ENGG1003 - Thursday Week 8

Numerical integration

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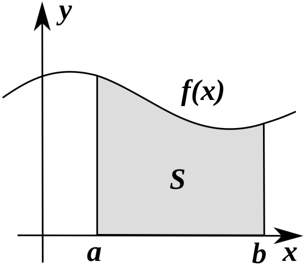
Lecture overview

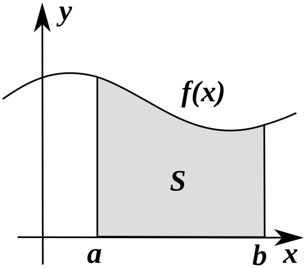
1. Basic ideas of integration §6.1
2. Trapezoidal method §6.2
3. Simpson's rule

1) Basic ideas of integration



$$\boxed{S = \int_a^b f(x)dx}$$

- interested in calculating shaded area S under function $f(x)$ between a and b
  - S is the *definite integral* of f over $[a, b]$



$$\boxed{S = \int_a^b f(x)dx}$$

- for some choices of f, possible to use calculus to compute S exactly
  - eg: MATH1002, MATH1110
  - numerical methods in this lecture apply even when f is hard (or impossible!) to integrate
- assume $f(x) \geq 0$

Many applications of integration

- area between curves
- **distance, velocity**, acceleration
- volume of solids
- average value of a function
- work done by a variable force
- center of mass
- kinetic energy
- probability
- arc length
- surface area

Distance = area under speed-time function

- accelerate a car from rest
- car speed v depends on time t, write: $v(t)$

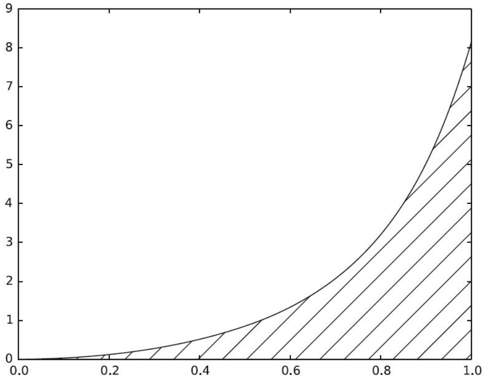
  Q: how far does the car travel in $T = 1$ seconds?

Distance traveled in T seconds given by the integral:

$$\int_0^T v(t)dt$$

Example:

$$v(t) = 3t^2 e^{t^3}$$



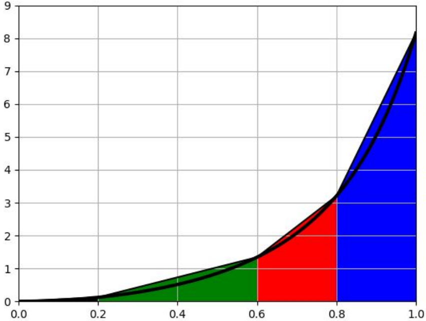
Distance traveled in first second is cross-hatched area:

$$\int_0^1 v(t)dt$$

- start at time 0, lower limit of integration
- end at time 1, upper limit of integration

2) Trapezoidal method

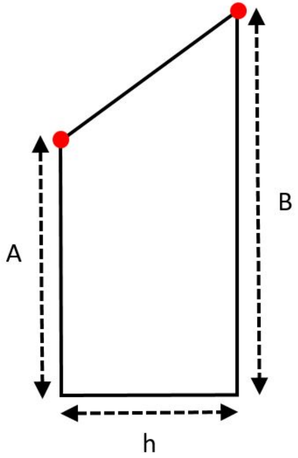
Example:



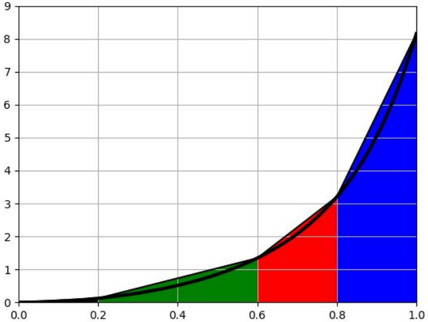
- approximate area under curve by total area of four trapezoids
  - black + green + red + blue
- area of each trapezoid is easy to calculate

Area of trapezoid

- area of trapezoid $= h \cdot \frac{A+B}{2}$

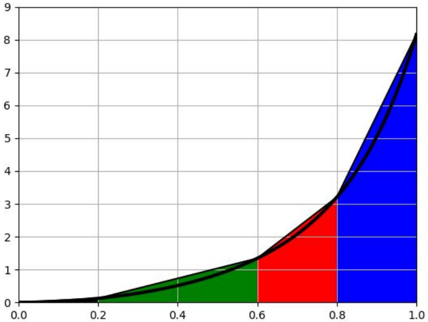


fourPanels.py



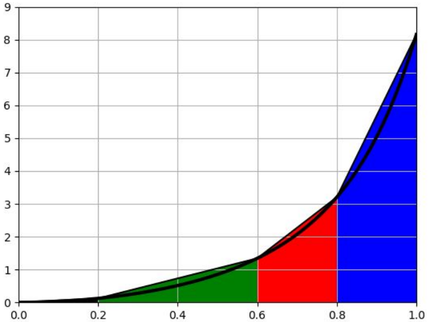
$$\int_0^1 v(t)dt \;=\; \int_0^{0.2} v(t)dt + \int_{0.2}^{0.6} v(t)dt + \int_{0.6}^{0.8} v(t)dt + \int_{0.8}^{1} v(t)dt$$

fourPanels.py



$$
\begin{aligned}
\int_0^1 v(t)dt &= \int_0^{0.2} v(t)dt + \int_{0.2}^{0.6} v(t)dt + \int_{0.6}^{0.8} v(t)dt + \int_{0.8}^{1} v(t)dt \\
&\approx h_1 \frac{v(0)+v(0.2)}{2} \quad + \quad h_2 \frac{v(0.2)+v(0.6)}{2} \\
&\quad + \quad h_3 \frac{v(0.6)+v(0.8)}{2} \quad + \quad h_4 \frac{v(0.8)+v(1)}{2}
\end{aligned}
$$

fourPanels.py



$$\int_0^1 v(t)dt = \int_0^{0.2} v(t)dt + \int_{0.2}^{0.6} v(t)dt + \int_{0.6}^{0.8} v(t)dt + \int_{0.8}^{1} v(t)dt$$

$$\approx h_1 \frac{v(0)+v(0.2)}{2} + h_2 \frac{v(0.2)+v(0.6)}{2}$$

$$+ h_3 \frac{v(0.6)+v(0.8)}{2} + h_4 \frac{v(0.8)+v(1)}{2}$$

$$h_1 = 0.2, \quad h_2 = 0.4, \quad h_3 = 0.2, \quad h_4 = 0.2$$

General trapezoidal method

- want to approximate integral $\int_a^b f(x)dx$ by n trapezoids *of equal width*
  - total of n intervals: $[x_0, x_1]$, $[x_1, x_2]$, $\ldots [x_{n-1}, x_n]$

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

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$$\approx h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + \cdots + h\frac{f(x_{n-1}) + f(x_n)}{2}$$

General trapezoidal method

- want to approximate integral $\int_a^b f(x)dx$ by n trapezoids *of equal width*
  - total of n intervals: $[x_0, x_1]$, $[x_1, x_2]$, $\ldots [x_{n-1}, x_n]$

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$\approx h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + \cdots + h\frac{f(x_{n-1}) + f(x_n)}{2}$$

In compact form:

$$\int_a^b f(x)dx \approx h\left[\frac{1}{2}f(x_0) + \{f(x_1) + \cdots + f(x_{n-1})\} + \frac{1}{2}f(x_n)\right]$$

Python code for trapezoidal method

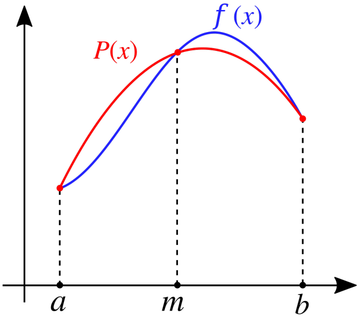
trapezoidal_method.py

```
import numpy as np

def v(t):
    return 3*t**2*np.exp(t**3)

def trapezoidal(f, a, b, n):
    h = (b-a)/n
    f_sum = 0
    for i in range(1, n, 1):
        x = a + i*h
        f_sum = f_sum + f(x)
    return h*(0.5*f(a) + f_sum + 0.5*f(b))

n = 4
trap = trapezoidal(v, 0, 1, n)
exact = np.exp(1) - 1

print('Trapezoidal, {} sub-intervals: {:.8f}'.format(n,trap))
print('Exact answer: {:.8f}'.format(exact))
```

Trapezoidal method: simulation results

- lines 3–4: function to be integrated
- lines 6–12: function to approximate integral using n trapezoids of equal width h
  - lines 8–11: compute $f(x_1) + \cdots f(x_{n-1})$
- line 16: exact result $\int_0^1 3t^2 e^{t^3} dt = e - 1$
- live demo: try $n = 4$, $n = 100$ and $n = 1000$ sub-intervals

3) Simpson's rule



- approximate $f(x)$ with parabola $P(x)$
- parabola $P(x)$ takes same values as $f(x)$ at end-points a and b, and midpoint $m = (a+b)/2$

Simpson's rule

- area under parabola $P(x)$ between a and b is:

$$\int_a^b P(x)dx$$

...which can be calculated *exactly* for a parabola (proof omitted):

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Applying Simpson's rule

- as for trapezoidal rule, apply Simpson's rule on each sub-interval of width $h = x_i - x_{i-1} = (b-a)/n$
  - total of n intervals: $[x_0, x_1]$, $[x_1, x_2]$, $\ldots [x_{n-1}, x_n]$

$$\int_a^b f(x)dx = \underbrace{\int_{x_0}^{x_1} f(x)dx}_{\text{apply Simpson's rule}} + \underbrace{\int_{x_1}^{x_2} f(x)dx}_{\text{apply Simpson's rule}} + \cdots + \underbrace{\int_{x_{n-1}}^{x_n} f(x)dx}_{\text{apply Simpson's rule}}$$

Python code for Simpson's rule

simpsons_rule.py

```
import numpy as np

def v(t):
    return 3*t**2*np.exp(t**3)

def simpson(f, a, b, n):
    h = (b-a)/n
    x0 = a
    summ = 0
    # i-th sub-interval (i=0,1,...,n-1) is [x_i,x_{i+1}]
    for i in range(0,n,1):
        xi = x0 + i*h
        summ += (f(xi) + 4*f((xi+xi+h)/2) + f(xi+h))*h/6
    return summ

n = 4
simp = simpson(v, 0, 1, n)
exact = np.exp(1) - 1

print('Simpson, {} sub-intervals: {:.8f}'.format(n,simp))
print('Exact answer: {:.8f}'.format(exact))
```

Simpson's rule: simulation results

- lines 3–4: function to be integrated
- lines 6–14: function to approximate integral using Simpson's rule on each of n sub-intervals $[x_i, x_{i+1}], i = 0, 1, 2, \ldots, n-1$
  - line 13: Simpson's rule with $a = x_i$ and $b = x_{i+1}$
- live demo: try $n = 4$, $n = 10$ and $n = 100$ sub-intervals

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