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faculty of science
department of mathematics

MACM 316 D100, Fall 2022
Midterm 1
October 19, 2022, 12:30 pm – 1:20 pm

SFU Email:	@SFU.CA	Signature:	

First Name:	
Last Name:	
SFU ID #:	

1. Do not open this booklet until told to do so.
2. Write your name, SFU student number and email ID in the space provided.
3. Write your answer in the space provided. If additional space is needed use the back of the previous page. Your final answer should be simplified as far as is reasonable.
4. To receive full credit for a particular question your solution must be complete and well presented.
5. **No** books, papers, or electronic devices with the exception of a basic scientific calculator shall be within the reach of a student during the examination.
6. **During the examination, copying from, communicating with, or deliberately exposing written papers to the view of, other examinees is forbidden.**

Question:	1	2	3	4	5	6	Total
Points:	11	8	10	9	10	12	60

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[1] 1. (a) For a general root finding problem, list the following three algorithms in order of increasing speed (where by faster we mean takes less steps to converge to an answer):

Secant method, Newton's method, Bisection method.

Bisection < Secant < Newton.

[2] (b) Write the equation for the tangent line to $y = f(x)$ at $x = p$.

$$y - f(p) = f'(p)(x-p)$$

[3] (c) Solve for the x-intercept of the line in part(b). What formula have you derived and with what roles for p and x?

X-intercept $\Rightarrow y = 0$

$$0 - f(p) = f'(p)(x_{int} - p)$$

$$x_{int} - p = \frac{-f(p)}{f'(p)}$$

$$x_{int} = p - \frac{f(p)}{f'(p)}$$ Newton's method

$p = x_n$, $x_{int} = x_{n+1}$ or $p = x_{n-1}$, $x_{int} = x_n$.

[2] (d) Write the equation of the line that intersects the curve $y = f(x)$ at $x = p$ and $x = q$.

two points $(p, f(p))$, $(q, f(q))$

$$\text{Slope} = \frac{f(q) - f(p)}{q-p}$$

$$y - f(p) = \frac{f(q) - f(p)}{q-p}(x-p)$$

[3] (e) Solve for the x-intercept of the line in part(d). What formula have your derived and with what roles for p, q and x?

$$0 - f(p) = \frac{f(q) - f(p)}{q-p}(x_{int} - p)$$

$$x_{int} = p + \frac{-f(p)(q-p)}{f(q) - f(p)}$$

Secant Method

$x_{int} = x_n$, $p = x_{n-1}$, $q = x_{n-2}$ OR $x_{int} = x_{n+1}$, $p = x_n$, $q = x_{n-1}$

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2. Answer **briefly** but include a **justification** for your answer.

[2] (a) A "Megaflop" stands for 10^6 floating point operations. I am computing the LU factorization of a 4000 by 4000 matrix on my laptop. It takes about 15 seconds. How many Megaflops per second is my laptop capable of performing?

LU : $2/3 (4000)^3$ flops $= 4.27 \times 10^{10}$ flops.

I can do $\dfrac{4.27 \times 10^{10}}{15} \dfrac{\text{flops}}{\text{sec}} \cdot \dfrac{1 \text{ megaflop}}{10^6 \text{ flops}} = 2.84 \times 10^3 \dfrac{\text{Megaflops}}{\text{sec}}$

[2] (b) If it takes 15 seconds to find the LU factorization of a 4000 by 4000 matrix, approximately how long will it take to find the LU factorization of a 8000 by 8000 matrix?

$O(n^3)$

n has doubled

$\therefore$ the time with be 8 times as much $\Rightarrow$ $8 \times 15 = 120$ seconds

[2] (c) I am experimentally investigating the relationship between the time to perform Gaussian elimation on a matrix and the matrix size. For the algorithm I have used, I expect to see performance of order $8n^2$. If I plot log(time) versus log(matrix size), how will I determine if my algorithm is performing as expected?

$t \propto 8n^2$

$\log t \propto \log(8n^2) = \log 8 + 2 \log n$

I expect to see a slope of 2 and a y-intercept of $\log 8$.

[2] (d) Show that for any positive integer k, the sequence defined by $p_n = 1/n^k$ converges linearly.

$\lim_{n \to \infty} \frac{1}{n^k} \ \forall k \in \mathbb{Z}^+ = 0$

$\lim_{n \to \infty} \dfrac{|p_{n+1} - p|}{|p_n - p|} = C$ is linear convergence

$p = 0 \quad p_{n+1} = \left(\frac{1}{n+1}\right)^k \quad p_n = \left(\frac{1}{n}\right)^k$

$\lim_{n \to \infty} \dfrac{|(\frac{1}{n+1})^k - 0|}{|(\frac{1}{n})^k - 0|} = \lim_{n \to \infty} \left(\frac{n}{n+1}\right)^k = \boxed{1}$

this is finite. $\therefore$ $\alpha = 1$ and in fact $C = 1$ as well.

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[5] 3. (a) Use 3-digit chopping arithmetic and the **most efficient method** to evaluate the polynomial $P(x) = 0.987x^2 + 11.2x + 0.246$ at $x = \sqrt{2}$. Clearly show your work.

$\sqrt{2} = 1.41$ in 3-digit chopping arithmetic

Most efficient is nested

$$p(x) = x\,(0.987x + 11.2) + 0.246$$

$$\begin{aligned} p(1.41) &= 1.41\,(0.987(1.41) + 11.2) + 0.246 \\ &= 1.41\,(1.39 + 11.2) + 0.246 \\ &= 1.41\,(12.5) + 0.246 \\ &= 17.6 + 0.246 = 17.8 \end{aligned}$$

Every calc is 3 digit chopped

[2] (b) What is the minimum number of flops needed to make this computation? Assume that you already have a decimal approximation to $\sqrt{2}$.

Nested form $\Rightarrow$ 2 multiplications, 2 additions } 4 operations.

[3] (c) Suppose that p^* must approximate $\sqrt{2}$ with relative error at most 10^{-3}. Find the largest interval in which p^* must lie.

$$\left| \frac{\sqrt{2} - p^*}{\sqrt{2}} \right| \leq 10^{-3}$$

$$|\sqrt{2} - p^*| \leq \sqrt{2} \times 10^{-3}$$

$$\sqrt{2} - \sqrt{2} \cdot 10^{-3} \leq p^* \leq \sqrt{2} \times 10^{-3} + \sqrt{2}$$ Good enough

OR//

$$1.4127993 \leq p^* \leq 1.4156277.$$

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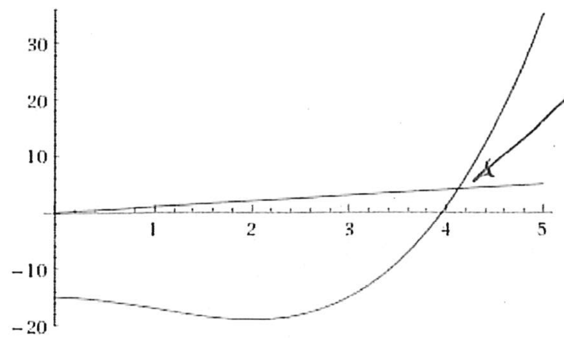
4. Consider the graphs of the two functions

$y = x$ and $y = x^3 - 3x^2 - 15$



this is the fixed point
It is near 4 on in $[4, 4.2]$.

$x = x^3 - 3x^2 - 15$

$x^3 - 3x^2 - x - 15 = 0$

$x^2(x-3) - x - 15 = 0$

$x(x^2 - 3x - 1) - 15 = 0$

[1] (a) Find a fixed point interation function that will find the intersection point. Any of these

$g_1(x) = x^3 - 3x^2 - 15$ $\quad g_3(x) = \sqrt{\dfrac{x+15}{x-3}}$ $\quad g_4(x) = \dfrac{15}{x^2 - 3x - 1}$

$g_2(x) = \sqrt[3]{3x^2 + x + 15}$

[2] (b) Clearly state a starting value and perform one iteration of the fixed point interation using your function. Start at $x = 4$

$g_1(4) = 1$ $\quad g_3(4) = 4.36$

$g_2(4) = 4.06$ $\quad g_4(4) = 5.$

[4] (c) Is your fixed point interation in part (a) guaranteed to converge? If so, show why. If not, find a different function that is guaranteed to converge and justify that it will do so.

$g_1(x)$ will NOT $g_1'(x) = 3x^2 - 6x > 1$ near 4

cube root is increasing function will converge

$\boxed{g_2'(x)} = \frac{1}{3}(3x^2 + x + 15)^{-2/3}(6x+1)$ $\quad g_2'(4) = 0.1243$

$g_3'(x) = \frac{1}{2}\left(\dfrac{x+15}{x-3}\right)^{-1/2}\left(\dfrac{x-3-(x+15)}{(x-3)^2}\right)$ $\quad g_3'(4) = -2.06$ No convergence

$g_4'(x) = -15(x^2 - 3x - 1)^{-2}(2x - 3)$ $\quad g_4'(4) = \dfrac{(-15)(5)}{(4^2 - 12 - 1)^2} = -8.3$ > 1 in absolute value

$\rightarrow$ derivative less than 1, guaranteed to converge.

[2] (d) State an equivalent root-finding problem which will determine the intersection point.

$f(x) = x^3 - 3x^2 - x - 15$

Find x such that $f(x) = 0$.

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[4] 5. (a) Factor the coefficient matrix, A into LU, using partial pivoting if required, such that $PA = LU$. Clearly identify P, L, and U.

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix} \xleftrightarrow{E_2 \leftrightarrow E_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix} \quad \begin{matrix} E_3 - E_1 \to E_3 \\ E_4 - E_1 \to E_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \quad \begin{matrix} E_3 + E_2 \to E_3 \\ E_4 + E_2 \to E_4 \end{matrix} \quad \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U.$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[2] (b) Use L and U to solve $LUx = b$ for $b = [1\ 8\ 4\ 5]^T$.

$LU\vec{x} = b$ $\quad y = U\vec{x}$ $\quad L\vec{y} = \vec{b}$

$L\vec{y} = \vec{b}$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 4 \\ 5 \end{bmatrix}$$

$y_1 = 1$
$y_2 = 8$
$y_1 - y_2 + y_3 = 4 \quad y_3 = 11$
$y_1 - y_2 + y_4 = 5 \quad y_4 = 12$

$$\vec{y} = \begin{bmatrix} 1 \\ 8 \\ 11 \\ 12 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} -16 \\ -15 \\ -11 \\ 12 \end{bmatrix}$$

$U\vec{x} = \vec{y}$
$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 11 \\ 12 \end{bmatrix}$$

$x_2 - x_3 + x_4 = 8 \quad x_2 = -15$
$x_3 = -11$
$x_4 = 12$
$x_1 + 2x_2 - x_3 + 3x_4 = 1$

$x_1 + 2(-15) + 11 + 3(12) = 1$
$x_1 - 30 + 11 + 36 = 1$
$x_1 = -16$

[2] (c) Algebraically state the system of linear equations (4 equations of the form $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = d$) for which you have found a solution.

You solved $LU\vec{x} = \vec{b}$ and $LU = PA$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 4 \\ 5 \end{bmatrix}$$

$x_1 + 2x_2 - x_3 + 3x_4 = 1$
$x_2 - x_3 + x_4 = 8$
$x_1 + x_2 - x_3 + 2x_4 = 4$
$x_1 + x_2 + 3x_4 = 5$

[2] (d) What is the determinant of A?

P swaps two rows $\Rightarrow$ mult det (LU) by -1

$\det(A) = -1 \det(L) \det(U) = (-1)(1)(-1) = \boxed{1}$

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3 data points $\Rightarrow$ degree 2 polynomial

6. Consider the function $f(x) = x^3 - e^{-x}$ and the nodes $x_0 = 0.5$, $x_1 = 0.7$, $x_2 = 1.0$.

[3] (a) Find the maximum degree Lagrange interpolating polynomial, $P_L(x)$, given the stated nodes. Do not do any arithmetic.

$$L_0(x) = \frac{(x-0.7)(x-1.0)}{(0.5-0.7)(0.5-1.0)} \qquad f(0.5) = (0.5)^3 - e^{-0.5}$$

$$L_1(x) = \frac{(x-0.5)(x-1.0)}{(0.7-0.5)(0.7-1.0)} \qquad f(0.7) = (0.7)^3 - e^{0.7}$$

$$f(1.0) = 1 - e^{-1}$$

$$L_2(x) = \frac{(x-0.5)(x-0.7)}{(1.0-0.5)(1.0-0.7)} \qquad P_L(x) = L_0(x)\cdot f(0.5) + L_1(x) f(0.7) + L_2(x) f(1.0)$$

[3] (b) Find the degree 2 Taylor polynomial, $T_2(x)$ centered about zero, for the function $f(x)$.

$$e^x = 1 + x + x^2/2! + \cdots \qquad e^{-x} = 1 - x + x^2/2! - x^3/3! + \cdots$$

$$x^3 - e^x = x^3 - [1 - x + x^2/2! - x^3/3! + \cdots] = x^3 - 1 + x - x^2/2 + x^3/3! + \cdots$$

$$T_2(x) = -1 + x - x^2/2$$

[2] (c) What is the maximum error incurred when using $P_L(x)$ on the interval $[0, 1]$?

$f(x) = x^3 - e^{-x}$ $\quad |f'''(x)| \le 7$ since e^{-x} is decreasing on $[0,1]$

$f'(x) = 3x^2 + e^{-x}$

$f''(x) = 6x - e^{-x}$ $\quad$ Error $\le 7/3!\,(x-0.5)(x-0.7)(x-1.0)$

$f'''(x) = 6 + e^{-x}$ $\quad \le 7/3!\,(0.5)(0.7)(1.0)$ $\quad$ Max occurs at $x=0$

[2] (d) What is the maximum error incurred when using $T_2(x)$ on the interval $[0, 1]$?

$f'''(x) \le 7$ as above

error $\le 7/3!\,(x)^3 \le 7/3!$ $\quad$ (Max of x^3 occurs at $x=1$)

[2] (e) On the interval $[0, 5]$, which polynomial approximation ($T_2(x)$ or $P_L(x)$) of $fx) = x^3 - e^{-x}$ is likely to give better accuracy. Why?

1) Generally $P_L(x)$ will be better since it is not centered at a point zero

2) However, in the interval $[1,5]$ we are not interpolating so we do not have an error bound.

3) On $[0,1]$ $P_L(x)$ is better. On $[1,5]$ $P_L(x)$ is likely better but we cannot tell.