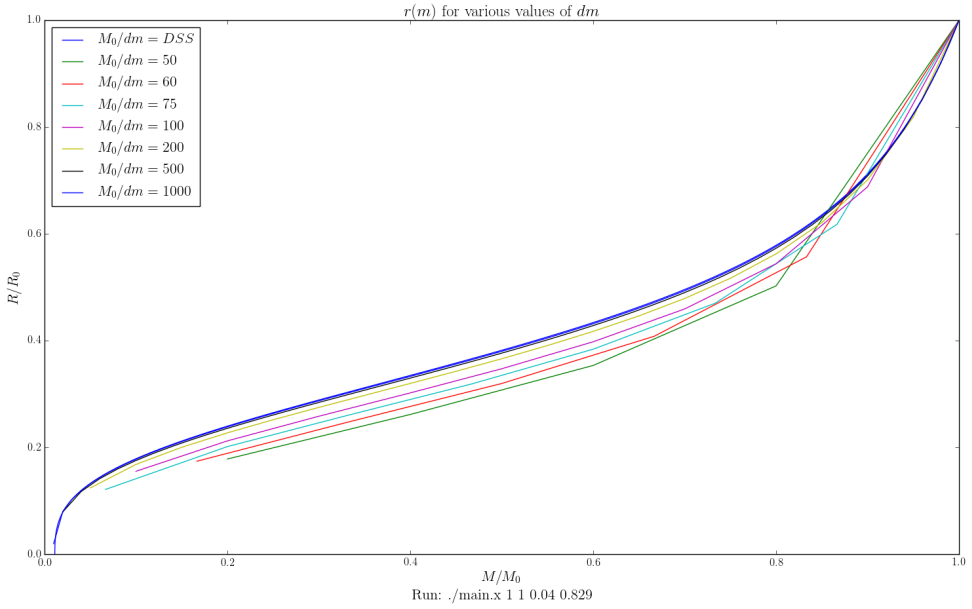

Figure 2: Plots of $r(m)$ using a range of different values for dm, including the solution obtained when using dynamic step size.

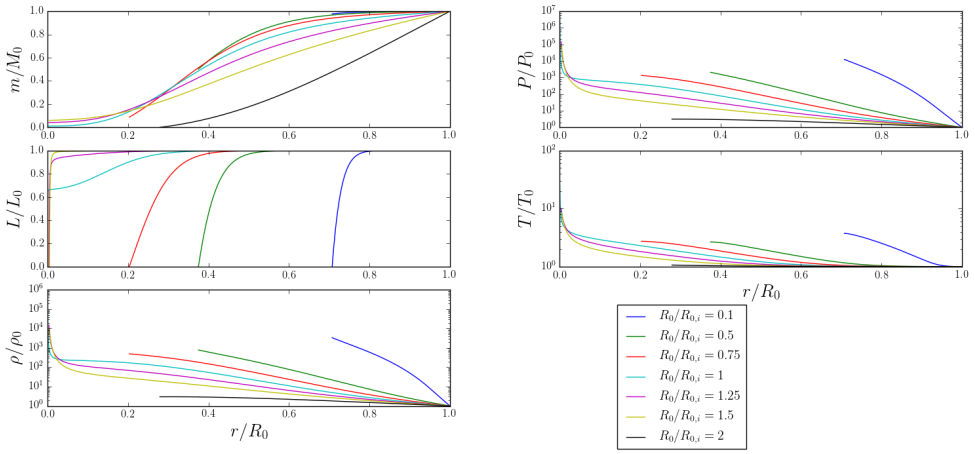

Figure 3: Plots for all parameters of the star showing the effects of changing the initial radius R_0.

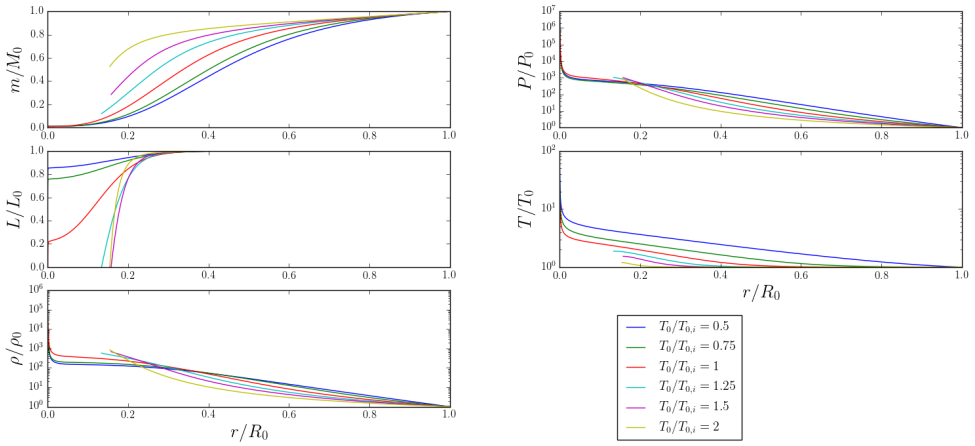

Figure 4: Plots for all parameters of the star showing the effects of changing the initial temperature T_0.

References

[1] B. Samseth. GitHub repository. URL: `https://github.com/bsamseth/ast3310.git`.

[2] B. V. Gudiksen. *AST3310: Astrophysical plasma and stellar interiors*. 2016.