

# Solving and estimating directional dynamic games and Multiplicity of equilibria

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# ROAD MAP for the lecture

1. Bertrand pricing and investment game (**Leapfrogging game**)
  - ▶ Collusion of Australian corrugated fibre packaging (CFP) producers
2. Directional dynamic games (DDGs)
  - ▶ State recursion algorithm
  - ▶ Recursive lexicographical search (RLS) algorithm
  - ▶ Full solution for the leapfrogging game
3. Structural estimation of directional dynamic games with **Nested RLS algorithm**

# Solving and estimating directional dynamic games: Bertrand pricing and investment (leapfrogging) game



## Amcor-Visy collusion case



- ▶ Australian market for **cardboard** is essentially a **duopoly**
- ▶ Between 2000 and 2005 *Visy* and *Amcor* **colluded** to divide the market of cardboard and to fix prices
- ▶ 2007: **Visy admits** to have been manipulating the market, issued with \$36 million fine
- ▶ July 2009: Cadbury vs. Amcor, **damages estimated at \$235.8 million**, settles out of court
- ▶ March 2011: **Class action suit** against both Amcor and Visy settles out of court for \$95 million

## Cardboard industry in Australia

Bertrand price competition in the short run, with leapfrogging investments by both firms in the longer run

- ▶ Cardboard is a highly standardized product
- ▶ Strong incentives for Bertrand-like price cutting
- ▶ Amcor and Visy do minimal amounts of R&D themselves,
- ▶ Spend considerable amounts on cost reducing investments
  - ▶ Amcor plans to build state-of-the-art paper mill in Botany Bay before the collusion took place
  - ▶ “B9” plant finally opened on February 1, 2013
- ▶ Amcor and Visy purchase new technology from other companies that specialize in doing the R&D and reduce cost of production of cardboard

# Leapfrogging

## Leapfrogging equilibrium

- ▶ Firms invest in alternating fashion and take turns in cost leadership
- ▶ Market price makes permanent downward shifts

## "The Bertrand Investment Paradox"

- ▶ Should Bertrand competitors undertake cost-reducing investments?
- ▶ If both firms acquire state-of-art technology simultaneously, the following Bertrand price competition leads to zero profits for each firm
- ▶ Since both firms have access to cost reducing technology, does either of them have any incentive to invest ex ante?

# Dynamic Bertrand price competition

## Stochastic dynamic game

- ▶ Two Bertrand competitors,  $n = 2$ , no entry or exit
- ▶ Discrete time, infinite horizon ( $t = 1, 2, \dots, \infty$ )
- ▶ Each firm maximizes expected discounted profits, common discount factor  $\beta \in (0, 1)$
- ▶ Each firm has two choices in each period:
  1. Price for the product
  2. Whether or not to buy the state of the art technology

## Static Bertrand price competition in each period

- ▶ Continuum of consumers make static purchase decision
- ▶ No switching costs: buy from the lower price supplier

# Cost-reducing investments

## State-of-the-art production cost $c$ process

- ▶ Initial value  $c_0$ , lowest value 0:  $0 \leq c \leq c_0$
- ▶ Discretized with  $n$  points
- ▶ Follows exogenous Markov process and only improves
- ▶ Markov transition probability  $\pi(c_{t+1}|c_t)$   
 $\pi(c_{t+1}|c_t) = 0$  if  $c_{t+1} > c_t$

## Investment choice: binary

- ▶ Investment cost of  $K(c)$  to obtain marginal cost  $c$
- ▶ One period construction time: production with technology obtained at  $t$  starts at  $t + 1$

# State space and information structure

## Common knowledge

- ▶ State of the game: cost structure  $(c_1, c_2, c)$
- ▶ State space is  $S = (c_1, c_2, c) \subset R^3: c_1 \geq c, c_2 \geq c$
- ▶ Actions are observable

## Private information

- ▶ In each period each firm incurs additive costs (benefits) from not investing and investing  $\eta \epsilon_{i,I}$  and  $\eta \epsilon_{i,N}$
- ▶  $\epsilon_{i,I}$  and  $\epsilon_{i,N}$  are extreme value distributed, independent across choice, time and firms
- ▶  $\eta \geq 0$  is a scaling parameter
- ▶ Investment choice probabilities have logit form for  $\eta > 0$

# Timing of moves

Pricing decisions are made simultaneously

Expected one period profit of firm  $i$  from Bertrand game ( $j \neq i$ )

$$r_i(c_1, c_2) = \begin{cases} 0 & \text{if } c_i \geq c_j \\ c_j - c_i & \text{if } c_i < c_j \end{cases}$$

Two versions regarding investment decisions

1. Simultaneous moves:

- ▶ Investment decisions are made simultaneously

2. Alternating moves:

- ▶ The “right to move” state variable  $m \in \{1, 2\}$ ,
- ▶ When  $m = i$ , only firm  $i$  can make a cost reducing investment
- ▶  $m$  follows an own Markov process  
(deterministic alternation as a special case).

# Actions and behavior strategies

## Two choices in each period

- ▶  $p_i(c_1, c_2, c) = \max(c_1, c_2)$  – Bertrand pricing decision
  - ▶  $P_i^I(c_1, c_2, c)$  – probability of firm  $i$  to invest in state-of-the-art production technology
- $$P_i^N(c_1, c_2, c) = 1 - P_i^I(c_1, c_2, c) \text{ – probability not to invest}$$

## Strategy profile

- ▶  $\sigma = (\sigma_1, \sigma_2)$  – pair of Markovian *behavior* strategies
- $$\sigma_i = \left( p_i(c_1, c_2, c), P_i^I(c_1, c_2, c) \right) \in \mathbb{R}_+ \times [0, 1]$$
- ▶ Pure strategies included as special case

# Definition of Markov Perfect Equilibrium

## Definition (Markov perfect equilibrium (MPE))

MPE of Bertrand investment stochastic game is a pair of

- ▶ strategy profile  $\sigma^* = (\sigma_1^*, \sigma_2^*)$ , and
- ▶ pair of *value functions*  $V(s) = (V_1(s), V_2(s))$ ,  $V_i : S \rightarrow R$ ,

such that

1. Bellman equations (below) are satisfied for each firm, and
2. strategies  $\sigma_1^*$  and  $\sigma_2^*$  constitute mutual best responses, and assign positive probabilities only to the actions in the set of maximizers of the Bellman equations.

## Bellman equations, firm $i = 1$ , simultaneous moves

$$V_i(c_1, c_2, c) = \max [v_i^I(c_1, c_2, c) + \eta \epsilon_{i,I}, v_i^N(c_1, c_2, c) + \eta \epsilon_{i,N}]$$

$$v_i^N(c_1, c_2, c) = r^i(c_1, c_2) + \beta EV_i(c_1, c_2, c|N)$$

$$v_i^I(c_1, c_2, c) = r^i(c_1, c_2) - K(c) + \beta EV_i(c_1, c_2, c|I)$$

With extreme value shocks, the investment probability is

$$P_i^I(c_1, c_2, c) = \frac{\exp\{v_i^I(c_1, c_2, c)/\eta\}}{\exp\{v_i^I(c_1, c_2, c)/\eta\} + \exp\{v_i^N(c_1, c_2, c)/\eta\}}$$

## Bellman equations, firm $i = 1$ , simultaneous moves

The expected values are given by

$$\begin{aligned}EV_i(c_1, c_2, c | \textcolor{blue}{N}) &= \int_0^c [P_j^I(c_1, c_2, c) H_i(\textcolor{blue}{c}_1, \textcolor{red}{c}, c') + \\&\quad P_j^N(c_1, c_2, c) H_i(\textcolor{blue}{c}_1, \textcolor{red}{c}_2, c')] \pi(dc' | c) \\EV_i(c_1, c_2, c | \textcolor{blue}{I}) &= \int_0^c [P_j^I(c_1, c_2, c) H_i(\textcolor{blue}{c}, \textcolor{red}{c}, c') + \\&\quad P_j^N(c_1, c_2, c) H_i(\textcolor{blue}{c}, \textcolor{red}{c}_2, c')] \pi(dc' | c)\end{aligned}$$

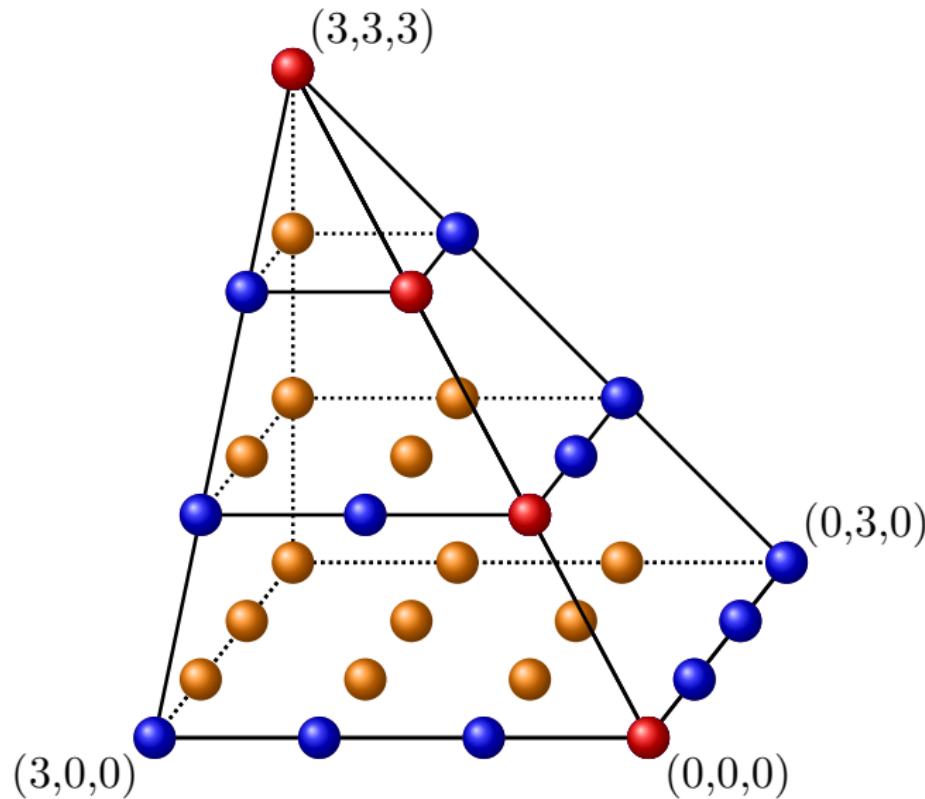
where

$$H_i(\textcolor{red}{c}_1, \textcolor{red}{c}_2, \textcolor{blue}{c}) = \eta \log [\exp(v_i^N(c_1, c_2, c)/\eta) + \exp(v_i^I(c_1, c_2, c)/\eta)]$$

is the “smoothed max” or logsum function

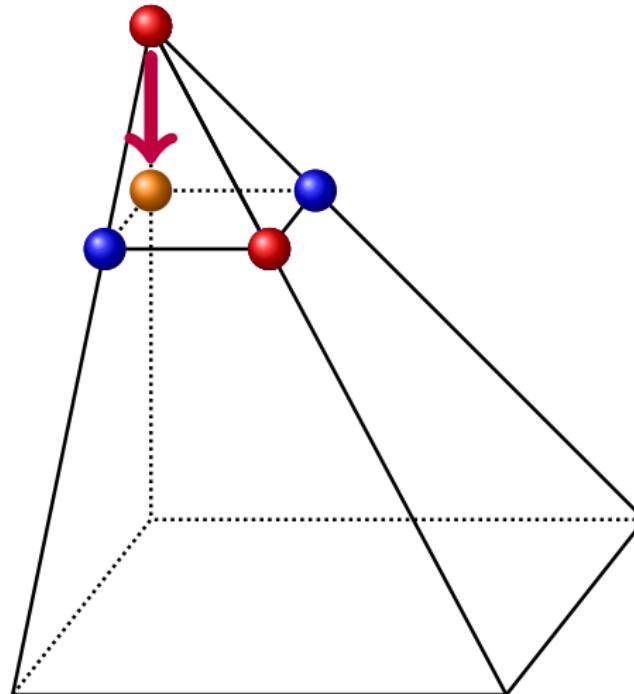
Discretized state space = a “quarter pyramid”

$$S = \{(c_1, c_2, c) | c_1 \geq c, c_2 \geq c, c \in [0, 3]\}, n = 4$$



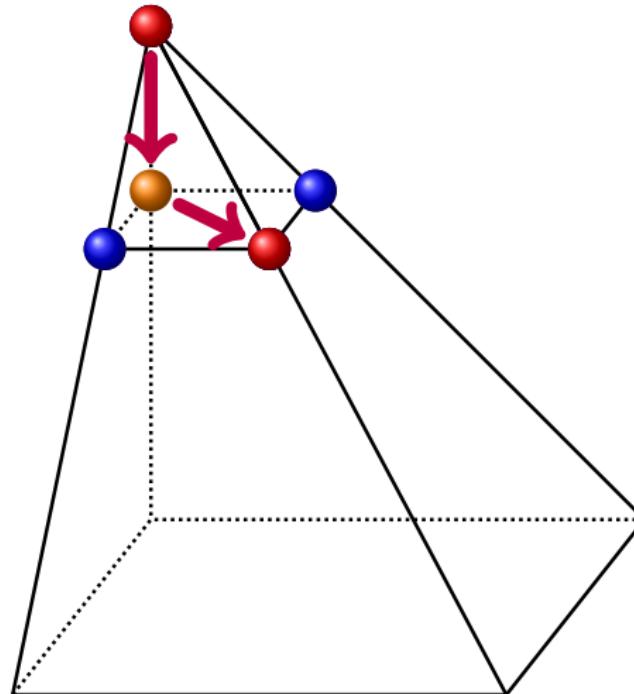
## Game dynamics: example

The game starts at the apex, as some point technology improves



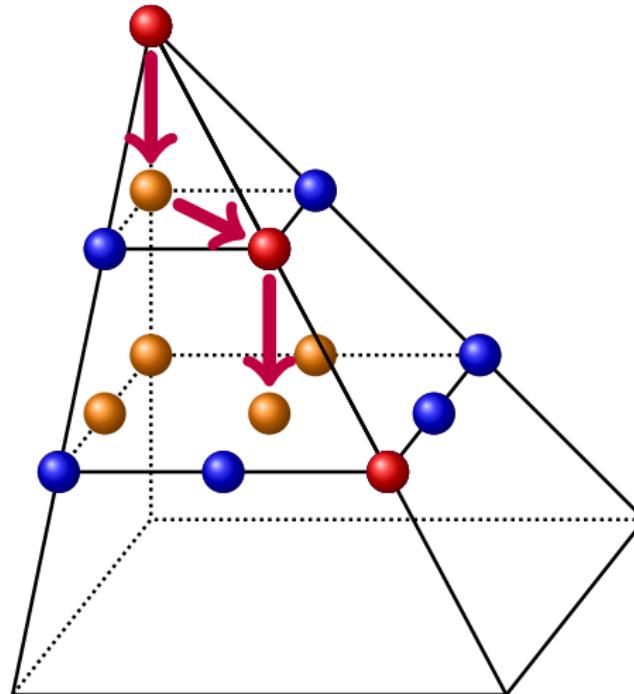
## Game dynamics: example

Both firms buy new technology  $c = 2 \rightsquigarrow (c_1, c_2, c) = (2, 2, 2)$



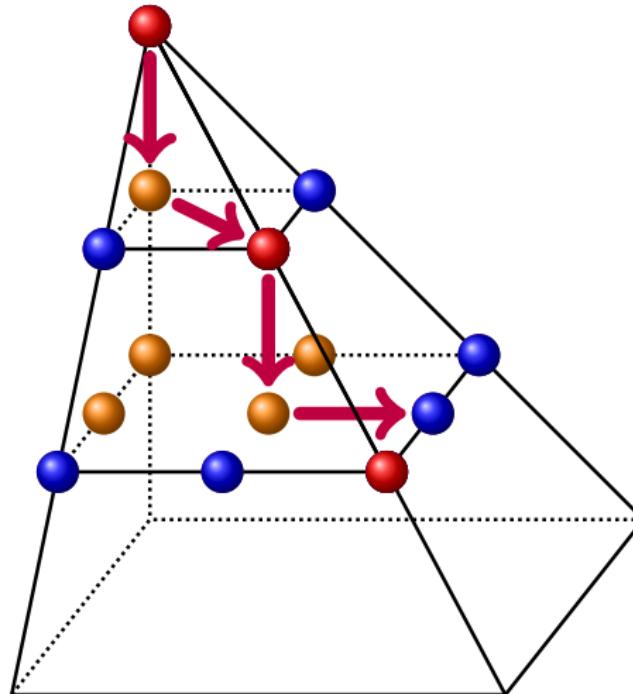
## Game dynamics: example

State-of-the-art technology becomes  $c = 1 \rightsquigarrow (c_1, c_2, c) = (2, 2, 1)$



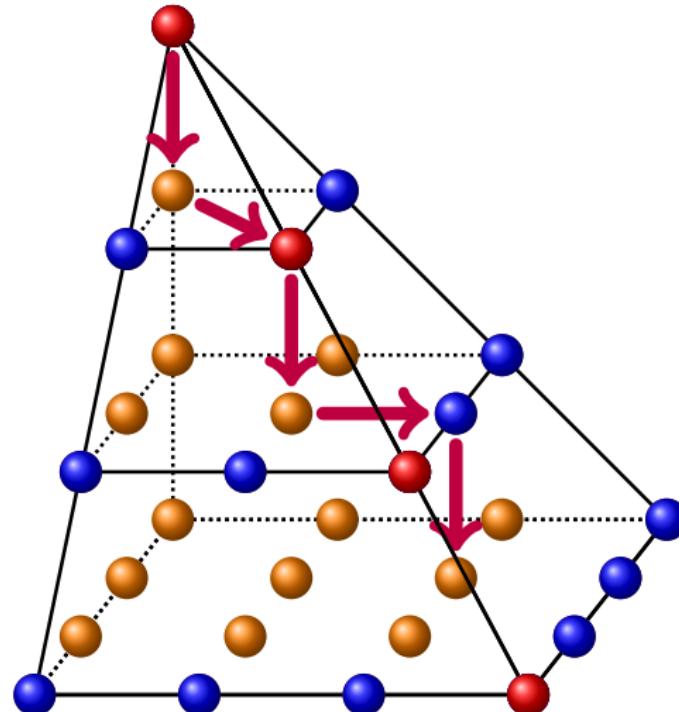
## Game dynamics: example

Firm 1 invests and becomes cost leader  $\rightsquigarrow (c_1, c_2, c) = (1, 2, 1)$



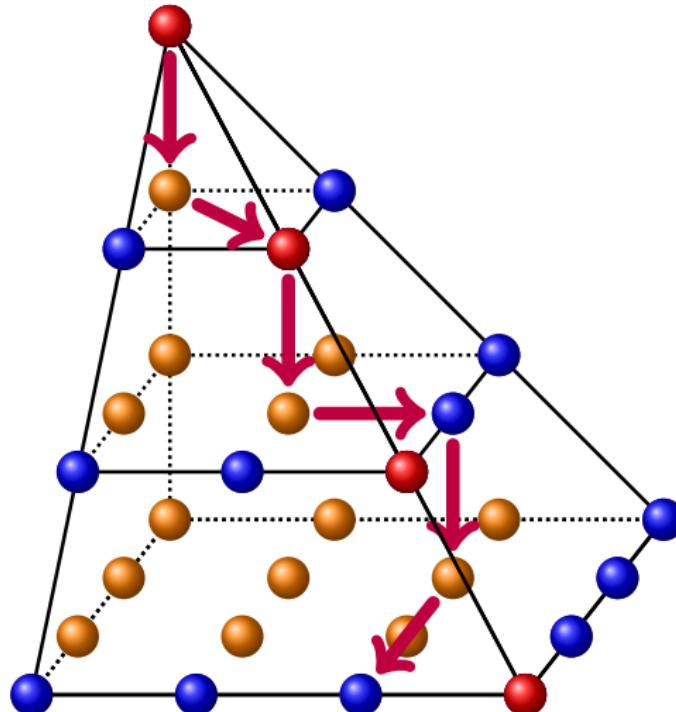
## Game dynamics: example

State-of-the-art technology becomes  $c = 0 \rightsquigarrow (c_1, c_2, c) = (1, 2, 0)$



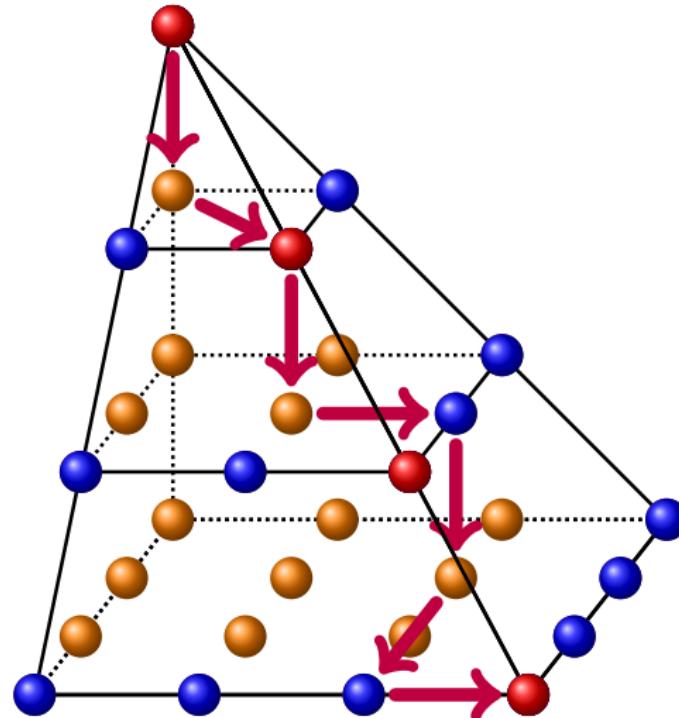
## Game dynamics: example

Firm 2 leapfrogs firm 1 to become new cost leader  $\rightsquigarrow (c_1, c_2, c) = (1, 0, 0)$



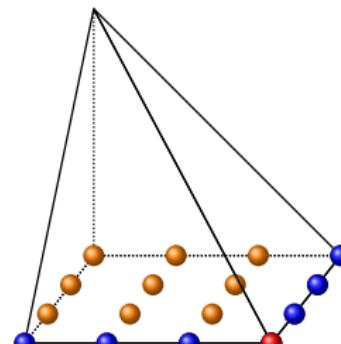
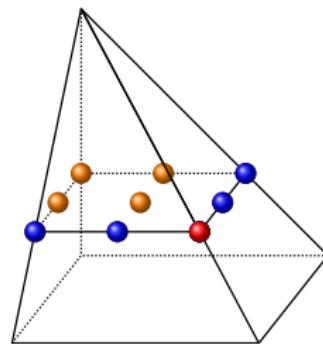
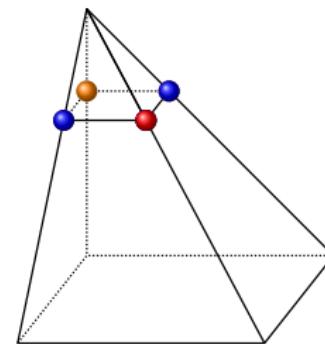
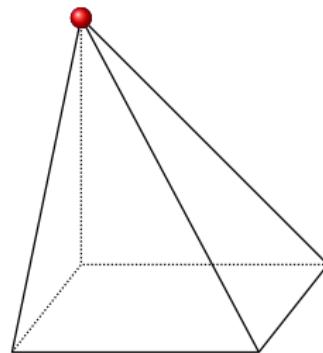
## Game dynamics: example

Firm 1 invests, and the game reaches terminal state  $\rightsquigarrow (c_1, c_2, c) = (0, 0, 0)$



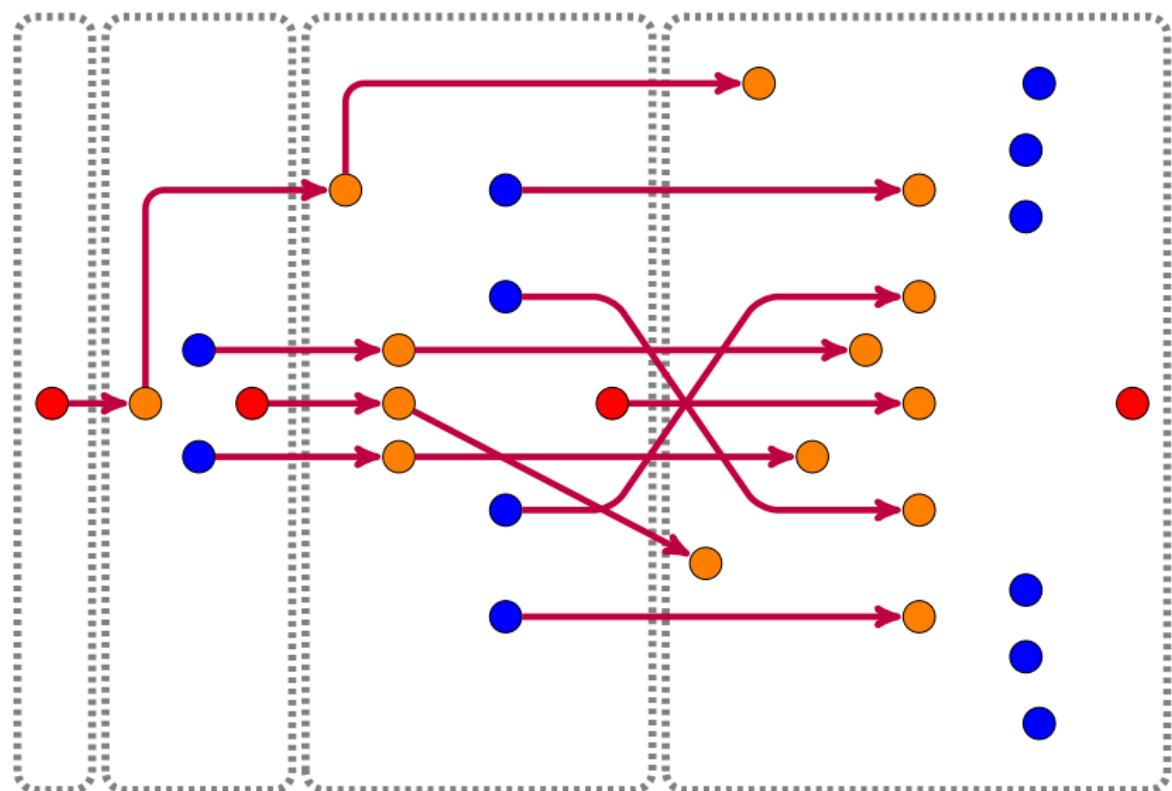
# Transitions due to technological progress

As  $c$  decreases, the game falls through the layers of the pyramid



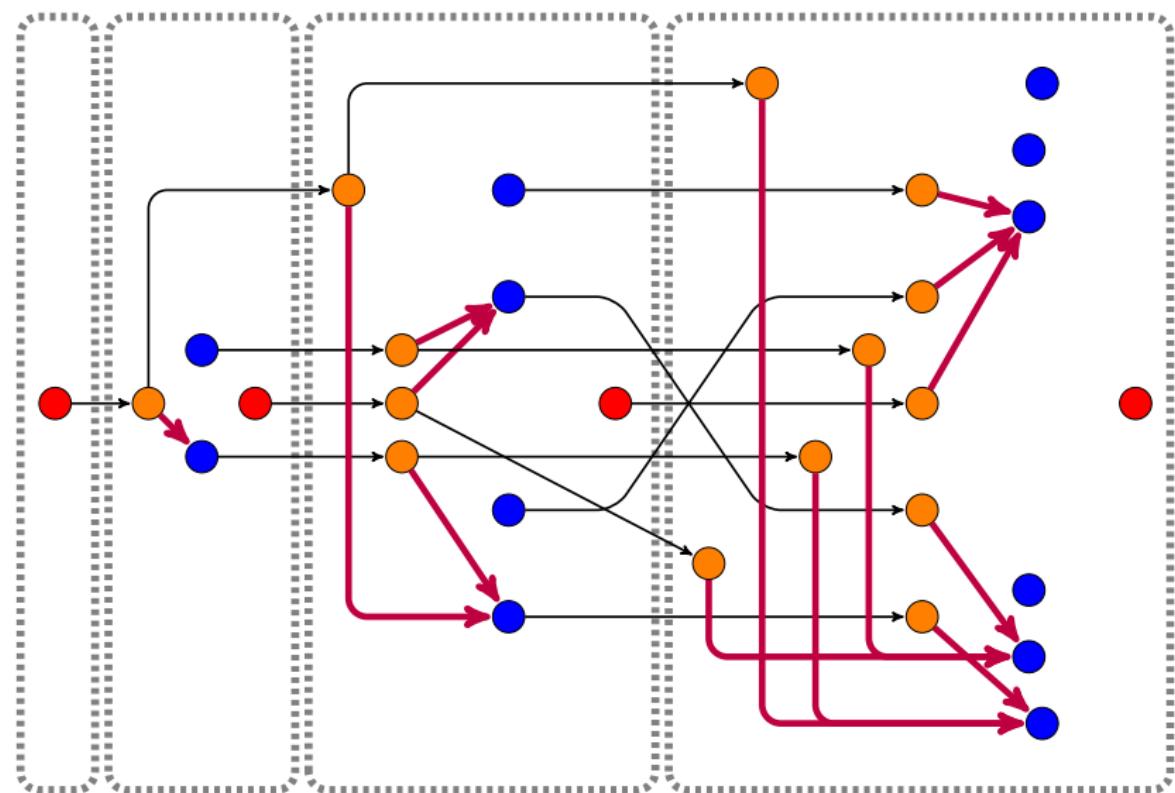
# Transitions due to technological progress

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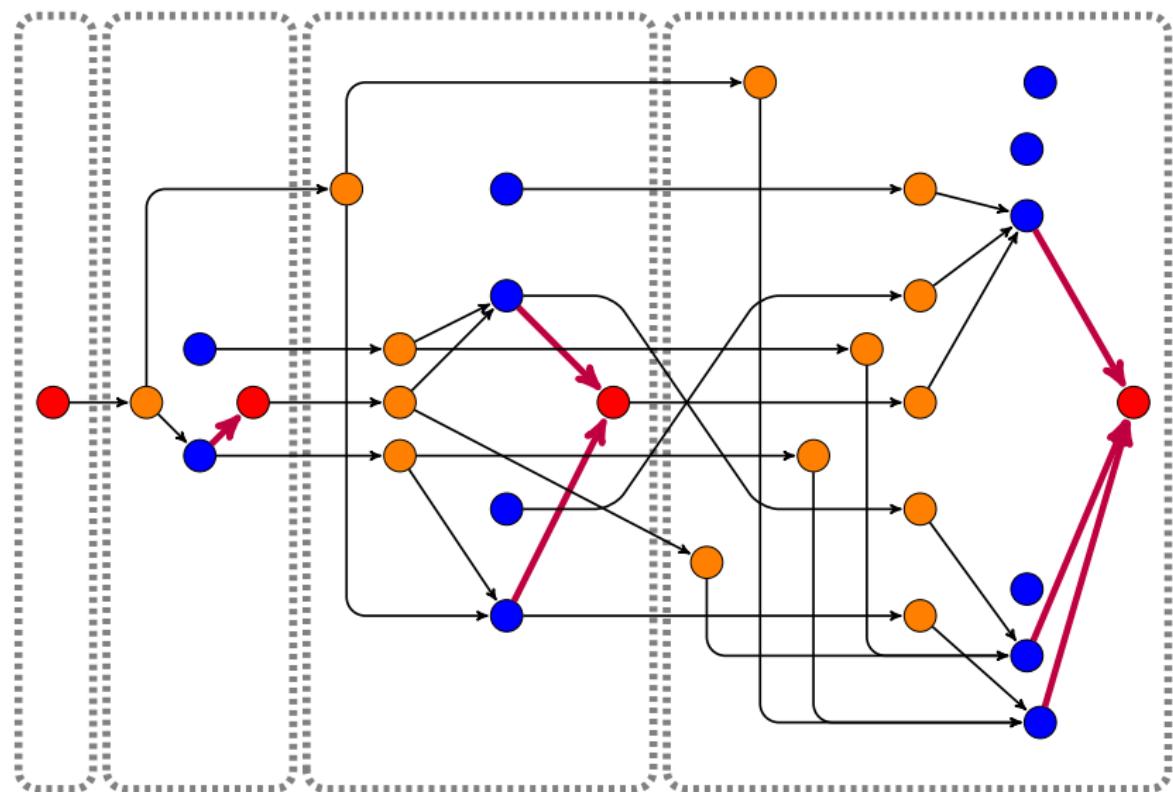
# Strategy-specific partial order on $S$

Strategy  $\sigma_1$  of firm 1: invest at all interior points



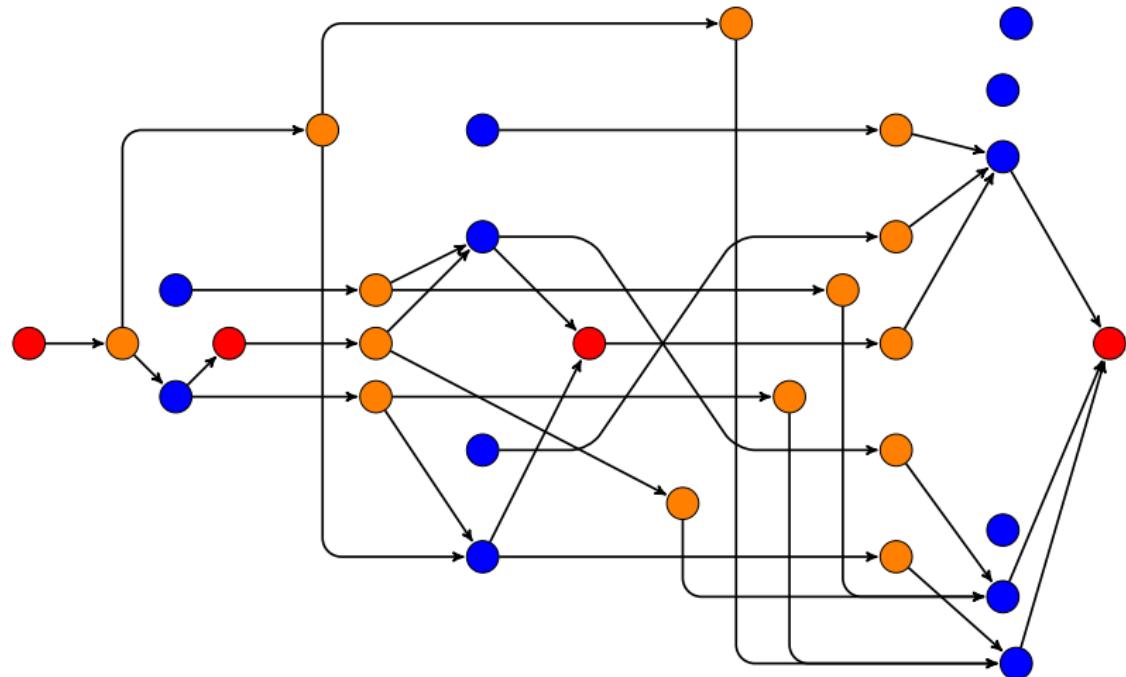
# Strategy-specific partial order on $S$

Strategy  $\sigma_2$  of firm 2: invest at all edge points



# Strategy-specific partial order on $S$

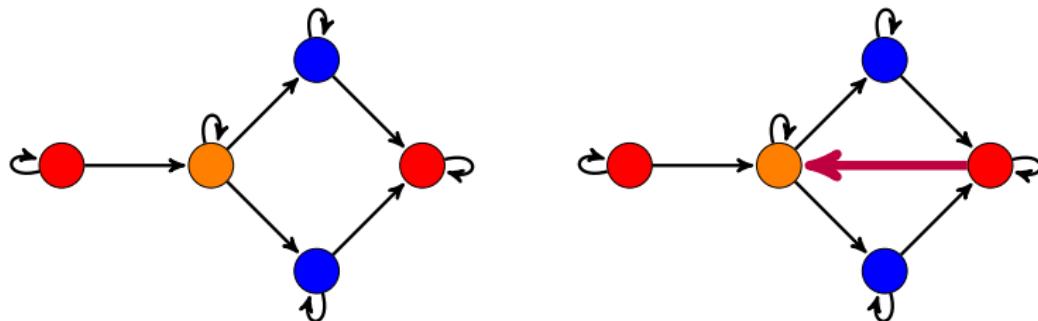
Strategy  $\sigma = (\sigma_1, \sigma_2)$  of both firms



## No loop (anti-cycling) condition

Hypothetical strategy profile inducing cycles

Self-loops appear when the game remains in the same state for two or more consecutive periods of time

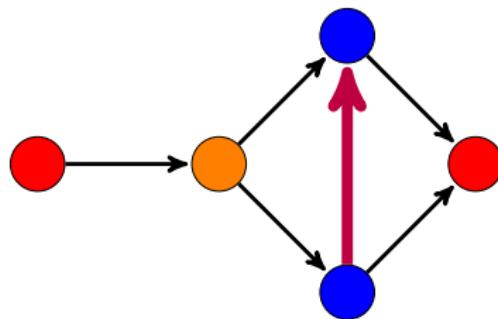
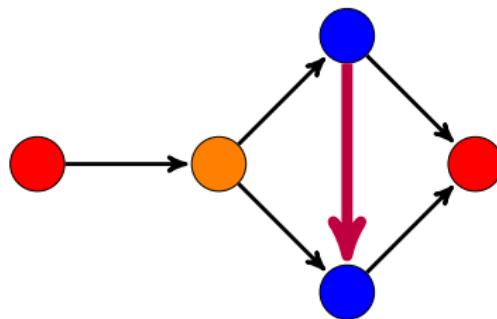


But loops between different states are not allowed

# Consistency of strategy specific partial orders

Two hypothetical inconsistent strategies

Two strategies that induce opposite transitions are **inconsistent**



Note that in both cases the no-loop condition is satisfied

# Definition of the Dynamic Directional Games

## Definition (Dynamic Directional Games, DDG)

Finite state Markovian stochastic game is a DDG if it holds:

1. Every feasible Markovian strategy  $\sigma$  satisfies the no loop condition.
2. Every pair of feasible Markovian strategies  $\sigma$  and  $\sigma'$  induce consistent partial orders on the state space.

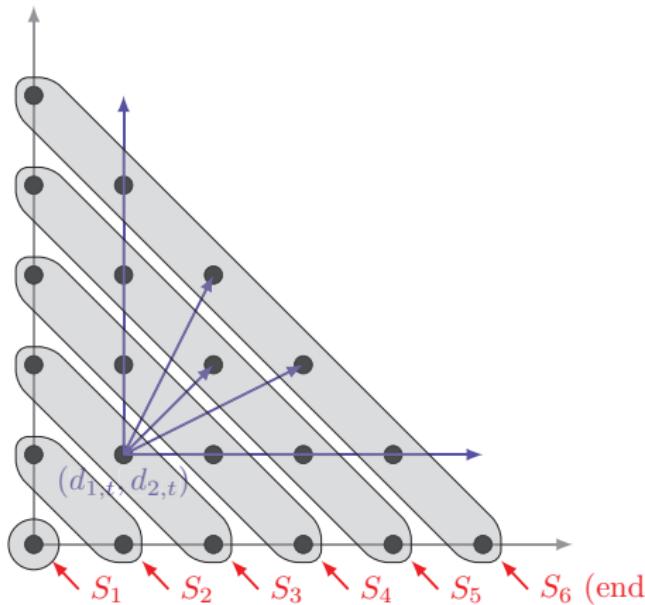
# Example of DDG: Market tipping game



Dubè, Hitsch and Chintagunta, 2010 *Marketing Science*

Tipping and Concentration in Markets with Indirect Network Effects

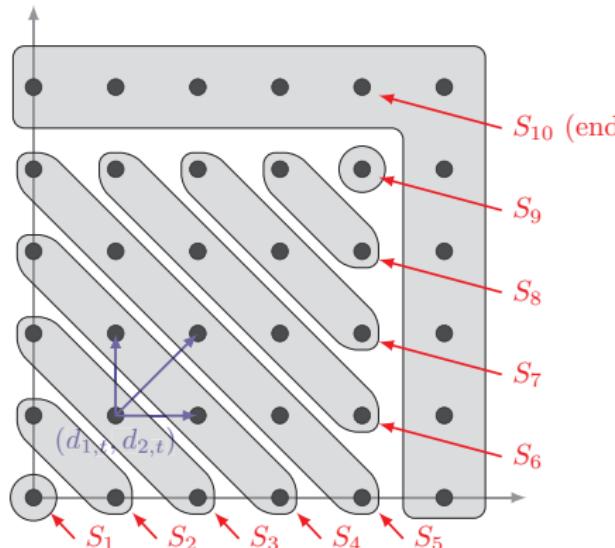
- ▶ Two competing gaming **platforms** (Xbox vs. Playstation)
- ▶ Number of games for each console depends on market share
- ▶ Consumers choose product they believe will win the war of the standards, or delay purchase
- ▶ Brand choices are absorbing: adoption base can only increase



# Example of DDG: Patent race game

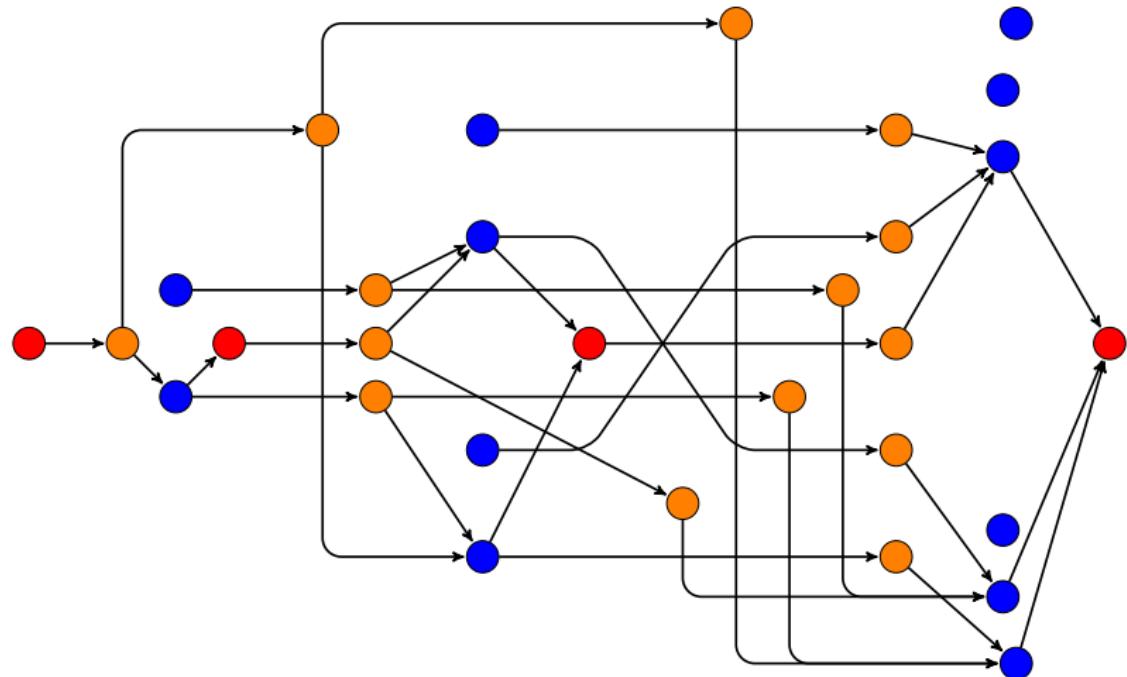
 Judd, Schmedders and Yeltekin, 2012 *IER*  
Optimal Rules for Patent Races

- ▶ Two firms are engaged in the  $N$ -steps race to acquire a patent
- ▶ Probability a step depends on the chosen R&D investment
- ▶ Only forward steps are possible, no “forgetting”
- ▶ Used state dependence structure to speed up computation of MPE by solving non-linear systems



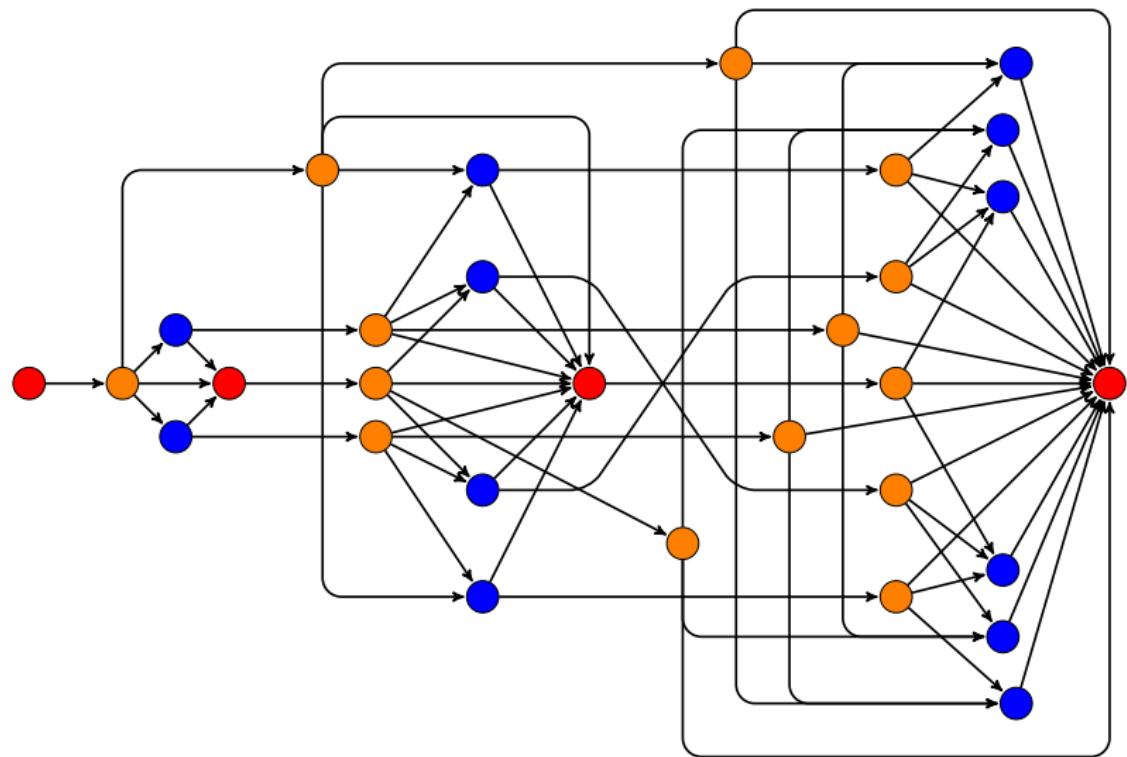
# Strategy-specific partial order on $S$

Strategy  $\sigma = (\sigma_1, \sigma_2)$  of both firms

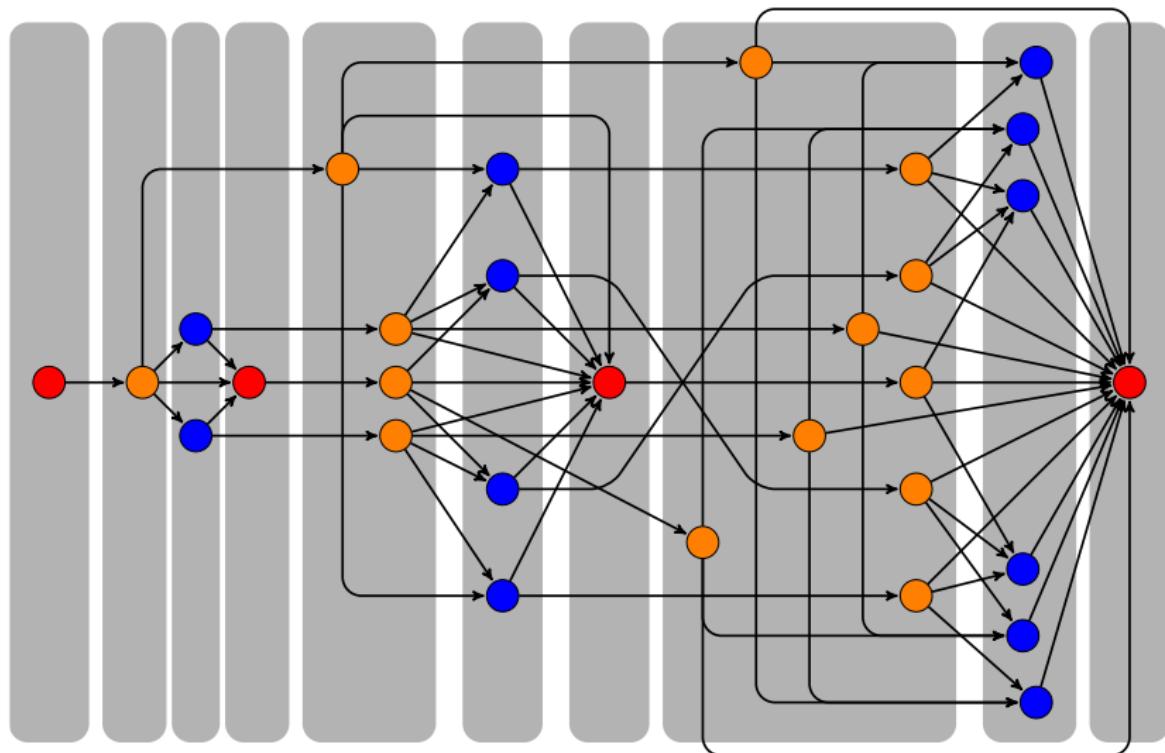


Strategy independent partial order on  $S$

Coarsest common refinement of partial orders induced by all strategies

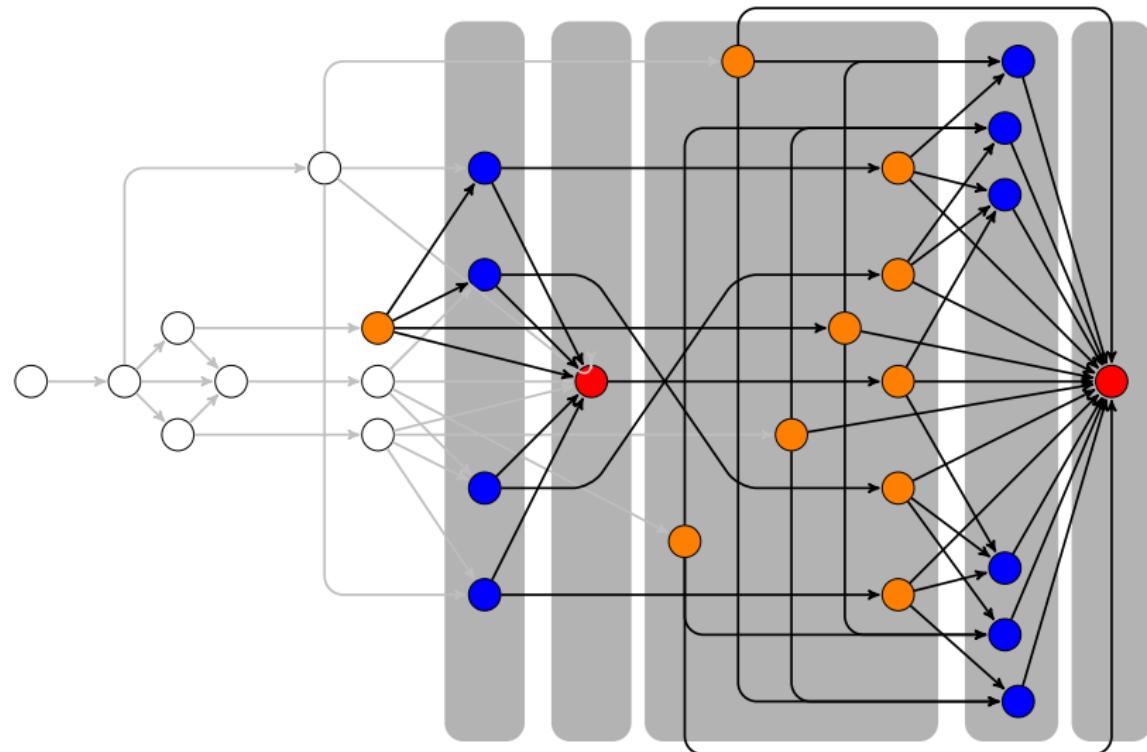


## Total order on the set of stages



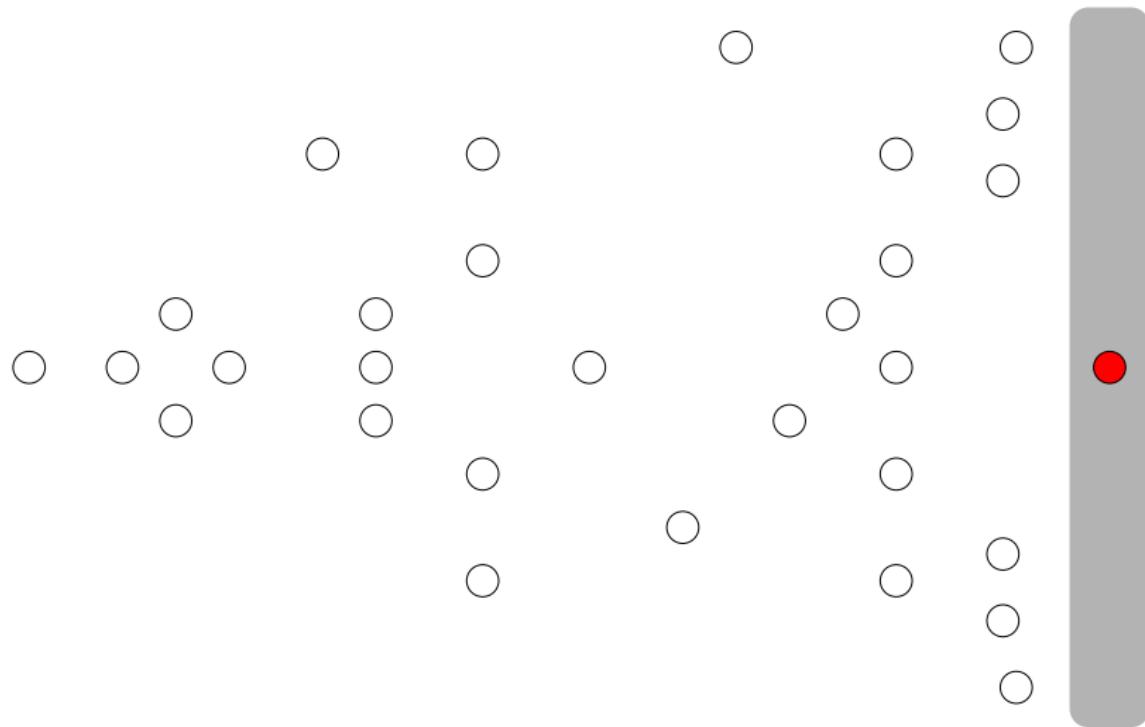
# Subgames of DDG and continuation strategies

## Subgames and continuation strategies



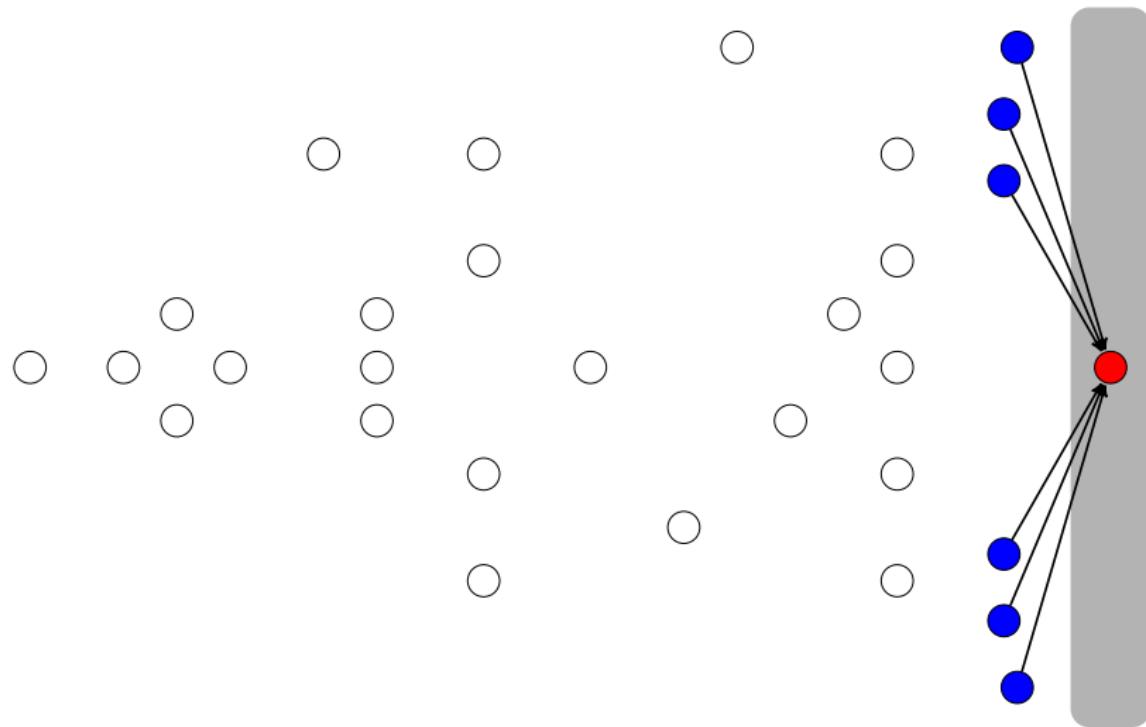
# State recursion algorithm

Backward induction on stages of DDG



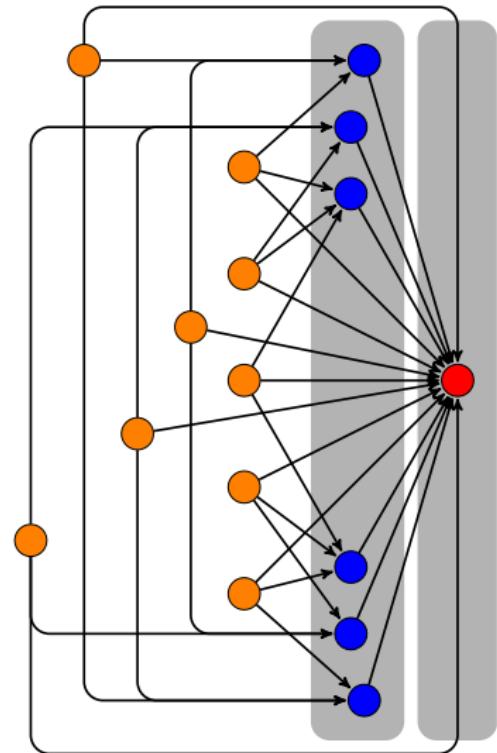
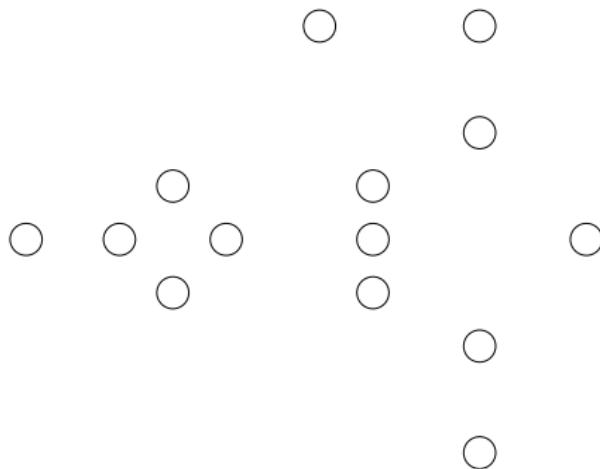
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Backward induction on stages of DDG



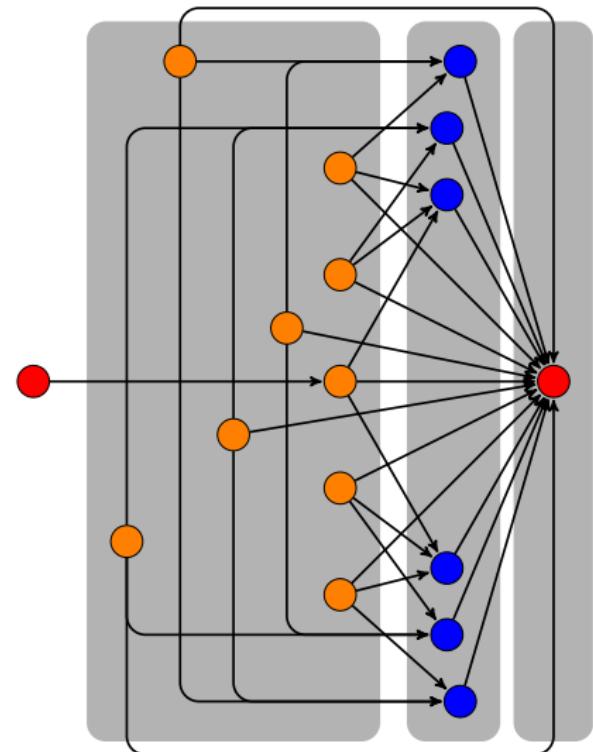
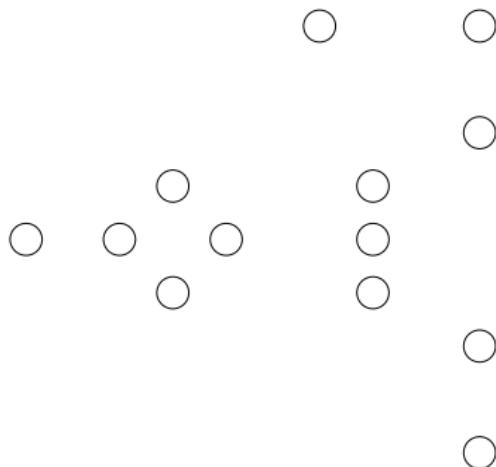
# State recursion algorithm

Backward induction on stages of DDG



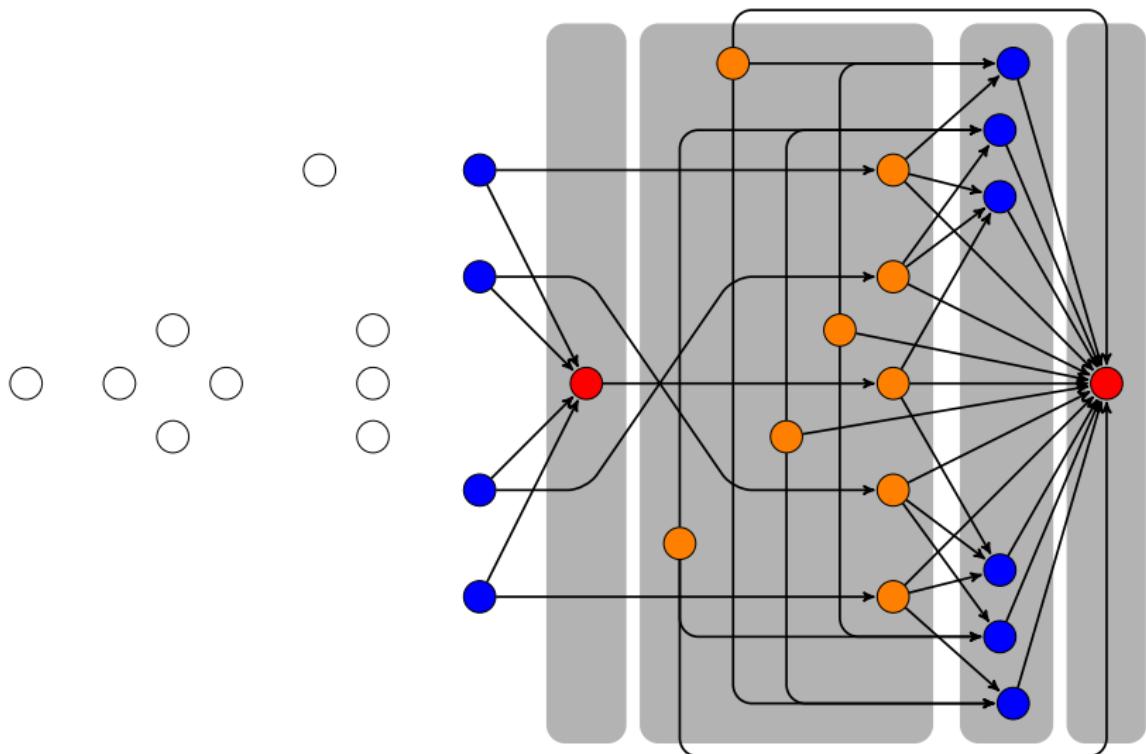
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Backward induction on stages of DDG



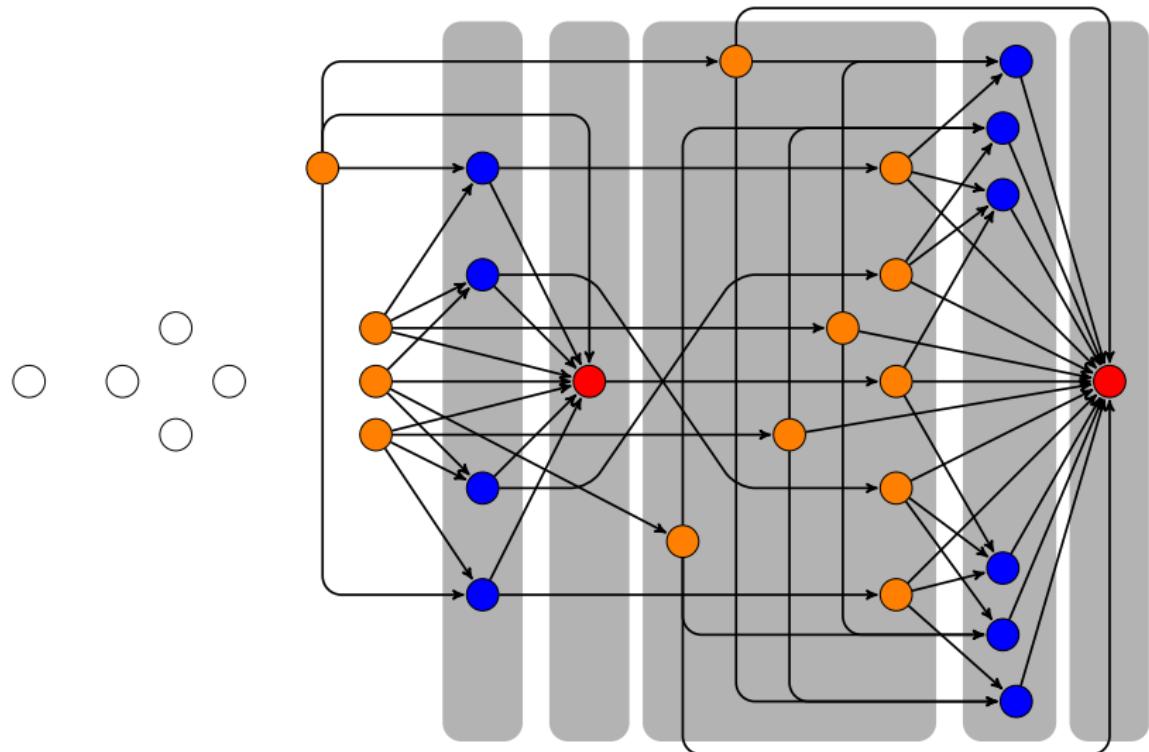
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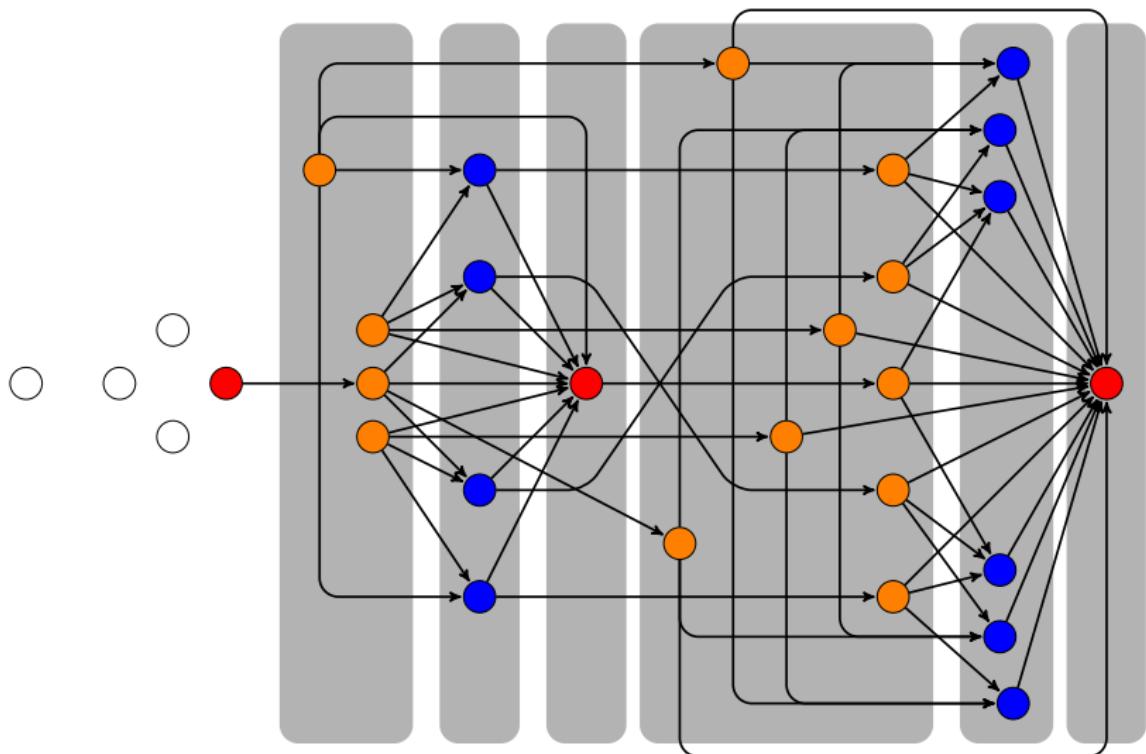
# State recursion algorithm

Backward induction on stages of DDG



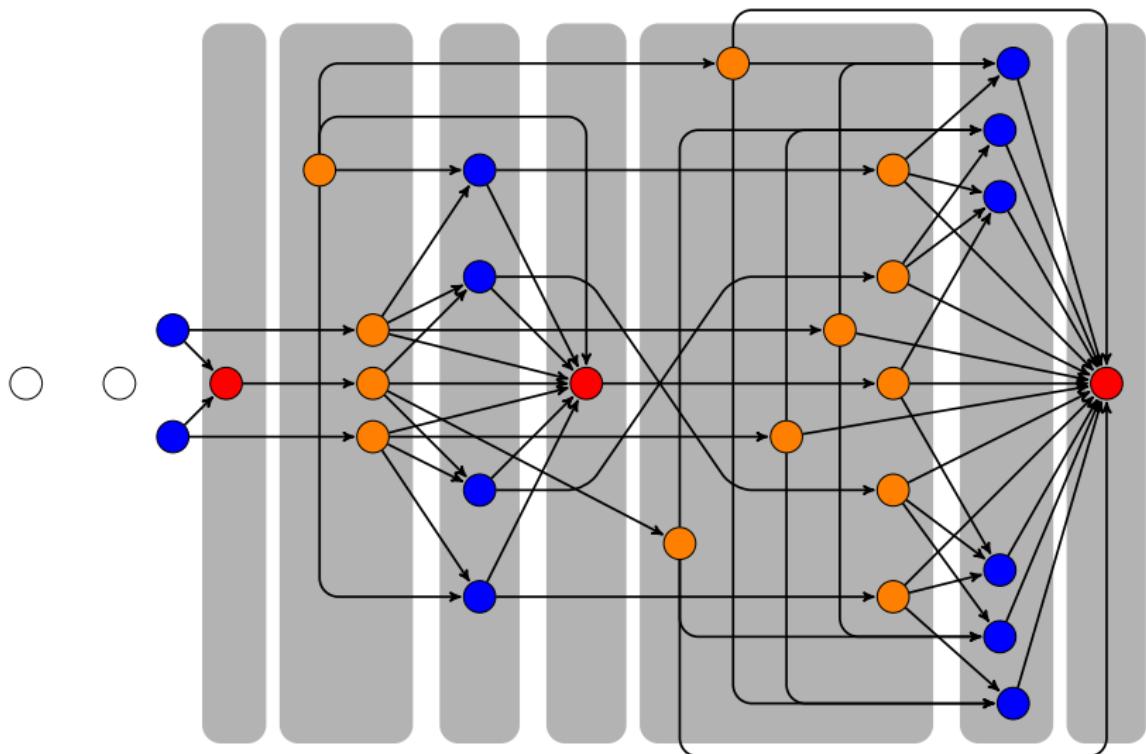
# State recursion algorithm

Backward induction on stages of DDG



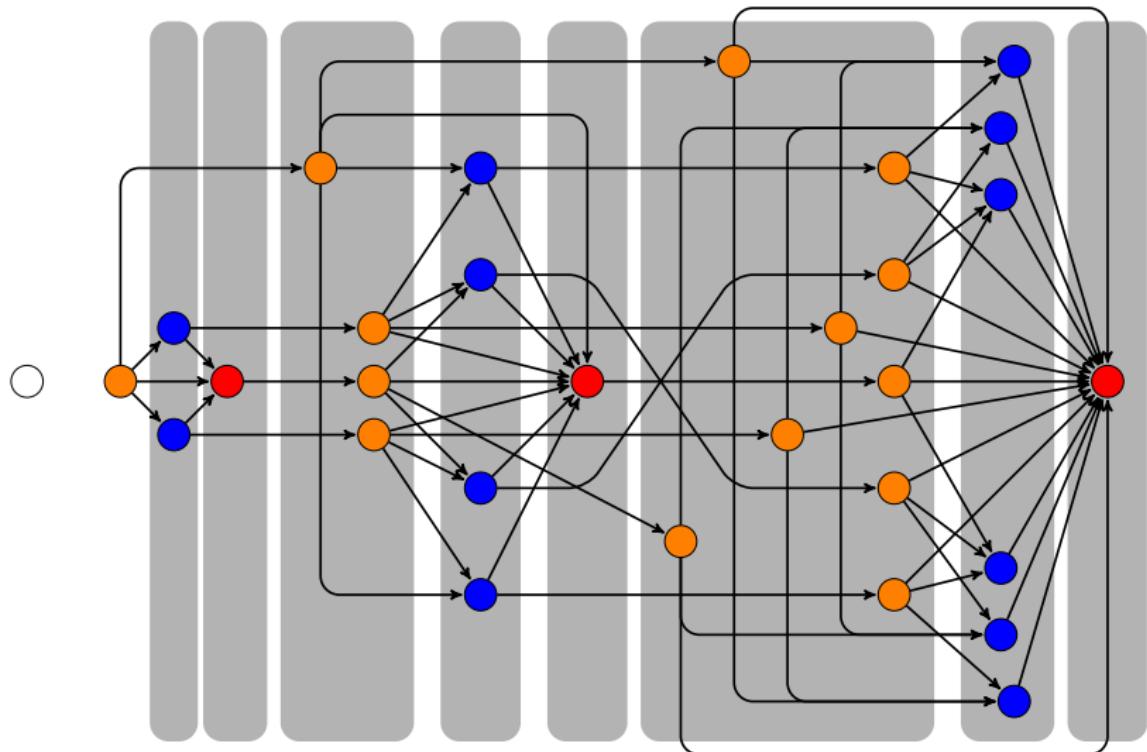
# State recursion algorithm

Backward induction on stages of DDG



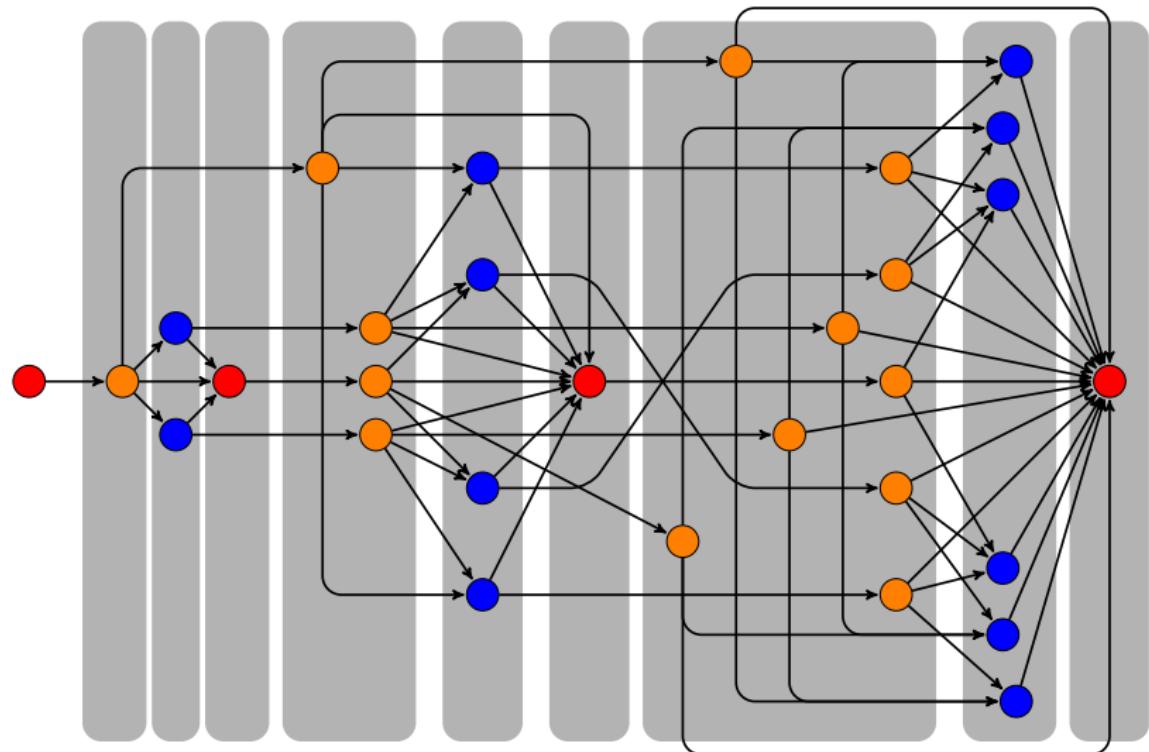
# State recursion algorithm

Backward induction on stages of DDG



# State recursion algorithm

Backward induction on stages of DDG

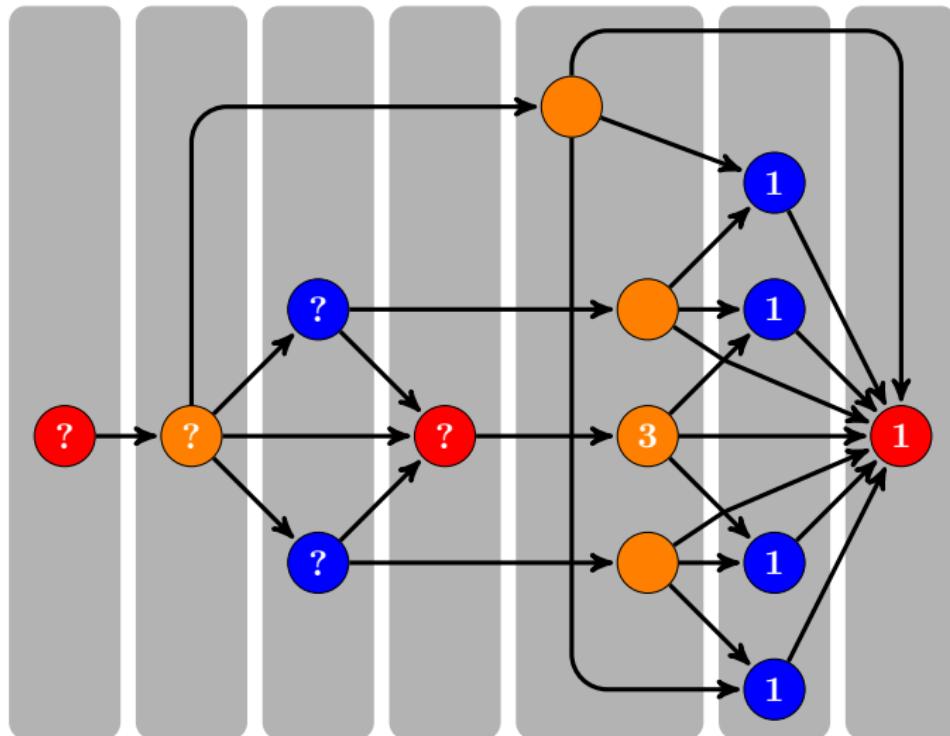


## State Recursion versus Backward Induction

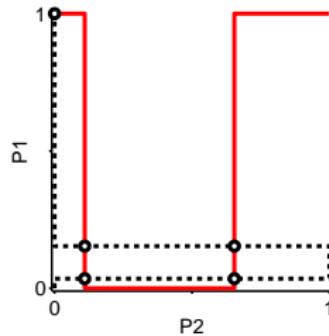
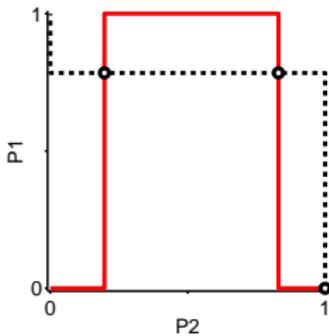
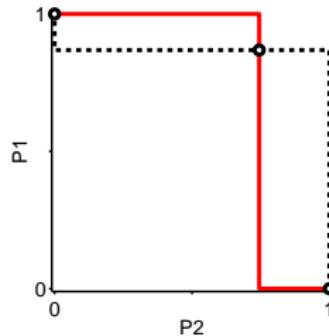
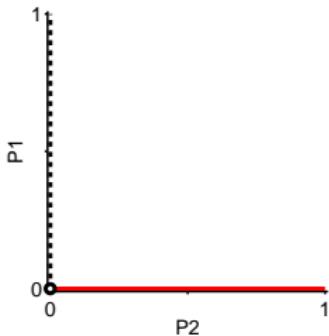
- ▶ State recursion – generalization of backward induction
- ▶ Runs on state space instead of time periods
- ▶ Time ( $t$ ) evolves as  $t \rightarrow t + 1$  with probability 1
- ▶ For stages of state space ( $\tau$ ) transitions are stochastic and not necessarily sequential
- ▶ Yet, probability of going  $\tau \rightarrow \tau'$  is zero when  $\tau' < \tau$
- ▶ With multiplicity, state recursion is performed **conditional** of a particular **equilibrium selection rule (ESR)**

# Multiplicity of stage equilibria

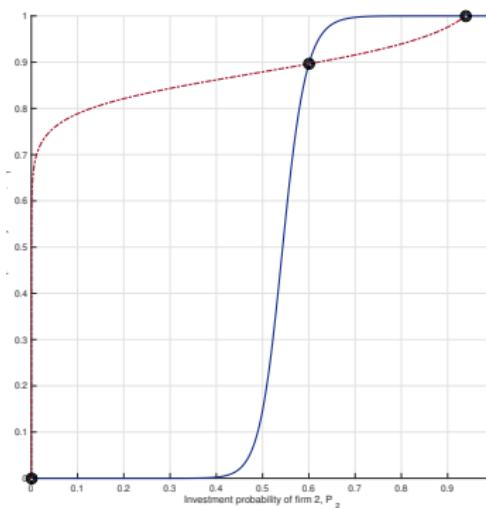
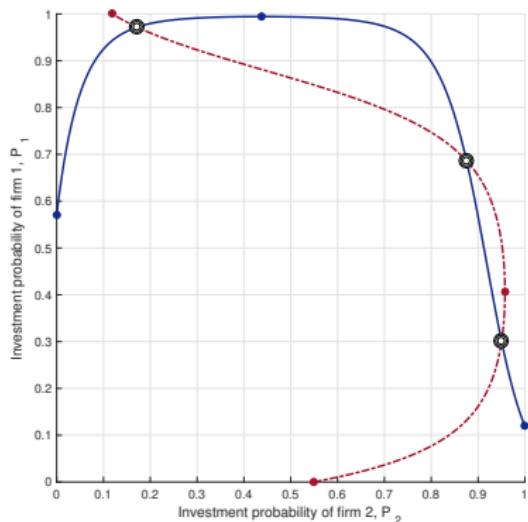
Number of equilibria in the higher stages depends on the selected equilibria



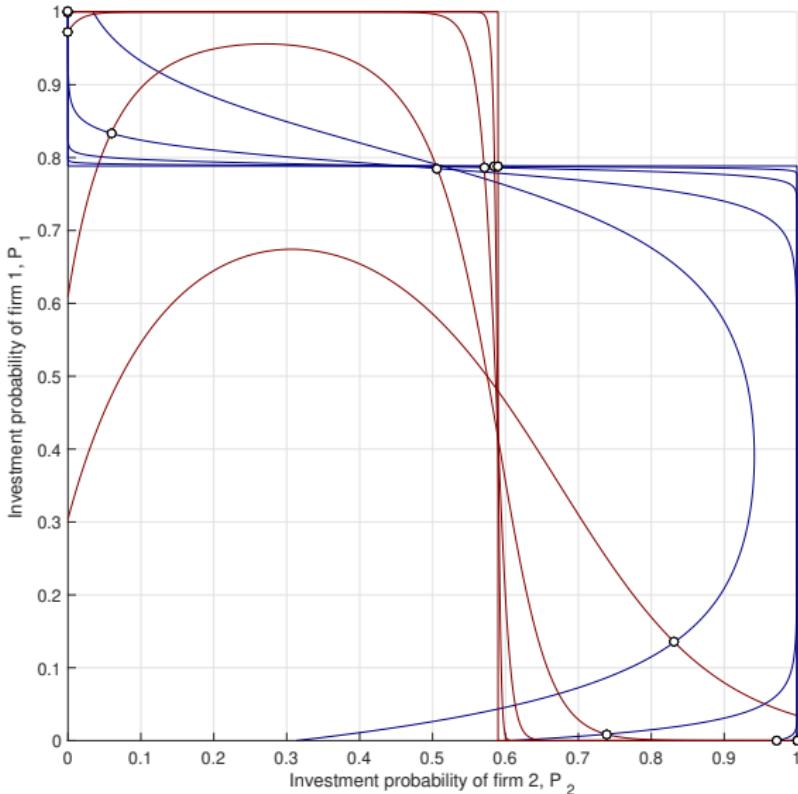
## Best response correspondences for $\eta = 0$



## Best response functions for $\eta > 0$



## Best response functions as $\eta \rightarrow 0$



# Recursive Lexicographic Search Algorithm

Building blocks of RLS algorithm:

1. State recursion algorithm solves the game **conditional on** equilibrium selection rule (ESR)
2. RLS algorithm efficiently cycles through **all feasible** ESRs

## Challenge:

- ▶ Choice of a particular MPE for any stage game at any stage
- ▶ may alter the **set** and even the **number** of stage equilibria at earlier stages

Need to find feasible ESRs

- ▶ ESR = **string of digits** that index the selected stage equilibrium in each point

## All possible ESR strings in lexicographic order

- ▶ Bound the maximum number of equilibria in each stage by  $K = 3$

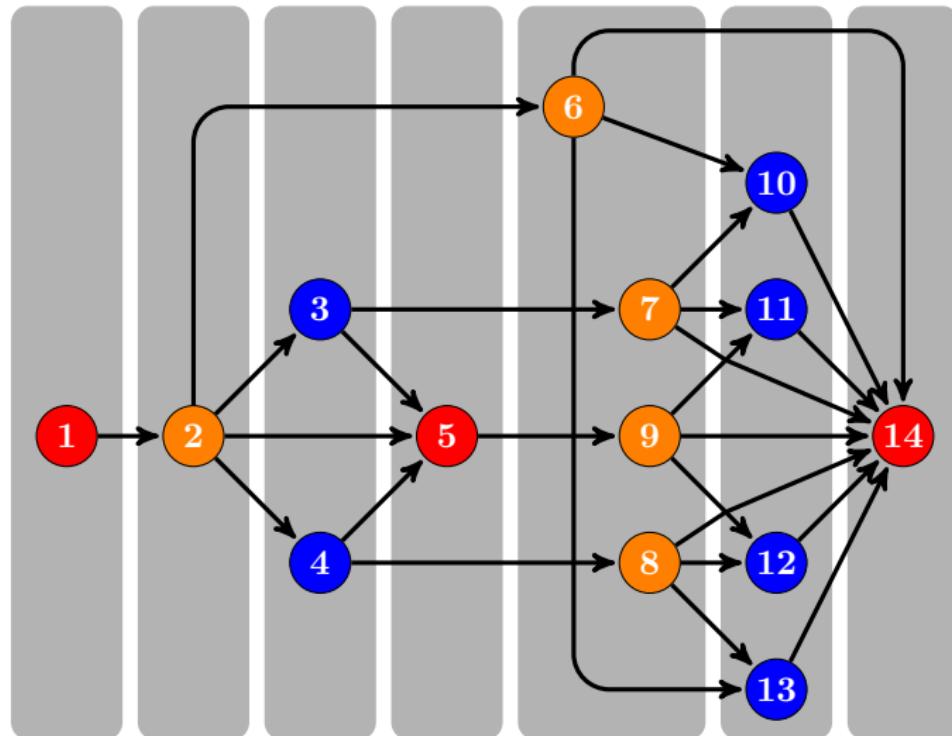
ESR string

Lexicograph

4,782,969

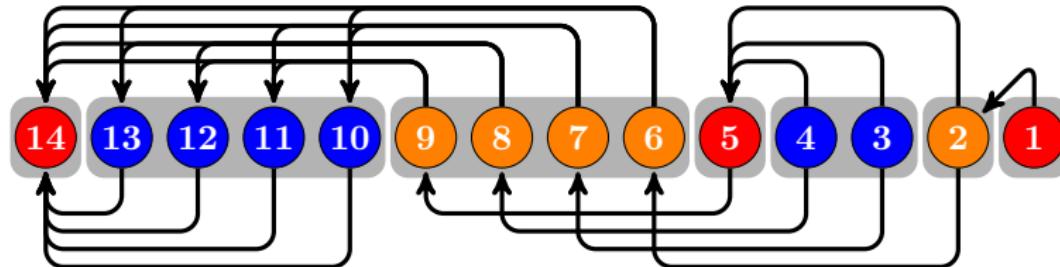
# Indexing of points in the state space

Lower index for dependent points, highest for terminal stage

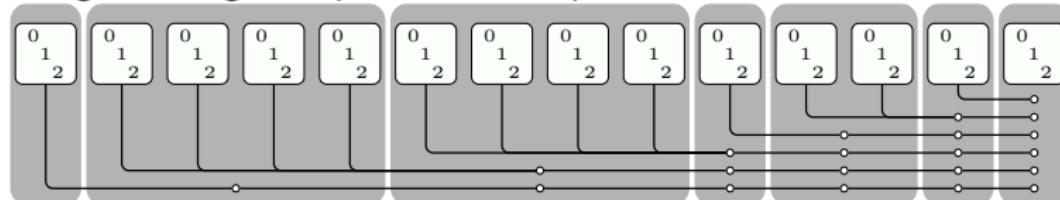


# Preserving stage order in ESR strings

Formalization of the ESR as strings of digits



- ▶ Digits arranged to preserve the dependence structure



# Recalculation of feasibility condition for new ESR

Avoid recalculation of subgames

ESR string	c	e	e	e	e	i	i	i	i	c	e	e	i	c
	14	13	12	11	10	9	8	7	6	5	4	3	2	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Nr of eqb	1	1	1	1	1	3	3	3	3	1	1	1	3	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	0	0	0	0	0	0	0	0	0	0	0	0	1	0
	1	1	1	1	1	3	3	3	3	1	1	1	3	*

No changes in the solution of the game including the number of stage equilibria

Might have changed

# Jumping over blocks of infeasible ESRs

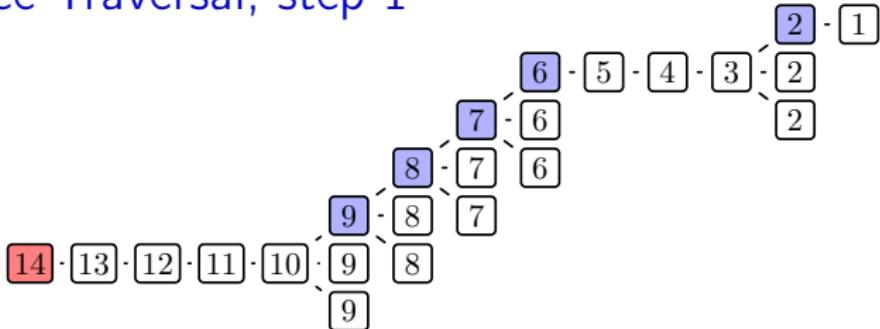
ESR string	c	e	e	e	e	i	i	i	i	c	e	e	i	c
	14	13	12	11	10	9	8	7	6	5	4	3	2	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	1	1	1	1	1	3	3	3	3	1	1	1	1	1a
	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	1	1	1	1	1	3	3	3	3	1	1	1	1	2a
	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	0	0	0	0	0	0	0	0	0	0	0	0	0	3
	1	1	1	1	1	3	3	3	3	1	1	1	1	3a
	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	0	0	0	0	0	0	0	0	0	0	0	0	0	3b
	0	0	0	0	0	0	0	0	0	0	0	0	0	...
	0	0	0	0	0	0	0	0	0	0	0	0	0	3c
	0	0	0	0	0	0	0	0	0	0	0	0	0	...
	0	0	0	0	0	0	0	0	0	0	0	0	0	3d
	0	0	0	0	0	0	0	0	0	1	0	0	0	4
	0	0	0	0	0	0	0	0	0	1	0	0	0	...

## Recursive Lexicographic Search (RLS) Algorithm

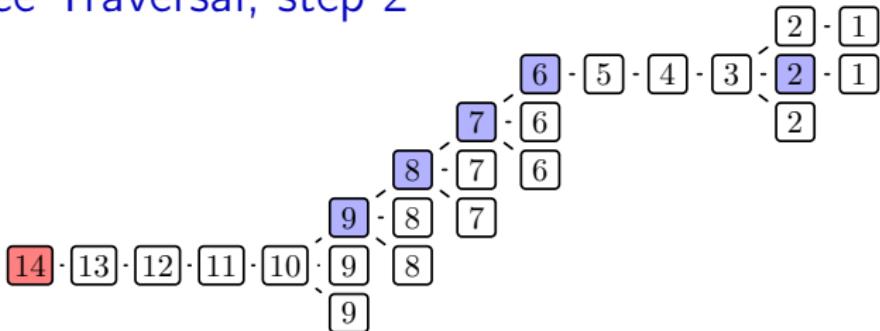
1. Set ESR =  $(0, \dots, 0)$
2. Run **State Recursion** using the current ESR
3. Save the number of equilibria in every stage game as  $ne(ESR)$
4. Add 1 to the ESR in bases  $ne(ESR)$  to obtain new feasible ESR
5. Stopping rule: successor function exceeds the maximum number with given number of digits
6. Return to step 2

**RLS = tree traversal!**

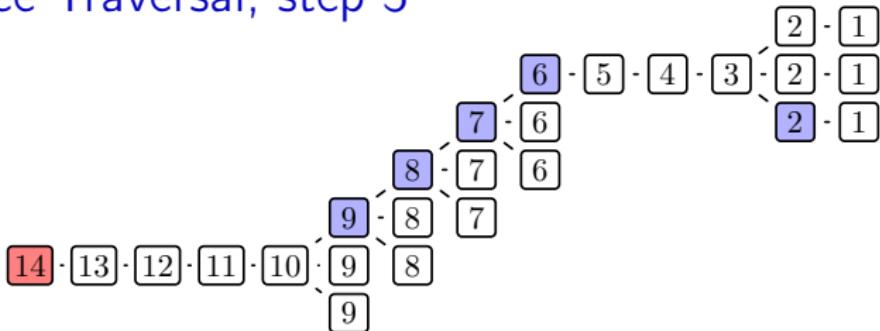
## RLS Tree Traversal, step 1



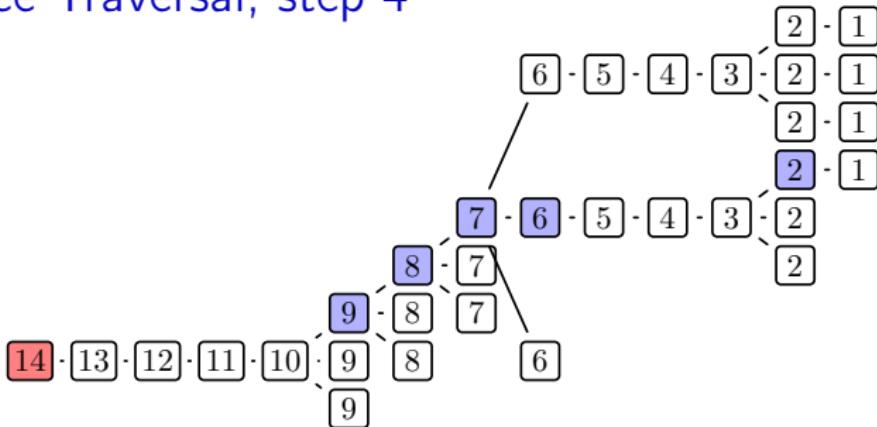
## RLS Tree Traversal, step 2



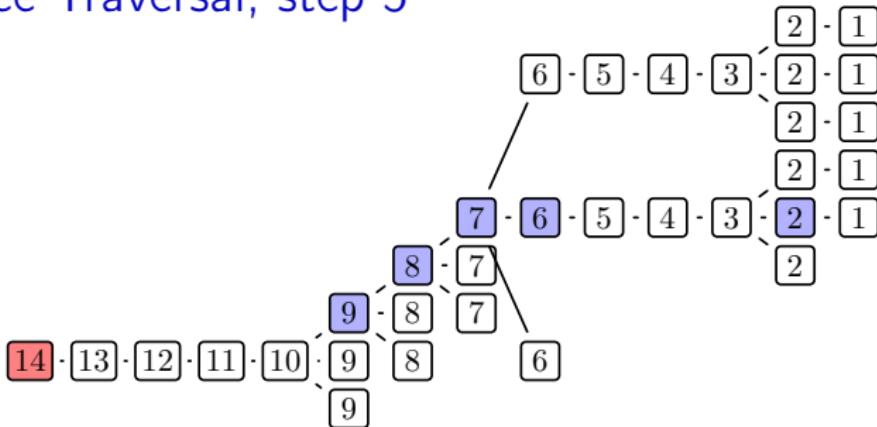
## RLS Tree Traversal, step 3



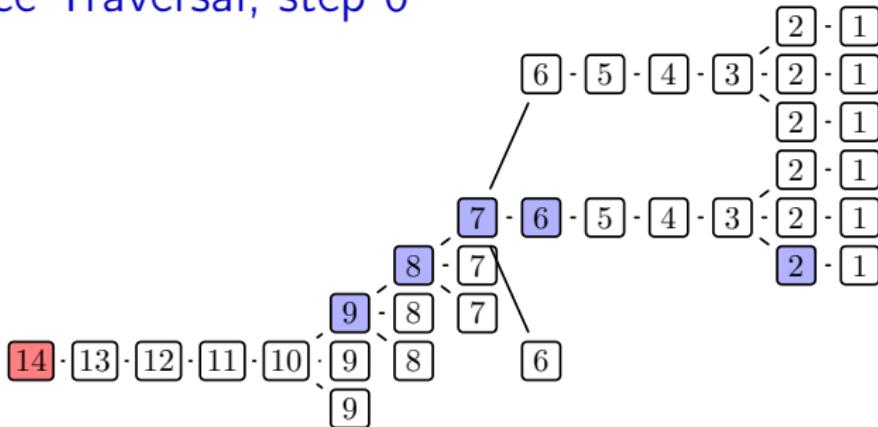
## RLS Tree Traversal, step 4



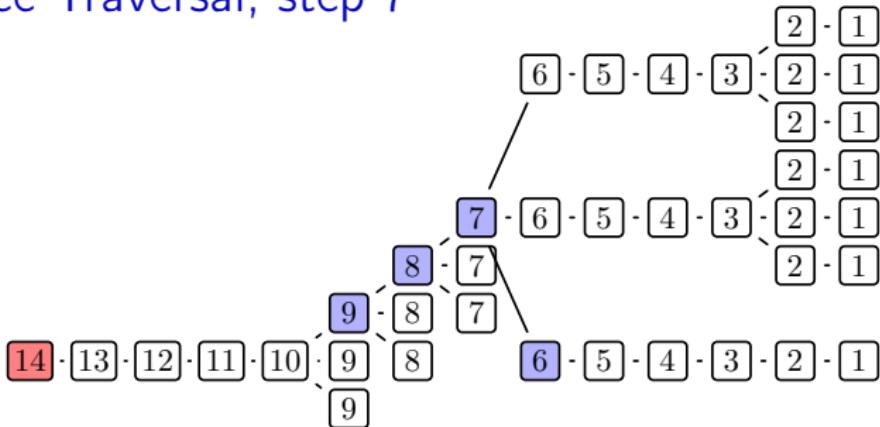
## RLS Tree Traversal, step 5



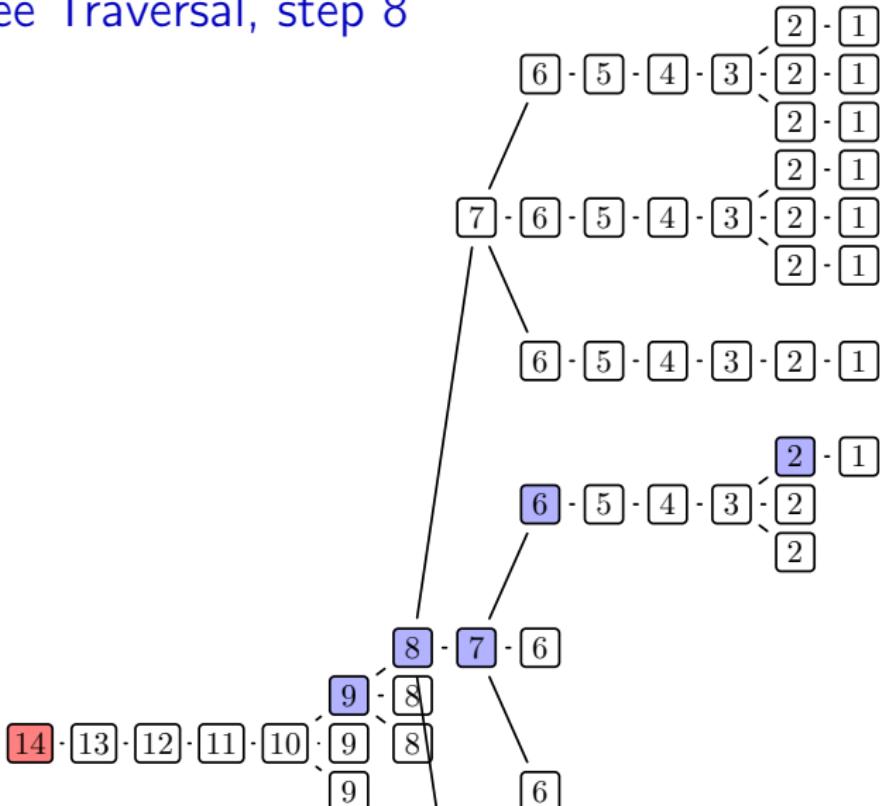
## RLS Tree Traversal, step 6



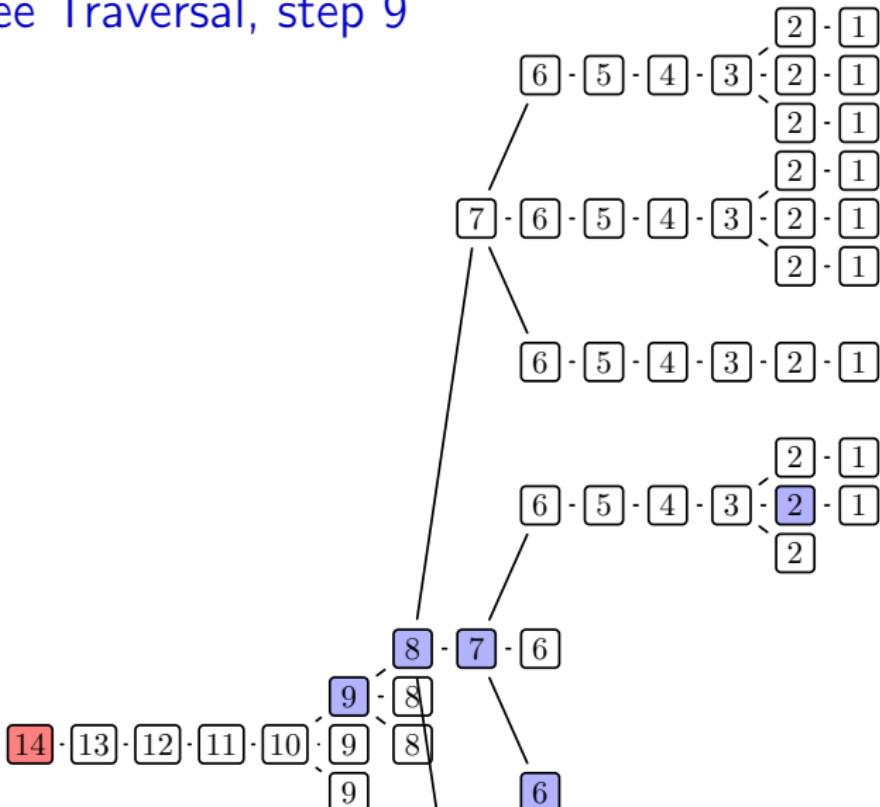
## RLS Tree Traversal, step 7



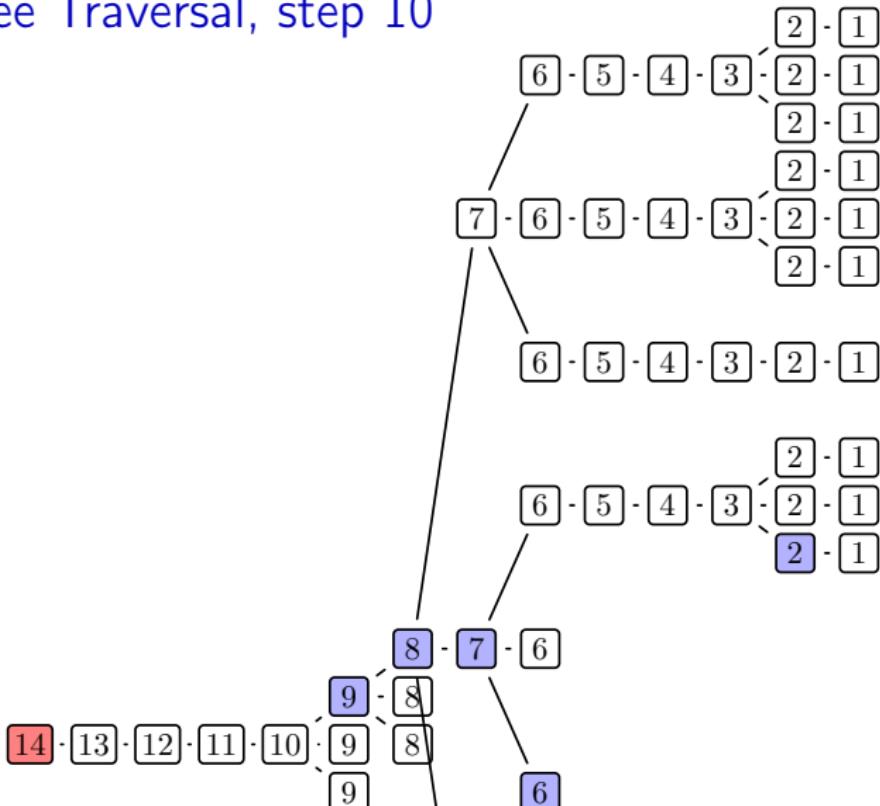
## RLS Tree Traversal, step 8



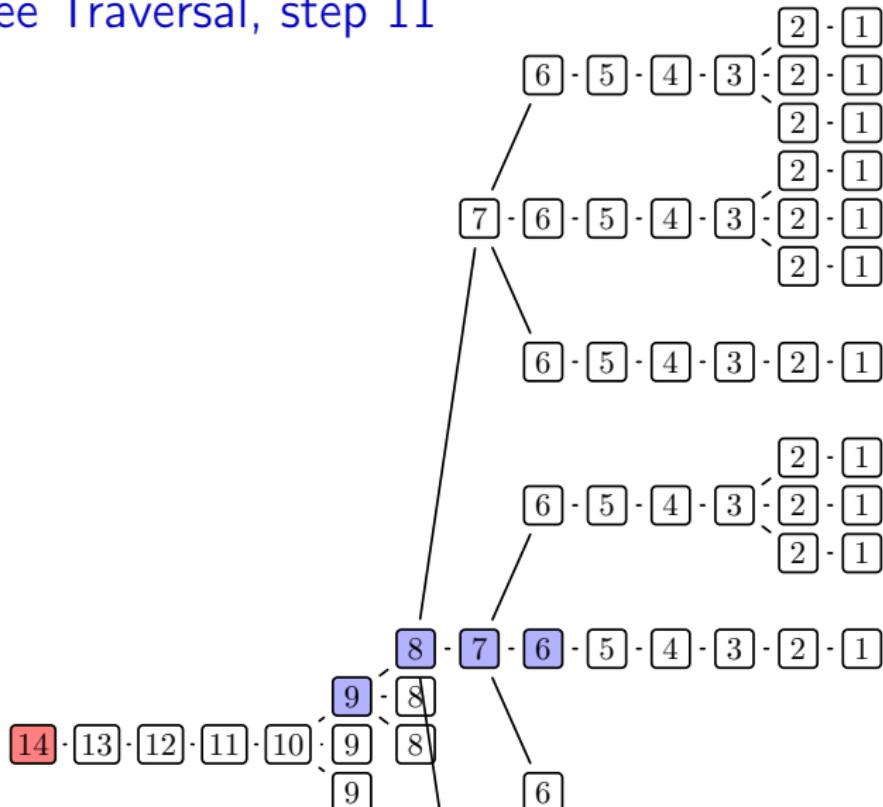
## RLS Tree Traversal, step 9



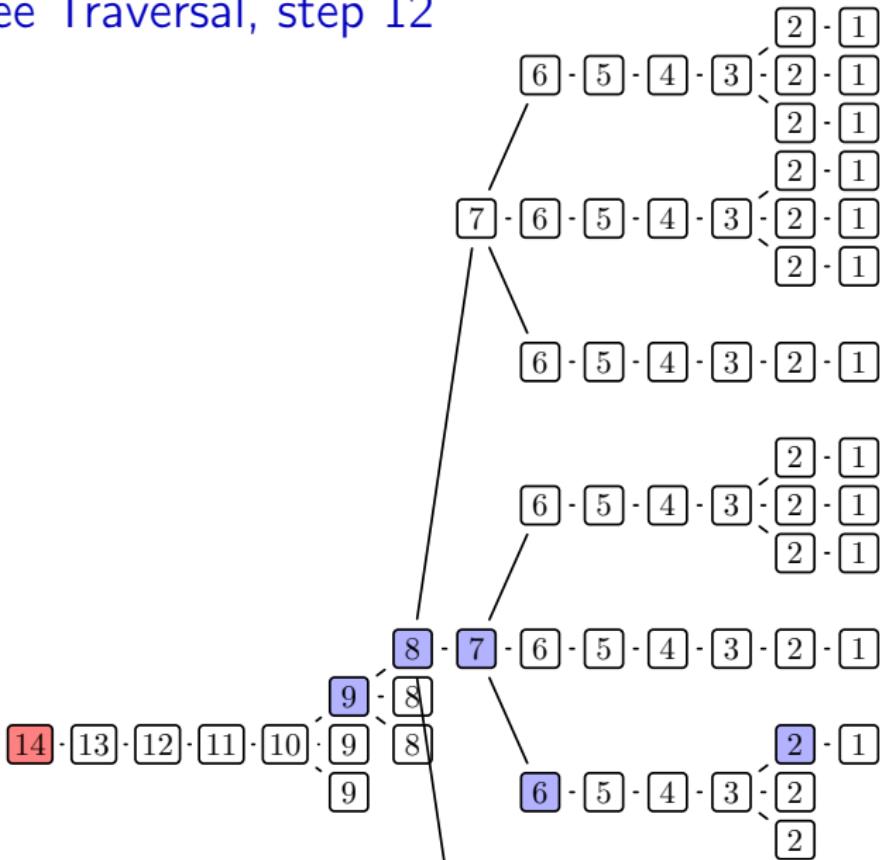
## RLS Tree Traversal, step 10



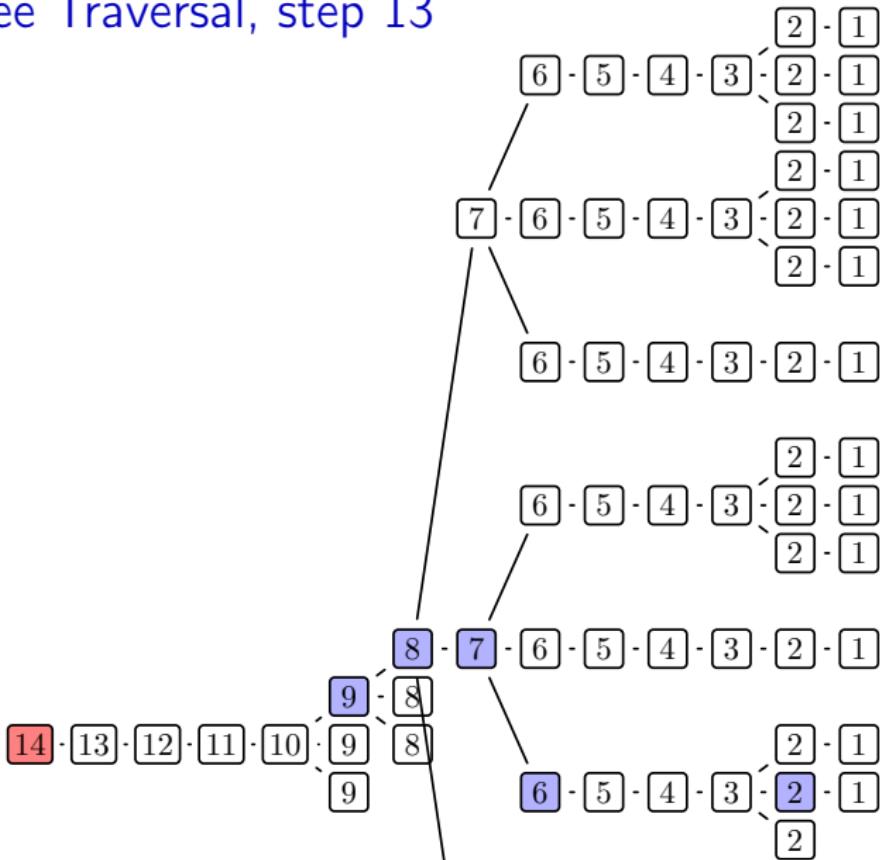
## RLS Tree Traversal, step 11



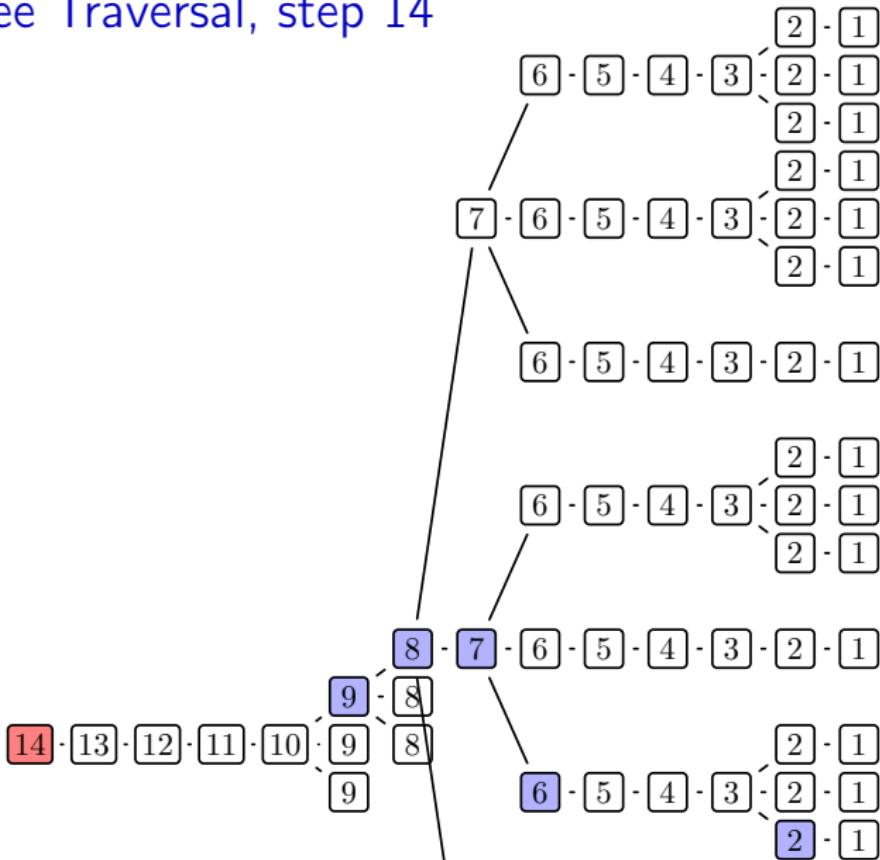
## RLS Tree Traversal, step 12



## RLS Tree Traversal, step 13



## RLS Tree Traversal, step 14



## Main result of the RLS Algorithm

Theorem (Decomposition theorem, strong)

*Assume there exists an algorithm that can find all MPE of every stage game of the DDG, and that the number of these equilibria is finite in every stage game.*

*Then the RLS algorithm finds all MPE of the DDG in a finite number of steps, which equals the total number of MPE.*



Iskhakov, Rust and Schjerning, 2016, ReStud

“Recursive lexicographical search: Finding all markov perfect equilibria of finite state directional dynamic games.”

## Main result of the RLS Algorithm

Theorem (Decomposition theorem, weak)

*Assume there exists an algorithm that can find at least one MPE of every stage game of the DDG, and that the number of these equilibria is finite in every stage game.*

*Then the RLS algorithm finds some (at least one) MPE of the DDG in a finite number of steps, which does not exceed the total number of MPE.*

## RLS algorithm: running times

$K = 3$

Simultaneous moves	$n = 3$	$n = 4$
Upper bound on number of MPE	4,782,969	3,948,865,611
Actual number of equilibria	127	46,707
Time used	0.008 sec.	0.334 sec.
Simultaneous moves	$n = 5$	
Upper bound on number of MPE	174,449,211,009,120,166,087,753,728	
Actual number of equilibria	192,736,405	
Time used	45 min.	
Alternating moves	$n = 5$	
Upper bound on number of MPE	174,449,211,009,120,166,087,753,728	
Actual number of equilibria	1	
Time used	0.006 sec.	

# Resolution to the Bertrand investment paradox

## Theorem (Solution to Bertrand investment paradox)

*If investment is socially optimal at a state point  $(c_1, c_2, c) \in S$ , then*

- ▶ *no investment by both firms cannot be an MPE outcome in the subgame starting from  $(c_1, c_2, c)$  in either the simultaneous or alternating move versions of the dynamic game.*

We show:

1. Many types of endog. coordination is possible in equilibrium
  - ▶ Leapfrogging (alternating investments)
  - ▶ Preemption (investment by cost leader)
  - ▶ Duplicative (simultaneous investments)
2. The equilibria are generally inefficient due to over-investment
  - ▶ Duplicative or excessively frequent investments

# Multiplicity of equilibria

## Theorem (Sufficient conditions for uniqueness)

*In the dynamic Bertrand investment and pricing game a sufficient condition for the MPE to be unique is that*

1. *firms move in alternating fashion (i.e.  $m \neq 0$ ), and,*
  2. *for each  $c > 0$  in the support of  $\pi$  we have  $\pi(c|c) = 0$ .*
- 
1. If firms move simultaneously,  
equilibrium is generally **not unique**.
  2. If technological change is stochastic,  
equilibrium is generally **not unique**.



Iskhakov, Rust and Schjerning, 2018, IER

“The dynamics of Bertrand price competition with cost-reducing investments.”

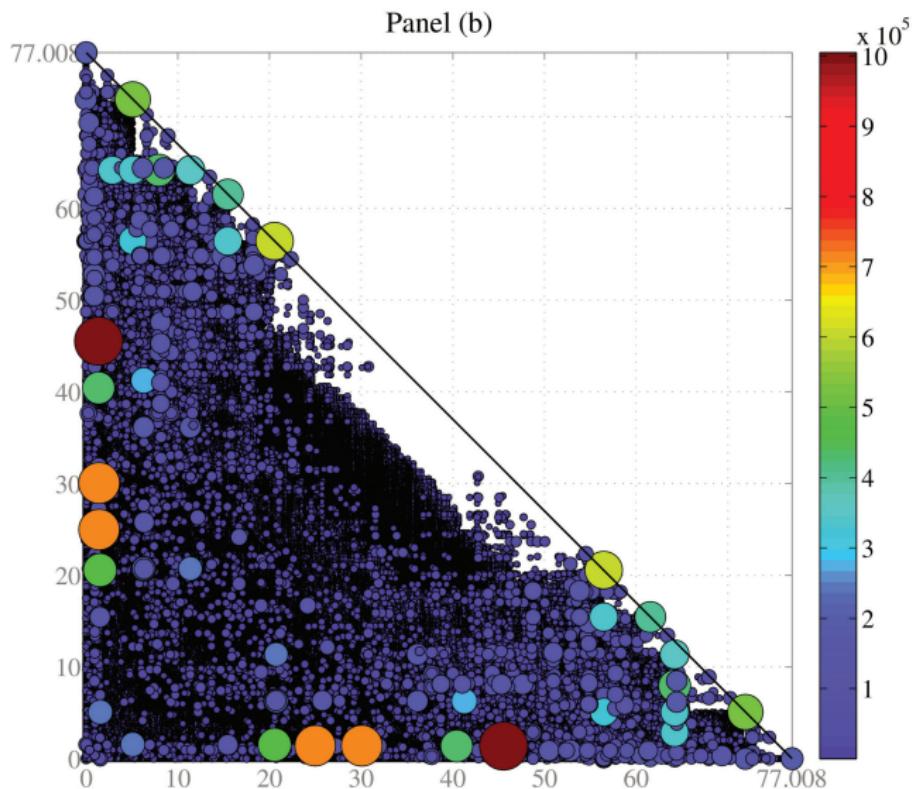
## Pay-offs in the simultaneous move game

### Theorem (Triangular payoffs in the simultaneous move game)

Suppose that the  $\{c_t\}$  process has finite support, that there are no idiosyncratic shocks to investment (i.e.  $\eta = 0$ ) and that firms move simultaneously

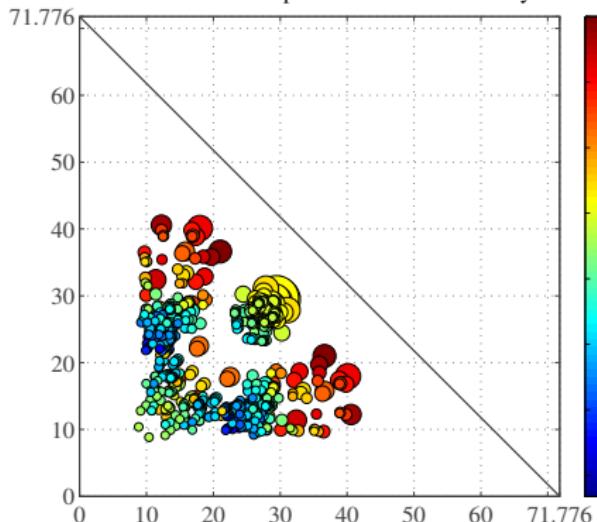
- ▶ The (convex hull of the) set of the expected discounted equilibrium payoffs at the apex state  $(c_0, c_0, c_0) \in S$  is a triangle
- ▶ The vertices of this triangle are at the points  $(0, 0)$ ,  $(0, V_M)$  and  $(V_M, 0)$  where  $V_M = v_{N,i}(c_0, c_0, c_0)$  is the expected discounted payoff to firm  $i$  in the monopoly equilibrium where firm  $i$  is the monopolist investor.

## Pay-off map

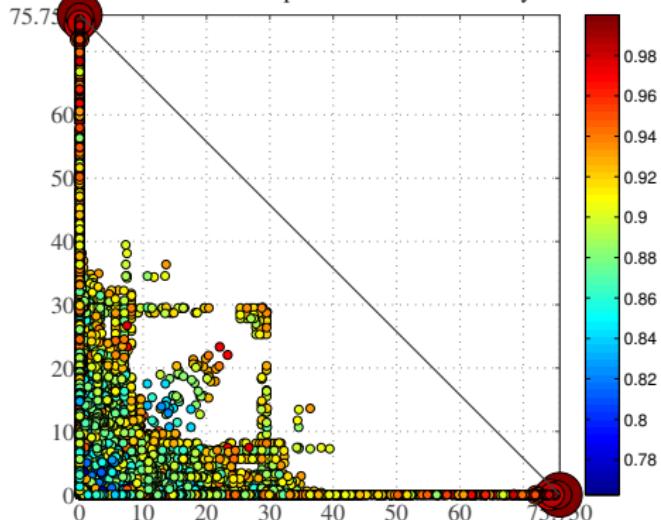


# Pay-offs: alternating vs simultaneous move games

Panel (a): Non-monotonic tech. progress  
17826 equilibria, 792 distinct pay-off points  
Size: number of repetitions Color: efficiency



Panel (b): Simultaneous move  
28528484 equilibria, 16510 distinct pay-off points  
Size: number of repetitions Color: efficiency



# Efficiency of equilibria

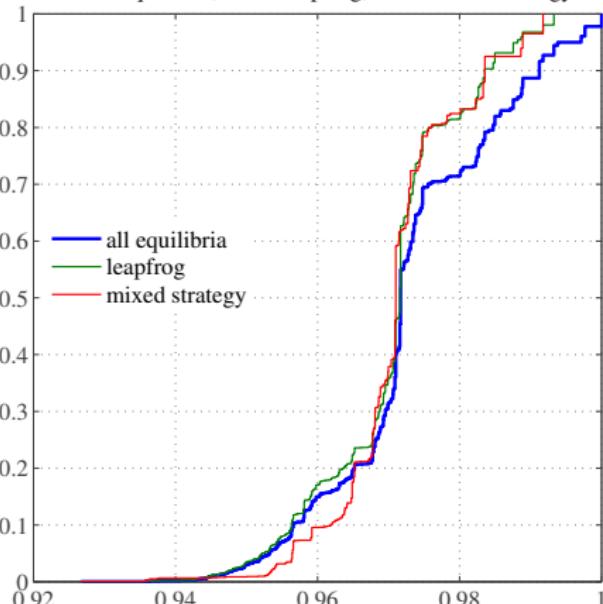
Simultaneous move game

## Theorem (Inefficiency of mixed strategy equilibria)

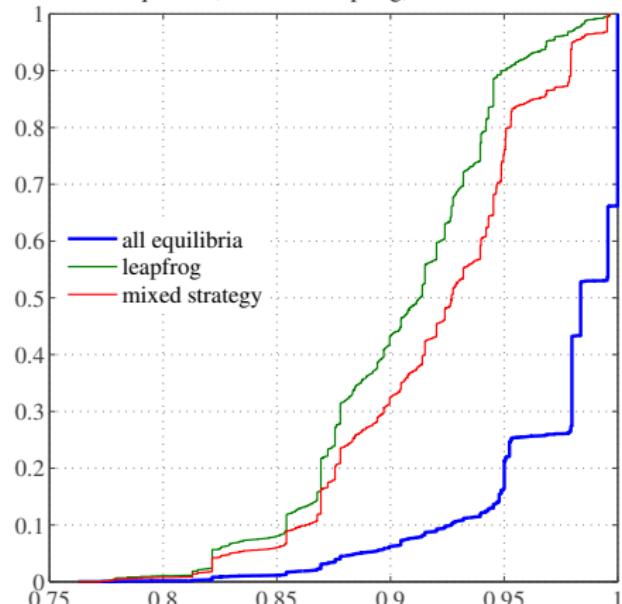
*A necessary condition for efficiency in the dynamic Bertrand investment and pricing game is that along MPE path only pure strategy stage equilibria are played.*

# Efficiency: alternating vs simultaneous move games

Panel (c): Non-monotonic tech. progress  
8913 equilibria, 7817 leapfrog, 2752 mixed strategy



Panel (d): Simultaneous move  
14264242 equilibria, 2040238 leapfrog, 2730910 mixed strategy



## Riordan and Salant: Full Preemption

### Theorem (Riordan and Salant, 1994)

*The continuous time investment game where*

1. *right to move alternates deterministically.*
2.  $K(c) = K$  *and is not prohibitively high.*
3. *technological progress is deterministic:  $c(t)$  is a continuous, decreasing function*

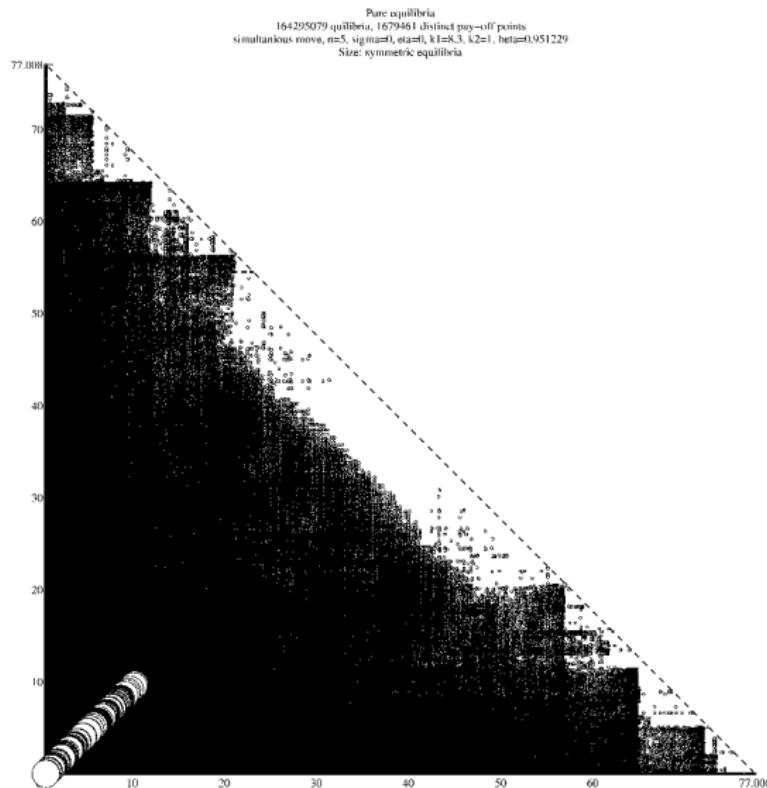
*has a unique MPE with*

- ▶ *preemptive investments: by only one firm and no investment in equilibrium by its opponent.*
- ▶ *rent dissipation: discounted payoffs of both firms in equilibrium is 0, so the entire surplus is wasted on excessively frequent investments by the preempting firm.*

We show by computing examples and counterexamples

1. Confirm R&S the result with high  $K$  and small  $dt$
2. Underinvestment: Rent dissipation is not a general outcome - disappears when  $K$  is low relative  $dt$
3. Leapfrogging: Preemption is not the general outcome - disappears when  $K$  is even lower
4. Random move alternation → Leapfrogging
5. Random onestep technology → Leapfrogging
6. Random multistep technology → Leapfrogging
7. Simultaneous moves → Leapfrogging

Symmetric equilibria:  $V_1(c_1, c_2, c) = V_2(c_2, c_1, c)$



# Failure of homotopy approach

## Homotopy parameter: $\eta$

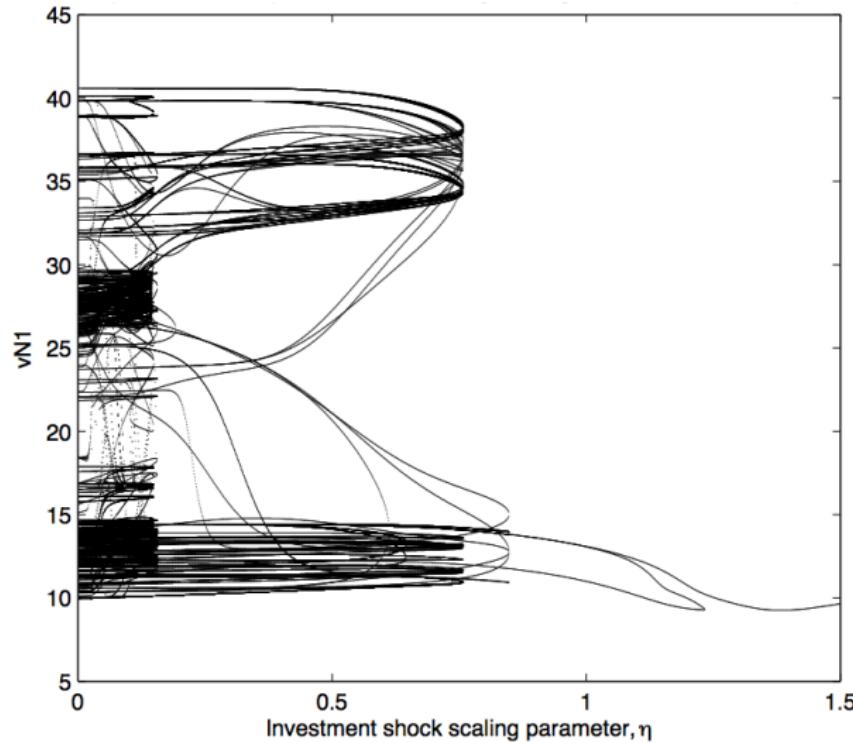
- ▶ In each period each firm incurs additive random costs/benefit from not investing and investing
- ▶  $\eta$  is a scaling parameter that index variance of idiosyncratic shocks to investment
- ▶ High  $\eta \rightarrow$  unique equilibrium  $\eta \rightarrow 0 \rightarrow$  multiple equilibria

## Problems:

- ▶ Multiplicity of equilibria  $\rightarrow$  too many bifurcations along the path
- ▶ Equilibrium correspondence is not lower hemi-continuous

# Failure of homotopy approach

Equilibrium correspondance, alternating move game:  $V_{N,1}(c_0, c_0, c_0)$  vs.  $\eta$



Estimation of directional dynamic games:  
Full solution nested MLE estimation

Nested Recursive Lexicographic Search  
algorithm

# Markov Perfect Equilibria

- ▶ MPE is a pair of **strategy profile** and **value functions**
- ▶ In compact notation

$$\begin{aligned}V &= \Psi^V(V, P, \theta) \\P &= \Psi^P(V, P, \theta)\end{aligned}$$

- ▶ Set of all Markov Perfect Equilibria

$$SOL(\Psi, \theta) = \left\{ (P, V) \mid \begin{array}{l} V = \Psi^V(V, P, \theta) \\ P = \Psi^P(V, P, \theta) \end{array} \right\}$$

- ▶  $\Psi^V : V, P \rightarrow V$  Bellman operator
- ▶  $\Psi^P : V, P \rightarrow P$  Choice probability formulas (logit)
- ▶  $\Gamma : P \rightarrow V$  Hotz-Miller inversion

# Estimation methods for *dynamic stochastic games*

## ► Two step (CCP) estimators

- Fast, potentially large finite sample biases

 Hotz, Miller (1993); Altug, Miller (1998); Pakes, Ostrovsky, and Berry (2007); Pesendorfer, Schmidt-Dengler (2008)

1. Estimate CCP  $\rightarrow \hat{P}$
2. Method of moments • Minimal distance • Pseudo likelihood

$$\min_{\theta} [\hat{P} - \Psi^P(\Gamma(\theta, \hat{P}), \hat{P}, \theta)]' W [\hat{P} - \Psi^P(\Gamma(\theta, \hat{P}), \hat{P}, \theta)] \\ \max_{\theta} \mathcal{L}(Z, \Psi^P(\Gamma(\theta, \hat{P}), \hat{P}, \theta))$$

## ► Nested pseudo-likelihood (NPL)

- Recursive two step pseudo-likelihood
- Bridges the gap between efficiency and tractability
- Unstable under multiplicity

 Aguirregabiria, Mira (2007); Pesendorfer, Schmidt-Dengler (2010); Kasahara and Shimotsu (2012); Aguirregabiria, Marcoux (2021)

# Estimation methods for *dynamic stochastic games*

- ▶ Equilibrium inequalities (BBL)

- ▶ Minimize the one-sided discrepancies
- ▶ Computationally feasible in large models

 Bajari, Benkard, Levin (2007)

- ▶ Math programming with equilibrium constraints (MPEC)

- ▶ MLE as constrained optimization
- ▶ Does not rely on the structure of the problem
- ▶ Much bigger computational problem

 Su (2013); Egesdal, Lai and Su (2015)

$$\max_{(\theta, P, V)} \mathcal{L}(Z, P) \text{ subject to } V = \Psi^V(V, P, \theta), P = \Psi^P(V, P, \theta)$$

- ▶ All solution homotopy MLE

 Borkovsky, Doraszelsky and Kryukov (2010)

## Overview of NRLS

- ▶ Robust and *computationally feasible*<sup>(?)</sup> MLE estimator for directional dynamic games (DDG)
- ▶ Rely of full solution algorithm that provably computes all MPE under certain regularity conditions
- ▶ Employ smart discrete programming method to maximize likelihood function over the finite set of equilibria
- ▶ Fully robust to multiplicity of MPE
- ▶ Relax single-equilibrium-in-data assumption
- ▶ Path to estimation of equilibrium selection rules

# Nested Recursive Lexicographical Search (NRLS)

- ▶ Data from  $M$  independent markets from  $T$  periods

$$Z = \{\bar{a}^{mt}, \bar{x}^{mt}\}_{m \in \mathcal{M}, t \in \mathcal{T}}$$

Usually assume only one equilibrium is played in the data.

- ▶ Denote  $\theta$   $(P^\ell(\theta), V^\ell(\theta)) \in SOL(\Psi, \theta)$  the  $\ell$ -the equilibrium

## 1. Outer loop

Maximization of the likelihood function w.r.t. to structural parameters  $\theta$

$$\theta^{ML} = \arg \max_{\theta} \mathcal{L}(Z, \theta)$$

## 2. Inner loop

Maximization of the likelihood function w.r.t. equilibrium selection

$$\mathcal{L}(Z, \theta) = \max_{(P^\ell(\theta), V^\ell(\theta)) \in SOL(\Psi, \theta)} \frac{1}{M} \sum_{i=1}^N \sum_{m=1}^M \sum_{t=1}^T \log P_i^\ell(\bar{a}_i^{mt} | \bar{x}^{mt}; \theta)$$

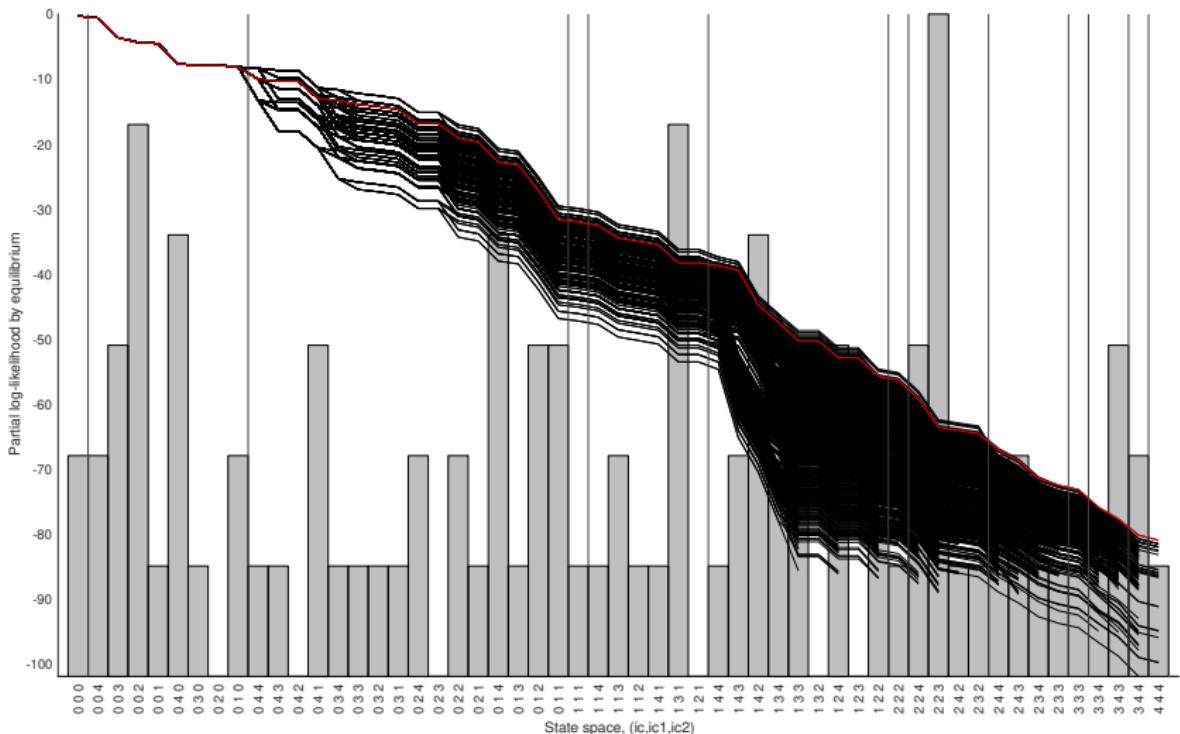
Max of a function on a discrete set organized into RLS tree

## Branch and bound (BnB) method

 Land and Doig, 1960 *Econometrica*

- ▶ Old method for solving **discrete programming** problems
- 1. Form a **tree** of subdivisions of the set of admissible plans
- 2. Specify a **bounding function** representing the best attainable objective on a given subset (branch)
- 3. Dismiss the subsets of the plans where the bound is below the current best attained value of the objective

# BnB and numerical performance of NRLS estimator



## Numerical performance and refinements of NRLS estimator

- ▶ Numerical performance of NRLS estimator depends crucially on how the data is able to distinguish between different equilibria
- ▶ Bounding criterion is **deterministic** → may use **statistical criterion** to decide whether to extend a given branch or not
- ▶ Have to assess **potential likelihood contribution** of the branches that are not fully extended → Vuong closeness test (LR-type test to assess how different two equilibria are given already computed partial likelihood)  
⇒ **Poly-algorithm** with statistical decision rule

# Monte Carlo simulations

A

---

Single equilibrium in the model  
Single equilibrium in the data

B

---

Multiple equilibria in the model  
Single equilibrium in the data

C

---

Multiple equilibria in the model  
Multiple equilibria in the data

1. Two-step CCP estimator
2. Nested pseudo-likelihood vs. NRLS estimator
3. MPEC

# Implementation details

- ▶ Two-step estimator and NPL
  - ▶ Matlab unconstraint optimizer (numerical derivatives)
  - ▶ CCPs from frequency estimators
  - ▶ For NPL max 30 iterations
- ▶ MPEC
  - ▶ Matlab constraint optimizer (interior-point algorithm)
  - ▶ MPEC-VP: Constraints on both values and choice probabilities (as in Egesdal, Lai and Su, 2015)
  - ▶ MPEC-P: Constraints in terms of choice probabilities + Hotz-Miller inversion
  - ▶ Starting values from two-step estimator
- ▶ Estimated parameters  $\theta = (k_1, k_2)$
- ▶ Sample size: 1000 markets in 5 time periods
- ▶ Initial state drawn uniformly over the state space

## Monte Carlo A, run 1: no multiplicity

Maximum number of equilibria in the model: 1

Number of equilibria in the data: 1

	PML2step	NPL	MPEC-VP	MPEC-P	NRLS
k1=3.5	3.51893	3.51022	3.50380	3.50380	3.50380
Bias	0.01893	0.01022	0.00380	0.00380	0.00380
MCSD	0.12087	0.12635	0.11573	0.11573	0.11573
k2=0.5	0.50860	0.50658	0.50452	0.50452	0.50452
Bias	0.00860	0.00658	0.00452	0.00452	0.00452
MCSD	0.06460	0.06247	0.05939	0.05939	0.05939
log-likelihood	-1958.176	-1953.406	-1953.327	-1953.327	-1953.327
$\ \Psi^P(P) - P\ $	0.25285	0.00001	0.00000	0.00000	0.00000
$\ \Psi^V(v) - v\ $	0.50038	0.00001	0.00000	0.00000	0.00000
Converged,%	100	100	100	100	100
K-L divergence	0.131139	0.005020	0.006770	0.006770	0.006770

- ▶ All MLE estimators identical to the last digit
- ▶ NPL estimator is approaching MLE

## Monte Carlo A, run 2: no multiplicity at true parameter

Maximum number of equilibria in the model: 3

Number of equilibria at true parameter value: 1

Number of equilibria in the data: 1

	PML2step	NPL	MPEC-VP	MPEC-P	NRLS
k1=3.5	3.50467	3.51307	3.49485	3.49318	3.49318
Bias	0.00467	0.01307	-0.00515	-0.00682	-0.00682
MCSD	0.11252	0.00000	0.10193	0.10177	0.10177
k2=0.5	0.50035	0.47394	0.50265	0.50157	0.50157
Bias	0.00035	-0.02606	0.00265	0.00157	0.00157
MCSD	0.05009	0.00000	0.04154	0.04205	0.04205
log-likelihood	-4106.771	-3940.158	-4091.873	-4093.040	-4093.04
$\ \Psi^P(P) - P\ $	0.41453	0.00001	0.00000	0.00000	0.00000
$\ \Psi^V(v) - v\ $	1.90182	0.00005	0.00000	0.00000	0.00000
Converged,%	100	1	98	100	100
K-L divergence	0.188551	0.004546	0.002921	0.002921	0.002920

- ▶ NPL estimator fails to converge
- ▶ MPEC is not affected by "nearby" equilibria with good starting values (PML2step)

## Monte Carlo B, run 1: moderate multiplicity

Number of equilibria in the model (at true parameter): 3

Number of equilibria in the data: 1

	PML2step	NPL	MPEC-VP	MPEC-P	NRLS
k1=3.5	3.50081	-	3.72713	3.94941	3.49624
Bias	0.00081	-	0.22713	0.44941	-0.00376
MCSD	0.12050	-	0.85934	1.16633	0.09537
k2=0.5	0.49478	-	0.56166	0.62361	0.49381
Bias	-0.00522	-	0.06166	0.12361	-0.00619
MCSD	0.04317	-	0.25552	0.32488	0.03510
log-likelihood	-4070.035	-	-4080.989	-4121.102	-4049.647
$\ \Psi^P(P) - P\ $	0.50375	-	0.00000	0.00000	0.00000
$\ \Psi^V(v) - v\ $	2.83611	-	0.00000	0.00000	0.00000
Converged,%	100	0	100	100	100
K-L divergence	0.304411	-	0.018636	2.302525	0.006314

- ▶ NPL estimator fails to converge
- ▶ MPEC fails to identify the equilibrium that generated the data (converges to a different MPE) as seen from MCSD and K-L divergence

## Monte Carlo B, run 2: higher multiplicity

Number of equilibria in the model (at true parameter): 81

Number of equilibria in the data: 1

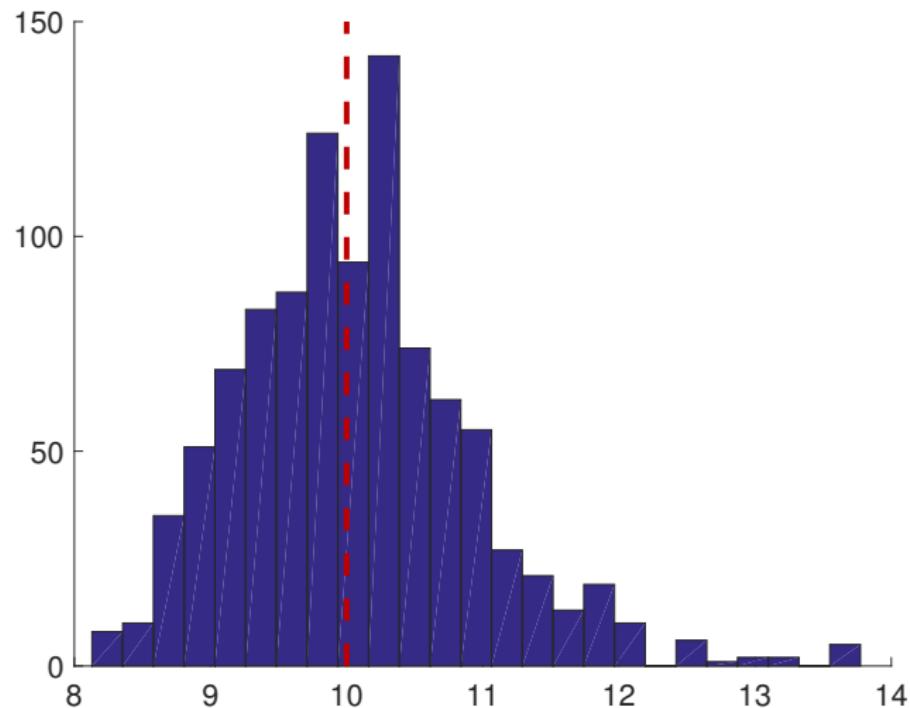
	PML2step	NPL	MPEC-VP	MPEC-P	NRLS
k1=3.5	3.51468	-	3.48740	3.49007	3.47786
Bias	0.01468	-	-0.01260	-0.00993	-0.02214
MCSD	0.04844	-	0.02802	0.02929	0.02731
k2=0.5	0.53780	-	0.49197	0.48944	0.49252
Bias	0.03780	-	-0.00803	-0.01056	-0.00748
MCSD	0.03894	-	0.00850	0.01033	0.00404
log-likelihood	-4038.78471	-	-4007.45663	-4010.18139	-3996.45223
$\ \Psi^P(P) - P\ $	0.68907	-	0.00000	0.00000	0.00000
$\ \Psi^V(v) - v\ $	5.44052	-	0.00000	0.00000	0.00000
Converged,%	100	0	100	100	100
K-L divergence	0.453917	-	0.278263	0.356678	0.000750

- ▶ NPL estimator fails to converge
- ▶ MPEC fails to identify the DGP equilibrium (converges to a different MPE)
- ▶ With good starting values, does not suffer more with higher multiplicity

## NRLS Monte Carlo setup (C)

- ▶  $n = 3$  points on the grid on the grid of costs
- ▶ 14 points in state space of the model
- ▶ 109 MPE in total
- ▶ 1000 random samples from 3 different equilibria (3 markets)
- ▶ 100 observations per market/equilibrium
- ▶ Uniform distribution over state space ↔ “ideal” data
- ▶ Estimating one parameter in cost function

## Distribution of estimated $k_1$ parameter



## MC results and numerical performance of NRLS

1. Average bias and RMSE of the estimates of the cost of investment parameter (**true value is 10.0**)

Bias = **0.0737**

RMSE = **0.8712**

2. Average fraction of MPE computed by BnB relative to RLS

**0.321** (std=0.11635)

3. Average fraction of stages solved by BnB relative to RLS

**0.263** (std=0.09725)

4. All 3 MPE correctly identified by BnB in

**98.4%** of runs

## Identification of multiple equilibria in the data (C)

- ▶ 100 random samples
- ▶ 3 market clusters with different equilibria
- ▶ 1000 observations per market cluster/equilibrium in 3 time periods
  
- ▶ Among all runs, 93% of equilibria were pin-pointed exactly
- ▶ Among the misidentified equilibria, all had deviation in one point of the state space

## Conclusions: Bertrand investments model

- ▶ Many types of endogenous coordination is possible in equilibrium
  - ▶ Leapfrogging (alternating investments)
  - ▶ Preemption (investment by cost leader)
  - ▶ Duplicative (simultaneous investments)
- ▶ Full rent dissipation and monopoly outcomes are supported as MPE.
- ▶ Numerous MPE equilibria and “Folk theorem”-like result
- ▶ The equilibria are generally inefficient due to over-investment
  - ▶ Duplicative or excessively frequent investments

## Conclusions: Solution of dynamic games

- ▶ When equilibrium is not unique the computation algorithm inadvertently acts as an **equilibrium selection mechanism**
- ▶ When directionality in the state space is present, state recursion algorithm is preferred to time iterations
- ▶ Plethora of Markov perfect equilibria poses new challenges:
  - ▶ How firms manage to coordinate on a particular equilibrium?
  - ▶ Increased difficulties for empirical applications.
  - ▶ Daunting perspectives for identification of equilibrium selection rule from the data.
- ▶ **Estimation of dynamic games with multiple equilibria**  
Nested Recursive Lexicographical Search (NRLS)

## Contributions and further developments

- ▶ NRLS is MLE estimator for dynamic games of a particular type, directional dynamic games (DDGs)
  - ▶ Fully robust to multiplicity of equilibria
  - ▶ Able to identify multiple equilibria in the data
- ▶ Further work on and tests of numerical performance
  - ▶ Refinements of the implementation of NRLS (optimization of BnB algorithm)
  - ▶ Statistical bounding criterion
- ▶ More detailed comparison of existing estimators using leapfrogging game
  - ▶ Refine the implementation of MPEC
  - ▶ Include recent estimators into the battery (Aguirregabiria and Marcoux, 2019, Bugni and Bunting, 2020)