Structural Models of Default: Merton Model

Overview of the Merton Model

- A model of default is known as a structural or firm-value model when it attempts to explain the mechanism by which default takes place.
- The model proposed in Merton (1974) is the prototype of all firm-value models.
- Many extensions of this model have been developed over the years, but Merton's original model remains an influential benchmark and is still popular with practitioners and researchers in credit risk analysis.

Setup

- The value at time t of equity and debt is denoted by S_t and B_t, respectively.
- The value of the firm's assets is simply the sum of these:

$$V_t = S_t + B_t, \quad 0 \le t \le T$$

- Default occurs if the firm with debt obligation B misses a payment to its debt holders, which in the Merton model can occur only at maturity T:
  - **No Default** ($V_T > B$): The firm's asset value exceeds the debt obligation. Debtholders receive the full face value B, and shareholders receive the residual value $S_T = V_T - B$.
  - **Default** ($V_T \le B$): The firm's asset value is insufficient to meet debt obligations. Shareholders exercise their limited liability and transfer control to bondholders, who liquidate the firm. The outcome is $B_T = V_T$ and $S_T = 0$.

Mathematical Setup

- Asset value V follows a geometric Brownian motion:

$$dV_t = V_t(\mu dt + \sigma dW_t).$$

- Default occurs at T if $V_T \leq B$.
- Equity and debt value at maturity:

$$S_T = \max\{V_T - B, 0\} = (V_T - B)^+,$$

and

$$B_T = \min\{V_T, B\} = B - (V_T - B)^-.$$

Assumptions

- We assume frictionless markets with continuous trading.
- The risk-free interest rate is constant and equal to $r \geq 0$.
- The firm's asset value process (V_t) is independent of the firm's capital structure, particularly the debt level B.

Default probability

- The asset value V_T is

$$V_T = V_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma W_T\right\}$$

- The log of the asset value at maturity is normally distributed:

$$\log V_T \sim \mathcal{N}\left(\log V_0 + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right)$$

- The probability of default is given by:

$$\begin{aligned} P(V_T \leq B) &= P\left(\log V_T \leq \log B\right) \\ &= P\left(W_T \leq \frac{\log B - \log V_0 - \left(\mu - \frac{\sigma^2}{2}\right) T}{\sigma}\right) \\ &= \Phi\left(\frac{\log(B/V_0) - \left(\mu - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}\right) \end{aligned}$$

Example: Default Probability Calculation

- Consider a firm with the following parameters:
  - Current asset value: $V_0 = \$100$ million
  - Debt obligation: $B = \$80$ million
  - Asset volatility: $\sigma = 0.2$ (20% per year)
  - Drift rate: $\mu = 0.05$ (5% per year)
  - Risk-free rate: $r = 0.03$ (3% per year)
  - Time to maturity: $T = 1$ year
- The default probability is:

$$
\begin{aligned}
P(V_T \leq B) &= \Phi\left(\frac{\log(80/100) - \left(0.05 - \frac{0.2^2}{2}\right) \times 1}{0.2\sqrt{1}}\right) \\
&= \Phi\left(\frac{-0.223 - 0.03}{0.2}\right) \\
&= \Phi(-1.265) \approx 0.103
\end{aligned}
$$

- The firm has approximately a 10.3% probability of defaulting within one year.

Pricing of equity

- The value of equity at maturity is given by:

$$S_T = (V_T - B)^+$$

- The present value of equity can be computed using the risk-neutral measure:

$$S_t = e^{-r(T-t)}\mathbb{E}^Q\left[(V_T - B)^+\right]$$

- By Black-Scholes formula, we have

$$S_t = C^{BS}(t, V_t; r, \sigma, B, T) = V_t\Phi(d_{t,1}) - Be^{-r(T-t)}\Phi(d_{t,2})$$

where

$$d_{t,1} = \frac{\log(V_t/B) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$
$$d_{t,2} = d_{t,1} - \sigma\sqrt{T-t}$$

Example: Equity Pricing

- Calculate the equity value using the Black-Scholes formula:

$$d_1 = \frac{\log(100/80) + (0.03 + 0.2^2/2) \times 1}{0.2\sqrt{1}} = \frac{0.223 + 0.05}{0.2} = 1.365$$
$$d_2 = 1.365 - 0.2 \times 1 = 1.165$$

- With $\Phi(1.365) \approx 0.914$ and $\Phi(1.165) \approx 0.878$:

$$\begin{aligned} S_0 &= 100 \times 0.914 - 80 \times e^{-0.03 \times 1} \times 0.878 \\ &= 91.4 - 80 \times 0.970 \times 0.878 \\ &= 91.4 - 68.1 = \$23.3 \text{ million} \end{aligned}$$

Pricing of Debt

- The value of debt at maturity is:

$$B_T = \min\{V_T, B\} = B - (V_T - B)^-$$

- The present value of debt is:

$$B_t = e^{-r(T-t)}\mathbb{E}^Q[B_T] = Be^{-r(T-t)} - e^{-r(T-t)}\mathbb{E}^Q[(V_T - B)^-]$$

- This can be written as:

$$B_t = Be^{-r(T-t)} - P^{BS}(t, V_t; r, \sigma, B, T)$$

  where P^{BS} is the Black-Scholes put option price.

- Alternatively: $B_t = V_t - S_t$
- For our example: $B_0 = 100 - 23.3 = \$76.7$ million

Credit Spread Calculation

- The credit spread is calculated as:

$$\text{Credit Spread} = \text{Yield on Risky Debt} - \text{Risk-free Rate}$$

- The yield on risky debt is:

$$y = \frac{1}{T} \log\left(\frac{B}{B_0}\right)$$

- Therefore, the credit spread is:

$$s = y - r = \frac{1}{T} \log\left(\frac{B}{B_0}\right) - r$$

- For our example:

$$\begin{aligned} s &= \frac{1}{1} \log\left(\frac{76.7}{80}\right) - 0.03 \\ &= -\log(0.959) - 0.03 \\ &= 0.042 - 0.03 = 0.012 = 120 \text{ basis points} \end{aligned}$$

Default probability under risk-neutral measure

- The default probability under the risk-neutral measure is given by:

$$Q(V_T \leq B) = \Phi\left(\frac{\log(B/V_0) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) = \Phi(-d_{0,2})$$

- For our example:

$$\begin{aligned} Q(V_T \leq 80) &= \Phi\left(\frac{\log(80/100) - (0.03 - 0.02) \times 1}{0.2\sqrt{1}}\right) \\ &= \Phi\left(\frac{-0.223 - 0.01}{0.2}\right) \\ &= \Phi(-1.165) \approx 0.122 \end{aligned}$$

- The firm has approximately a 12.2% probability of defaulting within one year under the risk-neutral measure.

Expected loss on the debt

- The expected loss on the debt can be calculated as:

$$\text{Expected Loss} = e^{-rT}B - B_0.$$

- For our example:

$$\text{Expected Loss} = e^{-0.03\times 1} \times 80 - 76.7 = 0.8596 \text{ million}$$

or

$$\frac{e^{-rT}B - B_0}{B_0} = \frac{0.8596}{76.7} \approx 0.0112 = 1.12\%$$

of its no-default value.

Delta of equity

- The delta of equity can be calculated as:

$$\Delta_S = \frac{\partial S}{\partial V} = \Phi(d_1)$$

- For our example:

$$\Delta_S = \Phi(1.365) \approx 0.914$$

The relationship between asset volatility and equity volatility

- The relationship between asset volatility (σ_V) and equity volatility (σ_S) can be expressed as:

$$\sigma_S S_0 = \Delta_S \sigma_V V_0$$

- (Proof) Using Itô's lemma and

$$dV_t = \mu_V V_t dt + \sigma_V V_t dW_t^V$$

to derive the dynamics of equity values,

$$\begin{aligned} dS_t &= \Delta_S dV_t + \frac{1}{2}\Gamma_S (dV_t)^2 \\ &= (\cdots)dt + \Delta_S \sigma_V V_t dW_t \end{aligned}$$

Example

- Asset volatility: $\sigma = 0.2$ (20% per year)
- Equity volatility: $\sigma_S = \frac{\Delta_S V_0}{S_0}\sigma = \frac{0.914 \times 100}{23.3} \times 0.2 \approx 0.785$ (78.5% per year)
- This shows that equity volatility is significantly higher than asset volatility due to the leverage effect.

Recovery Rate

- $$\text{Expected Loss Rate} = \text{Q-Default Probability} \times (1 - \text{Recovery Rate})$$
- For our example:

$$\begin{aligned}
\text{Expected Loss} &= 0.122 \times (1 - 0.91) \\
&= 0.122 \times 0.09 \\
&\approx 0.0110 \text{ million}
\end{aligned}$$

Summary of the Lecture

- The Merton model is a foundational framework for structural credit risk modeling, focusing on the relationship between a firm's asset value and its debt obligations.
- Key concepts include:
  - Default occurs when the firm's asset value falls below its debt obligation at maturity.
  - Equity can be viewed as a call option on the firm's assets, while debt can be viewed as a combination of risk-free debt and a short put option.
- The model provides formulas for default probability, equity pricing, debt pricing, credit spreads, and risk-neutral default probabilities.
- Extensions of the Merton model allow for more realistic assumptions, but the original model remains a cornerstone in credit risk analysis.