ENGG1003 - Thursday Week 9

Random numbers from normal distributions —aka random numbers from Gaussian distributions

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Lecture overview

1. normal distribution
   - also known as *Gaussian distribution* or "bell curve"
   - today: draw random samples from "standard" normal distribution
2. compute probabilities using normal distribution
   - uses numerical integration

1) Normal distribution

- intoduced *uniformly* distributed random numbers in week 4

- today: introduce "standard" normal distribution
  - extend next week to general form
  - also known as random numbers from *Gaussian distribution*
  - occur *very* widely in all branches of Engineering

- two goals today using standard normal distribution:
  1. use Python to draw random samples
  2. use Python to compute probability of random number falling in specified range

Standard normal distribution

- generate $100,000$ random numbers using `normal()` function in numpy's random library
  - standard normal: first two parameters in call to `normal()` are 0.0 and 1.0

```
x = np.random.normal(0.0, 1.0, size=100000)
```

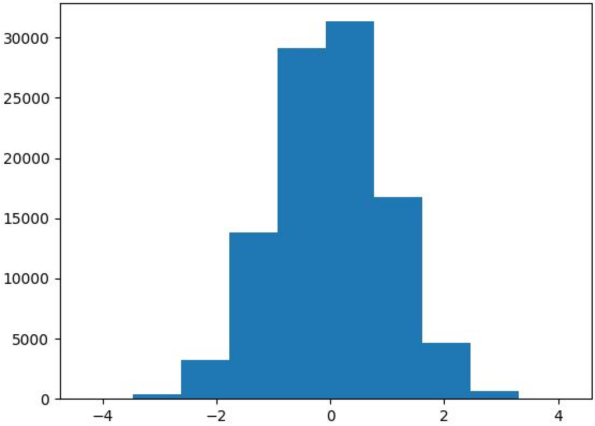
- use `hist()` function in matplotlib library to generate histograms of observed data
  - 10 bins
  - 100 bins

Python code: generate histograms

histdemo.py

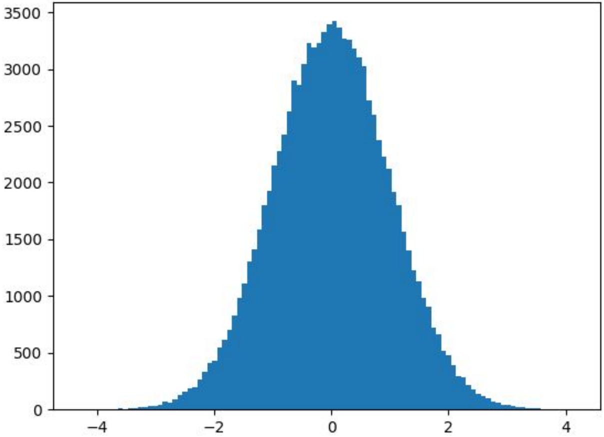
```
# histdemo
import numpy as np
import matplotlib.pyplot as plt

np.random.seed(1)
d = np.random.normal(0.0, 1.0, size=100000)

x = np.linspace(-5,5,num=1000)
f = 1/(np.sqrt(2 * np.pi)) * np.exp(-x**2 / 2)

plt.hist(d, 100, density=True)
plt.plot(x, f, color='r', linewidth=3)
#plt.hist(d, 100)

#plt.plot(d, 'o')
plt.show()
```

Histogram: 10 bins



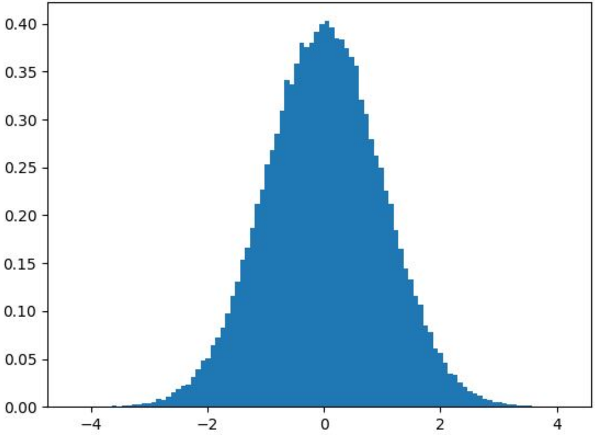
- height of each rectangle refects number of samples in each "bin"

Histogram: 100 bins



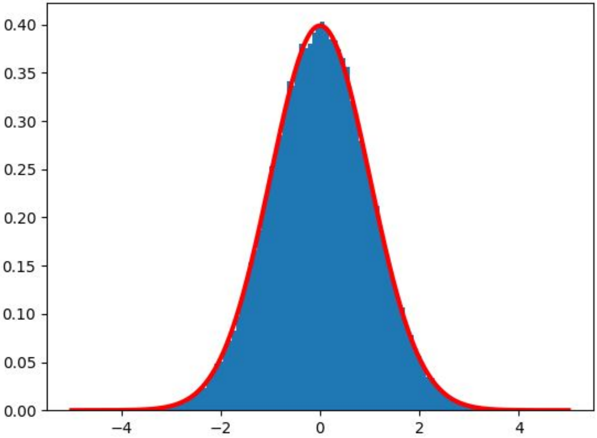
- identical data set as for 10 bins

Normalized histogram (area 1), 100 bins



- same histogram, except total area of rectangles is normalized to be 1

Normalized histogram with PDF



red curve is *probability density function (PDF)*

Standard normal distribution

Standard normal probability density function:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$$

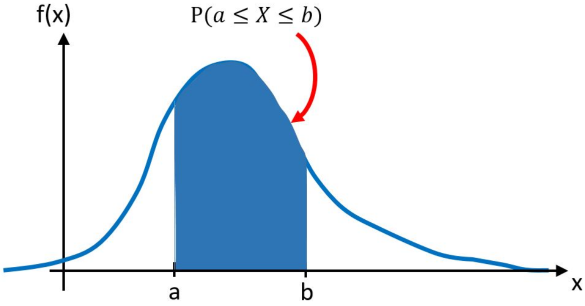
- *standard* normal distribution is a special case of distribution we'll see next week
- corresponds to parameters $\mu = 0$ and $\sigma = 1$

```
x = np.random.normal(0.0, 1.0, size=100000)
```

Probability density functions

If X is a random number drawn from a distribution with PDF $f(x)$, probability X takes a value in interval $[a, b]$ is

$$\boxed{\mathrm{P}(a \leq X \leq b) = \int_a^b f(x)dx}$$



Properties of PDFs

Two key properties of any probability density function:

1. **non-negativity**

   $f(x) \geq 0$ for all x

   - since probability can't be negative

2. **normalization**

   entire area under $f(x)$ must be equal to 1, since

   $$P(-\infty \leq X \leq \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$$

   - reason for the $1/\sqrt{2\pi}$ factor in standard normal PDF

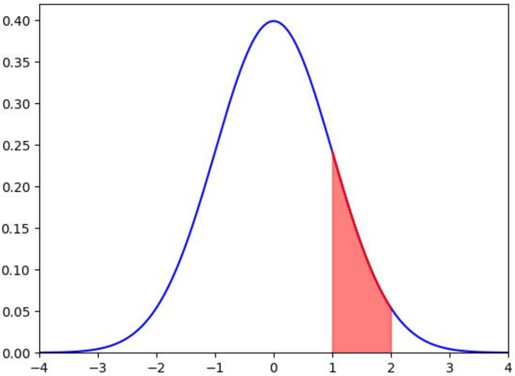
2) Computing probabilities using standard normal distribution

- probability of random number X drawn from standard normal distribution taking value in $[a, b]$:

$$\mathrm{P}(a \leq X \leq b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$$

- no exact expression exists for $\int_a^b e^{-x^2/2} dx$
  - need to use numerical integration!
- **Example:** use trapezoidal method to approximate $\mathrm{P}(1 \leq X \leq 2)$ when X is drawn from standard normal distribution

Example: fraction of standard normal numbers in $[1, 2]$



Red shaded area $= \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_1^2 e^{-x^2/2} dx \approx 0.1359$

Python code: fraction of numbers in $[1, 2]$

standardnormal.py

```
1  # standardnormal
2  import numpy as np
3  import matplotlib.pyplot as plt
4
5  def f(x):
6      return 1/(np.sqrt(2 * np.pi)) * np.exp(-x**2 / 2)
7
8  def trapezoidal(f, a, b, n):
9      h = (b-a)/n
10     f_sum = 0
11     for i in range(1, n, 1):
12         x = a + i*h
13         f_sum = f_sum + f(x)
14     return h*(0.5*f(a) + f_sum + 0.5*f(b))
```

- lines 5–6: PDF of standard normal distribution

Python code

`standardnormal.py`—continued

```
1 a = 1
2 b = 2
3 prob_ab = trapezoidal(f, a, b, 100)
4 print('Probability X in range [{},{}] is: {:.4f}'.format(a, b, prob_ab))
5
6 x = np.linspace(-4, 4, 1000)
7 xab = np.linspace(a, b, 100)
8
9 plt.plot(x, f(x), 'b')                    # standard normal pdf
10 plt.plot(xab,f(xab),'r')
11 plt.fill_between(xab,f(xab),color='r',alpha=0.5) #alpha=transparency
12 plt.axis([-4, 4, 0, 0.42])
13 plt.show()
```

- line 3: approximate $\frac{1}{\sqrt{2\pi}} \int_1^2 e^{-x^2/2} dx$, 100 panels
- line 7: $1 \leq x \leq 2$ for red shaded area plot

Live demo: standard normal generation

- draw $N = 10^6$ random numbers from standard normal distribution
- for large N, expect fraction of random numbers X in range $[1, 2]$ to be close to

$$\mathrm{P}(1 \leq X \leq 2) = \frac{1}{\sqrt{2\pi}} \int_1^2 e^{-x^2/2} dx \approx 0.1359$$

- live demo of `standardnormaldemo.py`
- observed fraction ≈ 0.136

Python code

standardnormaldemo.py

```
1 # standardnormaldemo
2 import numpy as np
3
4 N = 1000000
5 x = np.random.normal(0.0, 1.0, size=N)
6 a = 1
7 b = 2
8 num_ab = 0
9 for k in range(0, len(x)):
10     if a <= x[k] <= b:
11         num_ab += 1
12
13 print('{} standard normal random numbers'.format(N))
14 print('Fraction of random numbers in range [{},{}] = {:.4f}'.
       format(a, b, num_ab/N))
```

- lines 8–12: `for` loop counts number of random numbers x satisfying $1 \leq x \leq 2$

Lecture summary

1. standard normal distribution
   - drawing `N` random samples using `numpy.random.normal(0.0, 1.0, size=N)`
2. compute probability $\mathrm{P}(a \leq X \leq b)$ for standard normal distribution
   - needs numerical integration using PDF

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$$