Laws of Large Numbers

1) Bernoulli Law of Large Numbers

$$X_1, X_2, \ldots, X_n \sim BER(p) = BIN(1, p)$$

$$Y_n = \sum_{i=1}^{n} X_i \sim BIN(n, p)$$

$(q = 1-p)$

$$\bar{X} = \frac{1}{n} Y_n$$

$$M_{\bar{X}}(t) = E\left[e^{t\frac{1}{n}\sum x_i}\right] = \left(M\left(\frac{t}{n}\right)\right)^n = \left(pe^{\frac{t}{n}} + q\right)^n$$

$$= \left[p\left(1 + \frac{\left(\frac{t}{n}\right)^2}{2!} + \frac{\left(\frac{t}{n}\right)^3}{3!} + \ldots\right) + q\right]^n$$

$$= \left(1 + p\left(\frac{t}{n}\right) + p\frac{\left(\frac{t}{n}\right)^2}{2!} + \ldots\right)^n$$

$d(n) \to 0$ as $n \to \infty$

$$= \left[1 + p\left(\frac{t}{n}\right) + \frac{d(n)}{2}\right]^n$$

and

$$\left[1 + p\left(\frac{t}{n}\right) + \frac{d(n)}{2}\right]^n \longrightarrow e^{pt}$$

$*$ (ask for this identity)

So $\bar{X} \xrightarrow{p} p$

Stochastically

$$\left[1 + \frac{z}{n}\right]^n \underset{\text{as } n \to \infty}{\longrightarrow} e^z$$

$$\bar{X} \approx N\left(p, \frac{\sigma^2}{n}\right)$$

$\frac{\sigma^2}{n} \searrow 0$