Math 562 (Math. Stats)

Example 1:

$$\int \frac{1}{\theta} e^{-x/\theta}$$
$$\frac{1}{\theta}\int e^{-x/\theta}$$

1) $X \sim Exp(\theta)$

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0$$

and $Y = \ln X$.

Want to find pdf of Y.

Start by identifying domain: $Dom(Y) = (-\infty, \infty)$

$\ln(x) \in (-\infty, \infty), \ \forall x > 0$

Then for any $y \in (-\infty, \infty)$

$$P[Y \leq y] = P[\ln X \leq y]$$
$$= P[X \leq e^y]$$
$$= 1 - e^{(-1/\theta)(e^y)}$$

$$f_Y(y) = -e^{-1/\theta e^y}\left(-\frac{1}{\theta}e^y\right)$$
$$= \left(\frac{1}{\theta}\right)\left(e^{-\frac{1}{\theta}e^y}\right)\left(e^y\right)$$

Example 2:

If we have $X_1, \dots, X_n \sim Exp(\theta)$, iid.

and $Y = \sum_{i=1}^{n} X_i$.

What is the distribution of Y?

(Rule of thumb: with sum of ind. variables, easiest to work with mgf)

$$M_Y(t) = E[e^{tY}] = E[e^{t\Sigma x_i}] = E[e^{tx_1 + tx_2 + \dots}]$$
$$= E[e^{tx_1} e^{tx_2} \cdots e^{tx_n}] = E[e^{tx_1}]E[e^{tx_2}]\cdots E[e^{tx_n}]$$
$$= (E[e^{tx_1}])^n \quad \text{(because iid)}$$
$$= \frac{1}{(1-\theta t)^n} \sim GAM(\theta, n)$$

(See ☆2)

For $X_1 \sim Exp(\theta)$

$$M(t) = E[e^{tX_1}] = \int_0^\infty e^{tx_1} f(x_1)\,dx$$

$$= \int_0^\infty (e^{tx_1})\left(\frac{1}{\theta}\right)\left(e^{-\frac{x_1}{\theta}}\right) dx_1$$

☆2 (Derivation of MGF for Exponential distribution)

$$= \int_0^\infty \frac{1}{\theta} e^{-(1-\theta t)\frac{x_1}{\theta}}\, dx_1$$

$\left(s = \frac{x_1}{\theta} \quad ds = \frac{1}{\theta} dx_1 \right)$

$$= \int_0^\infty e^{-(1-\theta t)s}\, ds$$

$$= \frac{1}{1-\theta t}\left[-e^{-(1-\theta t)s}\right]_0^\infty$$

$$= \frac{1}{1-\theta t}$$

What's Gamma?

$X \sim GAM(\alpha, \theta)$,

if $f(\alpha) = \frac{1}{\Gamma(\alpha)\theta^\alpha}(x^{\alpha-1})(e^{-\frac{x}{\theta}})$

$\Gamma(\alpha)$ gamma function

$$M(t) = \frac{1}{(1-\theta t)^\alpha}$$

(when $\alpha = 1$, $Exp(\theta) = GAM(1, \theta)$)

$\Gamma(x) = \Gamma$

$\chi^2(r) \sim GAM(\frac{r}{2}, 2)$

$$M(t) = \frac{1}{(1-2t)^{r/2}}$$

$Y \sim GAM(\alpha, \theta)$

$X = \frac{2Y}{\theta} \sim \chi^2(2\alpha)$

$$E\left[e^{t(\frac{2Y}{\theta})}\right] = \frac{1}{(1-\theta(\frac{2t}{\theta}))^\alpha} = \frac{1}{(1-2t)^\alpha}$$

$\alpha = r/2$

$r = 2\alpha$

Example 3

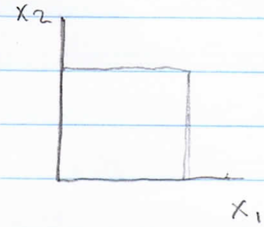
$X_1, X_2 \sim UNIF(0,1)$ and $X_1 \perp X_2$.

Consider $Y = X_1 + X_2$.

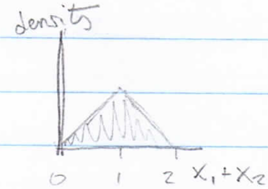
Determine the pdf of Y.

$Y \in (0,2)$. For $t \in (0,2)$,

$$P[Y \le t] = P[X_1 + X_2 \le t]$$



pdf of $X_1 + X_2$



imagine what happens as we increase n where we have $\sum_{i=1}^{n} X_i$

Example 4 Back to $N(\mu, \sigma^2)$

Take a random sample

$$X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2).$$

Then $\bar{X} = \frac{1}{n} \sum X_i \sim N(\quad , \quad)$

$$M_{\bar{X}}(t) = E\left[e^{t\bar{X}}\right] = E\left[e^{(1/n)(t)(x_1)} \cdots e^{(\frac{1}{n})(t)(x_n)}\right]$$

$$= E\left[e^{(1/n)(t)(x_1)}\right] \cdots E\left[e^{(1/n)(t)(x_1)}\right]$$

$$= \left(E\left[e^{(\frac{1}{n})(t)(x_1)}\right]\right)^n$$

$$= \left[e^{\mu(\frac{t}{n}) + \frac{\sigma^2 (\frac{t}{n})^2}{2}}\right]^n = e^{\left(\mu t + \frac{\left(\frac{\sigma^2}{n} \cdot t^2\right)}{2}\right)}$$

$$\sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Let's Standardize!

$$W_n = \frac{\bar{X} - \mu}{(\sigma/\sqrt{n})} \sim N(0,1)$$

$$\text{"lim"}(W_n) = Z \sim N(0,1)$$

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this is shorthand for i.i.d.

$\boxed{\text{The CLT}}$: Observe a random sample $X_1, X_2, \ldots, X_n$ from a population with mean μ and variance $\sigma^2 < \infty$.

Let $W_n = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \dfrac{(\sum X_i) - (n\mu)}{(\sqrt{n})(\sigma)}$

Then $W_n \xrightarrow{d} Z \sim N(0,1)$ as $n \to \infty$.

Note

1) "$\xrightarrow{d}$" converges in dist. means that

$$\lim_{n\to\infty} G_n(y) = G(y) \quad \text{for all } y$$

(i.e. the sequence of function converges pointwise)

2) If $M_n(t) \to M(t)$ on $(-\delta, \delta)$,
then $W_n \xrightarrow{d} W$

☆ Q1 What's "lim"?

Q2 What if X_i are not normal?

Note: For any distribution, $E[\bar{x}] = \mu$, and $Var(\bar{x}) = \frac{\sigma^2}{n}$
where $\bar{x}$ is the mean of sample size n.

Hence $E[W_n] = 0$, and $Var(W_n) = 1$

" $\xrightarrow{d}$ "

1) $\lim_{n \to \infty} G_n(x) = \Phi(x)$ for all x.

2) If $\lim_{n \to \infty} M_n(t) = M(t)$ on $(-\delta, \delta)$, ← (for some $\delta > 0$)

then $W_n \xrightarrow{d} W$.

this is sufficient, but not necessary

3) When n is large,

$W_n \approx Z$

$\bar{X}_n \approx N(\mu, \frac{\sigma^2}{n})$

(Since $W_n \xrightarrow{d} Z \sim N(0,1)$

$\boxed{\text{Example 1}}$

Let $X_1, X_2, \ldots, X_{100}$ be 100 observations from $\text{Exp}(1)$,

and $Y = \sum_{i=1}^{100} X_i$

$\frac{1}{(1-t)^n}$

Note: $Y \sim GAM(100, 1)$

Estimate $P[Y > 110]$.

$$Y = \sum_{i=1}^{100} x_i$$

Note: $\frac{110-100}{100} 10 = \frac{10}{100} \cdot 10 = \frac{100}{100} = 1$

$$P[Y > 110] = P\left[\frac{Y}{100} > \frac{110}{100}\right] = P\left[\frac{\frac{Y}{100} - 1}{1/\sqrt{100}} > \frac{\frac{110}{100} - 1}{1/\sqrt{100}}\right]$$

$\approx Z$ by CLT

$$\approx P[Z > 1]$$
$$= 0.1587$$

Example 2

Let $X_1, X_2, \dots X_{15}$ be a random sample from the distribution with pdf

$$f(x) = \begin{cases} (3/2) x^2 & x \in (-1,1) \\ 0 & \text{otherwise} \end{cases}$$

Use CLT to estimate $P\left[-0.3 \le \sum_{i=1}^{15} X_i \le 1.5\right]$

$\mu = 0$

$$\sigma^2 = E[X^2] - E[x]^2$$

$$= \int_{-1}^{1} x^2 \left(\tfrac{3}{2}\right) x^2 \, dx - 0^2 = \int_{-1}^{1} \left(\tfrac{3}{2}\right) x^4 \, dx = \frac{3}{2}\left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{3}{2}\left(\frac{1}{5} + \frac{1}{5}\right)$$

$$= \frac{3}{5}$$

(and so $\sigma = \sqrt{3/5} = \sqrt{0.6}$

So

$$P\left[-0.3 \le \sum_{i=1}^{15} X_i \le 1.5\right]$$

$$= P\left[\frac{-0.3}{\sqrt{15(0.6)}} \le \frac{\sum_{i=1}^{n} X_i - 0}{\sqrt{15 \cdot (0.6)}} \le \frac{1.5}{\sqrt{15 \cdot (0.6)}}\right]$$

(Recall $\dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \dfrac{\sum^n x_i - n\mu}{\sigma(\sqrt{n})}$

CLT

$$\approx P[-0.1 \le Z \le 0.5] = 0.6915 + 0.5398 - 1$$

$$= 0.2313$$

Sketch of the proof for CLT $\left(\begin{matrix}\text{When we have}\\ M_n(t)\end{matrix}\right)$.

Consider $X_1, X_2, \ldots, X_n \sim Exp(\theta)$ [Note: $\mu = \theta = \sigma$]

$$W_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{\sum^n x_i - n\mu}{\sigma(\sqrt{n})} = \frac{\sum x_i - n\theta}{(\sqrt{n})\theta}.$$

For each i, $X_i \sim Exp(\theta)$, $M_i(t) = (1-\theta t)^{-1}$

$$\sum_{i=1}^n x_i \sim M(t) = (1-\theta t)^{-n}$$

$$W_n = \frac{\sum x_i - n\theta}{(\sqrt{n})\theta}$$

$$M_{W_n}(t) = E[e^{tW_n}] = E\left[e^{t\left(\frac{\sum x_i - n\theta}{\sqrt{n}\theta}\right)}\right]$$

$$= E\left[e^{\frac{t\sum x_i}{\sqrt{n}\theta}} e^{-\frac{tn\theta}{\sqrt{n}\theta}}\right] = E\left[e^{\frac{t\sum x_i}{\sqrt{n}\theta}} e^{-t\sqrt{n}}\right]$$

$$= e^{-t\sqrt{n}} E\left[e^{\frac{t\sum x_i}{\sqrt{n}\theta}}\right] = e^{-t\sqrt{n}} \frac{1}{\left(1-\theta\cdot\left(\frac{t}{\sqrt{n}\theta}\right)\right)^n}$$

$$= e^{-t\sqrt{n}} \cdot \left(\frac{1}{1-\frac{t}{\sqrt{n}}}\right)^n = e^{-t\sqrt{n}}\left(1-\frac{t}{\sqrt{n}}\right)^{-n}$$

So $\ln M_{W_n}(t) = -\sqrt{n}\,t - n\ln\left(1-\frac{t}{\sqrt{n}}\right)$

$$= -\sqrt{n}\,t - n\left[-\left(\frac{t}{\sqrt{n}}\right) - \frac{\left(-\frac{t}{\sqrt{n}}\right)^2}{2} + \frac{\left(-t/\sqrt{n}\right)^3}{3} - \frac{\left(-t/\sqrt{n}\right)^4}{4} + \ldots\right]$$

$$= -\sqrt{n}\,t + \left[\sqrt{n}\,t + \frac{n\left(\frac{t^2}{n}\right)}{2} + \frac{n\left(\frac{t^3}{n(\sqrt{n})}\right)}{3} + \frac{n\left(\frac{t^4}{n^2}\right)}{4} + \ldots\right]$$

$$\star\left(\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right)$$

$$= 0 + \frac{t^2}{2} + \frac{1}{\sqrt{n}}\left[\frac{t^3}{3} + \frac{t^4}{(\sqrt{n})4} + \dots\right]$$

this is a convergent series

Recall if n is large, then $\bar{X} \approx N(\mu, \sigma^2/\sqrt{n})$

Q So what if $n \to \infty$?

In some sense, $\bar{X}$ "converges" to the single point μ.

LLN Laws of Large Numbers

1. Modes of convergence

1) Convergence in distribution

$$\lim_{n\to\infty} F_n(x) = F(x) \text{ for all points of continuity } x \text{ for } F(x).$$

2) Convergence in probability

For any $\varepsilon > 0$,

$$\lim_{n\to\infty} P[\,|Y_n - Y| < \varepsilon\,] = 1$$

3) Convergence almost surely

$$P\left[\omega \,\middle|\, \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\right] = 1$$

Strength:

Weaker — Convergence in distribution ← (often called "weak convergence")

Convergence in probability

Stronger — Convergence almost surely

<u>Remark</u> convergence almost surely
⇒ convergence in probability
⇒ convergence in distribution

<u>Def</u> $\boxed{\text{Stochastic Convergence}}$

$Y_n \longrightarrow c$ stochastically if $Y_n \xrightarrow{d} Y$ in probability where $P[Y=c]=1$.

$$\left(\text{i.e. } \forall \varepsilon > 0 \ \lim_{n\to\infty} P\left[|Y_n - c| < \varepsilon \right] = 1 \right)$$

Equivalently, $\lim_{n\to\infty} M_n(t) = e^{ct}$ on $(-\delta, \delta)$ (for some $\delta > 0$)

Laws of Large Numbers

1) Bernoulli Law of Large Numbers

$$X_1, X_2, \dots, X_n \sim BER(p) = BIN(1, p)$$

$$Y_n = \sum_{i=1}^{n} X_i \sim BIN(n, p)$$

$(q = 1-p)$

$$\bar{X} = \frac{1}{n} Y_n$$

$$M_{\bar{X}}(t) = E\left[e^{t \frac{1}{n} \sum x_i}\right] = \left(M\left(\tfrac{t}{n}\right)\right)^n = \left(pe^{\frac{t}{n}} + q\right)^n$$

$$= \left[p\left(1 + \frac{\left(\frac{t}{n}\right)^2}{2!} + \frac{\left(\frac{t}{n}\right)^3}{3!} + \dots\right) + q\right]^n$$

$$= \left(1 + p\left(\tfrac{t}{n}\right) + p\frac{\left(\frac{t}{n}\right)^2}{2!} + \dots\right)^n$$

$d(n) \to 0$ as $n \to \infty$

$$= \left[1 + p\left(\tfrac{t}{n}\right) + \frac{d(n)}{2}\right]^n$$

and

$$\left[1 + p\left(\tfrac{t}{n}\right) + \frac{d(n)}{2}\right]^n \longrightarrow e^{pt}$$

* (ask for this identity)

So $\bar{X} \xrightarrow{p} p$

Stochastically

$$\left[1 + \frac{z}{n}\right]^n \to e^z$$

as $n \to \infty$

$$\bar{X} \approx N\left(p, \frac{\sigma^2}{n}\right)$$

$\searrow 0$

2) Weak Law of Large Numbers

Let $X_1, X_2, \cdots, X_n$ be a random sample

from $f(x)$ with μ and $\sigma^2 < \infty$

Then $\bar{X} \xrightarrow{p} \mu$ (conv. in probability)

(i.e. $\lim_{n\to\infty} P\left[|\bar{X}-\mu| > \varepsilon\right] \le \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0$)

3) Strong Law of Large Number

If μ exists,

$$P\left[\omega \,\middle|\, \lim_{n\to\infty} \bar{X}(\omega) = \mu\right] = 1$$

i.e. $\bar{X} \to \mu$ almost surely

We will prove the weak LLN, but the strong LLN cannot be proven in this course (it's very difficult)

Proof of Weak LLN

(1) Let X be an RV., u a nonnegative function and $c > 0$.

Then
$$P[u(X) \geq c] \leq \frac{E[u(X)]}{c}$$

We only consider X has a pdf $f(x)$ (i.e. we ignore discrete case, whose proof is similar)

$$E[u(X)] = \int_{-\infty}^{\infty} u(x) f(x)\,dx = \int_{\{u(x) \geq c\}} u(x) f(x)\,dx + \int_{\{u(x) < c\}} u(x) f(x)\,dx$$

$$\geq \int_{\{u(x) \geq c\}} u(x) f(x)\,dx \geq c \int_{\{u(x) \geq c\}} f(x)\,dx$$

$$= c P[u(x) \geq c]$$

And so
$$P[u(x) \geq c] \leq \frac{E[u(x)]}{c}$$

(2) Markov Inequality

$u(x) = |X|^r$ and consider c^r in (1)

$$P\left[|X|^r \geq c^r\right] \leq \frac{E[|X|^r]}{c^r}$$

$$P\left[|X| \geq c\right] \leq \frac{E[|X|^r]}{c^r}$$

(3) Chebyshev inequality

$$P\left[|X-\mu| \geq k\sigma\right] \leq \frac{1}{k^2}$$

$$u(x) = (x-\mu)^2, \quad c = k^2\sigma^2$$

$$P\left[(x-\mu)^2 \geq k^2\sigma^2\right] \leq \frac{E[(x-\mu)^2]}{k^2\sigma^2} = \frac{\sigma^2}{k^2\cancel{\sigma^2}}$$

$\hookrightarrow = P\left[|x-\mu| > k\sigma\right]$

Now $P\left[|\bar{X}-\mu| < \varepsilon\right] \geq 1 - \sigma^2/\varepsilon^2 n \xrightarrow[n\to\infty]{as} 1$

and $P\left[|X-\mu| < k\sigma\right] \geq 1 - 1/k^2$

Weak LLN

$P\left[|x-\mu| < \varepsilon\right] \geq 1 - \frac{\sigma^2}{k^2}$ so $P\left[|x-\mu| \geq \varepsilon\right] = \frac{\sigma^2}{\varepsilon^2}$

Example of application of CLT to Analysis

Show that

1). $$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = 1/2$$

2) $$\lim_{n\to\infty} \int_0^n \frac{e^{-t} t^{n-1}}{(n-1)!} dt = 1/2$$

Proof of 1)

Let $X_1, X_2, \dots X_n$ be a random sample from $POI(1)$

and $Y_n = X_1 + X_2 + \dots + X_n \sim POI(n)$.

By CLT, $$W_n = \frac{Y_n - n}{\sqrt{n}} \xrightarrow{d} Z \sim N(0,1)$$

$$P\left[\frac{Y_n - n}{\sqrt{n}} \le 0\right] = P\left[Y_n \le n\right]$$

and this equals 1/2, which can be seen from CDF of Poisson

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{e^{-n} n^k}{k!}$$

Continued application of CLT to Analysis

Note:

$$\frac{\ln(n)}{n^r} \xrightarrow[n\to\infty]{} 0$$

Consider $n^n, e^n, n!$. Which grows fastest?

There are faster.

So

$$\lim_{n\to\infty} \frac{n^n}{e^n n!} = 0$$

Prove it!

August 29

Recall from last class:

1) $$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = 1/2.$$

by CLT

when $X_1, X_2, \dots, X_n \sim POI(1)$

$Y = \sum x_i \sim POI(n)$

$$P\left[\frac{Y-n}{\sqrt{n}} \le 0\right] = P\left[Y \le n\right] = \sum_{k=0}^{n} P[Y=k]$$

$$P[Z \le 0] = 1/2$$

$$= \sum_{k=0}^{n} \frac{e^{-n} n^k}{k!}$$

So $$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{e^{-n} n^k}{k!} = \lim_{n\to\infty} 1/2 = 1/2.$$

2) $$\lim_{n\to\infty} \int_0^n \frac{e^{-t} t^{n-1}}{(n-1)!} \, dt = 1/2$$

Let $X_1, X_2, \ldots, X_n \sim EXP(1)$

$$Y = \sum_{i=1}^{n} X_i \sim GAM(\underset{\alpha}{n}, \underset{\theta}{1})$$

$$f(t) = \frac{1}{\Gamma(n)\, 1^n} t^{n-1} e^{-t}$$

$$= \frac{1}{(n-1)!} t^{n-1} e^{-t}$$

So $P\left[\frac{Y-n}{\sqrt{n}} \le 0\right] = \lim_{n\to\infty} P[Y \le n] = \int_0^n \frac{e^{-t} t^{n-1}}{(n-1)!} dt$

$= P[Z \le 0]$

$= 1/2$

Taking the limit:

$$\lim_{n\to\infty} \int_0^n \frac{e^{-t} t^{n-1}}{(n-1)!} dt = \lim_{n\to\infty} 1/2 = 1/2$$

So we put 1) and 2) together

using IBP:

$u = \frac{t^{n-1}}{(n-1)!}$ $\quad dv = e^{-t} dt$

$du = \frac{t^{n-2}}{(n-2)!}$ $\quad -e^{-t}$

$$\int_0^n \frac{e^{-t} t^{n-1}}{(n-1)!} dt$$

$$= -e^{-t}\left(\frac{t^{n-1}}{(n-1)!}\right)\Big|_0^n + \int_0^n e^{-t} \frac{t^{n-2}}{(n-2)!} dt$$

$$= 1 + \int_0^n e^{-t}\left(\frac{t^{n-2}}{(n-2)!}\right) dt$$

$$= 1 - \sum_{k=0}^{n-1} \frac{e^{-n} n^k}{k!}$$

$$= 1 - \left(\sum_{k=0}^{n} \frac{e^{-n} n^k}{k!} \right) + \frac{e^{-n} n^n}{n!}$$

Take this limit:

Note:

$$\frac{1}{2} = 1 - \frac{1}{2} + \lim_{n \to \infty} \frac{e^{-n} n^n}{n!}$$

$$= \frac{1}{2} + \lim_{n \to \infty} \frac{e^{-n} n^n}{n!}$$

So

$$0 = \lim_{n \to \infty} \frac{e^{-n} n^n}{n!}$$

$\nabla\nabla\nabla$

$\triangle \triangle \triangle$

Sampling Distribution For $N(\mu, \sigma^2)$

Let $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$.

1) $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$

Then $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$$M_{\bar{x}}(t) = E[e^{t\bar{x}}] = E[e^{\frac{t}{n}\sum^{n} x_i}] = E[e^{\frac{t}{n}x_1} e^{\frac{t}{n}x_2} \cdots e^{(\frac{t}{n})x_n}]$$

$$= \left(E[e^{(\frac{t}{n})x_1}]\right)^n = \left[e^{\mu(\frac{t}{n}) + \frac{\sigma^2(\frac{t}{n})^2}{2}}\right]^n$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} = e^{\left(\mu t + \left(\frac{(\frac{\sigma^2}{n}) t^n}{2}\right)\right)} \sim N(\mu, \frac{\sigma^2}{n})$$

Special case of the "linear combination"

Example 1: Let $X_1, X_2, X_3 \sim N(\mu, 1/24)$ where $\mu \neq 0$ and indep.

Find a, b such that $Y = aX_1 + 4X_2 + bX_3 \sim N(0,1)$

$E[Y] = 0$

$E[Y] = a\mu + 4\mu + b\mu$

so

$a\mu + 4\mu + b\mu = 0$ and $\mu(a+b+4) = 0$ and

$a + b = -4$

$b = -(4+a)$ ☆$_1$

$Var(Y) = 1$

$Var(Y) = a^2\left(\frac{1}{24}\right) + 16\left(\frac{1}{24}\right) + b^2\left(\frac{1}{24}\right) = 1$

$a^2 + 16 + b^2 = 24$

$a^2 + b^2 = 8$ ☆$_2$

$a^2 + (4+a)^2 = 8$

$a^2 + a^2 + 8a + 16 = 8$

$2a^2 + 8a + 8 = 0$

$a^2 + 4a + 4 = 0$

$(a+2)^2 = 0$

So $a = -2$, $b = -6$

If $Z \sim N(0,1)$, then $Z^2 = V \sim \chi^2(1)$

— what we did at the beginning

If $\underbrace{Z_1, Z_2, \ldots, Z_n}$ $\sim N(0,1)$,

then $V = \sum_{i=1}^{n} Z_i^2 \sim \chi^2(n)$

! crucially ! these are assumed independent.

Third Question | Sample Variance

$$S^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - x)^2$$

Sampling distribution

Theorem Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $N(\mu, \sigma^2)$. $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

and $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$.

Then:

1) $\bar{X}$ and S^2 are independent

2) $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\sigma^2} \sim \chi^2(n-1)$

Proof (in 3 parts)

1) $$\underbrace{\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}_{\sim\chi^2(n)} = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^2} = \frac{\sum(x_i-\bar{x}+\bar{x}-\mu)^2}{\sigma^2}$$

by quadratic formula

$$= \sum\frac{(x_i-\bar{x})^2}{\sigma^2} + \sum\frac{(\bar{x}-\mu)^2}{\sigma^2} + 2\frac{\sum(x_i-\bar{x})(\bar{x}-\mu)}{\sigma^2}$$

$$= \sum\frac{(x_i-\bar{x})^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2} + \underbrace{2(\bar{x}-\mu)\frac{\sum(x_i-\bar{x})}{\sigma^2}}_{=0}$$

since $\sum x_i - n\bar{x} = 0$

$$= \underbrace{\frac{\sum(x_i-\bar{x})^2}{\sigma^2}}_{\text{we'll show } \sim\chi^2(n-1)} + \left(\frac{\bar{x}-\mu}{(\sigma/\sqrt{n})}\right)^2$$

Note: Class ended here

August 31

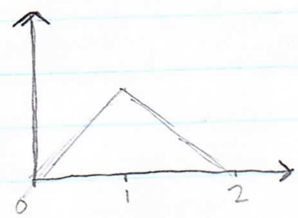
Note: Rule of thumb: When you're working with a sum of independent random variables, try to work with the MGF. Often clarifies things.

Let's talk about the sum of ind. unif. RVs.

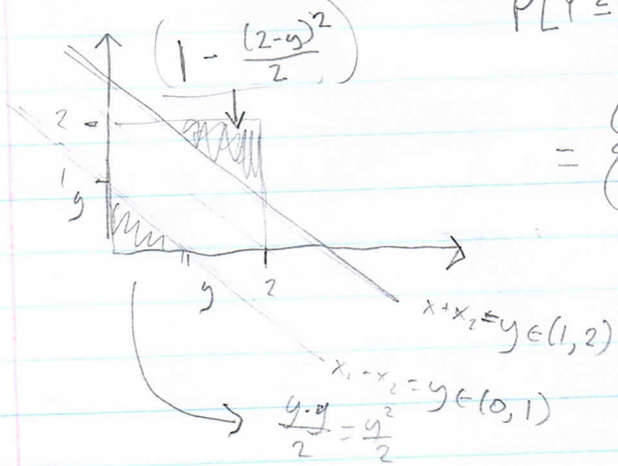
Consider $X_1, X_2 \sim UNIF(0,1)$ with $X_1 \perp X_2$.

$Y = X_1 + X_2$

Note the support of Y is $(0, 2)$



$$f(y) = \begin{cases} y & y \in (0,1) \\ 2-y & y \in (1,2) \end{cases}$$



$$P[Y \le y] = P[X_1 + X_2 \le y]$$

$$= \begin{cases} \frac{y^2}{2} \\ \end{cases}$$

$$f(y) = \frac{\partial}{\partial y} P[Y \le y] = \frac{\partial}{\partial y} F(y)$$

$$f(y) = \begin{cases} y & y \in (0,1) \\ 2-y & y \in (1,2) \end{cases}$$

$$P[Y \ge 1] = 1 - P[Y < 1]$$

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ $\qquad$ $S^2 = \frac{1}{(n-1)}\sum (x_i - \bar{x})^2$

Theorem

1) $\bar{X}$ and S^2 are independent

2) $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum (x_i - \bar{x})^2}{\sigma^2} \sim \chi^2(n-1)$

If $X_1, X_2, \ldots, X_n$ are a random sample of size n from $N(\mu, \sigma^2)$

$\boxed{\text{Proof}}$

1)
$$\underbrace{\sum \frac{(x_i-\mu)^2}{\sigma^2}}_{V \sim \chi^2(n)} = \underbrace{\sum \frac{(x_i-\bar{x})^2}{\sigma^2}}_{V_1 \sim ?} + \underbrace{\frac{n(\bar{x}-\mu)^2}{\sigma^2}}_{V_2 \ \chi^2(1)}$$

$$\Big(\underbrace{\frac{(x_i-\mu)}{\sigma}}_{z}\Big)^2 \sim \chi^2(1)$$

If $\bar{X}$ and S^2 are independent, then we have:

$$\underbrace{M_V(t)}_{\chi^2(n)} = M_{V_1}(t)\underbrace{M_{V_2}(t)}_{\chi^2(1)}$$

$$\frac{1}{(1-2t)^{n/2}} = M_{V_1}(t)\left(\frac{1}{(1-2t)^{1/2}}\right)$$

So

$$M_{V_1}(t) = \frac{(1-2t)^{1/2}}{(1-2t)^{n/2}} = \underbrace{\frac{1}{(1-2t)^{\frac{n-1}{2}}}}_{\text{Note this is the } \chi^2(n-1) \text{ mgf.}}$$

$V_1 \sim \chi^2(n-1)$

This shows that the second part of the theorem is true if the first part is true

Only need to show $\overline{X}$ and S^2 are indep.
In order to show that,
we only need to show X_i and S^2 are independent.

Second Step:
Show $\overline{X}$ and $(X_i - \overline{X})$ are independent.

For $X_1, X_2, \ldots, X_n$, the joint density function is

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{(\sqrt{2\pi})^n \sigma^n} e^{\left[-\frac{1}{2\sigma^2}\sum (x_i-\mu)^2\right]}$$

$$= \frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}} e^{\left[-\frac{1}{2\sigma^2}\left(\sum(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2\right)\right]}$$

Transformation:

$y_1 = \bar{x}$, $y_i = x_i - \bar{x}$, $2 \le i \le n$

$$x_1 - \bar{x} = -\sum_{i=2}^{n}(x_i - \bar{x}) = -\sum_{i=2}^{n} y_i$$

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 = \underbrace{\left(-\sum_{i=2}^{n} y_i\right)^2}_{(x_1-\bar{x})^2} + \underbrace{\sum_{i=2}^{n} y_i^2}_{\sum_{i=2}^{n}(x_i-\bar{x})^2}$$

$(x_i - \theta)$

$\alpha\theta^2$

The joint density for $Y_1 = \bar{X}$, $Y_i = X_i - \bar{X}$, $2 \le i \le n$

Jacobian, which is a constant since trans. is linear

$$g(y_1, y_2, \dots, y_n) = \frac{K}{(2\pi\sigma)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\left(\left(-\sum y_i\right)^2 + \sum_{i=2}^{n} y_i^2 + n(y_1-\mu)^2\right)\right]$$

$$= \frac{K}{(2\pi\sigma)^{n/2}} \exp\left[-\frac{1}{2\sigma^2} n(y_1-\mu)^2\right] \exp\left[-\frac{1}{2\sigma^2}(\quad)\right]$$

y_i $2 \le i \le n$

Remark:

Why take $S^2 = \frac{1}{(n-1)} \sum (x_i - \bar{x})^2$?

$(n-1)$?

Consider $E[S^2]$.

We know $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

Note if $V \sim \chi^2(r)$, then $E(V) = r$ and $Var(V) = 2r$

Recall if $X \sim GAM(\alpha, \theta)$
$E[X] = \alpha\theta$
$Var(X) = \alpha\theta^2$
$X \sim \chi^2(r) = GAM(\frac{r}{2}, 2)$
$E[X] = r$
$Var(X) = 2r$

So $E[S^2] = E\left[\frac{\sigma^2}{n-1} \frac{n-1}{\sigma^2} S^2\right]$

$$= \frac{\sigma^2}{n-1} E\left[\left(\frac{n-1}{\sigma^2}\right) S^2\right] = \frac{\sigma^2}{n-1} E\left[\frac{1}{\sigma^2} \sum (x_i - \bar{x})^2\right]$$

$$= \frac{\sigma^2}{(n-1)} (n-1) = \sigma^2$$

"unbiased"

$$Var(S^2) = Var\left(\frac{\sigma^2}{n-1}\frac{(n-1)}{\sigma^2}S^2\right)$$

$$= \frac{\sigma^4}{(n-1)^2} Var\left(\frac{n-1}{\sigma^2}S^2\right) = \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1)$$

$$= \frac{2\sigma^4}{(n-1)}$$

So far, we have from our theorem that

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \longrightarrow \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim Z \sim N(0,1)$$

$$S^2 \text{ ——— } \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

Two problems:

1) If σ^2 is unknown, we have to use S^2 to replace σ^2
What will be

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim ?$$

2) To compare σ_1^2 and σ_2^2 by using S_1^2 and S_2^2,
we do

$$\frac{\frac{S_1^2}{\sigma_1^2}}{\frac{S_2^2}{\sigma_2^2}} \sim ?$$