MACM 316 Tutorial 11

Q1. Consider the swinging pendulum problem

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = F(t, \theta, \theta'), \quad \theta(0) = \frac{\pi}{4}, \quad \theta'(0) = 0. \qquad \text{---} \ (1)$$

Re-write the above initial-value problem as a system of ODEs. Let $F = t^2$, $g = 9.81$ and $L = 1$. Use Euler's method to approximate the solution. Compute two steps by hand with $h = 0.01$.

Let $\phi_1 = \theta$, $\phi_2 = \frac{d\theta}{dt}$. We can rewrite (1) as

$$\begin{cases} \frac{d\phi_1}{dt} = \phi_2 \\ \frac{d\phi_2}{dt} = -g\sin(\phi_1) + F(t, \phi_1, \phi_2) \end{cases}$$

Euler's method:
$$\phi_1^{(n+1)} = \phi_1^{(n)} + h\phi_2^{(n)}$$
$$\phi_2^{(n+1)} = \phi_2^{(n)} + h\left(-g\sin\phi_1^{(n)} + F\left(t^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}\right)\right)$$

If $F = t^2$, $\phi_2^{n+1} = \phi_2^{n} + h\left(-g\sin\phi_1^{(n)} + \left(t^{(n)}\right)^2\right)$

Initial Condition: $\phi_1^{(0)} = \frac{\pi}{4}$, $\phi_2^{(0)} = 0$, $t^{(0)} = 0$, $h = 0.01$

$\phi_1^1 = \frac{\pi}{4} + (0.01)(0)$
$= 0.785398163\cancel{5}$
-0.0012433

$\phi_1^2 = \frac{\pi}{4} + (0.01)(\cancel{\ldots})$

$= 0.785398163\,5$
$-0.06436717\,52$

$= 0.7847044417476$

$\phi_2^1 = 0 + (0.01)(-9.81\sin(\frac{\pi}{4})) + (0.01)^2$

$= -0.0021866$

$= -0.064367175234$

$\phi_2^2 = -0.064367175234 + (0.01)(-9.81)(\sin(0.7853981)) + (0.01)^2$

$= -0.1386343504\,75$

$e^x = 1 + x + \frac{x^2}{2} + \cdots\cdots$

Review Questions

1. Find the rate of convergence of the following function as $h \to 0$:

$$\lim_{h\to 0} e^h + e^{-h} = 2.$$

$e^h + e^{-h} - 2$

$= \left(\cancel{1} + \cancel{h} + \frac{h^2}{2!} + \cancel{\frac{h^3}{3!}} + \cdots\right) + \left(\cancel{1} - \cancel{h} + \frac{h^2}{2!} - \cancel{\frac{h^3}{3!}} + \cdots\right) - \cancel{2}$

$= h^2 + O(h^4)$

$= O(h^4)$

2. Find the constants c_0, c_1 and x_1 so that the quadrature formula

$$\int_{-1}^{2} f(x)\,dx = c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of precision.

$f = 1: \int_{-1}^{2} dx = 3 = c_0 + c_1 \ldots (1)$

$f = x: \int_{-1}^{2} x\,dx = \left[\frac{1}{2}x^2\right]_{-1}^{2} = \frac{3}{2} = c_0 f(0) + c_1 f(x_1)$

$= c_0(0) + c_1 x_1$

$\Rightarrow c_1 x_1 = \frac{3}{2} \ldots (2)$

$f = x^2: \int_{-1}^{2} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{2} = \frac{1}{3}\left[8 + 1\right] = 3 = c_0 f(0) + c_1 x_1^2 =$

$c_1 x_1^2 = 3 \ldots (3)$

$3 \div 2: \boxed{x_1 = \frac{3}{3/2} = 2}$

$\frac{3}{2} = c_1 x_1 \Rightarrow 2c_1 = \frac{3}{2} \Rightarrow \boxed{c_1 = \frac{3}{4}}$

$c_0 + c_1 = 3 \Rightarrow \boxed{c_0 = 3 - \frac{3}{4} = \frac{9}{4}}$

$f = x^3: \int_{-1}^{2} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^{2} = 4 - \frac{1}{4}$

$= 3\frac{3}{4} \neq c_0 f(0) + c_1 f(x_1)$

$= \frac{3}{4} \cdot 2^3 = 6$

$x_1 = 2, c_1 = \frac{3}{4}, c_0 = \frac{9}{4}$ gives highest degree of precision which is 2