Example of application of CLT to Analysis

Show that

1). $\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = 1/2$

2) $\lim_{n\to\infty} \int_0^n \frac{e^{-t} t^{n-1}}{(n-1)!}\, dt = 1/2$

Proof of 1)

Let $X_1, X_2, \ldots X_n$ be a random sample from $POI(1)$

and $Y_n = X_1 + X_2 + \cdots + X_n \sim POI(n)$.

By CLT,

$$W_n = \frac{Y_n - n}{\sqrt{n}} \xrightarrow{d} Z \sim N(0,1)$$

$$P\left[\frac{Y_n - n}{\sqrt{n}} \leq 0\right] = P\left[Y_n \leq n\right]$$

and this equals $1/2$, which can be seen from CDF of Poisson

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{e^{-n} n^k}{k!}$$