Sampling Distribution For $N(\mu, \sigma^2)$

Let $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$.

1) $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Then $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$$M_{\bar{x}}(t) = E[e^{t\bar{x}}] = E[e^{\frac{t}{n}\sum^n X_i}] = E[e^{\frac{t}{n}X_1} e^{\frac{t}{n}X_2} \cdots e^{(\frac{t}{n})X_n}]$$

$$= \left(E[e^{(\frac{t}{n})X_1}]\right)^n = \left[e^{\mu(\frac{t}{n}) + \frac{\sigma^2(\frac{t}{n})^2}{2}}\right]^n$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} = e^{\left(\mu t + \left(\frac{(\frac{\sigma^2}{n}) t^n}{2}\right)\right)} \sim N(\mu, \frac{\sigma^2}{n})$$

Special case of the "linear combination"

Example 1: Let $X_1, X_2, X_3 \sim N(\mu, 1/24)$ where $\mu \neq 0$ and indep.

Find a, b such that $Y = aX_1 + 4X_2 + bX_3 \sim N(0,1)$

$E[Y] = 0$

$E[Y] = a\mu + 4\mu + b\mu$

so

$a\mu + 4\mu + b\mu = 0$ and $\mu(a+b+4) = 0$ and

$a + b = -4$

$b = -(4+a)$ ☆$_1$

$Var(Y) = 1$

$Var(Y) = a^2\left(\frac{1}{24}\right) + 16\left(\frac{1}{24}\right) + b^2\left(\frac{1}{24}\right) = 1$

$a^2 + 16 + b^2 = 24$

$a^2 + b^2 = 8$ ☆$_2$

$a^2 + (4+a)^2 = 8$

$a^2 + a^2 + 8a + 16 = 8$

$2a^2 + 8a + 8 = 0$

$a^2 + 4a + 4 = 0$

$(a+2)^2 = 0$

So $a = -2$, $b = -6$