4. $\frac{\bar{X}-\mu}{S/\sqrt{n}}$ ← We want to identify the distribution of this statistic.

$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$\sqrt{\frac{(n-1)S^2/(\sigma^2)}{(n-1)}}$ ← we know $\sim \chi^2(n-1)$

Def (Student's t-distribution)

Let $Z \sim N(0,1)$, $V \sim \chi^2(r)$, independent (i.e. $Z \perp V$)

↑ this is very important

and $T = \frac{Z}{\sqrt{V/r}}$

We call t a Student's t-distribution, with degree of freedom r denoted $t(r)$

To find pdf of this distribution:

Range of t: $(-\infty, \infty)$. Range of V: $(0, \infty)$

Joint Density of (Z, V)

$$\delta(z,u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{1}{\Gamma(\frac{r}{2}) 2^{r/2}} u^{\frac{r}{2}-1} e^{-\frac{u}{2}}$$

$z \in (-\infty, \infty)$, $u \in (0, \infty)$