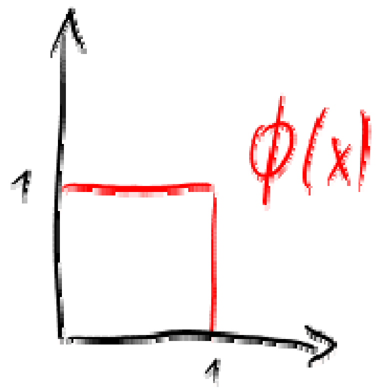
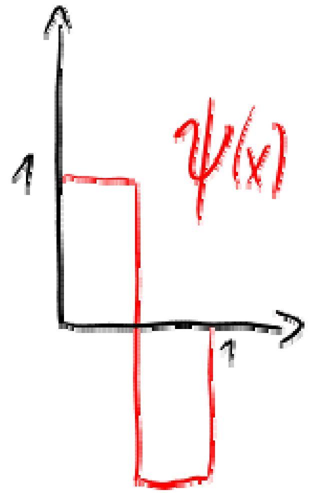
Haar wavelets

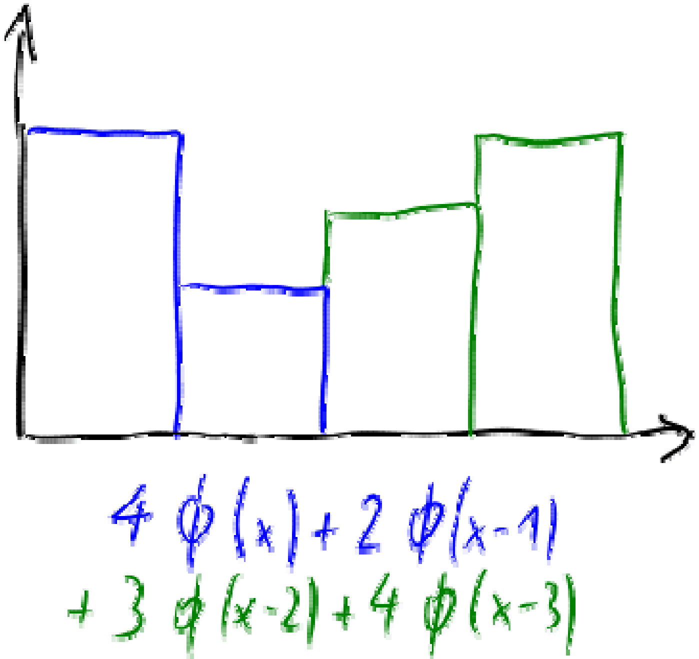
Basis function

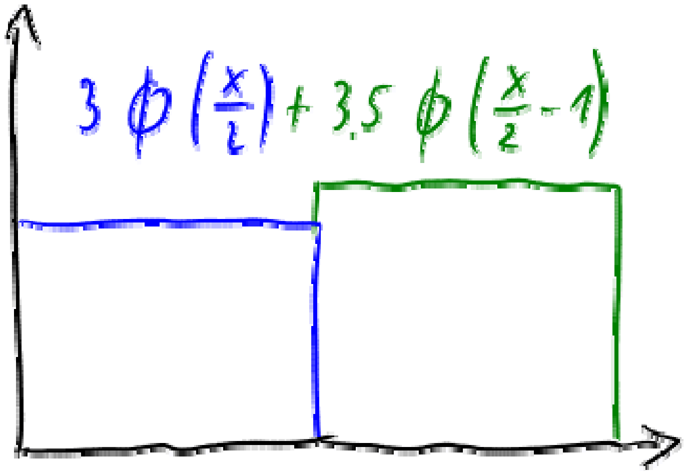


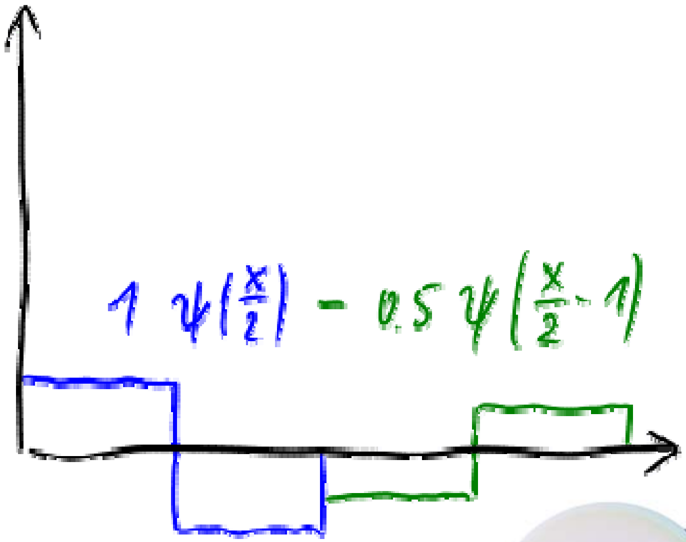
Wavelet function



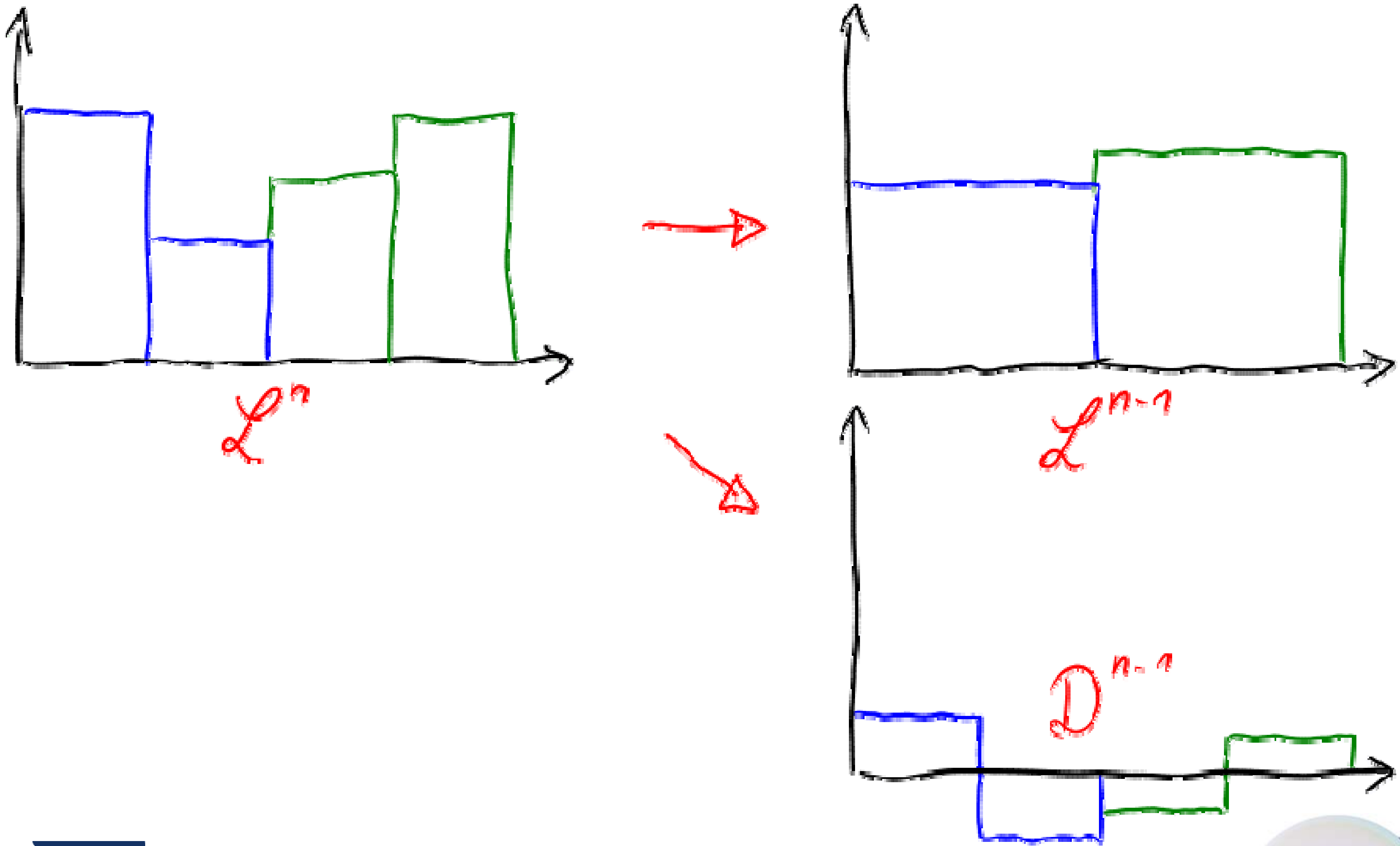
Haar wavelets







Multiresolution representation



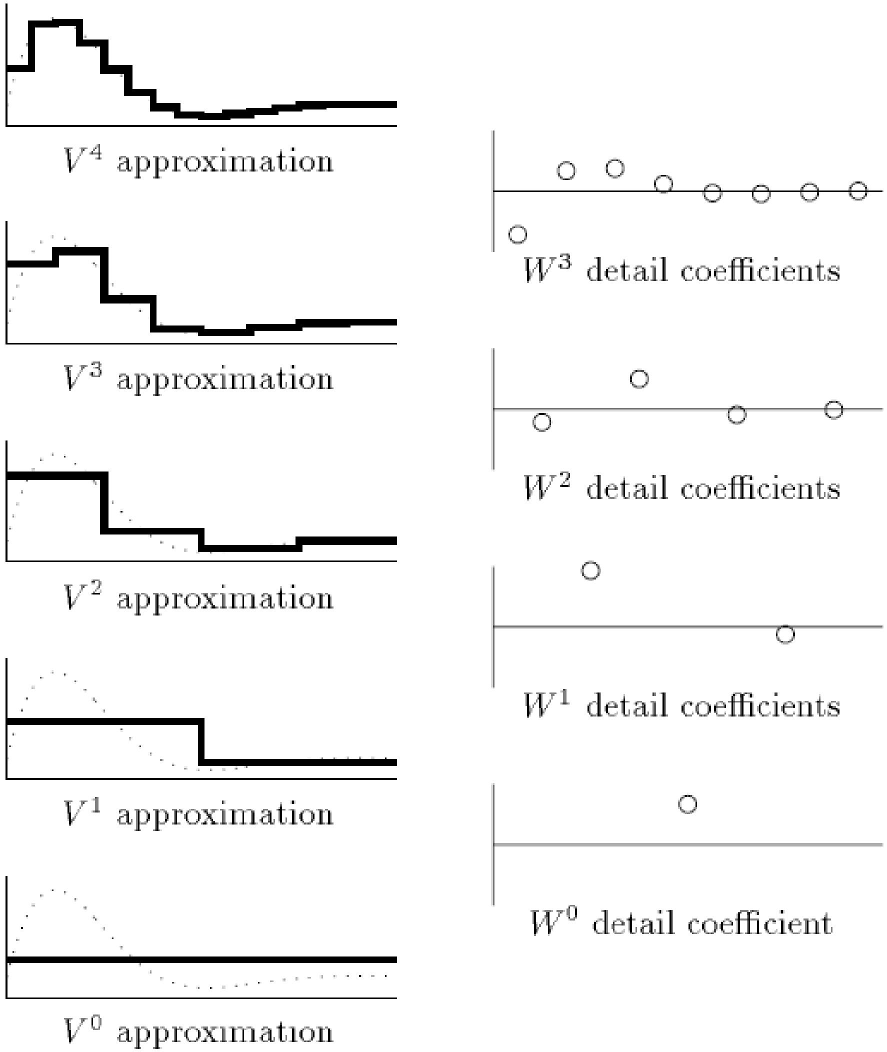
Multiresolution representation

- Object is represented as a sequence of resolutions.
- The resolutions $\mathcal{L}^n$ are called levels (levels of detail, LOD)
- The differences $\mathcal{D}^{n-1}$ are called detail coefficients.
- The levels build a multiresolution hierarchy:

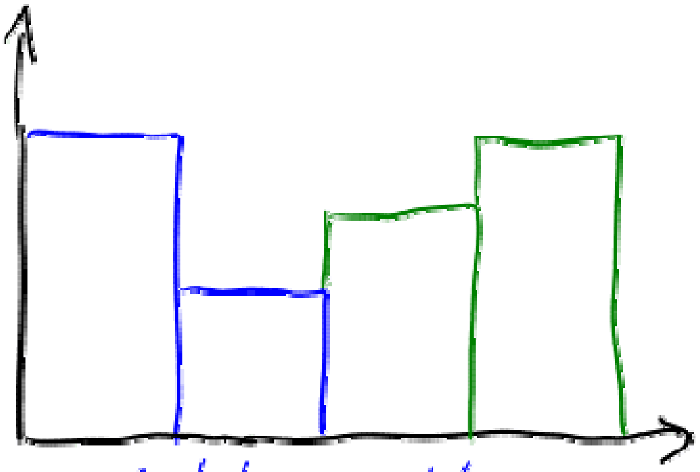
$$\mathcal{L}^n \supset \mathcal{L}^{n-1} \supset \ldots \supset \mathcal{L}^0$$

- The level $\mathcal{L}^0$ is the base level.
- The base level does not need to be represented by a regular mesh. All levels use then semi-regular meshes.

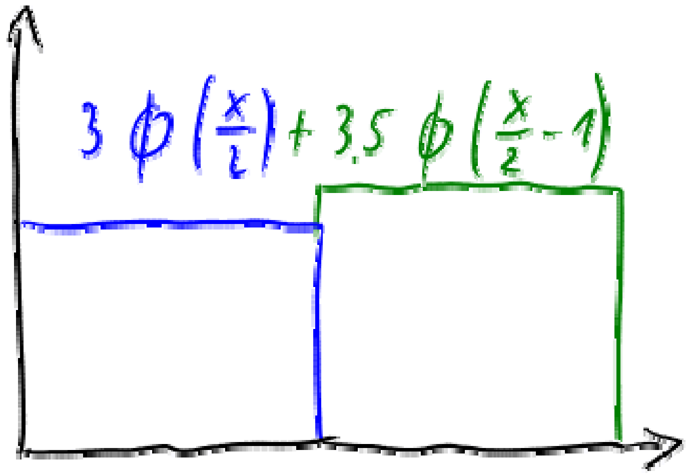
Multiresolution representation with Haar wavelets

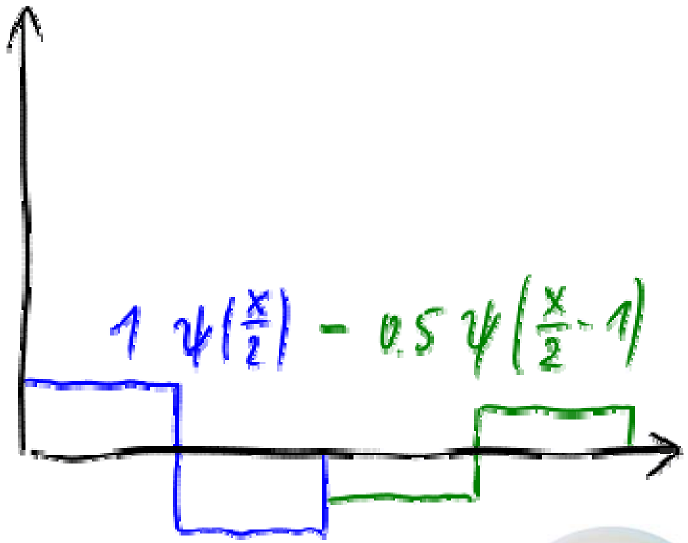


Memory requirements



$4\,\phi(x) + 2\,\phi(x-1)$
$+\,3\,\phi(x-2) + 4\,\phi(x-3)$





Storing the highest resolution $\mathcal{L}^n$ requires the same amount of storage as storing the coarsest resolution $\mathcal{L}^0$ and all detail coefficients $\mathcal{D}^{n-1}, \mathcal{D}^{n-2}, \ldots, \mathcal{D}^0$

1.4.2 Wavelets

References

- Wavelets for Computer Graphics: A Primer.

  Eric J. Stollnitz, Tony D. DeRose, David H. Salesin

  University of Washington, technical report, September 1994.

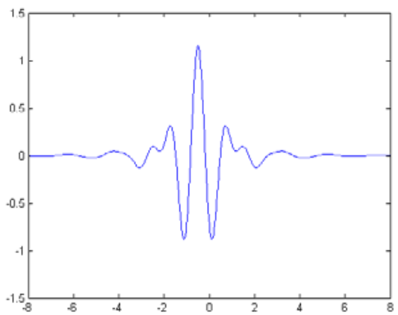
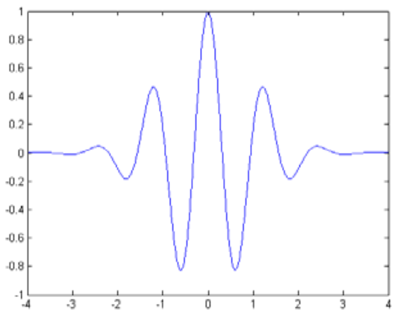
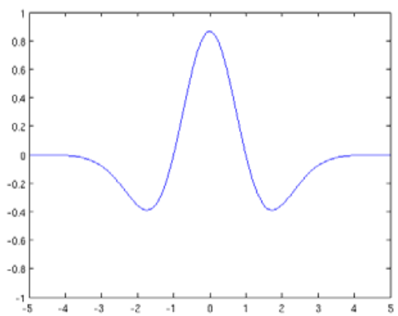
Wavelets

- Haar wavelets are the simplest and oldest wavelets (Alfred Haar, 1909).
- Wavelet theory started much later (boom in the 80s).
- Wavelet functions must fulfill the criteria:

$$\int_{-\infty}^{\infty} \psi(t)\, dt = 0$$

$$\int_{-\infty}^{\infty} |\psi(t)|^2\, dt = 1$$

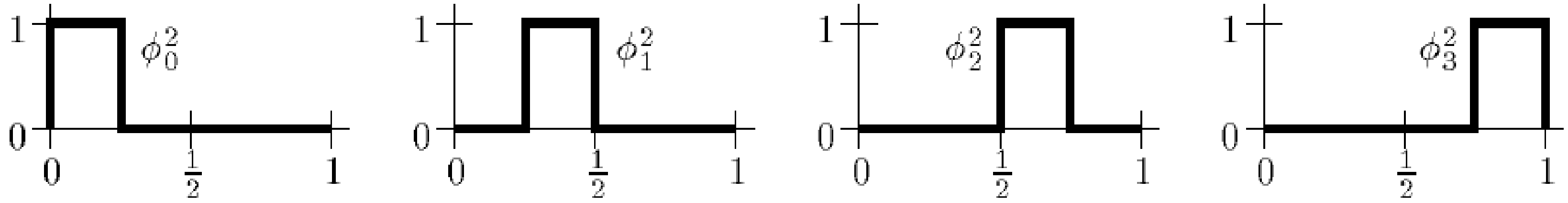
- Examples:

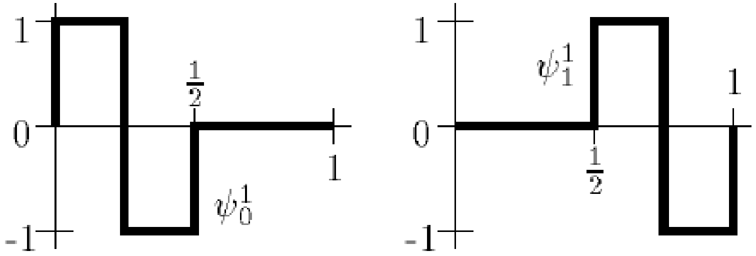
Multiresolution discrete wavelet transform

- Basis and wavelet functions span spaces:

$$V_m = \operatorname{span}(\phi_{m,n} : n \in \mathbb{Z}) \qquad \varphi_{m,n}(t) = 2^{-m/2}\varphi(2^{-m}t - n)$$



$$W_m = \operatorname{span}(\psi_{m,n} : n \in \mathbb{Z}) \qquad \psi_{m,n}(t) = 2^{-m/2}\psi(2^{-m}t - n)$$



Multiresolution discrete wavelet transform

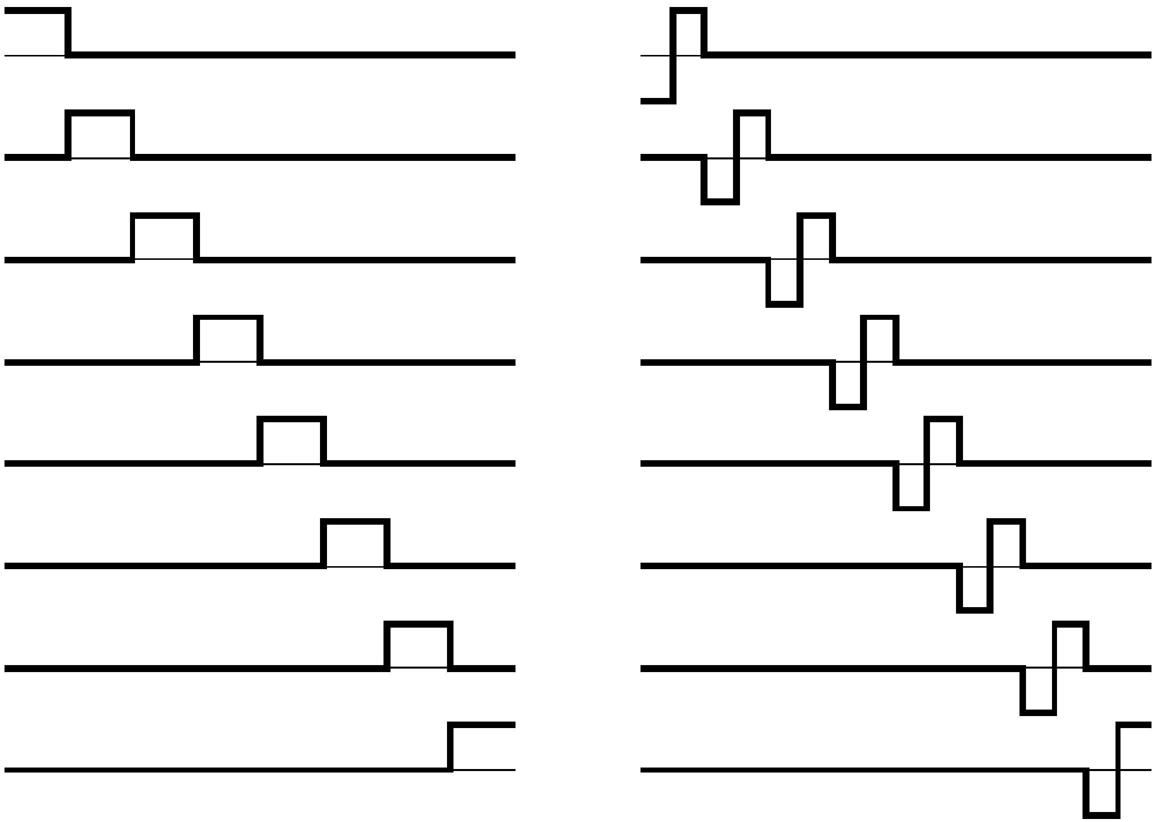
- The spaces V_m form a multiresolution analysis:

$$V_m \supset V_{m-1} \supset \ldots \supset V_0$$

- The spaces W_m are the orthogonal complements:

$$V_m \oplus W_m = V_{m+1}$$

Constant B-spline wavelets (Haar wavelets)



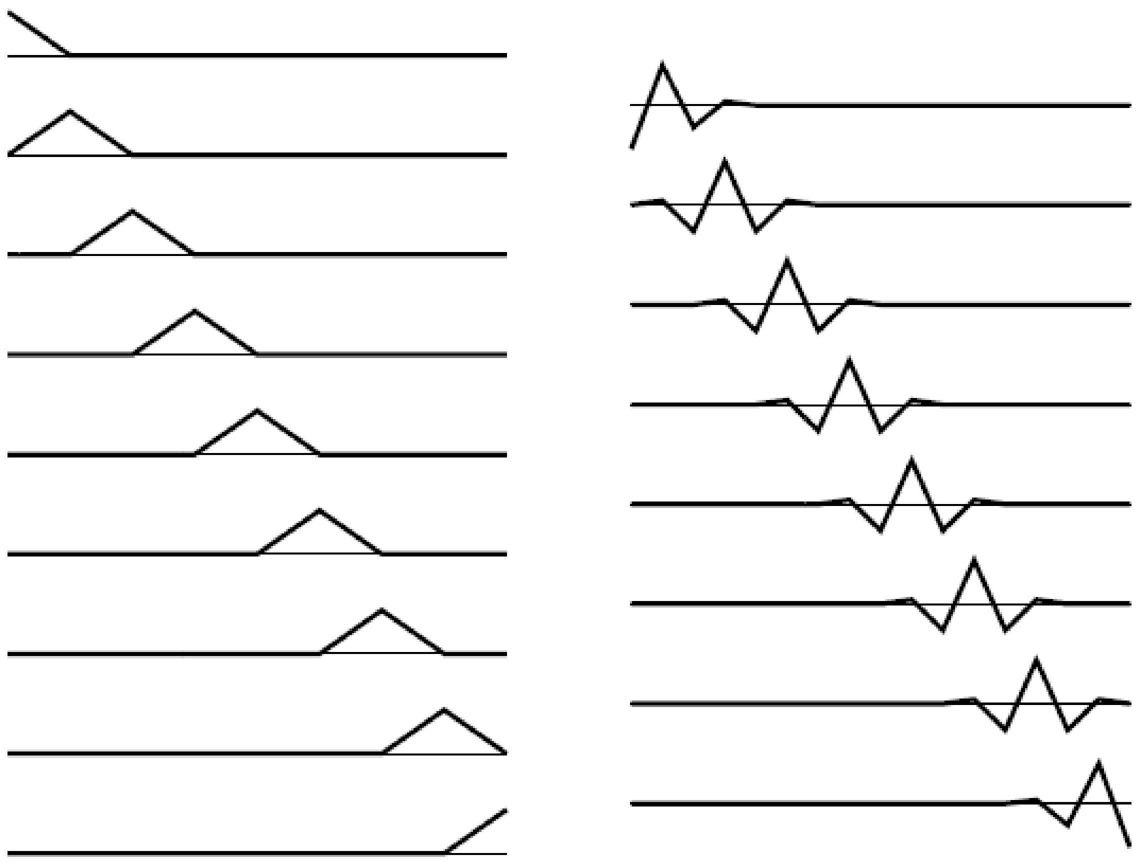
Constant B-spline wavelets (Haar wavelets)

The wavelet transform can also be written in matrix form:

$$P^j = \begin{bmatrix} 1 & & & & \\ 1 & & & & \\ & 1 & & & \\ & 1 & \cdot & & \\ & & & \cdot & & \\ & & & & \cdot & 1 & \\ & & & & & 1 & \\ & & & & & & 1 \\ & & & & & & 1 \end{bmatrix} \qquad Q^j = \begin{bmatrix} 1 & & & & \\ -1 & & & & \\ & 1 & & & \\ & -1 & \cdot & & \\ & & & \cdot & & \\ & & & & \cdot & 1 & \\ & & & & & -1 & \\ & & & & & & 1 \\ & & & & & & -1 \end{bmatrix}$$

The matrices are called synthesis matrices.

Linear B-spline wavelets



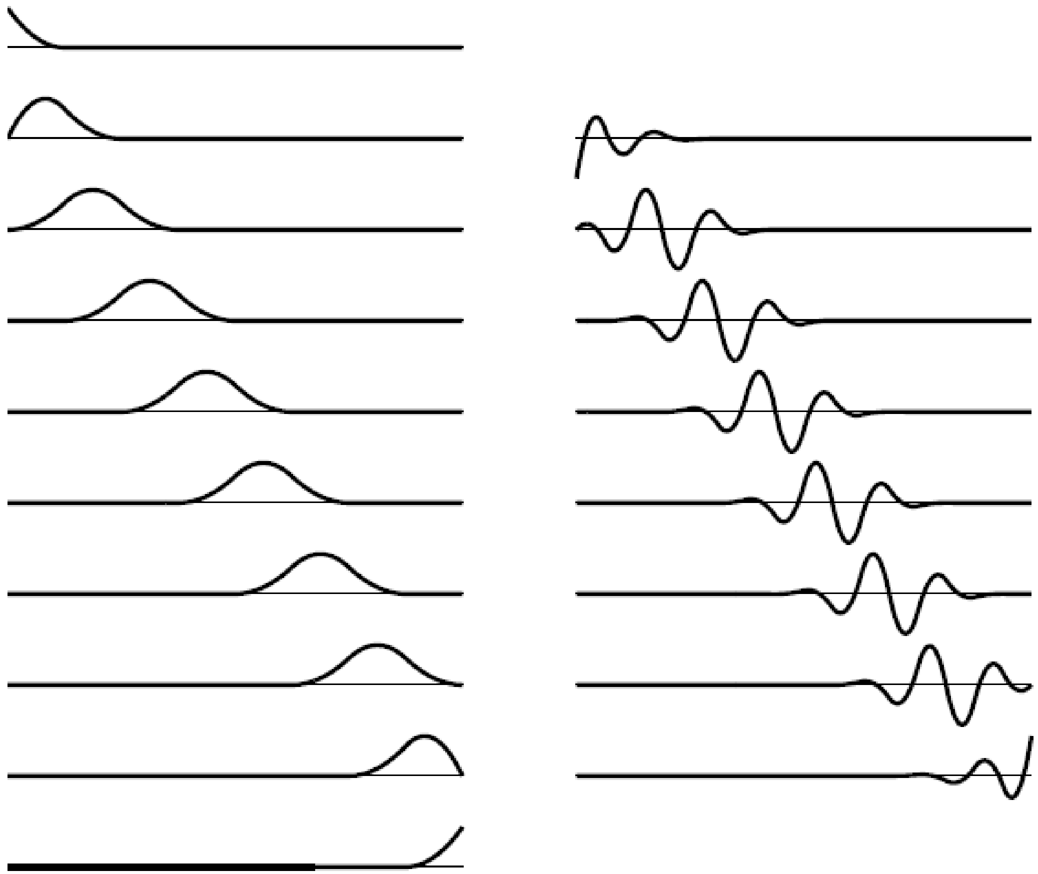
Linear B-spline wavelets

Synthesis matrices:

$$P^{j\geq 3} = \frac{1}{2}\begin{bmatrix} 2 & & & & & \\ 1 & 1 & & & & \\ & 2 & & & & \\ & 1 & 1 & & & \\ & & 2 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 2 & \\ & & & & 1 & 1 \\ & & & & & 2 \end{bmatrix}$$

$$Q^{j\geq 3} = \begin{bmatrix} -12 & & & & & \\ 11 & 1 & & & & \\ -6 & -6 & & & & \\ 1 & 10 & 1 & & & \\ & -6 & -6 & & & \\ & 1 & 10 & & & \\ & & -6 & \ddots & & \\ & & 1 & \ddots & 1 & \\ & & & \ddots & -6 & \\ & & & & 10 & 1 \\ & & & & -6 & -6 \\ & & & & 1 & 11 \\ & & & & & -12 \end{bmatrix}$$

Quadratic B-spline wavelets



Quadratic B-spline wavelets

Synthesis matrices:

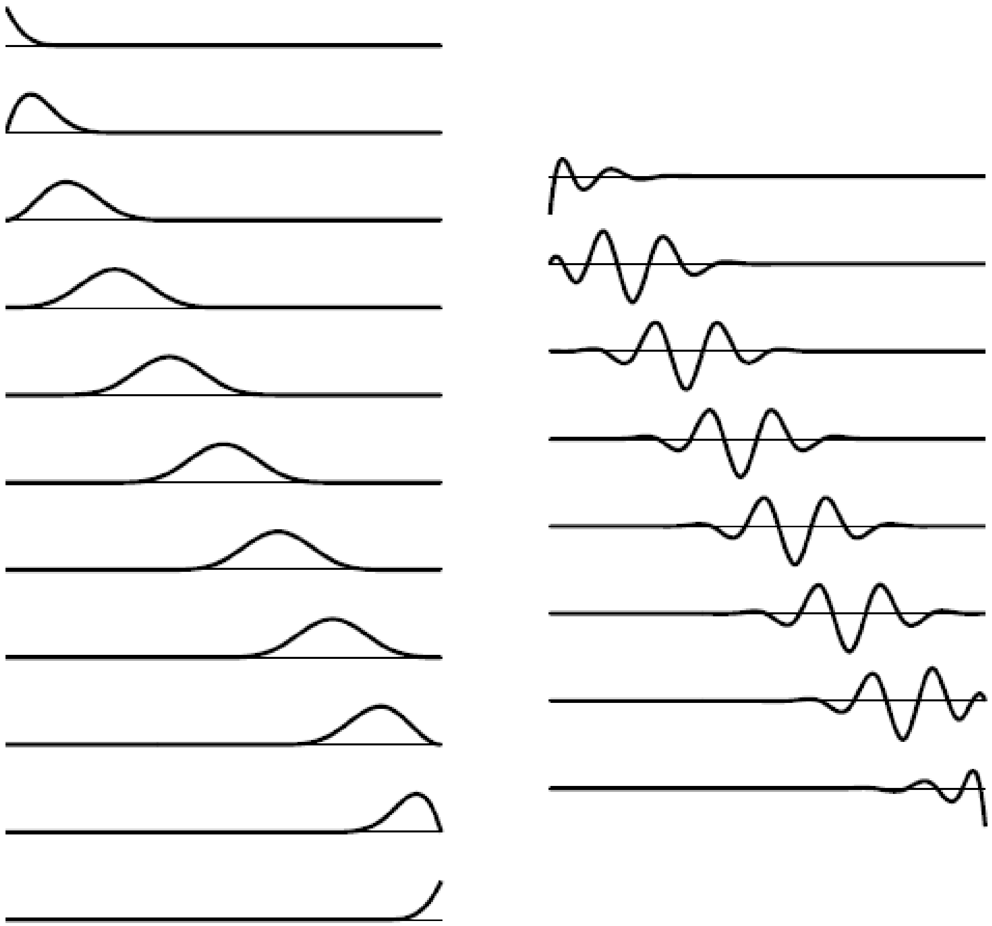
$$P^{j\geq 3} = \frac{1}{4}\begin{bmatrix}
4 & & & & & & & \\
2 & 2 & & & & & & \\
 & 3 & 1 & & & & & \\
 & 1 & 3 & & & & & \\
 & & 3 & 1 & & & & \\
 & & 1 & 3 & & & & \\
 & & & 3 & \ddots & & & \\
 & & & 1 & \ddots & 1 & & \\
 & & & & \ddots & 3 & & \\
 & & & & & 3 & 1 & \\
 & & & & & 1 & 3 & \\
 & & & & & & 2 & 2 \\
 & & & & & & & 4
\end{bmatrix}$$

Quadratic B-spline wavelets

Synthesis matrices:

$$
Q^{j \geq 4} = \begin{bmatrix}
-75504 & & & & & & & \\
91806 & 10920 & & & & & & \\
-54637 & -27286 & -154 & & & & & \\
22913 & 48734 & 4466 & & & & & \\
-4466 & -47068 & -22638 & -154 & & & & \\
154 & 22652 & 46662 & 4466 & & & & \\
 & -4466 & -46662 & -22638 & & & & \\
 & 154 & 22638 & 46662 & & -154 & & \\
 & & -4466 & -46662 & \cdot & 4466 & & \\
 & & 154 & 22638 & \cdot & -22638 & -154 & \\
 & & & -4466 & \cdot & 46662 & 4466 & \\
 & & & 154 & & -46662 & -22652 & -154 \\
 & & & & & 22638 & 47068 & 4466 \\
 & & & & & -4466 & -48734 & -22913 \\
 & & & & & 154 & 27286 & 54637 \\
 & & & & & & -10920 & -91806 \\
 & & & & & & & 75504
\end{bmatrix}
$$

Cubic B-spline wavelets



Cubic B-spline wavelets

Synthesis matrices:

$$P^{j\geq 3} = \frac{1}{16}\begin{bmatrix}
16 & & & & & & & & & \\
8 & 8 & & & & & & & & \\
 & 12 & 4 & & & & & & & \\
 & 3 & 11 & 2 & & & & & & \\
 & & 8 & 8 & & & & & & \\
 & & 2 & 12 & 2 & & & & & \\
 & & & 8 & 8 & & & & & \\
 & & & 2 & 12 & & & & & \\
 & & & & 8 & & & & & \\
 & & & & 2 & \ddots & 2 & & & \\
 & & & & & & 8 & & & \\
 & & & & & & 12 & 2 & & \\
 & & & & & & 8 & 8 & & \\
 & & & & & & 2 & 11 & 3 & \\
 & & & & & & & 4 & 12 & \\
 & & & & & & & & 8 & 8 \\
 & & & & & & & & & 16
\end{bmatrix}$$

Cubic B-spline wavelets

Synthesis matrices:

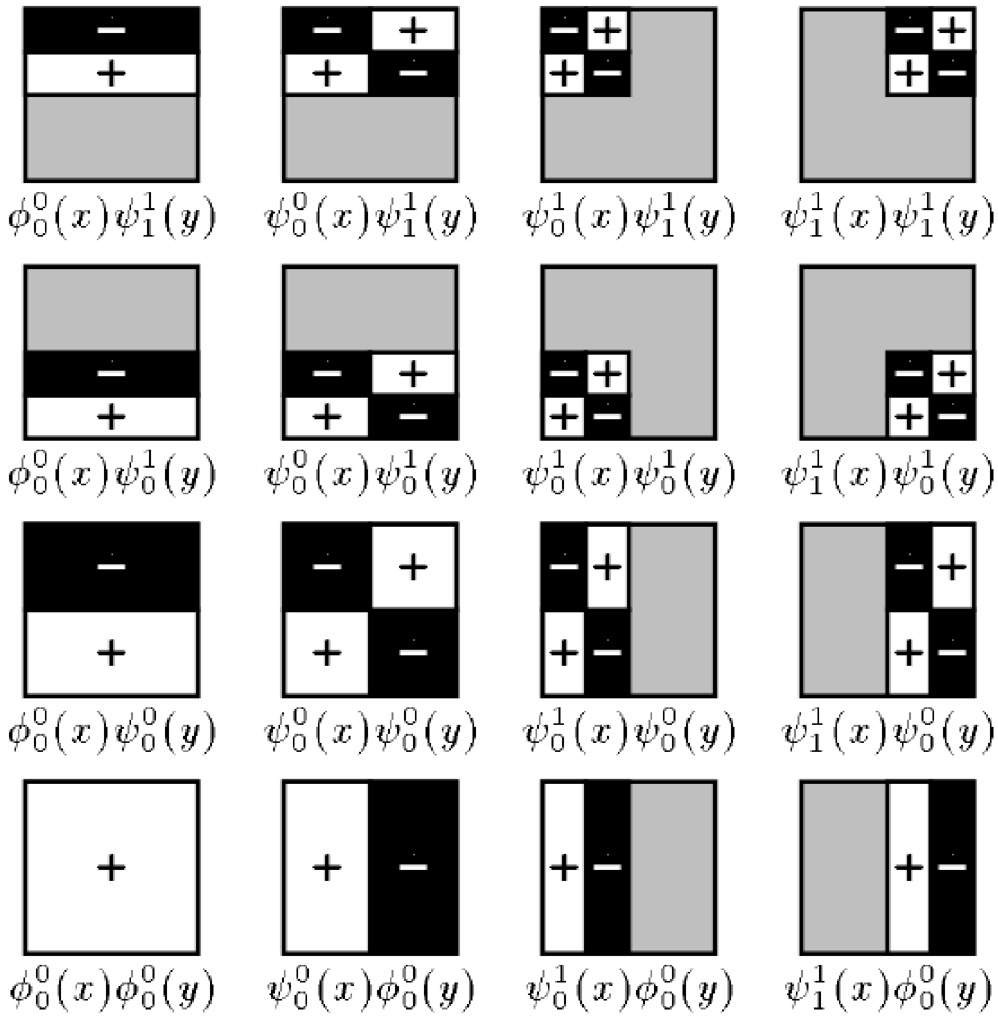
$$
Q^{j\geq 4} \approx \begin{bmatrix}
-686823 & & & & & & & & & \\
1000000 & 256326 & & & & & & & & \\
-798167 & -701422 & -15882 & & & & & & & \\
382460 & 925781 & 85895 & 41 & & & & & & \\
-138342 & -1000000 & -343712 & -5110 & & & & & & \\
28284 & 711467 & 771597 & 69115 & 41 & & & & & \\
-2088 & -297591 & -1000000 & -325750 & -5110 & & & & & \\
17 & 62856 & 758322 & 761705 & 69115 & & & & & \\
 & -4647 & -323843 & -1000000 & -325750 & & & & & \\
 & 37 & 68691 & 761705 & 761705 & & 41 & & & \\
 & & -5079 & -325750 & -1000000 & & -5110 & & & \\
 & & 41 & 69115 & 761705 & \cdot & 69115 & 41 & & \\
 & & & -5110 & -325750 & \cdot & -325750 & -5079 & & \\
 & & & 41 & 69115 & \cdot & 761705 & 68691 & 37 & \\
 & & & & -5110 & & -1000000 & -323843 & -4647 & \\
 & & & & 41 & & 761705 & 758322 & 62856 & 17 \\
 & & & & & & -325750 & -1000000 & -297591 & -2088 \\
 & & & & & & 69115 & 771597 & 711467 & 28284 \\
 & & & & & & -5110 & -343712 & -1000000 & -138342 \\
 & & & & & & 41 & 85895 & 925781 & 382460 \\
 & & & & & & & -15882 & -701422 & -798167 \\
 & & & & & & & & 256326 & 1000000 \\
 & & & & & & & & & -686823
\end{bmatrix}
$$

2D Haar wavelet transform

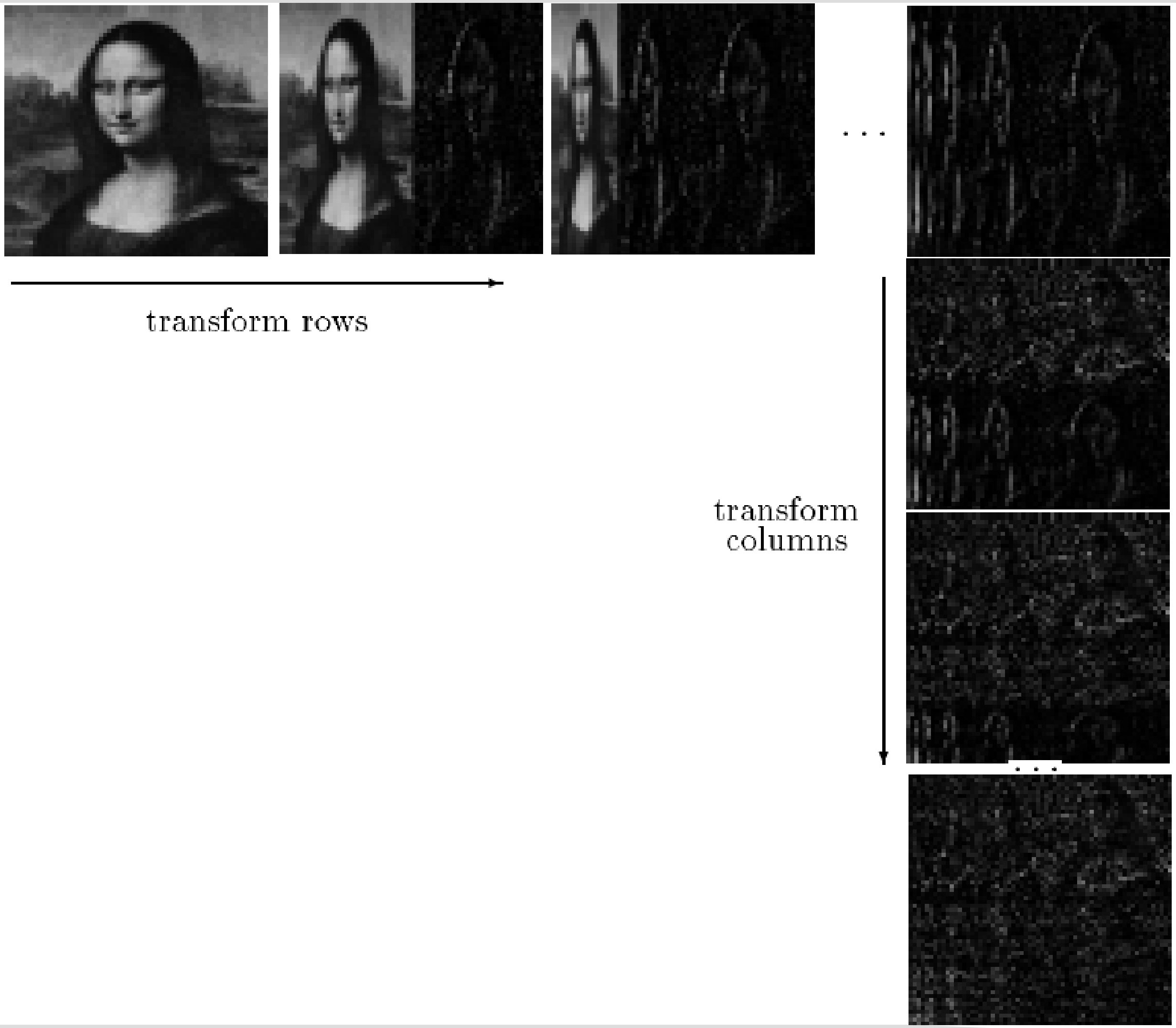
- 2D basis and wavelet functions are **tensor products** of 1D basis and wavelet functions.

2D Haar wavelet transform

Basis:



2D Haar wavelet transform



2D Haar wavelet transform

Alternative construction:

Use 2D basis function

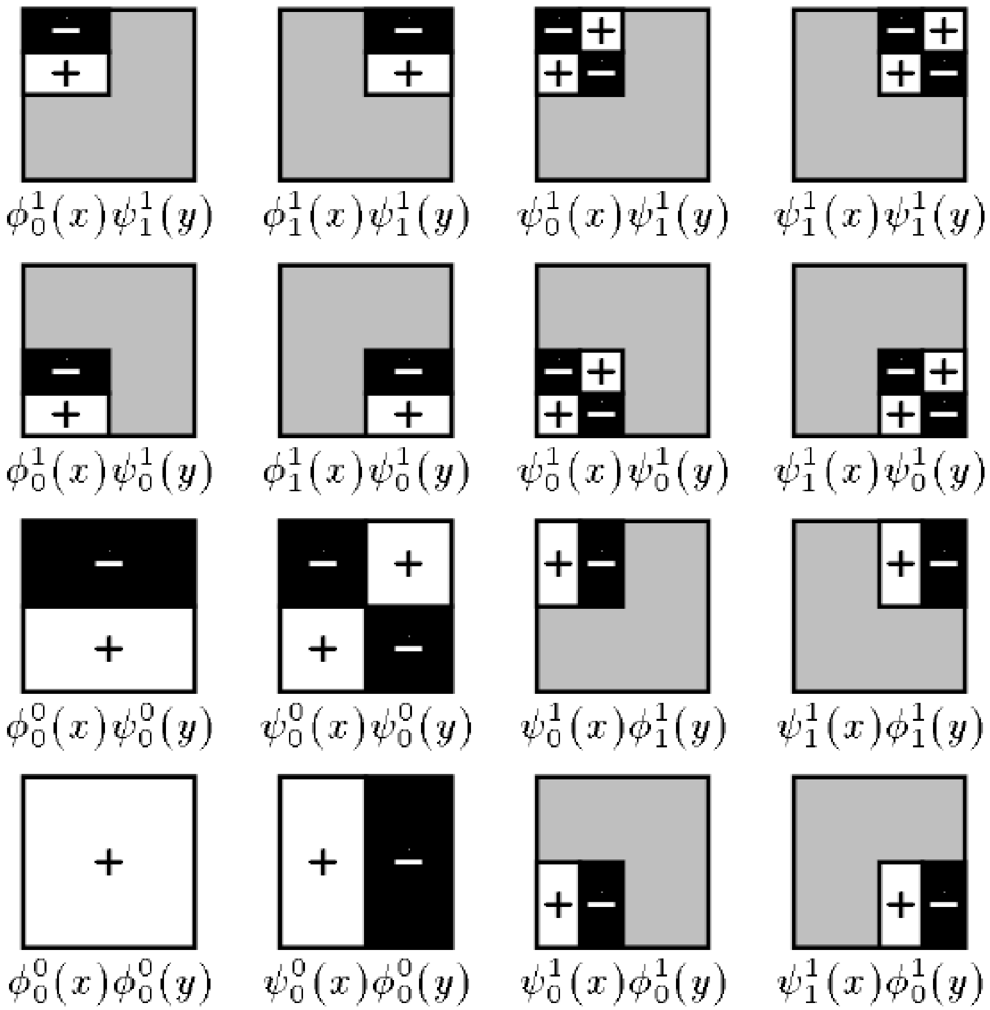
$$\phi\phi(x,y) := \phi(x)\,\phi(y)$$

and three 2D wavelet functions

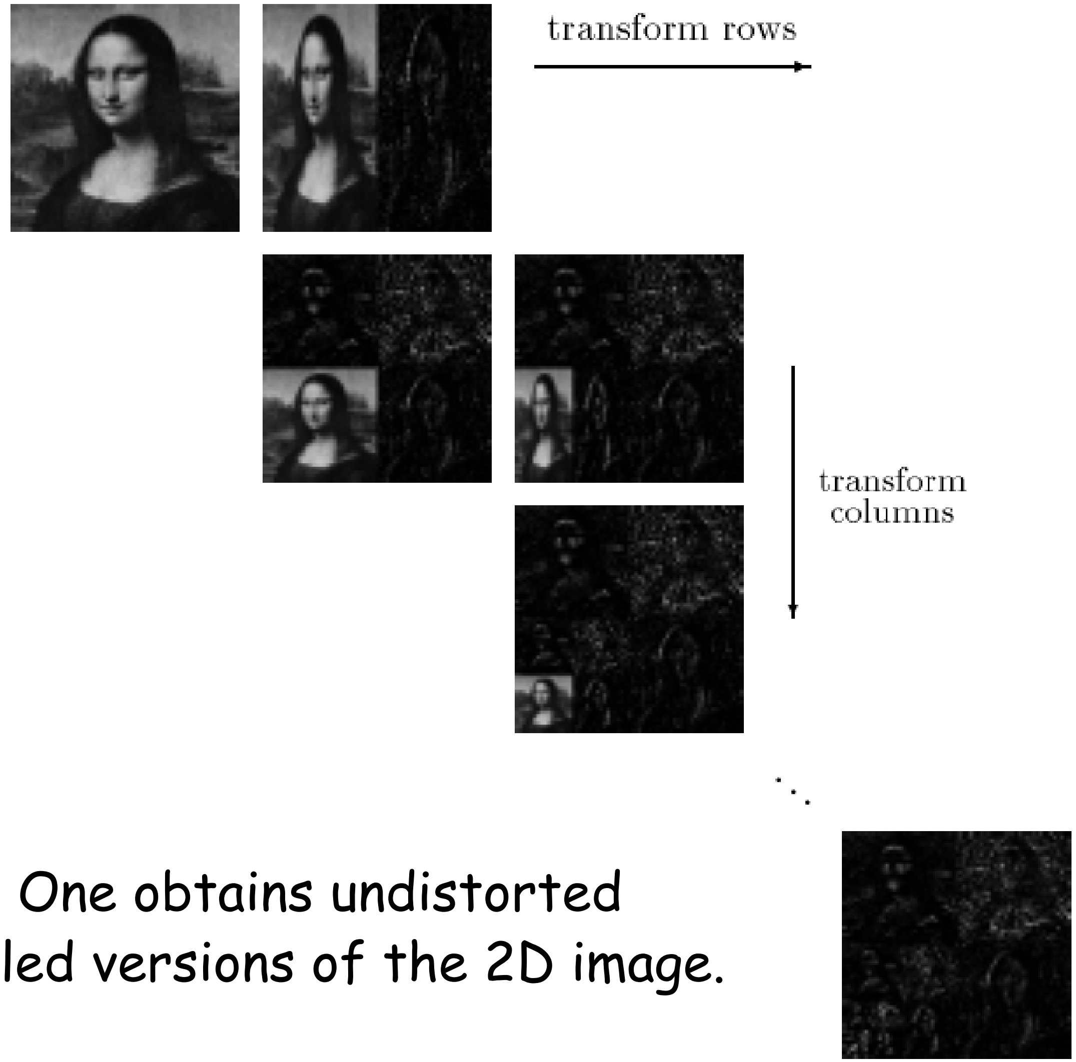
$$\begin{aligned}\phi\psi(x,y) &:= \phi(x)\,\psi(y)\\ \psi\phi(x,y) &:= \psi(x)\,\phi(y)\\ \psi\psi(x,y) &:= \psi(x)\,\psi(y)\end{aligned}$$

2D Haar wavelet transform

Basis:



2D Haar wavelet transform



Advantage: One obtains undistorted downscaled versions of the 2D image.

2D wavelet transform in RGB space

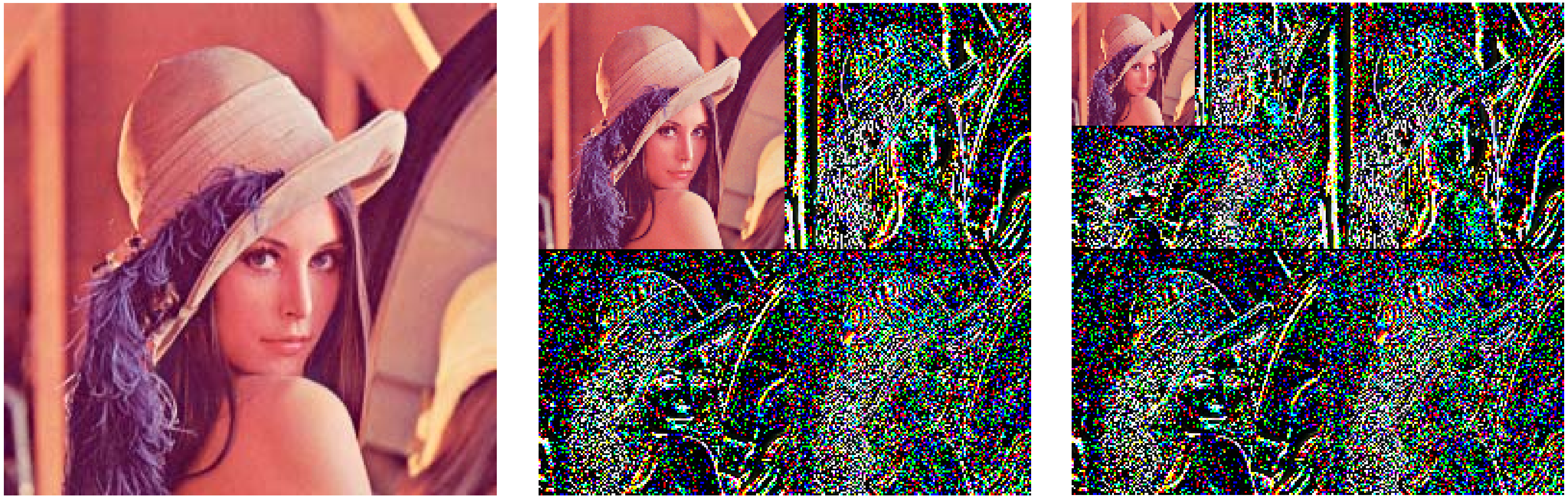


Image compression

- Loss-less compression
  - Do not store detail coefficients that are 0.
  - Constant regions are stored by 1 value only.

- Lossy compression
  - Set detail coefficients with small absolute values to 0.
  - A threshold determines the compression rate.

Image compression

Haar wavelets:

			
100%	21%	4%	1%
Error: 0%	5%	10%	15%

Image compression

JPEG 2000: Cohen-Daubechies-Feauveau wavelets

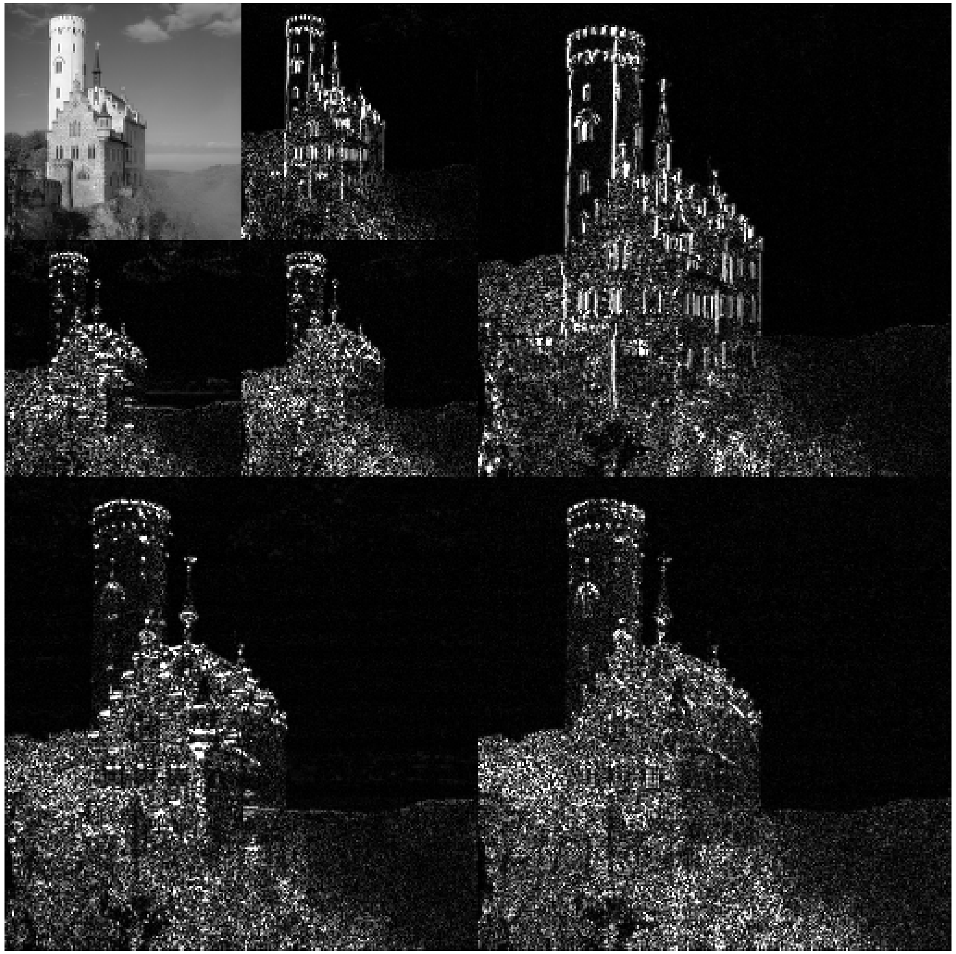
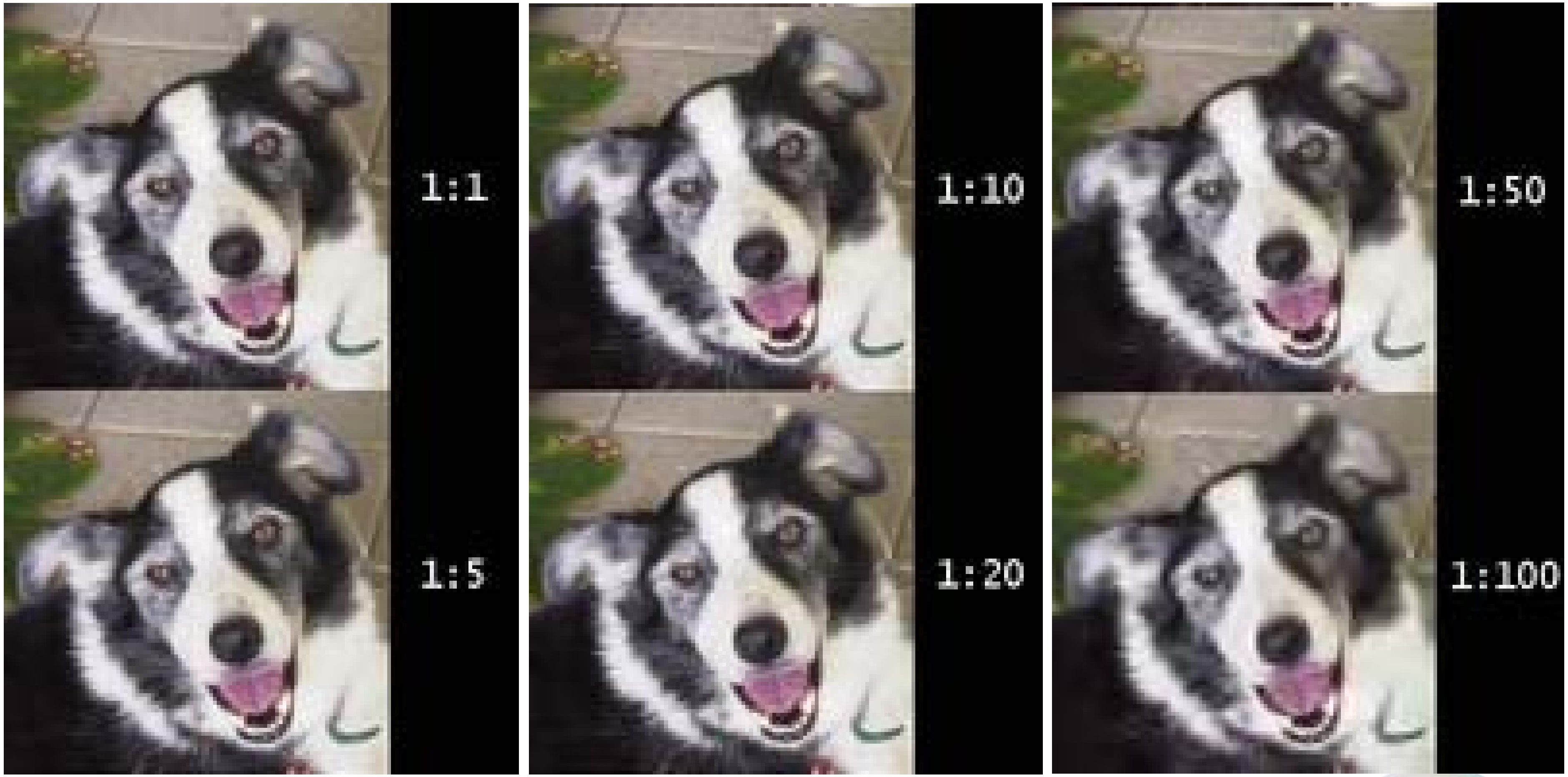


Image compression

JPEG 2000: lossy compression leads to blurring.



1.4.3 Multiresolution Modeling

References

- Multiresolution Techniques.
  Leif P. Kobbelt.
  In: Handbook of Computer Aided Geometric Design,
  G. Farin, J. Hoschek, M-S. Kim (eds.), Elsevier, 2002.

Multiresolution modeling

- Filter bank

$$
\begin{tikzcd}
\mathcal{L}^n \arrow[r, "P"] \arrow[dr, "Q"] & \mathcal{L}^{n-1} \arrow[r, "P"] \arrow[dr, "Q"] & \mathcal{L}^{n-2} \arrow[r, "P"] \arrow[dr, "Q"] & \cdots \arrow[r, "P"] \arrow[dr, "Q"] & \mathcal{L}^0 \\
& \mathcal{D}^{n-1} & \mathcal{D}^{n-2} & & \mathcal{D}^0
\end{tikzcd}
$$

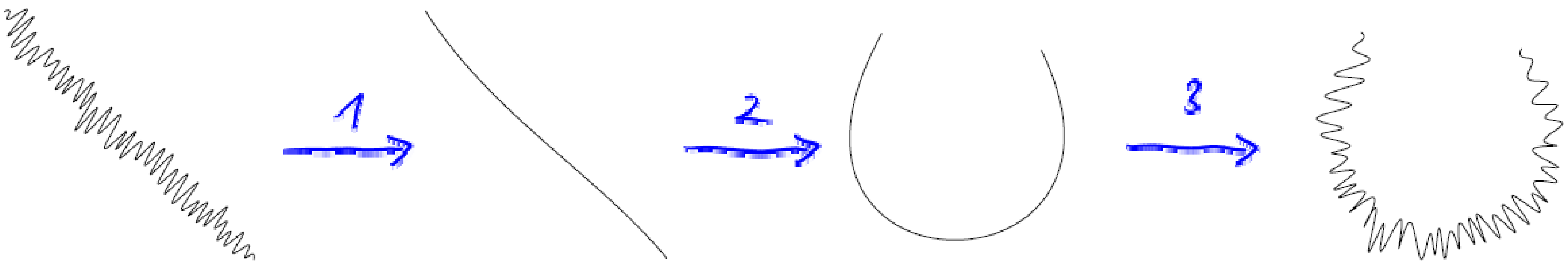
small details $\longrightarrow$ large details

high frequency ... low frequency

$\underbrace{\hspace{10em}}_{\text{frequency spectrum}}$

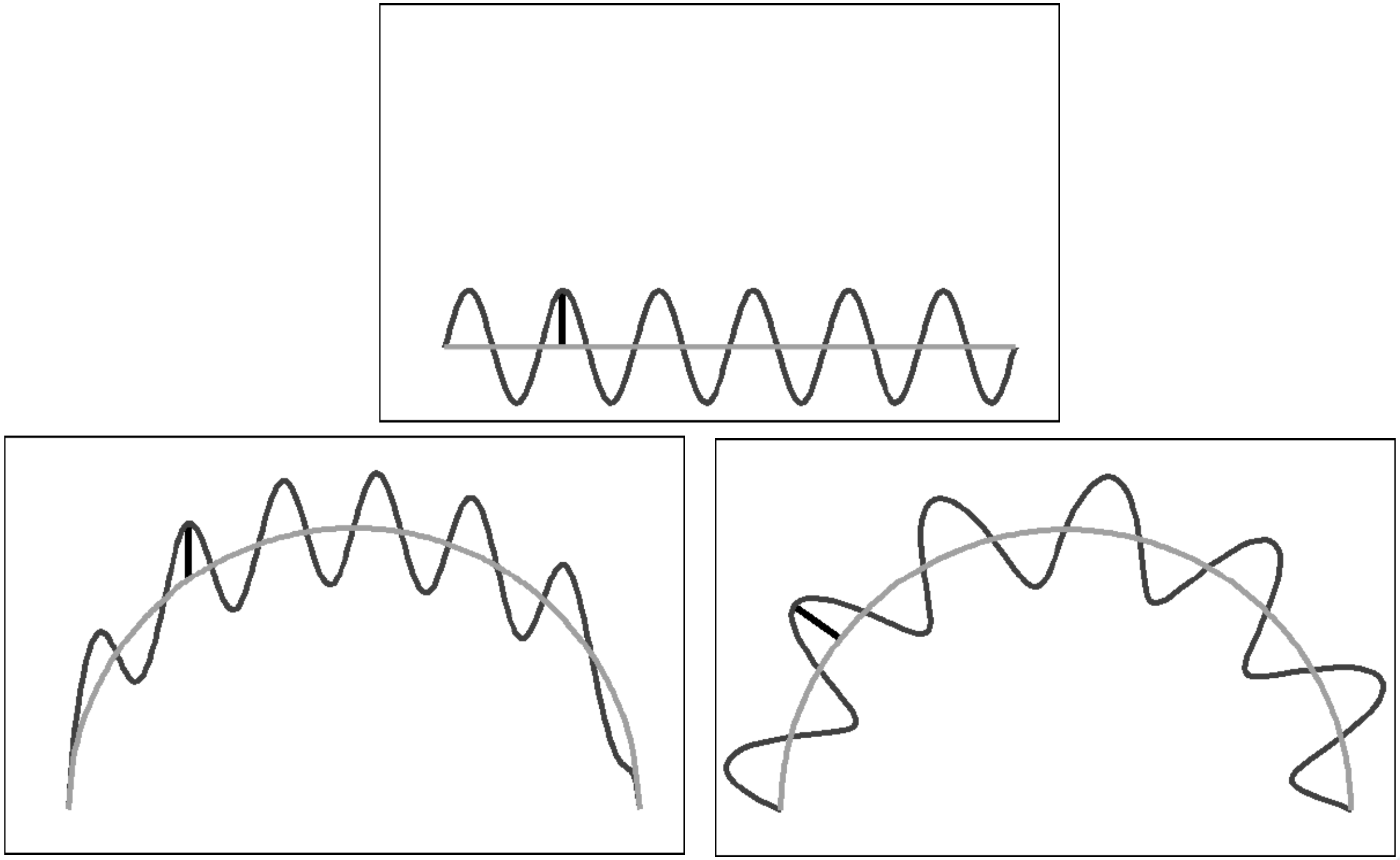
Multiresolution modeling: Example 1

1. Remove high-frequency detail by going to coarse resolution.
2. Perform modeling at coarse resolution by interactively changing the shape.
3. Reinsert high-frequency detail by going to fine resolution.



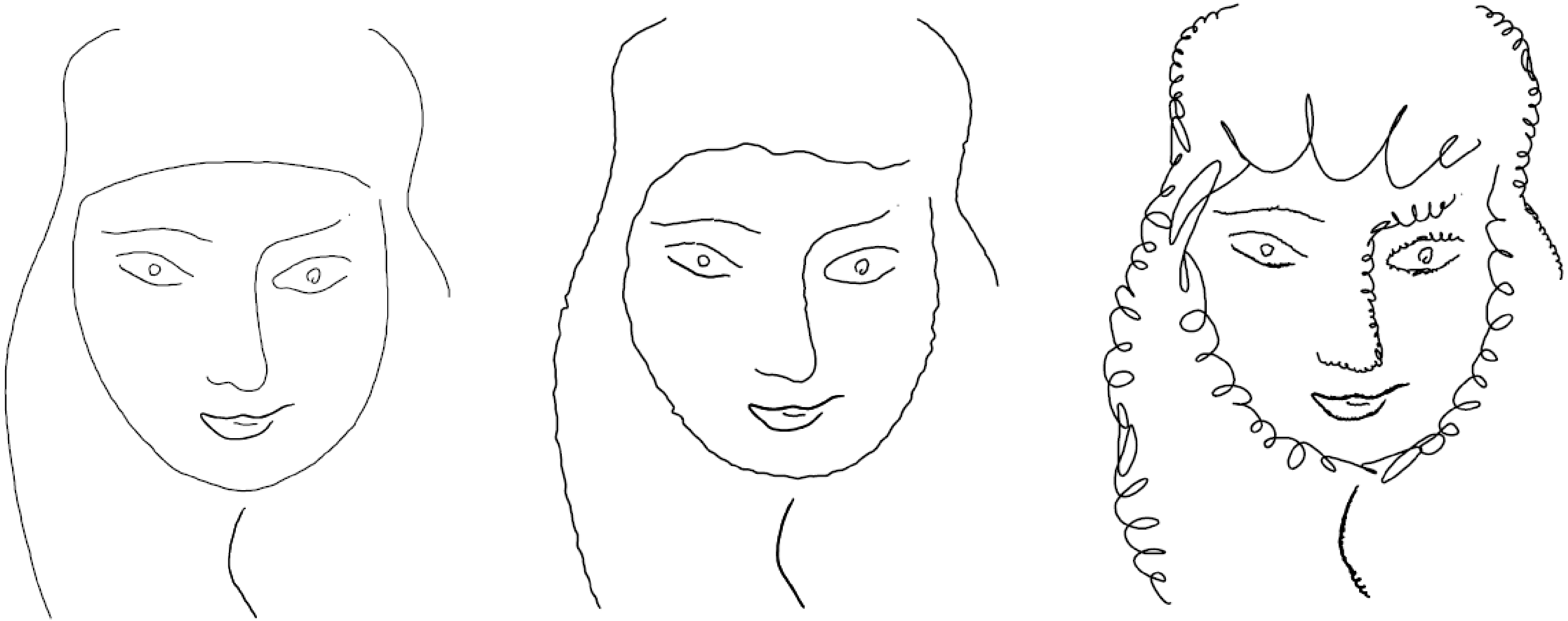
Multiresolution modeling: Example 1

- Global vs. local details:



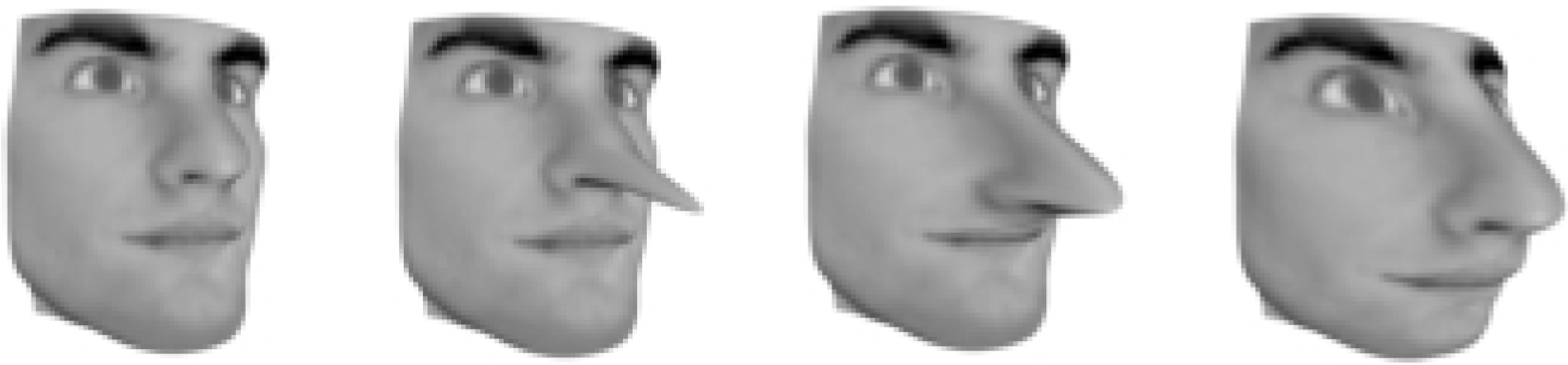
Multiresolution modeling: Example 2

- Change the character of an object by replacing detail coefficients with new ones, e.g., by replacing small high-frequency details with large ones.



Multiresolution modeling: Example 3

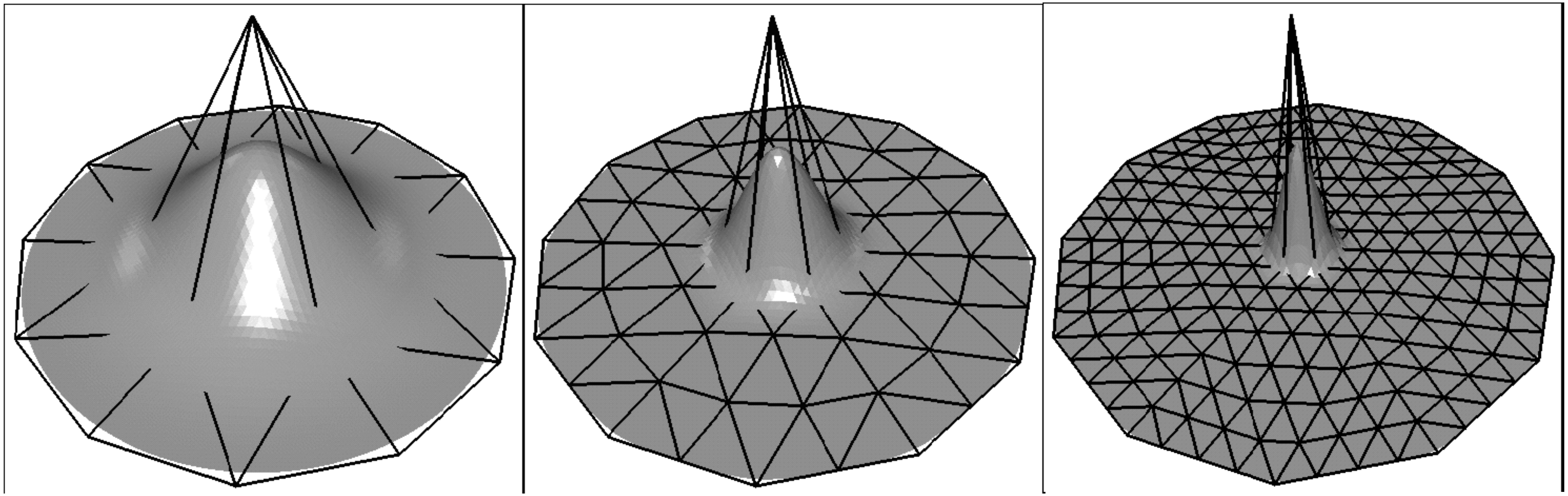
- Performing interactive changes of the shape at different levels of resolution.



fine level intermediate level coarse level

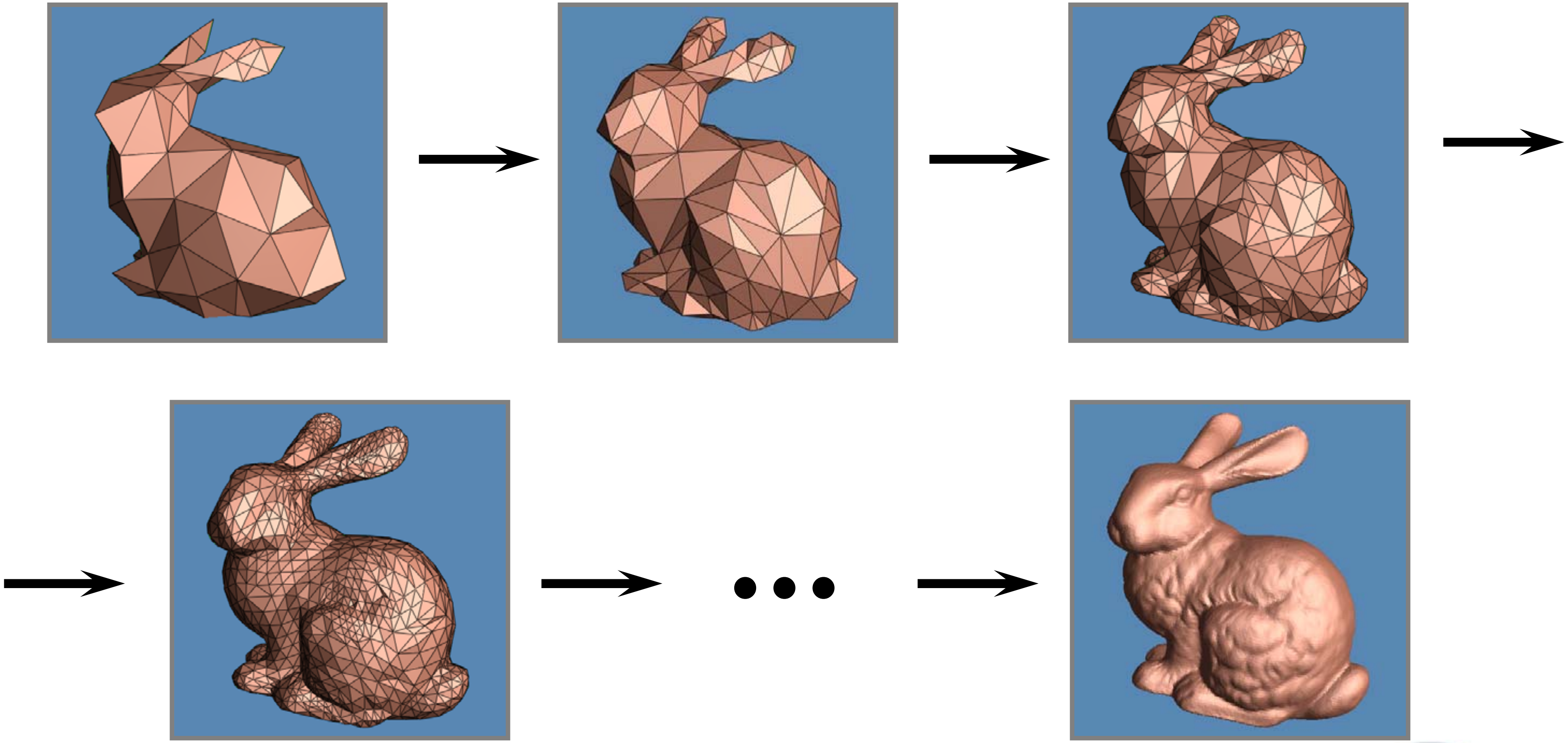
Multiresolution modeling: Example 3

- The finer the level of resolution, the more local the change.



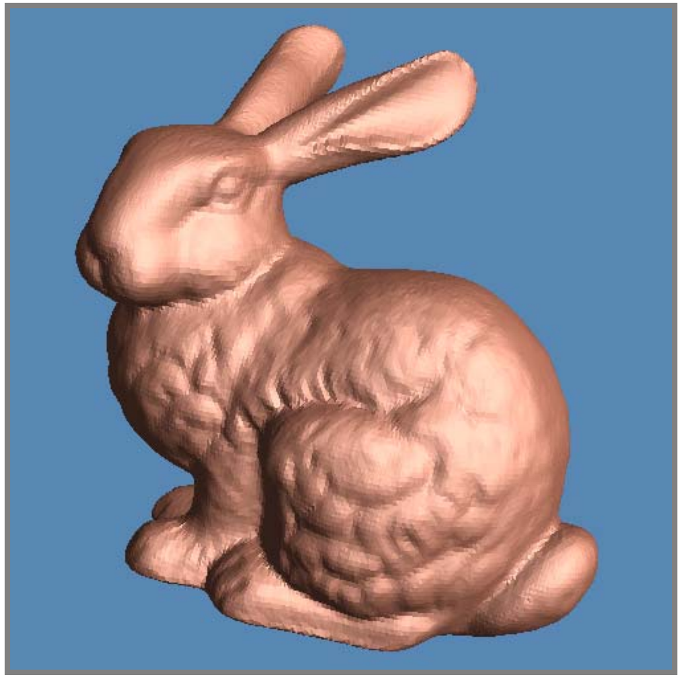
Applications

Hierarchy:



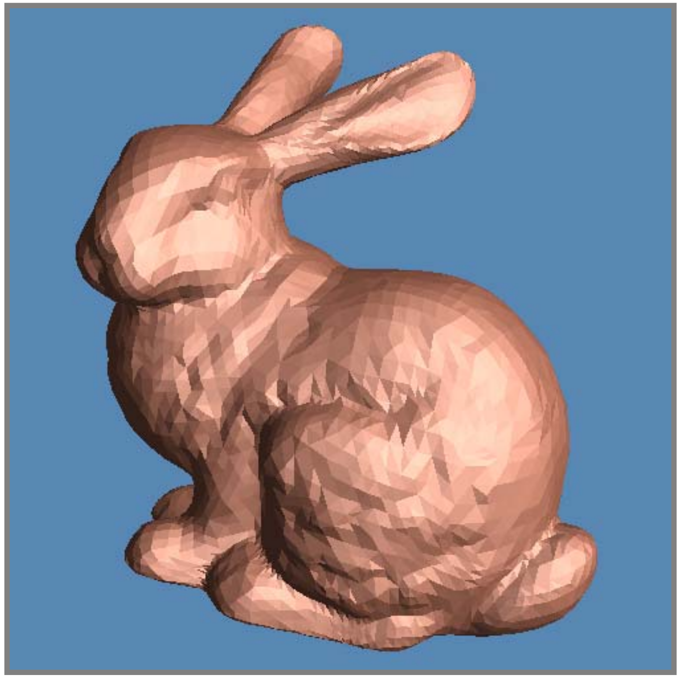
Compression

Reduction of storage requirements with error bounds:



$\varepsilon < 0.8\%$

$\longrightarrow$

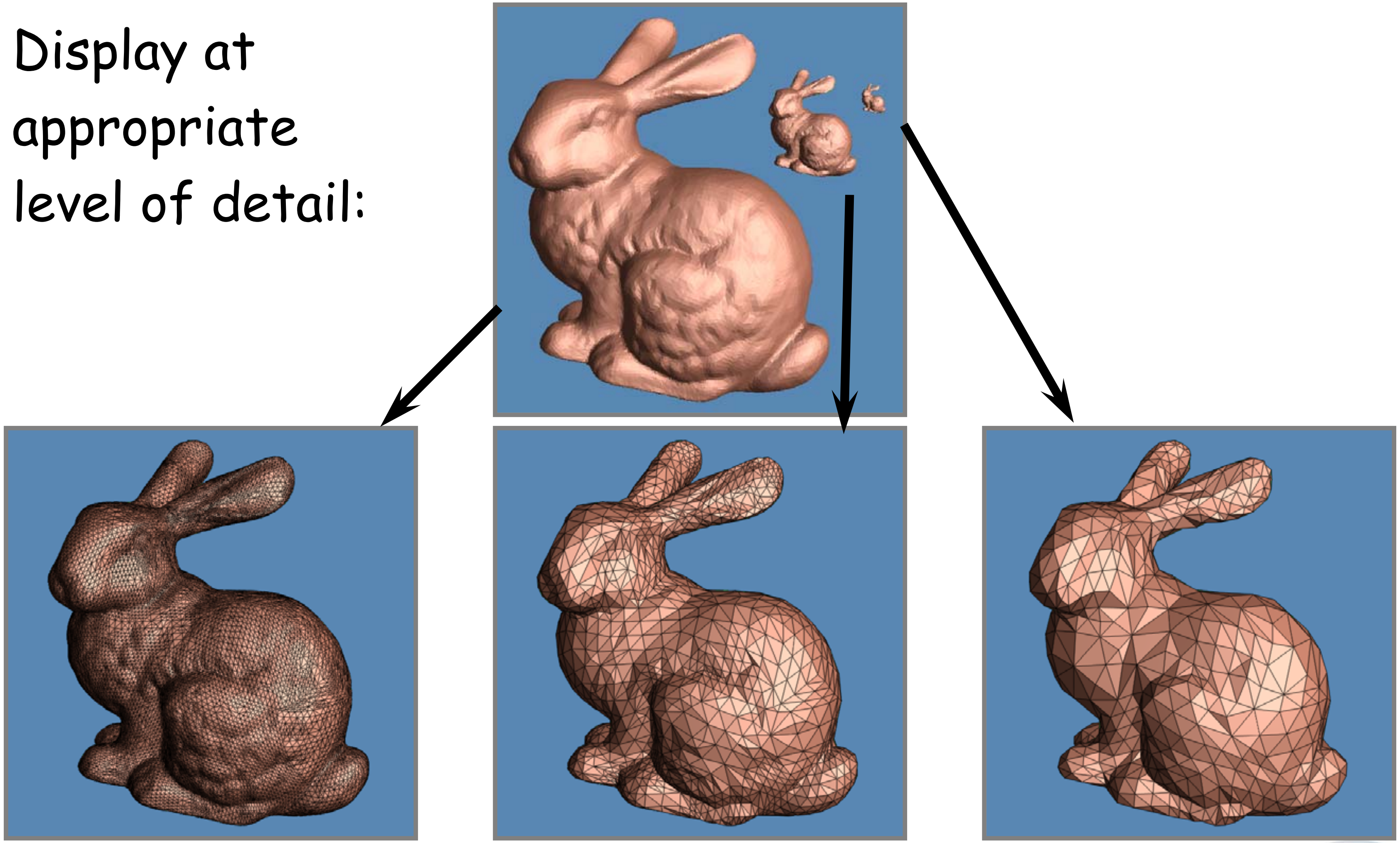


~70,000 faces

~11,000 faces

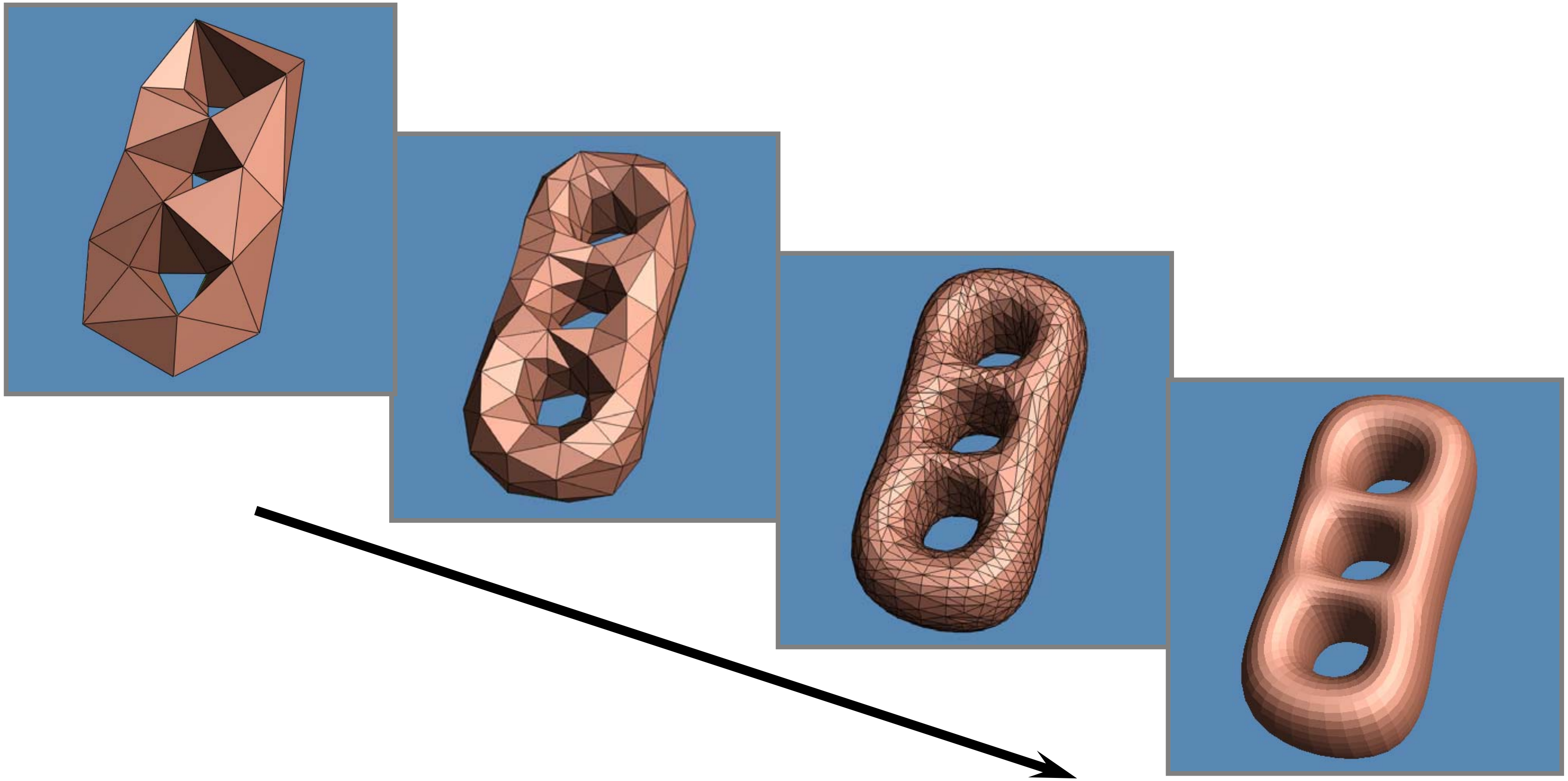
Level of detail control

Display at appropriate level of detail:



Progressive transmission and rendering

Progressive refinement of mesh over time:



Multiresolution editing

Shape editing at different LODs: