Default probability

Historical data method

Historical data provided by rating agencies can be used to estimate the probability of default.

Table: Average Cumulative Default Rates (%), 1970–2013

Rating	**1 yr**	**2 yr**	**3 yr**	**4 yr**	**5 yr**	**7 yr**	**10 yr**	**15 yr**	**20 yr**
Aaa	0.000	0.013	0.013	0.037	0.104	0.241	0.489	0.910	1.073
Aa	0.022	0.068	0.136	0.260	0.410	0.682	1.017	1.871	3.167
A	0.062	0.199	0.434	0.679	0.958	1.615	2.759	4.583	7.044
Baa	0.174	0.504	0.906	1.373	1.862	2.872	4.623	8.306	11.969
Ba	1.110	3.071	5.371	7.839	10.065	13.911	19.323	28.500	35.410
B	3.904	9.274	14.723	19.509	23.869	31.774	40.560	50.275	55.892
Caa-C	15.894	27.003	35.800	42.796	48.828	56.878	66.212	73.152	74.946

Interpretation

- The table shows the probability of default for companies starting with a particular credit rating.
- A company with an initial credit rating of Baa Bond has a probability of 0.174% of defaulting by the end of the first year, 0.504% by the end of the second year, and so on.
- Let τ be the time of default, then

$$\mathbb{P}(\tau \leq 1) = 0.00174, \quad \mathbb{P}(\tau \leq 2) = 0.00504, \ldots$$

Do Default Probabilities Increase with Time?

- For a company that starts with a good credit rating default probabilities tend to increase with time.
- For a company that starts with a poor credit rating default probabilities tend to decrease with time.

Example

- Consider a Caa-C bond. What is the probability that it will default during the third year conditional on no earlier default?
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$$\mathbb{P}(\tau \leq 3 | \tau > 2) = \frac{\mathbb{P}(\tau \leq 3) - \mathbb{P}(\tau \leq 2)}{\mathbb{P}(\tau > 2)} = \frac{0.358 - 0.270}{1 - 0.270} = 0.1205$$

Hazard Rate

The hazard rate, $\lambda(t)$, at time t is defined so that $\lambda(t)\Delta t$ is the probability of default between time t and $t + \Delta t$ conditional on no default between time zero and time t. That is,

$$\mathbb{P}(\tau \leq t + \Delta t | \tau > t) = \lambda(t)\Delta t + o(\Delta t).$$

Hazard Rate (cont.)

- Let $V(t) = \mathbb{P}(\tau > t)$ be the survival function. Then
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$$\mathbb{P}(\tau \leq t+\Delta t|\tau > t) = \frac{V(t) - V(t+\Delta t)}{V(t)} = \lambda(t)\Delta t+o(\Delta t).$$

- Rearranging gives

$$\dot{V}(t) = -\lambda(t)V(t).$$

Hazard Rate (cont.)

- Solving this differential equation, the survival function is given by
$$V(t) = \exp\left(-\int_0^t \lambda(s)ds\right) = e^{-\bar{\lambda}(0,t)t},$$
where $\bar{\lambda}(r,t) = \frac{1}{t-r}\int_r^t \lambda(s)ds$ is the average hazard rate.
- For $0 < r < t$,
$$\mathbb{P}(\tau > t | \tau > r) = e^{-\bar{\lambda}(r,t)(t-r)}.$$
- Defining $Q(t)$ as the probability of default by time t, so that $Q(t) = 1 - V(t)$ gives
$$Q(t) = 1 - \exp\left(-\int_0^t \lambda(s)ds\right) = 1 - e^{-\bar{\lambda}(0,t)t}.$$

Example

- Suppose that the hazard rate is a constant 1.5% per year.
- Find the probability of a default by the end of ith year, $i = 1, \ldots, 5$.
- Find the probability of default in the fourth year, conditional on no earlier default.

Example

- Suppose that the hazard rate is a constant 1.5% per year.
- The probability of a default by the end of the first year is $1 - e^{-0.015\times 1} = 0.0149$.
- The probability of a default by the end of the second year is $1 - e^{-0.015\times 2} = 0.0296$.
- The probability of a default by the end of the third, fourth, and fifth years are similarly 0.0440, 0.0582, and 0.0723.
- The unconditional probability of a default during the fourth year is $0.0582 - 0.0440 = 0.0142$.
- The probability of default in the fourth year, conditional on no earlier default is $0.0142/(1 - 0.0440) = 0.0149$.

Recovery Rate

- The recovery rate is the proportion of the value of a bond that is recovered by investors in 30 days after default.
- The recovery rate is usually expressed as a percentage of the face value of the bond.
- The recovery rate can vary significantly depending on the type of bond, the seniority of the bond in the capital structure, and the specific circumstances of the default.

Table: Recovery Rates on Corporate Bonds as a Percent of Face Value, 1982 to 2013, Issuer Weighted

Class	**Average Recovery Rate (%)**
Senior secured bond	52.2
Senior unsecured bond	37.2
Senior subordinated bond	31.0
Subordinated bond	31.4
Junior subordinated bond	24.7

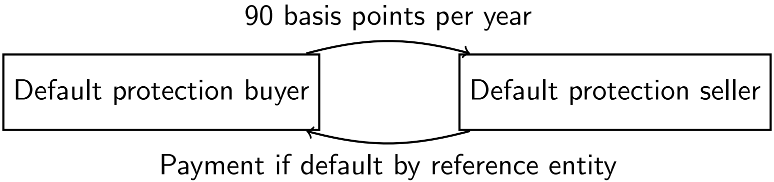
Source: Moody's

Credit Default Swap (CDS)

- A derivative that has become very important in credit markets is a credit default swap (CDS).
- The simplest type of CDS is an instrument that provides insurance against the risk of a default by a particular company, known as the **reference entity** and a default by the company is known as a **credit event**.
- The buyer of the insurance obtains the right to sell bonds issued by the company for their face value when a credit event occurs, while the seller of the insurance agrees to buy the bonds for their face value.
- The total face value of the bonds that can be exchanged is known as the credit default swap's **notional principal**.
- The buyer of a CDS makes periodic payments to the seller until the end of the life of the CDS or until a credit event occurs. These payments are typically made in arrears every quarter.

Example: Credit Default Swap Structure

- Suppose that two parties enter into a five-year credit default swap on December 20, 2015.
- Assume that the notional principal is $100 million and the buyer agrees to pay 90 basis points per year (quarterly in arrears) for protection against default by the reference entity.



Example: Credit Default Swap Structure (cont.)

- If no default, the buyer receives no payoff and pays approximately \$225,000 ($= 0.25 \times 0.0090 \times 100{,}000{,}000$) on March 20, June 20, September 20, and December 20 of each of the years 2016, 2017, 2018, 2019, and 2020.

Example: Credit Default Swap Structure (cont.)

- Suppose that the buyer notifies the seller of a credit event on May 20, 2018 (five months into the third year).
- If the contract specifies physical settlement, the buyer of protection has the right to sell to the seller of protection bonds issued by the reference entity with a face value of $100 million for $100 million.
- If, as is now usual, there is a cash settlement, a two-stage auction process is used to determine the mid-market value of the cheapest deliverable bond several days after the credit event.
- Suppose the auction indicates that the cheapest deliverable bond is worth $35 per $100 of face value. The cash payoff would be $65 million.
- The total amount paid per year, as a percent of the notional principal, to buy protection is known as the CDS spread. (In our example, the CDS spread is 90 basis points.)

CDS Pricing: Reduced-Form Model

- Default modeled as a Poisson process N_t with intensity λ
- **Default time** τ: First jump time of the Poisson process

$$\tau = \inf\{t \geq 0 : N_t = 1\}, \quad N_t \sim \text{Poisson}(\lambda)$$

- τ is exponentially distributed: $\mathbb{P}(\tau > t) = e^{-\lambda t}$

CDS Pricing: Reduced-Form Model

- Present value of premium leg:

$$PV_{\text{prem}} = S \sum_{i=1}^{N} \Delta t_i \mathbb{E}[e^{-rt_i} \mathbf{1}_{\tau > t_i}]$$

- Present value of protection leg:

$$PV_{\text{prot}} = (1 - R)\mathbb{E}[e^{-r\tau}]$$

- Fair CDS spread S^*: $PV_{\text{prem}} = PV_{\text{prot}}$

Example on CDS pricing

- Suppose that the hazard rate of a reference entity is 1.5% per annum for five years.
- The recovery rate is 40%.
- The risk-free interest rate is 5% per annum, continuously compounded.
- We will assume that defaults always happen halfway through a year and that payments on a five-year credit default swap are made annually at the end of each year.
- What is the fair CDS spread?

Example on CDS pricing (cont.)

- The survival function is $V(t) = e^{-\lambda t}$.
- The present value of regular payments is

$$PV_{\text{regular}} = S \sum_{i=1}^{5} e^{-ri} V(i) = 4.078S.$$

- The present value of the accrual payment when default occurs in a mid-year is

$$PV_{\text{accrual}} = S \sum_{i=1}^{5} e^{-r(i-0.5)} (V(i-1) - V(i)) \times 0.5 = 0.0284S.$$

- The present value of the premium leg is

$$PV_{\text{prem}} = PV_{\text{regular}} + PV_{\text{accrual}} = 4.078S + 0.0284S = 4.1064S.$$

- The present value of the protection leg at unit notional value is

$$PV_{\text{prot}} = (1-R) \sum_{i=1}^{5} (V(i-1) - V(i)) e^{-r(i-0.5)} = 0.0506.$$

- The fair CDS spread is $S^* = 0.0123$ or 123 basis points from $PV_{\text{prem}} = PV_{\text{prot}}$.