Math 562 Class Notes

Ben Schneider

Note: Since this class is not following the book, the notes shall follow a daily routine.

Example 1:

$$X \sim Exp(\theta)$$
$$f(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}},\ x > 0$$
$$Y = \ln X$$

Want to find the pdf of Y.

Start by identifying the domain:

$$Dom(Y) = (-\infty, \infty)$$
$$\ln(x) \in (-\infty, \infty),\ \forall x > 0$$

Then for any $y \in (-\infty, \infty)$:

$$P[Y \leq y] = P[\ln(X) \leq y]$$
$$= P[X \leq e^y]$$

8 January: Syllabus and "Warm-Up" Quiz

10 January
Recall: Cantor's Intersection Property
Given a descending sequence of nonempty sets $E_1 \supset E_2 \supset E_3 \supset \ldots$, when is $\cap_{n=1}^{\infty} E_n \neq \emptyset$?
If $E_n = (0, \frac{1}{n})$, $\forall n \in \mathbb{N}$, then $\cap_{n=1}^{\infty} E_n = \emptyset$. Note that each E_n is bounded, but not closed.
If $E_n = [n, \infty)$, $\forall n \in \mathbb{N}$, then $\cap_{n=1}^{\infty} E_n = \emptyset$. Note that each E_n is closed, but not bounded.

Theorem (Cantor): Let $E_1 \supset E_2 \supset E_3 \supset \ldots$ be a descending sequence of nonempty closed and bounded (compact) subsets of $\mathbb{R}$.
Then $E = \cap_{n=1}^{\infty} E_n$ is also closed and bounded (compact).

Corollary (Cantor's Intersection Theorem)
Define the diameter of a set by $diam(E_n) = \max(E_n) - \min(E_n)$.
Let $E_1 \supset E_2 \supset \ldots$ be a descending sequence of nonempty compact subsets of $\mathbb{R}$ such that $diam(E_n) \to 0$. Then $E = \cap_{n=1}^{\infty} E_n$ consists of a single point.

CHAPTER 8: THE INTEGRAL

This part of the course is a combination of sections 8.6 (Riemann Integral) and 8.2 (Cauchy's First Method), plus extra material.

The Darboux Integral

Definition: Let $f : [a, b] \to \mathbb{R}$ be a bounded function. (The image of f is bounded.)
For a nonempty set $S \subset [a, b]$, we denote **the maximum of f over S** by $M(f, S)$ and **the minimum of f over S** by $m(f, S)$.

$$M(f, S) = \sup\{f(x) : x \in S\}$$
$$m(f, S) = \inf\{f(x) : x \in S\}$$

A **partition** of $[a, b]$ is any finite ordered subset P having the form

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\}$$

not necessarily having equal width.

The **upper Darboux sum** $U(f, P)$ of f with respect to P is the sum

$$U(f, P) = \sum_{k=1}^{n} M(f, [x_{k-1}, x_k])(x_k - x_{k-1})$$

The **lower Darboux sum** $L(f, P)$ of f with respect to P is the sum

$$L(f, P) = \sum_{k=1}^{n} m(f, [x_{k-1}, x_k])(x_k - x_{k-1})$$

Remarks: (i) If f is continuous on $[a, b]$, then $m(f, [x_{k-1}, x_k]) = \min\{f(x) : x \in [x_{k-1}, x_k]\}$ and $M(f, [x_{k-1}, x_k]) = \max\{f(x) : x \in [x_{k-1}, x_k]\}$.
(ii) Note that $U(f, P) \leq \sum_{k=1}^{n} M(f, [a, b])(x_k - x_{k-1}) = M(f, [a, b])(b - a)$ and $L(f, P) \geq \sum_{k=1}^{n} m(f, [a, b])(x_k - x_{k-1}) = m(f, [a, b])(b - a)$.

$$m(f, [a, b])(b - a) \leq L(f, P) \leq U(f, P) \leq M(f, [a, b])(b - a) \tag{1}$$

Definition (Not in book): The **upper Darboux integral** $U(f)$ of f over $[a, b]$ is defined by

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

and the **lower Darboux integral** $L(f)$ of f over $[a,b]$ is defined by

$$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

12 January: Snow Day

15 January: Martin Luther King, Jr. Holiday

17 January
HW: $8.2 : \#2, 3, 4, 5, 14, 17a-d$

Definition: We say that f is **Darboux integrable on** $[a,b]$ if $U(f) = L(f)$.

Example: $[a,b] = [0,1]$. $P = \{0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1\}$. $U(f)$ is always larger than or equal to f, so it is the sum of the infima of our partitioned intervals. Similarly, $L(f)$ is always less than or equal to f, so it is the sum of the suprema of our partitioned intervals.

Example: Consider $f : [0,b] \to \mathbb{R}$, $b > 0$, defined by $f(x) = x^2$.
Let $P = \{0 = x_0 < x_1 < x_2 < \ldots < x_n = b\}$.

$$U(f,P) = \sum_{k=1}^{n} M(f,[x_{k-1},x_k])(x_k - x_{k-1}) = \sum_{k=1}^{n} {x_k}^2(x_k - x_{k-1})$$

Suppose $P = \{0 < \frac{b}{n} < \frac{2b}{n} < \ldots < \frac{nb}{n} = b\}$. So $x_k = \frac{kb}{n}$ and $x_k - x_{k-1} = \frac{b}{n}$. Now

$$U(f,P) = \sum_{k=1}^{n} \frac{k^2b^2}{n^2} \cdot \frac{b}{n} = \frac{b^3}{n^3} \sum_{k=1}^{n} k^2 = \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6} \to \frac{1}{3}b^3$$

So $U(f) \leq \frac{b^3}{3}$. Similarly,

$$L(f,P) = \sum_{k=1}^{n} {x_{k-1}}^2(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{(k-1)^2b^2}{n^2} \frac{b}{n} = \frac{b^3}{n^3} \sum_{k=1}^{n} (k-1)^2 = \frac{b^3}{n^3} \frac{n(n-1)(2n-1)}{6} \to \frac{1}{3}b^3$$

So $L(f) \geq \frac{b^3}{3}$. Since $\frac{b^3}{3} \leq L(f) \leq U(f) \leq \frac{b^3}{3}$, $f(x) = x^2$ is Darboux integrable and $\int_0^b x^2 dx = \frac{b^3}{3}$.

Example: Let $f : [0,b] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1, x \in \mathbb{Q} \cap [0,b] \\ 0, x \in \mathbb{I} \cap [0,b] \end{cases}$ for any partition $P = \{0 = x_0 < x_1 < \ldots < x_n = b\}$.
$U(f,P) = \sum_{k=1}^{n} 1 \cdot (x_k - x_{k-1}) = b$, since $\sup\{f(x) : x \in [x_{k-1}, x_k]\} = 1$. Hence $U(f) = 1$.
$L(f,P) = \sum_{k=1}^{n} 0 \cdot (x_k - x_{k-1}) = 0$, since $\inf\{f(x) : x \in [x_{k-1}, x_k]\} = 0$. Hence $L(f) = 0$.
Since $L(f) \neq U(f)$, f is not Darboux integrable.

Lemma 1: Let $f : [a, b] \to \mathbb{R}$ be bounded. If P and Q are partitions of $[a, b]$ and $P \subset Q$, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P) \tag{2}$$

Lemma 2: If $f : [a, b] \to \mathbb{R}$ is bounded and P and Q are partitions of $[a, b]$, then $L(f, P) \leq U(f, Q)$.

Theorem 1: If $f : [a, b] \to \mathbb{R}$ is bounded, then $L(f) \leq U(f)$.

Proof. Fix a partition P of $[a, b]$.
By Lemma 2, $L(f, P)$ is a lower bound of $\{U(f, Q) : Q \text{ is any partition of } [a, b]\}$.
Thus, $L(f, P) \leq U(f)$.
Similarly, $U(f)$ is an upper bound of $\{L(f, P) : P \text{ is any partition of } [a, b]\}$.
Hence, $L(f) \leq U(f)$. $\square$