Theoretical Global Seismology

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3.11 Hydrostatic Earth Model

A general Earth model of the type that we have been considering so far is specified by giving its density ρ^0, initial stress $\mathbf{T}^0$, rotation rate $\boldsymbol{\Omega}$, and elastic tensor $\boldsymbol{\Gamma}$. The first three parameters are related by the static equilibrium condition $\boldsymbol{\nabla}\cdot\mathbf{T}^0 = \rho^0\boldsymbol{\nabla}(\phi^0+\psi)$, together with the boundary conditions $\hat{\mathbf{n}}\cdot\mathbf{T}^0=\mathbf{0}$ on $\partial\oplus$ and $[\hat{\mathbf{n}}\cdot\mathbf{T}^0]_-^+=\mathbf{0}$ on Σ_{SS} and Σ_{FS}. For a given ρ^0 and Ω, these equations serve to constrain three of the six independent components of the static stress $\mathbf{T}^0=-p^0\mathbf{I}+\boldsymbol{\tau}^0$; the remaining three components must be treated as independently specified parameters.

The need to specify $\mathbf{T}^0$, and specifically the initial deviatoric stress $\boldsymbol{\tau}^0$, is obviated if we demand that $\mathbf{T}^0=-p^0\mathbf{I}$ everywhere in $\oplus$. The equilibrium condition reduces in this case to

$$\boldsymbol{\nabla}p^0+\rho^0\boldsymbol{\nabla}(\phi^0+\psi)=\mathbf{0}, \tag{3.255}$$

and the boundary conditions reduce to $p^0 = 0$ on $\partial\oplus$ and $[p^0]_-^+ = 0$ on Σ_{FS} and Σ_{SS}, where p^0 is the hydrostatic pressure. We derive the linearized equations of motion and boundary conditions governing such a *hydrostatic Earth model* in this section. The same symbols will be used to denote the Lagrangian and energy densities and associated action and energy integrals on a hydrostatic and a non-hydrostatic Earth, for simplicity.

3.11.1 Applicability of the theory

Upon taking the curl of equation (3.255) we deduce that

$$\boldsymbol{\nabla}\rho^0 \times \boldsymbol{\nabla}(\phi^0 + \psi) = \mathbf{0}. \tag{3.256}$$

Likewise, taking the cross product with $\boldsymbol{\nabla}p^0$, we find that

$$\boldsymbol{\nabla}p^0 \times \boldsymbol{\nabla}(\phi^0 + \psi) = \mathbf{0}. \tag{3.257}$$

These results show that the *level surfaces* of density ρ^0, pressure p^0 and geopotential $\phi^0 + \psi$ *must coincide* in a hydrostatic Earth model. Any boundary surface across which ρ^0 suffers a jump discontinuity, including the outer free surface, must also be a level surface; this means that we must have

$$\boldsymbol{\nabla}^\Sigma\rho^0 = \mathbf{0}, \qquad \boldsymbol{\nabla}^\Sigma(\phi^0 + \psi) = \mathbf{0}, \qquad \boldsymbol{\nabla}^\Sigma p^0 = \mathbf{0} \tag{3.258}$$

on $\Sigma = \partial\oplus \cup \Sigma_{\mathrm{SS}} \cup \Sigma_{\mathrm{FS}}$ as well. The conditions (3.256)–(3.258) place a very severe restriction upon the allowable density distributions ρ^0 that may be in a state of hydrostatic equilibrium; in fact, every level surface must be an axially symmetric ellipsoid, or, in the limit $\boldsymbol{\Omega} \to \mathbf{0}$, a sphere. To obtain the linearized equations and boundary conditions, it is not sufficient simply to set the deviatoric stress $\boldsymbol{\tau}^0$ equal to zero; as we shall see, we must also systematically utilize the constraints (3.258). The resulting theory is substantially simpler than the general theory; however, it is exactly valid only for a *rotating, ellipsoidal* Earth model or a *non-rotating, spherically symmetric* Earth model.

Any more general, laterally heterogeneous density distribution ρ^0 requires a non-zero initial deviatoric stress $\boldsymbol{\tau}^0$ for its support. To treat such an Earth model exactly, we must specify $\boldsymbol{\tau}^0$ and use the general equations of motion and boundary conditions derived above. At the present time the deviatoric stress within the Earth is not well enough known to be specified everywhere with any precision; because of this, $\boldsymbol{\tau}^0$ is commonly ignored in quantitative global seismology. The procedure for doing this is straightforward: the hydrostatic equations of motion and boundary conditions are simply assumed to apply more generally to a non-hydrostatic Earth model.

We shall refer to this subterfuge as the *quasi-hydrostatic approximation*. From a practical perspective, such an approximation is advantageous, because it rids the theory of any explicit dependence upon the initial stress. A quasi-hydrostatic, laterally heterogeneous Earth model is fully specified by giving its density ρ^0, rotation rate $\mathbf{\Omega}$, and elastic tensor $\mathbf{\Gamma}$. From a mathematical point of view, the best that can be said is that the procedure is self-consistent, and correct to zeroth order in the dimensionless ratio $\|\boldsymbol{\tau}^0\|/\mu$ where μ is the rigidity. The magnitude $\|\boldsymbol{\tau}^0\|$ of the deviatoric stress in the Earth's lithosphere is constrained by laboratory rock-strength measurements to be less than 0.5 GPa, whereas the lithospheric rigidity μ is of order 50 GPa, so that $\|\boldsymbol{\tau}^0\|/\mu \ll 10^{-2}$. The observed lateral heterogeneity of the Earth's rigidity $\delta\mu/\mu$, due to temperature and lithological variations, is several times greater than this; because of this, the quasi-hydrostatic approximation should be a reasonably good one.

14.1 Hydrostatic Ellipticity

The shape of a slowly rotating planet in hydrostatic equilibrium is described by the classical theory of Clairaut (1743). Modern accounts of this venerable topic may be found in Jeffreys (1970) and Bullen (1975). We give our own brief summary of hydrostatic equilibrium theory here, following the treatment by Chandrasekhar & Roberts (1963). All of our considerations are valid to first order in the centrifugal-to-gravitational-force ratio $\Omega^2 a^3/GM$, where Ω is the sidereal rate of rotation, a and M are the mean radius and mass of the Earth, and G is the gravitational constant. Higher-order hydrostatic perturbation theories have been developed for application to rapidly rotating planets such as Jupiter and Saturn (Zharkov 1978); however, this increased level of sophistication is not required for terrestrial seismological applications.

14.1.1 Clairaut's equation

We demonstrated in Section 3.11.1 that the surfaces of constant density and geopotential must coincide in a rotating body in hydrostatic equilibrium. Classical hydrostatic figure theory addresses the question—how are the level surfaces of an initially spherical non-rotating model perturbed by a slow rotation $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$? The initial internal gravitational potential $\Phi(r)$ is related to the initial density $\rho(r)$ by

$$\Phi(r) = -4\pi G \left(\frac{1}{r} \int_0^r \rho' \, r'^2 dr' + \int_r^a \rho' \, r' dr' \right), \tag{14.1}$$

where the prime denotes evaluation at the dummy integration variable r'. We temporarily express the perturbations in the form

$$\rho(r) \to \rho(r) + \delta\rho(r) P_2(\cos\theta), \tag{14.2}$$

$$\Phi(r) \to \Phi(r) + \delta\Phi(r) P_2(\cos\theta), \tag{14.3}$$

where θ is the colatitude and $P_2(\cos\theta)$ is the Legendre polynomial of degree two. The geopotential $\Phi(r) + \delta\Phi(r)P_2(\cos\theta) + \psi(r,\theta)$ in the rotating Earth model is the sum of the gravitational potential and the centrifugal potential $\psi = -\frac{1}{3}\Omega^2 r^2[1 - P_2(\cos\theta)]$. A point r, θ on an initially spherical level surface will move radially inward or outward to a new position:

$$r \to r[1 - \tfrac{2}{3}\varepsilon(r) P_2(\cos\theta)]. \tag{14.4}$$

The constancy of the density and geopotential on the perturbed level surfaces (14.4) is guaranteed by the first-order conditions

$$\delta\rho = \tfrac{2}{3} r\varepsilon\dot{\rho}, \qquad \delta\Phi = \tfrac{2}{3}(r\varepsilon g - \tfrac{1}{2}\Omega^2 r^2), \tag{14.5}$$

where a dot denotes differentiation with respect to radius r, and where $g(r) = \dot{\Phi}(r) = 4\pi G r^{-2} \int_0^r \rho' \, r'^2 dr'$ is the unperturbed, spherically symmetric acceleration of gravity. We seek to determine the radial dependence of the *hydrostatic ellipticity* or *flattening* $\varepsilon(r)$ of the perturbed Earth in the range $0 \leq r \leq a$. Geometrically, the quantity $\varepsilon(r)$ is the fractional difference $(r_{\mathrm{equator}} - r_{\mathrm{pole}})/r$ between the equatorial and polar radii of the level surface of mean radius r.

Equation (14.5) gives us one relation between the potential perturbation $\delta\Phi$ and the ellipticity; we may obtain another by solving the perturbed version of Poisson's equation

$$\delta\ddot{\Phi} + 2r^{-1}\delta\dot{\Phi} - 6r^{-2}\delta\Phi = \tfrac{8}{3}\pi G r \varepsilon \dot{\rho}, \tag{14.6}$$

subject to the boundary conditions

$$[\delta\Phi]_-^+ = 0 \quad \text{and} \quad [\delta\dot{\Phi} - \tfrac{8}{3}\pi G r \varepsilon \rho]_-^+ = 0 \tag{14.7}$$

at the internal and external discontinuity radii $r = d$. Equations (14.6) and (14.7) can be manipulated and combined into a single equation of the form

$$\left[\frac{d}{dr} - \frac{1}{r}\right]\left[\frac{1}{r^3}\frac{d}{dr}(r^3\delta\Phi)\right] = \tfrac{8}{3}\pi G\left\{ r\varepsilon\dot{\rho} + \sum_d d\varepsilon[\rho]_-^+\,\delta(r-d)\right\}, \tag{14.8}$$

where the sum is over all of the discontinuities, and $\delta(r-d)$ is the Dirac delta distribution. Upon integrating (14.8) once we deduce that

$$\frac{1}{r^3}\frac{d}{dr}(r^3\delta\Phi) = -\tfrac{8}{3}\pi G r\left\{\int_r^a \varepsilon'\dot{\rho}'\,dr' + \sum_{d>r}\varepsilon[\rho]_-^+\right\}, \tag{14.9}$$

where the sum is over only those discontinuities d lying above the radius r. An additional integration by parts yields the explicit result

$$\delta\Phi(r) = \tfrac{8}{15}\pi G\left[\frac{1}{r^3}\int_0^r \rho'(r'\dot{\varepsilon}' + 5\varepsilon')\,r'^4 dr' + r^2\int_r^a \rho'\dot{\varepsilon}'\,dr'\right]. \tag{14.10}$$

The similarity between the relations (14.1) and (14.10) is noteworthy; both Φ and $\delta\Phi$ contain a separate integral contribution from the shells of matter below and above the radius r.

Straightforward differentiation of the second hydrostatic relation (14.5) yields the result

$$\frac{1}{r^3}\frac{d}{dr}(r^3\delta\Phi) = \tfrac{2}{3}\left[\frac{1}{r^3}\frac{d}{dr}(r^4\varepsilon g) - \tfrac{5}{2}\Omega^2 r\right]. \tag{14.11}$$

Upon equating (14.11) and the intermediate result (14.9) we obtain

$$4\pi G\left\{\int_r^a \varepsilon'\dot{\rho}'\,dr' + \sum_{d>r}\varepsilon[\rho]_-^+\right\} = -\frac{1}{r^4}\frac{d}{dr}(r^4\varepsilon g) + \tfrac{5}{2}\Omega^2. \tag{14.12}$$

Differentiation of this relation leads, after some algebra, to a second-order ordinary differential equation known as *Clairaut's equation*:

$$\ddot{\varepsilon} + 8\pi G\rho g^{-1}(\dot{\varepsilon} + r^{-1}\varepsilon) - 6r^{-2}\varepsilon = 0. \tag{14.13}$$

Clairaut's equation can be solved for the ellipticity ε, subject to the two restrictions at the endpoints

$$\dot{\epsilon}_0 = 0, \qquad \dot{\epsilon}_a = a^{-1}(\tfrac{5}{2}\Omega^2 a^3/GM - 2\varepsilon_a), \tag{14.14}$$

where the subscripts denote evaluation at $r = 0$ and $r = a$, respectively. The first condition (14.14) is an obvious smoothness requirement at the center of the Earth, whereas the second is the limiting value of (14.12) at the Earth's surface.

14.1.2 Radau's approximation

It is possible to integrate Clairaut's equation numerically for a given density profile $\rho(r)$, using a shooting method. For seismological purposes, however, a remarkably accurate analytical approximation due to Radau (1885) is sufficient. Defining the auxiliary variable

$$\eta = (d\ln\varepsilon)/(d\ln r) = r\dot{\varepsilon}/\varepsilon, \tag{14.15}$$

we rewrite (14.13) in the alternative form

$$\frac{d}{dr}\left(r^4 g\sqrt{1+\eta}\right) = 5gr^3 f(\eta), \tag{14.16}$$

where

$$f(\eta) = \frac{1 + \frac{1}{2}\eta - \frac{1}{10}\eta^2}{\sqrt{1+\eta}}. \tag{14.17}$$

We focus our attention upon the variability of the function (14.17), noting first that it has a minimum value at the endpoint $f(0) = 1$ and a maximum value $f(1/3) = 1.00074$. The range of the dimensionless independent variable is $\eta_0 \leq \eta \leq \eta_a$ where $\eta_0 = 0$ and $\eta_a = \frac{5}{2}\varepsilon_a^{-1}(\Omega^2 a^3/GM) - 2$. Making use of the still-to-be-determined fact that the Earth's surface ellipticity is $\varepsilon_a \approx 1/300$, we find that $\eta_a \approx 0.59$ and $f(\eta_a) \approx 0.99961$. We conclude from this rudimentary analysis that, within the Earth, $f(\eta)$ never differs from unity by more than a few parts in 10,000. As a result, we can make *Radau's approximation* and replace the second-order differential equation (14.16) by

$$\frac{d}{dr}\left(r^4 g\sqrt{1+\eta}\right) \approx 5gr^3. \tag{14.18}$$

This approximate relation can be readily integrated once with the result

$$\eta(r) \approx \tfrac{25}{4}\left(1 - \frac{\displaystyle\int_0^r \rho'\, r'^4 dr'}{\displaystyle r^2 \int_0^r \rho'\, r'^2 dr'}\right)^2 - 1. \tag{14.19}$$

The ellipticity is given in terms of the function (14.19) by

$$\varepsilon(r) \approx \varepsilon_a \exp\left(-\int_r^a \eta'\, r'^{-1} dr'\right). \tag{14.20}$$

From equation (14.14) we deduce that the hydrostatic surface ellipticity is $\varepsilon_a = \frac{5}{2}\Omega^2 a g_a^{-1}(\eta_a + 2)^{-1}$, or, equivalently,

$$\varepsilon_a \approx \frac{10\,\Omega^2 a^3/GM}{4 + 25(1 - \frac{3}{2}I/Ma^2)^2}, \tag{14.21}$$

where

$$I = \tfrac{8}{3}\pi \int_0^a \rho\, r^4 dr. \tag{14.22}$$

The quantity (14.22) is readily recognized to be the *mean moment of inertia* of the Earth.

Spherically symmetric Earth models are constrained to have the observed mean radius $a = 6371$ km, mass $M = 5.974 \times 10^{24}$ kg and moment of inertia $I = 0.3308\, Ma^2$ (Romanowicz & Lambeck 1977). All such models have the same hydrostatic surface ellipticity:

$$\varepsilon_a^{\mathrm{hyd}} = 1/299.8. \tag{14.23}$$

This is 0.5 percent smaller than the observed flattening of the best-fitting ellipsoid:

$$\varepsilon_a^{\mathrm{obs}} = 1/298.3. \tag{14.24}$$

This discrepancy—often referred to as an "excess equatorial bulge" of the Earth—was the first major discovery of artificial satellite geodesy (Henriksen 1960; Jeffreys 1963). Figure 14.1 shows the radial variation of the Radau ellipticity (14.20) and its logarithmic derivative (14.19). The corresponding results obtained by numerical integration of Clairaut's equation are virtually indistinguishable on this scale.

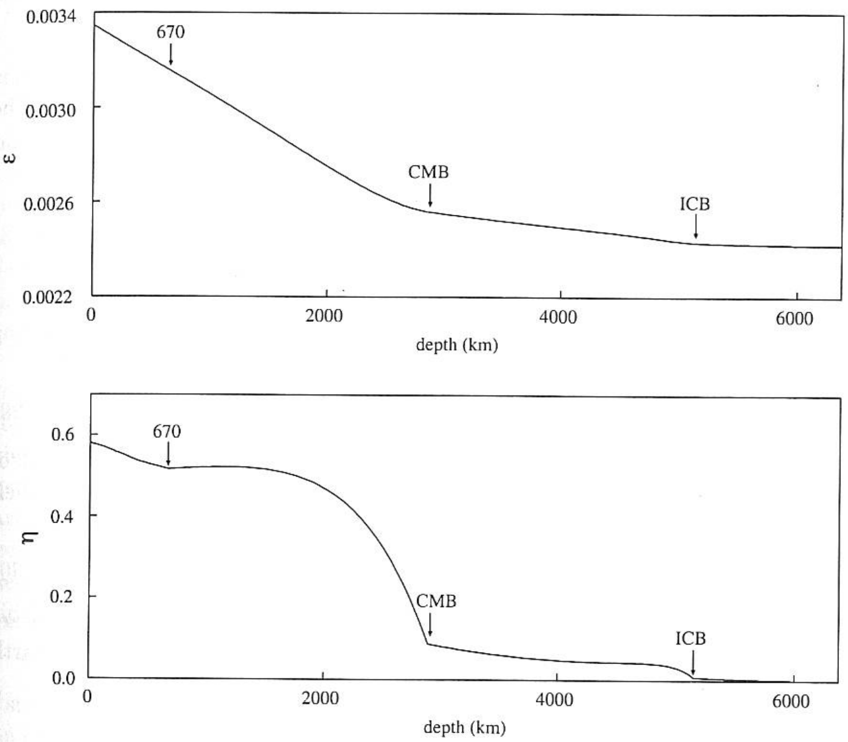

Figure 14.1. Depth dependence of the hydrostatic ellipticity ε (*top*) and its logarithmic derivative $\eta = r\dot{\varepsilon}/\varepsilon$ (*bottom*) within the PREM model. The locations of the 670 km discontinuity, the core-mantle boundary (CMB) and the inner-core boundary (ICB) are indicated.

14.1.5 Geographic versus geocentric colatitude

The location of a seismic station situated upon the Earth's surface is conventionally specified by giving its elevation e above the geoid, its longitude ϕ with respect to Greenwich, and its *geographical colatitude* θ', which is the angle between the normal $\hat{\mathbf{n}}$ to the reference ellipsoid and the normal $\hat{\mathbf{z}}$ to the equatorial plane. Prior to calculating a synthetic accelerogram or spectrum by mode summation, it is necessary to convert the geographic colatitude θ' to the corresponding *geocentric colatitude* θ, which is the angle between the radius vector $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$. Correct to first order in the ellipticity, the two colatitudes are related by

$$\tan\theta \approx (1 + 2\varepsilon_{\mathrm{a}})\tan\theta'. \tag{14.32}$$

Geometrically, the transformation $\theta' \to \theta$ projects a point on the reference ellipsoid to a point on the unperturbed sphere along a line through the origin. The original and projected points in Figure 14.2 are the feet of the unit normals $\hat{\mathbf{n}}$ and $\hat{\mathbf{r}}$, respectively. The hypocentral location of an earthquake source is likewise specified by giving its depth h beneath the geoid, its longitude ϕ_{s} and its geographical colatitude θ'_{s}. In calculating the real receiver and source vectors $r_k = \hat{\boldsymbol{\nu}} \cdot \mathbf{s}_k(\mathbf{x})$ and $s_k = \mathbf{M} : \boldsymbol{\varepsilon}_k(\mathbf{x}_{\mathrm{s}})$ we stipulate that

$$\mathbf{x} = (a, \theta, \phi), \qquad \mathbf{x}_{\mathrm{s}} = (a - h, \theta_{\mathrm{s}}, \phi_{\mathrm{s}}). \tag{14.33}$$

A similar geographic-to-geocentric coordinate transformation must precede the calculation of a spherical-Earth seismogram or spectrum, using equa-

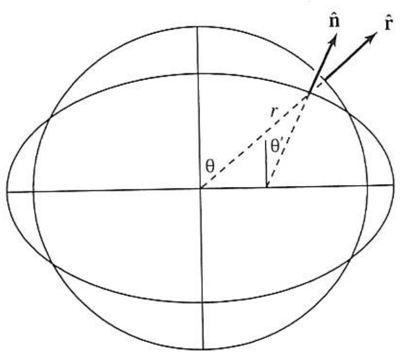

Figure 14.2. Cartoon illustrating the relation between the geographical colatitude θ and the geocentric colatitude θ'. The vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{r}}$ are the unit normals to the hydrostatic ellipsoid and the undeformed (equal-mass) sphere, respectively. Note that $\theta \geq \theta'$, with equality prevailing only at the poles and the equator.

tions (10.51)–(10.65). Station elevations e are generally ignored in normal-mode and long-period surface-wave seismology; however, they are routinely accounted for in short-period body-wave travel-time investigations.