Homework

(1) Consider two assets with returns $R_1 \sim N(0.1, 0.2^2)$ and $R_2 \sim N(0.05, 0.1^2)$, and correlation $\rho = 0.3$. Assume the portfolio return is $R = w_1 R_1 + w_2 R_2$ with $w_1 + w_2 = 1$. Find the weights w_1, w_2 that minimize the portfolio variance $\mathrm{Var}(R)$.

(2) Suppose X and Y are perfectly correlated with correlation $\rho = 1$. Prove that Y is linearly related to X, i.e., there exist constants a, b such that $Y = aX + b$.

Solution:

(1) Using the formulas for the optimal weights in a two-asset portfolio, we have

$$w_1^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \ w_2^* = 1 - w_1^*,$$

we have $\sigma_1 = 0.2, \sigma_2 = 0.1, \rho = 0.3$. Thus

$$w_1^* = \frac{0.1^2 - 0.3 \times 0.2 \times 0.1}{0.2^2 + 0.1^2 - 2 \times 0.3 \times 0.2 \times 0.1} = \frac{0.01 - 0.006}{0.04 + 0.01 - 0.012} = \frac{0.004}{0.038} \approx 0.1053,$$

$$w_2^* = 1 - w_1^* \approx 0.8947.$$

(2) Let $a = \frac{\sigma_Y}{\sigma_X} > 0$ and $b = \mu_Y - a\mu_X$. Let $Z = Y - aX - b$. Then $\mathbb{E}[Z] = 0$ and we know $\mathrm{Cov}(X, Y) = \rho\sigma_X\sigma_Y = \sigma_X\sigma_Y$ since $\rho = 1$. Thus Hence,

$$\mathrm{Var}(Z) = \mathrm{Var}(Y - aX) = \mathrm{Var}(Y) + a^2\mathrm{Var}(X) - 2a\mathrm{Cov}(X, Y) = (\sigma_Y - a\sigma_X)^2 = 0.$$