# Path Integrals and Instantons

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## Introduction

- Feynman's path integral approach considers the 'sum over all histories' of a quantum system i.e. integrating over all infintely many possible trajectories between two positions, or two states.
- This approach can be extended to fields, by integrating over all configurations of fields, that take us from an initial arrangement to a final one.
- This is incredibly useful and underpins much of quantum field theory, including the existence of instantons.

## Instantons

- Instantons describe how **fields** 'tunnel' through quantum barriers, which is analogous to quantum particle tunneling.
- A vacuum/field can 'tunnel' from one configuration to another even if classically it would lack the energy to do so.
- By definition, a vacuum has minimal potential energy, and yet it can tunnel to another configuration of a vacuum energy, through a double well potential barrier for example
- Figure 1 depicts the simplified one-dimensional version of this where the field could tunnel from the unstable minima to the stable for example
- Instantons are the solutions to the corresponding classical equations of motion but with imaginary time in the action and are given this name because they describe how at an arbitrary time, the field 'instantly' jumps from one vacuum to another
- The path integral can be used to demonstrate the mathematics
- This mathematical method can be extended to field solutions rather than potential wells where the solutions are given in Euclidean time, not real time. [2]

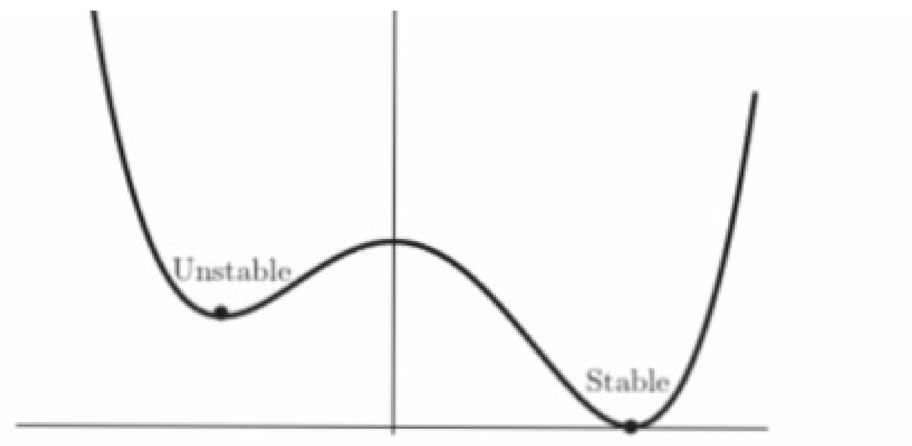


Figure 1: Showing an example potential energy with two separated minima

## Modern research

Instantons and their existence within quantum field theory is a very current and exciting area of research in theoretical physics.
As stated above, the Yang-Mills mass gap problem is a Millenium prize problem, but there is also a problem to solve for atomic nuclei energy levels, in order to use Quantum Field Theory to explain nuclear physics.

## Wick rotations and Euclidean Time

The path-integral formula can be adapted using a 'Wick Rotation'. This is a transformation in time from t -> -it which can be thought of as a $\pi/2$ rotation in the complex plane.

$$\int D(x)e^{\frac{iS}{\hbar}} \Rightarrow t \rightarrow -i\tau \Rightarrow \int dxD(x)e^{\frac{-S}{\hbar}} \tag{1}$$

This is known as making the path integral 'Euclidean', as opposed to 'Minkowski'. It is analytically continuing the integral into imaginary time. Interestingly, this method can be used to derive the statistical partition function equation. [1]

## The mathematics behind Instantons

Let's start with the path integral formula, and perform a wick rotation:

$$\int D(x)e^{\frac{iS}{\hbar}} \Rightarrow t \rightarrow -i\tau \Rightarrow \int dxD(x)e^{\frac{-S}{\hbar}} \tag{2}$$

Then manipulate the action by also changing it to imaginary time:

$$S = i\int dt(\frac{1}{2}(\frac{dx}{dt})^2 - V(x)) \tag{3}$$

$$S = -\int d\tau(\frac{1}{2}(\frac{dx}{dt})^2 + V(x)) \tag{4}$$

This is like considering an inverted potential energy. [3] Taking V(x) to be a double-well potential, for example, we end up with the solution:

$$x = a\tanh b(\tau - \tau_0) \tag{5}$$

A similar (but more complicated) calculation can be done with quantum fields, which result in the same concept. The fields can instantly jump from one configuration to another. [4]

## Yang-Mills Theory

The same concept can be applied the Yang-Mills theory, which generalises Maxwell's theory of electromagnetism, to include strong and weak forces. The strong force is described by QCD which can experience the instanton phenomena. [2]
The Yang-Mills theory is crucial to understanding the standard model and currently has many gaps yet to be filled. In fact, the Yang-Mills and mass gap problem (i.e. proving that Yang-Mills theory exists and that the mass of all particles predicted are positive) is a Millennium prize problem! [**Large**]

### References

[1]R. Feynman and A.Hibbs, *Quantum mechanics and path integrals* (McGraw-Hill, 1965), pp. 268–297.

[2]M. B. Paranjape, *The theory and applications of instanton calculations* (Cambridge University Press, 2017).

[3]M. Marino, *Instantons and large n an introduction to non-pertubrative methods in quantum field theory* (Cambridge University Press, 2015).

[4]H. Forkel, "A primer on instantons in qcd", (2002).