Copula Theory Fundamentals

Outline

Motivation

- Modeling dependence is crucial for risk aggregation and portfolio management.
- Correlation is limited; copulas provide a more general framework.
- Copulas allow separate modeling of marginals and dependence structure.

Definition: Copula

- **Definition:** An n-dimensional copula is a function $C : [0,1]^n \to [0,1]$ that is a joint cumulative distribution function (cdf) with uniform $[0,1]$ marginals.

Characterization of Copulas

- C is a copula if and only if
  - C is n-increasing,
  - For all i, $C(1,\ldots,1,u_i,1,\ldots,1)=u_i$,
  - (rectangular inequality) For all $(a_1,\ldots,a_d),(b_1,\ldots,b_d)\in[0,1]^d$ with with $a_i\le b_i$ for all i, we have

$$\sum_{i_1=1}^{2}\cdots\sum_{i_d=1}^{2}(-1)^{i_1+\cdots+i_d}C(u_1^{i_1},\ldots,u_d^{i_d})\ge 0,$$

where $u_j^1=a_j$ and $u_j^2=b_j$.

Sklar's Theorem (Rigorous Statement)

- **Theorem (Sklar, 1959):** Let F be an n-dimensional joint cdf with marginals $F_1, \dots, F_n$. Then there exists a copula C such that
$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n))$$
for all $x_i \in \mathbb{R}$.
- If $F_1, \dots, F_n$ are continuous, C is unique; otherwise, C is uniquely determined on $\text{Ran}\, F_1 \times \cdots \times \text{Ran}\, F_n$.
- Conversely, if C is a copula and F_i are cdfs, then F defined as above is a joint cdf with marginals F_i.

Proof of Sklar's Theorem (Sketch)

- **Existence:** Define $C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n))$
- For continuous marginals, this is well-defined and gives the unique copula.

Copula of F

- If $X \sim F$ with continuous marginal distributions $F_1, \ldots, F_d$, then the copula of F is the joint distribution of $(F_1(X_1), \ldots, F_d(X_d))$.

Example: Non-uniqueness of Copula (Discrete Case)

- Let (X, Y) have joint distribution:

$$P(X = 0, Y = 0) = P(X = 1, Y = 1) = 0.5$$

- Find two different copulas C_1, C_2 compatible with this joint law.

Solution: Non-uniqueness of Copula (Discrete Case)

- **Marginals:** $F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$; similarly for Y.
- **Joint distribution:** $F(x, y) = \begin{cases} 0 & \text{if } x \wedge y < 0 \\ 0.5 & \text{if } 0 \leq x \wedge y < 1 \\ 1 & \text{if } x \wedge y \geq 1 \end{cases}$
- By Sklar's theorem, any copula C such that $C(0.5, 0.5) = 0.5$ is a copula for this joint distribution.

Invariance of Copulas

- If $X_1, \ldots, X_d$ are random variables with continuous margins and copula C, then for any strictly increasing functions $g_1, \ldots, g_n$:

$$(g_1(X_1), \ldots, g_n(X_d)) \text{ has copula } C.$$

Proof: Invariance of Copulas

- Let $Y_i = g_i(X_i)$
- The marginal cdf of Y_i is $G_i(y) = P(Y_i \leq y) = P(g_i(X_i) \leq y) = P(X_i \leq g_i^{-1}(y)) = F_i(g_i^{-1}(y))$.
- For the joint distribution:

$$\begin{aligned}
P(Y_1 \leq y_1, \ldots, Y_d \leq y_d) &= P(g_1(X_1) \leq y_1, \ldots, g_d(X_d) \leq y_d) \\
&= P(X_1 \leq g_1^{-1}(y_1), \ldots, X_d \leq g_d^{-1}(y_d)) \\
&= C(F_1(g_1^{-1}(y_1)), \ldots, F_d(g_d^{-1}(y_d))) \\
&= C(G_1(y_1), \ldots, G_d(y_d))
\end{aligned}$$

- Therefore, $(Y_1, \ldots, Y_d)$ has the same copula C.

Fréchet–Hoeffding Bounds

- For any n-copula C and $\mathbf{u} \in [0,1]^n$:

$$W(\mathbf{u}) \leq C(\mathbf{u}) \leq M(\mathbf{u})$$

where

$$M(\mathbf{u}) = \min\{u_1, \ldots, u_n\}, \quad W(\mathbf{u}) = \max\left\{\sum_{i=1}^{n} u_i - n + 1, 0\right\}$$

- M is the comonotonic copula (upper bound),
- W is the countermonotonic copula (lower bound) if $n = 2$.

Proof: Fréchet–Hoeffding Bounds (Bivariate Case)

- Let C be any bivariate copula, $u, v \in [0, 1]$.
- Let U, V be uniform random variables on $[0, 1]$ with copula C.
- Upper bound:

$$\begin{aligned} C(u, v) &= \mathbb{P}(U \leq u, V \leq v) \\ &\leq \mathbb{P}(U \leq u) \wedge \mathbb{P}(V \leq v) \\ &= u \wedge v \end{aligned}$$

- Lower bound:

$$\begin{aligned} C(u, v) &= \mathbb{P}(U \leq u, V \leq v) \\ &= \mathbb{P}(U \leq u) + \mathbb{P}(V \leq v) - \mathbb{P}(U \leq u \text{ or } V \leq v) \\ &\geq u + v - 1 \end{aligned}$$

Independence Copula

- **Definition:** $\Pi(\mathbf{u}) = \prod_{i=1}^{n} u_i = u_1 \cdot u_2 \cdots u_n$
- Represents complete independence between random variables: $\Pi(\mathbf{u}) = F_{\mathbf{U}}(\mathbf{u})$, where $\mathbf{U} = (U_1, \ldots, U_n)$ is independent uniform random variables on $[0,1]^n$.
- **Properties:**
  - All variables are mutually independent
  - No linear or nonlinear dependence
  - Knowledge of one variable provides no information about others
- Most commonly assumed in classical finance models

Comonotonic Copula (Perfect Positive Dependence)

- **Definition:** $M(\mathbf{u}) = \min\{u_1, u_2, \ldots, u_n\}$
- Represents perfect positive dependence (upper Fréchet bound): $M(\mathbf{u}) = F_{\mathbf{U}}(\mathbf{u})$ where $\mathbf{U} = (U, U, \ldots, U)$ and $U \sim U(0, 1)$.
- **Properties:**
  - Variables move in the same direction
  - If one variable increases, all others increase
  - Strongest possible positive dependence
- **Interpretation:** There exists a random variable $U \sim \text{Uniform}[0, 1]$ such that all marginals can be written as increasing functions of U
- **Risk management:** Worst-case scenario for portfolio diversification

Countermonotonic Copula (Perfect Negative Dependence)

- **Definition:** $W(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}$ (only exists for $n = 2$)
- Represents perfect negative dependence (lower Fréchet bound): $W(u_1, u_2) = F_{\mathbf{U}}(u_1, u_2)$ where $\mathbf{U} = (U, 1 - U)$ and $U \sim U(0, 1)$.
- **Properties:**
  - Variables move in opposite directions
  - If one variable increases, the other decreases
  - Strongest possible negative dependence
- **Interpretation:** $Y = F_Y^{-1}(1 - F_X(X))$ for some strictly increasing functions
- **Risk management:** Perfect hedge - ideal for portfolio diversification
- **Note:** For $n > 2$, perfect negative dependence among all pairs is impossible

Example: Independence/Comonotonic/Countermonotonic Copulas

Let U and V be independent uniform random variables on $[0, 1]$.

- **Independence:** Find the copula C_1 for the joint distribution of (U, V).
- **Countermonotonic:** Find the copula C_2 for the joint distribution of $(U, 1 - U)$.
- **Comonotonic:** Find the copula C_3 for the joint distribution of (U, U).

Gaussian Copula

- **Definition:** If $Y \sim N_d(\mu, \Sigma)$, then its copula is called the Gaussian copula.
- The correlation matrix: $R = D^{-1/2}\Sigma D^{-1/2}$, where $D = \text{diag}(\Sigma)$.
- If $\Sigma_{ii} = 1$ for all i, then $R = \Sigma$.

Standardization

- **Standardization:** Let $X = D^{-1/2}(Y - \mu)$, then $X \sim N(0, R)$.
- **Invariance:** The copula of X and Y is the same. (why?)
- The copula of Y is given by:

$$C_{\Sigma}(u_1, \ldots, u_n) = \Phi_R\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)\right)$$

  where Φ is the standard normal cdf and Φ_R is the multivariate normal cdf with correlation matrix R.

Simulation with Gaussian Copula

- To generate random variables with marginal distributions $F_1, \dots, F_n$ and a Gaussian copula C_Σ:
  - Generate $(Z_1, \dots, Z_n) \sim N(0, \Sigma)$,
  - set $U_i = \Phi(Z_i)$.
  - Transform each U_i using the inverse of the desired marginal distribution F_i^{-1}:

$$X_i = F_i^{-1}(U_i), \quad i = 1, \dots, n$$

- **Limitation:** The Gaussian copula does not capture tail dependence (joint extreme events are underestimated).