Homework

(1) Suppose that the change in the value of a portfolio over a 10-day time horizon is normally distributed with a mean of zero and a standard deviation of 20 million. Compute the 10-day 99% VaR and ES for the portfolio.

(2) Consider two independent random variables L_1 and L_2 with the following distributions:
- $P(L_1 = 0) = 0.98$, $P(L_1 = 10) = 0.02$
- $P(L_2 = 0) = 0.98$, $P(L_2 = 10) = 0.02$

1. Prove that $\mathrm{VaR}_{0.98}$ is not subadditive.
2. Determine whether $\mathrm{VaR}_{0.99}$ is subadditive.
3. Find $\mathrm{ES}_{0.98}$ for $L_1, L_2, L_1 + L_2$, respectively.

(3) A bank has two 10 million one-year loans. The probabilities of default are as follows:
- $\mathbb{P}$(Neither loan defaults) = 0.975
- $\mathbb{P}$(Loan 1 defaults, Loan 2 does not default) = 1.25%
- $\mathbb{P}$(Loan 2 defaults, Loan 1 does not default) = 1.25%
- $\mathbb{P}$(Both loans default) = 0%

If a default occurs, all losses between 0% and 100% of the principal are equally likely. If the loan does not default, a profit of 0.2 million is made.

1. Compute one-year 99% VaR for the Loan 1.
2. Compute one-year 99% VaR for the bank loan portfolio. Is it subadditive?
3. Compute the one-year 99% expected shortfall (ES) for the Loan 1 and bank loan portfolio, respectively.

(4) VaR monotonicity says that If $L_1 \leq L_2$, then $\mathrm{VaR}_\alpha(L_1) \leq \mathrm{VaR}_\alpha(L_2)$. Justify the following statement:

If $L_1 < L_2$ almost surely, then $\mathrm{VaR}_\alpha(L_1) < \mathrm{VaR}_\alpha(L_2)$ for any $\alpha < 1$.

(5) Prove that if f is an increasing function and $X = f(Z)$ for some random variable Z, then $\mathrm{VaR}_\alpha(X) = f(\mathrm{VaR}_\alpha(Z))$.

(6) Prove the monotonicity of Expected Shortfall (ES).

(7) The loss variables X and Y are given by:

$$X = Z + \rho\epsilon, \quad Y = Z + 2\rho\epsilon$$

where $Z, \epsilon \sim N(0, 1)$ are independent, and ρ is some constant.

(a) Is X, Y positively or negatively dependent?

(b) For $\alpha \in (0, 1)$, prove that

$$\mathrm{VaR}_\alpha(X + Y) \leq \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y).$$

(c) Are X, Y comonotonic?

Solution:

(1) 1. The 10-day 99% VaR is given by

$$\mathrm{VaR}_{.99} = 20\Phi^{-1}(.99) = 46.5.$$

For the expected shortfall (ES),

$$\mathrm{ES}_{.99} = 20\frac{\phi(\Phi^{-1}(.99))}{1-.99} = 53.3.$$

2. To prove that $\mathrm{VaR}_{0.98}$ is not subadditive, consider the distributions of L_1 and L_2. The possible values of $L_1 + L_2$ are:

$$L_1 + L_2 = \begin{cases} 0 & \text{with probability } 0.98 \times 0.98 = 0.9604, \\ 10 & \text{with probability } 0.98 \times 0.02 + 0.02 \times 0.98 = 0.0392, \\ 20 & \text{with probability } 0.02 \times 0.02 = 0.0004. \end{cases}$$

The $\mathrm{VaR}_{0.98}$ for L_1 is 0, and similarly, the $\mathrm{VaR}_{0.98}$ for L_2 is 0. However, for $L_1 + L_2$, the $\mathrm{VaR}_{0.98}$ is 10 because the cumulative probability reaches 0.98 at 10. Thus: $\mathrm{VaR}_{0.98}(L_1 + L_2) = 10 > \mathrm{VaR}_{0.98}(L_1) + \mathrm{VaR}_{0.98}(L_2) = 0 + 0$. Therefore, $\mathrm{VaR}_{0.98}$ is not subadditive.

3. For $\mathrm{VaR}_{0.99}$, the cumulative probabilities for $L_1 + L_2$ are: $P(L_1 + L_2 \le 0) = 0.9604$, $P(L_1 + L_2 \le 10) = 0.9996$, $P(L_1 + L_2 \le 20) = 1$. The $\mathrm{VaR}_{0.99}$ for L_1 is 10, and similarly, the $\mathrm{VaR}_{0.99}$ for L_2 is 10. For $L_1 + L_2$, the $\mathrm{VaR}_{0.99}$ is also 10 because the cumulative probability reaches 0.99 at 10. Thus: $\mathrm{VaR}_{0.99}(L_1 + L_2) = 10 < \mathrm{VaR}_{0.99}(L_1) + \mathrm{VaR}_{0.99}(L_2)$. Therefore, $\mathrm{VaR}_{0.99}$ is subadditive in this case.

4. The expected shortfall $\mathrm{ES}_{0.98}$ is calculated as the average of losses exceeding the $\mathrm{VaR}_{0.98}$. For L_1 and L_2, the $\mathrm{VaR}_{0.98}$ is 0, so we consider the losses exceeding 0, which are 10 with probability 0.02. Thus: $\mathrm{ES}_{0.98}(L_1) = \mathrm{ES}_{0.98}(L_2) = \frac{10\times 0.02}{0.02} = 10$.
For $L_1 + L_2$, the $\mathrm{VaR}_{0.98}$ is 10. The tail loss is

$$\mathrm{ES}_{0.98}(L_1 + L_2) = \frac{20 \times 0.0004 + 10 \times (0.02 - 0.0004)}{0.02} = 10.2.$$

(2) 1. Note that

$$F_{L_1}(x) = \begin{cases} 0 & x < -.2 \\ 98.75\% & -.2 \le x < 0 \\ 1.25\%\frac{x}{10} + 98.75\% & 0 \le x < 10 \\ 100\% & x \ge 10 \end{cases}$$

We can find the inverse CDF $F_{L_1}^{-1}(u)$ for $u \in [0, 1]$:

$$F_{L_1}^{-1}(u) = \begin{cases} -0.2 & 0 \le u < 0.9875 \\ 80(u - 0.9875) & 0.9875 \le u < 1 \end{cases}$$

Hence, $F_{L_1}^{-1}(.99) = 800(.99 - 0.9875) = 800 \times 0.0025 = 2$.

2. Let $L = L_1 + L_2$. The CDF of L is:

$$F_L(x) = \begin{cases} 0 & x < -0.4 \\ 95.5\% & -0.4 \le x < -0.2 \\ 97.5\% + \frac{(x+.2)2.5}{10}\% & -.2 \le x < 9.8 \\ 100\% & x \ge 9.8 \end{cases}$$

So $F_L^{-1}(99\%) = 5.8$.

3. $L_1|D \sim U(2, 10)$, so $ES_{0.99}(L_1) = \frac{2+10}{2} = 6$.
   $L|D \sim U(5.8, 9.8)$, so $ES_{0.99}(L) = \frac{5.8+9.8}{2} = 7.8$.

(3) Counterexample: Let $\alpha \in (0, 1)$. Pick any $\epsilon \in (0, 1)$ and let $U \sim \mathcal{U}(0, 1)$. Define

$$L_1 = U, L_2 = g(U),$$

where $g(x) = x + \epsilon|x - \alpha|$. It can be shown that both VaR at level α is equal to α.

(4)

$$\begin{aligned}
\mathrm{VaR}_\alpha(X) &= \inf\{x : \mathbb{P}(X \leq x) \geq \alpha\} \\
&= \inf\{x : \mathbb{P}(f(Z) \leq x) \geq \alpha\} \\
&= \inf\{x : \mathbb{P}(Z \leq f^{-1}(x)) \geq \alpha\} \\
&= \inf\{f(z) : \mathbb{P}(Z \leq z) \geq \alpha\} \\
&= f(\inf\{z : \mathbb{P}(Z \leq z) \geq \alpha\}) = f(\mathrm{VaR}_\alpha(Z))
\end{aligned}$$

(5)
- Let $L_1 \leq L_2$ a.s. We want to show that $\mathrm{ES}_\alpha(L_1) \leq \mathrm{ES}_\alpha(L_2)$.

$$\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 F_{L_1}^{-1}(u)\,du \leq \frac{1}{1-\alpha}\int_\alpha^1 F_{L_2}^{-1}(u)\,du = \mathrm{ES}_\alpha(L_2)$$

- The inequality holds because $F_{L_1}^{-1}(u) \leq F_{L_2}^{-1}(u)$ for all $u \in [\alpha, 1]$ due to the stochastic dominance of L_2 over L_1.

(6) later

(7) (a) X and Y are positively dependent because $\mathrm{Cov}(X, Y) = 1 + 2\rho^2 > 0$.
(b) Note that

$$\mathrm{VaR}_\alpha(X) = \sqrt{1+\rho^2}\Phi^{-1}(\alpha), \quad \mathrm{VaR}_\alpha(Y) = \sqrt{1+4\rho^2}\Phi^{-1}(\alpha).$$

While,

$$\mathrm{VaR}_\alpha(X+Y) = \sqrt{4+9\rho^2}\Phi^{-1}(\alpha).$$

Thus, the inequality holds and equality holds if and only if $\rho = 0$.
(c) Comonotonicity holds if and only if $\rho = 0$.