Risk Factors and P&L Attribution

Outline

What are Risk Factors?

- Variables that drive changes in the value of financial instruments or portfolios.
- Examples: interest rates, equity prices, FX rates, credit spreads, volatility.
- Risk factor modeling is central to risk measurement and management.

Linear and Non-Linear Risk Factors

- **Linear risk factors:** Asset returns or price changes that affect portfolio value in a linear way.
- **Non-linear risk factors:** Derivatives, options, and portfolios with convexity or optionality.
- Non-linear exposures require advanced modeling (e.g., Greeks, Taylor expansion).

Linear Portfolio: Vector Notation

Let $\mathbf{w} \in \mathbb{R}^n$ be the vector of portfolio weights and $\mathbf{r} \in \mathbb{R}^n$ the vector of asset returns.

Portfolio return:

$$R_p = \mathbf{w}^\top \mathbf{r}$$

Attribution: The contribution of asset i is $w_i r_i$.

General Factor Model

Suppose asset returns are driven by k risk factors $\mathbf{f} \in \mathbb{R}^k$:

$$\mathbf{r} = \mathbf{B}\mathbf{f} + \boldsymbol{\epsilon}$$

where $\mathbf{B} \in \mathbb{R}^{n \times k}$ is the factor loading matrix, $\boldsymbol{\epsilon}$ is idiosyncratic noise.

Portfolio exposure to factor j:

$$\beta_j^{(p)} = \frac{\partial R_p}{\partial f_j} = \mathbf{w}^\top \frac{\partial \mathbf{r}}{\partial f_j} = \mathbf{w}^\top \mathbf{B}_{\cdot j}$$

P&L Attribution via Taylor Expansion

Let $V(\mathbf{x})$ be the portfolio value as a function of risk factors $\mathbf{x}$. For a small change $\Delta\mathbf{x}$:

$$\Delta V \approx \nabla V^\top \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^\top \nabla^2 V \Delta\mathbf{x}$$

where ∇V is the gradient (vector of first-order sensitivities), $\nabla^2 V$ is the Hessian (matrix of second-order sensitivities).

Greeks for Derivatives

For an option with value $V(S, t)$:

$$\Delta V \approx \Delta S \frac{\partial V}{\partial S} + \frac{1}{2}(\Delta S)^2 \frac{\partial^2 V}{\partial S^2}$$

- $\Delta = \frac{\partial V}{\partial S}$ (Delta)
- $\Gamma = \frac{\partial^2 V}{\partial S^2}$ (Gamma)

Example

Suppose that the gamma of a delta-neutral portfolio of options on an asset is $-10,000$. Suppose that a change of $+2$ in the price of the asset occurs over a short period of time (for which Δt can be assumed to be zero). The Taylor expansion shows that there is an unexpected decrease in the value of the portfolio of approximately:

$$0.5 \times 10,000 \times 2^2 = \$20,000$$

Note that the same unexpected decrease would occur if there were a change of -2.

Delta Hedging Concept

- Delta hedging involves adjusting the position in the underlying asset to offset changes in the option's value.

$$\Delta V - \Delta S \frac{\partial V}{\partial S} \approx 0$$

- The goal is to create a portfolio that is insensitive to small movements in the underlying asset price.
- Requires frequent rebalancing as delta changes with the underlying price and time.

Black-Scholes Formula

- The Black-Scholes formula for a European call option is given by:

$$C(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Black-Scholes Greeks

- The Greeks are measures of sensitivity of the option price to various factors.
- Common Greeks include:
  - Delta (Δ): Sensitivity to changes in the underlying asset price.
  - Gamma (Γ): Sensitivity to changes in Delta.
  - Vega (ν): Sensitivity to changes in volatility.
  - Theta (Θ): Sensitivity to the passage of time.
  - Rho (ρ): Sensitivity to changes in interest rates.

Greeks Formula of BS model

- The formulas in the Black-Scholes model are:

$$
\begin{aligned}
\Delta &= \Phi(d_1) \\
\Gamma &= \frac{\partial \Delta}{\partial S} = \frac{\phi(d_1)}{S\sigma\sqrt{T-t}} \\
\nu &= S\phi(d_1)\sqrt{T-t} \\
\Theta &= -\frac{S\phi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_2) \\
\rho &= K(T-t)e^{-r(T-t)}\Phi(d_2)
\end{aligned}
$$

where $\phi(\cdot)$ is the probability density function of the standard normal distribution.

Example: Delta Hedging with Black-Scholes

- Consider a trader who sells 100,000 European call options with the following parameters:
  - Spot price: $S_0 = \$49$
  - Strike price: $K = \$50$
  - Risk-free rate: $r = 5\%$ p.a.
  - Volatility: $\sigma = 20\%$ p.a.
  - Time to maturity: $T = 20$ weeks $= 0.3846$ years
- Premium received: $\$300,000$ (i.e., $3 per option)

Example: Delta Hedging with Black-Scholes (continued)

- Using the Black-Scholes formula, we calculate for $+1$ option:

$$\begin{aligned} d_1 &= 0.0542 \\ d_2 &= -0.0699 \\ C &= \$2.40 \text{ (option price)} \\ \Delta &= 0.5216 \\ \Gamma &= 0.0655 \end{aligned}$$

- Theoretical portfolio value is

$$V = -100,000 \times 2.40 + 300000 = 60,000$$

Example: Delta Hedging with Black-Scholes (continued)

- If the stock price at maturity is S_T, the unhedged portfolio value is:

$$V_{unhedged} = -100,000 \times (S_T - K)^+ + 300,000$$

- ex. If $S_T = 60$, then the portfolio value becomes:

$$V_{unhedged} = -100,000 \times (60 - 50)^+ + 300,000 = -700,000$$

- Can we lock this profit?

Example: Delta Hedging with Black-Scholes (continued)

Week 0: Stock price $S_0 = 49$.

- Short position: 100,000 calls and option delta $\Delta = 0.5216 \Rightarrow$ option book delta $= (-100{,}000) \times 0.5216 \approx -52{,}160$.
- Initial hedge: buy 52,160 shares of the underlying to neutralize delta.
- Funding: cash account updates to

$$\texttt{Cash} = 300{,}000 - 49 \times 52{,}160 = -2{,}255{,}840$$

- rebalancing weekly until expiration.

Example: Delta Hedging with Black-Scholes (continued)

Week	Stock Price	Delta	Shares to Buy/Sell	Cash
0	49.00	0.5216	52160.47	-2255862.84
1	51.45	0.6729	15129.97	-3034291.94
2	51.97	0.7037	3081.17	-3194435.44
⋮	⋮	⋮	⋮	⋮
17	50.77	0.6555	-6578.59	-3017333.22
18	50.38	0.6040	-5155.24	-2757592.67
19	52.76	0.9766	37261.67	-4723610.99

- Unhedged portfolio Value at Maturity: 23752.013
- Portfolio value after hedging: 152749.55146121606
- Hedging P&L: 128997.54