Gain

In ME2801 we cover a number of topics in linear systems and feedback control. In that context we use the term *gain* in a generic way and the term *DC gain* is used to refer to a specific property of linear systems.

1 Gain

The term *gain* is used generically to refer to a linear system with a constant transfer function. A common example is the transfer function for a PID controller which we can write as

$$\frac{U(s)}{E(s)} = K_p + \frac{K_i}{s} + K_d(s)$$

where K_p, K_i and K_d are referred to as the proportional, integral and derivative gain values. Another example is the simple block diagram shown in Figure 1 where the output is the product of the input and the gain value K.

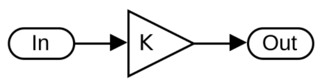


Figure 1: Block diagram of gain.

2 DC Gain

The *DC gain*, K_{dc}, is a property of a system. Considering a linear system described as a transfer function, the DC gain can be defined in either the time-domain or the frequency-domain.

2.1 Time-Domain

In the time-domain, the DC gain is the ratio of the magnitude of the steady-state step response to the magnitude of the step input. Furthermore, if we consider a unit step response (where the input has a magnitude of one), then the DC gain is the steady-state magnitude of the output. This property is illustrated in the second-order unity step response graphs of Figure 2. In the graph on the left, the input is amplified by a factor of 10 ($K_{dc} = 10$). In the graph on the right, the input is attenuated by a factor of 2 ($K_{dc} = 0.5$).

2.1.1 DC Gain from Transfer Function

Based on this time-domain definition of the DC gain, we can find the value directly from the transfer function

$$\frac{Y(s)}{X(s)} = G(s).$$

We consider the input to be a step input with a magnitude X_0, so $X(s) = \frac{X_0}{s}$. Assuming a stable response, we can solve for the steady-state magnitude of the output Y_0 using the final value theorem which yields

$$Y_0 = y(t \to \infty) = \lim_{s\to 0}[sY(s)] = \lim_{s\to 0}\left[sG(s)\frac{X_0}{s}\right] = \lim_{s\to 0}[G(s)]X_0 = G(s=0)X_0.$$

Since the DC gain is the ratio of output to input

$$K_{dc} = \frac{Y_0}{X_0} = \lim_{s\to 0}[G(s)] = G(s=0). \tag{1}$$

2.2 Frequency-Domain

In the frequency-domain, the DC gain is the magnitude of the ratio of output to input at zero frequency. The magnitude of the transfer function is expressed as $|G(s = j\omega)|$, so the DC gain is

$$K_{dc} = \lim_{\omega\to 0}[|G(j\omega)|] = G(s=0) \tag{2}$$

which is equivalent to (1).

Using the Bode plot of a system, the DC gain is the value of the magnitude portion of the plot as the frequency approaches zero as illustrated in Figure 3.

2.3 Canonical Form of First and Second Order Transfer Functions

In ME2801 we use standardized forms of the first and second order (underdamped) transfer functions:

$$G_1(s) = K_{dc}\frac{a}{s+a} = K_{dc}\frac{1}{\tau s+1}$$

and

$$G_2(s) = K_{dc}\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = K_{dc}\frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_d^2}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The reason for the particular forms of the expressions is that the gain term in each of the transfer functions above is equivalent to the DC gain (K_{dc}) as defined in (1).

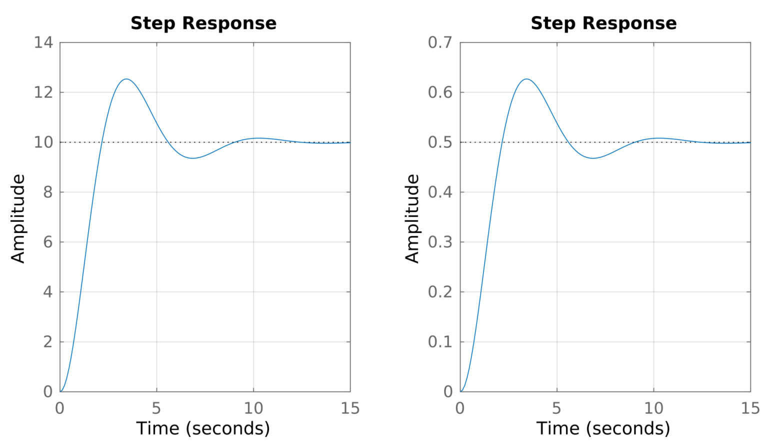
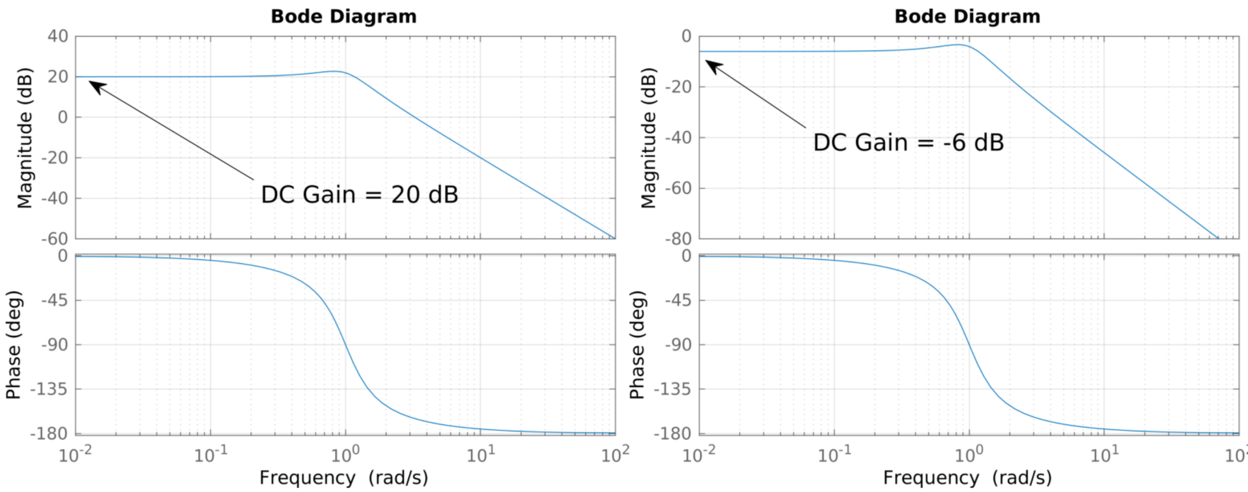


Figure 2: Illustration of DC gain visible in the unity step response. The system represented on the left has a DC gain of 10. The system on the right has a DC gain of 0.5.



(a) DC gain of 20 dB. (b) DC gain of -6 dB.

Figure 3: Illustration of DC gain visible in the frequency response (Bode plot) for the same two systems illustrated in Figure 2.

2.4 Exercises

1. Find K_{dc} for the the following transfer functions:

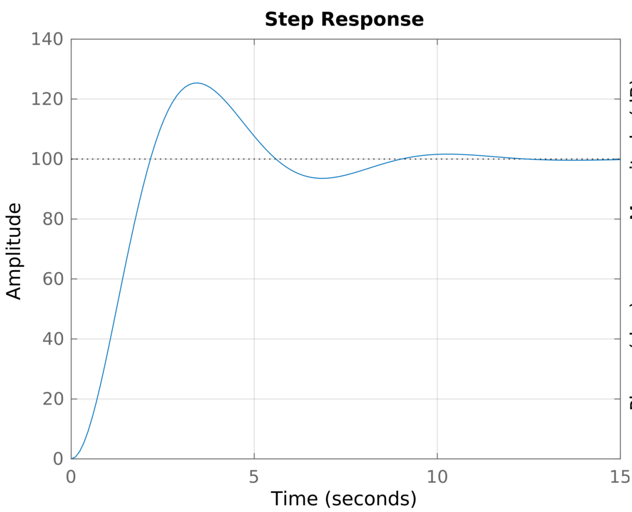
   (a) $G(s) = \dfrac{K}{s+a}$

   (b) $G(s) = \dfrac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

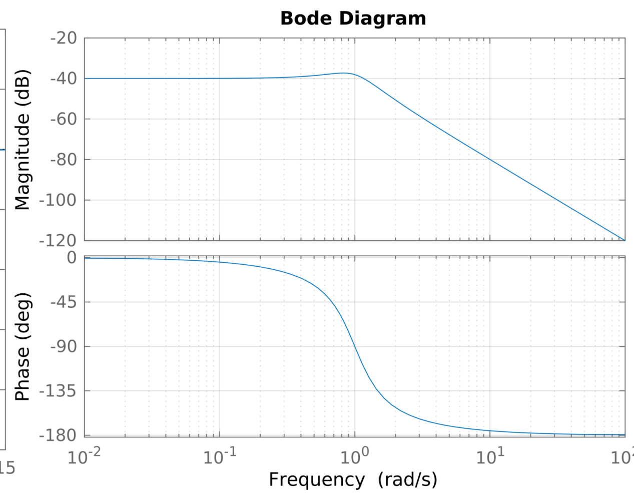
   (c) $G(s) = \dfrac{K}{(s+\zeta\omega_n)^2 + \omega_d^2}$

2. Given the second-order unit-step response in Figure 4a, find K_{dc}, sketch the Bode plot and approximate the system transfer function.

3. Given the Bode plot in Figure 4b, find K_{dc}, sketch the unit step response and approximate the system transfer function.



(a) Example unit-step response.



(b) Example Bode plot

Figure 4: Examples for exercises.