HOMEWORK

(1) Let $V(\mathbf{x})$ be a portfolio value as a function of d risk factors $\mathbf{x}$.
   (a) Write the second-order Taylor expansion for ΔV in terms of ∇V and $\nabla^2 V$.
   (b) Let $d = 2$, $\nabla V = \begin{bmatrix} .1 \\ -.2 \end{bmatrix}$, $\nabla^2 V = \begin{bmatrix} .3 & -.4 \\ -.4 & .5 \end{bmatrix}$. Write out the expansion explicitly.

(2) The delta of a derivatives portfolio dependent on an index is –2,100. The index is currently 1,000. Estimate what happens to the value of the portfolio when the index increases to 1,005.

(3) The gamma of a delta-neutral portfolio is 30 (per \$ per \$). Estimate what happens to the value of the portfolio when the price of the underlying asset (a) suddenly increases by \$2 and (b) suddenly decreases by \$2.

(4) Consider a one-year European call option on a stock when the stock price is \$30, the strike price is \$30, the risk-free rate is 5%, and the volatility is 25% per annum.
   (a) Use the Black-Scholes formula to derive the price, delta, gamma, vega, theta, and rho of the option.
   (b) If the stock price increases to \$30.1 at the end of one day, what is the actual difference in the option price?
   (c) Use the sensitivity analysis to estimate the change in the option price. Are they close?

(5) The Black-Scholes price of a European call is

$$C(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

where

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

(with $\Phi(\cdot)$ the standard normal CDF).
   (a) Show that

$$S\phi(d_1) = Ke^{-r(T-t)}\phi(d_2)$$

   where $\phi(\cdot)$ is the standard normal PDF.
   (b) Show that

$$\Delta = \frac{\partial C}{\partial S} = \Phi(d_1).$$

   (c) Show that

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\phi(d_1)}{S\sigma\sqrt{T-t}}.$$

(6) Consider a straddle portfolio of European call and put option with the following parameters:
   - Spot price: $S_0 = \$49$
   - Strike price: $K = \$50$
   - Risk-free rate: $r = 5\%$ p.a.
   - Volatility: $\sigma = 20\%$ p.a.
   - Time to maturity: $T = 20$ weeks $= 0.3846$ years

(a) Plot a payoff diagram for the straddle position.
(b) Compute the price of the call using the Black-Scholes formula.
(c) Using put-call parity $C - P = S_0 - Ke^{-rT}$, compute the price of the put.
(d) Compute the Greeks (Δ, Γ) for the straddle position.