An Ocean Turbulence Primer

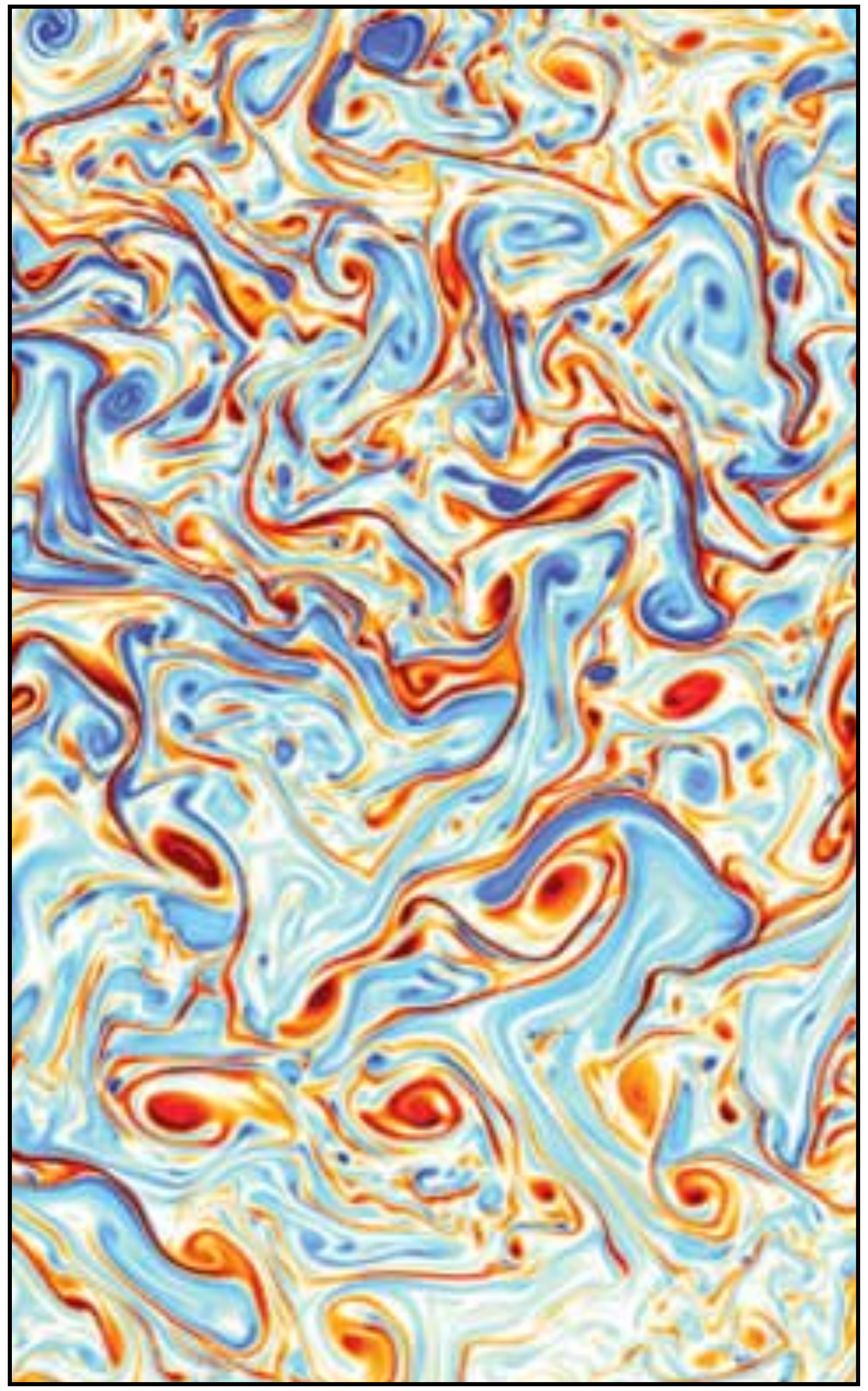
What, How, and Why

Definition: Turbulence

Turbulence in fluids is a "*flow regime characterized by chaotic property changes*".

Properties of turbulent flows:

- Randomness and nonlinearity
- Enhanced diffusivity
- Eddy-like structures and vorticity fluctuations
- Viscous dissipation



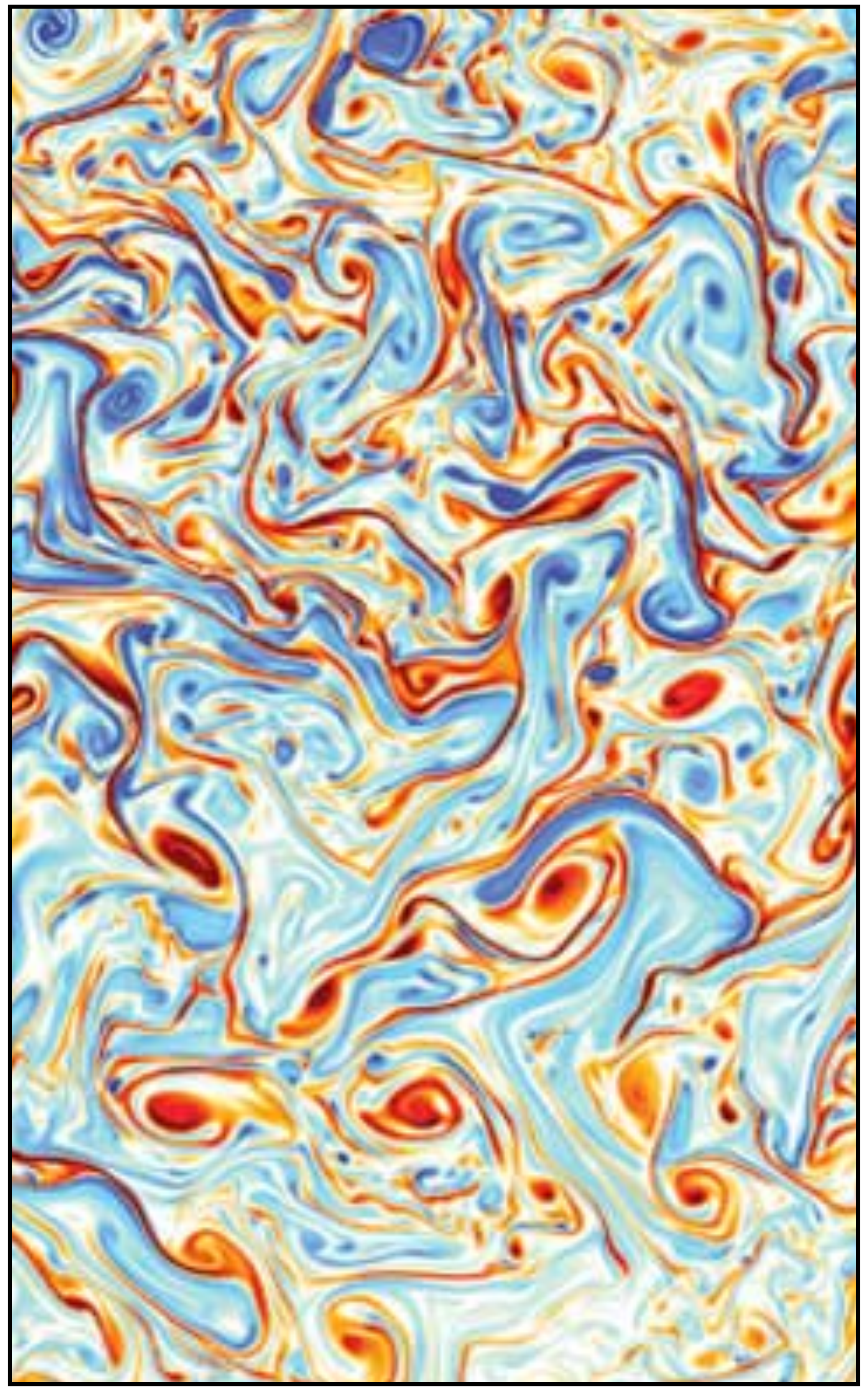
Cover: Physical Review Letters Volume 104, Issue 21

Definition: Turbulence

Turbulence in fluids is a “*flow regime characterized by chaotic property changes*”.

Examples of turbulent flows:

- Ocean and atmosphere mixed layers
- Engineering flows (car, airplane, golf-ball, …)
- Cardiology: turbulent blood flows in the heart
- Rising cigarette-smoke plume



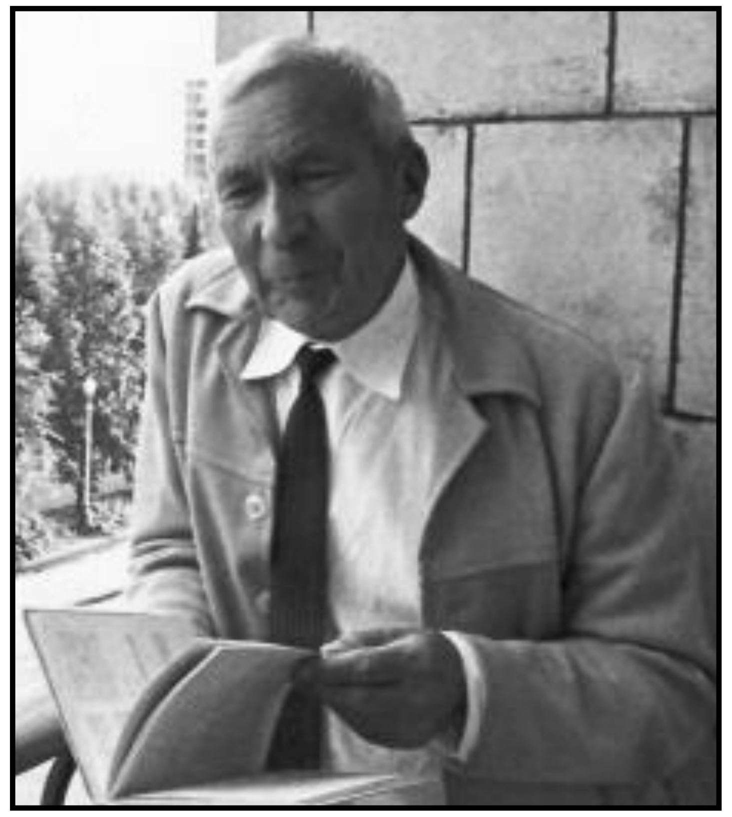
Cover: Physical Review Letters Volume 104, Issue 21

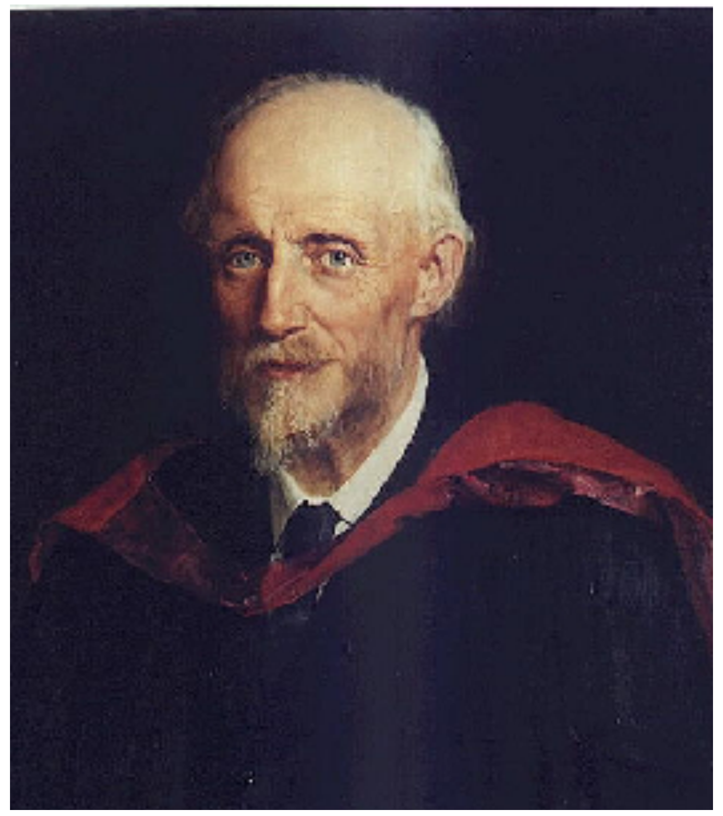
Turbulence: the unsolved problem

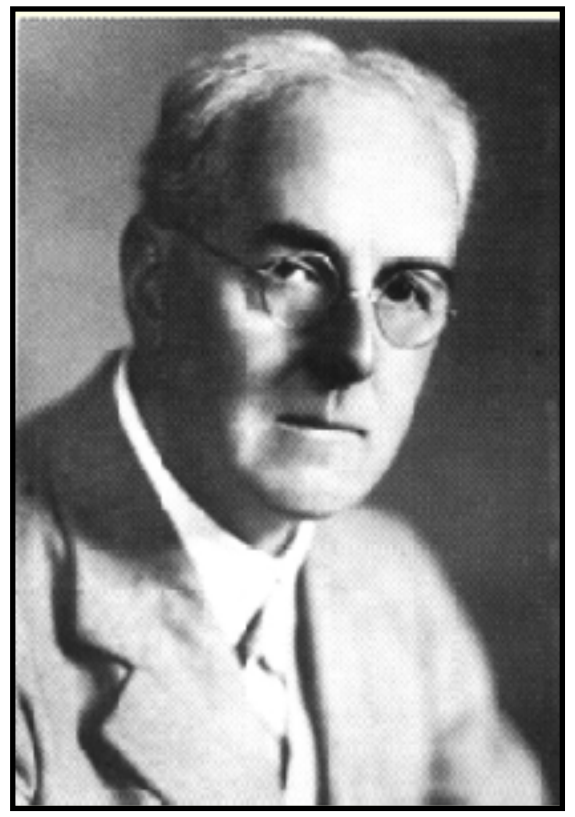
There is no model to predict the internal structure of turbulent flow.

Understanding of turbulent flows is based on:

- Lab experiments — O. Reynolds (1883)
- Statistical descriptions — G. I. Taylor (1936)
- Spectral cascades — L. Richardson (1922) & A. Kolmogorov (1941)
- Modern theoretical, lab, and numerical studies







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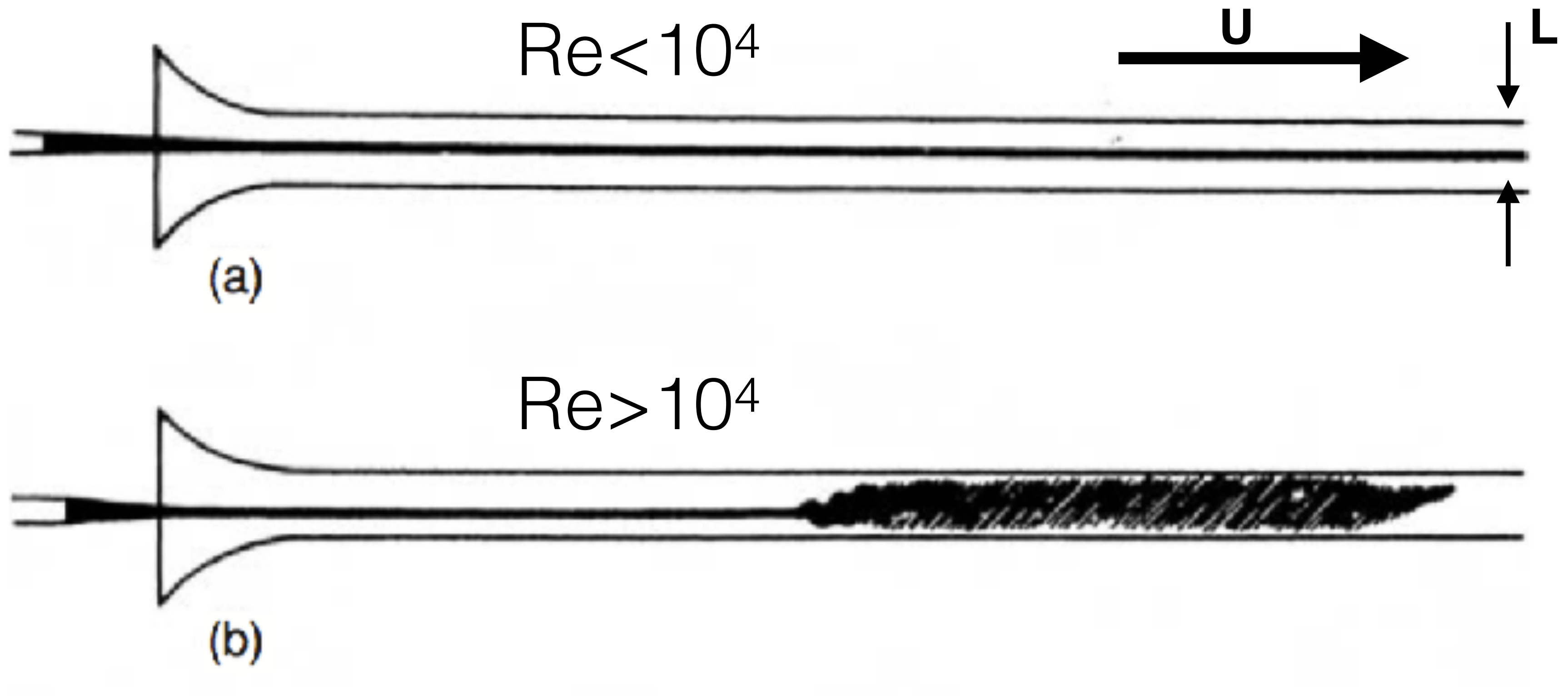
“Turbulence is the most important unsolved problem of classical physics.”

— Richard Feynman

Turbulence in the Ocean

Reynolds' Lab Experiment (1883)

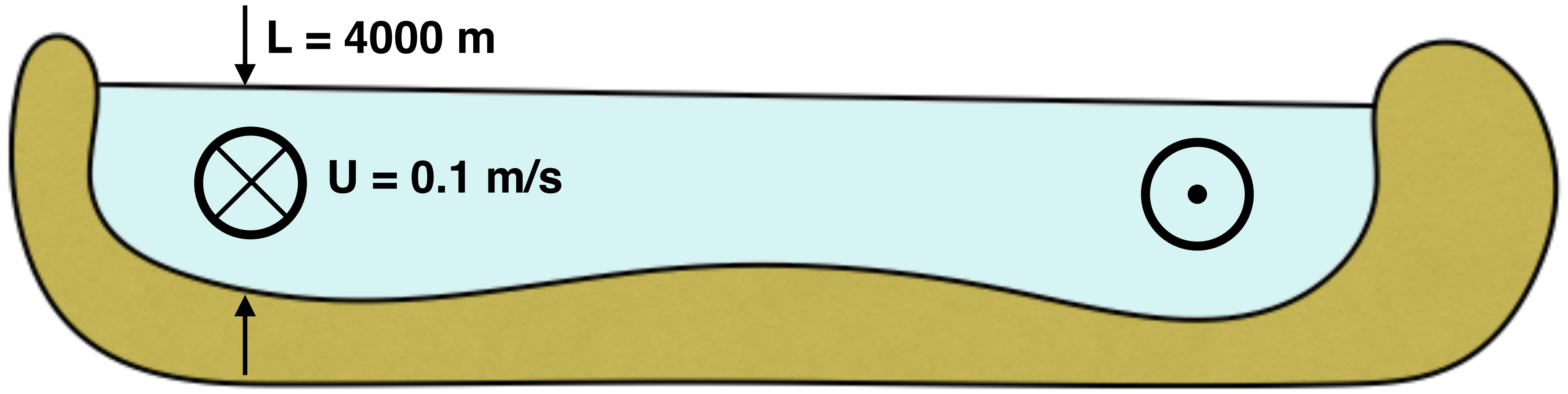
image: Thorpe (2007)



For a critical value of Re, the flow becomes turbulent

$$Re = \frac{\text{Non-linear Terms}}{\text{Viscous Terms}} = \frac{\left(u\,\dfrac{\partial u}{\partial x}\right)}{\left(\nu\,\dfrac{\partial^2 u}{\partial x^2}\right)} \approx \frac{U\,\dfrac{U}{L}}{\nu\,\dfrac{U}{L^2}} = \frac{UL}{\nu}$$

The Interior Ocean



$$Re = \frac{UL}{\nu} = \frac{0.1 \times 4000}{10^{-6}} \approx 4 \times 10^8$$

The ocean interior is prone to turbulence!

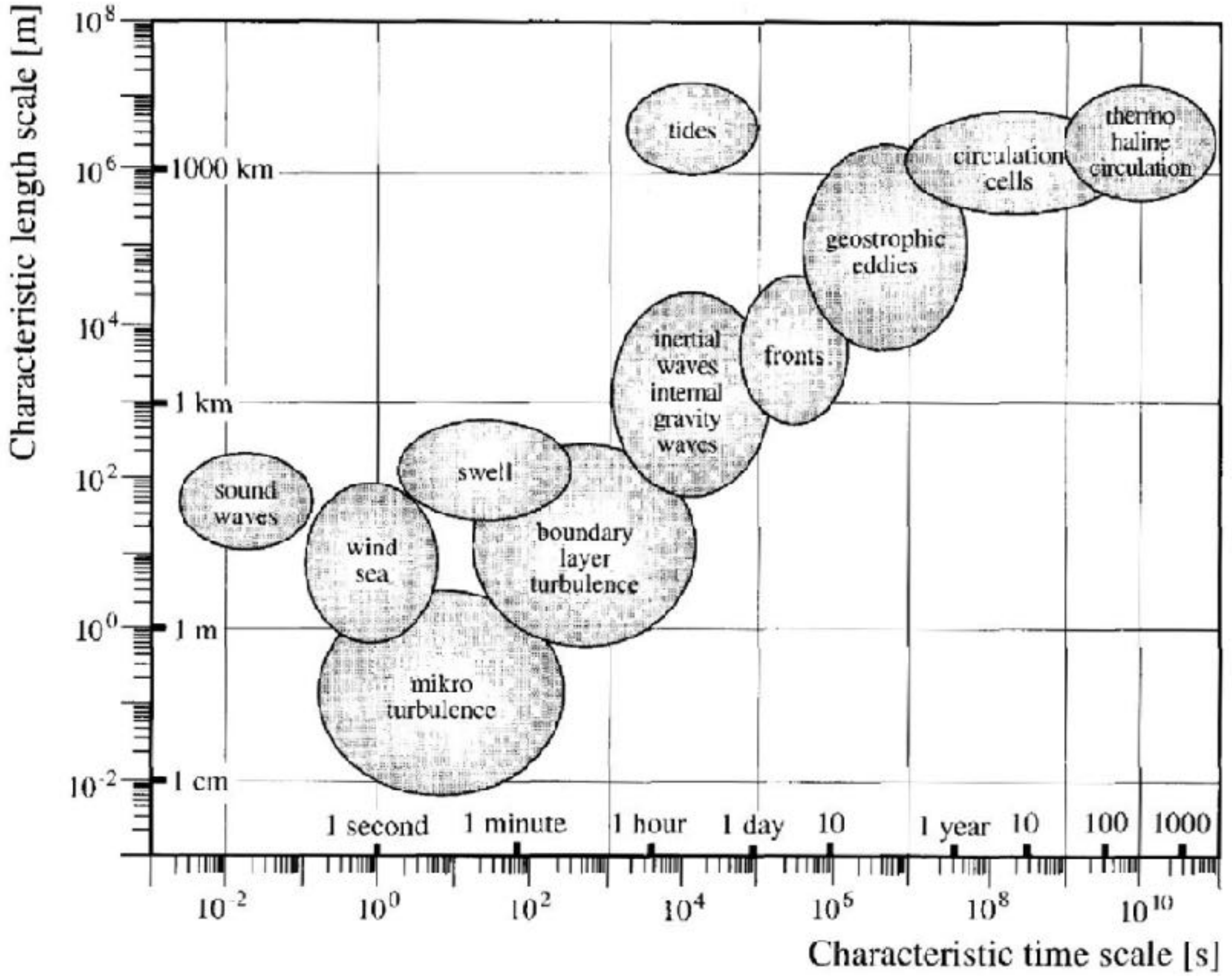
Two Effects of Ocean Turbulence

Turbulence has two important effects in the ocean:

- Cascades kinetic energy to smaller scales of motion, where that energy is eventually dissipated through viscosity
- Drives mixing between water masses with different properties

The Dissipation of Kinetic Energy in the Ocean

Energy is imparted to the ocean at large scales, but it is removed at small scales



Burchard (2002)

The Dissipation of Kinetic Energy in the Ocean

Energy is imparted to the ocean at large scales, but it is removed at small scales

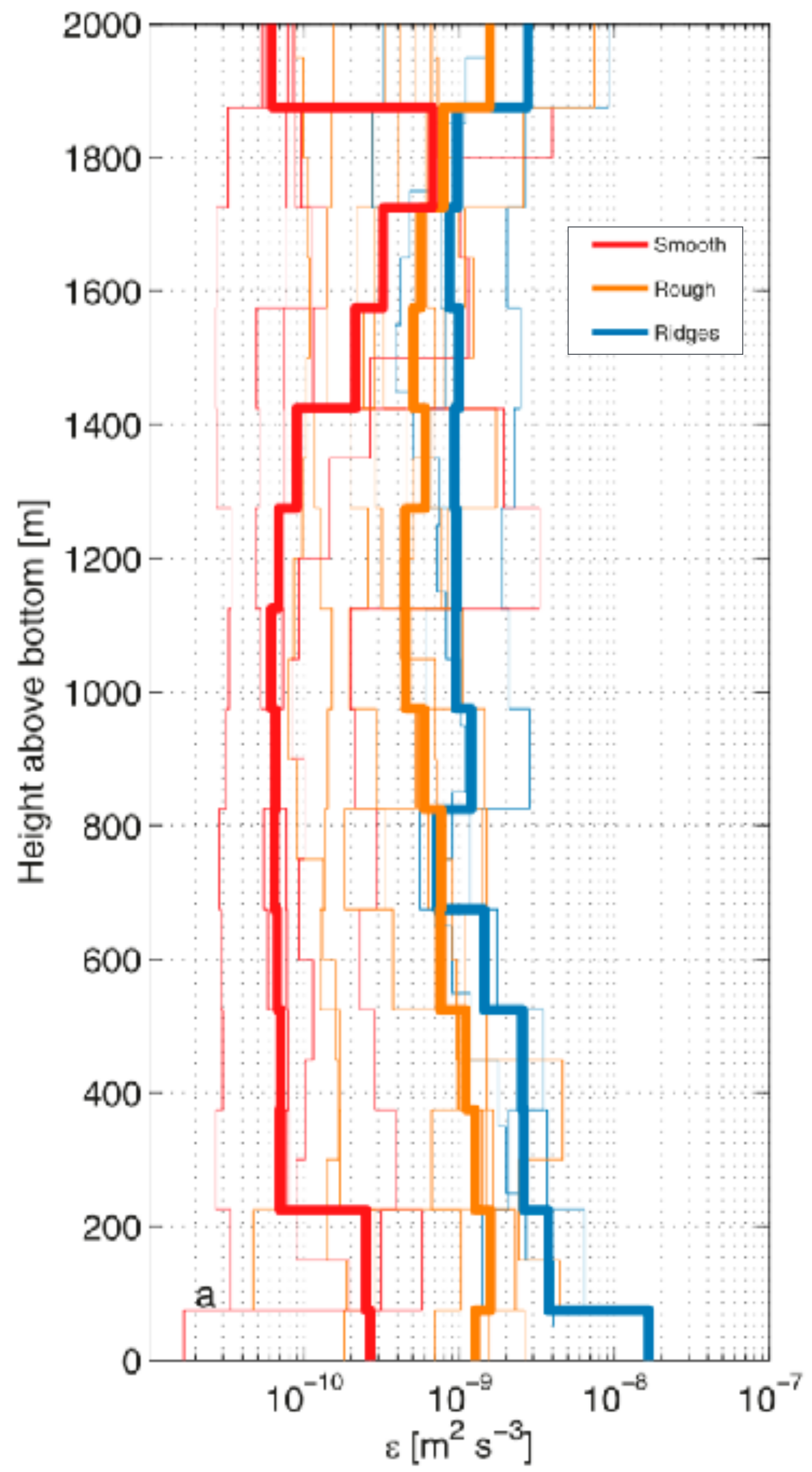
Steady state calculation:

- Rate of kinetic energy input (excl. tides) = 2.1×10^{12} W
- Mass of the ocean = 1.35×10^{21} kg
- Mean rate of energy dissipation = 1.5×10^{-9} W/kg

Important in kinetic energy budget and mixing parame-terizations, but NOT for ocean heat budget!

- Rate of temperature increase is $\varepsilon/c_p = 10^{-5}\ ^{o}C/yr$

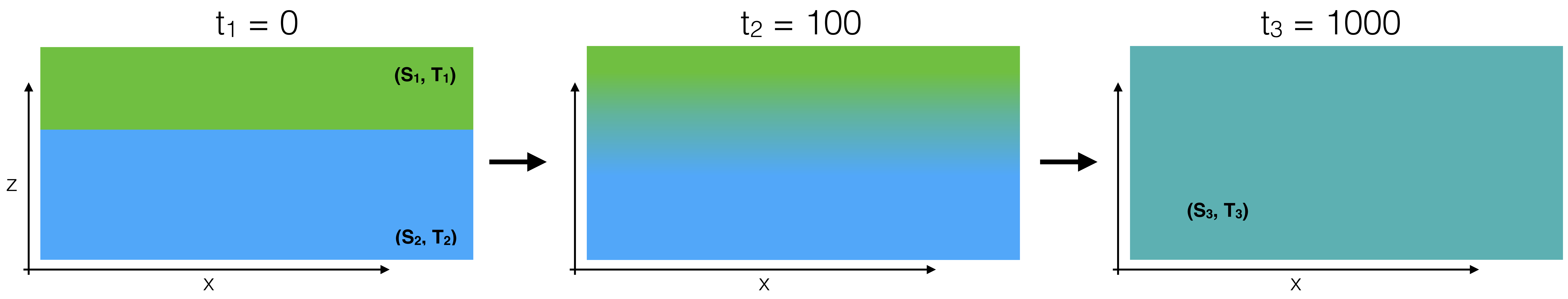
Waterhouse (2014)



Turbulence produces ocean mixing

Mixing is the exchange of properties between water parcels. Mixing is strongly enhanced by turbulence.

NO turbulent mixing:



$$\Phi_T = \kappa_T \frac{\mathrm{d}T}{\mathrm{d}z}$$

Temperature flux

$$\Phi_S = \kappa_S \frac{\mathrm{d}S}{\mathrm{d}z}$$

Salinity flux

Turbulence produces ocean mixing

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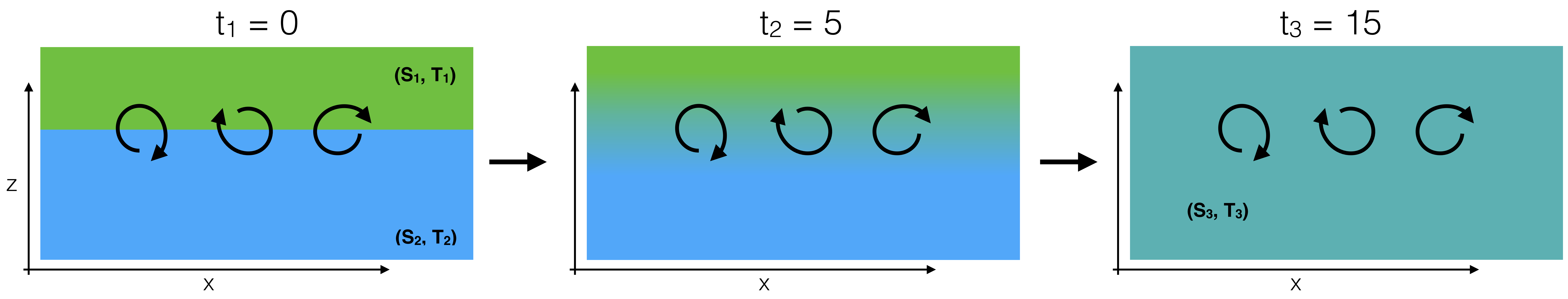
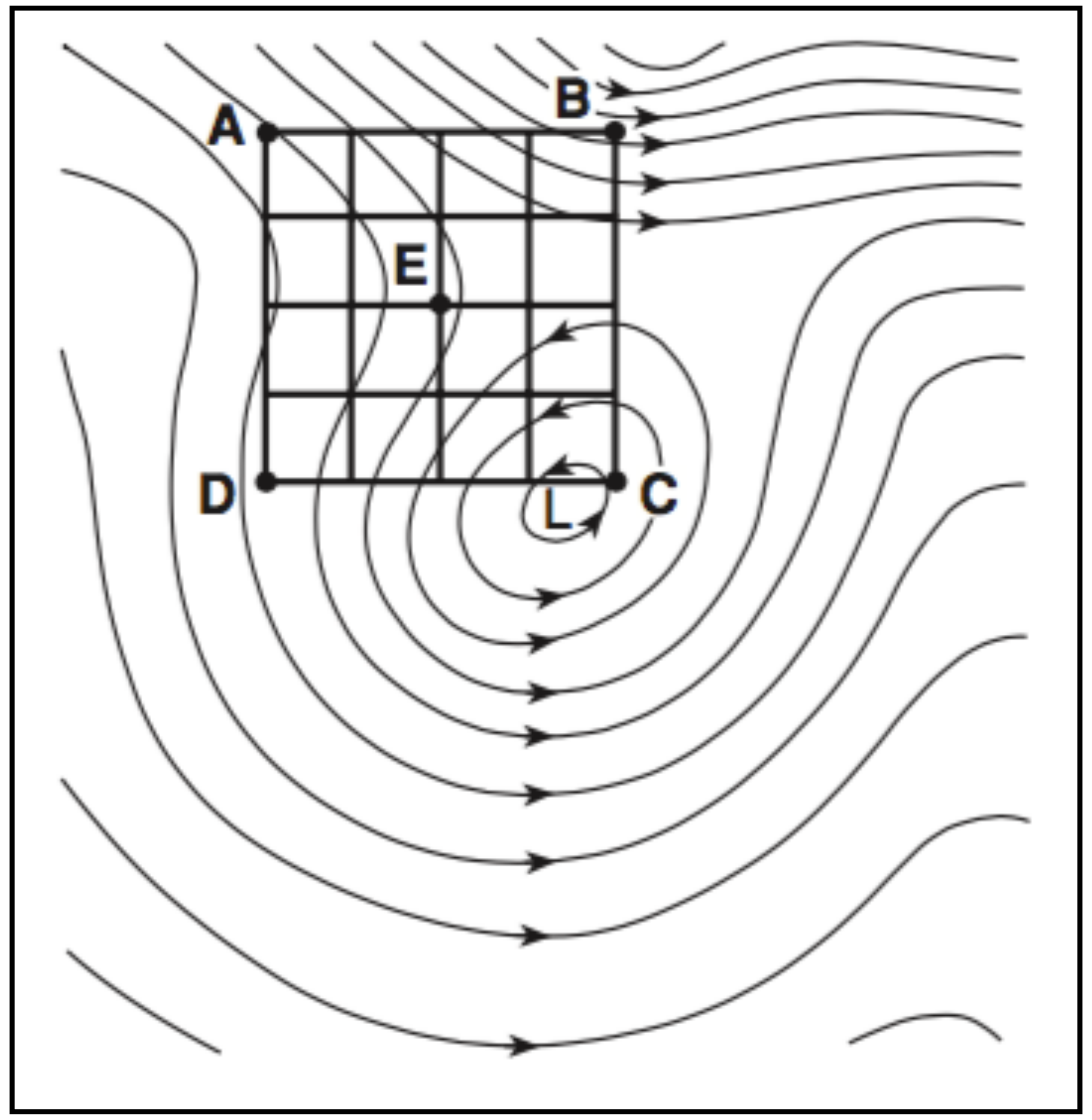
Turbulent mixing:



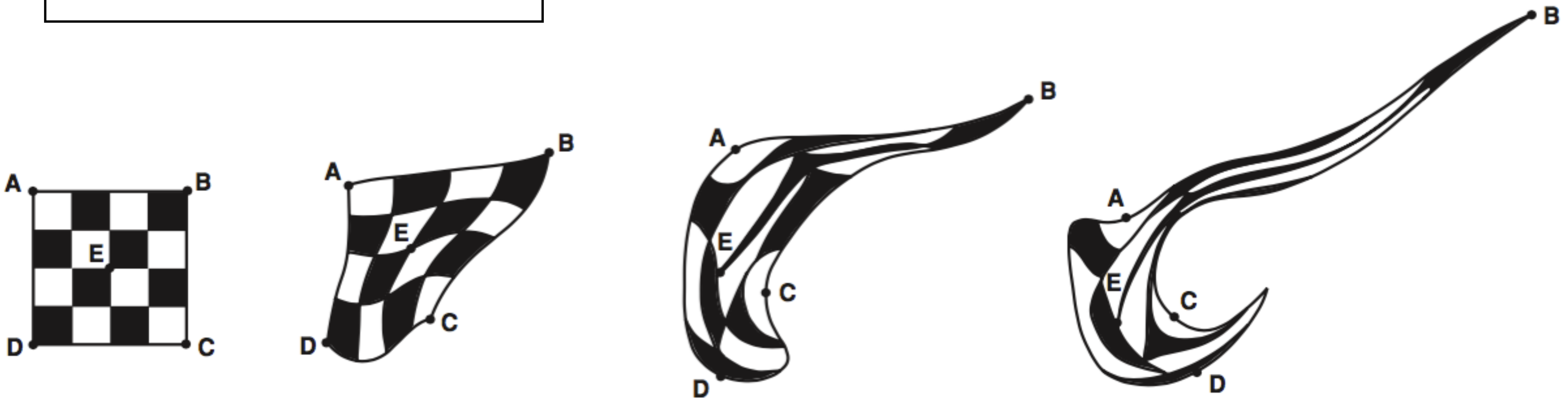
The rate of property exchange is much higher when actively mixed

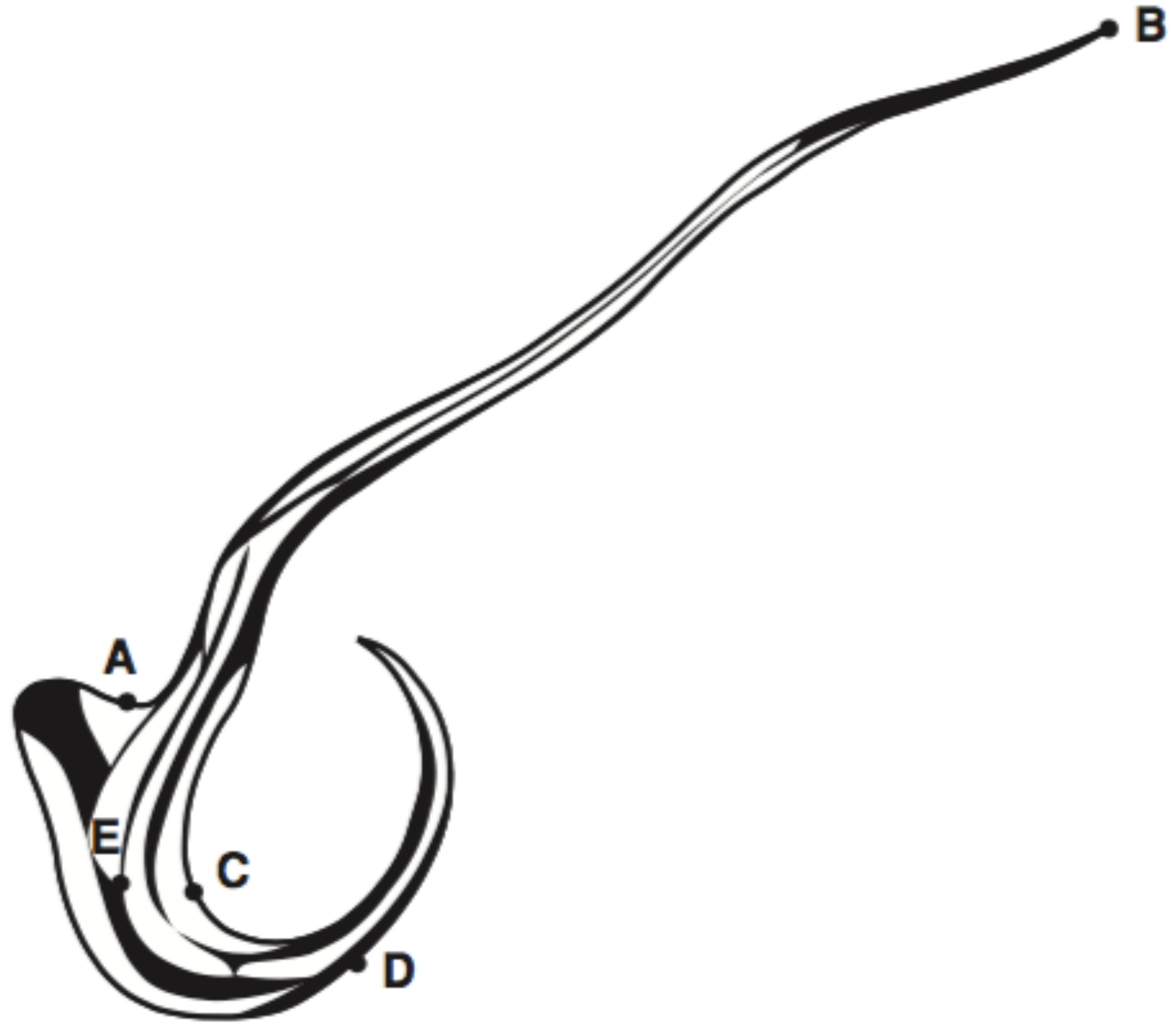
Turbulence produces ocean mixing



Mixing is the exchange of properties between water parcels. Mixing is strongly enhanced by turbulence.

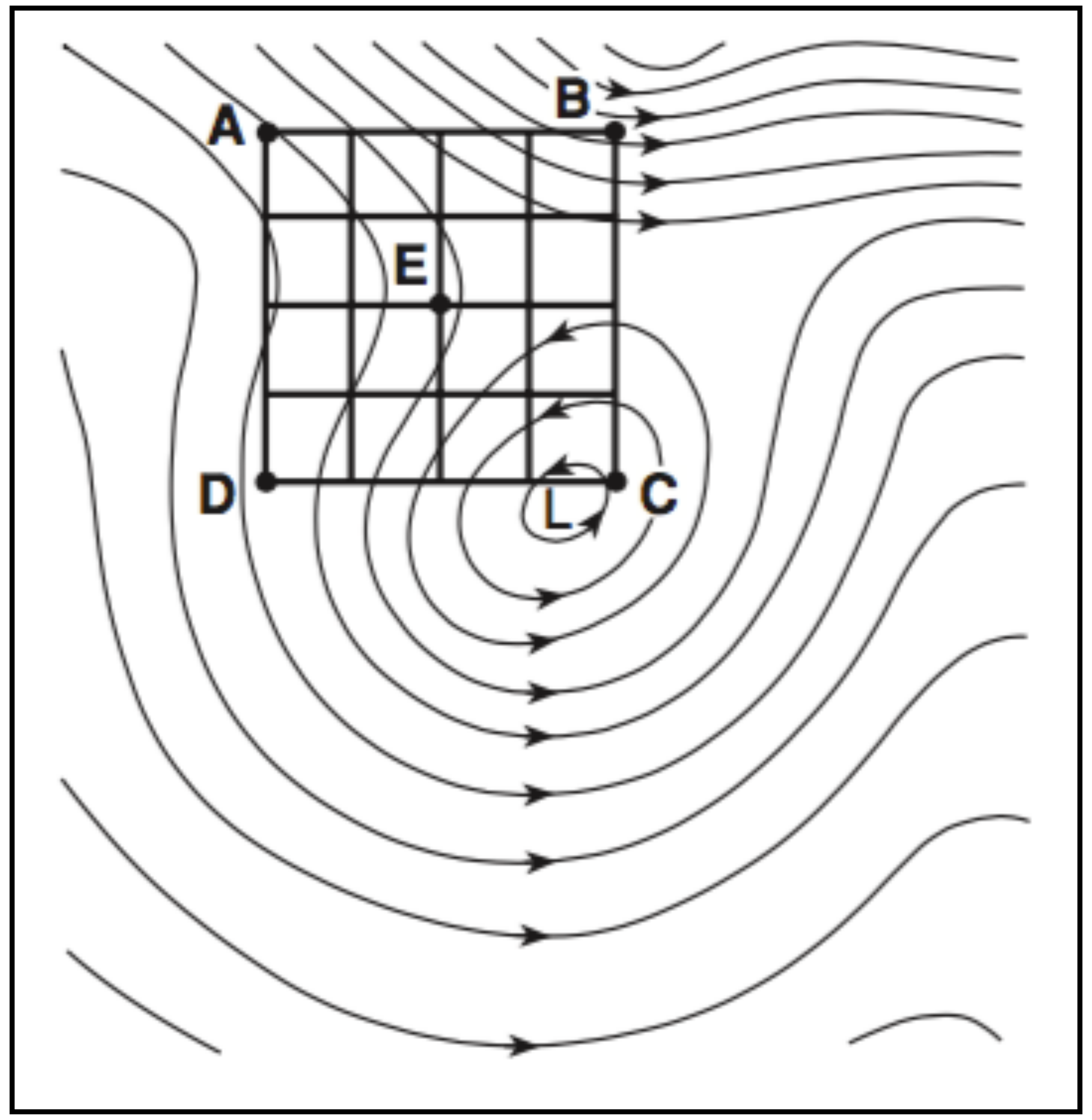
- stirring + diffusion = mixing
- stirring enhances gradients at turbulent scales
- property transfer is enhanced because gradients are enhanced → quicker diffusion





Thorpe (2007): An introduction to Ocean Turbulence

Turbulence produces ocean mixing



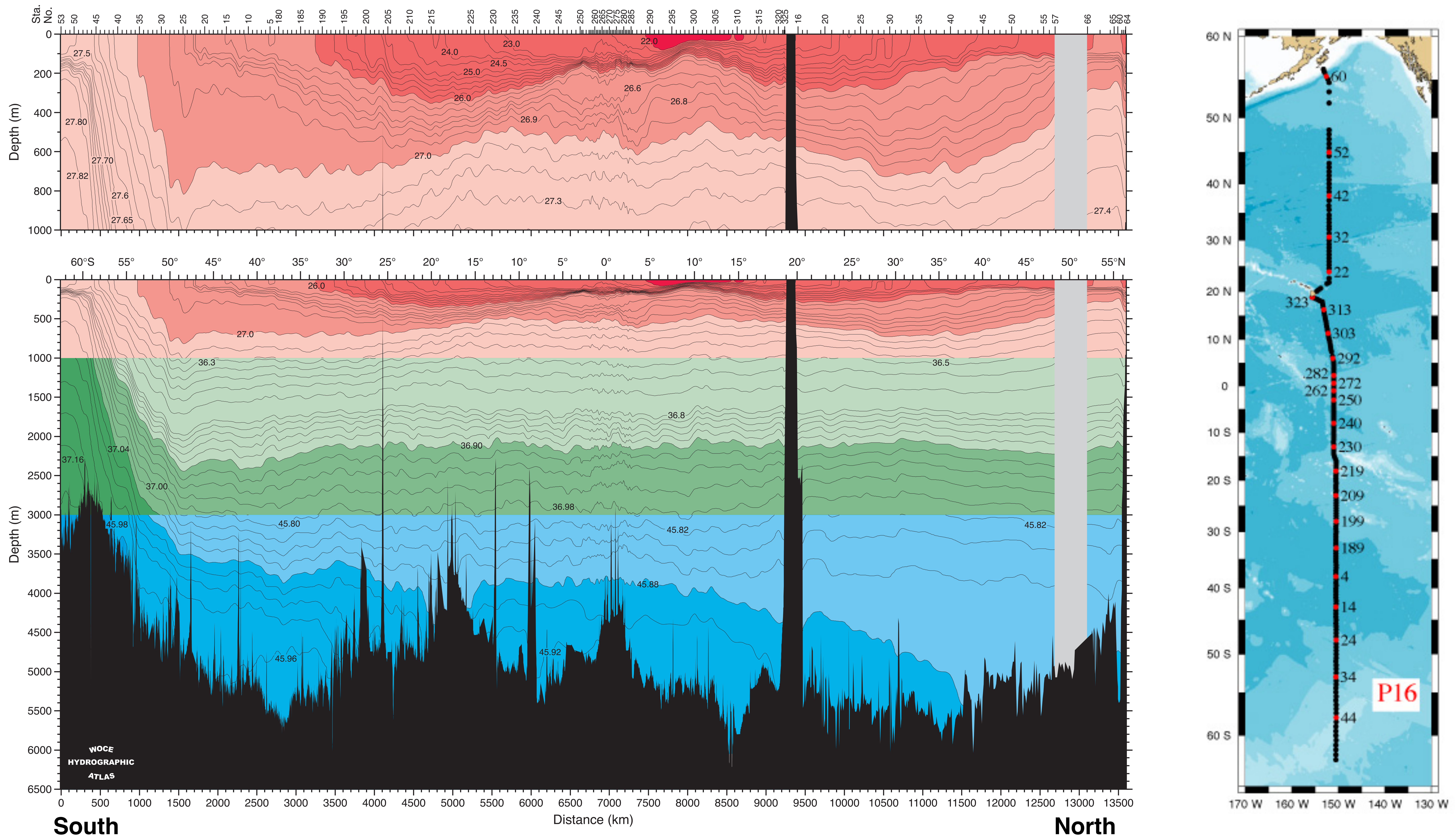
Mixing is the exchange of properties between water parcels. Mixing is strongly enhanced by turbulence.

- Parameterize the effect of "mixing" with the concept of an "eddy diffusivity" K_T:

$$\Phi_T = \kappa_T \frac{\mathrm{d}T}{\mathrm{d}z} + K_T \frac{\mathrm{d}T}{\mathrm{d}z}$$

- Molecular diffusivity → property of the **fluid**
- Eddy diffusivity → consequence of the **flow**

σ_0 (red), σ_2 (green), σ_4 (blue) for P16 150°W



Walter Munk: Abyssal Recipes (1966)

- The (1D) shape of the thermocline in the open ocean remains steady
- Surprising because we could expect mixing to deepen the thermocline
- Munk recognized that the downward mixing of heat must be balanced by (slow) upward advection of cold water:

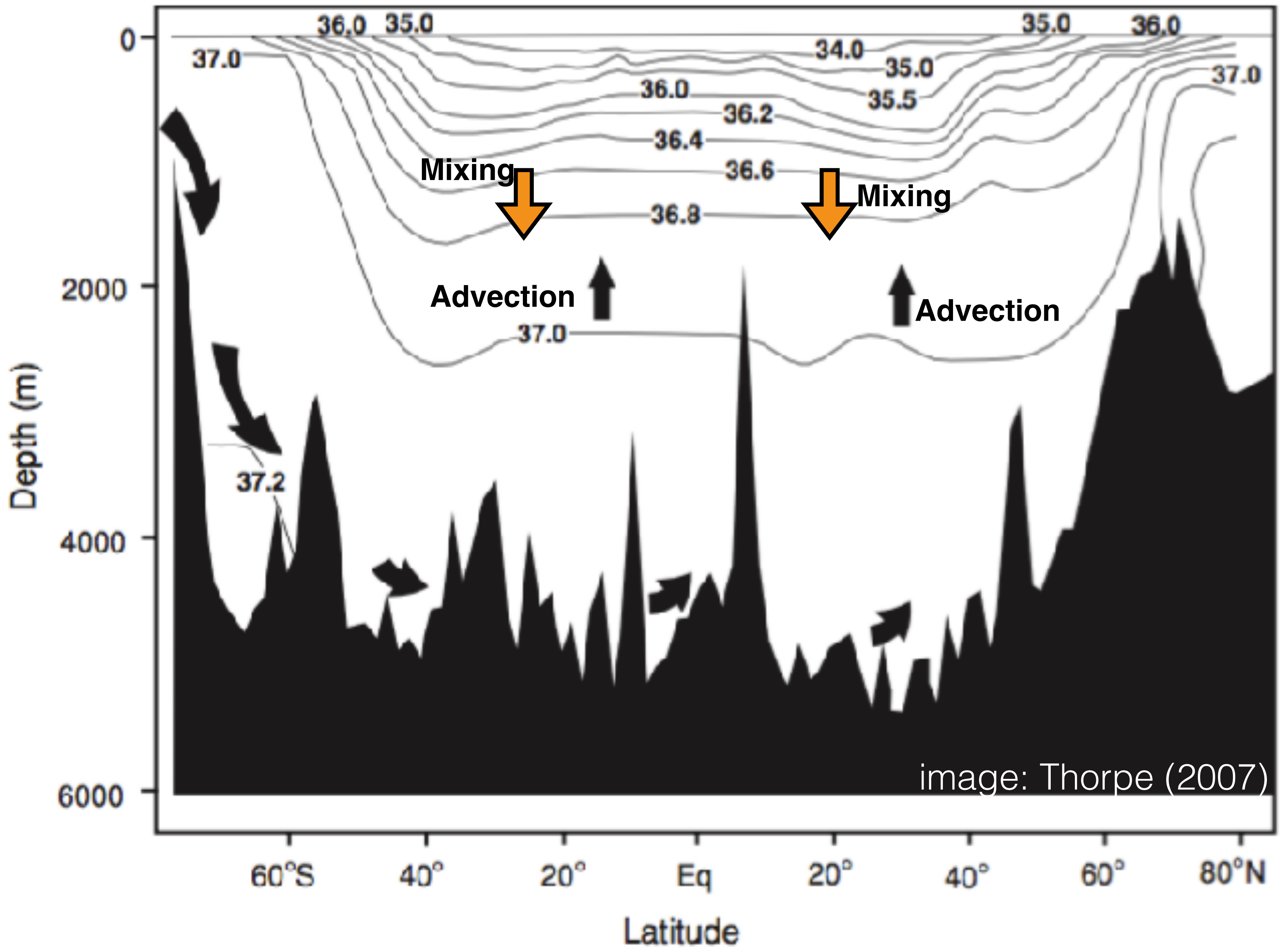


image: Thorpe (2007)

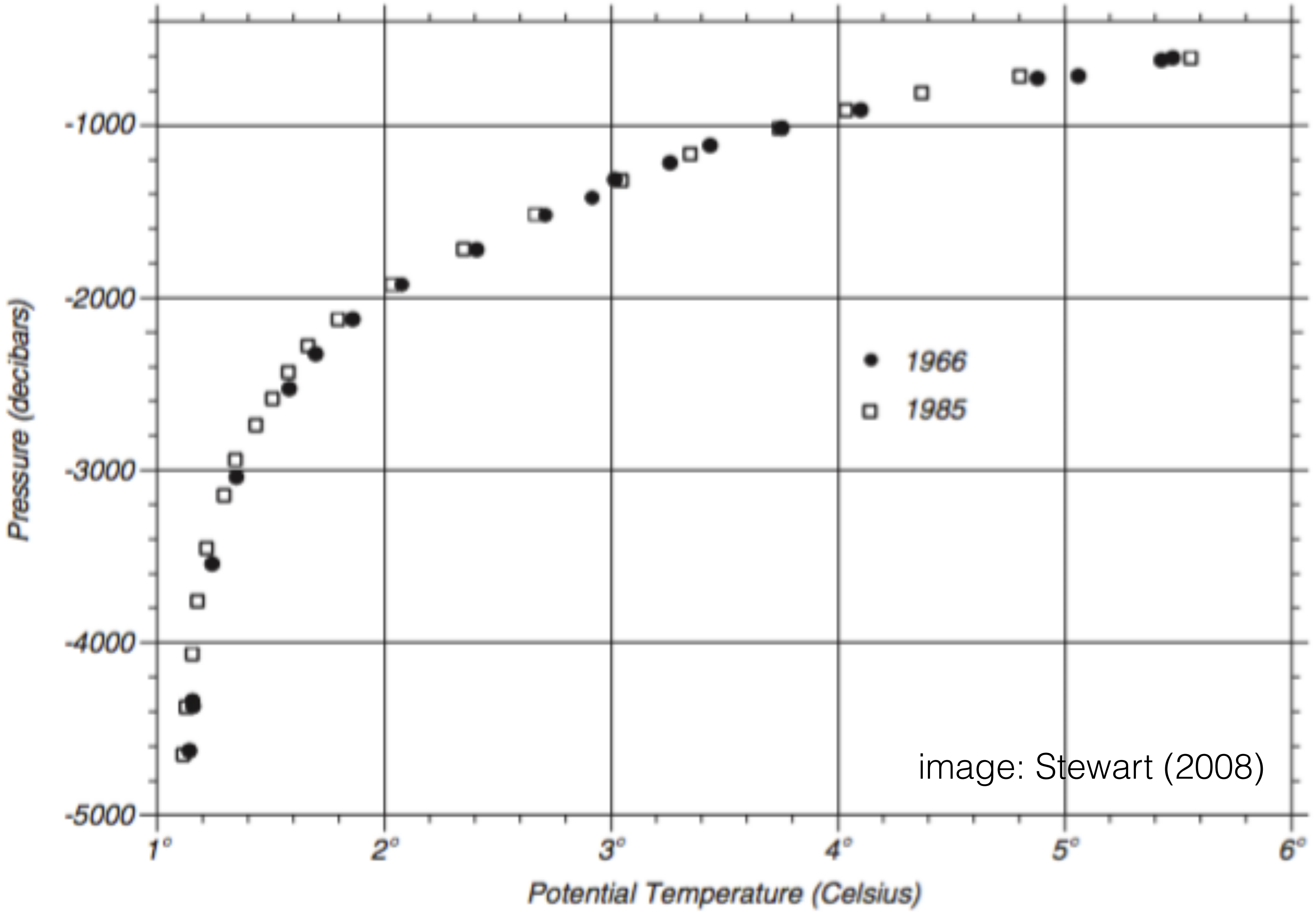


image: Stewart (2008)

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$$w\frac{\partial T}{\partial z} = \frac{\partial}{\partial z}\left(K_T\frac{\partial T}{\partial z}\right)$$

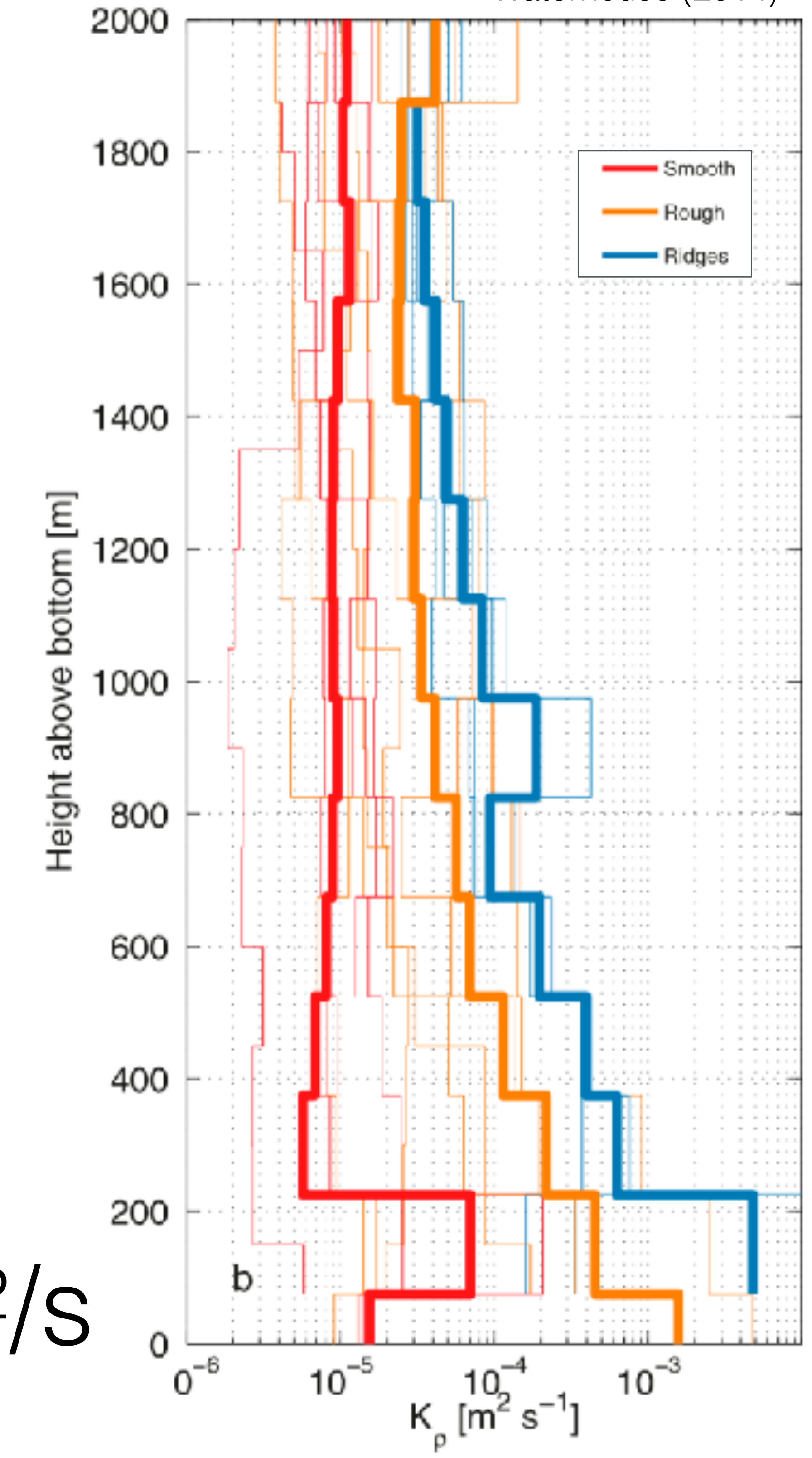
upward advection of temperature gradient

divergence of…

…the (downward) heat flux

- Solution is $T(z) = T_o \exp(\{w/K_T\}z)$ if K_T is constant
- Advection velocity $w \approx 4$ m/yr from radio-carbon isotopes
- Fitting to temperature profile, gives the prediction $K_T \approx 1.3 \times 10^{-4}$ m²/s

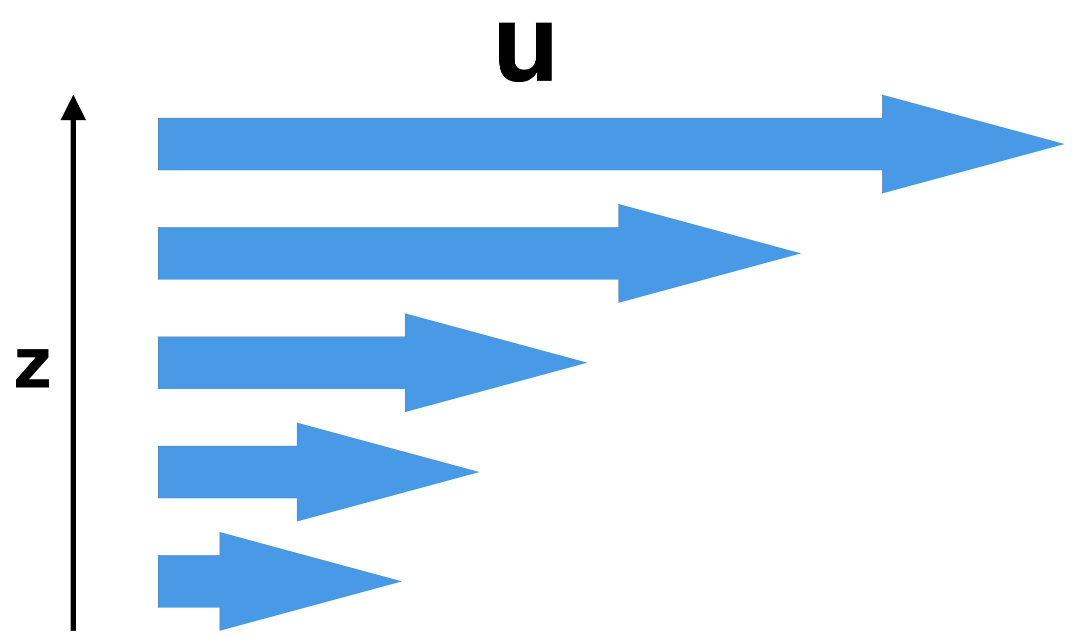
Waterhouse (2014)



How we measure mixing

- We characterize mixing with the eddy diffusivity K_ρ of density
  - If molecular diffusion can be ignored $K_\rho = K_T = K_S$
- Mixing is related to turbulent energetics by $K_\rho = \Gamma(\varepsilon/N^2)$
  - Coefficient Γ is a non-dimensional "mixing efficiency"
  - The factor ε is the "dissipation rate of turbulent kinetic energy"
- The rate ε is related to velocity shear at dissipative scales:

$$\epsilon = (15\,\nu/2)\left\langle(\partial u/\partial z)^2\right\rangle$$



A few concepts

The necessity to work with statistical measures:

- **Ensemble average** - the average result of a number of experiments performed under identical conditions. Denoted as $\langle U(t) \rangle$ or $\overline{U(t)}$. Turbulent velocity variables usually have a mean of zero.
- **Reynolds decomposition** - separate a variable into its "mean" and "fluctuating" components: $U = u + u'$ such that $\langle u' \rangle = 0$ and $\langle U \rangle = \langle u \rangle = u$
- **Variance** - a measure of the variability around the mean:

$$\mathrm{var}(U) = \left\langle (U - \bar{U})^2 \right\rangle \quad \text{or} \quad \mathrm{var}(u') = \left\langle u'^2 \right\rangle$$

Momentum Equation

Write the momentum equation with Boussinesq approximation using *index notation* because this simplifies working with three dimensions

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho_o}\frac{\partial P}{\partial x_i} - g\frac{\rho}{\rho_o}\delta_{i3} + \nu\frac{\partial^2 U_i}{\partial x_j \partial x_j}$$

$$U_i = \vec{U} = \begin{pmatrix} U \\ V \\ W \end{pmatrix}$$

$$\begin{pmatrix} \vec{U}\cdot\vec{\nabla}U \\ \vec{U}\cdot\vec{\nabla}V \\ \vec{U}\cdot\vec{\nabla}W \end{pmatrix}$$

$$\vec{\nabla}P$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \nabla^2 U \\ \nabla^2 V \\ \nabla^2 W \end{pmatrix}$$

$$\lambda = a_i b_i c_j = (a_1 b_1 + a_2 b_2 + a_3 b_3)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

- single (non-repeated) subscripts are vectors
- subscripts repeated twice are summed over all three dimensions

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Substitute the Reynolds-decomposed variables → $U = u + u'$ etc…

$$\frac{\partial (u_i + u_i')}{\partial t} + (u_i + u_i')\frac{\partial (u_i + u_i')}{\partial x_j} = -\frac{1}{\rho_o}\frac{\partial (p + p')}{\partial x_i} - g\frac{\rho + \rho'}{\rho_o}\delta_{i3} + \nu \frac{\partial^2 (u_i + u_i')}{\partial x_j \partial x_j}$$

And average…

Momentum Equation

The Reynolds-averaged "mean momentum" equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u_i' u_j'}) = -\frac{1}{\rho_o}\frac{\partial p}{\partial x_i} - g\frac{\rho}{\rho_o}\delta_{i3} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

The effect of turbulent fluctuations on the **mean** flow

Substitute the Reynolds-decomposed variables:

$$\frac{\partial (u_i + u_i')}{\partial t} + (u_i + u_i')\frac{\partial (u_i + u_i')}{\partial x_j} = -\frac{1}{\rho_o}\frac{\partial (p + p')}{\partial x_i} - g\frac{\rho + \rho'}{\rho_o}\delta_{i3} + \nu \frac{\partial^2 (u_i + u_i')}{\partial x_j \partial x_j}$$

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Momentum Equation

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which we can write as:

$$\frac{Du_i}{Dt} = \underbrace{\frac{1}{\rho_o}\frac{\partial \tau_{ij}}{\partial x_j}}_{\text{The divergence of a "stress"}} - g\frac{\rho}{\rho_o}\delta_{i3}$$

where

$$\tau_{ij} = -p\delta_{ij} + \underbrace{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{viscosity acting on shear in the mean flow}} - \underbrace{\rho_o \overbrace{\overline{u_i' u_j'}}^{\text{"Reynolds stress"}}}_{\text{stress from turbulent fluctuations acting on the mean flow}}$$

"Reynolds stress"

The divergence of a "stress"

viscosity acting on shear in the **mean** flow

stress from **turbulent fluctuations** acting on the mean flow

The Reynolds Stress Tensor

Writing explicitly the components of the Reynolds stress tensor:

$$-\rho_o \overline{u_i' u_j'} = -\rho_o \begin{pmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'v'} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'w'} \end{pmatrix}$$

The term $\rho_o \langle u'v' \rangle$ is the flux of x-momentum in the y-direction.

Equals the flux of y-momentum in the x-direction (ie. the tensor is symmetric).

In truly isotropic motion, the off-diagonal stresses are zero.

Mean Kinetic Energy of a Turbulent Flow

Reynolds-averaged momentum eq.:

$$\frac{Du_i}{Dt} = \frac{1}{\rho_o}\frac{\partial \tau_{ij}}{\partial x_j} - g\frac{\rho}{\rho_o}\delta_{i3}$$

$$\tau_{ij} = -p\delta_{ij} + 2\mu E_{ij} - \rho_o\overline{u_i'u_j'}$$

with $E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Now multiply by u_i for kinetic energy:

$$\frac{D}{Dt}\left(\frac{1}{2}u_i^2\right) = \frac{1}{\rho_o}\frac{\partial}{\partial x_j}(u_i\tau_{ij}) - \frac{1}{\rho_o}\tau_{ij}\frac{\partial u_i}{\partial x_j} - g\frac{\rho}{\rho_o}u_i\delta_{i3}$$

Substitute for τ_{ij} and E_{ij} and then more algebra:

$$\frac{D}{Dt}\left(\frac{1}{2}u_i^2\right) = \underbrace{\frac{\partial}{\partial x_j}\left(-\frac{pu_j}{\rho_o} + 2\nu u_i E_{ij} - \overline{u_i'u_j'}u_i\right)}_{\text{divergence of advection terms}} \underbrace{-2\nu E_{ij}E_{ij}}_{\text{dissipation by viscosity (small)}} + \underbrace{\overline{u_i'u_j'}\frac{\partial u_i}{\partial x_j}}_{\text{loss to turbulence}} \underbrace{- g\frac{\rho}{\rho_o}u_3}_{\text{loss to potential energy}}$$

Kinetic Energy Budget for the Turbulent Flow

Momentum equation for ***turbulent*** flow:

$$\frac{\partial(u_i + u_i')}{\partial t} + (u_i + u_i')\frac{\partial(u_i + u_i')}{\partial x_j} = -\frac{1}{\rho_o}\frac{\partial(p + p')}{\partial x_i} - g\frac{\rho + \rho'}{\rho_o}\delta_{i3} + \nu\frac{\partial^2(u_i + u_i')}{\partial x_j \partial x_j}$$

MINUS

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u_i' u_j'}) = -\frac{1}{\rho_o}\frac{\partial p}{\partial x_i} - g\frac{\rho}{\rho_o}\delta_{i3} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

EQUALS

$$\frac{\partial u_i'}{\partial t} + u_j\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial u_i}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} - \frac{\partial}{\partial x_j}(\overline{u_i' u_j'}) = -\frac{1}{\rho_o}\frac{\partial p'}{\partial x_i} - g\frac{\rho'}{\rho_o}\delta_{i3} + \nu\frac{\partial^2 u_i'}{\partial x_j \partial x_j}$$

NOW MULTIPLY BY u_i' AND AVERAGE → KINETIC ENERGY OF THE TURBULENT FLOW

Kinetic Energy Budget for the Turbulent Flow

$$\frac{D}{Dt}\overline{\left(\frac{1}{2}u_i'^2\right)} = \frac{\partial}{\partial x_j}\underbrace{\left(\frac{-1}{\rho_o}\overline{p'u_j'} + 2\nu\overline{u_i'e_{ij}} - \frac{1}{2}\overline{u_i'^2u_j'}\right)}_{\text{… the advection by the turbulent flow field}} - \overline{u_i'u_j'}\frac{\partial u_i}{\partial x_j} - \frac{g}{\rho_o}\overline{\rho'w'} - 2\nu\overline{e_{ij}e_{ij}}$$

Turbulent Kinetic Energy (TKE)

The divergence of …

Production of TKE by shear in the mean flow

-P

Loss of TKE to potential energy (buoyancy flux)

B

Loss of TKE to viscous dissipation (internal friction)

ε

with the turbulent "strain rate": $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$

Steady-state TKE equation: $P = B + \varepsilon$

Relating Turbulence to Mixing (Osborn 1980)

Steady-state TKE equation: $P = B + \rho_o \varepsilon$

$$P = -\rho_o \overline{u'w'} \frac{\partial u}{\partial z}$$

$$B = g\overline{\rho' w'}$$

$$\epsilon = 2\nu \overline{e_{ij} e_{ij}}$$

$$R \equiv \frac{B}{P} = \frac{g\overline{\rho' w'}}{P}$$

$$\downarrow$$

$$K_\rho = \frac{RP}{\rho_o N^2}$$

$$\downarrow$$

$$K_\rho = \frac{R}{(1-R)} \frac{\epsilon}{N^2} = \Gamma \frac{\epsilon}{N^2}$$

$$\Phi_\rho = K_\rho \frac{\mathrm{d}\rho}{\mathrm{d}z} = \overline{w'\rho'}$$

$$N^2 = \frac{g}{\rho_o} \frac{\partial \rho}{\partial z}$$

$$\overline{w'\rho'} = \frac{K_\rho N^2 \rho_o}{g}$$

$R = 0.15 \rightarrow \Gamma = 0.2$

Measuring the rate ε of TKE Dissipation

The definition of the dissipation rate of turbulent kinetic energy: $\epsilon = 2\nu\overline{e_{ij}e_{ij}}$

$e_{ij}e_{ij} = e_{11}e_{11} + e_{12}e_{12} + e_{13}e_{13} + e_{21}e_{21} + e_{22}e_{22} + e_{23}e_{23} + e_{31}e_{31} + e_{32}e_{32} + e_{33}e_{33}$ where $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$

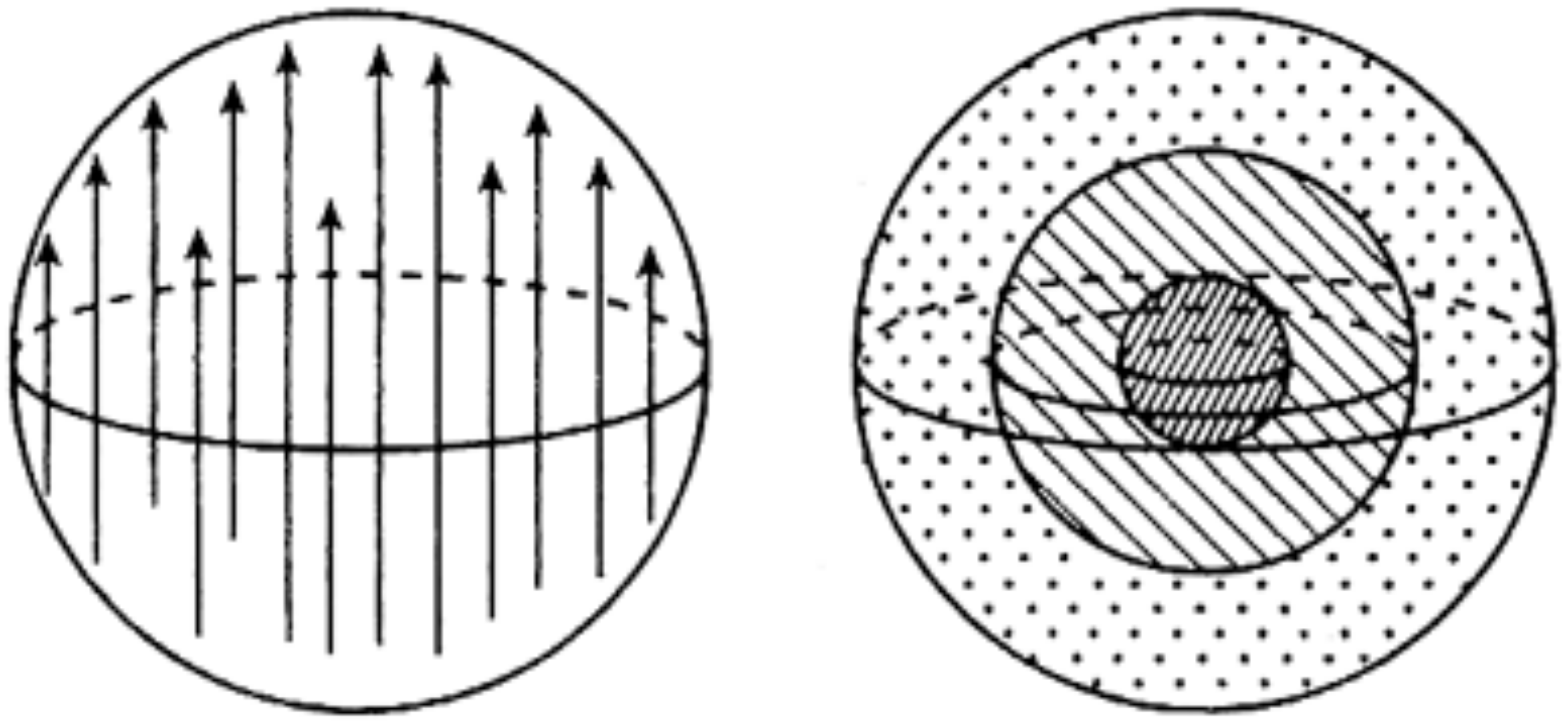
Requires measuring 9 tensor components (ie. 9 velocity shear components)

Simplify: assume the turbulent flow field is statistically isotropic

$$\epsilon = \frac{15\,\nu}{2}\overline{\left(\frac{\partial u(z)}{\partial z}\right)^2} = \frac{15\,\nu}{2}\int_{\mathrm{k_{min}}}^{\mathrm{k_{max}}} \Phi(k)\,\mathrm{d}k$$

1/s²

m/s²



anisotropic isotropic

Measuring the rate ε of TKE Dissipation

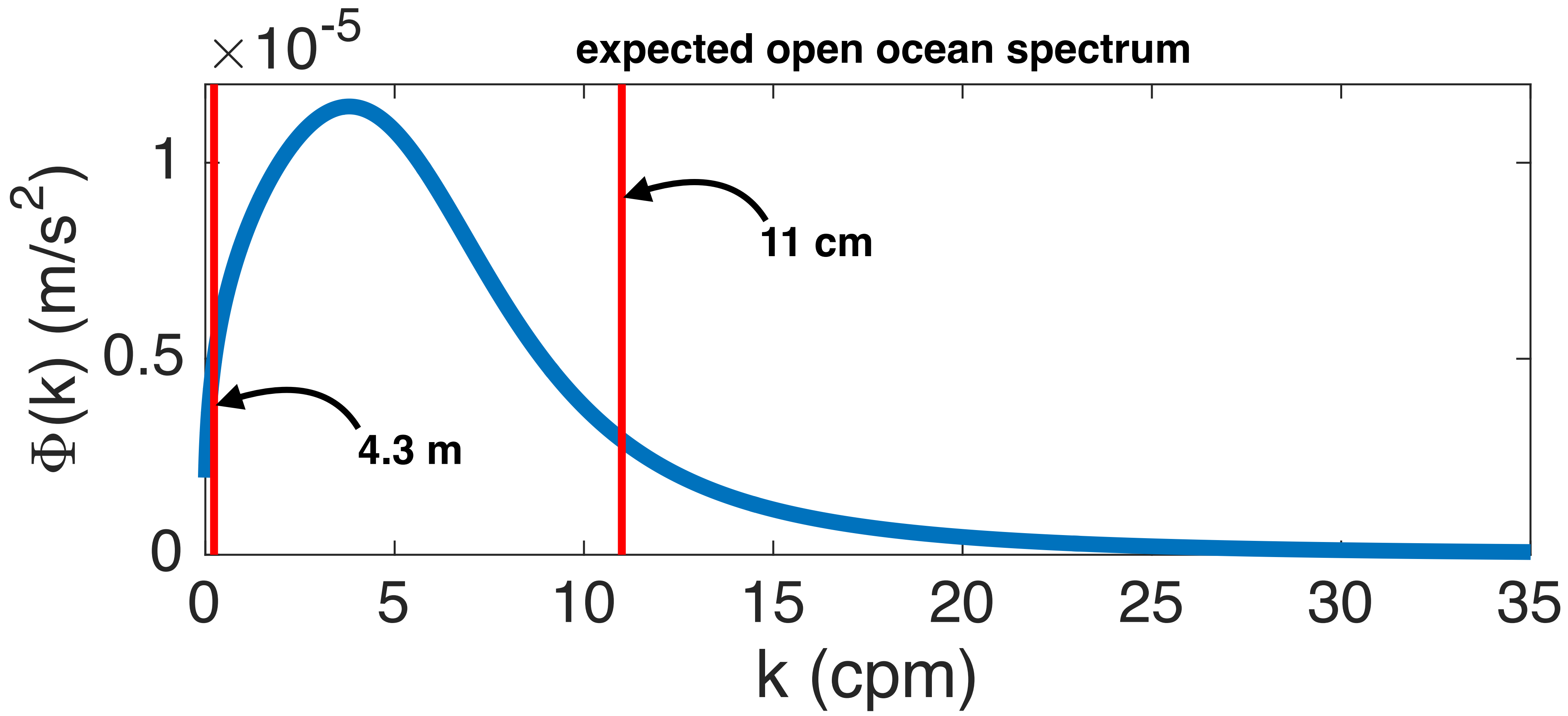
The definition of the dissipation rate of turbulent kinetic energy: $\epsilon = 2\nu\overline{e_{ij}e_{ij}}$

$$e_{ij}e_{ij} = e_{11}e_{11} + e_{12}e_{12} + e_{13}e_{13} + e_{21}e_{21} + e_{22}e_{22} + e_{23}e_{23} + e_{31}e_{31} + e_{32}e_{32} + e_{33}e_{33}$$ where $$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$$

Requires measuring 9 tensor components (ie. 9 velocity shear components)

Simplify: assume the turbulent flow field is statistically isotropic

$$\epsilon = \frac{15\,\nu}{2}\overline{\left(\underbrace{\frac{\partial u(z)}{\partial z}}_{1/s^2}\right)^2} = \frac{15\,\nu}{2}\int_{\mathrm{k_{min}}}^{\mathrm{k_{max}}} \underbrace{\Phi(k)}_{m/s^2}\,\mathrm{d}k$$



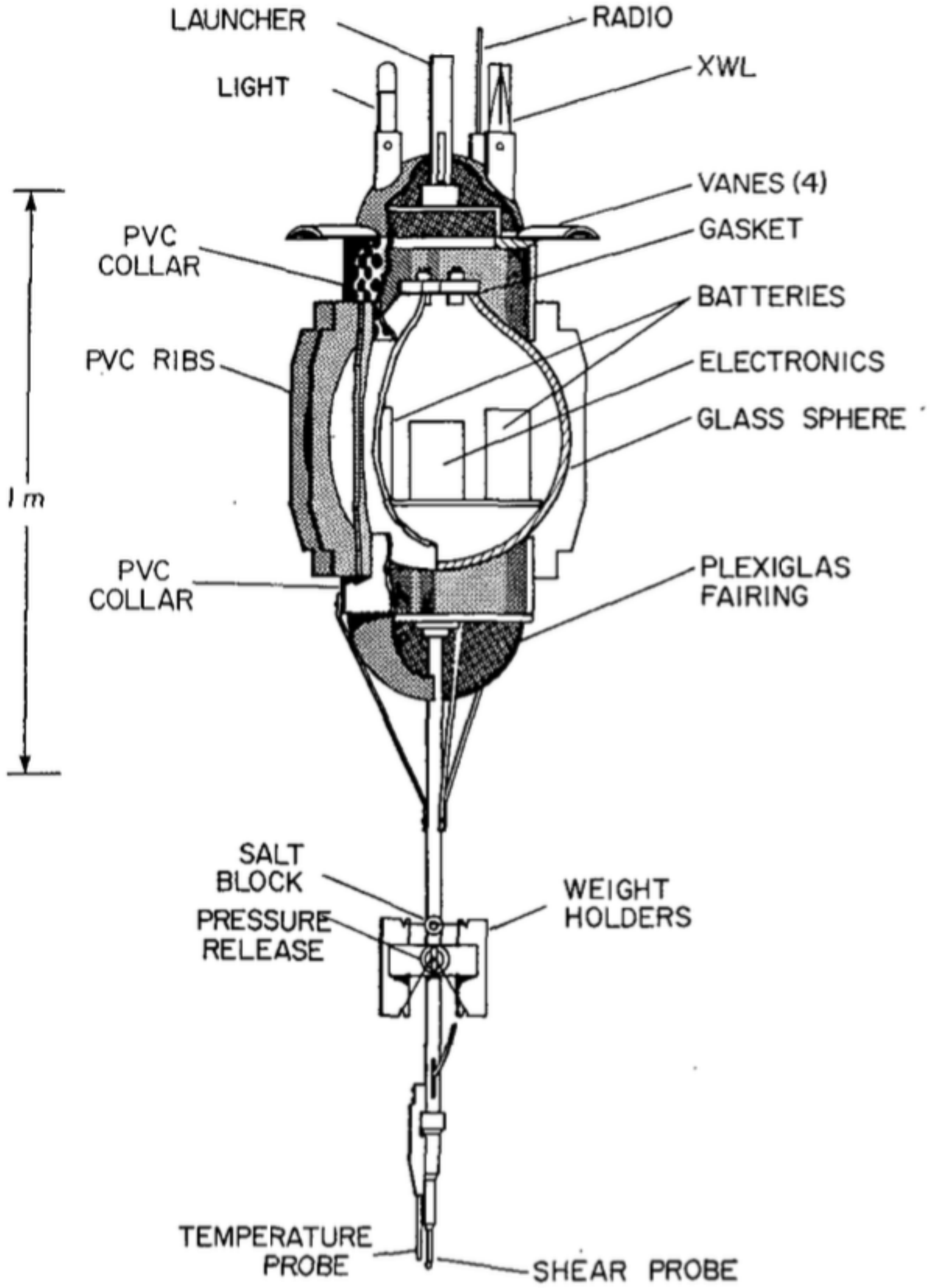


FIG. 1. Schematic diagram of temperature and velocity profiling instrument.

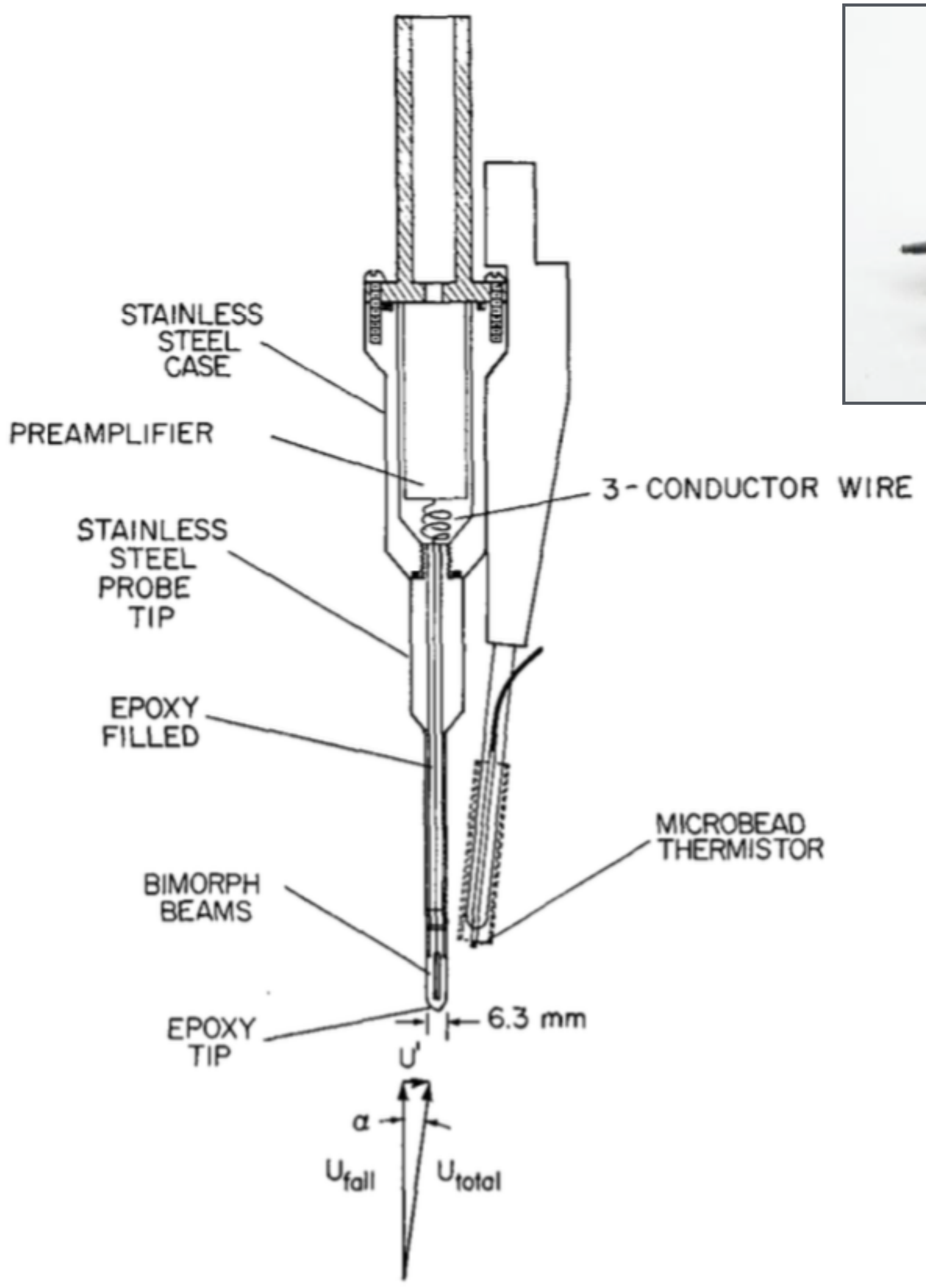
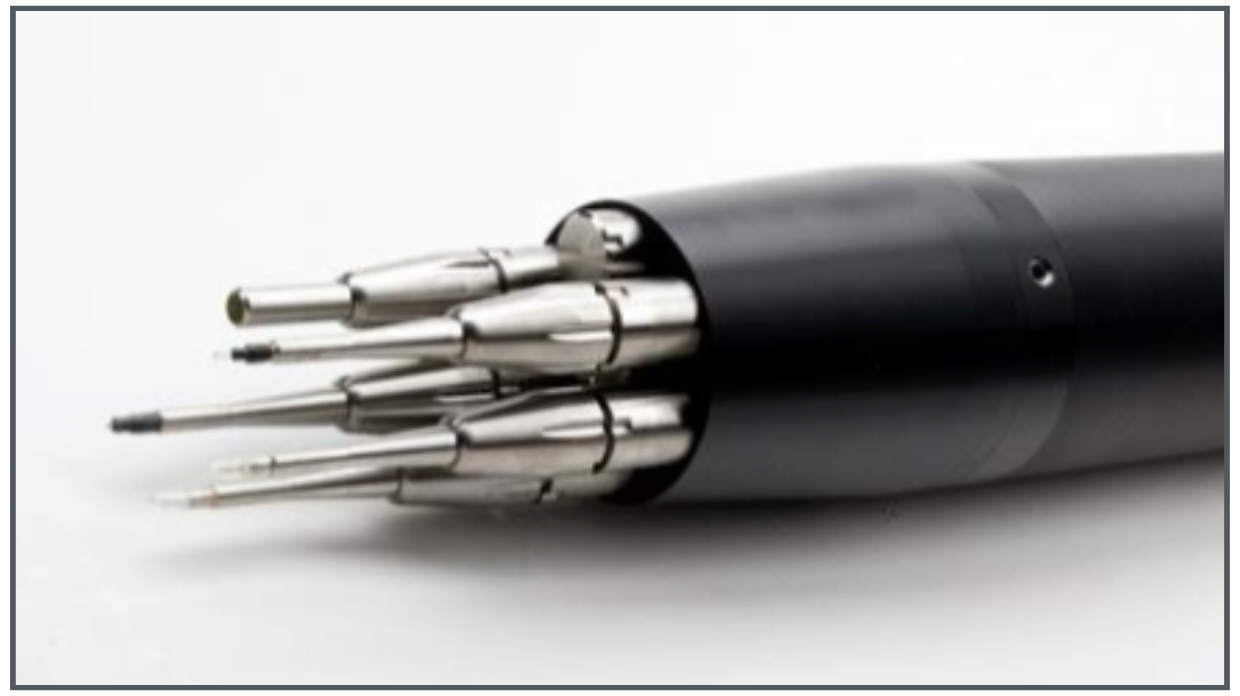
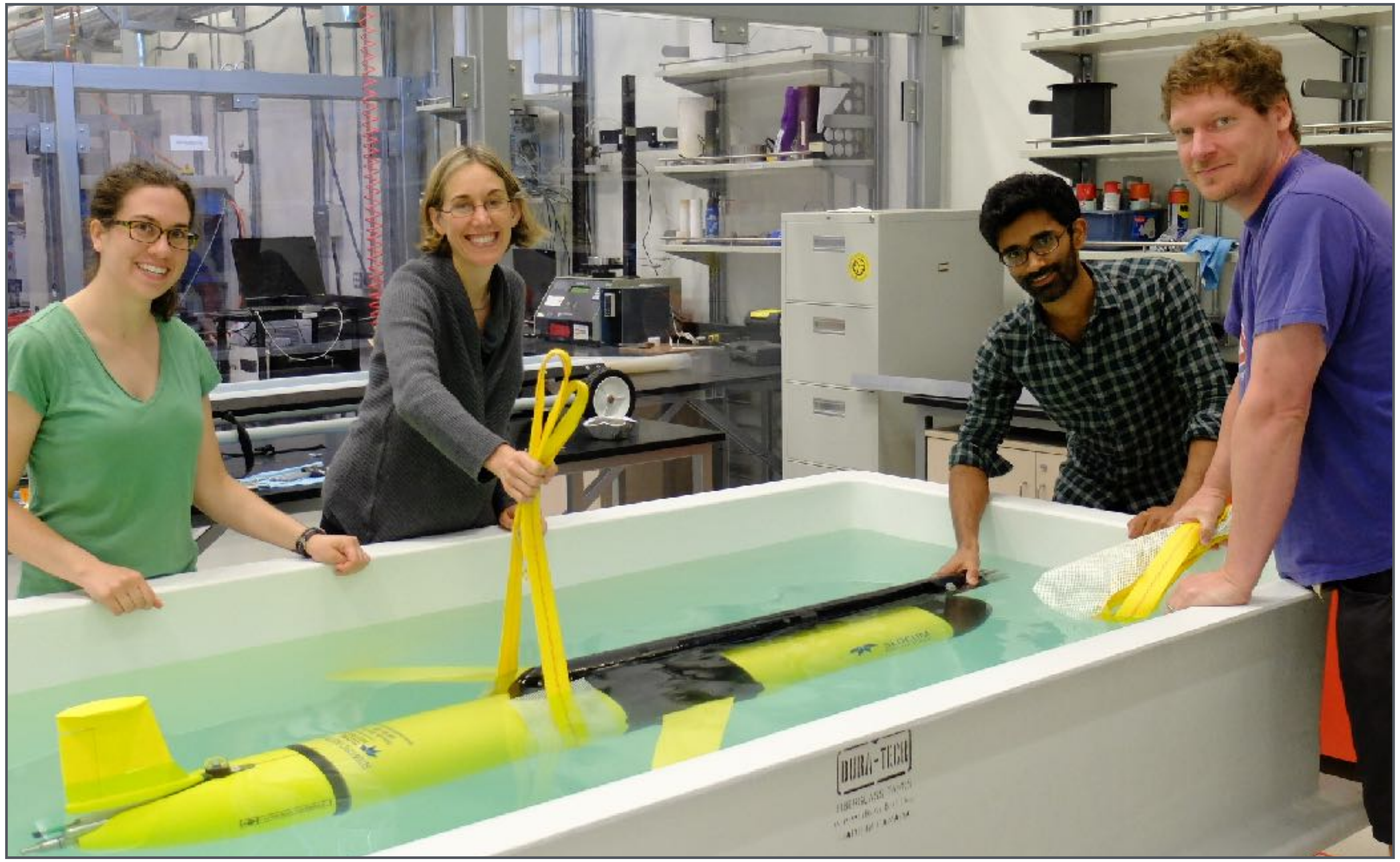
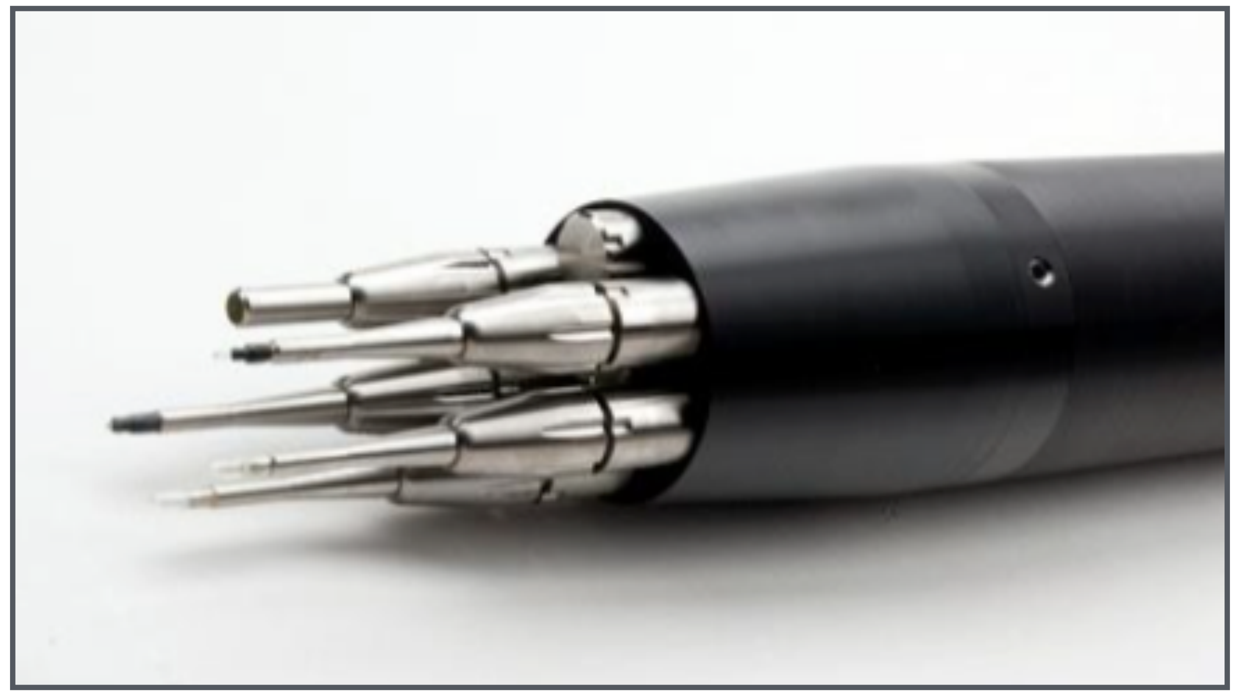


FIG. 2. Diagram of velocity and temperature probes. Vectors show that a change in the horizontal velocity relative to the probe tip appears as a change in the angle of attack of the overall velocity vector.



Osborn 1973

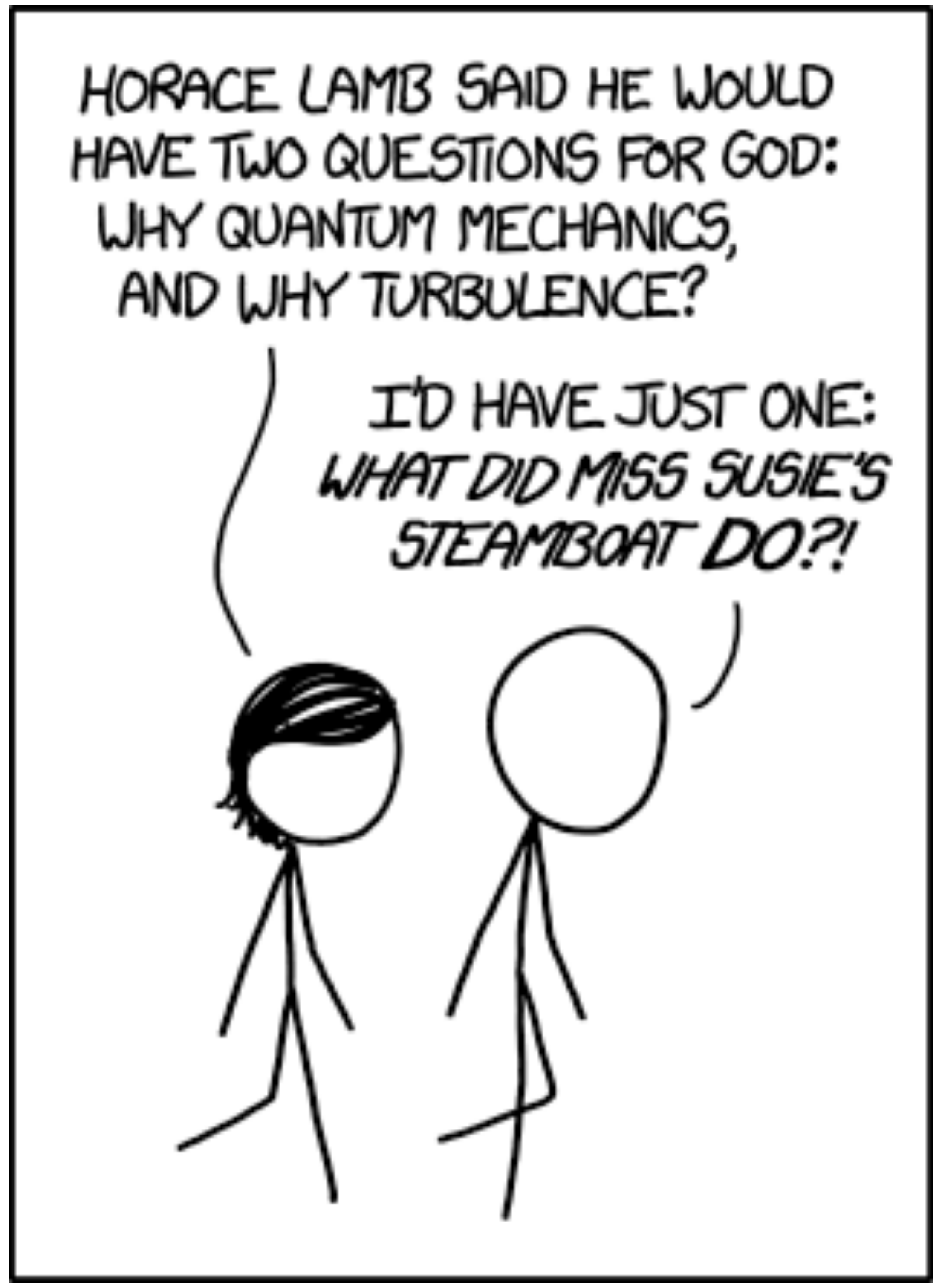




glider pictures: Tara

"I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic."

— Horace Lamb (1932)



xkcd