Interest Rate Risk

Outline

Compounding Frequencies for Interest Rates

- Suppose that an amount A is invested for n years at an interest rate of R per annum.
- If the rate is compounded m times per annum, the value becomes

$$A\left(1+\frac{R}{m}\right)^{mn}$$

- If the rate is compounded continuously, the value becomes

$$Ae^{Rn}$$

Example

- Consider an interest rate that is quoted as 10% per annum with semiannual compounding.
- The equivalent rate with continuous compounding is:

$$2 \ln\left(1 + \frac{0.1}{2}\right) = 0.09758$$

- This equals 9.758% per annum.

Zero rates

- n-year zero rate (or spot rate) is the rate of interest earned on an investment that starts today and lasts for n years.
- Zero rates are used to discount future cash flows to their present value.
- The zero rate as a function of maturity is referred to as the zero curve.

Bond Pricing

- Most bonds provide coupons periodically.
- The bond's principal (which is also known as its par value or face value) is received at the end of its life.
- If the coupon rate is zero, the bond is a zero-coupon bond (ZCB).
- The theoretical price of a bond can be calculated as the present value of all the cash flows that will be received by the owner of the bond.

Example: Pricing a 2-Year Coupon Bond

Suppose $i/2$-year zero rates (semi-annual) are quoted as

$$r_1 = 5\%, r_2 = 5.8\%, r_3 = 6.4\%, r_4 = 6.8\%.$$

- 2-year bond, principal = 100, semi-annual coupon rate = 6%
- Discount each cash flow using the corresponding zero rate:

$$\begin{aligned} P &= \frac{3}{(1+0.05/2)^1} + \frac{3}{(1+0.058/2)^2} \\ &\quad + \frac{3}{(1+0.064/2)^3} + \frac{103}{(1+0.068/2)^4} \\ &\approx 2.93 + 2.83 + 2.73 + 90.20 = 98.69 \end{aligned}$$

- **Theoretical price** ≈ 98.69

Yield to Maturity (YTM)

- The yield to maturity (YTM) is the internal rate of return (IRR) of the bond assuming it is held to maturity and all coupons are reinvested at the same rate.
- The YTM is the single discount rate that equates the present value of all future cash flows to the current market price of the bond.
- For a bond with price P, coupon payments C_i at times t_i, and principal F at maturity T, the (m-times compounded) YTM y satisfies:

$$P = \sum_i \frac{C_i}{(1 + y/m)^{mt_i}} + \frac{F}{(1 + y/m)^{mT}}$$

Motivation

- We need a way to measure the sensitivity of a bond's price to interest rates
- Measuring price sensitivity to the whole curve can be difficult
- We will focus on several key interest rate factors (e.g., short, medium, or long term rates)

Taylor's expansion of one-factor model

- Consider an asset price P as a function of a factor y:

$$P = f(y).$$

- First order approximation:

$$\Delta P \approx f'(y)\Delta y.$$

- Second order approximation:

$$\Delta P \approx f'(y)\Delta y + \frac{1}{2}f''(y)(\Delta y)^2.$$

ZCB as one single factor model

- A zero-coupon bond (ZCB) can be viewed as a single factor model where the only risk factor is the yield to maturity (YTM) y.
- The price of a semiannual ZCB with face value F and maturity T years is

$$P = f(y) = F\left(1 + \frac{y}{2}\right)^{-2T}.$$

Taylor Expansion for ZCB

- First order derivative is negative:
$$f'(y) = -\frac{T}{1+y/2} f(y)$$
- Second order derivative is positive:
$$f''(y) = \frac{T^2 + T/2}{(1+y/2)^2} f(y)$$
- Taylor approximation:
$$\Delta P \approx -\frac{T}{1+y/2} P \Delta y + \frac{1}{2} \frac{T^2 + T/2}{(1+y/2)^2} P (\Delta y)^2.$$
- $$\frac{\Delta P}{P} \approx -\frac{T}{1+y/2} \Delta y + \frac{1}{2} \frac{T^2 + T/2}{(1+y/2)^2} (\Delta y)^2.$$

DV01

- The DV01 (dollar value of 01) measures the change in the price of a bond for a 1 basis point (0.01%) change in yield.

$$DV01 = -\frac{\Delta P}{10000 \Delta y} \approx -\frac{f'(y)}{10000}.$$

- Prices usually go down if rates go up. Use minus sign to think of DV01 as a positive number.

Example

- Consider a bond with face value 100:
  - $P = 100$ when $y = 5\%$
  - $P = 99.973$ when $y = 5.02\%$
- Then, for a face F, its DV01 is

$$DV01 \approx -\frac{\frac{F}{100}(100 - 99.973)}{10^4 \times (0.05 - 0.0502)} = \frac{F}{100} 0.0135.$$

Example: DV01 Hedging

- Consider an option on a bond with face value \$100:
    - Option price is $P = 8.0866$ if bond YTM is 4.01%
    - Option price is $P = 8.2148$ if bond YTM is 3.99%
- Calculate DV01 for the option:

$$DV01(\text{option}) = -\frac{8.0866 - 8.2148}{10,000(0.0401 - 0.0399)} = 0.0641$$

- Consider another bond (\$100 face) with DV01 $= 0.0857$ at $y = 4\%$
- Question: If we have \$100 million face of the option, how can we hedge against small interest rate moves with the bond?

Example: DV01 Hedging (Continued)

- \$100 million face of the option has DV01:

$$\frac{100,000,000}{100} \times 0.0641 = 64,100$$

- Bond with face F has DV01:

$$\frac{F}{100} \times 0.0857$$

- To hedge, we need the total DV01 to be zero:

$$0 = 64,100 - \frac{F}{100} \times 0.0857$$

- Solving for F:

$$F = \frac{64,100 \times 100}{0.0857} \approx \$74,795,799$$

- We need to short approximately \$74.8 million face value of the bond.

Duration

- Duration is defined as:

$$D = -\frac{1}{P}\frac{\Delta P}{\Delta y} = \frac{10,000 \cdot DV01}{P}$$

- Measures sensitivity of the relative change in the price of the security to changes in y
- If $P = f(y)$, then for small changes in y:

$$\frac{\Delta P}{P} \approx -D\Delta y$$

- Duration can also be expressed as:

$$D = -\frac{f'(y)}{f(y)}$$

Convexity

- If $P = f(y)$, convexity is defined as:

$$C = \frac{1}{P}\frac{d^2 P}{dy^2} = \frac{f''(y)}{f(y)}$$

- Measures the sensitivity of interest rate sensitivity to changes in rates (it is a second derivative)
- Some textbooks define convexity as:

$$C = \frac{1}{2}\frac{f''(y)}{P}$$

Approximation Formulas

- 1st order approximation:

$$\frac{\Delta P}{P} = -D\Delta y$$

- 2nd order approximation:

$$\frac{\Delta P}{P} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

Economic Interpretation

- When $D > 0$: rates $\uparrow \Rightarrow P \downarrow$; rates $\downarrow \Rightarrow P \uparrow$
- When $D < 0$: the effect is opposite
- When $C > 0$: positive contribution to P when rates vary (either $\uparrow$ or $\downarrow$)
- When $C < 0$: negative contribution to P
- $C > 0$: long volatility or long convexity

Note on Scaling

- If $P(F) = F \times P(1)$ (i.e., price is linear in face value):
  - DV01 is linear in F
  - D and C do not depend on F

Coupon Bond Pricing

- Bond: annual coupon rate q, N remaining payments at times $\frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{N}{2}$
- Price (with F face value) satisfies:

$$\frac{P}{F} = \frac{q}{2}\sum_{i=1}^{N}\left(1+\frac{y}{2}\right)^{-i} + \left(1+\frac{y}{2}\right)^{-N}$$
$$= \frac{q}{y}\left(1-\frac{1}{(1+y/2)^N}\right) + \frac{1}{(1+y/2)^N}$$

- This can be written as:

$$\frac{P}{F} = f(y)$$

DV01 for Coupon Bond

- For a coupon bond, the DV01 is:

$$\text{DV01} = -\frac{F\cdot}{10,000} f'(y)$$

- Where:

$$-f'(y) = \frac{q}{y^2}\left(1-(1+y/2)^{-N}\right) + \frac{N}{2}(1-\frac{q}{y})(1+y/2)^{-N-1}$$

- Set $T = N/2$ (time to maturity in years), then:

$$\text{DV01} = \frac{F}{10000}\left[\frac{q}{y^2}\left(1-\frac{1}{(1+y/2)^{2T}}\right)+(1-\frac{q}{y})\frac{T}{(1+y/2)^{2T+1}}\right]$$

Duration for Coupon Bond

- Duration for a coupon bond is:

$$D = \frac{10000 \cdot \mathrm{DV01}}{P}$$

- Substituting the DV01 formula:

$$D = \frac{F}{P}\left[\frac{q}{y^2}\left(1 - \frac{1}{(1+y/2)^{2T}}\right) + \left(1 - \frac{q}{y}\right)\frac{T}{(1+y/2)^{2T+1}}\right]$$
$$= \frac{\frac{q}{y^2}\left(1 - \frac{1}{(1+y/2)^{2T}}\right) + \left(1 - \frac{q}{y}\right)\frac{T}{(1+y/2)^{2T+1}}}{\frac{q}{y}\left(1 - \frac{1}{(1+y/2)^{2T}}\right) + \frac{1}{(1+y/2)^{2T}}}$$

Zero Coupon Bond Summary

- ZCB with maturity T years:

$$P = F\left(1+\frac{y}{2}\right)^{-2T}$$

- Duration:

$$D = \frac{T}{1+y/2}$$

- Convexity:

$$C = \frac{T^2 + T/2}{(1+y/2)^2}$$

- DV01:

$$\mathrm{DV01} = \frac{1}{10,000}\frac{FT}{(1+y/2)^{2T+1}}$$

Effects of Maturity on ZCB Risk Measures

- For fixed $y > 0$:
  - $T \uparrow \Rightarrow P \downarrow$ (longer maturity bonds are cheaper)
  - $T \uparrow \Rightarrow D \uparrow$ (higher interest rate sensitivity)
  - $T \uparrow \Rightarrow C \uparrow$ (higher convexity)
  - $T \uparrow \Rightarrow$ DV01 first increases then decreases when T is sufficiently large

Fixed Income Portfolio

- Consider a portfolio of fixed income securities $S_1, \ldots, S_M$ with:
  - Prices $P_1, \ldots, P_M$
  - Face values $F_1, \ldots, F_M$
- Total value of portfolio:

$$P(\text{port}) = \sum_{i=1}^{M} F_i P_i$$

Portfolio DV01

- Let y be an interest rate factor, the portfolio DV01 is:

$$\text{DV01(port)} = -\frac{\Delta P(\text{port})}{10,000\Delta y}$$

- Note $\Delta P(\text{port}) = \sum_{i=1}^{M} F_i \Delta P_i$, then:

$$\text{DV01(port)} = \sum_{i=1}^{M} F_i \left(-\frac{\Delta P_i}{10,000\Delta y}\right) = \sum_{i=1}^{M} F_i \text{DV01}_i$$

- DV01_i is the DV01 for S_i
- DV01 is linear combination of individual DV01s

Portfolio duration and convexity

- Portfolio duration:

$$D(\text{port}) = -\frac{\Delta P(\text{port})}{P(\text{port})\Delta y} = \sum_{i=1}^{M} \frac{F_i P_i}{P(\text{port})} \cdot \frac{-\Delta P_i}{P_i \Delta y} = \sum_{i=1}^{M} \frac{F_i P_i}{P(\text{port})} D_i$$

- $D(\text{port})$ is a weighted average of the component durations D_i where the weight is the percentage of the portfolio total price in security S_i
- Similarly, portfolio convexity:

$$C(\text{port}) = \sum_{i=1}^{M} \frac{F_i P_i}{P(\text{port})} C_i$$

where C_i is the convexity of security S_i.

Hedging based on duration and convexity

- Basic idea: duration matching
- If portfolio A and B have the same total price $P_A = P_B$ and durations $D_A = D_B$, then for small changes in y, the price changes ΔP_A and ΔP_B are approximately the same:

$$\frac{\Delta P}{P} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

Example: Duration Matching

- 3 securities labeled S_1, S_2, S_3 with maturities 2, 5, 10 years
- Portfolio A: \$1M face in S_2
- Portfolio B: F_1 face in S_1, F_3 face in S_3
- Given: $\hat{r}(2) = 5.78\%$, $\hat{r}(5) = 6.02\%$, and $\hat{r}(10) = 6.26\%$
- Question: What values of F_1 and F_3 should be chosen so that the two portfolios have their values and duration matched?

Example: Duration Matching (Continued)

- Set up the equations for value matching:

$$P_A = P_B \Rightarrow F_2 P_2 = F_1 P_1 + F_3 P_3$$

- Set up the equations for duration matching:

$$D_A = D_B \Rightarrow D_2 = \frac{F_1 P_1 D_1 + F_3 P_3 D_3}{F_1 P_1 + F_3 P_3}$$

where

$$D_i = \frac{T_i}{1 + \hat{r}(T_i)/2}, P_i = \frac{1}{(1 + \hat{r}(T_i)/2)^{2T_i}}, F_2 = 1 \text{ million}$$

- Solution:

$$F_1 = 520,386.39, F_3 = 516,843.32, D_A = D_B = 4.8539$$

Case 1: Parallel Shift +35 bps

- New yields after shift: $y_1 = 0.0613$, $y_2 = 0.0637$, $y_3 = 0.0661$
- New prices: $P_1 = 0.8862$, $P_2 = 0.7309$, $P_3 = 0.5219$
- Exact change in portfolio A: -12511.6750
- Exact change in portfolio B: -12448.8441
- First order approximation are the same:

$$\Delta P_A \approx -P_A D_A \Delta y = -12628.8786$$

$$\Delta P_B \approx -P_B D_B \Delta y = -12628.8786$$

- Exact change in portfolio C (short A long B): 62.83

Case 2: Parallel shift -35 bps

- New yields after shift: $y_1 = 0.0543$, $y_2 = 0.0567$, $y_3 = 0.0591$
- New prices: $P_1 = 0.9008$, $P_2 = 0.7568$, $P_3 = 0.5513$
- Exact change in portfolio A: 12747.6861
- Exact change in portfolio B: 12813.2042
- First order approximation are the same:

$$\Delta P_A \approx -P_A D_A \Delta y = 12628.8786$$

$$\Delta P_B \approx -P_B D_B \Delta y = 12628.8786$$

- Exact change in portfolio C (short A long B): 65.5181

Barbell vs Bullet Strategy

- Long portfolio B and short portfolio A
- Initially, $P_B - P_A = 0$ and $D_B = D_A$
- +35bps shift, our gain/loss is (exactly):

$$\Delta P_B - \Delta P_A = -12,448.8 + 12,511.7 = 62.9$$

- -35bps shift, our gain/loss is (exactly):

$$\Delta P_B - \Delta P_A = 12,813.2 - 12,747.7 = 65.5$$

- Gain in both up and down shift!
- This is because we are long convexity!

$$C_A = 25.916, \quad C_B = 40.007$$

- Portfolio B is an example of a **barbell** - long short and long maturity bonds
- Portfolio A is an example of a **bullet** - long medium maturity bond