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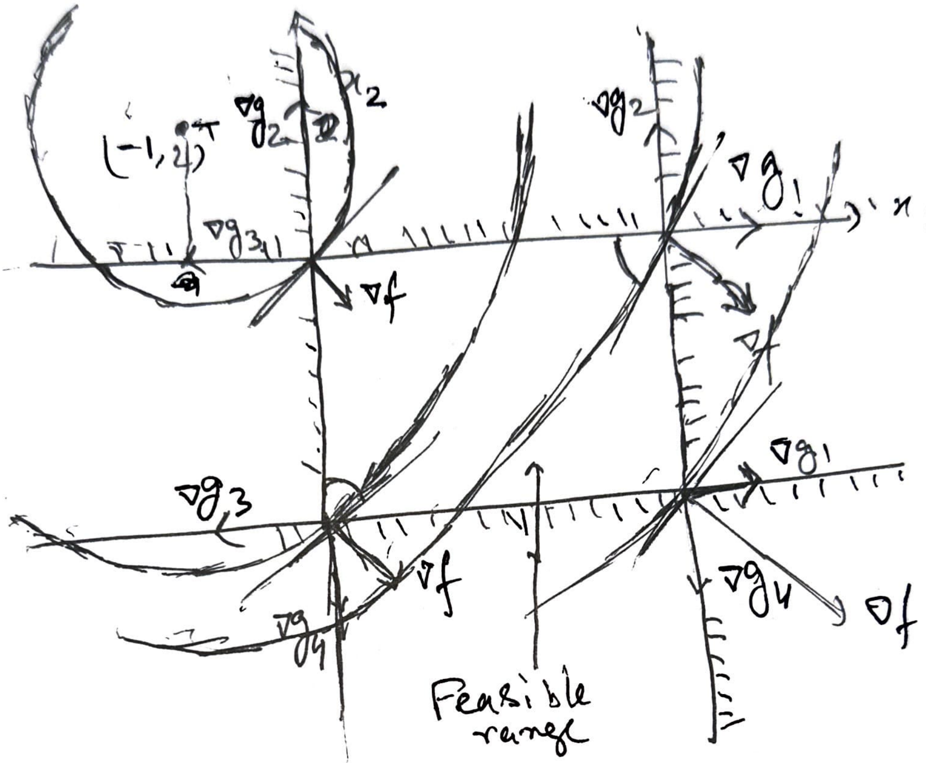
1) $f(x) = (x_1+1)^2 + (x_2-2)^2$

min $f(x)$ s.t $g_1 = x_1 - 2 \le 0$ $g_3 = -x_1 \le 0$

$g_2 = x_2 - 1 \le 0$ $g_4 = -x_2 \le 0$



The solution through the graph shows the optimum at $(0,1)^T$ with $f^* = 2$

At $(0,0)^T, (2,0)^T$ & $(2,1)^T$ feasible directions are available. No such direction for $(0,1)^T$

Applying KKT

$L(x,\mu) = f + \mu g = x_1^2 + 1 + 2x_1 - x_1\mu_3 + x_2^2 + 4 - 4x_2 - x_2\mu_4 + \mu_1(x_1-2) + \mu_2(x_2-1) - \mu_3(x_1)$

$(0,1)^T$ is on g_2 & g_3 $\therefore \mu_1 = \mu_4 = 0$ & $\mu_2 = \mu_3 > 0$

$\therefore \Delta F - \mu^T \nabla g_2 \; 0 \Rightarrow \begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix} + \begin{bmatrix} -\mu_3 \\ \mu_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Bigg|_{(0,1)}$

$\therefore \mu_2 = \mu_3 = 2$

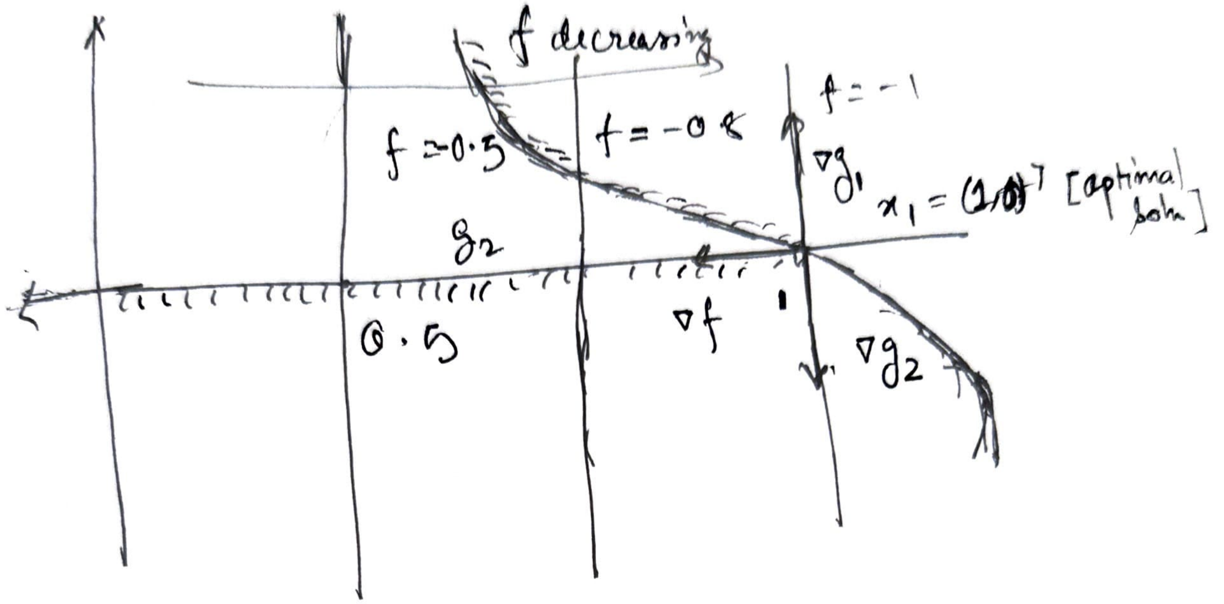
KKT condn satisfied at (0,1)

Hessian of Lagrangian $= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$: +ve definite ~~p.d~~ everywhere

$\therefore$ (0,1) = global minimum

2) min $f = -x_1$

s.t $g_1 = x_2 - (1-x_1)^3 \le 0$ & $x_2 \ge 0$ ∴ $-x_2 \le 0$



from graph, soln is $(1,0)$

$$L(x, u) = f(x) + u g(x)$$

$$= -x_1 + u_1[x_2 - (1-x_1)^3] - u_2(x_2)$$

$$= -x_1 + u_1 x_2 - u_1(1-x_1)^3 - u_2(x_2)$$

Taking $\frac{\partial L}{\partial x_i} = 0$

$$= \begin{bmatrix} -1 + 3u_1(1-x_1)^2 \\ u_1 - u_2 \end{bmatrix}$$

$\frac{\partial L}{\partial x}(1,0) = [\cancel{1}$

$u_1[x_2 - (1-x_1)^3] = 0$ $\quad u_1 \ge 0$ (1)

$u_2 \ge 0$ (2)

$-u_2 x_2 = 0$

at $(x_1^*, x_2^*) = (1,0)$

But solving (1) & (2) gives $u_1 = u_2$ & $\boxed{-1 = 0}$ Contradict

∴ Not a KKT point as not a regular point, ∴ No soln

3) using Lagrangian multipliers:

$$\lambda^T = -\left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial h}{\partial s}\right)^{-1}$$

$$\frac{\partial \mathcal{L}}{\partial d} = \frac{\partial f}{\partial d} + \lambda^T \left(\frac{\partial h}{\partial d}\right)$$

$$0 = \frac{\partial f}{\partial s} + \lambda^T \frac{\partial h}{\partial s}$$

$$\Rightarrow \frac{\partial f}{\partial x} + \lambda^T \frac{\partial f}{\partial x} = 0$$

general eqn at $\frac{\partial f}{\partial d} = 0$

$$h(x, \lambda) = -f(x) + \lambda^T h(x)$$

$$h(x_1, x_2, x_3, \lambda) = -f + \lambda h \quad (\text{max})$$

$$= -x_1x_2 - x_2x_3 - x_3x_1 + \lambda(x_1 + x_2 + x_3 - 3)$$

$$\frac{\partial L}{\partial x} = \begin{pmatrix} -x_2 - x_3 + \lambda \\ -x_1 - x_2 + \lambda \\ -x_1 - x_3 + \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $x_1^* = x_2^* = x_3^* = 1$ $\quad \lambda = 2$ $\quad$ & $f^* = 3$ is a global maxima.

#

4) min $F = x_1^2 + x_2^2 + x_3^2$

st $h_1 = x_1^2/4 + x_2^2/5 + x_3^2/25 - 1 = 0$

$h_2 = x_1 + x_2 - x_3 = 0$

let $d = x_1$

$s = [x_2, x_3] \neq (0, 0)$

$\frac{\partial f}{\partial d} = 2x_1$

$\frac{\partial f}{\partial s} = \begin{bmatrix} 2x_2 \\ 2x_3 \end{bmatrix}$

$\frac{\partial h}{\partial s} = \begin{bmatrix} 2x_2/5 & 2x_3/25 \\ 1 & -1 \end{bmatrix}$

$\frac{\partial h}{\partial d} = \begin{bmatrix} x_1/2 \\ 1 \end{bmatrix}$

5) Cities be labeled $1, 2, \ldots, n$ & ~~$0 < |C_{ij}| < \infty$~~

$0 < |C_{ij}| < \infty$ if ~~edge~~ exists

$|C_{ij}| = \infty$ if edge does not exist & $\min \sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} c_{ij} x_{ij}$

This can be reformulated as

$0 < |C_{ij}| < \infty$ & $x_{ij} = \begin{cases} 1 & \text{if edge exists} \\ 0 & \text{otherwise} \end{cases}$

Minimize tour cost $\sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} c_{ij} x_{ij}$

$\sum_{i=1, i \neq j}^{n} x_{ij} = 1 \quad \forall j = 1, 2, \ldots, n$ [only one edge going out of a node]

$\sum_{j=1, j \neq i}^{n} x_{ij} = 1$ for $i = 1, 2, \ldots, n$ [only 1 edge going in]

$\sum_{i \in S} \sum_{j \in S, j \neq i} x_{ij} \leq |S| - 1 \quad \forall S \subseteq \{1, 2, \ldots, n\}$, $|S| \geq 2$

This ensures no proper subset of Q can form a subtour (hamiltonian cycle)

This can be rewritten as

$c(S, j) = \min c(S - \{j\}, i) + c(i, j)$

where $i, j \in S$ & $j \neq i$