$(x_i - \theta)$

$\alpha\theta^2$

The joint density for $Y_1 = \bar{X}$, $Y_i = X_i - \bar{X}$, $2 \le i \le n$

Jacobian, which is a constant since trans. is linear

$$g(y_1, y_2, \dots, y_n) = \frac{K}{(2\pi\sigma)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\left(\left(-\sum y_i\right)^2 + \sum_{i=2}^{n} y_i^2 + n(y_1-\mu)^2\right)\right]$$

$$= \frac{K}{(2\pi\sigma)^{n/2}} \exp\left[-\frac{1}{2\sigma^2} n(y_1-\mu)^2\right] \exp\left[-\frac{1}{2\sigma^2}\left(\quad\right)\right]$$

y_i $2 \le i \le n$

Remark:

Why take $S^2 = \frac{1}{(n-1)} \sum (x_i - \bar{x})^2$?

$(n-1)$?

Consider $E[S^2]$.

We know $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

Note if $V \sim \chi^2(r)$, then $E(V) = r$ and $Var(V) = 2r$

Recall if
$X \sim GAM(\alpha, \theta)$
$E[X] = \alpha\theta$
$Var(X) = \alpha\theta^2$
$X \sim \chi^2(r) = GAM(\frac{r}{2}, 2)$
$E[X] = r$
$Var(X) = 2r$

So $E[S^2] = E\left[\frac{\sigma^2}{n-1} \frac{n-1}{\sigma^2} S^2\right]$

$$= \frac{\sigma^2}{n-1} E\left[\left(\frac{n-1}{\sigma^2}\right) S^2\right] = \frac{\sigma^2}{n-1} E\left[\frac{1}{\sigma^2} \sum (x_i - \bar{x})^2\right]$$

$$= \frac{\sigma^2}{(n-1)} (n-1) = \sigma^2$$

↑ "unbiased"