$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ $\qquad S^2 = \frac{1}{(n-1)}\sum(x_i - \bar{x})^2$

Theorem

1) $\bar{X}$ and S^2 are independent

2) $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum(x_i - \bar{x})^2}{\sigma^2} \sim \chi^2(n-1)$

If $X_1, X_2, \ldots, X_n$ are a random sample of size n from $N(\mu, \sigma^2)$

$\boxed{\text{Proof}}$

1) $$\underbrace{\sum \frac{(x_i-\mu)^2}{\sigma^2}}_{V \sim \chi^2(n)} = \underbrace{\sum \frac{(x_i-\bar{x})^2}{\sigma^2}}_{V_1 \sim ?} + \underbrace{\frac{n(x_i-\mu)^2}{\sigma^2}}_{V_2 \ \chi^2(1)}$$

$$\Big(\underbrace{\frac{(x_i-\mu)}{\sigma}}_{z}\Big)^2 \sim \chi^2(1)$$

If $\bar{x}$ and S^2 are independent, then we have:

$$\underbrace{M_V(t)}_{\chi^2(n)} = M_{V_1}(t) \underbrace{M_{V_2}(t)}_{\chi^2(1)}$$

$$\frac{1}{(1-2t)^{n/2}} = M_{V_1}(t)\left(\frac{1}{(1-2t)^{1/2}}\right)$$

So

$$M_{V_1}(t) = \frac{(1-2t)^{1/2}}{(1-2t)^{n/2}} = \underbrace{\frac{1}{(1-2t)^{\frac{n-1}{2}}}}$$

$V_1 \sim \chi^2(n-1)$

Note this is the $\chi^2(n-1)$ mgf.

This shows that the second part of the theorem is true if the first part is true