Wave Spectra Notes

1 Overview

The goal is to consolodate some of the existing literature and notes about wave spectra as they pertain to generating simple, but physically realizable, wave fields for simulation.

2 General Properties and Relationships

2.1 Properties

Consider a zero-mean, wide-sense-stationary, continuous time random process $x(t)$. The expected instantaneous power $E[x^2(t)]$ is equivalent to the variance in the signal σ_x^2, the autocorrelation function $R_{xx}(\tau)$, integral of the power spectral density (spectrum)

$$\sigma_x^2 = E[x^2(t)] = R_{xx}(\tau = 0) = \frac{1}{2\pi}\int_0^\infty S_{xx}(\omega)d\omega$$

The PSD is the Fourier transform of the autocorrelation

$$S_{xx}(\omega) = \mathcal{F}R_{xx}(\tau)$$

We can relate a spectrum expressed in angular frequency, $S(\omega)$, as a spectrum exrpessed in ordinary frequency, $S'(f)$ as [1]

$$S'(f) = S(\omega = 2\pi f)\frac{d\omega}{df} = (2\pi)S(\omega = 2\pi f) \tag{1}$$

where the factor of 2π insures that the area under the curve stays the same [2].

3 Relating Spectra in Space and Time

Techet notes[3] the general form of a fully-developed wave spectra, as a function of temporal frequency,

$$S(\omega) = \frac{C}{\omega^5}\exp\left[-D/\omega^4\right]. \tag{2}$$

A wave spectra can be equivalently expressed as a function of three different variables:

1. ω: *angular frequency* in rad/s,
2. f: *ordinary frequency* or just *frequency*, $\omega = 2\pi f$, in cycles/s or Hz, or
3. k: *wave number*, $k = 2\pi/\lambda$, in rad/m.

[1] We use the prime notation in $S(\omega)$, $S'(f)$, $S''(k)$ to differentiate between expressions of equivalent spectra with different independent variables.

[2] http://www.wikiwaves.org/Ocean-Wave_Spectra

[3] https://ocw.mit.edu/courses/mechanical-engineering/2-22-design-principles-for-ocean-vehicles-13-4
readings/lec6_wavespectra.pdf

Next we need to specify the dispersion relation[4]. For deep water

$$\omega = \sqrt{gk} \tag{3}$$
$$c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \tag{4}$$
$$c_g = \frac{\partial \omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{k}} \tag{5}$$

were c_p is the phase velocity and c_g is the group velocity.

The process of relating spectra expressed as functions of angular frequency (ω) or wavenumber (k) is described in [1]. The constraint is that total wave height variance must be indpendent of independent variable (area under the curve doesn't change), which is expressed as

$$\int S''(k) k dk = \int S'(f) df. \tag{6}$$

If we follow the argument of equating the energy density in spectral bands [], then

$$S''(k) = S'(f) \frac{1}{k} \frac{df}{dk} \tag{7}$$

and since

$$f = \frac{\omega}{2\pi} \rightarrow \frac{df}{dk} = \frac{d\omega}{dk} \frac{1}{2\pi} \tag{8}$$

we can substitute the definition of the group velocity

$$c_g = \frac{\partial \omega}{\partial k} \tag{9}$$

so that in general

$$S''(k) = S'(f) \frac{c_g}{2\pi k} = S(\omega) \frac{c_g}{k} \tag{10}$$

and specifically for the deep water definition of c_g

$$S''(k) = S'\left(f = \frac{\sqrt{gk}}{2\pi}\right) \frac{\sqrt{g}}{4\pi \left(k^{3/2}\right)} = S\left(\omega = \sqrt{gk}\right) \frac{\sqrt{g}}{2\left(k^{3/2}\right)}. \tag{11}$$

If we apply this to (2) we can express the general form of a fully-developed spectrum in deep-water as

$$S(\omega) = \frac{C}{\omega^5} \exp\left[-D/\omega^4\right] \tag{12}$$
$$S''(k) = \frac{C}{2g^2k^4} \exp\left[-D/(gk)^2\right] \tag{13}$$

4 Phillips Spectrum

The generic Phillips spectrum is reported by [2] as

$$S''_{Ph}(k) = \frac{A}{k^4} \exp\left[-\frac{1}{(kL)^2}\right] = \frac{A}{k^4} \exp\left[-\frac{g^2}{(k^2)(U^4)}\right]. \tag{14}$$

[4]`https://en.wikipedia.org/wiki/Dispersion_(water_waves)`

We can convert this to an expression in terms of angular frequency. Using (11) we find

$$S(\omega) = S''\left(k = \frac{\omega^2}{g}\right) 2\frac{\omega^3}{g^2} \tag{15}$$

which lead to

$$S_{Ph}(\omega) = \frac{A(2)(g^2)}{\omega^5} \exp\left[-\frac{(g/U)^4}{\omega^4}\right] \tag{16}$$

which agrees with the general form given in (2). This is close to, but not exactly the same as the expression derived in [3] (see equation (7)).

5 Pierson-Moskowitz

The Pierson-Moskowitz spectrum can be expressed as[5]

$$S_{PM}(\omega) = \frac{\alpha g^2}{\omega^5} \exp\left[-\beta\left(\frac{\omega_o}{\omega}\right)^4\right] = \frac{\alpha g^2}{\omega^5} \exp\left[-\beta\left(\frac{g}{U_{19.5}(\omega)}\right)^4\right] \tag{17}$$

where $\alpha = 8.1(10^{-3})$ and $\beta = 0.74$, $\omega_o = g/U_{19.5}$ and $U_{19.5}$ is the wind speed at 19.5 m above the sea surface.

The peak of the spectrum, where $dS/d\omega = 0$, occurs at $\omega_p = \omega_o/1.14$, therefore we can express the spectrum in terms of the peak angular frequency as

$$S_{PM}(\omega) = \frac{\alpha g^2}{\omega^5} \exp\left[-\beta\left(\frac{\omega_p(1.14)}{\omega}\right)^4\right] = \frac{\alpha g^2}{\omega^5} \exp\left[-\frac{5}{4}\left(\frac{\omega_p}{\omega}\right)^4\right] \tag{18}$$

Furthermore the relationship between significant wave height ($H_{1/3}$) and wind velocity is

$$H_{1/3} = 0.21\frac{(U_{19.5})^2}{g} \tag{19}$$

which lead to

$$S_{PM}(\omega) = \frac{\alpha g^2}{\omega^5} \exp\left[-\beta\left(\frac{(0.21(g))^2}{H_{1/3}^2}\right)\frac{1}{\omega^4}\right] \tag{20}$$

This agrees with the expression Techet provides the Pierson-Moskowitz spectrum

$$S_{PM}(\omega) = \frac{\alpha g^2}{\omega^5} \exp\left[-\frac{(0.032)g^2}{\left(H_{1/3}\right)^2}\frac{1}{\omega^4}\right]. \tag{21}$$

Applying (11) to express the spectrum in terms of wavenumber yields

$$S''_{PM}(k) = \frac{\alpha}{k^4} \exp\left[-\beta(0.21^2)\frac{1}{(H_{1/3})^2}\frac{1}{k^2}\right]. \tag{22}$$

[5] `http://www.wikiwaves.org/Ocean-Wave_Spectra`

6 Relating Phillips and Pierson-Moskowitz Spectra

If we want to equate the Phillips spectrum (14) with the P-M spectrum (22), the relationship between the constants is

$$A = \alpha \tag{23}$$

$$\left(\frac{g^2}{U^4}\right) = \frac{\beta(0.21)^2}{\left(H_{1/3}\right)^2} \tag{24}$$

which suggests

$$H_{1/3} = \frac{0.18(U)^2}{g} \tag{25}$$

which is consistent with (19). Therefore, if we wanted to use the Phillips spectrum as expressed in (14),

$$S''_{Ph}(k) == \frac{A}{k^4} \exp\left[-\frac{g^2}{(k^2)(U^4)}\right], \tag{26}$$

we would choose $A = \alpha = 8.1(10^{-3})$ and $U = 2.33\sqrt{H_{1/3}(g)}$ which would provide a Phillips spectrum that is equivalent to the PM spectrum (20) and (21).

References

[1] W. J. Plant, "The ocean wave height variance spectrum: Wavenumber peak versus frequency peak," *Journal of Physical Oceanography*, vol. 39, no. 9, pp. 2382–2383, 2009. [Online]. Available: https://doi.org/10.1175/2009JPO4268.1

[2] J. Tessendorf, C. C, and J. Tessendorf, "Simulating ocean water," 1999.

[3] L.-n. Chen, Y.-c. Jin, Y. Yin, and H.-x. Ren, *On the Wave Spectrum Selection in Ocean Wave Scene Simulation of the Maritime Simulator*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 453–465.