For $X_1 \sim Exp(\theta)$

$$M(t) = E[e^{tX_1}] = \int_0^\infty e^{tX_1} f(x_1)\,dx$$

$$= \int_0^\infty (e^{tx_1})\left(\frac{1}{\theta}\right)\left(e^{-\frac{x_1}{\theta}}\right) dx_1$$

$$= \int_0^\infty \frac{1}{\theta} e^{-(1-\theta t)\frac{x_1}{\theta}} dx_1$$

$$\left(\begin{aligned} s &= \frac{x_1}{\theta} \\ ds &= 1/\theta\, dx_1 \end{aligned}\right)$$

$$= \int_0^\infty e^{-(1-\theta t)s} ds$$

$$= \frac{1}{1-\theta t}\left[-e^{-(1-\theta t)s}\right]_0^\infty$$

$$= \frac{1}{1-\theta t}$$

★2 (Derivation of MGF For Exponential distribution)

What's Gamma?

$X \sim GAM(\alpha, \theta)$,

if $f(\alpha) = \frac{1}{\Gamma(\alpha)\theta^\alpha}(x^{\alpha-1})(e^{-\frac{x}{\theta}})$

$\Gamma(\alpha)$ Gamma function

$$M(t) = \frac{1}{(1-\theta t)^\alpha}$$

(when $\alpha = 1$, $Exp(\theta) = GAM(1, \theta)$)

$\Gamma(x) = \Gamma$

$$\chi^2(r) \sim GAM\left(\frac{r}{2}, 2\right)$$

$$M(t) = \frac{1}{(1-2t)^{r/2}}$$

$Y \sim GAM(\alpha, \theta)$

$X = \frac{2Y}{\theta} \sim \chi^2(2\alpha)$

$$E\left[e^{t\left(\frac{2Y}{\theta}\right)}\right] = \frac{1}{\left(1-\theta\left(\frac{2t}{\theta}\right)\right)^\alpha} = \frac{1}{(1-2t)^\alpha}$$

$\alpha = r/2$

$r = 2\alpha$