Copula Applications - Vasicek's Model

Outline

Overview

- Vasicek's model applies the one-factor Gaussian copula to credit risk.
- Used in Basel II for calculating capital requirements.
- Models default correlation in large loan portfolios.

Annual Default Rates (1970-2013)

What is the good estimate of default rate?

Year	Rate (%)	Year	Rate (%)	Year	Rate (%)
1970	2.621	1985	0.960	2000	2.852
1971	0.285	1986	1.875	2001	4.345
1972	0.451	1987	1.588	2002	3.319
1973	0.453	1988	1.372	2003	2.018
1974	0.274	1989	2.386	2004	0.939
1975	0.359	1990	3.750	2005	0.760
1976	0.175	1991	3.091	2006	0.721
1977	0.352	1992	1.500	2007	0.401
1978	0.352	1993	0.890	2008	2.252
1979	0.088	1994	0.663	2009	6.002
1980	0.342	1995	1.031	2010	1.408
1981	0.162	1996	0.588	2011	0.890
1982	1.032	1997	0.765	2012	1.381
1983	0.964	1998	1.317	2013	1.381
1984	0.934	1999	2.409		

Mathematical Setup

- T_i as the time when company i defaults, $i = 1, \ldots, n$ (large).
- Assume T_i has the same distribution,
- Define the probability of default (PD) by time $T = 1$:

$$p = \mathrm{PD} = \Pr(T_i < T).$$

- Goal: Good estimate of portfolios loss rate:

$$LR = \frac{\#\text{ of defaults}}{n}$$

Extreme cases

- Let $p = 2\%$.
- Comonotonic case: If $T_1 = \ldots = T_n$, then

$$\begin{aligned}\mathbb{P}(LR \leq 10\%) &= \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{T_i<T\}} \leq 10\%\right)\\ &= \mathbb{P}\left(\mathbf{1}_{\{T_1<T\}} \leq 0.1n\right) = 98\%.\end{aligned}$$

- Independent case: If T_i's are independent, then

$$\begin{aligned}\mathbb{P}(LR \leq 10\%) &= \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{T_i<T\}} \leq 10\%\right)\\ &\approx \mathbb{P}(.02 \leq .1) = 100\%.\end{aligned}$$

- Realistic case: T_i's are dependent, but not comonotonic.

One-factor Gaussian Copula model

- Gaussian Copula is assumed for the correlation structure of T_is, i.e.

$$T_i \sim F^{-1}(\Phi_\Sigma(Z_i)),$$

  where $Z = (Z_1, \ldots, Z_n) \sim N(0, \Sigma)$ and F is the cdf of T_i.

- One-factor model:

$$Z_i = \sqrt{\rho} Y + \sqrt{1-\rho}\epsilon_i,$$

  where $Y, \epsilon_i \sim N(0,1)$ are independent.

Analysis of One-factor Gaussian Copula model

- Let $c = \Phi^{-1}(p)$ be the default threshold.
- Each obligor i 'defaults' if $Z_i < c_i$,

$$p = \mathbb{P}(Z_i < c).$$

- Portfolio loss rate:

$$LR = \frac{\#\text{ of defaults}}{n} \sim \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{Z_i < c\}} := L.$$

Conditional default probability

- $$P(\text{default}_i | Y = y) = \Phi\left(\frac{c - \sqrt{\rho}y}{\sqrt{1-\rho}}\right) = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)$$

Default rate with Gaussian Copula

- Given $Y = y$, defaults are independent: $P(\text{default}_i | Y = y) = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)$
- Portfolio loss rate $L(y) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{Z_i < c\}}$
- For large n: $L(y) \to \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)$ by LLN
- We assume n is large enough so that the default rate can be approximated by

$$L(Y) \approx \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right)$$

Default Rate distribution with Gaussian Copula

- The loss rate can be approximated by

$$L(Y) \approx \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right).$$

- Therefore, the CDF of L is

$$F_L(x) = P(L < x) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}}\right)$$

Worst Case Default Rate (WCDR)

- For $q \in [0,1]$, $WCDR(q)$, as the default rate, is defined as the rate that will not be exceeded with probability q, i.e.,

$$WCDR(q) = F_L^{-1}(q)$$

  where $F_L^{-1}(q)$ is the quantile function of the default rate distribution.
- The WCDR can be computed as:

$$WCDR(q) = \Phi\left(\frac{\Phi^{-1}(p) + \sqrt{\rho}\Phi^{-1}(q)}{\sqrt{1-\rho}}\right).$$

Example: WCDR in a Loan Portfolio

Suppose that a bank has a large number of loans to retail customers. The one-year probability of default for each loan is 2% and the copula correlation parameter, ρ, in the Vasicek model is estimated as 0.1. Find the worst case default rate (WCDR) for the portfolio at the 99.9% confidence level.

- Given $p = 0.02$ and $\rho = 0.1$, we can compute the WCDR at $q = 0.999$:

$$WCDR(0.999) = \Phi\left(\frac{\Phi^{-1}(0.02) + \sqrt{0.1}\Phi^{-1}(0.999)}{\sqrt{1-0.1}}\right) \approx 0.129$$

- Therefore, the worst case default rate at the 99.9% confidence level is approximately 12.9%.

Estimating PD and ρ by MLE

- The maximum likelihood methods can be used to estimate PD and ρ from historical data on default rates.
- The CDF of L is

$$F_L(x) = P(L < x) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}}\right)$$

- The pdf of L is

$$f_L(x) = \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \exp\left\{\frac{1}{2}\left((\Phi^{-1}(x))^2 - \left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}}\right)^2\right)^2\right\}$$

- The estimates for ρ and PD given by the table are 0.108 and 1.41%, respectively.