8.1. **a)** Für die "äußere Entwicklung" verwenden wir den Reihenansatz wie bei regulär gestörten Problemen: $y = \sum_{i=0}^{\infty} \varepsilon^i y_i$, setzen in die Gleichung ein und machen einen Koeffizientenvergleich bezüglich ε:

$$\sum_{i=0}^{\infty} \varepsilon^{i+1} y_i'' + 2\sum_{i=0}^{\infty} \varepsilon^i y_i' + \sum_{i=0}^{\infty} \varepsilon^i y_i = e^x$$

$$2y_0' + y_0 = e^x, \quad y_0(1) = 0$$

$$2y_i' + y_i = -y_{i-1}'', \quad y_i(1) = 0, \quad \forall i \geq 1$$

b) Wir führen die Grenzschichtvariable $\xi = \frac{x}{\varepsilon}$ ein. (Warum nicht ε^α?)

$$Y(\xi) = y(x) \Rightarrow \frac{1}{\varepsilon} Y'(\xi) = y'(x), \frac{1}{\varepsilon^2} Y''(\xi) = y''(x)$$

$$\varepsilon^{-1} Y'' + 2\varepsilon^{-1} Y' + Y = e^{\varepsilon\xi} = \sum_{i=0}^{\infty} \frac{(\varepsilon\xi)^i}{i!}$$

$$Y(\xi) = \sum_{i=0}^{\infty} \varepsilon^i Y_i$$

$$\sum_{i=0}^{\infty} \varepsilon^{i-1} Y_i'' + 2\sum_{i=0}^{\infty} \varepsilon^{i-1} Y_i' + \sum_{i=0}^{\infty} \varepsilon^i Y_i = \sum_{i=0}^{\infty} \frac{(\varepsilon\xi)^i}{i!}$$

$$Y_0'' + 2Y_0' = 0, \quad Y_0(0) = 0$$

$$Y_1'' + 2Y_1' + Y_0 = 1, \quad Y_1(0) = 0$$

$$Y_i'' + 2Y_i' + Y_{i-1} = \frac{(\varepsilon\xi)^i}{i!}, \quad Y_i(0) = 0$$

c) Ansatz $y_0 = ae^x$:

$$2ae^x + ae^x = e^x \Rightarrow a = \frac{1}{3} \Rightarrow y_{0p} = \frac{1}{3} e^x$$

Charakteristisches Polynom für die homogene Lösung:

$$2\lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{2}$$

$$0 = y_0(1) = y_{0p}(1) + b y_{0h}(1) = \frac{1}{3} e + b e^{-\frac{1}{2}} \Rightarrow b = -\frac{1}{3} e^{\frac{3}{2}}$$

$$y_0 = \frac{1}{3} e^x - \frac{1}{3} e^{\frac{3}{2} - \frac{1}{2}x}$$

$$y_0'' = \frac{1}{3} e^x - \frac{1}{12} e^{\frac{3}{2} - \frac{1}{2}x}$$

$$2y_1' + y_1 = -\frac{1}{3}e^x + \frac{1}{12}e^{\frac{3}{2}-\frac{1}{2}x}$$

Ansatz:

$$y_{1p} = c_1 e^x + c_2 x e^{-\frac{1}{2}x}$$

$$2c_1 e^x - c_2 x e^{-\frac{1}{2}x} + 2c_2 e^{-\frac{1}{2}x} + c_1 e^x + c_2 x e^{-\frac{1}{2}x} = -\frac{1}{3}e^x + \frac{1}{12}e^{\frac{3}{2}-\frac{1}{2}x}$$

$$3c_1 = -\frac{1}{3} \Rightarrow c_1 = -\frac{1}{9}$$

$$2c_2 = \frac{1}{12}e^{\frac{3}{2}} \Rightarrow c_2 = \frac{1}{24}e^{\frac{3}{2}}$$

$$0 = y_1(1) = y_{1p}(1) + b y_{1h}(1) = -\frac{1}{9}e + \frac{1}{24}e + be^{-\frac{1}{2}}$$

$$\Rightarrow b = \frac{5}{72}e^{\frac{3}{2}}$$

$$y_1 = -\frac{1}{9}e^x + (3x+5)\frac{1}{72}e^{\frac{3}{2}-\frac{1}{2}x}$$

Lösungen für Y_0:

$$Y_{0,a} = a$$

Charakteristisches Polynom für Y_0:

$$\lambda^2 + 2\lambda = 0 \Rightarrow Y_{0,b} = e^{-2\xi}$$

$$0 = Y_0(0) = a + be^{-2\cdot 0} \Rightarrow Y_0 = a(1 - e^{-2\xi})$$

$$Y_1'' + 2Y_1' = 1 + a(e^{-2\xi} - 1)$$

$$Y_1(\xi) = -\frac{1}{2}ae^{-2\xi}\xi - \frac{a\xi}{2} - \frac{1}{4}c_1 e^{-2\xi} + c_2 + \frac{\xi}{2}$$

$$0 = Y_1(0) = -\frac{c_1}{4} + c_2 \Rightarrow c_2 = \frac{1}{4}c_1$$

Wolframalpha anscheinend:

$$Y_1 = -e^{-2\xi}\left(\frac{1}{2}a\xi - c_2\right) + \frac{1-a}{2}\xi + c_2$$

Matching:

$$\lim_{x\to 0} y_0(x) = \lim_{\xi\to\infty} Y_0(x) \Rightarrow \frac{1}{3}(1 - e^{\frac{3}{2}}) = a$$

$$\zeta = \frac{x}{\eta(\varepsilon)}$$

$$y_0(x) + \varepsilon y_1(x) = \frac{1}{3}e^{\eta\zeta} - \frac{1}{3}e^{\frac{3}{2}-\frac{1}{2}\eta\zeta} - \varepsilon\frac{1}{9}e^{-\zeta\eta} + \varepsilon(3\zeta\eta + 5)\frac{1}{72}e^{\frac{3}{2}-\frac{1}{2}\zeta\eta}$$

$$Y_0(\xi) + \varepsilon Y_1(\xi) = a(1 - e^{-2\frac{\zeta\eta}{\varepsilon}}) - \varepsilon e^{-2\frac{\zeta\eta}{\varepsilon}} \left(\frac{1}{2} a \frac{\zeta\eta}{\varepsilon} - c_2\right) + \varepsilon \frac{1-a}{2}\frac{\zeta\eta}{\varepsilon} + \varepsilon c_2$$

Jetzt $\eta \to 0$:

$$\lim_{\eta\to 0} y_0 + \varepsilon y_1 = \frac{1}{3} - \frac{1}{3}e^{\frac{3}{2}} - \varepsilon\frac{1}{9} + \varepsilon\frac{5}{72}e^{\frac{3}{2}}$$
$$\lim_{\eta\to 0} Y_0 + \varepsilon Y_1 = \frac{1}{3}(1 - e^{\frac{3}{2}}) + \varepsilon c_2$$
$$\Rightarrow c_2 = -\frac{1}{9} + \frac{5}{72}e^{\frac{3}{2}}$$

Uniforme Approximation: $y_0(x) + \varepsilon y_1(x) + Y_0(\frac{x}{\varepsilon}) + \varepsilon Y_1(\frac{x}{\varepsilon}) -$ gemeinsamer Teil.

8.2. **a)**

b)

c)

d)

8.3. **a)**

b)

8.4. **a)**

$$u_t = -cU', \quad u_x = U', \quad u_{xxx} = U'''$$
$$-cU' + UU' + U''' = -cU' + \frac{1}{2}(U^2)' + U''' = 0$$
$$-cU + \frac{1}{2}U^2 + U'' - A = 0$$

b)

$$\begin{pmatrix} U' \\ V' \end{pmatrix} = \begin{pmatrix} V \\ -\frac{1}{2}U^2 + cU \end{pmatrix}$$

Ruhelagen bei $(U', V') = 0$ folgt $V = 0$ und

$$U_{1,2} = c \pm c = 0, 2c$$

$$f'(U,V) = \begin{pmatrix} 0 & 1 \\ -U + c & 0 \end{pmatrix}$$

$$f'(0,0) = \begin{pmatrix} 0 & 1 \\ +c & 0 \end{pmatrix} \qquad \lambda^2 - c = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{|c|} \Rightarrow \text{Sattelpunkt}$$

$$f'(2c,0) = \begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix} \qquad \lambda^2 + c = 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{c} \Rightarrow \text{Zentrum (stabil)}$$

c)

d)

$$U'U'' + \frac{1}{2}U'U^2 - cU'U = 0$$
$$\frac{1}{2}((U')^2)' + \frac{1}{6}((U')^3)' - \frac{c}{2}(U^2)' = 0$$

8.5.

$$u_t = \left(-\frac{1}{2}u_x^2 - u_{xx}\right)_x$$
$$m_1(u)_t = \int_{-\infty}^{\infty} u_t \,\mathrm{d}x = \int_{\infty}^{\infty} \left(-\frac{1}{2}u_x^2 - u_{xx}\right) \mathrm{d}x = 0$$

$$2uu_t = -2u^2u_x - 2uu_{xxx}$$
$$\frac{1}{2}(u^2)_t = -\frac{2}{3}(u^3)_x - (2uu_{xx})_x + ((u_x)^2)_x$$