# Term project 2: Modelling a Stellar Core

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**Abstract**

In this report, this, that and more of this is discussed. All source material related to this report can be found at the projects GitHub repository [1].

## Contents



# I Introduction

When trying to understand the mechanics of a star, the most logical place to start is with what we can observe. Let's take a look at the properties of a star that we can measure with relative ease.

Firstly, we can look at how much radiation energy hits one particular area, i.e. a solar panel on earth. By assuming that the star radiates equally in all directions we can calculate the total amount of energy sent out by the star, the so-called luminosity (denoted as $L$).

In addition, we can find the surface temperature (denoted $T$) by looking at the color of the light emitted from the star and using Wien's displacements law.

Lastly, we might be able to look at the orbits of all the planets and do some number crunching and figure out the total mass of the star (denoted $M$). This last part may be easier said than done, for instance in the case of a far away solar system where we can't even see the planets orbiting, just their minuscule effect on the stars orbit. However, for the purposes of this report, let's assume that we are able to do so.

But that's about it for what we can figure out by just looking at the outside effects.

There are many other properties we would like to know about, e.g. the radius ($R$), density ($\rho$) and pressure ($P$) of the star. In particular, we would like to have all of these properties as a function of the radius, so that we can get an idea of what the cross section of the star looks like. In order to achieve this, we need to deploy many different branches of physics, including thermodynamics, hydrostatics and nuclear physics. And of course, a good number of assumptions to simplify things for us.

# II The Governing Equations

In this project, we will only be looking at the radiative core of the star. This means that we will not look at the outer convection layer, and assume that all energy transport is done through photon radiation. For reference, this corresponds to the inner $\sim 70\,\%$ of the sun. Furthermore, we will not consider time evolution; we look at one particular moment in time.

The following four differential equations govern the internal structure of radiative core of our star[1]:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{1}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \tag{2}$$

$$\frac{\partial L}{\partial m} = \epsilon \tag{3}$$

$$\frac{\partial T}{\partial m} = -\frac{3\kappa L}{256\pi^2 \sigma r^4 T^3} \tag{4}$$

The first thing to notice about these equations are that they are differentials with respect to mass, and not radius. Remember that what we want in the end is all the properties of the star as a function of radius, so it would make sense to write the equations with respect to $r$ instead[2]. It turns out that it is more numerically stable to use $dm$ compared to $dr$, so we are going to treat the radius as $r = r(m)$. When we plot the data later, however, we will return to using $r$ as the variable. Eq. (1) is simply stating the relation between taking an infinitesimal step in $r$, compared to an infinitesimal step in $m$.

Eq. (2) is the assumption of hydrostatic equilibrium, and states that if the gas in the star is to be at rest, then the outward pressure must exactly balance the force of gravity acting on the gas.

The $\epsilon$ in Eq. (3) represents the amount of energy produced by nuclear fusion per time and mass. This quantity will be treated more in section II.2.

Eq. (4) is the temperature gradient need to transport all the produced energy out of the star. I should be noted that this is only the correct temperature gradient when all the energy is transported by radiation.

## II.1 Equation of State

In the equations (1-4) we use $T, \rho$ and $P$. If we take a look at how many unknowns we have, we find that we need one more equation. To this end, we are going to assume that we can treat the gas in the star as an ideal gas, thus getting an equation for the gas pressure through the ideal gas law:

$$\begin{aligned} P_g V &= N k_B T \\ \Rightarrow P_g &= \frac{N}{V} k_B T = \frac{\rho}{\mu\, m_u}, \end{aligned} \tag{5}$$

where $P_g$ is the gas pressure, $V$ is the volume, $N$ is the number of particles and $T$ is the temperature of the gas. The quantity $\mu\, m_u$ is the average mass of all particles in the gas. $m_u$ is just a constant giving the average mass of a nucleon. $\mu$ is slightly more complicated, and represents the mass of the average particle in units of $m_u$ ($\mu$ is unit less). It can be calculated a couple of ways. From the above equation we have

$$\mu = \frac{\rho}{m_u} \frac{V}{N} = \frac{\rho}{m_u n_{\text{tot}}} \tag{6}$$

[1] Derivations of the equations are not given, as they are shown in the lecture notes [2]. A qualitative description of their meaning is given instead.

[2] I will use lower case $r$ and $m$ when referring to the radius and mass as variables, and upper case $R$ and $M$ for the total radius and mass.

where $n_{\rm tot}$ is the total number density of particles in the gas. $n_{\rm tot}$ can be taken as the sum of $n_i$ for all particles $i$,

$$n_{\rm tot} = \sum_i n_i = n_e + \sum_i \frac{X_i \rho}{C_i m_u} \tag{7}$$

where $X_i$ and $C_i$ is the mass fraction, and number of core elements of particle $i$, respectively. For instance, ${}^4_2$He gives $n_{{}^4_2\rm He} = Y\rho/4m_u$, when $Y$ is the mass fraction of Helium with four nucleons. The number density of the electron, $n_e$ is written separate and should only consider the number of *free* electrons in the gas. In the case of only fully ionized ${}^4_2$He, $n_e = 2n_{{}^4_2\rm He}$ because each ionized Helium core gives two electrons. A similar consideration should be taken for each species present in the gas.

Calculating $n_{\rm tot}$ is not necessarily so straight forward, but if we combine all the steps we have to take to compute $\mu$, we can end up with a convenient formula:

$$\mu = \frac{1}{\sum p_i X_i / C_i}. \tag{8}$$

where $p_i$ is the number of particles provided per nucleus (i.e. the core plus the number of ionized electrons).

This is the method used in the code, as a framework for summing over all particles is central to the structure of the code base.

Let's now return to the original problem of finding an extra equation. Eq. (5) gives us an additional equation, but it also introduces a new parameter, $P_g$. In addition to the gas pressure, we have radiation pressure $P_{\rm rad}$, so that the total pressure is $P = P_g + P_{\rm rad}$. Luckily, the expression for radiation pressure is quite simple, and depends only on temperature,

$$P_{\rm rad} = \frac{a}{3} T^4 \tag{9}$$

where $a = 4\sigma/c$ is a constant. This gives us finally the additional equation of state we need;

$$P = \frac{\rho}{\mu\, m_u} k_B T + \frac{a}{3} T^4 \tag{10}$$

$$\Leftrightarrow \rho = (P - \frac{a}{3}T^4)\frac{\mu\, m_u}{k_B T}. \tag{11}$$

I have listed the equation with respect to density as well; this relation will be used to find $P$ given $\rho$ and $T$, and vice versa.

## II.2 Energy Production

In Eq. (3), the quantity $\epsilon$ represents the amount of energy produced per time and mass. We derive the value of $\epsilon$ by looking at the reactions that produce energy inside a star. It is given by

$$\epsilon = \sum_{ij} r_{ij} Q_{ij}, \tag{12}$$

where $r_{ij}$ is the reaction rate for particles $i$ and $j$ (per time and mass), and $Q_{ij}$ is the energy produced for each such reaction. The exact values of $Q_{ij}$ can be found in the lecture notes [2], and also listed in the source code [1, `reaction_energies.h`]. More interesting is the rates, which are given as

$$r_{ij} = \frac{n_i n_j}{\rho(1+\delta_{ij})} \lambda_{ij} \tag{13}$$

where $n_i$ is the number density of particle $i$, and $\delta_{ij}$ is the Kronecker delta. $\lambda_{ij}$ is a complicated function of temperature, and we will use tabulated expressions for these functions, found once again in the lecture notes [2, Table 3.1].

There are a few different ways that fusion can happen, depending on the conditions of the star. In our case we will simplify things a bit and only consider the PPI and PPII chains (Proton-Proton based fusion).

One important thing to mention when calculating $\epsilon$ is that no reaction should happen more often (have larger value for $r_{ij}$) than the reaction(s) that produced the reactant(s) of the first reaction. For instance, in the PPII we have the two steps

$${}^3_2\text{He} + {}^4_2\text{He} \to {}^7_4\text{Be} \tag{14}$$

$${}^7_4\text{Be} + e^- \to {}^7_3\text{Li} + \nu_e. \tag{15}$$

Here, the second step can happen no more often than the first step; if this was not the case, the Beryllium would quickly disappear and we end up using Beryllium that does not exist. A similar considerations should be made to all dependencies of each step in the cycles.

The last assumption we add is that all Deuterium produced by Proton-Proton fusion, immediately fuses to ${}^3_2$He, thus effectively merging the two first steps in the PP chains into one reaction.

# III Assumptions

Up until this point, we've made several assumptions. In order to remember the limitations of our results, I have summarized them in a list:

- No time evolution, we only look at one moment in time.
- The gas of the star behaves as an ideal gas.
- All energy transport is done through radiation.
- Energy production:
  - Uniform and constant mass fractions. We use fixed values for all the mass fractions, independent of radius, and assume that they do not change.
  - Metals are not ionized at all, and all non-metals are fully ionized.
  - Only the PPI and PPII reaction chains produce energy.
  - Deuterium is produced and consumed at the same time, combining the first two steps of the PP chains into a single step.

# IV Solving the Equations

At this point we are ready to start solving the governing equations. For simplicity, I have chosen to deploy the standard Forward Euler method. One should then note that the simplicity comes at the cost of an global error proportional to the step size, $dm$. We will try to be smart about this step size in order to minimize the effects of this.

As a reminder, Forward Euler solves a differential equation $du/dx = f(u, x)$ by the explicit scheme:

$$u_{i+i} = u_i + du = u_i + f(u_i, x_i)\, dx.$$

In our case, with four differential equations, this corresponds to the following:

$$\begin{aligned}
r_{i+1} &= r_i + \frac{1}{4\pi r_i^2 \rho_i}\, dm \\
P_{i+1} &= P_i - \frac{Gm_i}{4\pi r_i^4}\, dm \\
L_{i+1} &= L_i + \epsilon\, dm \\
T_{i+1} &= T_i - \frac{3\kappa L_i}{256\pi^2 \sigma r_i^4 T_i^3}\, dm \\
m_{i+1} &= m_i + dm
\end{aligned}$$

I added the updating of the mass $m(r_i) = m_i$ for clarity.

## IV.1 Choosing a Suitable Step Size

Whenever a numerical solution to a differential equation is attempted, one has to think about what step size is suitable. In a perfect world, we would like to set it to an infinitesimal number, but the constraints of reality forces us to make a compromise between accuracy and speed. We need to find a value for $dm$ that gives an accurate enough result, but does so in a reasonable time.

In order to evaluate the effect of different step sizes, figure 1 shows $r(m)$ calculated for a number of different values of $dm$[3]. From the figure we can see a large variation in $r(m)$ for the lower values of $dm$. The solution seems to converge when $dm \geq M_0 \times 10^{-3}$, but even there it is not perfectly aligned. It turns out that the code runs very fast anyway (in large part because it is written in C++), so it is no problem to crank the step size down to $dm = M_0 \times 10^{-5}$, and still finish in under a second.

This is quite neat, but we could still try to be a bit smarter about this than just using brute force. The downside to setting a fixed value for $dm$ is that we might spend a lot

[3] The shape and values of $r(m)$ in figure 1 are not important. The graph is only meant to illustrate the different step sizes. Analysis of the shapes and input parameters will be discussed in detail later.

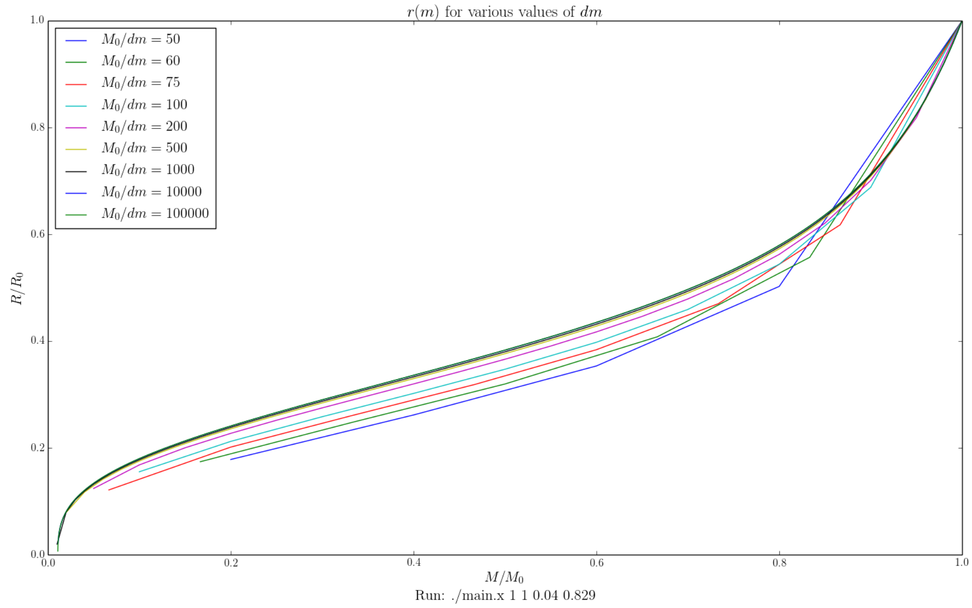


Figure 1: Plots of $r(m)$ using a range of different values for $dm$. We see that we need quite a large number of steps in order for the solution to converge.

of computational time on a part of our function(s) that is very stable, and on the other side, we might not have enough precision for where the function(s) is very steep. The alternative is *dynamic step size*, which involves defining the step size in such a way that the value of the function(s) does not change by more than a fixed percentage. In the case of a diff.eq. $du/dx = f(u, x)$, we define the step size in the following way:

$$\frac{du}{u} \leq p \Leftrightarrow \frac{f\,dx}{u} \leq p \tag{16}$$

$$\Rightarrow dx = \frac{pu}{f}, \tag{17}$$

where $p$ is the fraction that the solution is allowed to change by after one step. In our case we have more than one equation, so we impose a similar requirement to all equations and then simply choose the smallest required step size.

The only thing we then have to define is what the value of $p$ should be. As previously stated, the code is quite fast so we can afford to be quite strict. The code used in all subsequent figures is set to allow a maximum change of 5‰. Figure 2 shows the same plot as figure 1, but this time including dynamic step size as well. The figure shows that DSS gives good results.

# V Changing the Parameters

In this numerical model, we have the option to change the initial parameters to our liking. In particular, we would like to find the set of initial parameters that give the most realistic model of an actual star. To this end we are now going to look at how the solution is affected by changing $R_0, M_0, P_0$ etc. (In the following, $u_{0,i}$ will denote the default initial condition of quantity $u$, as defined in the exercise text. The defaults correspond to the values for the Sun at the bottom of the convection layer.)

## V.1 Changing $R_0$

In order to see the effects of changing the initial radius, figure 3 has been produced. From the figure we can conclude with the following for increasing initial radius:

- $m$ dies off faster
- $P$ becomes *much* smaller

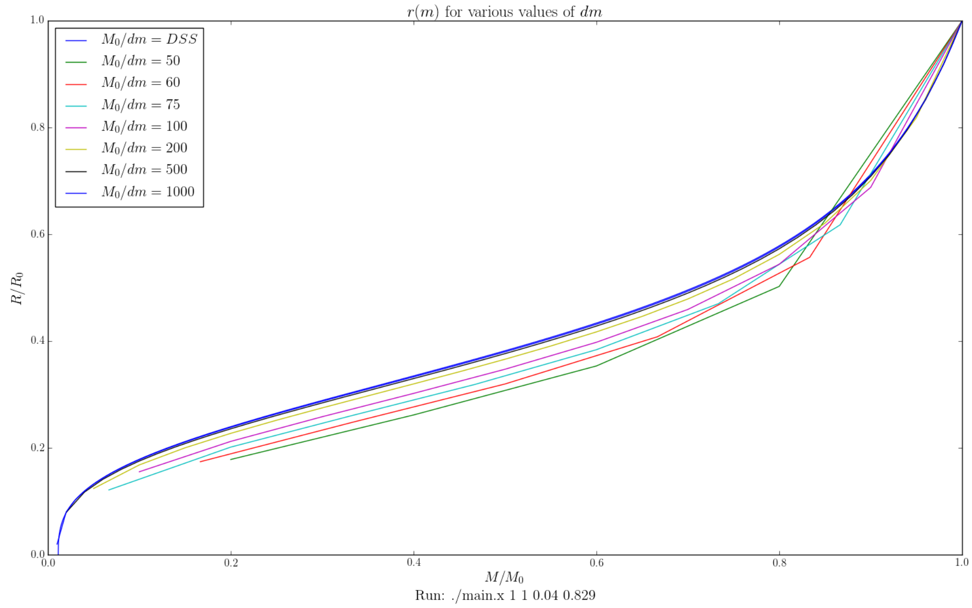


Figure 2: Plots of $r(m)$ using a range of different values for $dm$, including the solution obtained when using dynamic step size.

- $L$ remains constant for much longer (the core becomes a smaller fraction of the star)
- $T$ becomes smaller
- $\rho$ is similar to $T$, only that $\rho$ becomes *much* smaller with increasing $R_0$

These effects are all reasonable given an increase in the size of the star. We can compare this to what we know happens to a star towards the end of its life cycle, where it becomes a red giant. In that case, the radius increases, causing the surface temperature to drop (thus glowing more red). The core has no reason to expand or contract, so it becomes a smaller fraction of the total radius. Finally, with a constant total mass, it seems reasonable that the density would drop significantly, along with the pressure (all in accordance with Eq. (10)).

## V.2 Changing $T_0$

Figure 4 shows a similar comparison for a change in the initial temperature $T_0$. We conclude that for increasing initial temperature we have that:

- $L$ falls of more rapidly, causing the core to expand
- $m$ falls off more towards the center (larger core)
- $T$ is smaller relative to $T_0$, which hints at a more or less unchanged temperature graph.

The main change caused by increasing $T_0$ seems to be a larger core. This makes sense; a higher temperature in the outer parts would imply that the temperatures needed for fusion to happen, are present further out in star.

The plots for $\rho$ and $P$ also seem to comply with a more centrally dense star.

## V.3 Changing $\rho_0$

Figure 5 shows what happens to the solutions when we vary the initial density. We conclude that for increasing initial density we have that:

- $m$ falls off more rapidly
- $\rho$ and $P$ becomes much smaller compared their initial values.

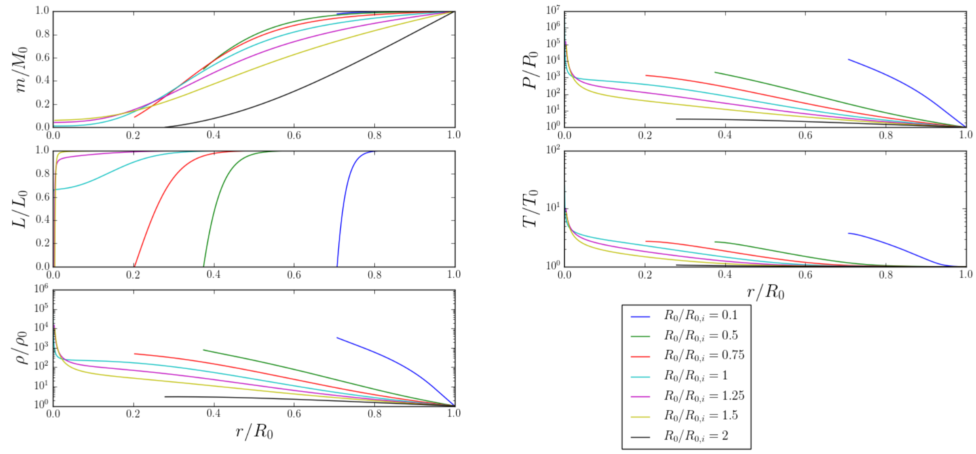


Figure 3: Plots for all parameters of the star showing the effects of changing the initial radius $R_0$.

The absolute main thing we gather from this figure is that the mass drops of more rapidly, which of course makes sense given a larger density further out in the star.

The other graphs are quite hard to extract anything else from given that the solution becomes unphysical very fast for large $\rho_0$'s. I will therefore not give much time to interpret the result for i.e. $P$, because they might as well be caused by very strange behaviours of other parameters.

### V.4 Changing $P_0$

Due to the connection between $P, \rho$ and $T$ shown in Eq. (10), the effects of changing $P_0$ is included in the above sections. In fact, the code sets the initial value for $P$ based on the values for $\rho$ and $T$. Producing a similar plot for variations in $P_0$ would be equivalent of changing $\rho_0$ or $T_0$, or some arbitrary combination of the two according to Eq. (10).

### V.5 Changing $M_0$

Figure 6 shows the solutions for a range of total masses. For increasing total mass we conclude that:

- Both $\rho$ and $P$ becomes *much* bigger in the outer parts, but get caught up by the smaller $M_0$'s towards the core
- $L$ starts to decline earlier (core becomes bigger), but it's not clear how increasing $M_0$ effects where $L \to 0$
- Low $M_0$'s causes $m$ to fall of faster in the beginning and slower towards the center, and the opposite is true for higher $M_0$'s

## VI Finding the most realistic model

Our end goal with this project was to determine how the parameters of the star evolves as a function of radius, given a set of initial conditions. As seen, what we chose to set as these conditions can have a great impact on the final solution. In particular, many of the conditions we have tried yielded very unphysical results. There are certain things we know need to happen at the center of the star, namely that the mass and luminosity both need to be zero (for $r \to 0$). We can therefore set out to find a combination of parameters that best

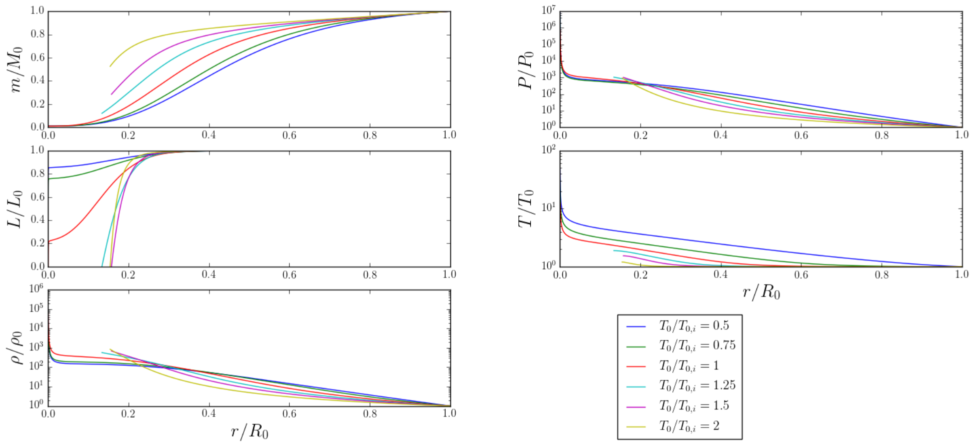


Figure 4: Plots for all parameters of the star showing the effects of changing the initial temperature $T_0$.

fulfills this requirement, using what we learned from the previous section.

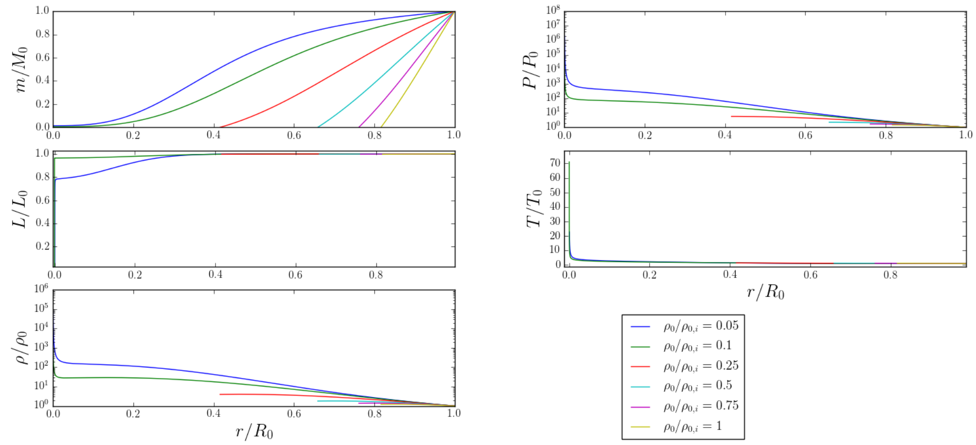


Figure 5: Plots for all parameters of the star showing the effects of changing the initial density $\rho_0$.

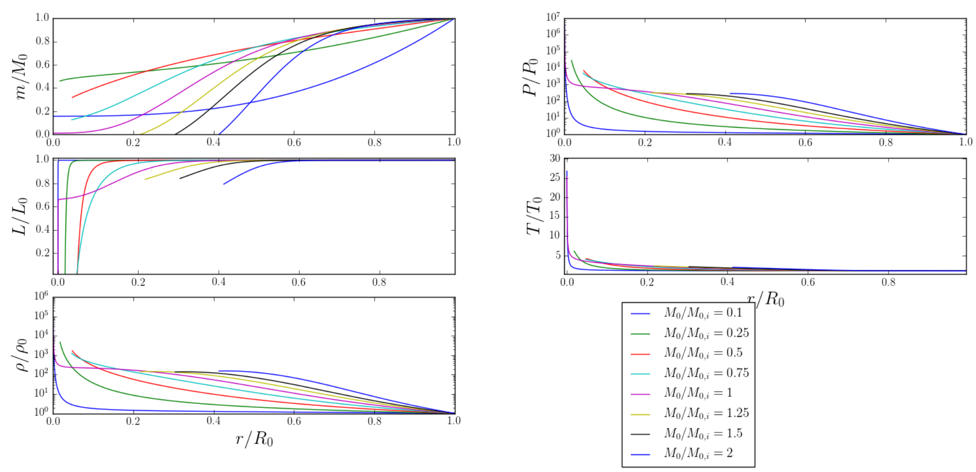


Figure 6: Plots for all parameters of the star showing the effects of changing the total mass $M_0$.

## References

[1] B. Samseth. GitHub repository. URL: `https://github.com/bsamseth/ast3310`.

[2] B. V. Gudiksen. *AST3310: Astrophysical plasma and stellar interiors*. 2016. URL: `http://www.uio.no/studier/emner/matnat/astro/AST3310/v16/noter/appd29-03-16.pdf` (visited on 03/31/2016).