# 1 Gröbner basics

## 1.1 Left ideals in factor algebras

**1.1 Fact.** We assume we have $\mathbb{K}\langle\mathbf{X}\rangle$ equipped with an arbitrary monomial ordering and we take an ideal $I$ with a finite Gröbner basis $G$. If we consider an ideal $J$ of the algebra $A := \mathbb{K}\langle\mathbf{X}\rangle/I$ we know that $I \subseteq J$ as an subset of $\mathbb{K}\langle\mathbf{X}\rangle$. If $J$ is an two-sided ideal generated by a set $E$ it is easy to see that that in order to get a Gröbner basis for $J$ we only need to compute a Gröbner basis of the set $E \cup G$ and then substract those elements in $G$. However, if we consider $J$ as an left ideal things get a little more complicated. Recall that we have $J = {}_A\langle E\rangle := \{\sum c_i e_i \mid c_i \in A, e_i \in E\}$. We identify elements of $A$ with their normal forms modulo $I$.

For notations: $\mathtt{NF}$ denotes the normal form, $\mathtt{lt}$ the leading term, both with respect to a fixed ordering.

**1.2 Definition.** Let $J \trianglelefteq \mathbb{K}\langle\mathbf{X}\rangle$ be an left ideal containing the two sided ideal $I$, and let $G \subset J$ be a set of non-zero polynomials, normal with respect to $I$. We call $G$ a (left) Gröbner basis of the ideal $J/I \trianglelefteq \mathbb{K}\langle\mathbf{X}\rangle/I$ if for every polynomial $p \in J/I$ there exists a polynomial $g \in G$ such that $\mathtt{lt}(g)$ is a suffix of $\mathtt{lt}(p)$. In other words $\mathtt{lt}(g)$ is a right divisor of $\mathtt{lt}(p)$, that is $\mathtt{lt}(p) = m\mathtt{lt}(g)$ for some $m \in \mathbb{K}\langle\mathbf{X}\rangle/I$.

Our first goal is to give an algorithm which allows us to do reduction with respect to $J/I$.

**1.3 Theorem.** Let $p \in \mathbb{K}\langle\mathbf{X}\rangle$, let $s \geq 1$, and let $g_1, \ldots, g_s \in \mathbb{K}\langle\mathbf{X}\rangle\setminus\{0\}$ be normal polynomials with respect to $I$. Consider the following sequence of instructions:

1. Set $q_1 = \ldots = q_s = 0$ and $v = \mathtt{NF}_I(p)$.
2. If there exists an index $j \in \{1, \ldots, s\}$ such that $\mathtt{lt}(v) \equiv w\mathtt{lt}(g_j)$ for some monomial $w \in \langle\mathbf{X}\rangle$, then replace $q_j$ by $q_j + \frac{\mathtt{lc}(v)}{\mathtt{lc}(g_j)}w$ and $v$ by $v - \frac{\mathtt{lc}(v)}{\mathtt{lc}(g_j)}w$.
3. Repeat step 2) until there is no more $j \in \{1, \ldots, s\}$ such that $\mathtt{lt}(g_j)$ is a suffix of $\mathtt{lt}(v)$. Return the tuple $(v, q_1, \ldots, q_s)$.

This is an algorithm which returns a tuple $(v, q_1, \ldots, q_s)$ such that $p - (\sum_{j_1}^{s} q_j g_j + v) \in I$ and such that the following conditions are satiesfied:

- The polynomial $v$ is in normal form with respect to $I$.
- For all $j \in \{1, \ldots, s\}$, $q_j$ is in normal form with respect to $I$. If $q_j \neq 0$ for some $j \in \{1, \ldots, s\}$, then $\mathtt{lt}(p) \geq \mathtt{lt}(q_j g_j)$.
- If $v \neq 0$, then $\mathtt{lt}(p) \geq \mathtt{lt}(v)$ and there is no $j \in \{1, \ldots, s\}$ such that $\mathtt{lt}(g_j)$ is a suffix of $\mathtt{lt}(v)$.

**Proof:** Omitted for the time being. q.e.d.

Note that this algorithm only reduces the leading term of $p$. For a complete reduction we have to iterate over all terms of $p$ after the leading term is completely reduced, that is restart the procedure with $p - \mathtt{lt}(p)$.

The polynomial $v$ obtained in the algorithm is called a *left normal form* of $p$ and we denote it by $\mathtt{lNF}(p)$.

**1.4 Proposition.** Let $\mathbf{J} \subseteq \mathbb{K}\langle \mathbf{X} \rangle$ be a left ideal containing $I$ and let $G \subset J$ be a set of polynomials in normal form with respect to $I$. The following conditions are equivalent:

- The set $G$ is a Gröbner basis of $J/I \trianglelefteq \mathbb{K}\langle \mathbf{X} \rangle / I$.
- Every normal polynomial $p \in J/I$ has a representation $p = \sum\limits_{j=1}^{s} q_j g_j + h$ with $q_i \in \mathbb{K}\langle \mathbf{X} \rangle \setminus \{0\}, h \in I$ such that $\mathtt{lt}(p) \geq \mathtt{lt}(q_i g_i) \quad \forall i \in \{1, \ldots, s\}$ and $\mathtt{lt}(p) > \mathtt{lt}(h)$.
- A polynomial $p \in \mathbb{K}\langle \mathbf{X} \rangle$ satisfies $p \in J$ if and only if $\mathtt{lNF}(p) = 0$.

**Proof:** as before q.e.d.

more blabla needed?

**1.5 Remark.** Taking a generating set $E$ for $J/I$ and a Gröbner basis $G_I$ for $I$, each polynomial $p \in J/I$ can be represented as $p = \sum\limits_{j=1}^{s} q_j e_j + \sum\limits_{i} c_i l_i g_i r_i$, where $e_k \in E, q_k \mathbb{K}\langle \mathbf{X} \rangle \setminus \{0\}, c_k \in \mathbb{K}, g_k \in G_I, l_k, r_k \in \langle \mathbf{X} \rangle$. It is easy to see that this representation does not necessarily satisfy the condition in 1.4, since there might be an index $j$ such that $\mathtt{lt}(q_j)\mathtt{lt}(e_j) \geq \mathtt{lt}(p)$ or $l_j \mathtt{lt}(g_j) r_j \geq \mathtt{lt}(p)$. There are three cases in which these terms may occur:

1. There exist $j, j' \in \{1, \ldots, s\}, j \neq j'$ such that $\mathtt{lt}(q_j)\mathtt{lt}(e_j) \equiv \mathtt{lt}(q_{j'})\mathtt{lt}(g_{j'}) > \mathtt{lt}(p)$, that is $e_j$ and $g_{j'}$ have an left overlap.
2. There exist $j, j' \in \{1, \ldots, s\}, j \neq j'$ such that $l_j \mathtt{lt}(g_j) r_j \equiv l'_{j'} \mathtt{lt}(g_{j'}) r_{j'} > \mathtt{lt}(p)$, that is $g_j$ and $g_{j'}$ have an overlap.

3. There $j \in \{1,\dots,s\}, j' \in \mathbb{N}$ such that $\mathtt{lt}(q_j)\mathtt{lt}(e_j) \equiv l_{j'}\mathtt{lt}(g_{j'})r_{j'}$. Since $e_j$ is normal with respect to $I$ and $G_I$ is a Gröbner basis this implies there is $w \in \langle \mathbf{X} \rangle \setminus \{1\}$ such that $\mathtt{lt}(q_j)\mathtt{lt}(e_j) \equiv \mathtt{lt}(g_{j'})w$.

Since $G_I$ is a Gröbner basis for $I$ one does not need to consider the second case, because one can simply use the Gröbner representation given by $G_I$.

**1.6 Proposition.** Let $G \subset J$ be a set of polynomials in normal form with respect to $I$ and let $J/I \trianglelefteq \mathbb{K}\langle \mathbf{X} \rangle / I$ be the left ideal generated by $G$. Say we have a finite Gröbner basis $G_I$ for $I$. Define two sets as follows:

$$O_G = \{\frac{1}{\mathtt{lc}(g)}g - \frac{1}{\mathtt{lc}(g')}wg' \mid g, g' \in G, g \neq g', w \in \langle \mathbf{X} \rangle, \mathtt{lt}(g) \equiv w\mathtt{lt}(g')\}$$

and

$$O_{GG_I} = \{\frac{1}{\mathtt{lc}(g)}wg - \frac{1}{\mathtt{lc}(g')}g'r' \mid g \in G, g' \in G_I, w, r' \in \langle \mathbf{X} \rangle, w\mathtt{lt}(g) \equiv \mathtt{lt}(g')r'\}.$$

Then $G$ is a Gröbner basis of $J/I$ if and only if $\mathtt{lNF}(p) = 0$ for all $p \in O_G \cup O_{GG_I}$.

**Proof:** Omitted. q.e.d.

**1.7 Algorithm.**

**Input:** $E \subseteq \mathbb{K}\langle \mathbf{X} \rangle \setminus \{0\}$, a generating set for $J$, normal with respect to $I$, $G_I$, a Gröbner basis for $I$

**Output:** $G$, a Gröbner basis for $J$

Start with $G = E$

Set $S = \{\frac{1}{\mathtt{lc}(g)}g - \frac{1}{\mathtt{lc}(g')}wg' \mid g, g' \in G, g \neq g', w \in \langle \mathbf{X} \rangle, \mathtt{lt}(g) \equiv w\mathtt{lt}(g')\} \cup \{\frac{1}{\mathtt{lc}(g)}wg - \frac{1}{\mathtt{lc}(g')}g'r' \mid g \in G, g' \in G_I, w, r' \in \langle \mathbf{X} \rangle, w\mathtt{lt}(g) \equiv \mathtt{lt}(g')r'\}$

**while** $S \neq \emptyset$ **do**

Take $s \in S$ and set $S = S \setminus \{s\}$

Compute $\overline{s} = \mathtt{lNF}(s)$.

**if** $\overline{s} \neq 0$ **then**

$S = S \cup \{\frac{1}{\mathtt{lc}(g)}g - \frac{1}{\mathtt{lc}(\overline{s})}w\overline{s} \mid g \in G, w \in \langle \mathbf{X} \rangle, \mathtt{lt}(g) \equiv w\mathtt{lt}(\overline{s})\} \cup \{\frac{1}{\mathtt{lc}(\overline{s})}w\overline{s} - \frac{1}{\mathtt{lc}(g')}g'r' \mid g' \in G_I, w, r' \in \langle \mathbf{X} \rangle, w\mathtt{lt}(\overline{s}) \equiv \mathtt{lt}(g')r'\}$

$G = G \cup \{\overline{s}\}$

**end if**

**end while**

**return** $G$

If $J/I$ has a finite (left) Gröbner basis this algorithm terminates after finitely many steps and returns a reduced Gröbner basis of $J/I$.

**Proof:** With Proposition above and analogous to the normal Gröbner basis algorithm q.e.d.

**1.8 Remark.** Gebauer-Möllers criteria can be applied in a similar fashion as before, one has just to take the special forms of obstructions into account.