Random Coefficients Logit Demand Models

Instrumental Variables, GMM, Contraction Mapping, and Structural Estimation

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Readings

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- None of these are required for the course, but I recommend taking a look afterwards.

Roadmap

Model Overview

- BLP is a full equilibrium model for both demand *and* supply of discrete goods
  - → We'll focus on demand side , but recall that prices are endogenous.
  - → You will see the full model in Micro B with Anders
    (Bertrand price competition of multi-product producers).

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- Each market has individuals with types denoted by $i \in \mathcal{I}_t$.
  - $\rightarrow$ Different demographics and preferences.
- Individuals are faced with choices denoted by $j \in \mathcal{J}_t$.
  - $\rightarrow$ Products, hospitals, candidates, etc.
  - $\rightarrow$ Outside option $j = 0$: no purchase, no treatment, no vote, etc.

Utility Maximization

$$\max_{j \in \mathcal{J}_t \cup \{0\}} u_{ijt}$$

- Individuals choose an alternative to maximize (indirect) utility u_{ijt}.
  - $\rightarrow$ We will specify a function for u_{ijt} and use revealed preference to estimate it.

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- We will parameterize δ_{jt} and μ_{ijt} and make a convenient assumption about ε_{ijt}.

Aggregate Market Shares

$$\max_{j \in \mathcal{J}_t \cup \{0\}} u_{ijt} = \delta_{jt} + \mu_{ijt} + \varepsilon_{ijt}$$

- Assume a convenient distribution for ε_{ijt}: iid type I extreme value.

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$$\max_{j \in \mathcal{J}_t \cup \{0\}} u_{ijt} = \delta_{jt} + \mu_{ijt} + \varepsilon_{ijt} \quad \implies \quad s_{ijt} = \mathbb{P}_{\varepsilon_{it}}\Big(u_{ijt} \geq u_{ikt} \text{ for all } k \in \mathcal{J}_t \cup \{0\}\Big)$$

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  Assume $\mu_{ijt} \sim F(\mu_{ijt}|\Sigma, \Pi)$ and aggregate over μ_{ijt}

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- We'll match these to observed quantities $\hat{s}_{jt}$ in our data.

Roadmap

Pure Logit Model

$$\max_{j \in \mathcal{J}_t \cup \{0\}} u_{ijt} = \delta_{jt} + \cancel{\mu_{ijt}}^{\,0} + \varepsilon_{ijt}$$

- Start with the simplest case: no heterogenous utility. This is the model we worked with in Econometrics A. We will put μ_{ijt} back soon.

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- Market shares simplify. No aggregation over individual types.
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- We can recover mean utilities from observed market shares (Berry, 1994).
  - $\rightarrow$ If we specify a function for δ_{jt}, we'll have a linear regression!

Pure Logit Estimating Equation

$$\log \frac{s_{jt}}{s_{0t}} = \delta_{jt}$$

- Running example: Grieco et al (2024)?
  - → In Grieco et al (2024), products j are combinations of makes and models of cars; markets t are simply time.
  - → If we estimate the model, we can change p_{jt} and estimate how consumers react.

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  - $\rightarrow$ If we estimate the model, we can change p_{jt} and estimate how consumers react.
- Specify δ_{jt} as a function of price p_{jt} and other product characteristics x_{jt}.
  - $\rightarrow$ So p_{jt} is price of a car; x_{jt} includes a constant, a “horse power”, ”fuel efficiency”, make dummies, etc.

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  - → So p_{jt} is price of a car; x_{jt} includes a constant, a “horse power”, ”fuel efficiency”, make dummies, etc.
- Interpret the regression error ξ_{jt} as unobserved product quality not in our data.
  - → Unobserved characteristics, advertising, average taste variation, “demand shocks,” etc.

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Endogeneity Concerns

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- Typically, we expect price to be strongly correlated with unobserved quality, ξ_{jt}
  - $\rightarrow$ Firms know more than us about demand when setting prices.
  - $\rightarrow$ Supply side lead to $\mathbb{C}(p_{jt}, \xi_{jt}) > 0$, so $\hat{\alpha} < 0$ is biased towards zero.

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- Potential solutions IV or Fixed Effects

Fixed Effects

$$\delta_{jt} = \alpha p_{jt} + x'_{jt}\beta + \xi_{jt}$$

- Adding product and market fixed effects to x_{jt} can eliminate a lot of bias.
  - $\rightarrow$ E.g. if p_{jt} is correlated with fixed effects ξ_j and/or ξ_t in $\xi_{jt} = \xi_j + \xi_t + \Delta\xi_{jt}$.
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- Modern grocery scanner datasets have many thousands of products/markets.
  - $\rightarrow$ Dummies take too much memory, so we "absorb" them, i.e. de-mean using within transformation.
- Helpful but insufficient: ξ_{jt} typically varies by product *and* market, e.g. $\mathbb{C}(p_{jt}, \Delta\xi_{jt}) > 0$.

Instrumental Variables

$$\delta_{jt} = \alpha p_{jt} + x'_{jt}\beta + \xi_{jt}$$

- With or without fixed effects, a carefully-chosen IV can be a good solution.
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  - $\rightarrow$ Is the instrument strong, or should you worry about weak instruments?

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- Many places to look. I'll discuss the most common ones.

Typical Instruments for Price

$$\delta_{jt} = \alpha p_{jt} + x_{jt}'\beta + \xi_{jt}$$

- Typically, prices are marginal costs plus a markup term.
  - $\rightarrow$ We want valid instruments that shift costs and/or markups.

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- Cost-shifters: Measures of input prices, tariffs, etc.
  $\rightarrow$ Consumers should only care about them through their effect on prices.

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- Hausman: Current price of the same product averaged across *other* locations.
  $\rightarrow$ Need costs to be correlated across locations, but not unobserved quality.

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- BLP: Average characteristics x_{kt} of *competing* products $k \neq j$.
  - $\rightarrow$ Characteristics of competing products affect markups.
  - $\rightarrow$ We'll come back to these later, since they can also serve a different purpose.
- I recommend starting with just one. A straightforward cost-shifter if you have it.

Empirical Example: Demand for cars (Grieco et al. QJE, 2024)

- Assume $\mu_{ijt} = 0$ and estimate pure logit demand model for cars with OLS and IV:

$$\log \frac{s_{jt}}{s_{0t}} = \delta_{jt} = \alpha p_{jt} + x_{jt}'\beta + \xi_{jt}$$

- x_{jt}: Observed product characteristics
  - $\rightarrow$ Variables in logs (height, hp, mpg, weight, footprint, number of trims)
  - $\rightarrow$ Variables in levels (release year, years since design)

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- Regression specifications:
  - $\rightarrow$ Baseline: $\delta_{jt} = \alpha p_{jt} + \text{const} + \xi_{jt}$
  - $\rightarrow$ Controls: Adding x_{jt}, car type, year, and make fixed effects (dummies)

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- Endogeneity problem:
  - $\rightarrow$ Price p_{jt} may be correlated with ξ_{jt}, biasing OLS estimates.
  - $\rightarrow$ Use IV regression with exchange rate instrument (RXR) for price.

Let's Code!

- **Jupyter Notebook:** `15_blp.ipynb`
- **Part 1:** IV estimation with $\mu_{ijt} = 0$ and real exchange rate as instrument for price

Roadmap

Random utility model for panel data

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- Extreme value shocks ε_{ijt} give logit market shares (conditional on individual types) $\rightarrow$ Mixed Logit aggregate market shares

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- Extreme value shocks ε_{ijt} give logit market shares (conditional on individual types)
  $\rightarrow$ Mixed Logit aggregate market shares
- Before we set $\mu_{ijt} = 0$ to get a conveniently linear estimating equation:

$$\log \frac{s_{jt}}{s_{0t}} = \delta_{jt} = \alpha p_{jt} + x'_{jt} \beta + \xi_{jt}$$

but now, δ_{jt} is implicitly given by s_{jt} as a fixed point on a non-linear equation.

Random Coefficients

$$u_{ijt} = x'_{jt}\beta + \xi_{jt} + \varepsilon_{ijt}$$

- How to add preference heterogeneity to our pure logit model?
  - $\rightarrow$ For simplicity, I'll just let x_{jt} denote all characteristics, including prices p_{jt}.

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- Intuitively, we want to replace β with *random coefficients* β_{it}.
  - $\rightarrow$ *Random* in that they're drawn from a distribution of consumer types $i \in \mathcal{I}_t$.
  - $\rightarrow$ For $x_{jt} = \text{car}_{jt}$ and $\mathcal{I}_t = \{\text{car-lovers}, \text{bus-lovers}\}$, want $\beta_{it} \gg 0$ for car-lovers.

Random Coefficients

$$u_{ijt} = x'_{jt} \underbrace{(\beta + \Pi y_{it} + \Sigma \nu_{it})}_{\beta_{it}} + \xi_{jt} + \varepsilon_{ijt}$$

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  - → For $x_{jt} = \text{car}_{jt}$ and $\mathcal{I}_t = \{\text{car-lovers, bus-lovers}\}$, want $\beta_{it} \gg 0$ for car-lovers.
- Most common specification is $\beta_{it} \sim N(\beta + \Pi y_{it}, \Sigma\Sigma')$.
  - → Π shifts preferences according to "observed" demographics $y_{it} \sim$ census.
  - → Σ shifts preferences according to "unobserved" preferences $\nu_{it} \sim N(0, I)$.
  - → Σ is the *Cholesky root* of the variance matrix. Usually diagonal with standard deviations.

Random Coefficients

$$u_{ijt} = \underbrace{x_{jt}'\beta + \xi_{jt}}_{\delta_{jt}} + \underbrace{x_{jt}'(\Sigma\nu_{it} + \Pi y_{it})}_{\mu_{ijt}} + \varepsilon_{ijt}$$

- How to add preference heterogeneity to our pure logit model?
  - $\rightarrow$ For simplicity, I'll just let x_{jt} denote all characteristics, including prices p_{jt}.
- Intuitively, we want to replace β with *random coefficients* β_{it}.
  - $\rightarrow$ *Random* in that they're drawn from a distribution of consumer types $i \in \mathcal{I}_t$.
  - $\rightarrow$ For $x_{jt} = \text{car}_{jt}$ and $\mathcal{I}_t = \{\text{car-lovers}, \text{bus-lovers}\}$, want $\beta_{it} \gg 0$ for car-lovers.
- Most common specification is $\beta_{it} \sim N(\beta + \Pi y_{it}, \Sigma\Sigma')$.
  - $\rightarrow$ Π shifts preferences according to "observed" demographics $y_{it} \sim$ census.
  - $\rightarrow$ Σ shifts preferences according to "unobserved" preferences $\nu_{it} \sim N(0, I)$.
  - $\rightarrow$ Σ is the *Cholesky root* of the variance matrix. Usually diagonal with standard deviations.

Roadmap

From Linear Regression to GMM

$$\log \frac{s_{jt}}{s_{0t}} = \delta_{jt} = x'_{jt}\beta + \xi_{jt}$$

- For $\mu_{ijt} = 0$ it is easy to estimate β by running the above regression.
  - $\rightarrow$ Again, let x_{jt} include price, a constant, any other characteristics.
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- Our exclusion restriction implies the moment condition $\mathbb{E}[\xi_{jt} \cdot z_{jt}] = 0$.
- We'd get the exact same $\hat{\beta}$ by optimizing the following GMM objective:

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}}\, g(\beta) W g(\beta)' \quad \text{where} \quad g(\beta) = \frac{1}{N} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}_t} (\delta_{jt} - x'_{jt}\beta) \cdot z_{jt}$$

Aggregate market shares

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- This effectively makes $\delta_{jt} = \delta_{jt}(\Sigma, \Pi)$ an *implicit function* of Σ and Π
- Need to approximate integral (Monte Carlo or Qadrature) and solve for δ_{jt}
- BLP's (1995) big advancement was how to incorporate flexible preference heterogeneity.
  - $\rightarrow$ Built on simulation estimator advancements (Pakes and Pollard, 1989; McFadden, 1989).

The BLP Contraction

- Given an estimate $\hat{s}_{jt}$ of s_{jt} and a guess of (Σ, Π) we could use Newton's Method to numerically solve the system of equations

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- Alternatively we can use the method of successive approximations to solve for the fixed point on the BLP contraction mapping, $\Gamma_{(\Sigma,\Pi)}$:

$$\delta_{jt}^{k+1} = \Gamma_{(\Sigma,\Pi)}(\delta_{jt}^{k}) = \delta_{jt}^{k} + \log(\hat{s}_{jt}) - \log(s_{jt}(\Sigma, \Pi)) \quad \text{for all} \quad j \in \mathcal{J}_t \text{ and } t \in \mathcal{T}$$

- Since $\Gamma_{(\Sigma,\Pi)}$ is a contraction, successive approximation will always find the *implicit function* $\delta_{jt}(\Sigma, \Pi)$ as the unique fixed point on the BLP contraction operator $\Gamma_{(\Sigma,\Pi)}$, i.e. where $\delta_{jt} = \Gamma_{(\Sigma,\Pi)}(\delta_{jt})$

Let's Code!

We start with the simple case where $\mu_{ijt} = 0$ (closed form for s_{jt} and δ_{jt})

- **Jupyter Notebook:** `15_blp.ipynb`
- **Part 2:** Solve for δ using the BLP contraction mapping and compare to $\log(\hat{s}_{jt}/\hat{s}_{0t})$
- **Part 3:** Estimate β using GMM with δ from Contraction Mapping and compare to IV

Approximating the Integral using Monte Carlo

$$s_{jt} = \int \frac{\exp(\delta_{jt} + \mu_{ijt})}{1 + \sum_{k \in \mathcal{J}_t} \exp(\delta_{kt} + \mu_{ikt})} \, \mathrm{d}F(\mu_{ijt}|\Sigma, \Pi)$$

1. Draw individual preferences from $F(\mu_{ijt}|\Sigma, \Pi)$:

$$\mu_{ijt}^{(r)} = x'_{jt}(\Sigma \nu_{it}^{(r)} + \Pi y_{it}^{(r)}), \quad r = 1, \ldots, R$$

   where $\nu_{it}^{(r)} \sim N(0, I)$ and $y_{it}^{(r)}$ comes from census data.

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- Better alternatives: Quasi-Monte Carlo (QMC) methods like **Sobol** or **Halton** sequences.
  - $\rightarrow$ Converge faster than standard random sampling.
  - $\rightarrow$ Reduce variance of integral approximation.

Let's Code!

We assume $\Pi = 0$, but $\Sigma \neq 0$ and thus have $\mu_{ijt} \neq 0$
(no closed form for s_{jt} and δ_{jt})

- **Jupyter Notebook:** `15_blp.ipynb`
- **Part 4:** Solve for δ using the BLP contraction given Σ

The BLP Estimator

$$\hat{\theta} = \operatorname*{argmin}_{\theta} g(\theta) W g(\theta)' \quad \text{where} \quad g(\theta) = \frac{1}{N} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}_t} (\delta_{jt}(\Sigma, \Pi) - x'_{jt}\beta) \cdot z_{jt}$$

- BLP estimation consists of two nested loops.
  1. In the "outer" loop, we optimize over $\theta = (\beta, \Sigma, \Pi)$.
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  - $\rightarrow$ Get $\hat{\beta}$ by running an IV regression of $\delta_{jt}(\Sigma, \Pi)$ on x_{jt}, like in the pure logit exercise.

- What about the GMM weighting matrix W?
  - $\rightarrow$ If you're just-identified ($\dim z_{jt} = \dim \theta$), it doesn't matter. You'll get a zero objective.
  - $\rightarrow$ Otherwise, you may want to repeat optimization with optimal the two-step GMM $\hat{W}$.

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$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}}\, Q(\theta)$$

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- Set box constraints $\theta \in [\underline{\theta}, \overline{\theta}]$ to preclude unrealistic and unstable guesses of θ.
  - $\rightarrow$ E.g. huge Σ values can make the inner loop unstable.
  - $\rightarrow$ Economic intuition and initial estimates will give a sense for reasonable bounds.

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- Check that 3-5 different starting values $\theta \sim U(\underline{\theta}, \overline{\theta})$ give the same $\hat{\theta}$.
  - $\rightarrow$ For 2-step GMM, do this twice, once for each step (6-10 jobs total).
  - $\rightarrow$ If you have access to a cluster, each can be a separate job, run in parallel.

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- Set box constraints $\theta \in [\underline{\theta}, \overline{\theta}]$ to preclude unrealistic and unstable guesses of θ.
- Check that 3-5 different starting values $\theta \sim U(\underline{\theta}, \overline{\theta})$ give the same $\hat{\theta}$.
- Prefer using gradient-based algorithms for "smooth" problems like BLP.
  - $\rightarrow$ Avoid derivative-free methods like Nelder-Mead/simplex, which tend to work worse.
  - $\rightarrow$ I prefer trust-region algorithms, e.g. SciPy's `trust-constr`

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- Prefer using gradient-based algorithms for "smooth" problems like BLP.
- Try to terminate on strict first-order conditions, e.g. $\|\text{gradient}\|_\infty <$ `1e-8`.
  - $\rightarrow$ Inner loop should be tighter to prevent error "bubbling up."
  - $\rightarrow$ Can also check second-order conditions, i.e. Hessian eigenvalues are positive.

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- Configure your optimizer! Defaults may not work for your setting.

Numerical Integration

$$s_{jt} = \int \frac{\exp(\delta_{jt} + \mu_{ijt})}{1 + \sum_{k \in \mathcal{J}_t} \exp(\delta_{kt} + \mu_{ikt})} \, \mathrm{d}F(\mu_{ijt})$$

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- Sometimes there are only a few types that we can integrate exactly.
  - $\rightarrow$ E.g. high- and low-income types $i \in \{1, 2\}$ with known shares w_{1t} and $w_{2t} = 1 - w_{1t}$.

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- But usually we approximate the distribution with Monte Carlo integration.
  - → Use a random number generator (RNG) to draw $M \approx$ 1,000 of (ν_{it}, y_{it})'s per market.
  - → Even better than your default RNG are quasi-Monte Carlo sequences.
  - → I recommend scrambled Sobol sequences. Python: Chaospy.

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- If you just need a few $\nu_{it} \sim N(0, I)$'s, try out Gauss-Hermite quadrature.
  - $\rightarrow$ 10-100$\times$ fewer carefully-chosen (w_{it}, ν_{it})'s that do just as well as Monte Carlo.
  - $\rightarrow$ Chosen to exactly integrate a polynomial expansion of the integrand.

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- Keep increasing M until your estimates stabilize across draws/starting values.

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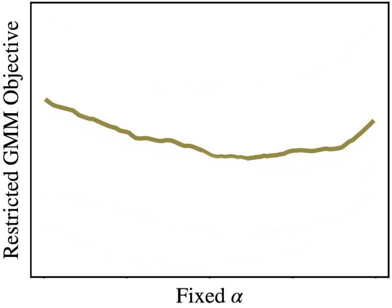
- Can see how $Q(\theta) = g(\theta)W g(\theta)'$ varies with θ.



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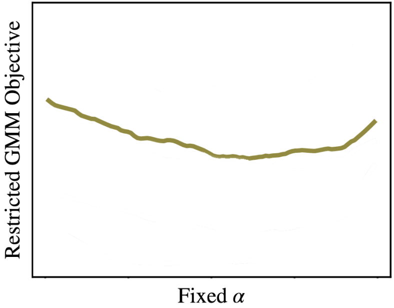

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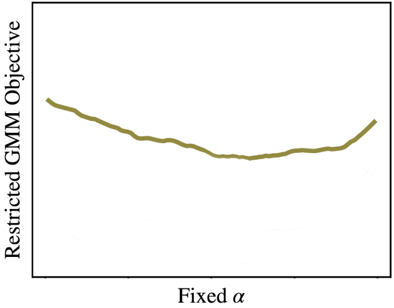

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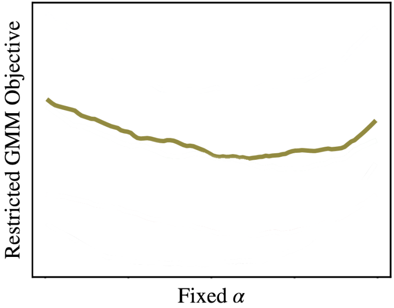

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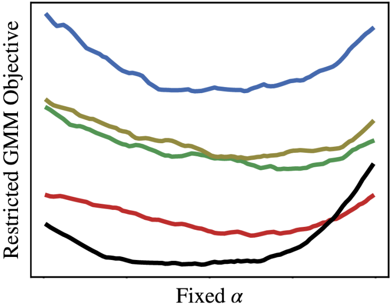

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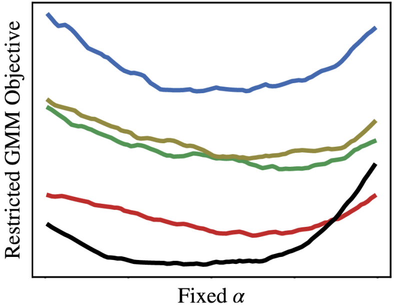
- Different instruments give different objectives.
  - $\rightarrow$ Even if they're all valid, some may be weaker.
  - $\rightarrow$ Weaker means flatter and harder to optimize.



Roadmap

Adding Instruments

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- Later, adding more can help with weakness and testing exclusion restrictions.

Linear Regression Approximation

- There's a lot of confusion about what instruments are needed for BLP estimation.
  - $\rightarrow$ Identification of nonlinear models like BLP can be challenging.
  - $\rightarrow$ See Berry and Haile (2014, 2023) for a more formal, nonparametric framework.

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- Salanié and Wolak (2022) approximate the BLP model around $\sigma, \pi \approx 0$:

$$\log \frac{s_{jt}}{s_{0t}} \approx \beta x_{jt} + \underbrace{\sigma^2 d^x_{jt} + \pi m^y_t x_{jt} + \pi^2 v^y_t d^x_{jt}}_{\text{Defined on the next slide.}} + \xi_{jt}$$

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- Let's use our stronger intuition about linear regression to think about instruments!

Linear Regression Intuition

$$\log \frac{s_{jt}}{s_{0t}} = \beta x_{jt} + \xi_{jt}$$

- If we set $\sigma = \pi = 0$ like on day 1, we get our familiar pure logit regression.

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- If we set $\sigma = \pi = 0$ like on day 1, we get our familiar pure logit regression.
  - $\rightarrow$ Use the same IV as before to target β: if $x_{jt} = p_{jt}$, a price IV; if exogenous, x_{jt} itself.

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$$\log \frac{s_{jt}}{s_{0t}} \approx \beta x_{jt} + \sigma^2 d_{jt}^x + \xi_{jt} \quad \text{where} \quad d_{jt}^x = \left(\frac{x_{jt}}{2} - \sum_{k \in \mathcal{J}_t} s_{kt} x_{kt}\right) x_{jt}$$

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- To target $\sigma \neq 0$, need a measure of how "differentiated" j is in terms of x_{jt} within t.
  - → Can't use d_{jt}^x itself because it depends on endogenous market shares s_{kt}.
  - → Conventional choice was $\sum_{k \neq j} x_{kt}$, the BLP instruments from day 1.

Linear Regression Intuition

$$\log \frac{s_{jt}}{s_{0t}} \approx \beta x_{jt} + \sigma^2 d_{jt}^x + \xi_{jt} \quad \text{where} \quad d_{jt}^x = \Big(\frac{x_{jt}}{2} - \sum_{k \in \mathcal{J}_t} s_{kt} x_{kt}\Big) x_{jt}$$

- If we set $\sigma = \pi = 0$ like on day 1, we get our familiar pure logit regression.
- To target $\sigma \neq 0$, need a measure of how “differentiated” j is in terms of x_{jt} within t.
  - → Can't use d_{jt}^x itself because it depends on endogenous market shares s_{kt}.
  - → Conventional choice was $\sum_{k \neq j} x_{kt}$, the BLP instruments from day 1.
  - → A stronger choice is $\sum_{k \neq j} (x_{jt} - x_{kt})^2$ or similar from Gandhi and Houde (2020).

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Linear Regression Intuition

$$\log \frac{s_{jt}}{s_{0t}} \approx \beta x_{jt} + \sigma^2 d_{jt}^x + \pi m_t^y x_{jt} + \pi^2 v_t^y d_{jt}^x + \xi_{jt} \quad \text{where} \quad d_{jt}^x = \Big(\frac{x_{jt}}{2} - \sum_{k \in \mathcal{J}_t} s_{kt} x_{kt}\Big) x_{jt}$$

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  - → We want cross-market demographic variation, otherwise $m_t^y x_{jt}$ is collinear with x_{jt}.
  - → Can technically identify π from higher-order variation, e.g. in variance v_t^y.

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- To target $\pi \neq 0$, we can interact x_{jt} with mean within-market income m_t^y.
- In your exercise, you'll target (β, σ, π) with $z_{jt} = (x_{jt}, \sum_{k \neq j} (x_{jt} - x_{kt})^2, m_t^y x_{jt})$.
  - $\rightarrow$ If $x_{jt} = p_{jt}$, can replace x_{jt} with fitted values $\hat{p}_{jt}$ from the price IV's first stage.

Optimal Instruments

- There are many valid instruments that satisfy exclusion restrictions $\mathbb{E}[\xi_{jt} \mid z_{jt}] = 0$.
  - $\rightarrow$ E.g. z_{jt} itself, z_{jt}^2, z_{jt}^3, or any function $f(z_{jt})$ of z_{jt}.

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- But adding a ton of instruments will bias your estimator.
  - $\rightarrow$ "Many weak IVs" problem is well-known for 2SLS (Angrist, Imbens and Krueger, 1999).
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- Can be a bit tricky to compute, but with PyBLP it's just one line of code.
  - $\rightarrow$ In practice, can update your IVs along with your weighting matrix for a second GMM step.

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