$$= \frac{1}{\sqrt{\pi}\,\Gamma(\frac{r}{2})} \int_0^\infty \frac{1}{2^{(\frac{r+1}{2})}\sqrt{r}} v^{\frac{r}{2}-1+\frac{1}{2}} e^{-\frac{v}{2}(1+\frac{x^2}{r})} \, dv$$

$$= \frac{1}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})} \int_0^\infty \frac{1}{2^{\frac{r+1}{2}}} v^{\frac{r+1}{2}-1} e^{-(1+\frac{x^2}{r})\frac{v}{2}} \, dv$$

$$y = \left(1+\frac{x^2}{r}\right) v$$

$$dy = \left(1+\frac{x^2}{r}\right) dv$$

$$\frac{1}{\left(1+\frac{x^2}{r}\right)} dy = dv$$

$$= \frac{1}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})} \int_0^\infty \frac{1}{2^{(\frac{r+1}{2})}} \left(\frac{y}{1+\frac{x^2}{r}}\right)^{\frac{r+1}{2}-1} e^{-\frac{y}{2}} \left(\frac{1}{1+\frac{x^2}{r}}\right) dy$$

$$= \frac{1}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})} \left(\frac{1}{1+\frac{x^2}{r}}\right)^{\frac{r+1}{2}} \int_0^\infty \frac{1}{2^{(\frac{r+1}{2})}} y^{(\frac{r+1}{2}-1)} e^{-\frac{y}{2}} \, dy$$

multiply by $\left(\frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r+1}{2})}\right)$

Note this looks very similar to $\chi^2(r+1)$ pdf

$$= \frac{1}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})} \left(\frac{1}{1+\frac{x^2}{r}}\right)^{\frac{r+1}{2}} \Gamma\left(\frac{r+1}{2}\right) \underbrace{\int_0^\infty \frac{1}{2^{(\frac{r+1}{2})}\Gamma(\frac{r+1}{2})} y^{(\frac{r+1}{2}-1)} e^{-\frac{y}{2}} \, dy}_{\text{pdf of } \chi^2(r+1) \; = 1}$$

$$= \underbrace{\frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2})\sqrt{\pi r}} \left(1+\frac{x^2}{r}\right)^{-(\frac{r+1}{2})}}_{\text{this is the PDF of } t(r)}$$

(yay!)