# Risk Aggregation Principles

# Outline

# Motivation

- Why aggregate risks?
- Portfolio effects: diversification and risk reduction.
- Understanding dependence is crucial for risk management.

# Portfolio Risk

- Total portfolio loss: $L = L_1 + L_2 + \cdots + L_n$
- Variance of sum: $\mathrm{Var}(L) = \sum_i \mathrm{Var}(L_i) + 2\sum_{i<j} \mathrm{Cov}(L_i, L_j)$

## Diversification Benefit on Standard Deviation

- $\rho = 1$: perfect positive dependence; $\rho = 0$: independence (for normal); $\rho = -1$: perfect negative dependence.
- Let $\sigma = \mathrm{SD}(L)$, $\sigma_i = \mathrm{SD}(L_i)$ and $\rho_{ij} = \mathrm{Cor}(L_i, L_j)$, then

$$\begin{aligned} \sigma^2 &= \sum_i \sigma_i^2 + 2 \sum_{i<j} \rho_{ij}\sigma_i\sigma_j \\ &\leq (\sum_i \sigma_i)^2 \quad \text{since } |\rho_{ij}| \leq 1. \end{aligned}$$

- Diversification reduces risk (SD) if losses are not perfectly correlated.

# Diversification Benefit on Variance

- If $L_1, L_2$ are independent: $\mathrm{Var}(L_1 + L_2) = \mathrm{Var}(L_1) + \mathrm{Var}(L_2)$
- **Diversification benefit:**
  $\mathrm{Var}(L_1 + L_2) < \mathrm{Var}(L_1) + \mathrm{Var}(L_2)$ if they are negatively correlated: $\mathrm{Cov}(L_1, L_2) < 0$

# Risk Aggregation with Dependence

- Aggregated risk depends on the joint distribution of losses.
- Comonotonicity: worst-case dependence, no diversification.
- Copulas provide a general framework for modeling dependence (see later lectures).

# Example: Two-Asset Portfolio

- $L_1, L_2$ with means $\mu_1, \mu_2$, variances $\sigma_1^2, \sigma_2^2$, correlation $\rho$
- $L = w_1 L_1 + w_2 L_2$, **with** $w_1 + w_2 = 1$ **(fully invested portfolio)**
- Minimum variance portfolio: choose $w_1, w_2$ (with $w_1 + w_2 = 1$) to minimize $\text{Var}(L)$

# Solution: Minimum Variance Portfolio Weights

$$\begin{aligned}
\mathrm{Var}(L) &= w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\rho\sigma_1\sigma_2 \\
&= (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)w_1^2 + 2(\rho\sigma_1\sigma_2 - \sigma_2^2)w_1 + \sigma_2^2 \\
&= aw_1^2 + bw_1 + c
\end{aligned}$$

- (No diversificatio) If $a = 0$, then $\rho = 1$ and $\sigma_1 = \sigma_2$, and $\mathrm{Var}(L) = \sigma_2^2$ for any choice of $w_1, w_2$.
- If $a > 0$,

$$w_1^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, w_2^* = 1 - w_1^*.$$

This gives the lowest possible portfolio variance for given $\sigma_1, \sigma_2, \rho$.