Sketch of the proof for CLT $\left(\begin{matrix}\text{When we have}\\ M_n(t)\end{matrix}\right)$.

Consider $X_1, X_2, \ldots, X_n \sim Exp(\theta)$ [Note: $\mu = \theta = \sigma$]

$$W_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{\sum^n x_i - n\mu}{\sigma(\sqrt{n})} = \frac{\sum x_i - n\theta}{(\sqrt{n})\theta}$$

For each i, $X_i \sim Exp(\theta)$, $M_i(t) = (1-\theta t)^{-1}$

$$\sum_{i=1}^n x_i \sim M(t) = (1-\theta t)^{-n}$$

$$W_n = \frac{\sum x_i - n\theta}{(\sqrt{n})\theta}$$

$$M_{W_n}(t) = E\left[e^{tW_n}\right] = E\left[e^{t\left(\frac{\sum x_i - n\theta}{\sqrt{n}\theta}\right)}\right]$$

$$= E\left[e^{\frac{t\sum x_i}{\sqrt{n}\theta}} e^{-\frac{tn\theta}{\sqrt{n}\theta}}\right] = E\left[e^{\frac{t\sum x_i}{\sqrt{n}\theta}} e^{-t\sqrt{n}}\right]$$

$$= e^{-t\sqrt{n}} E\left[e^{\frac{t\sum x_i}{\sqrt{n}\theta}}\right] = e^{-t\sqrt{n}} \frac{1}{\left(1-\theta\cdot\left(\frac{t}{\sqrt{n}\theta}\right)\right)^n}$$

$$= e^{-t\sqrt{n}} \cdot \left(\frac{1}{1-\frac{t}{\sqrt{n}}}\right)^n = e^{-t\sqrt{n}}\left(1-\frac{t}{\sqrt{n}}\right)^{-n}$$

So $\ln M_{W_n}(t) = -\sqrt{n}\,t - n\ln\left(1-\frac{t}{\sqrt{n}}\right)$

$$= -\sqrt{n}\,t - n\left[-\left(\frac{t}{\sqrt{n}}\right) - \frac{\left(-\frac{t}{\sqrt{n}}\right)^2}{2} + \frac{\left(-t/\sqrt{n}\right)^3}{3} - \frac{\left(-t/\sqrt{n}\right)^4}{4} + \cdots\right]$$

$$= -\sqrt{n}\,t + \left[\sqrt{n}\,t + \frac{n\left(\frac{t^2}{n}\right)}{2} + \frac{n\left(\frac{t^3}{n(\sqrt{n})}\right)}{3} + \frac{n\left(\frac{t^4}{n^2}\right)}{4} + \cdots\right]$$