Path Integrals with Linking-Number Weighted Paths

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Path Integrals - An Introduction

Path integrals are a powerful tool in quantum mechanics as they give us the probability amplitude for a particle to move from an initial state to a final state. This is calculated by summing over all possible paths the particle could take, each weighted by the action. An example of such a path integral is [1]

$$K(q_i, 0; q_f, T) = \int_{q(0)=q_i}^{q(T)=q_f} \mathcal{D}q e^{\frac{i\mathcal{S}[q]}{\hbar}} \quad \text{Where,} \quad \mathcal{S}(q) = \int_{t_i=0}^{t_f=T} \mathcal{L}(q)dt \tag{1}$$

In some cases the paths are further weighted by something called a linking number. The linking number (LN) is a measure of how intertwined two strands of a closed curve are. Some of these cases include the Aharonov-Bohm, polymer entanglement and more. [2]

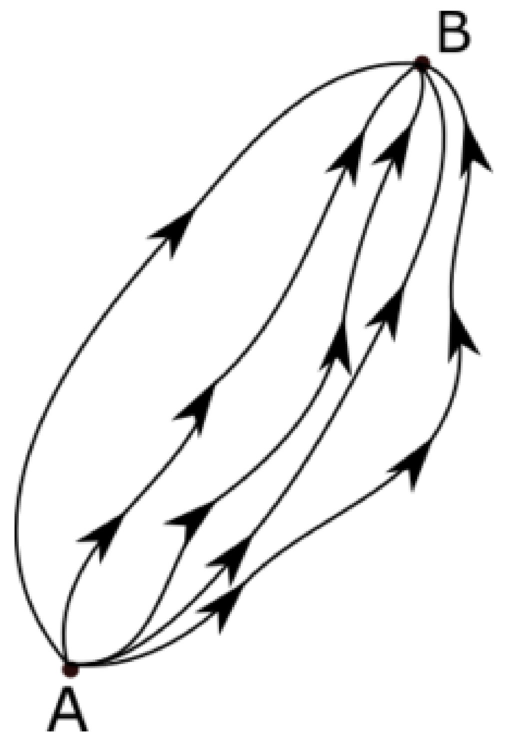

Fig. 1: Paths a particle could take from A to B [3]

Aharonov-Bohm Effect

To briefly summarize the results of this effect, an external vector potential induces a measurable phase shift (AB phase) in a charged particles wave function, even when the particle is in a region where the magnetic field is 0 [4]. Meaning that a once purely mathematical concept has an effect on the physical quantum world.

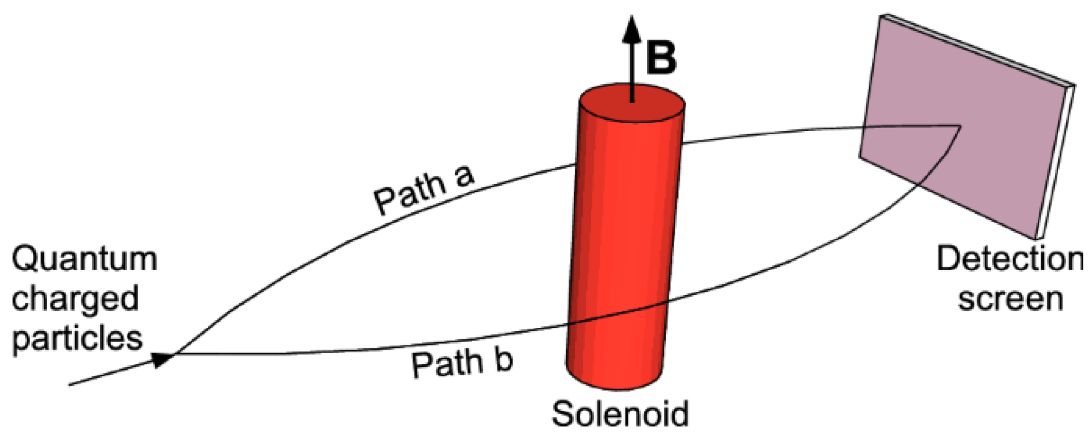

Fig. 2: Local description of the Aharonov-Bohm effect [5]

- AB phase is proportional to the LN between the paths of electrons and the magnetic flux lines [7]
- The phase shift describes of each path describes the interference pattern, and hence gives insight to the probability of the particle reaching a certain point on the screen

Formulation via Path Integrals

Consider a solenoid with electrons being fired at it (Fig. 2). We have an external field with vector potential $\boldsymbol{A(r)}$. The Lagrangian for this system will include the vector potential term. [2]

It can be found that after we substitute this Lagrangian into (1) that there will be a term in the exponential (below),

$$\int_{t_i}^{t_f} \dot{\boldsymbol{r}} \cdot \boldsymbol{A} dt = \int_C \boldsymbol{A} \cdot d\boldsymbol{r} = \int_{C_0} \boldsymbol{A} \cdot d\boldsymbol{r} + n\Phi \tag{2}$$

Where C is the path of the electron. The final equality comes from properties of $\boldsymbol{A(r)}$, we have that

$$\oint \boldsymbol{A} \cdot d\boldsymbol{r} = \Phi \tag{3}$$

for every contour which winds around an element of D - the solenoid once. Hence for n, (the linking number) loops around the solenoid we have (2). With C_0 as some standard connecting contour. We have shown that the paths are weighted by both the action and the linking number.

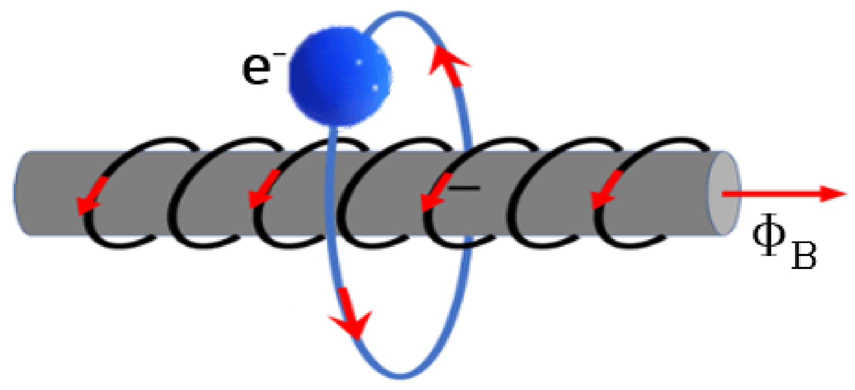

Fig. 3: Schematic of Aharonov-Bohm effect, showing the loop of (3) [6]

Polymer Entanglement

This phenomenon occurs when two or more polymer chains become intertwined with each other forming a network of knots and links.

In this case the paths are those which connect the ends of the polymer chains in the system. The weight of which is based on the linking number of the chains it passes through, creating knots. [2],

By performing the path integral, the partition function of the polymer system, which gives us the probability distribution of different entangled states can be calculated.[8]

Fig. 4: Examples of knots the polymers can find themselves in [9]

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