Proof of Weak LLN

(1) Let X be an R.V., u a nonnegative function and $c > 0$.

Then
$$P[u(X) \geq c] \leq \frac{E[u(X)]}{c}$$

We only consider X has a pdf $f(x)$ (i.e. we ignore discrete case, whose proof is similar)

$$E[u(X)] = \int_{-\infty}^{\infty} u(x) f(x)\, dx = \int_{\{u(x) \geq c\}} u(x) f(x)\, dx + \int_{\{u(x) < c\}} u(x) f(x)\, dx$$

$$\geq \int_{\{u(x) \geq c\}} u(x) f(x)\, dx \geq c \int_{\{u(x) \geq c\}} f(x)\, dx$$

$$= c P[u(x) \geq c]$$

And so
$$P[u(x) \geq c] \leq \frac{E[u(x)]}{c}$$