Volatility

Outline

Volatility

- Volatility is a statistical measure of the dispersion of returns for a given security or market index.
- It is commonly defined as the standard deviation of returns.
- High volatility indicates a greater range of potential returns, while low volatility suggests a more stable investment.

Definition of Volatility

- Suppose that S_i is the value of a variable on day i. The volatility per day is the standard deviation of $\ln \frac{S_i}{S_{i-1}}$.
- The annualized volatility is given by:

$$\sigma_{annual} = \sigma_{daily}\sqrt{T}$$

  where T is the number of trading days in a year (typically 252).
- Variance is the square of volatility:

$$\sigma^2_{annual} = \sigma^2_{daily} \cdot T$$

Implied Volatility

- Implied volatility is the market's forecast of a likely movement in a security's price.
- It is derived from the market price of an option using models like Black-Scholes.
- Implied volatility reflects the market's expectations of future volatility and is often used in options pricing.

Implied Volatility Calculation

- The Black-Scholes formula can be rearranged to solve for implied volatility.
- Given the market price of an option, we can use numerical methods to find the implied volatility that matches the market price.
- Implied volatility is not constant; it varies with the strike price and expiration date of the option.

Implied Volatility Example

- Consider a European call option with:
  - Current stock price: $S_0 = \$100$
  - Strike price: $K = \$105$
  - Time to expiration: $T = 0.25$ years (3 months)
  - Risk-free rate: $r = 5\%$
  - Market price of option: $C_{market} = \$2.50$
- **What is the implied volatility?**

Implied Volatility Example (continued)

- Using the Black-Scholes formula:

$$C = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

where $d_1 = \frac{\ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

- We solve numerically for σ such that $C(\sigma) = \$2.50$
- The implied volatility is approximately $\sigma_{implied} = 20\%$ per annum

VIX - The Volatility Index

- The VIX (Volatility Index) is a real-time market index that represents the market's expectations for volatility over the coming 30 days.
- Often called the "fear gauge," it is calculated from S&P 500 index options.
- VIX values typically range from 10 to 80:
  - Low VIX (below 20): Market complacency
  - High VIX (above 30): Market fear and uncertainty
- The VIX tends to spike during market downturns and decline during bull markets.

A table of daily returns:

	Real World (%)	Normal Model (%)
> 1 SD	25.04	31.73
> 2 SD	5.27	4.55
> 3 SD	1.34	0.27
> 4 SD	0.29	0.01
> 5 SD	0.08	0.00
> 6 SD	0.03	0.00

Table: Comparison of Real World vs Normal Model Daily Returns: Table 10.1, page 205

Heavy Tails

- Daily exchange rate changes are not normally distributed
- The distribution has heavier tails than the normal distribution
- It is more peaked than the normal distribution
- This means that small changes and large changes are more likely than the normal distribution would suggest
- Many market variables have this property, known as excess kurtosis

Heavy Tails Visualization

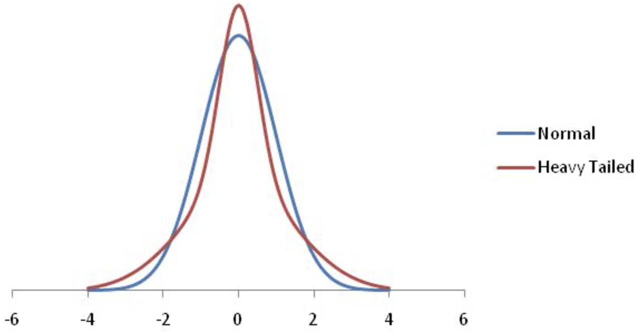

Figure: Comparison of Normal Distribution vs Heavy-Tailed Distribution

Power Law Distributions

- Power law distributions are given by the formula:

$$\mathbb{P}(X > x) = Kx^{-\alpha}$$

  where K is a constant and $\alpha > 1$ is the exponent.
- They are characterized by a straight line on a log-log plot.
- Common in natural phenomena, economics, and finance.
- Examples include wealth distribution, city sizes, and earthquake magnitudes.
- Power law distributions have "heavy tails," meaning they have a higher probability of extreme events compared to normal distributions.

Examples of Power Law Distributions

- Assume a Loss L follows a power law distribution with $\alpha = 3$.
- We know $\mathbb{P}(L > 10) = 0.05$.
- What is $\mathbb{P}(L > 100)$?

Solution to Power Law Example

- Given $\mathbb{P}(L > 10) = 0.05$,
- We can express this as:

$$\mathbb{P}(L > 10) = K \cdot 10^{-3} = 0.05$$

- Therefore, $K = 0.05 \cdot 10^3 = 50$.
- Now, to find $\mathbb{P}(L > 100)$:

$$\mathbb{P}(L > 100) = K \cdot 100^{-3} = 50 \cdot 10^{-6} = 0.00005$$

- Thus, $\mathbb{P}(L > 100) = 0.00005$.

Estimating Daily Volatility

- Define σ_n as the volatility per day between day $n-1$ and day n, as estimated at end of day $n-1$.
- Let S_n be the value of a market variable at the end of day n.
- Define the daily log return as:

$$u_n = \ln \frac{S_n}{S_{n-1}}$$

- The daily volatility is estimated as:

$$\sigma_n^2 = \frac{1}{m-1} \sum_{i=1}^{m} (u_{n-i} - \bar{u})^2$$

where $\bar{u}$ is the average of the log returns.

Simplification of Volatility Estimation

- Define the daily return as:

$$u_n = \frac{S_n - S_{n-1}}{S_{n-1}}$$

- Assume mean return is zero, i.e., $\bar{u} = 0$.
- Replace $m - 1$ by m in the denominator.
- The simplified formula for daily volatility becomes:

$$\sigma_n^2 = \frac{1}{m} \sum_{i=1}^{m} u_{n-i}^2$$

- This is a more straightforward way to estimate daily volatility using past returns.

Weighted Volatility Estimation

- An alternative method is to use a weighted average of past squared returns.
- Define weights w_i such that $\sum_{i=1}^{m} w_i = 1$.
- The weighted volatility is given by:

$$\sigma_n^2 = \sum_{i=1}^{m} w_i u_{n-i}^2$$

- This method allows more recent returns to have a greater influence on the volatility estimate.

Exponential Weighted Moving Average (EWMA)

- The EWMA method gives more weight to recent observations.
- The formula is:

$$\sigma_n^2 = (1-\lambda)u_{n-1}^2 + \lambda\sigma_{n-1}^2$$

  where σ_{n-1}^2 is the previous day's volatility and λ is the decay factor.
- The weights decrease exponentially:

$$w_i = (1-\lambda)\lambda^{i-1}$$

- This method is particularly useful for financial time series where recent volatility is more relevant.
- The sum of weights:

$$(1-\lambda)\sum_{i=1}^{m}\lambda^{i-1} = 1-\lambda^m \approx 1$$

Attraction of EWMA

- Relatively little data needs to be stored
- We need only remember the current estimate of the variance rate and the most recent observation on the market variable
- Tracks volatility changes
- $\lambda = 0.94$ has been found to be a good choice across a wide range of market variables

ARCH(m) Model

- The Autoregressive Conditional Heteroskedasticity (ARCH) model is defined as:

$$\sigma_n^2 = \omega + \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where ω is a constant and α_i are parameters that determine the influence of past squared returns on current volatility.

- $\omega = \gamma V_L$ is scaled by the long-term variance V_L and total weight is:

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

Illustration of clustering volatility: ARCH(1) model

- The ARCH(1) model is a special case of the ARCH model with $m = 1$:

$$\sigma_n^2 = \omega + \alpha_1 u_{n-1}^2$$

- If there was a large shock in the previous period, then the current volatility will be high, capturing the clustering of volatility observed in financial markets.

GARCH(1,1) Model

- The GARCH(1,1) model is a specific case of the GARCH model:
$$\sigma_n^2 = \omega + \alpha_1 u_{n-1}^2 + \beta_1 \sigma_{n-1}^2$$
where $\omega = \gamma V_L$, and $\gamma + \alpha_1 + \beta_1 = 1$.
- $V_L = \frac{\omega}{1-\alpha_1-\beta_1}$ is the long-term variance.
- Here, if $\omega = 0$, it simplifies to EWMA, and if $\beta_1 = 0$, it simplifies to ARCH(1).
- The parameters α_1 and β_1 capture the effects of past squared returns and past volatility, respectively.

GARCH(1,1) Model Example

- Consider a GARCH(1,1) model:

$$\sigma_n^2 = 0.000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2$$

- The long-term variance is:

$$V_L = \frac{0.000002}{1 - 0.13 - 0.86} = 0.0002$$

- The long-term volatility per day is:

$$\sigma_L = \sqrt{V_L} = \sqrt{0.0002} \approx 0.01414 \text{ or } 1.414\%$$

Example continued

- Suppose the estimate of the volatility on day $n-1$ is 1.6% per day
- and on day $n-1$ the market variable decreased by 1%.
- The update for the new variance estimate is:

$$\sigma_n^2 = 0.000002 + 0.13(-0.01)^2 + 0.86(0.016)^2 = 0.000235$$

- The new volatility estimate is:

$$\sigma_n = \sqrt{0.000235} \approx 0.01533 \text{ or } 1.533\%$$

GARCH(p, q) Model

- The GARCH(p, q) model generalizes the GARCH(1,1) model to include p lags of past squared returns and q lags of past volatility:

$$\sigma_n^2 = \omega + \sum_{i=1}^{p} \alpha_i u_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2$$

- The long-term variance is given by:

$$V_L = \frac{\omega}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j}$$

Maximum Likelihood Estimation (MLE)

- MLE is a method for estimating the parameters of a statistical model.
- It finds the parameter values that maximize the likelihood of observing the given data.

MLE Example

- We observe that a certain event happens one time in ten trials. What is our estimate of success probability p?
- The likelihood function is:

$$L(p) = p^1(1-p)^9$$

- To maximize this, we take the derivative and set it to zero:

$$\frac{dL}{dp} = 1 \cdot p^0(1-p)^9 - 9p^1(1-p)^8 = 0$$

- Solving gives us:

$$p = \frac{1}{10} = 0.1$$

- Thus, our estimate of the success probability is 0.1.

Likelihood function

- Given data $x_1, x_2, \ldots, x_n$ and a probability model $f(x;\theta)$:

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

  where $f(x;\theta)$ is the probability density function (pdf) or probability mass function (pmf) depending on whether x is continuous or discrete.

- The log-likelihood is:

$$\ell(\theta) = \log L(\theta) = \sum_{i=1}^{n} \log f(x_i;\theta)$$

MLE for normal Distribution

- Suppose we have a sample of n observations from a Gaussian distribution with mean $\mu = 0$ and unknown variance σ^2.
- The likelihood function is:

$$L(\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_i^2}{2\sigma^2}}$$

where x_i are the observed values.

MLE for normal Distribution (cont.)

- Taking the logarithm of the likelihood function gives:

$$\log L(\sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2$$

- To find the maximum likelihood estimate, it's sufficient to maximize:

$$\sum_{i=1}^{n} -\log(\sigma^2) - \frac{x_i^2}{\sigma^2}$$

- Taking the derivative with respect to σ^2 and setting it to zero, the MLE of the variance gives the sample variance:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

- This is equivalent to the unbiased sample variance when using $n-1$ in the denominator.

MLE for GARCH(1,1) Model

- Goal: Estimate parameters $\omega, \alpha_1, \beta_1$ of the GARCH(1,1) model from time series data $\{S_i : i = 1, \ldots, n\}$.
- Define the daily returns: $u_i = \frac{S_i - S_{i-1}}{S_{i-1}}$
- Update the variance recursively:

$$\sigma_i^2 = \omega + \alpha_1 u_{i-1}^2 + \beta_1 \sigma_{i-1}^2$$

- Use numerical optimization methods (e.g., ‘scipy.optimize.minimize‘) to maximize the likelihood function:

$$L(\omega, \alpha_1, \beta_1) = \sum_{i=1}^{n} -\log(\sigma_i^2) - \frac{x_i^2}{\sigma_i^2}.$$

S&P500 application

- Apply the GARCH(1,1) model to volatility of the daily returns of the S&P 500 index from 2005-07-18 to 2010-08-13.
- Estimate the parameters $\omega, \alpha_1, \beta_1$ using maximum likelihood estimation.
- Analyze the estimated volatility and its implications for risk management.

S&P500 application (cont.)

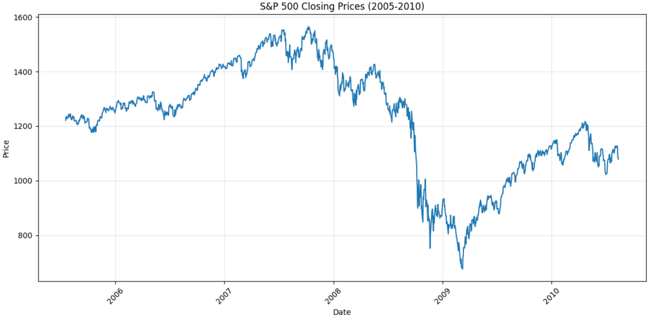

Figure: S&P 500 Closing Prices

S&P500 application (cont.)

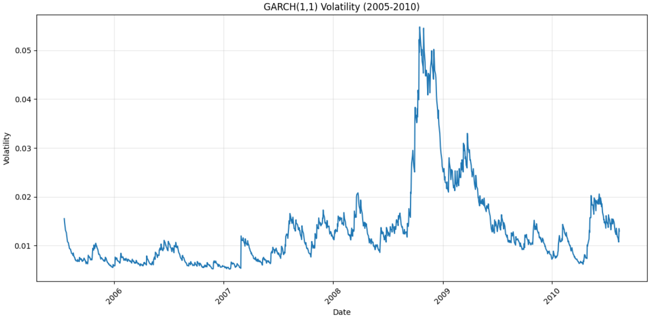

Figure: S&P 500 Volatility with Fitted parameters: $\omega = 0.00000139, \alpha = 0.09647, \beta = 0.90039$

Forecasting volatility

- GARCH(1,1) model:

$$\sigma_n^2 = (1-\alpha-\beta)V_L + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2$$

  or equivalently

$$\sigma_n^2 - V_L = \alpha(u_{n-1}^2 - V_L) + \beta(\sigma_{n-1}^2 - V_L)$$

- On day $n+t$ in the future, we have

$$\sigma_{n+t}^2 - V_L = \alpha(u_{n+t-1}^2 - V_L) + \beta(\sigma_{n+t-1}^2 - V_L)$$

- Assuming $\mathbb{E}u_n = 0$, then $\mathbb{E}u_n^2 = \sigma_n^2$. Then,

$$\mathbb{E}[\sigma_{n+t}^2] = V_L + (\alpha+\beta)^t(\sigma_n^2 - V_L)$$

Forecasting volatility (cont.)

- $\alpha + \beta = 0.997$
- Long-term variance rate: $V_L = 0.00044$
- Current volatility rate: $\sigma_n = 1.295\%$
- 1-day ahead expected volatility forecast: 1.299%
- 10-day ahead expected volatility forecast: 1.328%
- Long-term expected volatility: 2.106%

Volatility term structure

- Define
$$V(t) = \mathbb{E}[\sigma_{n+t}^2], \;\; a = \ln\left(\frac{1}{\alpha+\beta}\right)$$
- $$V(t) = V_L + e^{-at}[V(0) - V_L]$$
- $$\frac{1}{T}\int_0^T V(t)dt = V_L + \frac{1-e^{-aT}}{aT}[V(0) - V_L]$$
- Define $\sigma(T)$ as the volatility per annum that should be used to price a T-day option. Assuming 252 days per year,
$$\sigma(T)^2 = 252\left\{V_L + \frac{1-e^{-aT}}{aT}[\sigma_n^2 - V_L]\right\}$$

Impact of volatility changes

- If $\hat{\sigma}_n$ is also the volatility per annum, then

$$\sigma(T)^2 = 252\left\{V_L + \frac{1-e^{-aT}}{aT}[\frac{\hat{\sigma}_n^2}{252} - V_L]\right\}$$

- If $\hat{\sigma}_n$ increases by $\Delta\hat{\sigma}_n$ per annum, then $\sigma(T)$ increases approximately by

$$\Delta\sigma(T) \approx \frac{1-e^{-aT}}{aT}\frac{\hat{\sigma}_n}{\sigma(T)}\Delta\hat{\sigma}_n$$