Stats 230, Homework 4

Due date: March 11

1. Let $X \sim \text{beta}(4, 2)$. Compute the second of moment of X, $\text{E}(X^2)$, using Monte Carlo (not MCMC), with 1,000 realizations from this distribution. Compute the Monte Carlo error and 95% confidence interval of you approximation.

2. Consider a Markov chain, on a countable state-space E, that given its current state i proceeds by randomly drawing another state j with proposal probability q_{ij} and then accepting/rejecting this proposed state with probability

$$a_{ij} = \frac{\pi_j q_{ji}}{\pi_j q_{ji} + \pi_i q_{ij}}, \tag{1}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots)$ is a probability mass function on E. This MCMC construction is called Barker's algorithm. Notice that $0 \le a_{ij} \le 1$ by definition.

   (a) What are the transition probabilities of this Markov chain, p_{ij}, for $i \neq j$?

   (b) Show that $\boldsymbol{\pi}$ is a stationary distribution of Barker's Markov chain.

3. Let's return to the beta distribution example. Let $X \sim \text{beta}(4, 2)$. Use a Metropolis-Hastings algorithm and a multiplicative log-normal proposal to approximate $\text{E}(X^2)$. More specifically, given x^{cur}, we generate $x^{prop} = x^{cur} e^X$, where $X \sim \mathcal{N}(0, \sigma^2)$.

   (a) What is the corresponding proposal density? Recall that if $\ln Y \sim \mathcal{N}(\mu, \sigma^2)$, then Y has a log-normal distribution with density

$$g(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}.$$

   (b) What is the Metropolis-Hastings ratio corresponding to the above proposal?

   (c) In addition to the $\text{E}(X^2)$ approximation, provide a trace plot and a histogram of the samples.