HOMEWORK

(1) Let $Y = X^{2n}$ for some positive integer n, where $X \sim \mathcal{N}(0,1)$. Prove that
  (a) X and Y are uncorrelated, i.e. $\mathbb{E}[XY] = 0$.
  (b) X and Y are not independent.

(2) Let X and Y be random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

  (a) Find the marginal densities $f_X(x)$ and $f_Y(y)$.
  (b) Are X and Y independent? Justify your answer.
  (c) Compute $\mathrm{Cov}(X,Y)$ and $\mathrm{Corr}(X,Y)$.

(3) Consider two random variables X and Y such that:

$$X \sim \text{Uniform}(-1,1)$$

$$Y = \begin{cases} 1 & \text{if } |X| < 0.5 \\ -1 & \text{if } |X| \ge 0.5 \end{cases}$$

  (a) Show that $\mathbb{E}[X] = 0$ and $\mathbb{E}[Y] = 0$.
  (b) Compute $\mathrm{Cov}(X,Y)$ and show that X and Y are uncorrelated.
  (c) Are X and Y independent? Explain why or why not.

(4) Suppose that the current daily volatilities of asset A and asset B are 1.6% and 2.5%, respectively. The prices of the assets at close of trading yesterday were \$20 and \$40 and the estimate of the coefficient of correlation between the returns on the two assets made at that time was 0.25. The parameter λ used in the EWMA model is 0.95.
  (a) Calculate the current estimate of the covariance between the assets.
  (b) On the assumption that the prices of the assets at close of trading today are \$20.50 and \$40.50, update the correlation estimate.

(5) Let (X,Y) follow a bivariate normal distribution with parameters $\mu_X = 0$, $\mu_Y = 0$, $\sigma_X^2 = 1$, $\sigma_Y^2 = 4$, and correlation coefficient $\rho = 0.6$. Find $\mathbb{P}(Y > 2X + 1)$.

Solutions

(1)
- To show that X and Y are uncorrelated, we compute:
$$\mathbb{E}[XY] = \mathbb{E}[X \cdot X^2] = \mathbb{E}[X^3]$$
Since $X \sim \mathcal{N}(0,1)$, all odd moments are zero, thus $\mathbb{E}[X^3] = 0$. Therefore, $\mathbb{E}[XY] = 0$ and X and Y are uncorrelated.
- To show that X and Y are not independent,
$$\mathbb{P}(X \leq 1)\mathbb{P}(Y \leq 1) < \mathbb{P}(Y \leq 1) = \mathbb{P}(X \leq 1, Y \leq 1).$$

(2) (a) To find the marginal densities, we integrate over the appropriate regions:
For $f_X(x)$: For $0 \leq x \leq 1$,
$$f_X(x) = \int_x^1 2\,dy = 2(1-x)$$
So $f_X(x) = \begin{cases} 2(1-x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$
For $f_Y(y)$: For $0 \leq y \leq 1$,
$$f_Y(y) = \int_0^y 2\,dx = 2y$$
So $f_Y(y) = \begin{cases} 2y & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

(b) X and Y are not independent because $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$. For example, at $(x,y) = (0.5, 0.5)$:
$$f_{X,Y}(0.5, 0.5) = 2 \neq 1 \cdot 1 = f_X(0.5)f_Y(0.5)$$

(c) First, we compute the means:
$$\mathbb{E}[X] = \int_0^1 x \cdot 2(1-x)\,dx = 2\int_0^1 (x - x^2)\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$
$$\mathbb{E}[Y] = \int_0^1 y \cdot 2y\,dy = 2\int_0^1 y^2\,dy = \frac{2}{3}$$
For $\mathbb{E}[XY]$:
$$\mathbb{E}[XY] = \int_0^1 \int_x^1 xy \cdot 2\,dy\,dx = 2\int_0^1 x\left[\frac{y^2}{2}\right]_x^1 dx = \int_0^1 x(1-x^2)\,dx = \frac{1}{4}$$
Therefore:
$$\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{4} - \frac{1}{3} \cdot \frac{2}{3} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$$
For the correlation, we need $\text{Var}(X)$ and $\text{Var}(Y)$:
$$\mathbb{E}[X^2] = \int_0^1 x^2 \cdot 2(1-x)\,dx = \frac{1}{6}, \quad \text{Var}(X) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$$

$$\mathbb{E}[Y^2] = \int_0^1 y^2 \cdot 2y\, dy = \frac{1}{2}, \quad \mathrm{Var}(Y) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

Thus: $\mathrm{Corr}(X,Y) = \frac{1/36}{\sqrt{1/18 \cdot 1/18}} = \frac{1/36}{1/18} = \frac{1}{2}$

(3) (a) Since $X \sim \text{Uniform}(-1,1)$, we have $\mathbb{E}[X] = 0$ by symmetry.
For $\mathbb{E}[Y]$:

$$\mathbb{P}(Y=1) = \mathbb{P}(|X| < 0.5) = \mathbb{P}(-0.5 < X < 0.5) = \frac{1}{2}$$

$$\mathbb{P}(Y=-1) = \mathbb{P}(|X| \geq 0.5) = \frac{1}{2}$$

Therefore: $\mathbb{E}[Y] = 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$

(b) To compute $\mathrm{Cov}(X,Y) = \mathbb{E}[XY]$ (since both means are 0):

$$\mathbb{E}[XY] = \mathbb{E}[X \cdot 1 \cdot \mathbf{1}_{|X|<0.5}] + \mathbb{E}[X \cdot (-1) \cdot \mathbf{1}_{|X|\geq 0.5}]$$
$$= \int_{-0.5}^{0.5} x \cdot \frac{1}{2}\, dx - \int_{-1}^{-0.5} x \cdot \frac{1}{2}\, dx - \int_{0.5}^{1} x \cdot \frac{1}{2}\, dx = 0$$

By symmetry, each integral equals zero, so $\mathrm{Cov}(X,Y) = 0$.

(c) X and Y are not independent. For example:

$$\mathbb{P}(X \in (0.6, 0.7), Y = 1) = 0$$

but

$$\mathbb{P}(X \in (0.6, 0.7)) \cdot \mathbb{P}(Y=1) = 0.1 \cdot 0.5 = 0.05 > 0$$

(4) (a) The volatilities and correlation imply that the current estimate of the covariance is $0.25 \times 0.016 \times 0.025 = 0.0001$.

(b) If the prices of the assets at close of trading are \$20.50 and \$40.50, the proportional changes are $0.5/20 = 0.025$ and $0.5/40 = 0.0125$. The new covariance estimate is:

$$0.95 \times 0.0001 + 0.05 \times 0.025 \times 0.0125 = 0.0001106$$

The new variance estimate for asset A is:

$$0.95 \times 0.016^2 + 0.05 \times 0.025^2 = 0.00027445$$

so that the new volatility is $\sqrt{0.00027445} = 0.0166$.
The new variance estimate for asset B is:

$$0.95 \times 0.025^2 + 0.05 \times 0.0125^2 = 0.000601562$$

so that the new volatility is $\sqrt{0.000601562} = 0.0245$.
The new correlation estimate is:

$$\frac{0.0001106}{0.0166 \times 0.0245} = 0.272$$

(5) Note that $Cov(X,Y) = \rho\sigma_X\sigma_Y = 0.6 \cdot 1 \cdot 2 = 1.2$. Hence, $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N(\mathbf{0}, \begin{bmatrix} 1 & 1.2 \\ 1.2 & 4 \end{bmatrix})$
So $Z = -2X + Y \sim N(0, 4 + 4 - 4.8) = N(0, 3.2)$.

$$\mathbb{P}(Z > 1) = \mathbb{P}(\frac{Z-0}{\sqrt{3.2}} > \frac{1-0}{\sqrt{3.2}}) = 1 - \Phi\left(\frac{1}{\sqrt{3.2}}\right) = \Phi\left(-\frac{1}{\sqrt{3.2}}\right)$$